Find the value c that satisfies the Mean Value Theorem.

Mean value theorem:

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

7. See graph pg. 295

$$f'(c) = \frac{f(0) - f(5)}{0 - 5}$$
$$f'(c) = \frac{1 - 3}{0 - 5}$$
$$f'(c) = \frac{2}{5}$$

We know the graph has a slope of $\frac{2}{5}$ at c, therefore $3 < c < 4$. How does the book narrow to 3.8?

8. See graph pg. 295. Mean value theorem does not apply. Function is non-differentiable at 4.

Verify the function satisfies Rolle's Theorem and find all numbers c.

11. $f(x) = \sin(x/2), \ [\pi/2, \ 3\pi/2]$
Function is continuous on the open interval, differentiable on the closed interval, and $f(\pi/2) = f(3\pi/2)$.
c $= \pi$ because $f(\pi) = 0$ and π is in the interval.

Find c with Mean Value Theorem.

21. $f(x) = (x - 3)^{-2}, \ [1, \ 4]$

Function is not continuous at 3, therefore the MVT does not apply on this interval.

Verify the function has only one real solution.

23. $2x + \cos x = 0$
Function is differentiable for all real numbers. We apply Intermediate Value Theorem to show there is at least one solution.

$$2\left(-\frac{\pi}{2}\right) + \cos\left(-\frac{\pi}{2}\right) = -\frac{2\pi}{2} - 1 \qquad 2\left(\frac{\pi}{2}\right) + \cos\left(\frac{\pi}{2}\right) = \frac{2\pi}{2} + 1$$

We get one positive and one negative number. Since $f(x)$ is continuous, we know that there must be at least one root. Then, we show that the derivative has NO real roots. Hence, the graph is always decreasing or increasing and never "turns" back towards the x axis.

$$2 + -\sin x = 0 \qquad \text{implicit differentiation}$$

$-1 \leq sinx \leq 1$, so there are no real solutions to the above equation.
Therefore, there can only be one solution to the original function.

25. $x^3 - 15x + c = 0$ on the interval [-2, 2]

$$(-2)^3 - 15(-2) + c = 22 + c$$
$$(2)^3 - 15(2) + c = -22 + c$$

$-2 \leq c \leq 2$, so regardless of what c is, one solution is positive and one is negative.

$$3x^2 - 15 = 0 \qquad \text{implicit differentiation}$$
$$3x^2 = 15$$
$$x^2 = 5$$
$$x = \pm\sqrt{5}$$

The solutions to the derivative falls outside the interval. Thus, since the graph "turns" back towards the x axis after the interval, the interval can not have more than one real solution.

Show that a polynomial of degree n has at most n real roots.

27. Proof by induction. First, we define a one degree polynomial. Assuming $a \neq 0$

$$0 = ax + b$$
$$x = -\frac{b}{a}$$

Because a can't be 0, this fraction always evaluates to a real number. Therefore this polynomial has one real root. Next, we make this polynomial a degree higher.

$$f(x) = (ax + b)(x - c)$$

We know that $(x - c)$ has no more than 1 real solution, c. This second degree polynomial can't have more than two solutions (one from the first degree polynomial, and at most one from our factor). We can continue this process to an n degree polynomial. Therefore, an n degree polynomial has at most n real solution(s).

Prove.

33. $\sin x < x$ if $0 \leq x \leq 2\pi$

Both functions are differentiable on the open interval. We can use the MVT to compare mean instantaneous rates of change.

$$f'(c) = \frac{f(2\pi) - f(0)}{2\pi - 0} \qquad \text{MVT for } f(x) = sinx$$
$$f'(c) = \frac{1 - 0}{2\pi - 0}$$
$$f'(c) = \frac{1}{2\pi}$$

$$f'(c) = 1 \qquad \text{MVT for } f(x) = x$$

This tells us the mean slope for $sin(x)$ is less or equal to x on the interval.

Next, find derivatives.

$$\cos x \leq 1$$

$f'(x) = \cos x$ has only one real root, $f'(x) = 1$ has no real roots. We know that the slope for both functions was the same initially (plug in start of interval to both derivatives), the slope of x NEVER decreases, and the slope of $\sin(x)$ ONLY decreases on this interval. Therefore $\sin x < x$ on the interval.