1 Background in Hermitian Geometry

In this section, we'll define several items:

- Let A be a local ring with identity.
- Let $\mathfrak{r}$ be the Jacobsen radical of A. Because A is local, $\mathfrak{r}$ is maximal, and contains all non-units of A.
- Let $*$ be an involution of A. Assume that elements fixed by $*$ are in the center of A, forming a ring $R = \{a \in A : a^* = a\}$. Note that R is local, with maximal ideal $R \cap \mathfrak{r}$. This is because any element of R that is not in $R \cap \mathfrak{r}$ is invertible by definition. Thus the only maximal ideal is $R \cap \mathfrak{r}$.
- Let $Q : A^* \to R^* : a \mapsto aa^*$ (the norm map) be a group homomorphism with kernel N.

Now let V be a right A-module and $h : V \times V \to A$ be a *-hermitian form. By definition, h is linear in the second variable and $h(v, u) = h(u, v)*$ for $u, v \in V$. Then $h(u, u) = h(u, u)*$ and $h(u, u) \in R \subseteq Z(A)$ for all $u \in V$.

Now consider the dual space V^*. We can define an operation $V^* \times A \to V^*$ by $(\alpha a)(v) = a^*\alpha(v)$ where $\alpha \in V^*, a \in A, v \in V$. Under this operation, V^* is a right A-module. Now we can define a homomorphism of right A-modules $\gamma_h : V \to V^*$ associated with h given by $\gamma_h(u) = h(u, -)$.

Some more assumptions:

- Assume that h is non-degenerate; γ_h is an isomorphism.
- Let U be the subgroup of $GL(V)$ preserving h. I'm guessing that this means for $\varphi \in U, u, v \in V$, $h(\varphi(u), \varphi(v)) = h(u, v)$.
- Assume the existence of an element $d \in A$ such that $d + d^* = 1$.
- Assume that V is a free A-module of rank $m \geq 1$. According to paper: "This is well defined, as can be seen by reducing modulo $\mathfrak{r}$.

Now let $\{v_1, v_2, \ldots, v_m\}$ be a basis of V throughout the section.

Lemma 1.1. *There is a vector $u \in V$ such that $h(u, u) \in R^*$.*

Proof. Assume otherwise; that for some $h(u, u) \in \mathfrak{m}$ for all $u \in V$. Then using the linearity of h:

$$h(u, v) + h(u, v)^* = h(u + v, u + v) - h(u, u) - h(v, v) \in \mathfrak{m}$$

for all $u, v \in V$. Let $\alpha \in V*$ be the linear functional such that $\alpha(v_1) = d$ and $\alpha(v_i) = 0$ for all $i > 1$. Because h is assumed to be non-degenerate, there exists $u \in V$ such that $h(u, -) = \alpha$. Then $d = \alpha(v_1) = h(u, v_1)$ and $1 = d + d^* = h(u, v_1) + h(u, v_1)^* \notin \mathfrak{m}$, contradicting the original hypothesis. $\square$

Lemma 1.2. *V has an orthogonal basis $u_1, u_2, \ldots u_m$. Any such basis satisfies $h(u_i, u_i) \in R^*$.*

Proof. Prove with induction on m. Assume that $m = 1$. By the previous lemma, there exists $u \in V$ such that $h(u, u) \in R^*$. Then $u = v_1 a_1$ for some $a_1 \in A^*$, and $h(u, u) = h(v_1 a_1, v_1 a_1) = a_1^* h(v_1, v_1) a_1 \in R*$ implying that $h(v_1, v_1) \in R^*$. Now assume that $m > 1$ and that the hypothesis holds for $m - 1$. Once again, there exists $u \in V$ such that $h(u, u) \in R^*$. Then $u = v_1 a_1 + \cdots + v_m a_m$ with $a_i \in A$. If all $a_i \in \mathfrak{r}$, then $h(u, u) \in \mathfrak{m}$, a contradiction. Without loss of generality, assume that $a_1 \notin \mathfrak{r}$. Then if $u_1 = v_1 a_1$, the set $\{u_1, v_2, \ldots, v_m\}$ is a basis of V. For $1 < i \leq m$, set

$$u_i = v_i - u_1[h(u_1, v_i)/h(u_1, u_1)]$$

Then $u_1, u_2, \ldots, u_m$ is a basis of V satisfying $h(u_1, u_i) = 0$ for $1 < i \leq m$. Let $V_1 = u_1 A$ and $V_2 = \text{span}\{u_2, \ldots, u_m\}$. Then $V = V_1 \perp V_2$ and the restriction of h to V_2 induces an isomorphism $V_2 \to V_2^*$. Applying the inductive hypothesis to this space completes the proof. $\square$

Lemma 1.3.

- *(a) Suppose $u_1, \ldots, u_s \in V$ are orthogonal and satisfy $h(u_i, u_i) \in R^*$. Then $u_1, \ldots, u_s \in V$ can be extended to an orthogonal basis of V with the same property.*
- *(b) If V_1 is a submodule of V such that the restriction of h to V_1 is non-degenerate there is another such submodule V_2 of V such that $V = V_1 \perp V_2$.*

Proof. (a) Can represent $u_1 = v_1 a_1 + \cdots + v_m a_m$ for some $a_i \in A$. Since $h(u_1, u_1) \in R^*$ (by previous lemma), one of the scalars must be a unit. WLOG assume $a_i \in A*$. Thus $u_1, v_2, \ldots, v_m$ is a basis of V. Suppose $1 \leq t \leq s$ and the list $u_1, \ldots, u_t, u_{t+1}, \ldots, v_m$ is a basis of V. Then

$$u_{t+1} = u_1 b_1 + \cdots + u_t b_t + v_{t+1} b_{t+1} + \cdots + v_m b_m$$

for some $b_i \in A$. Suppose, if possible, that $b_i \in \mathfrak{r}$ for all $i \geq t + 1$. Then for every $i \leq t$,

$$0 = h(u_i, u_{t+1}) = h(u_i, u_i) b_i + h(u_i, v_{t+1}) b_{t+1} + \cdots + h(u_i, v_m) b_m$$

implying that $b_i \in \mathfrak{r}$ for all $1 \leq i \leq t$, contradicting the assumptino that $h(u_{t+1}, u_{t+1}) \in R^*$. Thus at least one of $b_{t+1}, \ldots, b_m$ is a unit (assume b_{t+1} and $u_1, \ldots u_t, u_{t+1}, v_{t+2}, \ldots, v_m$ is a basis of V.

This process can be repeated to extend $u_1, \ldots, u_s$ to a basis $u_1, \ldots, u_s, u_{s+1}, \ldots u_m$ of V. For $s < i \leq m$, let

$$z_i = u_i - ([u_1 h(u_1, u_i)/h(u_1, u_1)] + \cdots + u_s h(u_s, u_i)/h(u_s, u_s)].$$

Then $u_1, \ldots, u_s, z_1, \ldots, z_{m-s}$ is a basis of V satisfying $h(u_i, z_j) = 0$. If follows that the restriction of h to $M = \text{span}\{z_1, \ldots, z_{m-s}\}$ is non-degenerate and by lemma 1.2 that M has an orthogonal basis with $h(z_i, z_i) \in R^*$ for any $i \leq m - s$.

(b) Follows from (a) and lemma 1.2 □

Lemma 1.4. *Let $u_1, \ldots u_s \in V$, with corresponding Gram matrix $M \in M_s(A)$, defined by $M_{ij} = h(u_i, u_j)$. If $M \in GL_m(A)$, then $u_1, \ldots, u_s$ are linearly independent.*

Proof. Suppose $a_1, \ldots, a_s$ satisfy $u_1 a_1 + \cdots + u_s a_s = 0$. Then for $1 \leq i \leq s$

$$0 = h(u_i, u_1 a_1 + \cdots + u_s a_s) = h(u_i, u_1) a_1 + \cdots + h(u_i, u_s) a_s$$

implying that

$$M \begin{pmatrix} a_1 \\ \vdots \\ c_1 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Since M is inverible, the desired result follows. □