Computer Vision, Assignment 3, Epipolar Geometry

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1 Exercise 1

The fundamental matrix is given by:

$$F = [e_2] \times A = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ -2 & -2 & 0 \end{bmatrix}$$

The fundamental line is given by

$$\lambda Ax + t = \lambda \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$$

We can check if a point is on the line by checking $l^T x = 0$.

$$l = t \times (Ax) = \begin{bmatrix} 2 \\ 0 \\ -4 \end{bmatrix}$$

All three of the points (2, 0),(2, 1) and (4, 2) satisfy that.

Exercise 2

The camera center for P_1 has the trivial solution:

$$C_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

To find C_2, we solve:

$$P_2 C_2 = 0 \iff \begin{cases} x_1 + x_2 + x_3 + 2x_4 = 0 \\ 2x_2 + 2x_4 = 0 \\ x_3 = 0 \end{cases}$$

which gives:

$$C_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}$$

An epipole is the center of a camera in another camera's projection given by $e_i = P_i E_j$

$$e_1 = P_1 C_2 = \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$e_2 = P_2 C_1 = \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$$

The fundamental matrix is computed as:

$$F = [t]_\times A$$

where:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad t = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}, \quad [t] = \begin{bmatrix} 0 & -t_3 & t_2 \\ t_3 & 0 & -t_1 \\ -t_2 & t_1 & 0 \end{bmatrix}$$

$$F = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & -2 \\ -2 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & -2 \\ -2 & 2 & -2 \end{bmatrix}$$

The determinant of F is:

$$\det(F) = det(\begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & -2 \\ -2 & 2 & -2 \end{bmatrix})$$
$$2det(\begin{bmatrix} 0 & 0 \\ -2 & 2 \end{bmatrix}) = 0$$

For $e_2^T F$ and Fe_1 we get:

$$e_2^T F = \begin{bmatrix} 2 & 2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & -2 \\ -2 & 2 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$$

$$Fe_1 = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & -2 \\ -2 & 2 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Th confirming that e_2, e_1 satisfies the epipolar constraint.

Exercise 2 - Optional

For a general camera pair:

$$P_1 = \begin{bmatrix} I & 0 \end{bmatrix}, \quad P_2 = \begin{bmatrix} A & t \end{bmatrix}$$

where A is invertible.

The camera center of a camera matrix P is the point C that satisfies:

$$PC = 0.$$

- For the first camera, $P_1 = \begin{bmatrix} I & 0 \end{bmatrix}$, the camera center C_1 satisfies:

$$\begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} C_1 \\ 1 \end{bmatrix} = 0$$

  which implies $C_1 = (0, 0, 0)^T$.

- For the second camera, $P_2 = \begin{bmatrix} A & t \end{bmatrix}$, the camera center C_2 satisfies:

$$\begin{bmatrix} A & t \end{bmatrix} \begin{bmatrix} C_2 \\ 1 \end{bmatrix} = 0.$$

  which solving for C_2 gives,

$$AC_2 + t = 0 \quad \iff \quad C_2 = -A^{-1}t.$$

The epipoles are as mentioned the image projections of the other camera's centers.

- The epipole in the first image:

$$e_1 = P_1 C_2 = \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} -A^{-1}t \\ 1 \end{bmatrix} = -A^{-1}t.$$

- The epipole in the second image:

$$e_2 = P_2 C_1 = \begin{bmatrix} A & t \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = t.$$

To verify the fundamental matrix

$$F = [t]_\times A,$$

we need to check whether it satisfies the key epipolar constraints:

$$e_2^T F = 0$$

$$F e_1 = 0$$

where e_1 and e_2 are the epipoles, the projections of the camera centers.

$[t]_\times$ is the skew-symmetric matrix:

$$[t]_\times = \begin{bmatrix} 0 & -t_3 & t_2 \\ t_3 & 0 & -t_1 \\ -t_2 & t_1 & 0 \end{bmatrix}.$$

with the unique property that

$$t^T[t]_\times = \begin{bmatrix} t_1 & t_2 & t_3 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -t_3 & t_2 \\ t_3 & 0 & -t_1 \\ -t_2 & t_1 & 0 \end{bmatrix}.$$

$$= \begin{bmatrix} t_2 \cdot t_3 + t_3 \cdot (-t_2) \\ t_1 \cdot (-t_3) + t_3 \cdot t_1 \\ t_1 \cdot t_2 + t_2 \cdot (-t_1) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}.$$

We saw earlier that $e_2 = t$ which gives the first equation the result

$$e_2^T F = \underbrace{t^T [t]_\times}_{0} A = 0$$

We also saw earlier that $e_1 = -A^{-1}t$, which for the second equation gives

$$Fe_1 = [t]_\times A(-A^{-1}t)$$
$$= [t]_\times(-t) = -\underbrace{[t]_\times t}_{0} = 0$$

For determining the determinant of F we know $F = [t]_\times A$ and that $\det([t]_\times)= 0$ since all the diagonal elements of $[t]_\times$ is 0 meaning that the matrix $[t]_\times$ is a of rank 2 or less. The fundamental matrix is formed by multiplying a rank-2 (or less) matrix $[t]_\times$ with A, meaning:

$$\operatorname{rank}([t]_\times) \leq 2 \quad \Longrightarrow \quad \operatorname{rank}([t]_\times A) = \operatorname{rank}(F) \leq 2.$$

Since a 3×3 matrix with rank ≤ 2 must have determinant zero, we conclude:

$$\det(F) = 0.$$

Therefore the fundamental matrix always has determinant zero.

2 Exersice 3

$$F = N_2^T \tilde{F} N_1$$

Computer Exersice 1

I got the un-normalized fundamental matrix (assuming less than 10^{-5} is 0):

$$F = \begin{bmatrix} 0 & 0 & 0.0058 \\ 0 & 0 & -0.0267 \\ -0.0072 & 0 & 1 \end{bmatrix}$$

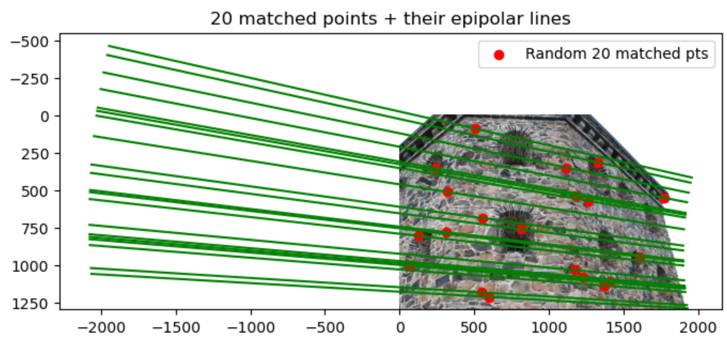

Figure 1: 20 points and their epipolar lines with normalization.

They do seem to be fairly close to each other, at least that you can't tell any difference by your eye.

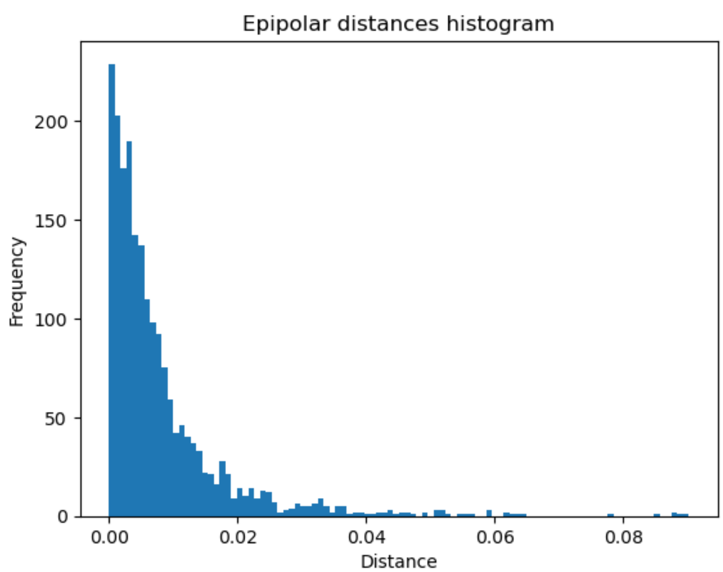

Figure 2: Distance between points and their corresponding line.

The mean distance is: 0.008124251966516033.

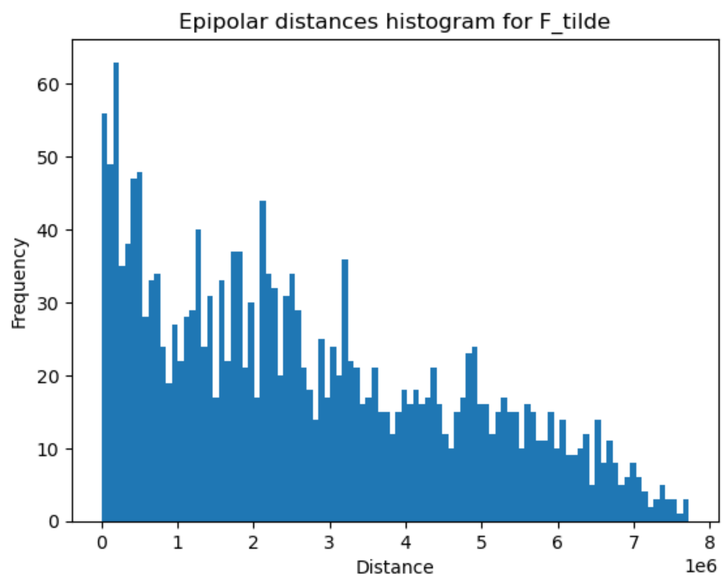

Figure 3: Distance between points and their corresponding line without normalization using only $\tilde{F}$.

Exercise 4

To find e_2 we use the epipolar constraint:

$$e_2^T F = 0.$$

$$\iff e_2^T F = \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} = 0$$

$$\iff e_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

The skew-symmetric matrix $[e_2]_\times$ is then given by:

$$[e_2]_\times = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}.$$

We can then get $[e_2]_\times F$:

$$[e_2]_\times F = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & -2 \\ 1 & 0 & 0 \end{bmatrix}.$$

Put together we get that the second camera matrix is:

$$P_2 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & -2 & -2 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix}.$$

For verifying the epipolar constraint of the projection of the 3D points we compute:

$$x_1 = P_1 X = X, \tag{1}$$

$$x_2 = P_2 X = [e_2]_\times F X + e_2 \tag{2}$$

For $X_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$$\begin{aligned}
x_1 &= \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, && (3)\\
x_2 &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & -2 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, && (4)\\
&= \begin{bmatrix} 1 \\ -10 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, && (5)\\
&= \begin{bmatrix} 2 \\ -10 \\ 0 \end{bmatrix}. && (6)
\end{aligned}$$

For $X_2 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$

$$\begin{aligned}
x_1 &= \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, && (7)\\
x_2 &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & -2 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, && (8)\\
&= \begin{bmatrix} 3 \\ -6 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, && (9)\\
&= \begin{bmatrix} 4 \\ -6 \\ 2 \end{bmatrix}. && (10)
\end{aligned}$$

For $X_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$$x_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \tag{11}$$

$$x_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & -2 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \tag{12}$$

$$= \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \tag{13}$$

$$= \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix}. \tag{14}$$

The epipolar constraint states that:

$$x_2^T F x_1 = 0.$$

For X_1 we get:

$$x_{2,1}^T F x_{1,1} = \begin{bmatrix} 2 & -10 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \tag{15}$$

$$= \begin{bmatrix} -10 & 2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \tag{16}$$

$$= (-10 \cdot 1) + (2 \cdot 2) + (2 \cdot 3) = -10 + 4 + 6 = 0. \tag{17}$$

For X_2 we get:

$$x_{2,2}^T F x_{1,2} = \begin{bmatrix} 4 & -6 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \tag{18}$$

$$= \begin{bmatrix} -6 & 6 & 6 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \tag{19}$$

$$= (-6 \cdot 3) + (6 \cdot 2) + (6 \cdot 1) = -18 + 12 + 6 = 0. \tag{20}$$

For X_3 we get:

$$x_{2,3}^T F x_{1,3} = \begin{bmatrix} 2 & -2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \tag{21}$$

$$= \begin{bmatrix} -2 & 2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \tag{22}$$

$$= (-2 \cdot 1) + (2 \cdot 0) + (2 \cdot 1) = -2 + 0 + 2 = 0. \tag{23}$$

Thus, the epipolar constraint holds for all three points.

For finding the camera center of P_2 we know tha it should satisfy the equation:

$$P_2 C_2 = 0.$$

$$\iff P_2 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & -2 & -2 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0$$

$$x_1 = -x_4, \quad x_2 = -x_3, \quad x_1 = x_4$$

$$\iff x_1 = x_4 = 0, \quad x_2 = -x_3$$

Solving for C_2, we can quickly find a solution and with $x_4 = 1$ we get:

$$C_2 = (0, -1, 1, 0)^T.$$

as a solution. This means the second camera center is at infinity along the direction $(0, -1, 1)$.

Computer Exercise 2

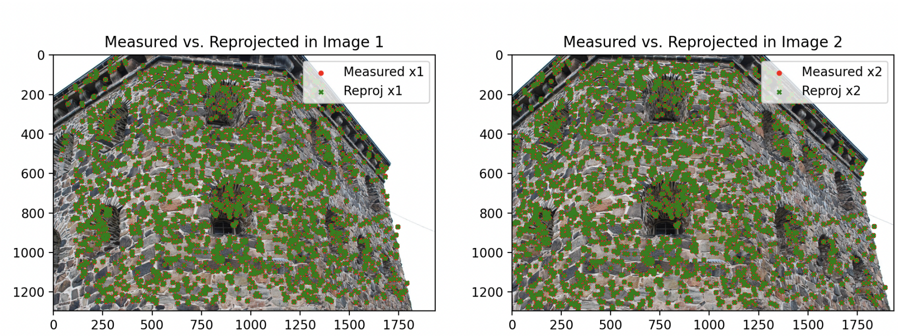

Figure 4: Measured image points and projected 3D points

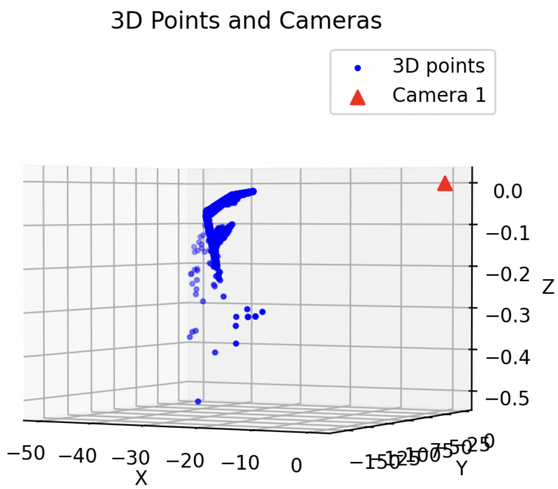

Figure 5: Front view of points and Camera 1. Camera 2 is an infinity point and therefore not shown.

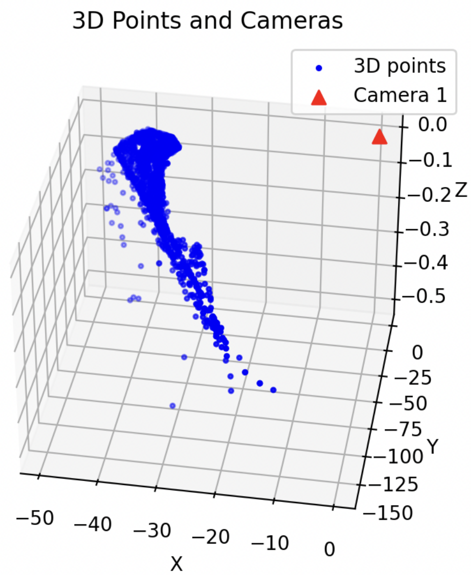

Figure 6: Top view of points and Camera 1. Camera 2 is an infinity point and therefore not shown.

Using

$$P_2 = \begin{bmatrix}[e_2]_\times F & e_2\end{bmatrix}$$

where F is of rank(2) we know A is going to be singular and when calculating C_2 it to be expected that the camera center is an infinity point. Regarding the points themselves they look not exactly as expected since I hoped them to turn around more like the tower.

Computer Exercise 3

The E-matrix

$$\begin{bmatrix} -8.89 & -1005.81 & 377.08 \\ 1252.52 & 78.37 & -2448.17 \\ -472.79 & 2550.19 & 1 \end{bmatrix}$$

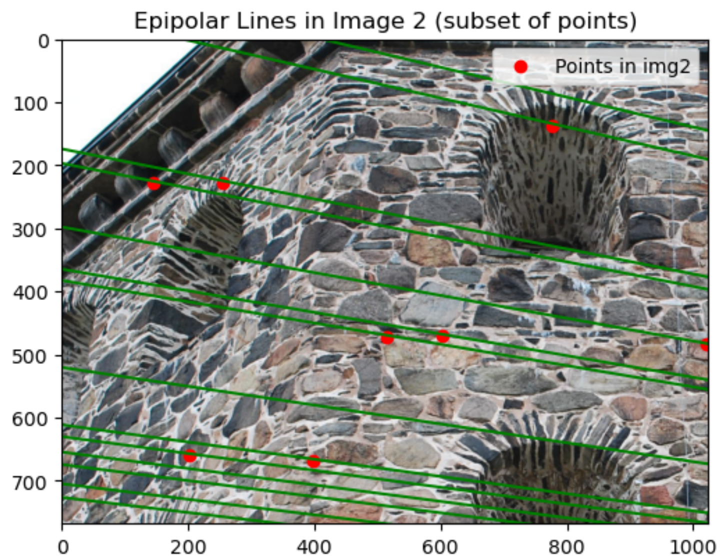

Figure 7: Epipolar lines in image 2

The mean distance is 2.23 which is way over what we got in assignment 1. The result is about 300 times worse now.

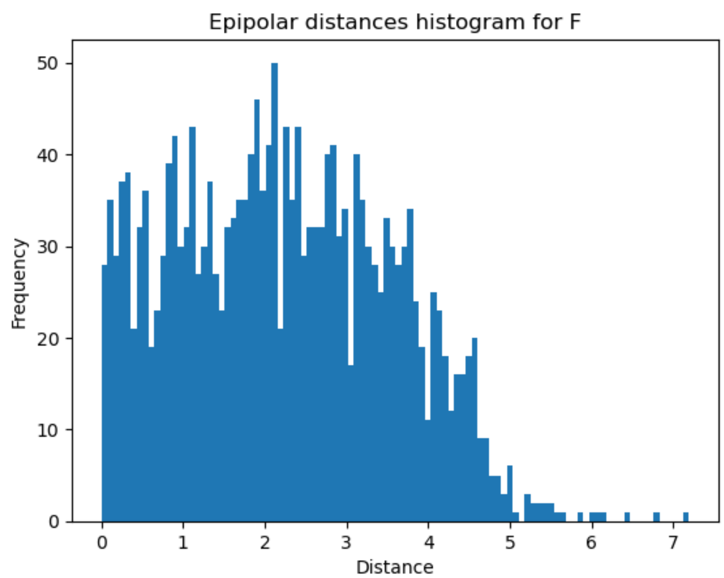

Figure 8: Distance from points in x2 to its corresponding epipolar lines.

Singular values of $[t]_\times$ and Eeigenvalues of $[t]_\times^T[t]_\times$

Let $[t]_\times$ be the 3×3 skew-symmetric matrix with singular value decomposition (SVD):

$$[t]_\times = USV^T,$$

where U and V are orthogonal matrices (inverse is it's transpose) and S is a diagonal matrix with non-negative singular values $\sigma_1, \sigma_2, \sigma_3$.

The eigenvalues of $[t]_\times^T[t]_\times$ are given by:

$$[t]_\times^T[t]_\times = (VS^TU^T)(USV^T) = VS^2V^T,$$

where $V^T = V^{-1}$ giving us a diagonalization form of $[t]_\times^T[t]_\times$ and $S^2 = \text{diag}(\sigma_1^2, \sigma_2^2, \sigma_3^2)$. Therefore, the eigenvalues of $[t]_\times^T[t]_\times$ are precisely the squares of the singular values of $[t]_\times$.

Verifying vector product property

$[t]_\times^T[t]_\times w = \lambda w$

Since $[t]_\times$ is skew-symmetric, we have:

$$[t]_\times^T = -[t]_\times.$$

We can therefore rewrite our problem and then since they are skew-symmetric use the property of it's similarity to vector products and get:

$$[t]_\times^T[t]_\times w = -[t]_\times([t]_\times w) = -t \times (t \times w).$$

Showing which eigenvectors and eigenvalues

Using the vector triple product identity we can then get the eigenvalues:

$$u \times (v \times w) = (u \cdot w)v - (u \cdot v)w,$$

with $u = t$, we get:

$$\begin{aligned} t \times (t \times w) &= (t \cdot w)t - \|t\|^2 w \\ \iff -t \times (t \times w) &= \|t\|^2 w - (t \cdot w)t \\ \iff [t]_\times^T[t]_\times w &= \|t\|^2 w - (t \cdot w)t \end{aligned}$$

- If $w = t$, then $t \times t = 0$, so t is an eigenvector with eigenvalue 0.
- If w is perpendicular to t, then $t \cdot w = 0$, and we obtain:

$$[t]_\times^T[t]_\times w = \|t\|^2 w,$$

showing that any w orthogonal to t is an eigenvector with eigenvalue $\|t\|^2$.

This means that every vector in the plane perpendicular to t is an eigenvector with eigenvalue $\|t\|^2$, forming a two-dimensional eigenspace, showing that $\|t\|^2$ is a double eigenvalue since it spans up a plane and not only a line.

Therefore, $0, \quad \|t\|^2, \|t\|^2$ are all of the eigenvalues of $[t]_\times^T [t]_\times$ (since dimension 3 it can be max 3). **Singular values of** $[t]_\times$

Since we showed that the eigenvalues of $[t]_\times^T [t]_\times$ are the singular values squared, we get the singular values from the square root of the nonzero eigenvalues of $[t]_\times^T [t]_\times$. It's also given that the singular values are non-negative why we conclude that the singular values of $[t]_\times$ are $\sqrt{0}$ and $\sqrt{\|t\|^2}$:

$$0, \quad \|t\|, \quad \|t\|.$$

SVD of $E = [t]_\times R$

If $[t]_\times = USV^T$, then multiplying by a rotation matrix R gives:

$$E = [t]_\times R = USV^T R.$$

Define $W^T = V^T R$, which is an orthogonal matrix. Then:

$$E = USW^T,$$

which is an SVD of E. Since the diagonal singular values remain unchanged, the singular values of E are also:

$$0, \quad \|t\|, \quad \|t\|.$$

Exercise 6

Verifying $\det(UV) = 1$

$$U \;=\; \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad V \;=\; \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\det(U\,V^T) \;=\; 1$$

Develop from the third row and you get:

$$\det(U) \;=\; 1 \cdot (\tfrac{1}{2} + \tfrac{1}{2}) \;=\; 1$$

Develop from the first row and you get:

$$\det(V) \;=\; 1 \cdot -(-1) \;=\; 1$$

$$\det(UV) = \det(U)\det(V) = 1 \cdot 1 = 1$$

Computing Essential matrix

Using SVD-decomposition we know the essential matrix is defined as:

$$E = U\,\mathrm{diag}(1,1,0)\,V^T.$$

Where:

$$S = \mathrm{diag}(1,1,0),$$

meaning for SV^T we can just remove the third row from V^T and compute USV^T:

$$E = U(SV^T) = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 \end{bmatrix}$$

Epipolar Constraint Verification

A plausible correspondence refers to a pair of points, one in each image, that satisfy the epipolar constraint.

In homogeneous coordinates:

$$x_1 = (0,0,1)^T, \quad x_2 = (1,1,1)^T.$$

We verify that:

$$x_2^T E x_1 = 0.$$

First, compute Ex_1. Since $x_1 = (0, 0, 1)$, the result is simply the third column of E:

$$E(0,0,1)^T = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}.$$

Next,

$$x_2^T (Ex_1) = (1,1,1) \cdot \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + 0 = 0.$$

Therefore, $x_2^T E x_1 = 0$, confirming that the pair $(0, 0)$ and $(1, 1)$ satisfies the epipolar constraint and is a plausible correspondence under E.

Show X(s) is the solution for $x_1 = PX$

The first camera matrix is:

$$P_1 = [I\ 0] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

A 3D point X in homogeneous coordinates is (X_1, X_2, X_3, X_4). Under P_1, the image point is:

$$x_1 \sim P_1 X = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}.$$

In inhomogeneous coordinates, this maps to:

$$\left(\frac{X_1}{X_3}, \frac{X_2}{X_3} \right).$$

If $x_1 = (0, 0)$, we must have $X_1 = 0$, $X_2 = 0$, and $X_3 \neq 0$,

$$X = \begin{pmatrix} 0 \\ 0 \\ X_3 \\ X_4 \end{pmatrix}.$$

Since we often normalize by setting $X_3 = 1$, we define:

$$X(s) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ s \end{pmatrix},$$

which still projects to $(0,0)$ for any $s \neq 0$. Assuming normalized X_3 all solutions will be on the form $X(s) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ s \end{pmatrix}$.

Solving for s under the four possible P_2

We have 4 alternatives:

1. $P_2 = [UWV^T \; u_3 \;]$
2. $P_2 = [UWV^T \; -u_3]$
3. $P_2 = [UW^TV^T \; u_3]$
4. $P_2 = [UW^TV^T \; -u_3]$

where

$$W = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad W^T = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad u_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

We want $X(s)$ to project to $x_2 = (1,1)$ in the second image, that will say:

$$P_2X(s) \sim (1,1).$$

In homogeneous form, this means (after the 3D-to-2D division) we want

$$\frac{(P_2X(s))_1}{(P_2X(s))_3} = 1, \quad \frac{(P_2X(s))_2}{(P_2X(s))_3} = 1.$$

$$\iff (P_2X(s))_1 = (P_2X(s))_3, \quad (P_2X(s))_2 = (P_2X(s))_3.$$

We precalculate some stuff:

$$WV^T = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$

$$W^TV^T = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$

$$UWV^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & -1 & 0 \end{bmatrix}.$$

$$UW^TV^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \end{bmatrix}.$$

Since $X(s) = \begin{bmatrix}0 & 0 & 1 & s\end{bmatrix}^T$ we only need to look at third column and add $\pm u_{3,3}\ cdot X(s)_4 = \pm s$ on the third row. We then get the different cases to:

Substituting into each case:

- Case 1: $P_2X(s) = \begin{bmatrix}-1/\sqrt{2} \\ -1/\sqrt{2} \\ s\end{bmatrix}$, solving for s gives $s = -\frac{1}{\sqrt{2}}$ (behind the first camera).
- Case 2: $P_2X(s) = \begin{bmatrix}-1/\sqrt{2} \\ -1/\sqrt{2} \\ -s\end{bmatrix}$, solving for s gives $s = \frac{1}{\sqrt{2}}$ (in front of both cameras).
- Case 3: $P_2X(s) = \begin{bmatrix}1/\sqrt{2} \\ 1/\sqrt{2} \\ s\end{bmatrix}$, solving for s gives $s = \frac{1}{\sqrt{2}}$ (in front of both cameras).
- Case 4: $P_2X(s) = \begin{bmatrix}1/\sqrt{2} \\ 1/\sqrt{2} \\ -s\end{bmatrix}$, solving for s gives $s = -\frac{1}{\sqrt{2}}$ (behind the first camera).

Therefore both m$P_2 = [UWV^T\ - u_3]$ and $P_2 = [UW^TV^T\ u_3]$ gives a solution that is in front of both cameras.