Answer questions about function properties with graphs.

1. See pg. 305
   a. (1,3), (4, 6)
   b. (0, 1), (3, 4)
   c. (0, 2)
   d. (2, 4), (4, 6) Note: intervals are separate even though they are adjacent.
   e. [2, 3] Note: non-differentiable doesn't mean there will be an inflection point

5. See pg. 305 a. increasing: (0, 1), (3, 5) decreasing: (1, 3), (5, 6) b. We evaluate roots and undefined inputs, not endpoints, for local extrema. If $f''(x) < 0$, extrema is a local max. The opposite holds true for local min.

   | max: $f(1)$ and $f(5)$, min: $f(3)$ |

Find intervals, extrema, concavity (requirements vary)

9. $2x^3 - 15x^2 + 24x - 5$

   Polynomials are differentiable for all real numbers.

   $$f'(x) = 6x^2 - 30x + 24 \qquad\qquad \text{find derivative}$$
   $$0 = 6x^2 - 30x + 24$$

   $$0 = x^2 - 5x + 4 \qquad\qquad \text{find root(s)}$$
   $$0 = (x-1)(x-4)$$

   Critical values are 1 and 4. These are fence-posts. Test a number from each of the three "fences".

   $$f(0) = (0-1)(0-4) \qquad\qquad \text{interval } (-\infty, 1)$$
   $$f(0) = (-)(-) \qquad\qquad\qquad \text{increasing}$$

   $$f(2) = (2-1)(2-4) \qquad\qquad \text{interval } (1, 4)$$
   $$f(2) = (+)(-) \qquad\qquad\qquad \text{decreasing}$$

   $$f(5) = (5-1)(5-4) \qquad\qquad \text{interval } (4, \infty)$$
   $$f(5) = (+)(+) \qquad\qquad\qquad \text{increasing}$$

   | Therefore, function is increasing on $(-\infty, 1) \cup (4, \infty)$ and decreasing on $(1, 4)$. |

   Increasing to decreasing is a local max, while decreasing to increasing is a local min. We can take the second derivative to show this.

   $$f''(x) = 12x - 30 \qquad\qquad \text{find second derivative}$$

   $$f''(1) = 12(1) - 30 \qquad\qquad \text{critical value \#1}$$
   $$f''(1) = -18 \qquad\qquad\qquad \text{max}$$

   $$f''(4) = 12(4) - 30 \qquad\qquad \text{critical value \#2}$$
   $$f''(4) = 18 \qquad\qquad\qquad \text{min}$$

Next, plug in both critical values to find values of local extrema at the critical values.

$$f(1) = 2(1)^3 - 15(1)^2 + 24(1) - 5 \qquad\qquad \text{critical value \#1}$$
$$f(1) = 2 - 15 + 24 - 5$$
$$f(1) = 6$$

$$f(4) = 2(4)^3 - 15(4)^2 + 24(4) - 5 \qquad\qquad \text{critical value \#2}$$
$$f(4) = 128 - 240 + 96 - 5$$
$$f(4) = -21$$

$$\boxed{\text{max: } f(1) = 6, \text{ min: } f(4) = -21}$$

13. $\dfrac{x^2 - 24}{x - 5}$

$$f'(x) = \frac{(x-5)(2x) - (x^2 - 24)(1)}{(x-5)^2} \qquad\qquad \text{find derivative with quotient rule}$$
$$f'(x) = \frac{2x^2 - 10x - x^2 + 24}{(x-5)^2}$$
$$f'(x) = \frac{x^2 - 10x + 24}{(x-5)^2}$$

Set factor in denominator to 0 and evaluate to find undefined inputs. Set numerator to 0 and evaluate to find roots. These will be the critical numbers.

$$0 = x^2 - 10x + 24 \qquad\qquad \text{denominator}$$
$$x = (x-4)(x-6)$$
$$x = 4 \text{ and } 6$$

$$0 = x - 5 \qquad\qquad \text{numerator}$$
$$x = 5$$

Critical numbers are 4, 5, 6. Use inputs that fall in the intervals we want to verify. We only need sign, so

separate into factors if calculator is not an option. Plug into derivative.

$$f'(0) = \frac{0^2 - 10(0) + 24}{(0-5)^2} \qquad\qquad \text{interval } (-\infty, 4)$$

$$f'(0) = \frac{+}{+} \qquad\qquad \text{increasing}$$

$$f'(4.5) = \frac{(4.5)^2 - 10(4.5) + 24}{((4.5)-5)^2} \qquad\qquad \text{interval } (4,5)$$

$$f'(4.5) = \frac{-}{+} \qquad\qquad \text{decreasing}$$

$$f'(5.5) = \frac{(5.5)^2 - 10(5.5) + 24}{((5.5)-5)^2} \qquad\qquad \text{interval } (5,6)$$

$$f'(5.5) = \frac{-}{+} \qquad\qquad \text{decreasing}$$

$$f'(10) = \frac{10^2 - 10(10) + 24}{(10-5)^2} \qquad\qquad \text{interval } (6,\infty)$$

$$f'(10) = \frac{+}{+} \qquad\qquad \text{increasing}$$

Points where sign of derivative changes include a local max: $f(4)$ and a local min: $f(6)$

$$f(4) = \frac{(4)^2 - 24}{4 - 5}$$
$$f(4) = \frac{-4}{-1}$$
$$f(4) = 4$$

$$f(6) = \frac{(6)^2 - 24}{6 - 5}$$
$$f(6) = \frac{12}{1}$$
$$f(6) = 12$$

$\boxed{\text{local max: } f(4) = 8, \text{ local min: } f(6) = 12}$

17. $x^3 - 3x^2 - 9x + 4$

Polynomials are differentiable for all real numbers.

$$f'(x) = 3x^2 - 6x - 9 \qquad\qquad \text{find derivative}$$

$$0 = 3x^2 - 6x - 9 \qquad\qquad \text{find root(s)}$$
$$0 = x^2 - 2x - 3$$
$$0 = (x+1)(x-3)$$

Critical numbers are -1 and 3

$$f'(-2) = 3(-2)^2 - 6(-2) - 9$$
$$f'(-2) = 15$$

interval $(-\infty, -1)$
increasing

$$f'(0) = 3(0)^2 - 6(0) - 9$$
$$f'(0) = -9$$

interval $(-1, 3)$
decreasing

$$f'(4) = 3(4)^2 - 6(4) - 9$$
$$f'(4) = 24$$

interval $(3, \infty)$
increasing

Local max: $f(-1)$, Local min: $f(3)$. Plug back into original function. We can skip this step if just trying to find concavity and inflection point.

$$f(-1) = (-1)^3 - 3(-1)^2 - 9(-1) + 4$$
$$f(-1) = -1 - 3 + 9 + 4$$
$$f(-1) = 9$$

$$f(3) = (3)^3 - 3(3)^2 - 9(3) + 4$$
$$f(3) = 27 - 27 - 27 + 4$$
$$f(3) = -23$$

max: $f(-1) = 9$, min: $f(3) = -23$
Next, we will work with second derivative.

$$f'(x) = 3x^2 - 6x - 9$$
$$f''(x) = 6x - 6$$

find second derivative

$$0 = 6x - 6$$
$$x = 1$$

find roots

Fence posts are the roots. Evaluate inputs in the "fences". Negative means concave down and positive means concave up.

$$f''(0) = 6(0) - 6$$
$$f''(0) = -6$$

interval $(-\infty, 1)$
concave down

$$f''(2) = 6(2) - 6$$
$$f''(2) = 12$$

interval $(-1, \infty)$
concave up

Find inflection point by plugging in root of second derivative into original equation.

$$f(1) = (1)^3 - 3(1)^2 - 9(1) + 4$$
$$f(1) = 1 - 3 - 9 + 4$$
$$f(1) = -7$$

Inflection point: $(1, -7)$

21. $\ln(x^2 + 5)$

Function has natural log so domain is all real numbers greater than 0.

$$f'(x) = \frac{1}{x^2 + 5}(2x)$$ find derivative

$$f'(x) = \frac{2x}{x^2 + 5}$$

$$f''(x) = \frac{(x^2 + 5)(2) - (2x)(2x)}{(x^2 + 5)^2}$$ find second derivative

$$f''(x) = \frac{2x^2 + 10 - 4x^2}{(x^2 + 5)^2}$$

$$f''(x) = \frac{-2x^2 + 10}{(x^2 + 5)^2}$$

Denominator can never be 0. Set numerator to 0 to find roots, aka the x-values of inflection points.

$$0 = -2x^2 + 10$$
$$2x^2 = 10$$
$$x^2 = 5$$
$$x = \pm\sqrt{5}$$

Evaluate the "fences".

$$f''(-3) = \frac{-2(-3)^2 + 10}{((-3)^2 + 5)^2}$$ interval $(-\infty, -\sqrt{5})$

$$f''(-3) = \frac{-}{+}$$ concave down

$$f''(0) = \frac{-2(0)^2 + 10}{(0^2 + 5)^2}$$ interval $(-\sqrt{5}, \sqrt{5})$

$$f''(0) = \frac{+}{+}$$ concave up

$$f''(3) = \frac{-2(3)^2 + 10}{(3^2 + 5)^2}$$ interval $(\sqrt{5}, \infty)$

$$f''(3) = \frac{-}{+}$$ concave down

Plug solutions back into original equation to find inflection points.

$$f(-\sqrt{5}) = \ln\left((-\sqrt{5})^2 + 5\right)$$
$$f(-\sqrt{5}) = \ln(5 + 5)$$
$$f(-\sqrt{5}) = \ln 10$$

$$f(\sqrt{5}) = \ln\left(\sqrt{5}^2 + 5\right)$$
$$f(\sqrt{5}) = \ln(5 + 5)$$
$$f(\sqrt{5}) = \ln 10$$

Inflection points: $(\pm\sqrt{5}, \ln 10)$

25. $x^2 - x - \ln x$

Function has natural log so domain is all real numbers greater than 0.

$$f'(x) = 2x - 1 - \frac{1}{x}$$ find derivative

$$0 = 2x - 1 - \frac{1}{x}$$ find root(s) of first derivative

$$0 = 2x * \frac{x}{x} - 1 * \frac{x}{x} - \frac{1}{x}$$ get common denominators

$$0 = \frac{2x^2 - x - 1}{x}$$

$$0 = 2x^2 - x - 1$$ set numerator $= 0$

$$0 = 2x(x - 1) + (1)(x - 1)$$

$$0 = (2x + 1)(x - 1)$$ factors

By evaluating roots and inputs that make derivative undefined, the only critical value is shown to be 1. $-\frac{1}{2}$ and 0 are not included because they aren't in domain of original function. Next, evaluate inputs on intervals surrounding the critical value.

$$f'(\frac{1}{2}) = 2\left(\frac{1}{2}\right) - 1 - \frac{1}{\frac{1}{2}}$$ interval $(0, 1)$

$$f'(\frac{1}{2}) = 1 - 1 - 2$$

$$f'(\frac{1}{2}) = -2$$ decreasing

$$f'(2) = 2(2) - 1 - \frac{1}{2}$$ interval $(1, \infty)$

$$f'(2) = 4 - 1 - \frac{1}{2}$$

$$f'(2) = -2.5$$ increasing

Plug in critical values to original function.

$$f(0) = 0^2 - 0 - \ln 0$$

No local max. local min: $f(1) = 0$

$$f''(x) = 2 + x^{-2}$$ find second derivative

$$f''(x) = 2 + \frac{1}{x^2}$$

Solution is always positive. There are no real roots, so no real inflection points. Therefore the function is concave upwards everywhere on its domain.

Find local max and min with first and second derivative test

29. $f(x) = 1 + 3x^2 - 2x^3$

Function is a polynomial so it is differentiable for all real numbers.

$$f'(x) = -6x^2 + 6x \qquad \text{find derivative}$$

$$0 = -6x^2 + 6x \qquad \text{find root(s)}$$
$$0 = -6x(x-1)$$

Critical numbers are 0 and 1.

First derivative test:

$$f'(-1) = -6(-1)^2 + 6(-1) \qquad \text{interval } (-\infty, 0)$$
$$f'(-1) = -6 - 6 \qquad \text{decreasing}$$

$$f'(\tfrac{1}{2}) = -6\left(\frac{1}{2}\right)^2 + 6\left(\frac{1}{2}\right) \qquad \text{interval } (0,1)$$

$$f'(\tfrac{1}{2}) = -2\frac{1}{2} + 3 \qquad \text{increasing}$$

$$f'(10) = -6(10)^2 + 6(10) \qquad \text{interval } (1,\infty)$$
$$f'(10) = -600 + 60 \qquad \text{decreasing}$$

local min at $f(0)$, local max at $f(1)$.

Second derivative test:

$$f''(x) = -12x + 6 \qquad \text{find derivative}$$

$$f''(0) = -12(0) + 6 \qquad \text{critical value \#1}$$
$$f''(0) = 6 \qquad \text{positive so this is occurs at a local min}$$

$$f''(1) = -12(1) + 6 \qquad \text{critical value \#2}$$
$$f''(1) = -12 + 6 \qquad \text{negative so this occurs at a local max}$$

We've found the local min and max locations. Now plug into original function to get extrema values.

$$f(0) = 1 + 3(0)^2 - 2(0)^3$$
$$f(0) = 1$$

$$f(1) = 1 + 3(1)^2 - 2(1)^3$$
$$f(1) = 1 + 3 - 2$$
$$f(1) = 2$$

$\boxed{\text{local max: } f(1) = 2, \text{ local min: } f(0) = 1}$