

Lecture 3

Forward price



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Where does the forward price come from?

WTI CRUDE FUTURE Feb20 Comdty

WEIF

Related Functions Menu

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World Equity Index Futures

1) Americas			2Day	Last	Net Chg	Time	High	Low
11) Dow Jones mini	Mar20	DMHO		28856.00	+79.00	12:08	28907.00	28775.00
12) S&P 500 mini	Mar20	ESH0		3282.75	+18.00	12:08	3283.00	3265.50
13) NASDAQ 100 mini	Mar20	NQH0		9060.75	+82.50	12:08	9064.75	8979.00
14) S&P/TSX 60	Mar20	PTH0		1024.40	+0.00	11:58	1026.60	1021.80
15) MEX IPC	Mar20	ISH0		45150.00	-4.00	11:54	45445.00	45125.00
16) IBOVESPA	Feb20	BZGO		117530	+1804	12:03	117610	116120
2) EMEA								
21) EURO STOXX 50	Mar20	VGHO		3769.00	-11.00	12:03	3788.00	3756.00
22) FTSE 100	Mar20	Z H0		7556.0	+26.5	12:08	7572.0	7516.0
23) CAC 40	Jan20	CFF0		6034.00	-1.50	12:02	6058.00	6017.00
24) DAX	Mar20	GXHO		13452.50	-38.50	12:03	13527.00	13398.50
25) IBEX 35	Jan20	IBFO		9556.0	-24.0	12:03	9583.0	9501.0
26) FTSE MIB	Mar20	STHO		23825	-106	12:02	23965	23720
27) AEX	Jan20	EOF0		609.45	-0.96	12:03	611.35	606.30
28) OMX STKH30	Jan20	QCFO		1789.00	+4.00	11:59	1798.25	1780.25
29) SWISS MKT	Mar20	SMHO		10530.00	-26.00	12:02	10583.00	10493.00
3) Asia/Pacific								
31) NIKKEI 225 (OSE)	Mar20	NKHO		23670	-130	15:30	23830	23670
32) HANG SENG	Jan20	HIFO		29072	+110	12:03	29073	28887
33) CSI 300	Jan20	IFBFO		4197.40	+30.20	02:00	4212.60	4149.60
34) S&P/ASX 200	Mar20	XPH0		6853.0	+12.0	12:08	6860.0	6825.0

Suggested Functions

WCR See global currency rates in real time

FIT Watch FI market moves & execute trades

Roadmap: the forward price

1. No-arbitrage and the LoP.

2. Forward price.

3. Synthetic positions.

4. Value of off-market forwards.

5. Summary

Arbitrage

- ▶ A **portfolio** is a collection of securities.
- ▶ An **arbitrage** is a portfolio that either:
 1. Costs nothing today, never has a negative payoff, and sometimes has a positive payoff.
 2. Costs a negative amount today (you get paid) and never has a negative payoff.

Illustrating arbitrage with no uncertainty

- ▶ Let V_0 denote the price of a portfolio today.
- ▶ Let V_T denote the value of the portfolio in the future.
- ▶ Suppose that V_T is known today – i.e., no uncertainty.
- ▶ When is the portfolio V an arbitrage?

A portfolio with no uncertainty

today

the future

V_0  V_T

Arbitrage conditions with no uncertainty

► With no uncertainty, V is an arbitrage if either:

1. $V_0 = 0$ and $V_T > 0$.

2. $V_0 < 0$ and $V_T \geq 0$.

Practice problem

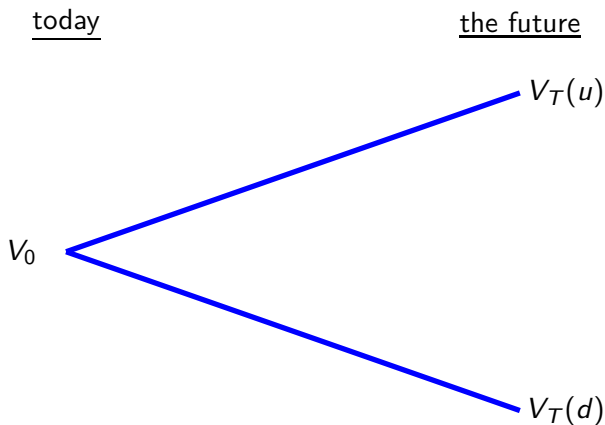
Consider two securities X and Y . Security X has a price of $X_0 = 1$ and a certain payoff of $X_T = 3$. Security Y has a price of $Y_0 = 1$ and a payoff of $Y_T = 2$. Is there an arbitrage?

Practice problem scratch paper

Illustrating arbitrage with uncertainty

- ▶ Now suppose there is uncertainty about the payoff V_T .
- ▶ Specifically, two possible states of the world in the future:
 - State u (“up”) with payoff $V_T(u)$.
 - State d (“down”) with payoff $V_T(d)$.
- ▶ When is the portfolio V an arbitrage?

A portfolio with uncertainty



Arbitrage conditions with uncertainty

- ▶ With uncertainty over state $\{u, d\}$, V is an arbitrage if either:
 1. $V_0 = 0$ and either:
 - a. $V_T(u) > 0$ and $V_T(d) \geq 0$.
 - b. $V_T(u) \geq 0$ and $V_T(d) > 0$.
 2. $V_0 < 0$ and both $V_T(u) \geq 0$ and $V_T(d) \geq 0$.

Practice problem

Consider two securities: X and Y . Security X has a price of $X_0 = 1$ and payoffs $X_T(u) = 3$ and $X_T(d) = 0$. Security Y has a price of $Y_0 = 1$ and payoffs $Y_T(u) = 2$ and $Y_T(d) = 0$. Is there an arbitrage?

Practice problem scratch paper

Practice problem

Consider two securities: X and Y . Security X has a price of $X_0 = 1$ and payoffs $X_T(u) = 3$ and $X_T(d) = 0$. Security Y has a price of $Y_0 = 2$ and payoffs $Y_T(u) = 0$ and $Y_T(d) = 3$. The probability of the “up” state occurring is $\Pr(u) = 0.5$. The probability of the “down” state occurring is $\Pr(d) = 0.5$. Is there an arbitrage? If so, construct a portfolio to capitalize upon it.

Practice problem scratch paper

No-arbitrage

- ▶ What happens if a portfolio is an arbitrage?
- ▶ Free money is on the table! Arbitrageurs buy low and sell high.
- ▶ No-arbitrage means that market prices preclude arbitrage.

The law of one price (LoP)

- ▶ The law of one price (LoP) states that portfolios with the same payoffs must have the same price.
- ▶ If the LoP does not hold, then there is an arbitrage.
- ▶ The LoP is the key concept underlying derivatives pricing.

Practice problem

Security X has a price of $X_0 = 10$ and payoffs $X_T(u) = 20$ and $X_T(d) = 5$. Assume there is no arbitrage.

1. What is the price of a security Y with payoffs $Y_T(u) = 20$ and $Y_T(d) = 5$?
2. What is the price of a security Z with payoffs $Z_T(u) = 10$ and $Z_T(d) = 2.5$?
3. What is the price of a security W with payoffs $W_T(u) = 40$ and $W_T(d) = 10$?

Practice problem scratch paper

Practice problem scratch paper

Roadmap: the forward price

1. No-arbitrage and the LoP.
2. **Forward price.**
3. Synthetic positions.
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5. Summary

Back to cocoa

► Setup:

1. Two dates: 0 and T .
2. C.c. risk-free interest rate of $r \geq 0$.
3. Cocoa spot price at date t of S_t .

► Consider the payoff to a long forward:

$$X_T = S_T - F_{0,T}$$

► What other portfolio has this same payoff?

Replicating portfolio

- ▶ Form the following portfolio at date 0:

1. Long one unit of cocoa at cost S_0 .
2. Short a risk-free bond with face-value of $F_{0,T}$.

- ▶ The payoff to the portfolio at date T is:

$$Y_T = S_T - F_{0,T}$$

- ▶ This is the same payoff as the long forward!

The forward price today

- ▶ Apply LoP: value of forward today=value of replicating portfolio.
- ▶ Value of forward today is zero by construction:

$$X_0 = 0.$$

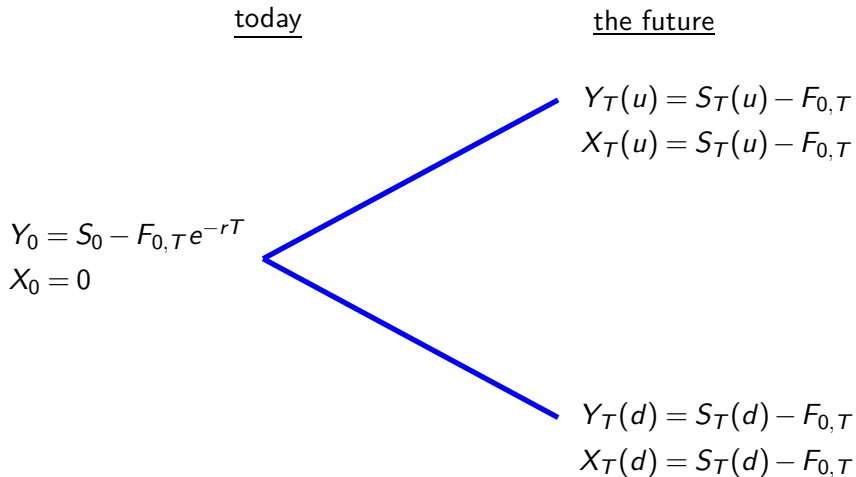
- ▶ Value of long cocoa, short bond portfolio today is:

$$Y_0 = S_0 - F_{0,T}e^{-rT}.$$

- ▶ Equating $X_0 = Y_0$ and solving for the forward price yields:

$$F_{0,T} = S_0e^{rT}.$$

Example of state-by-state replication



Practice problem

The spot price of cocoa is $S_0 = \$2,500$. The c.c. risk-free interest rate is 0.05. Assume there is no arbitrage. What is the one-year forward price of cocoa? Demonstrate that payoffs to the long forward and the replicating portfolio are equivalent at date T .

Practice problem scratch paper

Exploiting arbitrage opportunities: “buy low, sell high”

- ▶ What would you do if $F_{0,T} > S_0 e^{rT}$?
 1. Enter short forward contract.
 2. Purchase synthetic forward: long cocoa and short bond.

- ▶ What would you do if $F_{0,T} < S_0 e^{rT}$?
 1. Enter long forward contract.
 2. Sell synthetic forward: short cocoa and long bond.

Practice problem

The spot price of cocoa is $S_0 = \$2,500$. The c.c. risk-free interest rate is 0.05. Suppose the one year forward price is $F_{0,1} = \$2,700$. Is there an arbitrage? If so, construct a strategy to capitalize upon it.

Practice problem scratch paper

What is the meaning of $F_{0,T} = S_0 e^{rT}$?

- Is the forward price = expected future spot price? **No!** Instead:

$$F_{0,T} = S_0 + \text{cost of carry}.$$

- The **cost of carry** is the net cash flow associated with holding a long position in the underlying. In our setting:

$$\text{cost of carry} = \underbrace{S_0 (e^{rT} - 1)}_{\text{interest foregone by long}}.$$

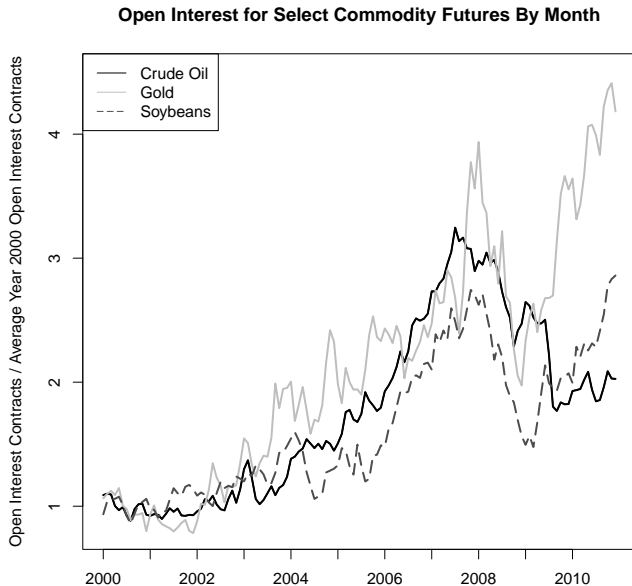
Roadmap: the forward price

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Synthetic positions

- ▶ Replication can be used to create synthetic positions:
 1. $\text{Forward} = \text{stock} - \text{bond}.$
 2. $\text{Stock} = \text{forward} + \text{bond}.$
 3. $\text{Bond} = \text{stock} - \text{forward}.$
- ▶ Synthetics are the most basic form of [financial engineering](#).

Recall synthetics drove commodity futures growth



Roadmap: the forward price

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The forward price changes over time

- What is the value of your original long forward at date τ ?

origination

tomorrow

the future

$$F_{0,T} \text{ ————— } F_{\tau,T} \text{ ————— } F_{T,T} = S_T$$

Value of off-market forward

- ▶ The value of your original long forward at date τ is:

$$(F_{\tau,T} - F_{0,T})e^{-r(T-\tau)}.$$

- ▶ Intuition:
 - You can lock-in a payoff by shorting a new forward.
 - Because the payoff is risk-free, we discount with rate r .
 - If the forward price has risen, we have made a profit.

Practice problem

Suppose the spot price of the S&P 500 index is $S_0 = \$3,300$ and risk-free interest rate is $r = 0.10$. At date 0, you enter into a long, three-year forward contract on the S&P 500 index. One year passes and the S&P index has risen to $S_1 = \$3,500$. What is the value of your forward contract at date $\tau = 1$?

Practice problem scratch paper

Practice problem scratch paper

Roadmap: the forward price

1. No-arbitrage and the LoP.
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- 5. Summary**

Summary

- ▶ The **law of one price** states that portfolios with the same payoffs must have the same prices. This is true if **no-arbitrage** holds.
- ▶ No-arbitrage forward price = spot price + cost of carry:

$$F_{0,T} = S_0 e^{rT}$$

- ▶ Value of off-market forward at date $\tau \in (0, T)$ is:

$$V_\tau = (F_{\tau,T} - F_{0,T}) e^{-r(T-\tau)}.$$

References

- ▶ Textbook chapters 5.1, 5.2, and 5.3.
- ▶ Equity index forward prices are from Bloomberg.
- ▶ Slides are created using code on my [Github](#).

Why is $F_{0,T} \neq$ expected future spot price?

- ▶ If the underlying has a positive risk premium, then the forward price must be a downward biased predictor of future spot price:

$$F_{0,T} < \mathbb{E}(S_T).$$

- ▶ Rationale:
 - Despite no investment, forward retains risks of underlying.
 - To bear this risk, buyer must earn at least the risk premium.
 - This can only occur if $X_T = S_T - F_{0,T} > 0$, on average.