Answer questions about function properties with graphs.

1. See pg. 305
   a. (1,3), (4, 6)
   b. (0, 1), (3, 4)
   c. (0, 2)
   d. (2, 4), (4, 6) Note: intervals are separate even though they are adjacent.
   e. [2, 3] Note: non-differentiable doesn't mean there will be an inflection point

5. See pg. 305 a. increasing: (0, 1), (3, 5) decreasing: (1, 3), (5, 6) b. We evaluate roots and undefined inputs, not endpoints, for local extrema. If $f''(x) < 0$, extrema is a local max. The opposite holds true for local min.

   $\boxed{\text{max: } f(1) \text{ and } f(5), \text{ min: } f(3)}$

Find intervals, extrema, concavity (requirements vary)

9. $2x^3 - 15x^2 + 24x - 5$

   Polynomials are differentiable for all real numbers.

   $$f'(x) = 6x^2 - 30x + 24 \qquad \text{find derivative}$$
   $$0 = 6x^2 - 30x + 24$$
   $$0 = x^2 - 5x + 4 \qquad \text{find root(s)}$$
   $$0 = (x - 1)(x - 4)$$

   Critical values are 1 and 4. These are fence-posts. Test a number from each of the three "fences".

   $$f(0) = (0 - 1)(0 - 4) \qquad \text{interval } (-\infty, 1)$$
   $$f(0) = (-)(-) \qquad \text{increasing}$$

   $$f(2) = (2 - 1)(2 - 4) \qquad \text{interval } (1, 4)$$
   $$f(2) = (+)(-) \qquad \text{decreasing}$$

   $$f(5) = (5 - 1)(5 - 4) \qquad \text{interval } (4, \infty)$$
   $$f(5) = (+)(+) \qquad \text{increasing}$$

   $\boxed{\text{Therefore, function is increasing on } (-\infty, 1) \cup (4, \infty) \text{ and decreasing on } (1, 4).}$

   Increasing to decreasing is a local max, while decreasing to increasing is a local min. We can take the second derivative to show this.

   $$f''(x) = 12x - 30 \qquad \text{find second derivative}$$

   $$f''(1) = 12(1) - 30 \qquad \text{critical value \#1}$$
   $$f''(1) = -18 \qquad \text{max}$$

   $$f''(4) = 12(4) - 30 \qquad \text{critical value \#2}$$
   $$f''(4) = 18 \qquad \text{min}$$

Next, plug in both critical values to find values of local extrema at the critical values.

$$f(1) = 2(1)^3 - 15(1)^2 + 24(1) - 5 \hspace{2cm} \text{critical value \#1}$$
$$f(1) = 2 - 15 + 24 - 5$$
$$f(1) = 6$$

$$f(4) = 2(4)^3 - 15(4)^2 + 24(4) - 5 \hspace{2cm} \text{critical value \#2}$$
$$f(4) = 128 - 240 + 96 - 5$$
$$f(4) = -21$$

$$\boxed{\text{max: } f(1) = 6, \text{ min: } f(4) = -21}$$

13. $\dfrac{x^2 - 24}{x - 5}$

$$f'(x) = \frac{(x-5)(2x) - (x^2 - 24)(1)}{(x-5)^2} \hspace{2cm} \text{find derivative with quotient rule}$$
$$f'(x) = \frac{2x^2 - 10x - x^2 + 24}{(x-5)^2}$$
$$f'(x) = \frac{x^2 - 10x + 24}{(x-5)^2}$$

Set factor in denominator to 0 and evaluate to find undefined inputs. Set numerator to 0 and evaluate to find roots. These will be the critical numbers.

$$0 = x^2 - 10x + 24 \hspace{2cm} \text{denominator}$$
$$x = (x-4)(x-6)$$
$$x = 4 \text{ and } 6$$

$$0 = x - 5 \hspace{2cm} \text{numerator}$$
$$x = 5$$

Critical numbers are 4, 5, 6. Use inputs that fall in the intervals we want to verify. We only need sign, so

separate into factors if calculator is not an option. Plug into derivative.

$$f'(0) = \frac{0^2 - 10(0) + 24}{(0-5)^2}$$ interval $(-\infty, 4)$

$$f'(0) = \frac{+}{+}$$ increasing

$$f'(4.5) = \frac{(4.5)^2 - 10(4.5) + 24}{((4.5)-5)^2}$$ interval $(4, 5)$

$$f'(4.5) = \frac{-}{+}$$ decreasing

$$f'(5.5) = \frac{(5.5)^2 - 10(5.5) + 24}{((5.5)-5)^2}$$ interval $(5, 6)$

$$f'(5.5) = \frac{-}{+}$$ decreasing

$$f'(10) = \frac{10^2 - 10(10) + 24}{(10-5)^2}$$ interval $(6, \infty)$

$$f'(10) = \frac{+}{+}$$ increasing

Points where sign of derivative changes include a local max: $f(4)$ and a local min: $f(6)$

$$f(4) = \frac{(4)^2 - 24}{4 - 5}$$

$$f(4) = \frac{-4}{-1}$$

$$f(4) = 4$$

$$f(6) = \frac{(6)^2 - 24}{6 - 5}$$

$$f(6) = \frac{12}{1}$$

$$f(6) = 12$$

$\boxed{\text{Local max: } f(4) = 8 \text{ and Local min: } f(6) = 12}$

17. $x^3 - 3x^2 - 9x + 4$ Polynomials are differentiable for all real numbers.

$$f'(x) = 3x^2 - 6x - 9$$ find derivative

$$0 = 3x^2 - 6x - 9$$ find root(s)

$$0 = x^2 - 2x - 3$$

$$0 = (x+1)(x-3)$$

Critical numbers are -1 and 3

$$f'(-2) = 3(-2)^2 - 6(-2) - 9 \qquad\qquad \text{interval } (-\infty, -1)$$
$$f'(-2) = 15 \qquad\qquad\qquad\qquad\qquad \text{increasing}$$

$$f'(0) = 3(0)^2 - 6(0) - 9 \qquad\qquad \text{interval } (-1, 3)$$
$$f'(0) = -9 \qquad\qquad\qquad\qquad\qquad \text{decreasing}$$

$$f'(4) = 3(4)^2 - 6(4) - 9 \qquad\qquad \text{interval } (3, infty)$$
$$f'(4) = 24 \qquad\qquad\qquad\qquad\qquad \text{increasing}$$

Local max: $f(-1)$, Local min: $f(3)$. Plug back into original function. We can skip this step if just trying to find concavity and inflection point.

$$f(-1) = (-1)^3 - 3(-1)^2 - 9(-1) + 4$$
$$f(-1) = -1 - 3 + 9 + 4$$
$$f(-1) = 9$$

$$f(3) = (3)^3 - 3(3)^2 - 9(3) + 4$$
$$f(3) = 27 - 27 - 27 + 4$$
$$f(3) = -23$$

max: $f(-1) = 9$, min: $f(3) = -23$

Next, we will work with second derivative.

$$f'(x) = 3x^2 - 6x - 9$$
$$f''(x) = 6x - 6 \qquad\qquad\qquad\qquad \text{find second derivative}$$

$$0 = 6x - 6 \qquad\qquad\qquad\qquad \text{find roots}$$
$$x = 1$$

Fence posts are the roots. Evaluate inputs in the "fences". Negative means concave down and positive means concave up.

$$f''(0) = 6(0) - 6 \qquad\qquad\qquad \text{interval } (-\infty, 1)$$
$$f''(0) = -6 \qquad\qquad\qquad\qquad\qquad \text{concave down}$$

$$f''(2) = 6(2) - 6 \qquad\qquad\qquad \text{interval } (-1, \infty)$$
$$f''(2) = 12 \qquad\qquad\qquad\qquad\qquad \text{concave up}$$

Find inflection point by plugging in root of second derivative into original equation.

$$f(1) = (1)^3 - 3(1)^2 - 9(1) + 4$$
$$f(1) = 1 - 3 - 9 + 4$$
$$f(1) = -7$$

Inflection point: (1, -7)

21. $\ln(x^2 + 5)$

$$f'(x) = \frac{1}{x^2 + 5}(2x) \qquad\qquad\qquad \text{find derivative}$$

$$f'(x) = \frac{2x}{x^2 + 5}$$

$$f''(x) = \frac{(x^2+5)(2) - (2x)(2x)}{(x^2+5)^2} \qquad\qquad \text{find second derivative}$$

$$f''(x) = \frac{2x^2 + 10 - 4x^2}{(x^2+5)^2}$$

$$f''(x) = \frac{-2x^2 + 10}{(x^2+5)^2}$$

Denominator can never be 0. Set numerator to 0 to find roots, aka the x-values of inflection points.

$$0 = -2x^2 + 10$$
$$2x^2 = 10$$
$$x^2 = 5$$
$$x = \pm\sqrt{5}$$

Evaluate the "fences".

$$f''(-3) = \frac{-2(-3)^2 + 10}{((-3)^2 + 5)^2} \qquad\qquad \text{interval } (-\infty, -\sqrt{5})$$

$$f''(-3) = \frac{-}{+} \qquad\qquad\qquad\qquad \text{concave down}$$

$$f''(0) = \frac{-2(0)^2 + 10}{(0^2 + 5)^2} \qquad\qquad \text{interval } (-\sqrt{5}, \sqrt{5})$$

$$f''(0) = \frac{+}{+} \qquad\qquad\qquad\qquad \text{concave up}$$

$$f''(3) = \frac{-2(3)^2 + 10}{(3^2 + 5)^2} \qquad\qquad \text{interval } (\sqrt{5}, \infty)$$

$$f''(3) = \frac{-}{+} \qquad\qquad\qquad\qquad \text{concave down}$$

Plug solutions back into original equation to find inflection points.