Lecture 3
Forward price

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Where does the forward price come from?

< > WTI CRUDE FUTURE Feb20 Comdty ▾ | WEIF ▾ | Related Functions Menu ▾ | Message

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News ▾ Settings ▾ World Equity Index Futures

				2Day	Last	Net Chg	Time	High	Low
1) Americas									
11) Dow Jones mini	Mar20	DMH0	d		28856.00	+79.00	12:08	28907.00	28775.00
12) S&P 500 mini	Mar20	ESH0	d		3282.75	+18.00	12:08	3283.00	3265.50
13) NASDAQ 100 mini	Mar20	NQH0	d		9060.75	+82.50	12:08	9064.75	8979.00
14) S&P/TSX 60	Mar20	PTH0	d		1024.40	+0.00	11:58	1026.60	1021.80
15) MEX IPC	Mar20	ISH0	d		45150.00	-4.00	11:54	45445.00	45125.00
16) IBOVESPA	Feb20	BZG0	d		117530	+1804	12:03	117610	116120
2) EMEA									
21) EURO STOXX 50	Mar20	VGH0	d		3769.00	-11.00	12:03	3788.00	3756.00
22) FTSE 100	Mar20	Z H0	d		7556.0	+26.5	12:08	7572.0	7516.0
23) CAC 40	Jan20	CFF0	d		6034.00	-1.50	12:02	6058.00	6017.00
24) DAX	Mar20	GXH0	d		13452.50	-38.50	12:03	13527.00	13398.50
25) IBEX 35	Jan20	IBF0	d		9556.0	-24.0	12:03	9583.0	9501.0
26) FTSE MIB	Mar20	STH0	d		23825	-106	12:02 c	23965	23720
27) AEX	Jan20	EOF0	d		609.45	-0.96	12:03	611.35	606.30
28) OMX STKH30	Jan20	QCF0	d		1789.00	+4.00	11:59 c	1798.25	1780.25
29) SWISS MKT	Mar20	SMH0	d		10530.00	-26.00	12:02	10583.00	10493.00
3) Asia/Pacific									
31) NIKKEI 225 (OSE)	Mar20	NKH0	d		23670	-130	15:30	23830	23670
32) HANG SENG	Jan20	HIF0	d		29072	+110	12:03	29073	28887
33) CSI 300	Jan20	IFBF0	d		4197.40 s	+30.20	02:00 c	4212.60	4149.60
34) S&P/ASX 200	Mar20	XPH0	d		6853.0	+12.0	12:08	6860.0	6825.0

Suggested Functions | **WCR** See global currency rates in real time | **FIT** Watch FI market moves & execute trades

Roadmap: the forward price

Arbitrage

- A portfolio is a collection of securities.
- An arbitrage is a portfolio that either:
  1. Costs nothing today, never yields negative cash flows, and sometimes yields positive cash flows.
  2. Costs a negative amount today (you get paid to buy it) and never yields negative cash flows.

Illustrating arbitrage with no uncertainty

- Let V_0 denote the price of a portfolio today.
- Let V_T denote the value of the portfolio in the future.
- Suppose that V_T is known today – i.e., no uncertainty.
- When is the portfolio V an arbitrage?

A portfolio with no uncertainty

today

the future

V_0

V_T

Arbitrage conditions with no uncertainty

- With no uncertainty, V is an arbitrage if either:

  1. $V_0 = 0$ and $V_T > 0$.

  2. $V_0 < 0$ and $V_T \geq 0$.

- The first case is "get something for paying nothing".

- The second case is "get paid to potentially get something".

Practice problem

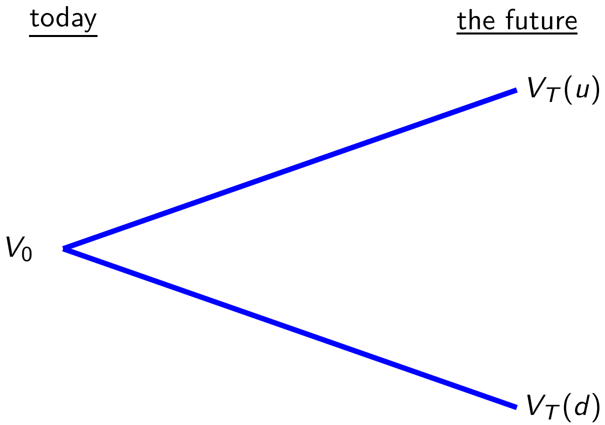
Consider two securities X and Y. Security X has a price of $X_0 = 1$ and a certain payoff of $X_T = 3$. Security Y has a price of $Y_0 = 1$ and a payoff of $Y_T = 2$. Is there an arbitrage? If so, construct a portfolio to capitalize upon it.

Practice problem scratch paper

Illustrating arbitrage with uncertainty

- Now suppose there is uncertainty about the payoff V_T.
- Specifically, two possible states of the world in the future:
  - State u ("up") with payoff $V_T(u)$.
  - State d ("down") with payoff $V_T(d)$.
- When is the portfolio V an arbitrage?

A portfolio with uncertainty



Arbitrage conditions with uncertainty

- With uncertainty over state $\{u, d\}$, V is an arbitrage if either:

  1. $V_0 = 0$ and either:

     a. $V_T(u) > 0$ and $V_T(d) \geq 0$.

     b. $V_T(u) \geq 0$ and $V_T(d) > 0$.

  2. $V_0 < 0$ and both $V_T(u) \geq 0$ and $V_T(d) \geq 0$.

Practice problem

Consider two securities: X and Y. Security X has a price of $X_0 = 1$ and payoffs $X_T(u) = 3$ and $X_T(d) = 0$. Security Y has a price of $Y_0 = 1$ and payoffs $Y_T(u) = 2$ and $Y_T(d) = 0$. Is there an arbitrage? If so, construct a portfolio to capitalize upon it.

Practice problem scratch paper

Practice problem

Consider two securities: X and Y. Security X has a price of $X_0 = 1$ and payoffs $X_T(u) = 3$ and $X_T(d) = 0$. Security Y has a price of $Y_0 = 2$ and payoffs $Y_T(u) = 0$ and $Y_T(d) = 3$. The probability of the "up" state occurring is $\Pr(u) = 0.5$. The probability of the "down" state occurring is $\Pr(d) = 0.5$. Is there an arbitrage? If so, construct a portfolio to capitalize upon it.

Practice problem scratch paper

No-arbitrage

- What happens if a portfolio is an arbitrage?
- Free money is on the table! Arbitrageurs buy low and sell high.
- Demand will be infinite; markets will not clear.
- No-arbitrage means that market prices preclude arbitrage.

The law of one price (LoP)

- The law of one price (LoP) states that portfolios with the same payoffs (or terminal values) must have the same price.

- The LoP is directly implied by no-arbitrage.

- The LoP is the fundamental concept behind all derivatives pricing. We will use it to derive forward and options prices.

Practice problem

Security X has a price of $X_0 = 10$ and payoffs $X_T(u) = 20$ and $X_T(d) = 5$. Assume there is no arbitrage.

1. What is the price of a security Y with payoffs $Y_T(u) = 20$ and $Y_T(d) = 5$?

2. What is the price of a security Z with payoffs $Z_T(u) = 10$ and $Z_T(d) = 2.5$?

3. What is the price of a security W with payoffs $W_T(u) = 40$ and $W_T(d) = 10$?

Practice problem scratch paper

Practice problem scratch paper

Roadmap: the forward price

1. No-arbitrage and the LoP.
2. **Forward price.**
3. Synthetic positions.
4. Value of off-market forwards.
5. Summary

Back to cocoa

- Setup:

  1. Two dates: 0 and T.
  2. C.c. risk-free interest rate of $r \geq 0$.
  3. Cocoa spot price at date t of S_t.

- Consider the payoff to a long forward:

$$X_T = S_T - F_{0,T}$$

- What other portfolio has this same payoff?

Replicating portfolio

- Form the following portfolio at date 0:

  1. Long one unit of cocoa at cost S_0.
  2. Short a risk-free bond with face-value of $F_{0,T}$.

- The payoff to the portfolio at date T is:

$$Y_T = S_T - F_{0,T}$$

- This is the same payoff as the long forward!

Example of state-by-state replication

today

$Y_0 = S_0 - F_{0,T}e^{-rT}$

$X_0 = 0$

the future

$Y_T(u) = S_T(u) - F_{0,T}$

$X_T(u) = S_T(u) - F_{0,T}$

$Y_T(d) = S_T(d) - F_{0,T}$

$X_T(d) = S_T(d) - F_{0,T}$

The forward price today

- Apply LoP: value of forward today=value of replicating portfolio.
- Value of forward today is zero by construction:

$$X_0 = 0.$$

- Value of long cocoa, short bond portfolio today is:

$$Y_0 = S_0 - F_{0,T}e^{-rT}.$$

- Equating $X_0 = Y_0$ and solving for the forward price yields:

$$F_{0,T} = S_0 e^{rT}.$$

Practice problem

The spot price of cocoa is $S_0 = \$2,500$. The c.c. risk-free interest rate is 0.05. Assume there is no arbitrage. What is the one-year forward price of cocoa? Demonstrate that payoffs to the long forward and the replicating portfolio are equivalent at date T.

Practice problem scratch paper

Exploiting arbitrage opportunities: "buy low, sell high"

- What would you do if $F_{0,T} > S_0 e^{rT}$?
  1. Enter short forward contract.
  2. Purchase synthetic forward: long cocoa and short bond.
- What would you do if $F_{0,T} < S_0 e^{rT}$?
  1. Enter long forward contract.
  2. Sell synthetic forward: short cocoa and long bond.

Practice problem

The spot price of cocoa is $S_0 = \$2,500$. The c.c. risk-free interest rate is 0.05. Suppose the one year forward price is $F_{0,1} = \$2,700$. Is there an arbitrage? If so, construct a strategy to capitalize upon it.

Practice problem scratch paper

What is the meaning of $F_{0,T} = S_0 e^{rT}$?

- Is the forward price = expected future spot price? No! Instead:

$$F_{0,T} = S_0 + \text{cost of carry}.$$

- The cost of carry is the net cash flow associated with holding a long position in the underlying. In our setting:

$$\text{cost of carry} = \underbrace{S_0\left(e^{rT} - 1\right)}_{\text{interest foregone by long}}.$$

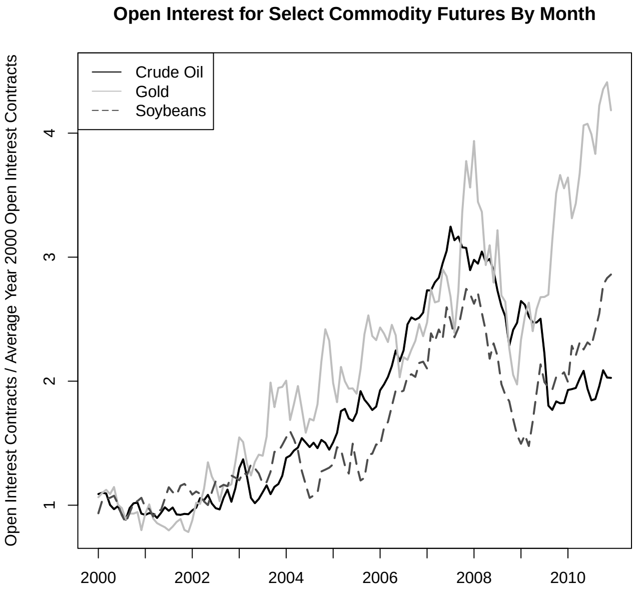
Roadmap: the forward price

1. No-arbitrage and the LoP.
2. Forward price.
3. **Synthetic positions.**
4. Value of off-market forwards.
5. Summary

Synthetic positions

- Replication can be used to create synthetic positions:
  1. Forward = stock - bond.
  2. Stock = forward + bond.
  3. Bond = stock - forward.
- Synthetics are the most basic form of financial engineering.

Recall synthetics drove commodity futures growth



Roadmap: the forward price

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The forward price changes over time

- What is the value of your original long forward at date τ?

<u>origination</u> <u>tomorrow</u> <u>the future</u>

$F_{0,T}$ ——— $F_{\tau,T}$ ——— $F_{T,T} = S_T$

Value of off-market forward

- The value of your original long forward at date τ is:

$$\left(F_{\tau,T} - F_{0,T}\right) e^{-r(T-\tau)}.$$

- Intuition:
  - You can lock-in a payoff by shorting a new forward.
  - Because the payoff is risk-free, we discount with rate r.
  - If the forward price has risen, we have made a profit.

Practice problem

Suppose the spot price of the S&P 500 index is $S_0 = \$3,300$ and risk-free interest rate is $r = 0.10$. At date 0, you enter into a long, three-year forward contract on the S&P 500 index. One year passes and the S&P index has risen to $S_1 = \$3,500$. What is the value of your forward contract at date $\tau = 1$?

Practice problem scratch paper

Practice problem scratch paper

Roadmap: the forward price

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Summary

- The law of one price states that portfolios with the same payoffs must have the same prices. This is true if no-arbitrage holds.

- No-arbitrage forward price = spot price + cost of carry:

$$F_{0,T} = S_0 e^{rT}$$

- Value of off-market forward at date $\tau \in (0, T)$ is:

$$V_\tau = (F_{\tau,T} - F_{0,T})\, e^{-r(T-\tau)}.$$

References

- Textbook chapters 5.1, 5.2, and 5.3.
- Equity index forward prices are from Bloomberg.
- Slides are created using code on my Github.

Why is $F_{0,T} \neq$ expected future spot price?

- If the underlying has a positive risk premium, then the forward price must be a downward biased predictor of future spot price:

$$F_{0,T} < \mathbb{E}(S_T).$$

- Rationale:
  - Despite no investment, forward retains risks of underlying.
  - To bear this risk, buyer must earn at least the risk premium.
  - This can only occur if $X_T = S_T - F_{0,T} > 0$, on average.

Return