Portland State University

Math 501 Project

Properties of Hermitian Forms on Modules over Local Rings

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1.   Introduction

For my 501 project, I worked with with the paper "Unitary Groups Over Local Rings"[1]. The paper contains a discussion of hermitian forms and unitary groups over a local ring. In my project, I'll provide an introduction to the properties of local rings, and a detailed discussion of hermitian forms defined on modules over a local ring. Several results are presented that investigate properties of such modules, as well as specific properties of hermitian forms on these modules, and relationships between norm maps and hermitian geometry.

1.1   Local Fields and the p-Adic Numbers

This introduction gives a brief discussion of local fields and the construction of a local ring from a local field. First, some definitions:

- Let R be a ring and K be the field of fractions for R. A function $\nu : K \to \mathbb{Z}$ is a **discrete valuation** provided that for $x, y \in K$:

    1. $\nu(xy) = \nu(x) + \nu(y)$
    2. $\nu(x + y) \geq \inf\{\nu(x), \nu(y)\}$

  By convention, $\nu(0) = \infty$. Now let $0 \leq a \leq 1$ and define a function $\| \cdot \| : K \to \mathbb{R}$ by
  $$\|x\| = \begin{cases} a^{\nu(x)} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$
  Then $\| \cdot \|$ can be shown the be an absolute value on K.

  If $p \in R$ is a unique irreducible element (up to multiplication by a unit), then each element $a \in K$ can be written uniquely in the form $a = p^n b$, where b is invertible. Define $\nu_p : K \to \mathbb{R}$ by $\nu_p(a) = n$. Then ν_p is a discrete valuation. Taking $a = \frac{1}{p}$, $\| \cdot \|_p$ is the absolute value defined with respect to ν_p. If K is complete with respect to the topology induced by $\| \cdot \|_p$, then we say that K is complete with respect to ν_p.

- A field K is a **local field** if it is complete with respect to a discrete valuation and a has finite residue field. An equivalent definition is that a field is locally compact with respect to a non-discrete topology (see [8] for a detailed explanation). The real numbers $\mathbb{R}$ and rational numbers $\mathbb{Q}$ are examples of local fields, with the standard topology and absolute values.

- A local field K is said to be **archimedean** if for any element $x \in K$, there exists $n \in \mathbb{N}$ such that $|nx| > 1$. A local field is said to be **non-archimedean** if this property does not hold.

- In any field, the **ring of integers** $\mathcal{O}$ refers to the set of all integral elements, that is, elements that are the root of a monic polynomial with integer coefficients (that is, coefficients from the set $\{z \cdot 1_K : z \in \mathbb{Z}\}$, where 1_k is the multiplicative identity of K).

- Given a ring A, an automorphism $* : A \to A$ (denoted $a \mapsto a^*$ for $a \in A$) is an **involution** if it satisfies $(a^*)^* = a$ for all $a \in A$ and $(ab)^* = b^* a^*$ for $a, b \in A$.

Example 1.1. Consider the integers $\mathbb{Z}$ with field of fractions $\mathbb{Q}$. Let $p \in \mathbb{Z}$ be prime. Then p is an irreducible element. Let ν_p be the discrete valuation (as previously defined) with respect to p, $\| \cdot \|_p$ be the absolute value derived from ν_p. Define $\mathbb{Q}_p$ to be the completion of $\mathbb{Q}$ under $\| \cdot \|_p$. Then it can be shown that each $a \in Q_p$ can be written uniquely in the form

$$a = \sum_{i=k}^{\infty} a_i p^i \quad (k \text{ not necessarily positive}).$$

$\mathbb{Q}_p$ is known as the **p-adic numbers**. The standard integers $\mathbb{Z} \subset \mathbb{Q}_p$ form a ring of integers, denoted $\mathcal{O}$, with maximal ideal (p) and residue field $\mathbb{F}_p = \mathcal{O}/(p)$ of characteristic p. Unlike the standard integers, elements of $\mathcal{O} \backslash (p)$ are invertible (see [4] for an explanation). Thus $\mathbb{Q}_p$ is a complete with respect to $\| \cdot \|_p$ and has a finite residue field, and is thus a (non-archemedian) local field.

Define $F = \mathbb{Q}_p[\sqrt{p}]$. Depending on $p \mod 4$, the ring of integers of F is slightly different (see [6] or [7] for a more detailed discussion). Define

$$\omega = \begin{cases} \sqrt{p} & \text{if } p \equiv 2,3 \pmod 4 \\ \frac{1+\sqrt{p}}{2} & \text{if } p \equiv 1 \pmod 4; \end{cases}$$

then the ring of integers of $\mathbb{Q}_p$ is $\mathcal{R} = \mathcal{O}[\omega]$. Then $\mathcal{R}$ is a free $\mathcal{O}$-module of rank 2. $\mathcal{R}$ is a principle ideal domain with a unique prime element ω (up to associates), and $\mathcal{R}/(\omega) \cong \mathbb{F}_p$. There is an involution of F defined by $a + b\omega \mapsto a - b\omega$ that fixes $\mathbb{Q}_p$, with $\mathcal{R}$ invariant under this involution. Given $\ell \geq 1$, let $A = \mathcal{R}/(\omega)^{2\ell}$. Then A is a finite, commutative, principal ideal ring, with an involution inherited from $\mathcal{R}$. A is also a local ring, a concept that will be introduced in detail in the next section.

2. Background in Hermitian Geometry

In this section, we'll define and introduce introduce several objects that will be used throughout the paper. Some definitions:

- A ring is said to be **local** if it has a unique maximal left ideal or unique maximal right ideal.

- The **Jacobson radical** of a ring R, denoted $J(R)$ is the intersection of all maximal left (right) ideals. In a local ring, $J(R)$ coincides with the unique maximal left ideal and unique maximal right ideal, showing that the maximal ideal is two sided.

Throughout the paper, the following objects will be fixed:

- Let A be a local ring with identity.

- Let $\mathfrak{r}$ be the Jacobson radical of A. Because A is local, $\mathfrak{r}$ is maximal, two-sided, and contains all non-units of A.

- Let $A^{\times}$ denote the set of all units of A.

- Let $*$ be an involution of A.

- Let $R = \{a \in A : a^* = a\}$. It will be assumed throughout the paper that R is a subset of the center of A.

- Let $\mathfrak{m} = R \cap \mathfrak{r}$. Then R is local as well, with maximal ideal $\mathfrak{m}$. This is because any element of R that is not in $\mathfrak{m}$ is invertible by definition, and cannot be contained in any proper ideal.

- Let $R^{\times}$ denote the set of all units of R.

- Let $Q : A \to R : a \mapsto aa^*$ denote the norm-map.

Example 2.1. Returning to the objects defined in the introduction, we can see an example of each of these objects using the p-adic integers. Let $\mathbb{Q}_p, \mathcal{O}, \mathcal{R}$ and ω be the same as defined in the previous section. For simplicity, assume that $p > 2$ and $p \equiv 3 \mod 4$. Then $\omega = \sqrt{p}$ and $\mathcal{R} = \mathbb{Z}[\sqrt{p}]$. For $a + b\sqrt{p} \in \mathcal{R}$, let $\overline{a + b\sqrt{p}}$ denote the coset $a + b\sqrt{p} + (p)^{2\ell}$ in $A = \mathcal{R}/(\sqrt{p})^{2\ell}$. Then

- $A = \mathcal{R}/(\sqrt{p})^{2\ell}$ is a local ring with maximal ideal $\mathfrak{r} = (\sqrt{p})$.

- $A^{\times} = A \backslash (\sqrt{p}) = \{\overline{a + b\sqrt{p}} : a \notin (p)\}$.

- $* : A \to A$ is defined by $\overline{a + b\sqrt{p}} \mapsto \overline{a - b\sqrt{p}}$.

- $R = \{\overline{a + b\sqrt{p}} : b = 0\}$ with maximal ideal $(\sqrt{p})^2 = (p)$.

- $R^\times = \{\overline{a + b\sqrt{p}} : b = 0 \text{ and } a \notin (p)\}$.

- $Q : A^\times \to R^\times : \overline{a + b\sqrt{p}} \mapsto \overline{a^2 - b^2 p}$.

The definition of these objects will remain constant through examples for the remainder of the paper.

Let V be a right A-module. A function $h : V \times V \to A$ is a **Hermitian form** if

- h is linear in the second variable. That is, for $u, v, w \in V$ and $a \in A$, $h(u, v + w) = h(u, v) + h(u, w)$ and $h(u, va) = h(u, v)a$

- $h(u, v) = h(v, u)^*$.

Combining these facts shows that $h(ua, v) = a^* h(u, v)$ for all $u, v \in V$ and $a \in A$. Also, $h(u, u) = h(u, u)^*$ and $h(u, u) \in R \subseteq Z(A)$ for all $u \in V$.

Now consider the dual space V^*. Define an operation $V^* \times A \to V^*$ by $(\alpha a)(v) = a^* \alpha(v)$ where $\alpha \in V^*, a \in A, v \in V$. Under this operation, V^* is a right A-module. Now we can define a homomorphism of right A-modules $\gamma_h : V \to V^*$ associated with h given by $\gamma_h(u) = h(u, -)$. Additionally, for the remainder of the paper:

- Assume that h is non-degenerate; By definition, γ_h is an isomorphism.

- Assume the existence of an element $d \in A$ such that $d + d^* = 1$.

- Assume that V is a free A-module of rank $m \geq 1$.

For the remainder of this section, let $\{v_1, v_2, \ldots, v_m\}$ be a basis of V.

Example 2.2. Continuing with the A as defined in Example 2.1 and assuming that V is a right A-module, let $a = v_1 a_1 + \cdots v_m a_m$ and $b = v_1 b_1 + \cdots + v_m b_m$ with $a_i, b_i \in A$ for $1 \leq i \leq m$. Define $h : V \times V \to A$ by $h(a, b) = a_1^* b_1 + \ldots + a_m^* b_m$. This is a Hermitian form, and can be verified to be non-degenerate. For the rest of the paper, assume that this is how h is defined in all examples.

Lemma 2.3. *There is a vector $u \in V$ such that $h(u, u) \in R^\times$.*

Proof. Assume otherwise; that $h(u, u) \in \mathfrak{m}$ for all $u \in V$. Then using the linearity of h:

$$h(u, v) + h(u, v)^* = h(u + v, u + v) - h(u, u) - h(v, v) \in \mathfrak{m}$$

for all $u, v \in V$. Recall that $d \in A$ is assumed to be an element such that $d + d^* = 1$. Let $\alpha \in V^*$ be the linear functional such that $\alpha(v_1) = d$ and $\alpha(v_i) = 0$ for all $i > 1$. Because h is assumed to be non-degenerate, there exists $u \in V$ such that $h(u, -) = \alpha$. Then $d = \alpha(v_1) = h(u, v_1)$ and $1 = d + d^* = h(u, v_1) + h(u, v_1)^* \notin \mathfrak{m}$, contradicting the original hypothesis. $\square$

Example 2.4. Note that in the $\mathbb{Q}_p$, 2^{-1} is a series satisfying $2^{-1} + 2^{-1} = 1$. Because 2^{-1} is a p-adic integer, $2^{-1} \in R$ and is fixed under the involution. Thus $d = 2^{-1}$. Using h as defined in Example 2.2, pick $u = v_1 2^{-1}$. Then $h(u, v_1) = 2^{-1}$ and $h(u, v_i) = 0$ for $i \neq 1$, as desired.

Lemma 2.5. *V has an orthogonal basis (with respect to h) $u_1, u_2, \ldots u_m$. Any such basis satisfies $h(u_i, u_i) \in R^\times$.*

Proof. Prove with induction on m. Assume that $m = 1$. By Lemma 2.3, there exists $u \in V$ such that $h(u, u) \in R^\times$. Then $u = v_1 a_1$ for some $a_1 \in A^\times$, and $h(u, u) = h(v_1 a_1, v_1 a_1) = a_1^* h(v_1, v_1) a_1 \in R^\times$ implying that $h(v_1, v_1) \in R^\times$. Now assume that $m > 1$ and that the hypothesis holds for $m - 1$. Once again, there exists $u \in V$ such that $h(u, u) \in R^\times$. Then $u = v_1 a_1 + \cdots + v_m a_m$ with $a_i \in A$. If all $a_i \in \mathfrak{r}$, then using the properties of h:

$$h(u, u) = \sum_{i=1}^{m} \sum_{j=1}^{m} a_i^* h(v_i, v_j) a_j$$

and each term of this sum is an element of $\mathfrak{r}$. Because $\mathfrak{r}$ is an ideal, $h(u, u) \in \mathfrak{r}$ implying that $h(u, u) \in \mathfrak{r} \cap R = \mathfrak{m}$, a contradiction. Without loss of generality, assume that $a_1 \notin \mathfrak{r}$. Then if $u_1 = v_1 a_1$, the set $\{u_1, v_2, \ldots, v_m\}$ is a basis of V. For $1 < i \leq m$, set

$$u_i = v_i - u_1[h(u_1, v_i)/h(u_1, u_1)]$$

Then $u_1, u_2, \ldots, u_m$ can be shown to be a basis of V satisfying $h(u_1, u_i) = 0$ for $1 < i \leq m$. Let $V_1 = u_1 A$ and $V_2 = \mathrm{span}\{u_2, \ldots, u_m\}$. Then $V = V_1 \perp V_2$ and the restriction of h to V_2 induces an isomorphism $V_2 \to V_2^*$. Applying the inductive hypothesis to this space completes the proof. $\square$

Lemma 2.6. *(a) Suppose $u_1, \ldots, u_s \in V$ are orthogonal and satisfy $h(u_i, u_i) \in R^\times$. Then $u_1, \ldots, u_s \in V$ can be extended to an orthogonal basis of V with the same property.*

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(b) If V_1 is a submodule of V such that the restriction of h to V_1 is non-degenerate there is another such submodule V_2 of V such that $V = V_1 \perp V_2$.

Proof. (a) Because $\{v_1, \ldots, v_m\}$ is a basis of v, $u_1 = v_1 a_1 + \cdots + v_m a_m$ for some $a_i \in A$. Since $h(u_1, u_1) \in R^\times$ (by hypothesis), one of the scalars must be a unit (see proof of Lemma 2.5). Without loss of generality, assume $a_1 \in A^\times$. Thus $u_1, v_2, \ldots, v_m$ is a basis of V. Suppose $1 \le t \le s$ and the list $u_1, \ldots, u_t, v_{t+1}, \ldots, v_m$ is a basis of V. Then

$$u_{t+1} = u_1 b_1 + \cdots + u_t b_t + v_{t+1} b_{t+1} + \cdots + v_m b_m$$

for some $b_i \in A$. Suppose, if possible, that $b_i \in \mathfrak{r}$ for all $i \ge t+1$. Then for every $i \le t$,

$$0 = h(u_i, u_{t+1}) = h(u_i, u_i) b_i + h(u_i, v_{t+1}) b_{t+1} + \cdots + h(u_i, v_m) b_m$$

implying that $b_i \in \mathfrak{r}$ for all $1 \le i \le t$, contradicting the assumption that $h(u_{t+1}, u_{t+1}) \in R^\times$. Thus at least one of $b_{t+1}, \ldots, b_m$ is a unit (assume b_{t+1} and $u_1, \ldots u_t, u_{t+1}, v_{t+2}, \ldots, v_m$ is a basis of V.

This process can be repeated to extend $u_1, \ldots, u_s$ to a basis $u_1, \ldots, u_s, u_{s+1}, \ldots u_m$ of V. For $s < i \le m$, let

$$z_i = u_i - ([u_1 h(u_1, u_i)/h(u_1, u_1)] + \cdots + u_s h(u_s, u_i)/h(u_s, u_s)].$$

Then $u_1, \ldots, u_s, z_1, \ldots, z_{m-s}$ is a basis of V satisfying $h(u_i, z_j) = 0$. If follows that the restriction of h to $M = \mathrm{span}\{z_1, \ldots, z_{m-s}\}$ is non-degenerate and by Lemma 2.5 that M has an orthogonal basis $\{y_1, \ldots, y_{m-s}\}$. with $h(z_i, z_i) \in R^\times$ for any $i \le m - s$. Because $M = \mathrm{span}\{z_1 \ldots z_m\}$ and $h(u_i, z_j) = 0$ for all i, j, it follows that $h(u_i, y_k) = 0$ for all k. Thus $\{u_1, \ldots u_s, y_1, \ldots y_{m-s}\}$ is an orthogonal basis for V satisfying $h(u_i, u_i) \in R^\times$ for $i \le s$ and $h(y_i, y_i) \in R^\times$ for $i \le m - s$.

(b) Follows from (a) and Lemma 2.5 $\qquad \square$

Given a set $U = \{u_1, \ldots, u_s\} \subset V$, for $s \le m$, the **Gram matrix** H of the set U with respect to h is defined by $H_{ij} = h(u_i, u_j)$. If U is a basis, then for any $u, v \in V$, $h(u, v) = u^* H v$, where u^* is the transpose of u under $*$.

Lemma 2.7. *Let* $u_1, \ldots u_s \in V$, *with corresponding Gram matrix* $H \in M_s(A)$, *defined by* $H_{ij} = h(u_i, u_j)$. *If* $H \in GL_m(A)$, *then* $u_1, \ldots, u_s$ *are linearly independent.*

Proof. Suppose $a_1, \ldots, a_s$ satisfy $u_1 a_1 + \cdots + u_s a_s = 0$. Then for $1 \le i \le s$

$$0 = h(u_i, u_1 a_1 + \cdots + u_s a_s) = h(u_i, u_1) a_1 + \cdots + h(u_i, u_s) a_s$$

implying that

$$H \begin{pmatrix} a_1 \\ \vdots \\ c_1 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Since H is invertible, the desired result follows. $\qquad\square$

3. Classification of Hermitian Forms

Let B be a subset of A. Define $VB = \{vb : v \in V, b \in B\}$. If B is an ideal, then VB is a submodule of V. A vector $v \in V$ is said to be **primitive** if $v \notin V\mathfrak{r}$. This is equivalent to saying that v belongs to a basis of V. We say that h is **isotropic** if there is a primitive vector $v \in V$ such that $h(v,v) = 0$.

Lemma 3.1. *Suppose h is isotropic. Then, given any $r \in R$ there is a primitive vector v satisfying $h(v,v) = r$.*

Proof. By assumption, h is isotropic so there is a primitive vector $u \in V$ such that $h(u,u) = 0$. Because h is assumed to be non-degenerate, there exists $w \in V$ such that $h(u,w) = d$. Set $s = r - h(w,w) \in R$ and $v = us + w$. Then

$$
\begin{aligned}
h(v,v) &= h(us + w, us + w) \\
&= sh(u,w) + sh(w,u) + h(w,w) \\
&= s(d + d^*) + h(w,w) \\
&= s + h(w,w) \\
&= r - h(w,w) + h(w,w) \\
&= r
\end{aligned}
$$

$\square$

We assume for the remainder of the paper that the squaring map of the 1-group $1 + \mathfrak{m}$ is an epimorphism and that $R/\mathfrak{m} = F_q$ is a field of finite order q and odd characteristic. Thus $[F_q^* : F_q^{*2}] = 2$. To see this, pick $r \in F_q^* \backslash F_q^{*2}$. Then the minimal polynomial of r is $t^2 - r^2 \in F_q^{*2}[t]$. A similar argument shows that $[R^* : R^{*2}] = 2$. For the remainder of the paper, fix an element $\varepsilon \in R^* \backslash R^{*2}$. Since $R^{*2} \subseteq Q(A^*)$, we infer $Q(A^*) = R^*$ if Q is surjective and $Q(A^*) = R^{*2}$ otherwise.

Proposition 3.2. *The division ring $A/\mathfrak{r}$ is commutative. Moreover,*

(a) *If the involution that $*$ induces on $A/\mathfrak{r}$ is the identity then Q is not surjective and $A/\mathfrak{r} \cong F_q$.*

(b) *If the involution that $*$ induces on $A/\mathfrak{r}$ is not the identity then Q is surjective and $A/\mathfrak{r} \cong F_{q^2}$.*

Proof. We begin embedding $R/\mathfrak{m}$ in $A/\mathfrak{r}$ using the mapping $x + \mathfrak{m} \mapsto x + \mathfrak{r}$ for $x \in R$. Thus $R/\mathfrak{m}$ can be viewed as a subfield of $A/\mathfrak{r}$. Now let $\circ$ be the involution that $*$ induces on $A/\mathfrak{r}$ by $a + \mathfrak{r} \mapsto a^* + \mathfrak{r}$. Let $k = \{a \in A/\mathfrak{r} : a^\circ = a\}$ be the set of all elements of $A/\mathfrak{r}$ that are fixed by $\circ$. Then $R/\mathfrak{m} \subseteq k$ (by definition, R is fixed under $*$). Conversely, assume that $a + \mathfrak{r} \in k$. Then $a - a^* \in \mathfrak{r}$, so

$$a = \frac{a + a^*}{2} + \frac{a - a*}{2} \in R + \mathfrak{r}$$

and $k \subseteq (R + \mathfrak{r})/\mathfrak{r} = R/\mathfrak{m}$. Thus $k = R/\mathfrak{m}$. Consider two cases:

(a) The involution that $*$ induces on $A/\mathfrak{r}$ is the identity: In this case, $A/\mathfrak{r} = k = R/\mathfrak{m}$ and the norm map $(A/\mathfrak{r})^* \to (R/\mathfrak{m})^*$ (induced by Q) is the squaring map of F_q^*. This map is not surjective, so the norm map Q is not surjective. Because R is commutative and $\mathfrak{m}$ is a maximal ideal, $R/\mathfrak{m}$ is a field, implying that $A/\mathfrak{r}$ is commutative. This completes the proof of this case. (b) The involution that $*$ induces on $A/\mathfrak{r}$ is not the identity: In this case, we assume that $A/\mathfrak{r}$ properly contains k. Then for any $f \in A/\mathfrak{r} \backslash k$, the minimal polynomial of f is $(t - f)(t - f^\circ) = t^2 - (f + f^\circ)t + ff^\circ \in k[t]$.

Let $f, e \in A/\mathfrak{r}$. The goal is to show that f and e commute. Let $f_1 = f - (f + f^\circ)/2$ and $e_1 = e - (e + e^\circ)/2$. Since $(f + f^\circ)/2, (e + e^\circ)/2 \in k$ (which is a field), it is sufficient to show that f_1 and e_1 commute. Note that $f_1^\circ = -f_1$ and $e_1^\circ = -e_1$. Then

$$(e_1 f_1 + f_1 e_1)^\circ = f_1^\circ e_1^\circ + e_1^\circ f_1^\circ = f_1 e_1 + e_1 f_1$$

and $e_1 f_1 + f_1 e_1 \in k$. Thus $k\langle f_1, e_1 \rangle$ is the k-span of $1, f_1, e_1, f_1 e_1$ and $k\langle f_1, e_1 \rangle$ is a finite dimensional division algebra over k. Thus by Wedderburn's theorem, $k\langle f_1, e_1 \rangle$ is a field, implying that f and e commute.

Thus $A/\mathfrak{r}$ is a field, algebraic over $k = R/\mathfrak{m}$, where every element of $A/\mathfrak{r} \backslash k$ has degree 2 over k. Since every algebraic extension of k is separable, the primitive element theorem implies that $[A/\mathfrak{r} : k] = 2$.

Let $\hat{Q} : (A/\mathfrak{r})^* \to k^*$ be the norm map induced by Q. We now want to show that $\hat{Q}$ is surjective. Because $k \subseteq A/\mathfrak{r}$, if $r \in k^2$, then there exists $s \in k$ with s fixed under $*$ and $s^2 = r$. Thus $\hat{Q}(s) = s^2 = r$. Now pick $x \in k \backslash k^2$. Then $\sqrt{x} \notin k$.

Consider two cases:

Case 1: Assume $-x \notin k^2$. Then $A/\mathfrak{r} \cong F_{q^2} = k(\sqrt{-x})$. Every element of $A/\mathfrak{r}$ can be written in the form $a + b\sqrt{-x}$ with $a, b \in k$ and

$$\hat{Q}(a + b\sqrt{-x}) = a^2 - b^2 \cdot -x = a^2 + b^2 x.$$

Then taking $s = \sqrt{-x}$ gives $\hat{Q}(s) = x$ and $x \in \hat{Q}(A/\mathfrak{r})$.

Case 2: Assume that $-x \in k^2$. Because exactly half of the elements in k^* have square roots, there must exist some element $z \in k^*$ where $\sqrt{z} \in k^*$ and $\sqrt{z+1} \notin k^*$. Then $F_{q^2} = k(\sqrt{z+1})$. Now take $s = \sqrt{-x}\sqrt{z} + \sqrt{-x}\sqrt{z+1}$. Then

$$\hat{Q}(s) = (-xz) - (-x(z+1)) = (z+1)x - zx = x$$

and $\hat{Q}$ is surjective.

Using the fact that the squaring map of $1 + \mathfrak{m}$ is surjective, this can be used to show that $Q : A^\times \to R^\times$ is surjective. $\qquad\square$

Example 3.3. Recall that in our example, $\mathfrak{r} = (\sqrt{p})$ and $\mathfrak{m} = (p)$. Thus $A/\mathfrak{r} \cong R/\mathfrak{m} \cong F_p$. Furthermore, the involution $*$ induces on $A/\mathfrak{r}$ is the identity. Pick $c \in F_p \backslash F_p^2$. Then c cannot be in the image of $*$, because for any $a + b\sqrt{p} \in A$, $c \not\equiv a^2 \mod p$, and therefore $c \neq a^2 - b^2 p$. Thus Q is not surjective.

Proposition 3.4. *Suppose $m \geq 2$. Then given any unit $r \in R$ there is a primitive vector $v \in V$ satisfying $h(v, v) = r$.*

Proof. Consider two cases:

Case 1: h is isotropic. Then 3.1 applies.

Case 2: h is non-isotropic.

By Lemma 2.5, there is an orthogonal basis $u_1, u_2, \ldots u_m$ of V such that $h(u_i, u_i) \in R^\times$. Let $a = h(u_1, u_1) \in R^\times$ and $b = h(u_2, u_2) \in R^\times$. Let $c = b/a$. If $t_1, t_2 \in R^\times$, then $v = u_1 t_1 + u_2 t_2$ is primitive (because it is part of a basis), so

$$0 \neq h(v, v) = at_1^2 + bt_2^2.$$

Dividing by a gives

$$0 \neq t_1^2 + ct_2^2.$$

Because the choices of t_1 and t_2 were arbitrary, $-c$ cannot be a square in $R^\times$.

Let $S = R[t]/(t^2 + c)$ and $\delta = t + (t^2 + c) \in S$. Then $S = R[\delta], \delta^2 = -c$ and every element of S can be uniquely written in the form $t_1 + t_2\delta$ with $t_1, t_2 \in R$. We have an involution $s \mapsto \hat{s}$ defined by $t_1 + t_2\delta \mapsto t_1 - t_2\delta$, whose corresponding norm map $J : S \to R$ is given by $s \mapsto s\hat{s}$, or $t_1 + t_2\delta \mapsto t_1^2 + ct_2^2$. Note that for $s \in S^\times$, $J(s) \in R^\times$. Then for an element $z \in S$, if $J(z) \in \mathfrak{m}$, $z \notin S^\times$.

We claim that S is local with maximal ideal $S\mathfrak{m}$. To show this, we want to show that every non-invertible element of S is contained in $S\mathfrak{m}$. To do this, we'll show that for $s \in S$, $J(s) \notin \mathfrak{m}$ if $s \notin S\mathfrak{m}$. Let $t_1, t_2 \in R$, not both in $\mathfrak{m}$, and consider $J(t_1 + t_2\delta) = t_1^2 + ct_2^2$. This breaks into two cases:

Case 1: One of $t_1, t_2 \in \mathfrak{m}$. Then if $t_1^2 + ct_2^2 \in \mathfrak{m}$, $t_1^2 = -ct_2^2 + m$ and $ct_2^2 = -t_1^2 + m$ for some $m \in \mathfrak{m}$. This implies that both $t_1, t_2 \in \mathfrak{m}$, a contradiction. Thus $t_1^2 + ct_2^2 \in R^\times$ and $t_1 + t_2\delta \in S^*$.

Case 2: Both $t_1, t_2 \notin \mathfrak{m}$. Suppose that $t_1 + t_2\delta \notin S^*$. Then $t_1^2 + ct_2^2 = f \in \mathfrak{m}$, and
$$-c = (t_2^{-1})^2(t_1^2 - f) = (t_2^{-1})^2 t_1^2 (1 - (t_1^{-1})^2 f).$$

Local rings have the property that if m is a nonunit, $1 - m$ must be a unit. Then $1 - (t_1^{-1})^2 f \in R^\times$. Using the assumption that the squaring map of $1 + \mathfrak{m}$ is an epimorphism, we see that $1 - (t_1^{-1})^2 f \in R^{\times^2}$, implying that $-c \in R^{\times^2}$, a contradiction. Thus $t_1 + t_2\delta \in S^\times$.

Therefore $S\mathfrak{m}$ is an ideal of S, containing every non-unit, implying that it is maximal and that S is local.

Because S is commutative, $S/S\mathfrak{m}$ is a field. The embedding $R/\mathfrak{m} \to S/S\mathfrak{m}$ allows us to view $S/S\mathfrak{m}$ as a vector space over $R/\mathfrak{m}$, with $\{1 + S\mathfrak{m}, \delta + S\mathfrak{m}\}$ as a basis. Thus $S/S\mathfrak{m}$ is a quadratic extension of $R/\mathfrak{m}$. The involution of S induces the $R/\mathfrak{m}$-automorphism of $S/S\mathfrak{m}$ of order 2 and the norm map J induces the norm map $(S/S\mathfrak{m})^\times \to (R/\mathfrak{m})^\times$.

Since $R/\mathfrak{m}$ is known to be F_q, this map is known to be surjective. We claim that J is surjective. Indeed, pick $e \in R^\times$. Then by the surjectivity of the norm map Q, which induces J, there is $s \in S$ and $f \in \mathfrak{m}$ such that
$$J(s) = e + f = e(1 + e^{-1}f).$$

Since $1 + e^{-1}f \in R^{\times^2}$ (by surjectivity of squaring map of $1 + \mathfrak{m}$), it follows that e is in the image of J, as claimed. Thus J is surjective.

By the claim there are $t_1, t_2 \in R$ with at least one in R^*, such that $t_1^2 + t_2^2 c = r/a$. Then $v = u_1 t_1 + u_2 t_2$ is primitive and
$$h(v, v) = at_1^2 + bt_2^2 = r.$$

This completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Recall that ε is an arbitrary fixed element of $R^\times \backslash R^{\times^2}$.

Theorem 3.5. *There is an orthogonal basis $v_1, v_2, \ldots, v_m$ of V satisfying*

$$h(v_1, v_1) = \cdots = h(v_{m-1}, v_{m-1}) = 1 \ and$$
$$h(v_m, v_m) = 1 \ if \ Q(A^\times) = R^\times$$
$$h(v_m, v_m) \in \{1, \varepsilon\} \ if \ Q(A^\times) = R^{\times 2}$$

Proof. To prove this, we'll use induction on m. Assume that $m = 1$. By Lemma 2.5, V has a basis $\{u_1\}$ such that $h(u_1, u_1) \in R^\times$. If $Q(A^\times) = R^\times$, then $h(u_1, u_1) = aa^*$ for some $a \in A^\times$. Let $v_1 = u_1 a^{-1}$. Then

$$h(v_1, v_1) = (a^{-1})^* a^{-1} h(u, u) = 1.$$

If $Q(A^\times) = R^{\times 2}$, then consider two cases.

Case 1: $h(u_1, u_1) \in R^{\times 2}$. Then there exists v such that $h(v, v) = 1$, as shown.

Case 2: $h(u_1, u_1) \in R^\times \backslash R^{\times 2}$. Then $h(u_1, u_1) = \varepsilon b$ for some $b \in R^{\times 2}$ (note that from earlier result, $[R^\times : R^{\times 2}] = 2$). Then $b = r^2$ for some $r \in R^\times$. Let $v = r^{-1} u_1$. Then

$$h(v, v) = (r^{-1})^2 h(u, u) = b^{-1} \varepsilon b = \varepsilon.$$

Taking $v_1 = v$ gives $h(v_1, v_1) \in \{1, \varepsilon\}$.

Now assume that $m > 1$ and that the hypothesis is true for $m - 1$. By Proposition 3.4, there exists v_1 such that $h(v_1, v_1) = 1$. Using Lemma 2.6, this can be extended to an orthogonal basis $v_1, u_2, \ldots, u_m$. Let $V' = \text{span}\{u_2, \ldots, u_m\}$. Then V' has dimension $m - 1$ and by the inductive hypothesis there exits is a basis $v_2, \ldots v_m$ of V'. Then $\{v_1, v_2, \ldots, v_m\}$ is a basis of V with the desired property. $\square$

Example 3.6. In our example, because h was defined in terms of $v_1, \ldots, v_m$, this basis will be orthogonal by the definition of h, and $h(v_i, v_i) = 1$ for all i.

Let $\mathfrak{i}$ be a $*$-invariant ideal of A and let $\overline{A} = A/\mathfrak{i}$. Then $*$ induces an involution on $\overline{A}$. Moreover, $\overline{V} = V/V\mathfrak{i}$ is a free $\overline{A}$ module of rank m. Define $\overline{h} : \overline{V} \times \overline{V} \to \overline{A}$, given by $\overline{h}(v + V\mathfrak{i}, w + V\mathfrak{i}) = h(v, w)$. $\overline{h}$ can be verified to be well-defined and non-degenerate.

Recall that when A is commutative the discriminant of h is the element of $R^\times / Q(A^\times)$ obtained by taking the determinant of the Gram matrix of h relative to any basis of V. Thus by Theorem 3.5 the discriminant of h is

1 or ε. Two hermitian forms h_1 and h_2 are said to be **equivalent** if there exists an isomorphism $T : V \to V$ such that $h_1(v, w) = h_2(Tv, Tw)$ for all $v, w \in V$.

Corollary 3.7. *Let h_1 and h_2 be non-degenerate hermitian forms on V. Then the following conditions are equivalent:*

  (a) h_1 and h_2 are equivalent.

  (b) The reductions $\overline{h_1}$ and $\overline{h_2}$ modulo $\mathfrak{r}$ are equivalent.

  (c) The discriminants of $\overline{h_1}$ and $\overline{h_2}$ are the same.

Proof. (a) implies (b): Assume that h_1 and h_2 are equivalent. Then there exists some isomorphism $T : V \to V$ such that $h_1(v, w) = h_2(Tv, Tw)$ for all $v, w \in V$. Then

$$\overline{h_1}(v + V\mathfrak{r}, w + V\mathfrak{r}) = h_1(v, w) = h_2(Tv, Tw) = \overline{h_2}(Tv + V\mathfrak{r}, Tw + V\mathfrak{r})$$

and $\overline{h_1}$ is equivalent to $\overline{h_2}$.

(b) implies (c): Assume that $v_1 + V\mathfrak{r}, \ldots, v_m + V\mathfrak{r}$ is the orthogonal basis for $\overline{V}$ given in Theorem 3.5 and that $\overline{h_1}(v + V\mathfrak{r}, w + V\mathfrak{r}) = \overline{h_2}(T(v + V\mathfrak{r}), T(w + V\mathfrak{r}))$ for some invertible $T : \overline{V} \to \overline{V}$ and all $v \in V$. Then $T(v_1 + V\mathfrak{r}), \ldots, T(v_m + V\mathfrak{r})$ is a basis of $\overline{V}$; and is orthogonal with respect to $\overline{h_2}$. Let Δ represent the discriminant function. Using the fact that the determinant is invariant under choice of basis:

$$\Delta(\overline{h_1}) = \prod_{i=1}^{m} \overline{h_1}(v_i + V\mathfrak{r}, v_i + V\mathfrak{r})$$
$$= \prod_{i=1}^{m} \overline{h_2}(T(v_i + V\mathfrak{r}), T(v_i + V\mathfrak{r}))$$
$$= \Delta(\overline{h2})$$

(c) implies (a): Assume that the discriminants of $\overline{h_1}$ and $\overline{h_2}$ are the same. Let $v_1, \ldots, v_m$ and $w_1, \ldots, w_m$ be orthogonal bases satisfying Theorem 3.5 for h_1 and h_2 respectively. Then $h_1(v_i, v_i) = h_2(w_i, w_i)$ for all i (because the discriminants are equal, it is ensured that $h_1(v_m, v_m) = h_2(w_m, w_m)$). Let

$T : V \to V$ be defined by $v_i \mapsto w_i$. Then for $x, y \in V$:

$$
\begin{aligned}
h_1(x, y) &= \sum_{i=1}^{m} h_1(v_i x_i, v_i y_i) \\
&= \sum_{i=1}^{m} x_i^* y_i h_1(v_i, v_i) \\
&= \sum_{i=1}^{m} x_i^* y_i h_2(T v_i, T v_i) \\
&= \sum_{i=1}^{m} h_2(T v_i x_i, T v_i y_i) \\
&= h_2(T x, T y)
\end{aligned}
$$

and h_1 and h_2 are equivalent.

$\square$

Example 3.8. Let $h_1 = h$ as defined in previous examples (where $h(v_i, v_i) = 1$ for basis $v_1, \ldots, v_m$) and define h_2 similarly for some basis $u_1, \ldots, u_m$. Let $T : V \to V$ be defined by $u_i \mapsto v_i$. Then h_1 and h_2 are equivalent, with $h_2(u_i, u_i) = 1 = h_1(v_i, v_i) = h_1(T u_i, T u_i)$ for all i. Similarly, the reductions of h_1 and h_2 modulo $\mathfrak{r}$ are equivalent, because each has a Gram matrix of the identity matrix with respect to the basis by which they are defined. This also implies that the discriminants of $\overline{h_1}$ and $\overline{h_2}$ are equal. In fact, in our example, because A is commutative, the notion of a determinant is well defined for h_1 and h_2. Thus we can see that h_1 and h_2 have the same discriminant (1 in this example) by taking the determinant of the Gram matrix with for h_1 with respect to the basis $v_1, \ldots, v_m$ and for h_2 with respect to the basis $u_1, \ldots, u_m$.

Given $r_1, \ldots, r_m \in R^\times$ we say that h is of type $\{r_1, \ldots, r_m\}$ if there is a basis B of V relative to which h has matrix $\text{diag}\{r_1, \ldots, r_m\}$. Note that because these matrices contain only elements of $R^\times$, which is commutative, the notion of a determinant is well defined.

Lemma 3.9. *The Hermitian form h is of type $\{r_1, \ldots, r_m\}$ and $\{s_1, \ldots, s_m\}$ if and only if*
$(r_1 \cdots r_m)(s_1 \cdots s_m)^{-1} \in Q(A^\times)$.

Proof. Assume that h is of type $\{r_1, \ldots, r_m\}$ and of type $\{s_1, \ldots s_m\}$. Because the determinant is invariant under the choice of basis, $r_1 \cdots r_m = s_1 \cdots s_m$ and $(r_1 \cdots r_m)(s_1 \cdots s_m)^{-1} = 1 \in Q(A^\times)$.

Now assume that h is of type $\{r_1, \ldots, r_m\}$ with respect to basis R and $(r_1 \cdots r_m)(s_1 \cdots s_m)^{-1} \in Q(A^\times)$. It's clear that if $m = 1$ that this implies that h is of type $\{s_1\}$. Consider the case that $m > 1$ and assume also that h is not of type $\{s_1, s_2, \ldots, s_m\}$. Then for every orthogonal basis B of V where $h(v_i, v_i) \in R^\times$ for $v_i \in B$, let k_B denote the number of $v_i \in V$ such that $h(v_i, v_i) \neq s_i$, and let $k = \min\{k_B : B \text{ is a basis of } V\}$. Because h is not of type $\{s_1, \ldots, s_m\}$, $k > 0$. Similarly, by proposition 3.4, $k \leqslant m - 1$. Without loss of generality, assume that $h(v_i, v_i) = s_i$ when $i \leqslant m - k$ and $h(v_i, v_i) = d_i s_i$ where $d_i \neq 1 \pmod{Q(A^\times)}$ when $i > m - k$. Because h is of type $\{h(v_i, v_i) : 1 \leq i \leq m\}$, and $\det(h)$ is invariant under choice of basis,

$$\prod_{i=1}^{m} r_i = \prod_{i=1}^{m} h(v_i, v_i) = \left(\prod_{i=1}^{m-k} s_i \right) \left(\prod_{i=m-k+1}^{m} d_i s_i \right)$$

and by assumption,

$$\prod_{i=m-k+1}^{m} d_i = 1 \mod Q(A^\times).$$

This result shows that $k > 1$. Now let $V_1 = \text{span}\{v_i : i \leq m - k\}$ and $V_2 = \text{span}\{v_i : i > m - k\}$. Because $k \geq 2$, by Proposition 3.4 there exists an orthogonal basis $\{w_1, \ldots, w_k\}$ with $h(w_1, w_1) = s_{m-k+1}$. But then $V' = \{v_1, v_2, \ldots, v_{m-k}, w_1, \ldots, w_k\}$ is a basis for V with $k_{V'} < k$, contradicting the assumption that k was minimized. Thus $d_i = 1 \pmod{Q(A^\times)}$ for all i, and for some basis V, h is of type $\{s_1, s_2, \ldots, s_m\}$. $\qquad\square$

Lemma 3.10. *When m is even then h is of type $\{1, -1, \ldots, 1, -1\}$ (define this as kind I) or $\{1, -1, \ldots, 1, -\varepsilon\}$ (kind II). When m is odd then h is of type $\{1, -1, \ldots, 1, -1, -1\}$ (kind I) or of type $\{1, -1, \ldots, 1, -1, -\varepsilon\}$ (kind II).*

Proof. By Theorem 3.5, we know that h is of type $\{1, 1, \ldots, 1\}$ or $\{1, 1, \ldots, \varepsilon\}$. If $-1 \in Q(A^\times)$, then the result is immediate. Assume that Q is not surjective and that $-1 \notin Q(A^\times)$. Let $r = 1$ if h is of type $\{1, 1, \ldots, 1\}$ and $r = \varepsilon$ if h is of type $\{1, 1, \ldots, \varepsilon\}$. Let $k = \frac{m}{2}$ if m is even, and $k = \frac{m+1}{2}$ if m is odd. Recall that ε is a fixed element of $R^\times \backslash R^{\times^2}$, and let $\delta \in \{1, \varepsilon\}$. By Lemma 3.9, if $r(-1)^k \delta^{-1} \in Q(A^\times)$, then h is of type $\{1, -1, \ldots, 1, -\delta\}$ (m even) or $\{1, -1, \ldots, 1, -1, -\delta\}$ (m odd). Note that because $-1 \notin Q(A^\times) = R^{\times^2}$ and $\varepsilon \notin Q(A^\times)$, then $-\varepsilon \in Q(A^\times)$ (because $[R^\times : R^{\times^2}] = 2$), and $-\varepsilon^{-1} \in Q(A^\times)$ as well. Consider 4 cases:

Case 1: $r = 1$ and k even. Let $\delta = 1$. Then $r(-1)^k \delta^{-1} = 1 \in Q(A^\times)$ and h is of type $\{1, -1, \ldots, 1, -1\}$.

Case 2: $r = 1$ and k odd. Let $\delta = \varepsilon$. Then $r(-1)^k \delta^{-1} = -\varepsilon^{-1} \in Q(A)$ and h is of type $\{1, -1, \ldots, -1, -\varepsilon\}$.

Case 3: $r = \varepsilon$ and k even. Let $\delta = \varepsilon$. Result follows similarly to case 2, and h is of type $\{1, -1, \ldots, 1, -\varepsilon\}$

Case 4: $r = \varepsilon$ and k odd. Let $\delta = 1$. Result follows similarly to case 1, and h is of type $\{1, -1, \ldots, -1, -1\}$.

$\square$

Additionally, it is clear from these prior results that h is of kind I and kind II if and only if $Q(A^\times) = R^\times$. The following result classifies the range of h based on the dimension of m and the type of h.

Theorem 3.11. *Let Λ be the set of all values $h(u, u)$ with $u \in V$ primitive. Assume that the involution $*$ induces on $A/\mathfrak{r}$ is the identity.*

(a) *Suppose $m = 1$. If h is of type $\{1\}$ then $\Lambda = R^{\times 2}$ and if h is of type $\{\varepsilon\}$ then $\Lambda = R^\times \backslash R^{\times 2}$.*

(b) *Suppose $m = 2$. If h is of type $\{1, -1\}$, then $\Lambda = R$ and if h is of type $\{1, -\varepsilon\}$ then $\Lambda = R^\times$.*

(c) *If $m > 2$ then $\Lambda = R$.*

Proof. (a) Assume $m = 1$. Assume that h is of type $\{1\}$. Then $\{u_1\}$ is a basis of V with $h(u_1, u_1) = 1$. Pick $r \in R^{\times 2}$. Because $*$ is the identity on $A/\mathfrak{r}$, $Q(A^\times) = R^{\times 2}$. Pick $r \in R^{\times 2}$. Then $r = Q(a) = aa^*$ for some $a \in A^\times$ and $h(u_1 a^*, u_1 a^*) = aa^* = r$. Thus $R^{\times 2} \subseteq \Lambda$. Now let $v \in V$ be primitive. Because $m = 1$, $v = u_1 a$ for some $a \in A^\times$ and $h(v, v) \in Q(A) = R^{\times 2}$. Thus $\Lambda = R^{\times 2}$. A similar argument shows that $\Lambda = R \backslash R^{\times 2}$ when h is of type $\{\varepsilon\}$.

(b) Assume $m = 2$ and h is of type $\{1, -1\}$ with corresponding basis vectors u_1, u_2. Then $u_1 + u_2$ is primitive and $h(u_1 + u_2, u_1 + u_2) = 0$. Applying Lemma 3.1 shows that $\Lambda = R$.

Suppose instead that h is of type $\{1, -\varepsilon\}$. Assume that $v = u_1 a_1 + u_2 a_2$ is primitive and $h(v, v) \in \mathfrak{m}$. That is, $a_1 a_1^* - \varepsilon a_2 a_2^* = f \in \mathfrak{m}$. Because v is primitive, at least one a_1, a_2 is a unit. Without loss of generality, assume that $a_1 \in A^\times$. Then a_2 is also a unit because $\varepsilon a_2 a_2^* \neq 0$ in $A/\mathfrak{r}$.

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Because $Q(A) = R^{\times^2}$, $a_1 a_1^* = b_1^2$ and $a_2 a_2^* = b_2^2$ for some $b_1, b_2 \in R^\times$. Then $b_1^2 - \varepsilon b_2^2 = f$. Let $c_1 = b_1 + \mathfrak{r}, c_2 = b_2 + \mathfrak{r}, \delta = \varepsilon + \mathfrak{r} \in A/\mathfrak{r}$ Then $c_1^2 - \delta c_2^2 = 0$ in $A/\mathfrak{r}$ with $c_1, c_2, \delta \neq 0$. But $\delta = c_1^2(c_2^{-1})^2 = (c_1 c_2^{-1})^2$, contradicting the assumption that $\varepsilon \notin R^{\times^2}$. Thus $h(v,v) \in R^\times$ for all primitive v. Because h is of type $\{-1, \varepsilon\}$ as well as $\{1, -\varepsilon\}$ there are primitive vectors u and v with $h(u,u) = 1$ and $h(v,v) = \varepsilon$. Thus $\Lambda = R^\times$.

(c) Assume that $u_1, u_2, \ldots u_m$ is an orthogonal basis of V with $h(u_i, u_i) \in R^\times$. Then $-h(u_3, u_3) \in R^\times$ and by Proposition 3.4, there exists a primitive vector $v \in u_1 A \oplus u_2 A$ with $h(v,v) = -h(v_3, v_3)$. Then $u = v + u_3$ is primitive with $h(u,u) = 0$, and applying lemma 3.1 shows that $\Lambda = R$. $\qquad\square$

Corollary 3.12. *When $m = 1$, h cannot be isotropic. When $m = 2$, h is isotropic if and only if it is of type $\{1, -1\}$. When $m \geq 3$, h must be isotropic.*

Proof. Follows directly from preceding result. $\qquad\square$

Example 3.13. Continue with h as defined previously. In our example, the involution that $*$ induces on $A/\mathfrak{r}$ is the identity. Then h is of type $\{1\}$ with respect to the basis $v_1, \ldots, v_m$.

(a) If $m = 1$, then $v_1 a$ is primitive for all $a \notin \mathfrak{r}$. Then $h(v_1 a, v_1 a) = a^* a \in Q(A^\times)$. Then $\Lambda = Q(A^\times) = R^{\times^2}$

(b) If $m = 2$, then by Lemma 3.10, h is of type $\{1, -\varepsilon\}$. Then h is also of type $\{-1, \varepsilon\}$ and there exist primitive vectors u and v such that $h(u,u) = 1$ and $h(v,v) = \varepsilon$. Then for $x \in R^{\times^2}, x = Q(a)$ for some $a \in A^\times$ and $x = aa^* = h(va^*, va^*)$. Thus $R^{\times^2} \subset \Lambda$. Similarly, if $y \notin R^{\times^2}$, then $y = \varepsilon b^2$ for some $b \in R^\times$ (note this is because $[R^\times : R^{\times^2}] = 2$). Then $y = h(vb, vb)$ and $R^\times \backslash R^{\times^2} \subset \Lambda$, implying that $\Lambda = R^\times$.

(c) By assumption, $h(v_3, v_3) = 1$. Applying Proposition 3.4, there exists a primitive $u \in v_1 A \oplus v_2 A$ such that $h(u,u) = -1$. Then $u + v_3$ is primitive and $h(u + v_3, u + v_3) = 0$, allowing Lemma 3.1 to be applied and shows that $\Lambda = R$.

This shows, that by Corollary 3.12, h from our example is not isotropic if $m = 1$ or $m = 2$.

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