1 Introduction Workthrough with p-adic Numbers

K is a non-archimedian local field. In this case K will be the p-adic numbers $\mathbb{Q}_p$, where $p > 1$ is prime. Let $\mathcal{O}$ be the ring of integers (integral elements) of K. These are elements of the form

$$a = \sum_{i>0} a_i p^i.$$

with each $0 \leq a_i \leq p - 1$ for all i. See [2], page 28 for more explanation of this representation. It is important to note that because of the metric imposed on the field of p-adic numbers, adding terms with higher powers of p does not increase the magnitude of the integer. Thus for every p-adic integer a, $|a| \leq 1$. As in the rational numbers (can be shown with rational root theorem) these are the integers. To see that (p) is the maximal ideal, note that for any $a \in \mathbb{O}$, if $|a| = 1$, then a is invertible. Thus the only non-invertible elements of $\mathbb{O}$ have p as a divisor, and (p) is the unique maximal ideal of $\mathbb{O}$.

Now let $F_q = \mathbb{O}/(p)$. Then this is a field of characteristic p. Let $F = K[\sqrt{p}]$ be a ramified quadratic extension of K with ring of integers R. From wikipedia [1]: Define

$$\omega = \begin{cases} \sqrt{p} & \text{if } p \equiv 2, 3 \pmod 4 \\ \frac{1+\sqrt{p}}{2} & \text{if } p \equiv 1 \pmod 4 \end{cases}$$

Then $R = \{a + b\omega\} = \mathbb{O}[\sqrt{p}]$.

References

[1] Wikipedia contributors. Quadratic integer, July 2013.

[2] Svetlana Katok. *p-adic Analysis Compared with Real.* American Mathematical Society, Providence, Rhode Island, 2007.