Mathematical Modeling of Plastic Beaching

An introduction to mathematical modeling with MATLAB

Name: ________________

This outreach activity was prepared by Benjamin Davidson and reviewed by Nimish Pujara. The activity is created for use by the Fluid Mechanics in Environmental Processes group at the University of Wisconsin - Madison. This activity was first conducted with students from Madison East High School, on April 9, 2024. The activity is based upon the code and data from:

Davidson, B., Brenner, J., Pujara, N., 2023. Beaching model for buoyant marine debris in bore-driven swash. Flow 3. E35 https://doi.org/10.1017/flo.2023.31.

Pre-survey

- What math classes have you taken (or are currently taken)?
- What science classes have you taken (or are currently taking)?
- Why did you want to come on the trip today?
- Write down anything you currently know about mathematical modeling:

1 Beach Flow Model

We will start by opening the file 'Beach_Flow_Model_1.m' in MATLAB.

Beach Set-up

In order to visualize our flow model, we will need to set up our simulated beach. We start by making a figure and adding in the beach surface with the specific slope. Click in the section 'Beach Set-up' and press 'Run Section.'

You will see a cross-profile plot of the beach.

1-1. What happens when you change the value of s?

1-2. What happens when you change the value of x_max?

Shoreline Motion

For the flow of a wave on a beach, we can start with the movement of the shoreline, which can be expressed as

$$x_s = t - \frac{1}{2}st^2. \tag{1}$$

Here, x_s is the shoreline position, t is the time, and s is the slope of the beach. Let's start by visualizing how the shoreline runs up and down the beach with an incoming wave.

We will need to fill in the equation for x_s in our code, using the equation above. Fill in line 28 with the equation of the shoreline. *If you are having trouble, or want to check your answer, go to the bottom of the page of code.*

Click in the section 'Shoreline Motion' and press 'Run Section.'

1-3. Does a steeper slope (larger s) increase or decrease the distance that water runs up the beach? Why do you think this is?

Water Surface

The water depth at any point on the beach can be found with the equation:

$$h = \frac{1}{9}\left(1 - \frac{1}{2}st - \frac{x}{t}\right)^2. \tag{2}$$

We can add the water surface by adding this equation into the code from above. Fill in the equation for h into the code, then click in the section 'Water Surface' and press 'Run Section.'

You can change the scale on the y-axis by changing 'y-limit' and the slope by changing 's.'

1-4. If you decrease the slope to $s = 0.05$, what else must you do so that you see the wave run back down, off of the beach?

Water Velocity

The water velocity on the beach is given by the equation:

$$u = \frac{1}{3}\left(1 - 2st + 2\frac{x}{t}\right). \tag{3}$$

We can show the water velocity by adding this equation into the code and plotting the values as arrows. Fill in the equation for u into the code, then click in the section 'Water Velocity' and press 'Run Section.'

1-5. Where is the largest velocity?

1-6. When does the water turn around? Is it at the same time everywhere on the beach?

1-7. Assumptions

List at least three assumptions of the flow model that we are using:

1.
2.
3.

How might each of these assumptions impact how the model compares to the 'real-life' scenario?

1.
2.
3.

2 Beach Debris Modeling

In order to model beach debris transport and beaching, we need an equation for the motion that we are simulating, and we need to carefully consider the assumptions of the model. We are considering a 'point particle,' that is a particle that takes up no space. We can imagine it as a physical particle, but in the model we do not consider it to take up any actual space.

Vertical Force Balance

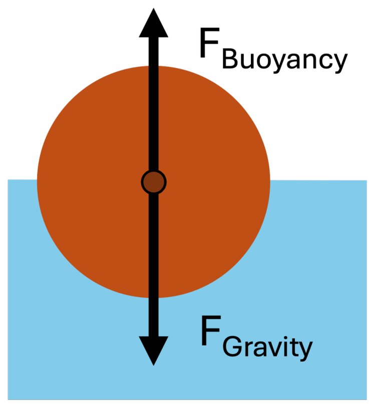


Figure 1: Vertical.

We start by thinking about a particle floating in water. By Newton's 2nd law, the force acting on this particle is going to be the mass of the particle times its acceleration: $F = ma$. In the vertical direction, the force of gravity that is pulling the particle down is exactly opposed by the force of buoyancy keeping the particle at the surface. Since the particle is always floating at the surface, and these forces are always equal, we can ignore the vertical force balance.

Horizontal Force Balance

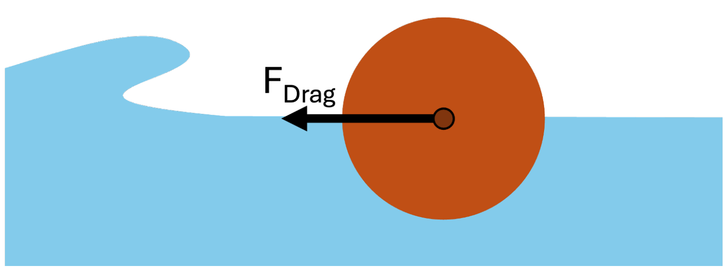


Figure 2: Horizontal.

In the horizontal direction, there is the force of drag that is pulling the particle backwards, working to keep the particle in one place. This force of drag is what is causing the velocity of the particle to change.

Equation of Motion

We can express the horizontal force as an equation:

$$F_{\text{drag}} = m_p a_p. \tag{4}$$

Here, m_p is the mass of the particle, and a_p is the acceleration of the particle. Using notation from calculus (which is A-OKAY if you haven't learned yet), we can write the acceleration of the particle as $\frac{dv_p}{dt}$, which we read as "the change of the particle velocity with time." We may also see written as $\frac{\Delta v_p}{\Delta t}$. The particle acceleration is

$$\frac{dv_p}{dt} = \frac{u - v_p}{\text{St}}. \tag{5}$$

The fluid velocity is u and the particle velocity is v_p. St represents the particle inertia - or how big the particle is.

Solve the particle position

The simplest way to solve the particle equation of motion is with a method called finite differences. To solve, we will need to know some extra information about the particle, like where it starts and at what time the model begins. After this, we will be able to find the motion of the particle as it moves up the beach. Let's start by saying that the particle will move with 90% of the initial shoreline velocity (we can change this later). This point where the wave originally crosses the shoreline is going to be t=0. At the next time step (t = 1), the particle will move by the equation

$$\frac{dx_p}{dt} = \frac{x_1 - x_0}{t_1 - t_0} = v_p. \tag{6}$$

This is saying that the distance the particle moves divided by the time it takes will equal the particle velocity (over a small length of time). We know the initial velocity, so we can use that to solve for the new particle position.

2-1. Rearrange Equation (6) to solve for x_1.

Open the file 'Beach_Debris_Modeling_2.m.' In the first section, 'Solve the Particle Position,' fill in the equation for x_1. Run this section to get the particle position at time 1. You should find $x_1 = 0.9$.

Solve the particle velocity

We now have the particle position at time 1, but we can't find x_2 without v_1. We now need to know how fast the particle is moving at this new point. We can solve Equation (5) using finite differences. First we will set this equation up over our two time steps, time t_0 and time t_1:

$$\frac{\Delta v_p}{\Delta t} = \frac{v_1 - v_0}{t_1 - t_0} = \frac{u_0 - v_0}{\mathrm{St}} \tag{7}$$

We know: $v_0, t_0, t_1, u_0, \text{and} u_1$, so if we define St, we can solve for v_1.

2-2. Re-arrange this equation for v_1.

Run the section 'Solve the particle velocity' after solving for v_1. You should find $v_1 = 0.8885$.

Iterate the solution to every step of run-up

Do you notice the pattern here? We are solving the same equation at each step, only changing the starting numbers based upon the previous step. We can set up our code so that we using the answer from the last step for the start of the next step. We will make a 'loop' through our time vector, and at each step we will have an initial value for x, v, and t, and solve for the final value of x and v. At the next time step, the final values from the step before will become the new initial values.

Run the section 'Iterate the solution to every step of run-up,' and answer the following:

2-3. What happens if we change dt?

2-4. What happens if we change v_0?

2-5. What happens if we change t_0? What happens if $t_0 = 0$? Why?

2-6. What happens after the maximum run-up? What happens when $t_{\mathbf{end}} > 10$?

3 Debris Beaching

We saw from before that the water only goes up the beach a maximum of $x = 5$. However, in the last case we saw the particle move well past this point. This is because we have not yet defined a parameter in the model to cause the particle to get stuck on the beach.

Open the file 'Debris_Beaching_3.m,' first keeping all of the variables set as they are. Let's add a simple parameter for beaching: once the water has turned around, if the particle is further up the beach than the shoreline, then the particle stops moving. Can you add this beaching condition to the code? (Again, the solution is at the bottom of the code).

Once you see the particle trajectory and that the particle is beaching, run the next section called 'animate particle on beach,' to see an animation of the particle moving up the beach.

3-1. What happens when you set $v_0 = 0.2$ and $t_0 = 1$?

Compare to Data

In the 'Debris Beaching' section, there was a line of code that reads *save('model_data.mat','t','x_f').* This line saved our particle trajectory in the file 'model_data.mat,' which we can use to compare to some experimental data.

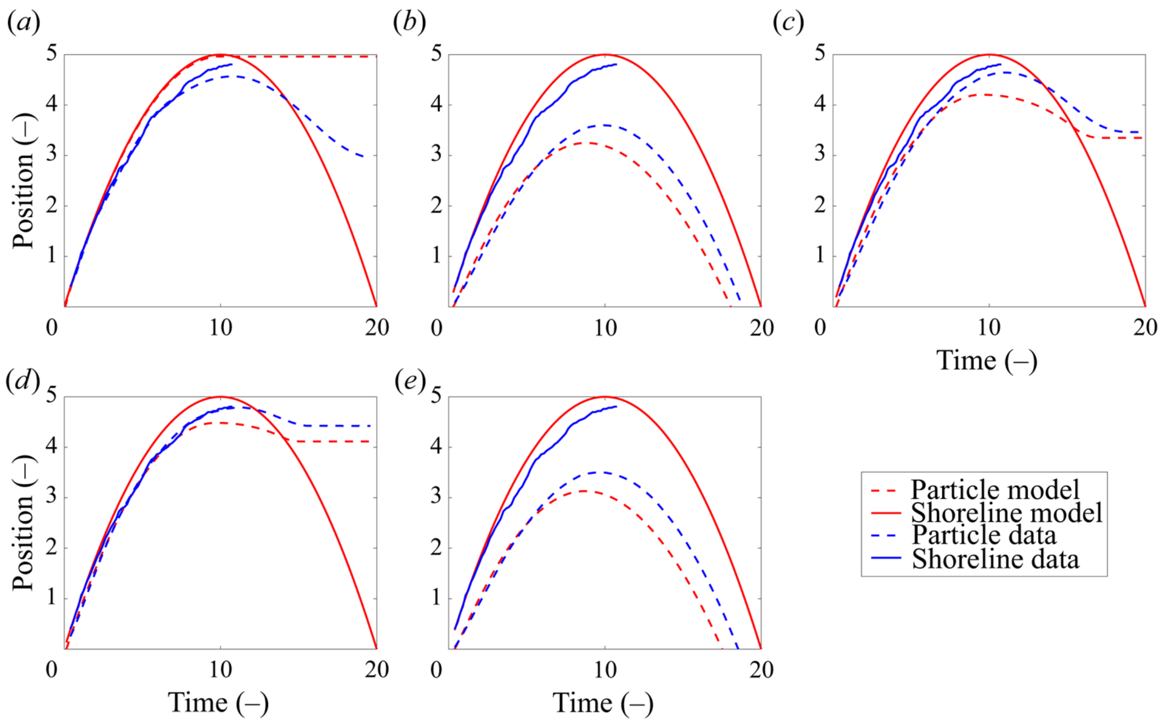
Go to the next section, 'Compare to Experiments.' Let's fill in the lines to plot the experimental data and the model data. Once you get this code working compare the five experiments with your model. In each experiment, the slope, St, and particle parameters stayed the same, but the initial velocity and time changed. You can find the initial conditions for the five particles in the table below:

Particle	v_0	t_0
1	1.03	0.0028
2	0.57	0.015
3	0.77	0.0096
4	0.86	0.0067
5	0.58	0.019

3-2. Compare the path of each of these particles with your model. Do they match well? Why or why not?

4 Full Model

The results of the full version of the model are depicted below in Figure 3.



4-1. What differences do you notice between the results of the simple model we coded today and the full version of the model shown here?

In the model we looked at today, we picked out a few of the important parameters needed to create the model, but still ignoring many important considerations that influence the results of the model. Below is a list of other considerations we needed to account for in the full model to get the results shown above (including some of the relevant parameters used in the model).

- Forcing from the fluid (β)
- Force of added mass of the particle moving through the fluid (C_m)
- Force of friction with the beach (μ)
- Characteristics of the swash tip (C_{St})
- Beaching - compare the water depth and particle height (μ_{calc})

In the folder 'Full_Model_4' is a script called 'figure8.m.' This is the code used to generate the figure above from the full model.

4-2. Can you find a place in this full model where each of the above parameters is used?

4-3. Use the space below to draw an outline or road map of this code. Focus on looking at the big picture and structure - how will the computer move through the code.

Wrap Up Questions

- What is one BIG IDEA that I am taking away from today?
- What is another example of a problem that could be considered using mathematical modeling?
- Has this activity changed the way I think about modeling? If so, how?
- What part of today was the most interesting?