Lecture 4
Frictions

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What assumptions have we made so far?

- Zero cost of storage.
  - Financial asset: no dividends.
  - Commodity: no physical storage costs.
- Investor can short the underlying.
- Investment asset – i.e., no production.
- No transactions costs, can borrow at risk-free rate, ...

What happens if we introduce frictions?

Roadmap: the forward price with frictions

1. **Dividends.**
2. Commodity storage and convenience.
3. Limits to arbitrage.

Dividends

- Financial assets (e.g., stocks) commonly pay dividends.
- Let $d > 0$ be the dividend paid on a stock at time T.
- Will the dividend increase or decrease the forward price?
- Think about the forward price decomposition:

$$F_{0,T} = S_0 + \text{cost of carry.}$$

Not-so replicating portfolio

- Dividends do not affect the payoff to a long forward:

$$X_T = S_T - F_{0,T}.$$

- Stock payoff now includes dividend (price is ex-dividend):

$$S_T + d.$$

- Leveraged stock portfolio no longer replicates forward:

$$(S_T + d) - F_{0,T} > S_T - F_{0,T}.$$

Downward adjusted replicating portfolio

- Form the following portfolio at date 0:

  1. Long one share of stock.

  2. Short a risk-free bond with face-value $F_{0,T} + d$.

- The payoff to this portfolio at date T is:

$$\begin{aligned} Y_T &= (S_T + d) - (F_{0,T} + d) \\ &= S_T - F_{0,T}. \end{aligned}$$

- This is the same payoff as the long forward!

The forward price with a dividend

- Apply LoP: value of forward today=value of replicating portfolio.
- Value of forward today is zero by construction:

$$X_0 = 0.$$

- Value of long stock, short bond portfolio today is:

$$Y_0 = S_0 - (F_{0,T} + d)\, e^{-rT}.$$

- Equating $X_0 = Y_0$ and solving for the forward price yields:

$$F_{0,T} = S_0 e^{rT} - d.$$

Practice problem

The price of AAPL stock is \$300 per share. Apple has comitted with certainty to make a \$5 divided payment per share in 6 months. The c.c. risk-free rate is one percent.

1. What is the six-month forward price of AAPL stock?
2. Is there an arbitrage if $F_{0,0.5} = \$310$? If so, construct a portfolio to capitalize upon it.
3. Is there an arbitrage if $F_{0,0.5} = \$290$? If so, construct a portfolio to capitalize upon it.

Practice problem scratch paper

Practice problem scratch paper

Cost of carry with dividends

- The asset holder enjoys the dividend while the forward does not.
- Thus, forward must be price-compensated for deferred purchase.
- Another way to say this is dividends reduce the cost of carry:

$$\text{cost of carry} = S_0 \left(e^{rT} - 1 \right) - d.$$

- Forward curve is up (down) sloping if interest $>$ ($<$) dividends.

E-mini forward curve (01/15/19)

Description	Last	Chg Settle	Time	Bid	Ask	Open Int	Volume	Yest Settle
21) Spot	s	--	Close	2572.16	2641.69			
22) Mar19	2605.50 s	+25.00	Close	2605.50	2606.00	2596574	1494266	2580.50
23) Jun19	2610.50 s	+25.25	Close	2602.00	2620.00	55892	8530	2585.25
24) Sep19	2615.25 s	+25.00	Close	2550.00	2662.00	2673	156	2590.25
25) Dec19	2617.50 s	+22.75	Close		2650.00	66		2594.75
26) Mar20	2624.00 s	+25.75	Close			5	5	2598.25

S&P 500 dividend yield vs one year LIBOR

E-mini forward curve (01/15/2010)

S&P 500 dividend yield vs one year LIBOR

General formulas for dividends

- For an asset with dividends $\{d_1, d_2, ..., d_n\}$ at dates $\{t_1, t_2, ..., t_n\}$ such that $0 \leq t_1 \leq ... \leq t_n \leq T$, the no-arbitrage forward price is:

$$F_{0,T} = S_0 e^{rT} - \sum_{i=1}^{n} d_i e^{r(T-t_i)}.$$

- For a financial asset with a continuous dividend yield $\delta > 0$:

$$F_{0,T} = S_0 e^{(r-\delta)T}.$$

- This latter formula is commonly used to price index forwards.

Roadmap: the forward price with frictions

1. Dividends.
2. **Commodity storage and convenience.**
3. Limits to arbitrage.

Commodity storage costs

- Commodities are costly to store (e.g., grain silos), if at all.
- Let $c > 0$ be the storage cost for date 0 to T, paid at T.
- Will the storage cost increase or decrease the forward price?
- Think about the forward price decomposition:

$$F_{0,T} = S_0 + \text{cost of carry}.$$

Not-so replicating portfolio

- Storage costs do not affect the payoff to a long forward:

$$X_T = S_T - F_{0,T}.$$

- Commodity payoff now includes storage cost:

$$S_T - c.$$

- Leveraged commodity portfolio no longer replicates forward:

$$(S_T - c) - F_{0,T} < S_T - F_{0,T}.$$

Upward adjusted replicating portfolio

- Form the following portfolio at date 0:

  1. Long one unit of the commodity.
  2. Short a risk-free bond with face-value $F_{0,T} - c$.

- The payoff to this portfolio at date T is:

$$\begin{aligned} Y_T &= (S_T - c) - (F_{0,T} - c) \\ &= S_T - F_{0,T}. \end{aligned}$$

- This is the same payoff as the long forward!

The forward price with storage costs

- Apply LoP: value of forward today=value of replicating portfolio.
- Value of forward today is zero by construction:

$$X_0 = 0.$$

- Value of long commodity, short bond portfolio today is:

$$Y_0 = S_0 - (F_{0,T} - c)\, e^{-rT}.$$

- Equating $X_0 = Y_0$ and solving for the forward price yields:

$$F_{0,T} = S_0 e^{rT} + c.$$

Practice problem

The price of AAPL stock is \$300 per share. Apple has comitted with certainty to make a \$5 divided payment per share in 6 months. The c.c. risk-free rate is one percent.

1. What is the six-month forward price of AAPL stock?
2. Is there an arbitrage if $F_{0,0.5} = \$310$? If so, construct a portfolio to capitalize upon it.
3. Is there an arbitrage if $F_{0,0.5} = \$290$? If so, construct a portfolio to capitalize upon it.

Practice problem scratch paper

Practice problem scratch paper

Cost of carry with storage costs

- The asset holder pays storage cost while the forward does not.
- Thus, forward must provide price compensation for deferring.
- Another way to say this is storage costs increase cost of carry:

$$\text{cost of carry} = S_0\left(e^{rT} - 1\right) + c.$$

- Forward curve is upward sloping based on this formula.

Gold forward curve (01/13/20)

Description	Last	Chg Settle	Time	Bid	Ask	Open Int	Volume	Yest Settle
21) Feb20	↓1550.10	-10.00	12:33	1550.00	1550.10	460509	256646	1560.10
22) Apr20	1556.40	-10.00	12:33	1556.30	1556.40	201731	54281	1566.40
23) Jun20	1562.00	-10.10	12:30	1561.90	1562.10	68517	4240	1572.10
24) Aug20	1568.00	-9.50	12:06	1567.30	1567.40	22226	1342	1577.50
25) Oct20	1575.70	-6.60	8:31	1572.10	1572.30	7513	29	1582.30
26) Dec20	1577.90	-9.10	12:19	1576.80	1576.90	18646	2433	1587.00
27) Feb21	1581.90	-9.60	10:07	1581.00	1581.20	4572	217	1591.50
28) Apr21		--	1/10	1584.60	1585.30	530	111	1595.40
29) Jun21		--	1/10	1588.40	1589.30	1757	21	1599.10
30) Aug21		--	1/10	1585.10	1600.90	1		1603.30
31) Oct21		--	1/10	1589.30	1605.60			1607.20
32) Dec21	1606.20	-5.00	8:52	1600.50	1601.10	806	1	1611.20
33) Jun22		--	1/10	1598.10	1623.10	4		1620.10
34) Dec22		--	1/10	1620.00	1627.50	47	1	1629.00
35) Jun23		--	1/10	1610.00	1644.10	5		1638.40

A portfolio with no uncertainty

today	the future
V_t	V_T

Arbitrage with no uncertainty

- With no uncertainty, V is an arbitrage if either:
  1. $V_t = 0$ and $V_T > 0$ or
  2. $V_t < 0$ and $V_T \geq 0$.
- The first is "get something for paying nothing".
- The second is "get paid to potentially get something".

Practice problem #1

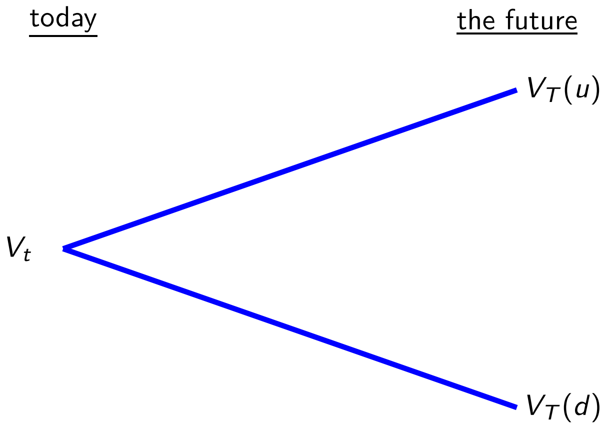
Consider two securities X and Y. Security X has a price of $X_t = 1$ and a certain payoff of $X_T = 3$. Security Y has a price of $Y_t = 1$ and a certain payoff of $Y_T = 2$. Is there an arbitrage? If so, construct a portfolio to capitalize upon the arbitrage.

Practice problem #1 solutions

Arbitrage with uncertainty

- Now suppose there is uncertainty about the payoff V_T.
- Specifically, two possible states of the world in the future:
  - State u ("up").
  - State d ("down").
- Let $V_T(\omega)$ be the payoff of the portfolio in state ω.
- When is the portfolio V an arbitrage?

A portfolio with uncertainty



Arbitrage with uncertainty

- With uncertainty over the state ω, V is an arbitrage if either:
  1. $V_t = 0$ and either:
     a. $V_T(u) > 0$ and $V_T(d) \geq 0$ or
     b. $V_T(u) \geq 0$ and $V_T(d) > 0$ or
  2. $V_t < 0$ and both $V_T(u) \geq 0$ and $V_T(d) \geq 0$.

Practice problem #2

1. Consider two securities: X and Y. Security X has a price of $X_t = 1$ and payoffs $X_T(u) = 3$ and $X_T(d) = 0$. Security Y has a price of $Y_t = 1$ and payoffs $Y_T(u) = 2$ and $Y_T(d) = 0$. Is there an arbitrage? If so, construct a portfolio to capitalize upon it.

2. Consider two securities: X and Y. Security X has a price of $X_t = 1$ and payoffs $X_T(u) = 3$ and $X_T(d) = 0$. Security Y has a price of $Y_t = 1$ and payoffs $Y_T(u) = 0$ and $Y_T(d) = 3$. Is there an arbitrage? If so, construct a portfolio to capitalize upon it.

Practice problem #2 solutions

No-arbitrage

- What happens if a portfolio is an arbitrage?
- Free money is on the table! Arbitrageurs buy low and sell high.
- Demand will be infinite; markets will not clear.
- No-arbitrage means that market prices preclude arbitrage.

The law of one price (LoP)

- The law of one price (LoP) states that portfolios with the same payoffs must have the same price.
- The LoP is directly implied by no-arbitrage.
- The LoP is the fundamental concept behind all derivatives pricing. We will use it to derive forward and options prices.

Practice problem #3

Security X has a price of $X_t = 10$ and payoffs $X_T(u) = 20$ and $X_T(d) = 5$. Assume there is no arbitrage. Answer the following:

1. What is the price of a security Y with payoffs $Y_T(u) = 20$ and $Y_T(d) = 5$?

2. What is the price of a security Z with payoffs $Z_T(u) = 10$ and $Z_T(d) = 2.5$?

3. What is the price of a security W with payoffs $W_T(u) = 40$ and $W_T(d) = 10$?

Practice problem #3 solutions

Practice problem #3 solutions

Roadmap: the forward price

1. No-arbitrage and the law of one price
2. **Deriving the no-arbitrage forward price.**
3. Value of off-market forwards.
4. Summary

Back to cocoa

- Setup:
  1. Two dates: t (today) and T (the future).
  2. Discrete risk-free interest rate of $r \geq 0$.
  3. Cocoa price of S_t and random payoff S_T.
- Consider the payoff to a long forward:

$$X_T = S_T - F_{t,T}$$

- What other portfolio has this same payoff?

Replicating portfolio

- Form the following portfolio at date t:

  1. Long one unit of cocoa at cost S_t.
  2. Short a zero-coupon bond with face value of $F_{t,T}$.

- The payoff to the portfolio at date T is:

$$Y_T = S_T - F_{t,T}$$

- This is the same payoff as the long forward in every state.

Example of state-by-state replication

today

$Y_t = S_t - \frac{F_{t,T}}{(1+r)^{T-t}}$

$X_t = 0$

the future

$Y_T(u) = S_T(u) - F_{t,T}$

$X_T(u) = S_T(u) - F_{t,T}$

$Y_T(d) = S_T(d) - F_{t,T}$

$X_T(d) = S_T(d) - F_{t,T}$

The forward price today

- By the LoP, the value of the forward contract today must equal the value of the portfolio today.

- Price of portfolio of long cocoa, short bond today is:

$$Y_t = S_t - F_{t,T}/(1+r)^{T-t}$$

- Value of forward today is $X_t = 0$ by definition.

- Thus, value of forward = price of portfolio implies:

$$F_{t,T} = S_t(1+r)^{T-t}$$

Practice problem #4

The spot price of cocoa at date 0 is $S_0 = \$2,500$. The risk-free interest rate is 0.05. What is the one-year forward price of cocoa? Demonstrate that the payoff to the long forward and the replicating portfolio are equivalent at date T.

Practice problem #4

Exploiting arbitrage opportunities

- What would you do if $F_{t,T} > S_t(1+r)^{T-t}$?
  1. Enter short forward contract.
  2. Purchase synthetic forward: long cocoa and short bond.
- What would you do if $F_{t,T} < S_t(1+r)^{T-t}$?
  1. Enter long forward contract.
  2. Sell synthetic forward: short cocoa and long bond.

Roadmap: the basics of forward contracts

1. Definitions
2. Payoffs
3. **Applications of forwards**
4. Interest rates

Application of forwards

- Two applications: Risk management and speculation.
- Risk management give the example.
- Speculation is because of the lvereage.
- In our previous example, Hershey uses cocoa as an input in production. Hence, it has a natural short position in cocoa.

- On the other hand, farmers output cocoa and have natural long positions.

- A forward contract can be used to transfer risks between Hershey and cocoa farmers and capture certain surplus.

Roadmap: the basics of forward contracts

1. Definitions
2. Payoffs
3. Application of forwards
4. **Interest rates**

Technical note on interest rates

- Throughout the semester, we will assume there is a single risk-free interest rate of $r \geq 0$.

- Risk-free cash flows should be discounted at the risk-free rate.

- If $B_T \geq 0$ is a risk-free cash flow at time T, then the price of this cash flow at date $t < T$ is given by:

$$B_t = \begin{cases} \frac{B_T}{(1+r)^{-(T-t)}} & \text{if } r \text{ is discretely compounded} \\ \\ B_T e^{-r(T-t)} & \text{if } r \text{ is continuously compounded} \end{cases}$$

Practice problem #3

Suppose $r = 0.05$ and today is date $t = 0$. Compute the prices of the following securities assuming discrete compounding:

1. A risk-free security that pays \$1 at date $T = 1$.
2. A risk-free zero-coupon bond with a face value of \$100 and maturity of $T = 5$.
3. A $T = 3$ year risk-free coupon bond with annual coupons of \$5 and face value of \$100.

Practice problem #3 solutions

1. The price is $\$1 \cdot (1 + 0.05)^{-1} = \$0.952$.

2. The price is $\$100 \cdot (1 + 0.05)^{-5} = \$78.353$.

3. The price is $\sum_{i=1}^{3} 5 \cdot (1 + 0.05)^{-i} + \$100 \cdot (1 + 0.05)^{-3} = \$100$.

Practice problem #4

Suppose $r = 0.05$ and today is date $t = 0$. Compute the prices of the following securities assuming continuous compounding:

1. A risk-free security that pays \$1 at date $T = 1$.
2. A risk-free zero-coupon bond with a face value of \$100 and maturity of $T = 5$.
3. A $T = 3$ year risk-free coupon bond with annual coupons of \$5 and face value of \$100.

Practice problem #3 solutions

1. The price is $\$1 \cdot e^{-0.05} = \$0.951$.

2. The price is $\$100 \cdot e^{-0.05 \cdot 5} = \$77.88$.

3. The price is $\sum_{i=1}^{3} 5 \cdot e^{-0.05 \cdot i} + \$100 \cdot e^{-0.05 \cdot 3} = \$99.43$.

Summary

- A forward contract is an agreement to buy or sell an asset at a future date at the forward price.
- Date T payoff of long forward originated at date 0:

$$X_T = S_T - F_{0,T}.$$

- Forwards can be used to hedge input and output price risk.
- Date 0 price risk-free \$1 payoff at date T is $(1+r)^{-T}$ or e^{-rT}.

References

- Textbook chapters XXX, XXX, and XXX.
- Hershey chocolate article is in the Wall Street Journal.
- Cocoa prices are from Bloomberg screen XXX.
- Graphs are created using code on my Github.