Lecture 4
Frictions

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What assumptions have we made so far?

- Zero cost of storage.
  - Financial asset: no dividends.
  - Commodity: no physical storage costs.
- Investor can short the underlying.
- Investment asset – i.e., no production.
- No transactions costs, can borrow at risk-free rate, ...

What happens if we introduce frictions?

Roadmap: the forward price with frictions

1. **Dividends.**
2. Commodity storage and convenience.
3. Limits to arbitrage.

Dividends

- Financial assets (e.g., stocks) commonly pay dividends.
- Let $d > 0$ be the dividend paid on a stock at time T.
- Will the dividend increase or decrease the forward price?
- Think about the forward price decomposition:

$$F_{0,T} = S_0 + \text{cost of carry}.$$

Not-so replicating portfolio

- Dividends do not affect the payoff to a long forward:

$$X_T = S_T - F_{0,T}.$$

- Stock payoff now includes dividend (price is ex-dividend):

$$S_T + d.$$

- Leveraged stock portfolio no longer replicates forward:

$$(S_T + d) - F_{0,T} > S_T - F_{0,T}.$$

Downward adjusted replicating portfolio

- Form the following portfolio at date 0:

  1. Long one share of stock.
  2. Short a risk-free bond with face-value $F_{0,T} + d$.

- The payoff to this portfolio at date T is:

$$\begin{aligned} Y_T &= (S_T + d) - (F_{0,T} + d) \\ &= S_T - F_{0,T}. \end{aligned}$$

- This is the same payoff as the long forward!

The forward price with a dividend

- Apply LoP: value of forward today=value of replicating portfolio.
- Value of forward today is zero by construction:

$$X_0 = 0.$$

- Value of long stock, short bond portfolio today is:

$$Y_0 = S_0 - (F_{0,T} + d)\, e^{-rT}.$$

- Equating $X_0 = Y_0$ and solving for the forward price yields:

$$F_{0,T} = S_0 e^{rT} - d.$$

Practice problem

The price of AAPL stock is \$300 per share. Apple has comitted with certainty to make a \$5 divided payment per share in 6 months. The c.c. risk-free rate is one percent.

1. What is the six-month forward price of AAPL stock?
2. Is there an arbitrage if $F_{0,0.5} = \$310$? If so, construct a portfolio to capitalize upon it.
3. Is there an arbitrage if $F_{0,0.5} = \$290$? If so, construct a portfolio to capitalize upon it.

Practice problem scratch paper

Practice problem scratch paper

Cost of carry with dividends

- The asset holder enjoys the dividend while the forward does not.
- Thus, forward must be price-compensated for deferred purchase.
- Another way to say this is dividends reduce the cost of carry:

$$\text{cost of carry} = S_0\left(e^{rT} - 1\right) - d.$$

- Forward curve is up (down) sloping if interest $>$ ($<$) dividends.

E-mini forward curve (01/15/19)

Description	Last	Chg Settle	Time	Bid	Ask	Open Int	Volume	Yest Settle
21) Spot	s	--	Close	2572.16	2641.69			
22) Mar19	2605.50 s	+25.00	Close	2605.50	2606.00	2596574	1494266	2580.50
23) Jun19	2610.50 s	+25.25	Close	2602.00	2620.00	55892	8530	2585.25
24) Sep19	2615.25 s	+25.00	Close	2550.00	2662.00	2673	156	2590.25
25) Dec19	2617.50 s	+22.75	Close		2650.00	66		2594.75
26) Mar20	2624.00 s	+25.75	Close			5	5	2598.25

S&P 500 dividend yield vs one year LIBOR

E-mini forward curve (01/15/2010)

Description	Last	Chg Settle	Time	Bid	Ask	Open Int	Volume	Yest Settle
21) Spot	s	--	Close	1135.60	1136.43			
22) Mar10	1132.25s	-13.00	Close	1131.75	1132.50	2454453	2031715	1145.25
23) Jun10	1127.50s	-12.75	Close	1120.00	1128.50	5936	1432	1140.25
24) Sep10	1122.75s	-13.00	Close	1075.00	1127.75	1394	19	1135.75
25) Dec10	1119.75s	-13.00	Close	1044.00	1154.75	11	3	1132.75
26) Mar11	1117.75s	-13.00	Close					1130.75

S&P 500 dividend yield vs one year LIBOR

General formulas for dividends

- For an asset with dividends $\{d_1, d_2, ..., d_n\}$ at dates $\{t_1, t_2, ..., t_n\}$ such that $0 \leq t_1 \leq ... \leq t_n \leq T$, the no-arbitrage forward price is:

$$F_{0,T} = S_0 e^{rT} - \sum_{i=1}^{n} d_i e^{r(T-t_i)}.$$

- For a financial asset with a continuous dividend yield $\delta > 0$:

$$F_{0,T} = S_0 e^{(r-\delta)T}.$$

- This latter formula is commonly used to price index forwards.

Roadmap: the forward price with frictions

1. Dividends.
2. **Commodity storage and convenience.**
3. Limits to arbitrage.

Commodity storage costs

- Commodities are costly to store (e.g., grain silos), if at all.
- Let $c > 0$ be the storage cost for date 0 to T, paid at T.
- Will the storage cost increase or decrease the forward price?
- Think about the forward price decomposition:

$$F_{0,T} = S_0 + \text{cost of carry}.$$

Not-so replicating portfolio

- Storage costs do not affect the payoff to a long forward:

$$X_T = S_T - F_{0,T}.$$

- Commodity payoff now includes storage cost:

$$S_T - c.$$

- Leveraged commodity portfolio no longer replicates forward:

$$(S_T - c) - F_{0,T} < S_T - F_{0,T}.$$

Upward adjusted replicating portfolio

- Form the following portfolio at date 0:
    1. Long one unit of the commodity.
    2. Short a risk-free bond with face-value $F_{0,T} - c$.
- The payoff to this portfolio at date T is:

$$\begin{aligned} Y_T &= (S_T - c) - (F_{0,T} - c) \\ &= S_T - F_{0,T}. \end{aligned}$$

- This is the same payoff as the long forward!

The forward price with storage costs

- Apply LoP: value of forward today=value of replicating portfolio.
- Value of forward today is zero by construction:

$$X_0 = 0.$$

- Value of long commodity, short bond portfolio today is:

$$Y_0 = S_0 - (F_{0,T} - c)\, e^{-rT}.$$

- Equating $X_0 = Y_0$ and solving for the forward price yields:

$$F_{0,T} = S_0 e^{rT} + c.$$

Practice problem

The price of gold is $\$1,500$ per ounce. For safety, gold must be stored in Fort Knox at a cost of \$45 per ounce per year. The c.c. risk-free rate is five percent and the expected future spot price in four years is $\$1,550$.

1. What is the four-year forward price of gold?

2. What is the cost of carry for a long gold position?

2. Is there an arbitrage if $F_{0,4} = \$1,550$? If so, construct a portfolio to capitalize upon it.

3. Is there an arbitrage if $F_{0,4} = \$2,000$? If so, construct a portfolio to capitalize upon it.

Practice problem scratch paper

Practice problem scratch paper

Practice problem scratch paper

Cost of carry with storage costs

- The asset holder pays storage cost while the forward does not.
- Thus, forward must provide price compensation for deferring.
- Another way to say this is storage costs increase cost of carry:

$$\text{cost of carry} = S_0 \left(e^{rT} - 1 \right) + c.$$

- Forward curve is upward sloping based on this formula.

Gold forward curve (01/13/20)

Generic 1st 'GC' Future Comdty | CT | Related Functions Menu | Message

GC1 ↓1550.10 -10.00 ic1550.00/1550.10 26x11 Prev 1560.10

At 12:33d Vol 256646 Op 1563.00 Hi 1563.10 Lo 1547.00 OpenInt 460509

GC1 COMB Comdty | 1) Actions | 2) Settings | Upgrade Available | Futures Contract Table

Generic 1st 'GC' F | As of 01/13/20 | Session COMB | First Leg All

7) COMEX | CEM » | Contracts 20 | Aggr Vol 319,322 | Aggr Open Int 786,871 | Length 0

Intraday | Daily | 8) Curve | CCRV »

Display | Quoted Val | Yield

Description	Last	Chg Settle	Time	Bid	Ask	Open Int	Volume	Yest Settle
21) Feb20	↓1550.10	-10.00	12:33	1550.00	1550.10	460509	256646	1560.10
22) Apr20	1556.40	-10.00	12:33	1556.30	1556.40	201731	54281	1566.40
23) Jun20	1562.00	-10.10	12:30	1561.90	1562.10	68517	4240	1572.10
24) Aug20	1568.00	-9.50	12:06	1567.30	1567.40	22226	1342	1577.50
25) Oct20	1575.70	-6.60	8:31	1572.10	1572.30	7513	29	1582.30
26) Dec20	1577.90	-9.10	12:19	1576.80	1576.90	18646	2433	1587.00
27) Feb21	1581.90	-9.60	10:07	1581.00	1581.20	4572	217	1591.50
28) Apr21		--	1/10	1584.60	1585.30	530	111	1595.40
29) Jun21		--	1/10	1588.40	1589.30	1757	21	1599.10
30) Aug21		--	1/10	1585.10	1600.90	1		1603.30
31) Oct21		--	1/10	1589.30	1605.60			1607.20
32) Dec21	1606.20	-5.00	8:52	1600.50	1601.10	806	1	1611.20
33) Jun22		--	1/10	1598.10	1623.10	4		1620.10
34) Dec22		--	1/10	1620.00	1627.50	47	1	1629.00
35) Jun23		--	1/10	1610.00	1644.10	5		1638.40

People actually perform this arbitrage in real-life

The Wall Street Journal, “Oil Stockpile on Ship Shrinks”

THE WALL STREET JOURNAL.

Oil Stockpile on Ships Shrinks

Less oil is floating at sea on supertankers like the Maran Centaurus. AGENCE FRANCE-PRESSE/GETTY IMAGES

By Guy Chazan

Updated Feb. 1, 2010 12:01 am ET

General formulas for storage

- For storage costs $\{c_1, c_2, ..., c_n\}$ at dates $\{t_1, t_2, ..., t_n\}$ such that $0 \leq t_1 \leq ... \leq t_n \leq T$, the no-arbitrage forward price is:

$$F_{0,T} = S_0 e^{rT} + \sum_{i=1}^{n} c_i e^{r(T-t_i)}.$$

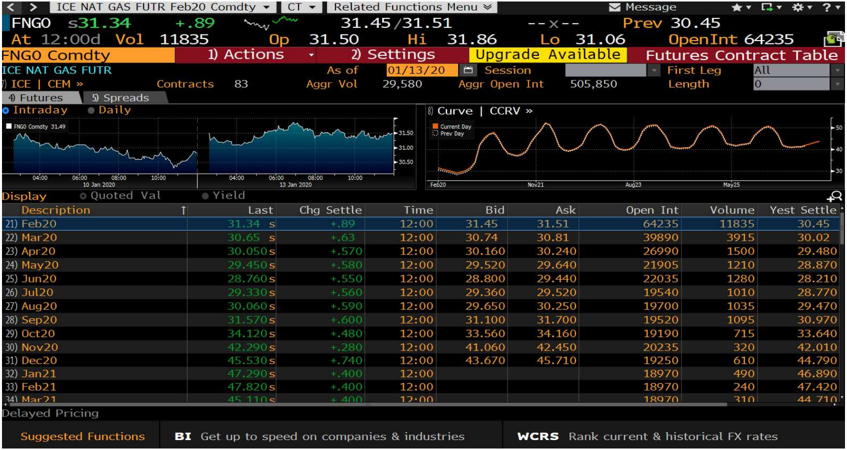
- For a commodity with a continuous storage cost $\gamma > 0$:

$$F_{0,T} = S_0 e^{(r+\gamma)T}.$$

An arbitrage opportunity?

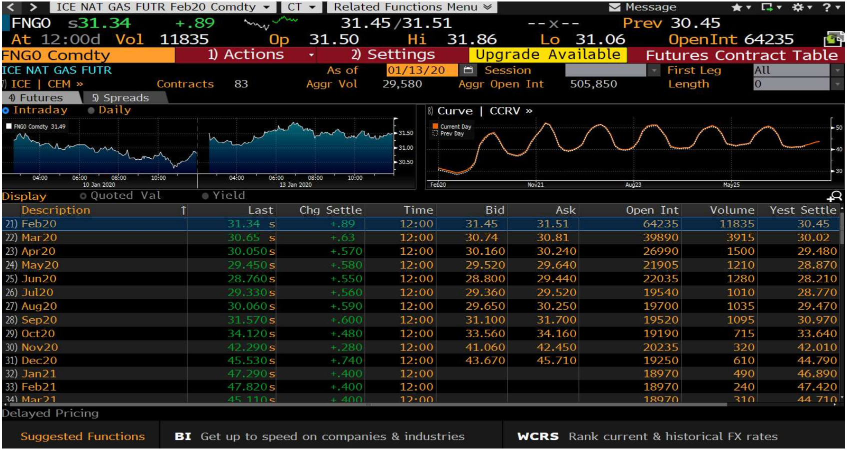
- It seems storable commodities should always be in contango.
  - Contango means an upward-sloping forward curve.
- But sometimes they display backwardation or seasonality.
  - Backwardation means a downward-sloping forward curve.
- Does this imply that arbitrage opportunities are plentiful?

The Sumitomo copper affair! (circa 1990 / 1995)



Description	Last	Chg Settle	Time	Bid	Ask	Open Int	Volume	Yest Settle
21) Feb20	31.34 s	+.89	12:00	31.45	31.51	64235	11835	30.45
22) Mar20	30.65 s	+.63	12:00	30.74	30.81	39890	3915	30.02
23) Apr20	30.050 s	+.570	12:00	30.160	30.240	26990	1500	29.480
24) May20	29.450 s	+.580	12:00	29.520	29.640	21905	1210	28.870
25) Jun20	28.760 s	+.550	12:00	28.800	29.440	22035	1280	28.210
26) Jul20	29.330 s	+.560	12:00	29.360	29.520	19540	1010	28.770
27) Aug20	30.060 s	+.590	12:00	29.650	30.250	19700	1035	29.470
28) Sep20	31.570 s	+.600	12:00	31.100	31.700	19520	1095	30.970
29) Oct20	34.120 s	+.480	12:00	33.560	34.160	19190	715	33.640
30) Nov20	42.290 s	+.280	12:00	41.060	42.450	20235	320	42.010
31) Dec20	45.530 s	+.740	12:00	43.670	45.710	19250	610	44.790
32) Jan21	47.290 s	+.400	12:00			18970	490	46.890
33) Feb21	47.820 s	+.400	12:00			18970	240	47.420
34) Mar21	45.110 s	+.400	12:00			18970	310	44.710

Natural gas forward curve (01/13/20)



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30) Nov20	42.290 s	+.280	12:00	41.060	42.450	20235	320	42.010
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The convenience yield

- Commodities sometimes yield non-monetary benefits to holders.
- *Ex:* Hershey needs cocoa inventory to produce chocolate.
- The convenience yield is the non-monetary benefit of ownership.
- Convenience yields push down forward prices (like dividends).

Forward pricing with storage and convenience

- Let $y > 0$ be the convenience yield for a commodity.
- The no-arbitrage forward price is given by:

$$F_{0,T} = S_0 e^{(r+c-y)T}.$$

- The forward curve will be in contango (backwardation) whenever storage costs and interest are $>$ ($<$) the convenience yield..

Practice problem

The spot price for cocoa is $\$2,000$ per ton. The c.c. risk-free rate is one percent and the c.c. storage cost is two percent. The six-month futures price of cocoa is $\$1,900$ per ton. What is the convenience yield for cocoa over this six month period?

Practice problem scratch paper

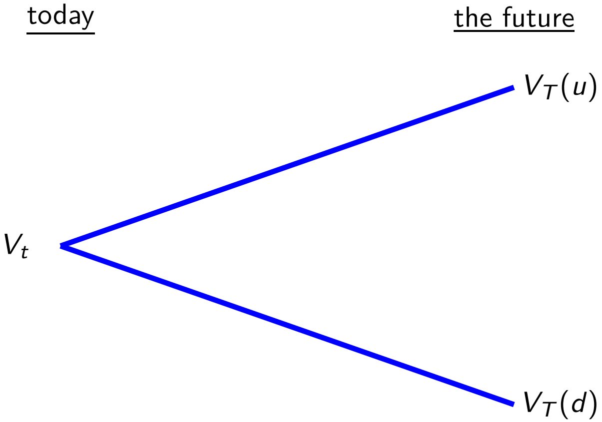
Roadmap: the forward price with frictions

1. Dividends.
2. Commodity storage and convenience.
3. **Limits to arbitrage.**

Arbitrage with uncertainty

- Now suppose there is uncertainty about the payoff V_T.
- Specifically, two possible states of the world in the future:
  - State u ("up").
  - State d ("down").
- Let $V_T(\omega)$ be the payoff of the portfolio in state ω.
- When is the portfolio V an arbitrage?

A portfolio with uncertainty



Arbitrage with uncertainty

- With uncertainty over the state ω, V is an arbitrage if either:
  1. $V_t = 0$ and either:
     a. $V_T(u) > 0$ and $V_T(d) \geq 0$ or
     b. $V_T(u) \geq 0$ and $V_T(d) > 0$ or
  2. $V_t < 0$ and both $V_T(u) \geq 0$ and $V_T(d) \geq 0$.

Practice problem #2

1. Consider two securities: X and Y. Security X has a price of $X_t = 1$ and payoffs $X_T(u) = 3$ and $X_T(d) = 0$. Security Y has a price of $Y_t = 1$ and payoffs $Y_T(u) = 2$ and $Y_T(d) = 0$. Is there an arbitrage? If so, construct a portfolio to capitalize upon it.

2. Consider two securities: X and Y. Security X has a price of $X_t = 1$ and payoffs $X_T(u) = 3$ and $X_T(d) = 0$. Security Y has a price of $Y_t = 1$ and payoffs $Y_T(u) = 0$ and $Y_T(d) = 3$. Is there an arbitrage? If so, construct a portfolio to capitalize upon it.

Practice problem #2 solutions

No-arbitrage

- What happens if a portfolio is an arbitrage?
- Free money is on the table! Arbitrageurs buy low and sell high.
- Demand will be infinite; markets will not clear.
- No-arbitrage means that market prices preclude arbitrage.

The law of one price (LoP)

- The law of one price (LoP) states that portfolios with the same payoffs must have the same price.
- The LoP is directly implied by no-arbitrage.
- The LoP is the fundamental concept behind all derivatives pricing. We will use it to derive forward and options prices.

Practice problem #3

Security X has a price of $X_t = 10$ and payoffs $X_T(u) = 20$ and $X_T(d) = 5$. Assume there is no arbitrage. Answer the following:

1. What is the price of a security Y with payoffs $Y_T(u) = 20$ and $Y_T(d) = 5$?
2. What is the price of a security Z with payoffs $Z_T(u) = 10$ and $Z_T(d) = 2.5$?
3. What is the price of a security W with payoffs $W_T(u) = 40$ and $W_T(d) = 10$?

Practice problem #3 solutions

Practice problem #3 solutions

Roadmap: the forward price

1. No-arbitrage and the law of one price
2. **Deriving the no-arbitrage forward price.**
3. Value of off-market forwards.
4. Summary

Back to cocoa

- Setup:
  1. Two dates: t (today) and T (the future).
  2. Discrete risk-free interest rate of $r \geq 0$.
  3. Cocoa price of S_t and random payoff S_T.
- Consider the payoff to a long forward:

$$X_T = S_T - F_{t,T}$$

- What other portfolio has this same payoff?

Replicating portfolio

- Form the following portfolio at date t:

  1. Long one unit of cocoa at cost S_t.
  2. Short a zero-coupon bond with face value of $F_{t,T}$.

- The payoff to the portfolio at date T is:

$$Y_T = S_T - F_{t,T}$$

- This is the same payoff as the long forward in every state.

Example of state-by-state replication

<u>today</u>

$$Y_t = S_t - \frac{F_{t,T}}{(1+r)^{T-t}}$$

$$X_t = 0$$

<u>the future</u>

$$Y_T(u) = S_T(u) - F_{t,T}$$

$$X_T(u) = S_T(u) - F_{t,T}$$

$$Y_T(d) = S_T(d) - F_{t,T}$$

$$X_T(d) = S_T(d) - F_{t,T}$$

The forward price today

- By the LoP, the value of the forward contract today must equal the value of the portfolio today.

- Price of portfolio of long cocoa, short bond today is:

$$Y_t = S_t - F_{t,T}/(1+r)^{T-t}$$

- Value of forward today is $X_t = 0$ by definition.

- Thus, value of forward = price of portfolio implies:

$$F_{t,T} = S_t(1+r)^{T-t}$$

Practice problem #4

The spot price of cocoa at date 0 is $S_0 = \$2,500$. The risk-free interest rate is 0.05. What is the one-year forward price of cocoa? Demonstrate that the payoff to the long forward and the replicating portfolio are equivalent at date T.

Practice problem #4

Exploiting arbitrage opportunities

- What would you do if $F_{t,T} > S_t(1+r)^{T-t}$?

  1. Enter short forward contract.
  2. Purchase synthetic forward: long cocoa and short bond.

- What would you do if $F_{t,T} < S_t(1+r)^{T-t}$?

  1. Enter long forward contract.
  2. Sell synthetic forward: short cocoa and long bond.

Roadmap: the basics of forward contracts

1. Definitions
2. Payoffs
3. **Applications of forwards**
4. Interest rates

Application of forwards

- Two applications: Risk management and speculation.
- Risk management give the example.
- Speculation is because of the lvereage.
- In our previous example, Hershey uses cocoa as an input in production. Hence, it has a natural short position in cocoa.
- On the other hand, farmers output cocoa and have natural long positions.
- A forward contract can be used to transfer risks between Hershey and cocoa farmers and capture certain surplus.

Roadmap: the basics of forward contracts

1. Definitions
2. Payoffs
3. Application of forwards
4. **Interest rates**

Technical note on interest rates

- Throughout the semester, we will assume there is a single risk-free interest rate of $r \geq 0$.
- Risk-free cash flows should be discounted at the risk-free rate.
- If $B_T \geq 0$ is a risk-free cash flow at time T, then the price of this cash flow at date $t < T$ is given by:

$$B_t = \begin{cases} \frac{B_T}{(1+r)^{-(T-t)}} & \text{if } r \text{ is discretely compounded} \\ \\ B_T e^{-r(T-t)} & \text{if } r \text{ is continuously compounded} \end{cases}$$

Practice problem #3

Suppose $r = 0.05$ and today is date $t = 0$. Compute the prices of the following securities assuming discrete compounding:

1. A risk-free security that pays \$1 at date $T = 1$.
2. A risk-free zero-coupon bond with a face value of \$100 and maturity of $T = 5$.
3. A $T = 3$ year risk-free coupon bond with annual coupons of \$5 and face value of \$100.

Practice problem #3 solutions

1. The price is $\$1 \cdot (1+0.05)^{-1} = \$0.952$.

2. The price is $\$100 \cdot (1+0.05)^{-5} = \$78.353$.

3. The price is $\sum_{i=1}^{3} 5 \cdot (1+0.05)^{-i} + \$100 \cdot (1+0.05)^{-3} = \$100$.

Practice problem #4

Suppose $r = 0.05$ and today is date $t = 0$. Compute the prices of the following securities assuming continuous compounding:

1. A risk-free security that pays \$1 at date $T = 1$.
2. A risk-free zero-coupon bond with a face value of \$100 and maturity of $T = 5$.
3. A $T = 3$ year risk-free coupon bond with annual coupons of \$5 and face value of \$100.

Practice problem #3 solutions

1. The price is $\$1 \cdot e^{-0.05} = \$0.951$.

2. The price is $\$100 \cdot e^{-0.05 \cdot 5} = \$77.88$.

3. The price is $\sum_{i=1}^{3} 5 \cdot e^{-0.05 \cdot i} + \$100 \cdot e^{-0.05 \cdot 3} = \$99.43$.

Summary

- A forward contract is an agreement to buy or sell an asset at a future date at the forward price.
- Date T payoff of long forward originated at date 0:

$$X_T = S_T - F_{0,T}.$$

- Forwards can be used to hedge input and output price risk.
- Date 0 price risk-free \$1 payoff at date T is $(1+r)^{-T}$ or e^{-rT}.

A final note on commodity forward pricing

- FILL IN HERE FROM GOODNOTES SHEET.

References

- Textbook chapters 5.2, 5.3, 6.1, 6.3, 6.4, and 6.6.
- Hershey chocolate article is in the Wall Street Journal.
- Cocoa prices are from Bloomberg screen XXX.
- Graphs are created using code on my Github.