Nested Radical

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Let

$$x = \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + ...}}}}$$

Define the sequence $(a_n)_{n=1}^{\infty}$

$$a_1 = \sqrt{1} = 1 \tag{1}$$

$$a_2 = \sqrt{1 + \sqrt{1}} = \sqrt{2} \tag{2}$$

$$a_3 = \sqrt{1 + \sqrt{1 + \sqrt{1}}} = \sqrt{1 + \sqrt{2}} \tag{3}$$

$$a_4 = \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1}}}} = \sqrt{1 + \sqrt{1 + \sqrt{2}}} \tag{4}$$

$$\tag{5}$$

The pattern continues and satisfies the recurrence relation

$$a_n = \sqrt{1 + a_{n-1}}, \forall n \geq 2$$

This begs the question, Does $x = \lim_{n\to\infty} a_n$ exist? If so, what is a closed form of x?

Proof. We show that a_n is increasing monotonically. Let's proceed by induction Base Case: $n = 1$

$$a_1 < a_2$$

Induction step: Assume the induction hypothesis holds $\forall n \leq k - 1$.

$$a_{k+1} = \sqrt{1 + a_k}$$

$$a_k = \sqrt{1 + a_{k-1}}$$

Since the square root function is montonically increasing and $a_{k-1} < a_k$ by the induction hypothesis, it follows that a_n is a monotonically increasing sequence. We conclude the sequence is monotonically increasing $\forall n \in \mathbb{N}$ by the Principle of Mathematical Induction. So the sequence is increasing monotonically, but how do we know it does not trail off to infinity? We claim the sequence is bounded by 2 We prove this by mathematical indcution.
Base case: $n = 1$

$$a_1 = 1 < 2$$

Induction Step:
Assume the induction hypothesis holds $\forall n \leq k - 1$.

$$a_k = \sqrt{1 + a_{k-1}} \leq \sqrt{1 + 2} = \sqrt{3} < \sqrt{4} = 2$$

The induction hypothesis holds and we conclude that a_n is bounded for all $n \in \ltimes$.

By the Monotonic Convergence Theorem, limit exists.

$$x = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{1 + a_n} = \sqrt{\lim_{n\to\infty} 1 + a_n} = \sqrt{1 + \lim_{n\to\infty} a_n} = \sqrt{1 + x}$$

The reader is encouraged to stop at this point and ask what allowed for the interchange of the limit and the sqaure root function. Hint: think continuity. This yields:

$$x^2 - x - 1 = 0$$

Solving gives two roots, $x_1 = \frac{1-\sqrt{5}}{2}$ and $x_2 = \frac{1+\sqrt{5}}{2}$. Since a_n is strictly positive, it follows that

$$x = \frac{1 + \sqrt{5}}{2}$$

This limit is known as the "Golden Ratio". □