Nested Radical

Define

$$x := \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+...}}}}$$

Define the sequence:

$$a_1 = \sqrt{1} = 1 \tag{1}$$

$$a_2 = \sqrt{1+\sqrt{1}} = \sqrt{2} \tag{2}$$

$$a_3 = \sqrt{1+\sqrt{1+\sqrt{1}}} = \sqrt{1+\sqrt{2}} \tag{3}$$

$$a_4 = \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1}}}} = \sqrt{1+\sqrt{1+\sqrt{2}}} \tag{4}$$

The pattern continues and satisfies the recurrence relation:

$$a_n = \sqrt{1+a_{n-1}}, \forall n \geq 2$$

This begs the question, does $x = \lim_{n\to\infty} a_n$ exist? If so, what is a closed form of x?

Proof. We show that a_n is increasing monotonically. Let's proceed by induction
Base Case: $n = 1$

$$a_1 < a_2$$

Induction step: Assume the induction hypothesis holds $\forall n \leq k-1$.

$$a_{k+1} = \sqrt{1+a_k}$$

$$a_k = \sqrt{1+a_{k-1}}$$

Since the square root function is montonically increasing and $a_{k-1} < a_k$ by the induction hypothesis, it follows that a_n is a monotonically increasing sequence. We conclude the sequence is monotonically increasing $\forall n \in \mathbb{N}$ by the Principle of Mathematical Induction. The sequence is increasing monotonically, but how do we know it does not trail off to infinity? We claim the sequence is bounded by 2. We prove this by mathematical induction.
Base case: $n = 1$

$$a_1 = 1 < 2$$

Induction Step:
Assume the induction hypothesis holds $\forall n \leq k-1$.

$$a_k = \sqrt{1 + a_{k-1}} \leq \sqrt{1+2} = \sqrt{3} < \sqrt{4} = 2$$

The induction hypothesis holds and we conclude that a_n is bounded for all $n \in \mathbb{N}$.
By the Monotonic Convergence Theorem, the limit exists.

$$x = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{1+a_n} = \sqrt{\lim_{n\to\infty} 1 + a_n} = \sqrt{1 + \lim_{n\to\infty} a_n} = \sqrt{1+x}$$

The reader is encouraged to stop at this point and justify the passing of the limit inside the square root function. Hint: Think continuity. This yields:

$$x^2 - x - 1 = 0$$

Solving gives two roots, $x_1 = \frac{1-\sqrt{5}}{2}$ and $x_2 = \frac{1+\sqrt{5}}{2}$. Since a_n is strictly positive, it follows that

$$x = \frac{1+\sqrt{5}}{2}$$

This limit is known as the "Golden Ratio". □