Computer Vision, Assignment 1 Elements of Projective Geometry

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Exercise 1

Exercise 1.1

For a point in homogeneous coordinates $\mathbf{x} = \begin{pmatrix} x \\ y \\ w \end{pmatrix}$ it can be converted to 2D Cartesian coordinates as $\left(\frac{x}{w}, \frac{y}{w}\right)$.

- $x_1 = \begin{pmatrix} 4 \\ -2 \\ 2 \end{pmatrix}$: $\left(\frac{4}{2}, \frac{-2}{2}\right) = (2, -1)$.
- $x_2 = \begin{pmatrix} 3 \\ -2 \\ -1 \end{pmatrix}$: $= \left(\frac{3}{-1}, \frac{-2}{-1}\right) = (-3, 2)$.
- $x_3 = \begin{pmatrix} 4\lambda \\ -2\lambda \\ 2\lambda \end{pmatrix}$, where $\lambda \neq 0$: $\left(\frac{4\lambda}{2\lambda}, \frac{-2\lambda}{2\lambda}\right) = (2, -1)$.
- $x_4 = \begin{pmatrix} 4 \\ -2 \\ 0 \end{pmatrix}$, we get division by 0 according to our formula: $\left(\frac{4}{0}, \frac{-2}{0}\right) = (\inf, -\inf)$.

  We could interpret x_4 as representing a projection in its direction that continues indefinitely toward infinity. In the lecture note also refer to as a vanishing point.

Computer Exercise 1

The code can be found in the file "plat.m".

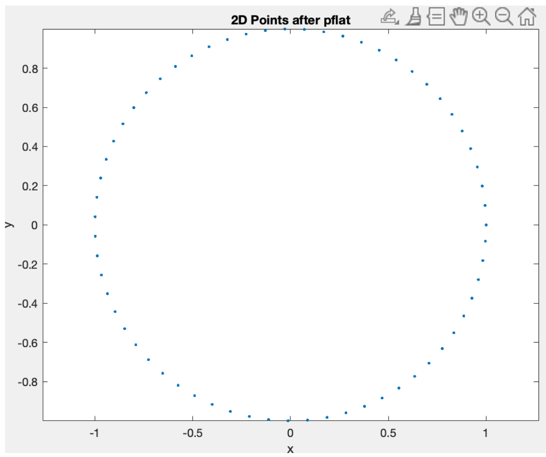

Figure 1: Plot of x2D points after applying the `pflat` function.

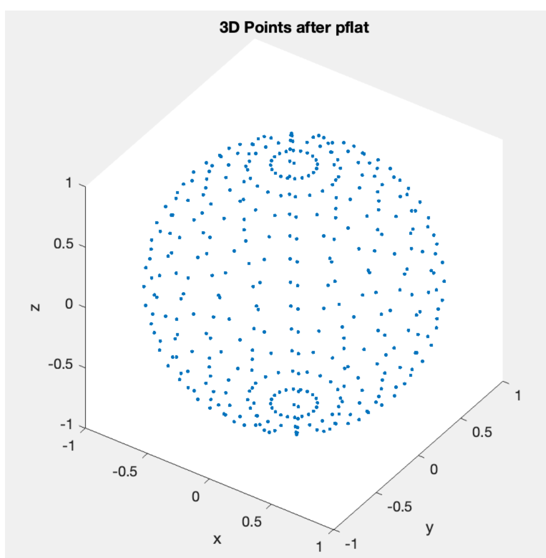

Figure 2: Plot of x3D points after applying the `pflat` function.

Exercise 2

Exercise 2.1

We can describe a line passing through the origin as $l^T x = 0$ where l^T is a perpendicular vector from any point on the line according to our lecture notes and our basic course in linear algebra at LTH.

The intersection between l_1 and l_2 can therefore be found at

$$l_1^T x = l_2^T x = 0 \iff \begin{cases} \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \\ \\ \begin{pmatrix} 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \end{cases} \iff \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} s \\ -2s \\ s \end{pmatrix}, \text{ for } s \in \mathbb{R}$$

In $\mathbb{R}^2$ the corresponding point is $\begin{pmatrix} s/s \\ -2s/s \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

The intersection between l_3 and l_4 can similarly be found at

$$l_3^T x = l_4^T x = 0 \iff \begin{cases} \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \\ \\ \begin{pmatrix} 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \end{cases} \iff \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2s \\ s \\ 0 \end{pmatrix}, \text{ for } s \in \mathbb{R}$$

Since $z = 0$ we can interpret this point as a vanishing point like the one in Exercise 1). In this case it means that l_1 and l_2 are parallel and therefore will never intersect.

Exercise 2.2

For the last sub-assignment we notice that the Cartesian points $x1$ and $x2$ correspond in homogeneous points to $l_1 = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$ and $l_2 = \begin{pmatrix} 3 & 2 & 1 \end{pmatrix}$.

The equations we solved on the form $l_i^T x_j = 0$ transposed is $x_j^T l_i = 0$, which is almost the same, just that the line and point vector have changed places. Since we now instead know 2 points x_1 and x_2, and are trying to fit a single line l through both points, the algebraic problem becomes the same equation system as we solved first with the same solution $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ with $s = z = 1$. We then get the line on normal from $l^T x = 0 \iff \begin{pmatrix} 1 & -2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = x - 2y + 1 = 0$ The line equation

Exercise 3

The nullspace $N(A) = \{x \in \mathbb{R}^n \mid Ax = 0\}$ of our matrix

$$M = \begin{pmatrix} 3 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

gives the same equation system as finding the intersection of l_1 and l_2 in homogeneous coordinates. We note that we have more variables than equations, meaning an underdetermined system that gives us infinitely many solutions. We get the solution

$$\begin{pmatrix} s \\ -2s \\ s \end{pmatrix}, \text{ for } s \in \mathbb{R}$$

For l_1 and l_2 in $\mathbb{P}^2$, that will say their Cartesian coordinates we notice that we have an additional equation $\begin{pmatrix} s/s \\ -2s/s \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ that now gives us the same amount of equations as variables leading to a determined system and thereby only one solution.

I visualize it as if we have three linearly independent vectors spanning a full $\mathbb{R}^3$, if we want to go to a projected plane spanned by only two of these vectors, all points on the same line with the direction of the third linearly independent vector all gets projected into this the same point on the projected plane. Dimension reduction, is what causes this and why it's not a contradiction, you could also algebraically see it as the homogeneous representation has an extra equation and thereby reduces the degree of freedome by 1 when you solve out that equation, which we do when we go over to our cartesian coordinates.

Computer Exercise 2

To easier calculate the line between two points, we use the fact that the line two points in homogeneous coordinates can be calculated with the help of the cross-product since we always have $\begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$ we can interpret this as the variables to the line equation $ax + by + c = 0$ that defines our sought line.

As shown before, it is the same equation system and algebraical problem to find the intersection of two lines as finding the line between two points, it is also mentioned in Chapter 2. Therefore we can also use the cross-product to calculate the intersection between two lines.

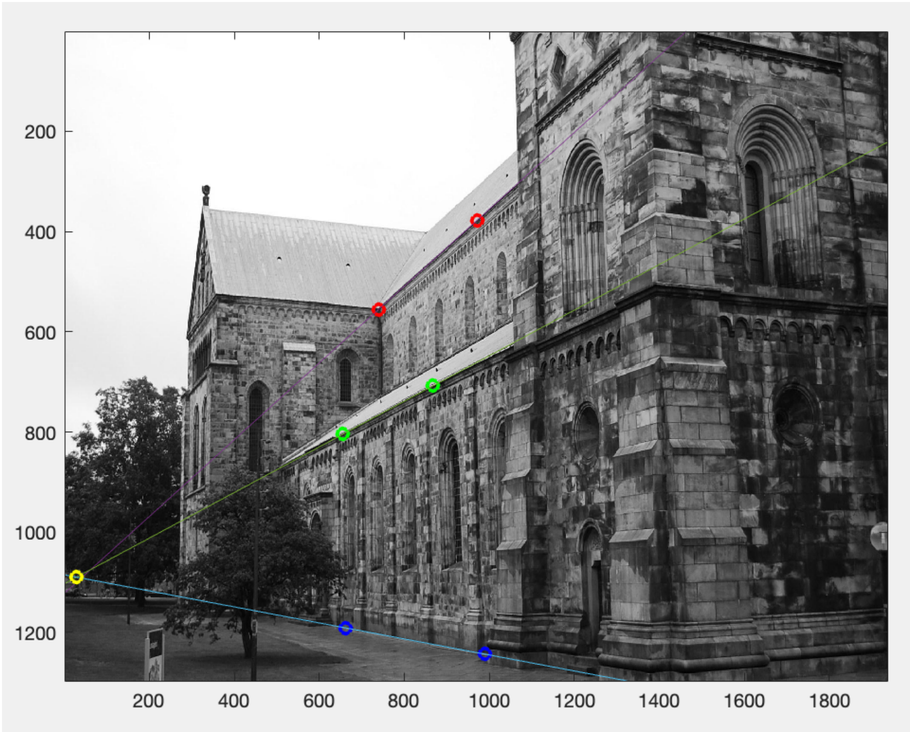

Figure 3: Line 1 is red, line 2 green, and line 3 blue. The yellow dot is the intersection of line 2 and line 3. The red, green, and blue dots are the corresponding points we got the lines from.

I got that the distance from between the intersection for line 2 and line 3 at (x, y) = (28.6, 1089,6) was 8,2. Since the picture is scaled 0-1200 on the y-axis and 0-1800 on the x-axis I would consider a distance of 8,2 units representing less than 1% of the height of the total picture in distance from each other. For the naked eye and thin lines it almost looks like they all three intersect.

Remark: After hand, I realized that I could also have used the null-space function instead of struggling to find a smart implementation to get the line passing through two points. The null space of a matrix consisting of our two points would give us the line since it would construct the line equation $ax + by + c = 0$ and solve it for us giving us our line $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ with need to .

Exercise 4

The projections are given by,

$$y_1 = Hx_1 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$y_2 = Hx_2 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

in homogeneous coordinates.

For computing the lines we use the cross product and get

$$l_1 = x_1 \times x_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$$

$$l_2 = y_1 \times y_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

For computing $(H^{-1})^T l_1$

$$H^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix}$$

$$(H^{-1})^T = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$(H^{-1})^T l_1 = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} = l_2$$

We see that $(H^{-1})^T l_1 = l_2$.

For a given line $l_1^T x = 0$ we know that for an invertible projective transformation H,

$$l_1^T x = 0 \iff \underbrace{l_1^T H^{-1}}_{(H^{-1})^T l_1} Hx = 0 \iff$$

$$\underbrace{(H^{-1})^T l_1}_{l_2^T} \underbrace{Hx}_{y} = 0 \iff l_2^T y = 0$$

We see that for each line l_1 there is a corresponding line l_2 such that if x belongs to l_1 then there is a transformation $y \sim Hx$ that belongs to l_2 since

$$l_1^T x = 0 \iff l_2^T \underbrace{y}_{Hx} = 0$$

.

Computer Exercise 3

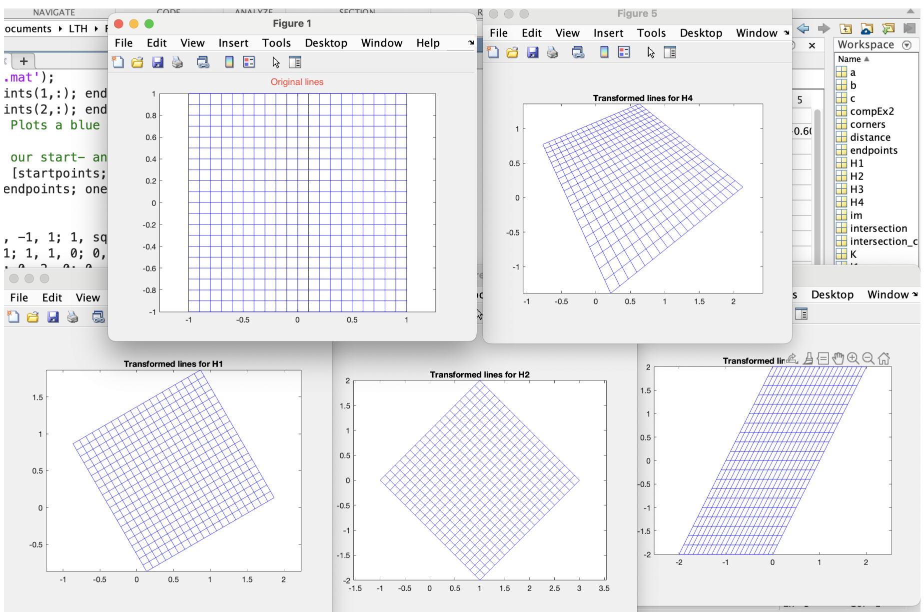

Figure 4: The top left plot is the original lines, top right is H4, bottom left is H1, bottom middle is H2, and bottom right is H3.

A projective transformation is just a mapping $\mathbb{P}^n \to \mathbb{P}^n$ defined as:

$$\mathbf{y} \sim H\mathbf{x},$$

where $\mathbf{x}, \mathbf{y}$ are homogeneous coordinates in projective space $\mathbb{P}^n$. Some properties for some different projective transformations are:

- As we showed in Exercise 4, an arbitrary projective transformation preserves collinearity, that will say points that are on the same line will after the transformation still remain points on the same line.
- An Euclidean transformation also preserves both lengths between points and angles between lines.
- A similarity transformation preserves angles between lines but not lengths between points.

- An affine transformation preserves parallelism, that will say that two parallel lines remain parallel after the transformation, but not lengths or angles.

For our transformation H1 to H4 we can see that:

- H1 looks like an Euclidean transformation. It preserves lengths between points, angles between lines and parallel lines stay parallel.
- H2 is a similarity transformation. We see angles and lines are preserved, but everything has been scaled up by a factor 2 meaning that the distance between points has changed.
- H3 is an affine transformation. Parallel lines stay parallel, but lengths and angles are changed.
- H4 is just a projective transformation. It does not preserve parallel lines, lengths, nor angles.

Exercise 5

Projection of X_1:

$$x_1 \sim PX_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3+1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix},$$

and in Cartesian coordinates:

$$x_1 = \begin{pmatrix} \frac{1}{4} \\ \frac{2}{4} \end{pmatrix} = \begin{pmatrix} 0.25 \\ 0.5 \end{pmatrix}.$$

Projection of X_2:

$$x_2 \sim PX_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1+1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix},$$

and in Cartesian coordinates:

$$x_2 = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}.$$

Projection of X_3:

$$x_3 \sim PX_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1+1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

Since $z = 0$ this means that the Cartesian coordinates for X_3 project to a point at infinity, which is a vanishing point that geometrically can be interpreted as a point at infinity the direction (1, 1).

Camera center & principal axis

The camera center C is the point in 3D space that is mapped to the origin of our projection. That is equivalent to the null space of our camera matrix. Using our camera equation $\lambda x = PX$ where $\lambda x = 0$ we get,

$$PX = 0 \iff \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\implies \begin{cases} x_1 = 0 \\ x_2 = 0 \\ x_3 = -s \\ x_4 = s \end{cases}$$

Dividing by x_4 we get that the camera center is in $\begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$.

In the lecture notes the viewing direction, principal axis, is defined by e_z of our projective matrix excluding the last translation column. In our case that would be $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

From exercise 2 on page 10 in the textbook we also learn how we can find both the camera center and the viewing direction through the equations

$$PX = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \text{ for camera center and } PX = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \text{ for principal axis}$$

since the last column in P just becomes $t \times 1$ where t is the translation .

1 Computer Exercise 4

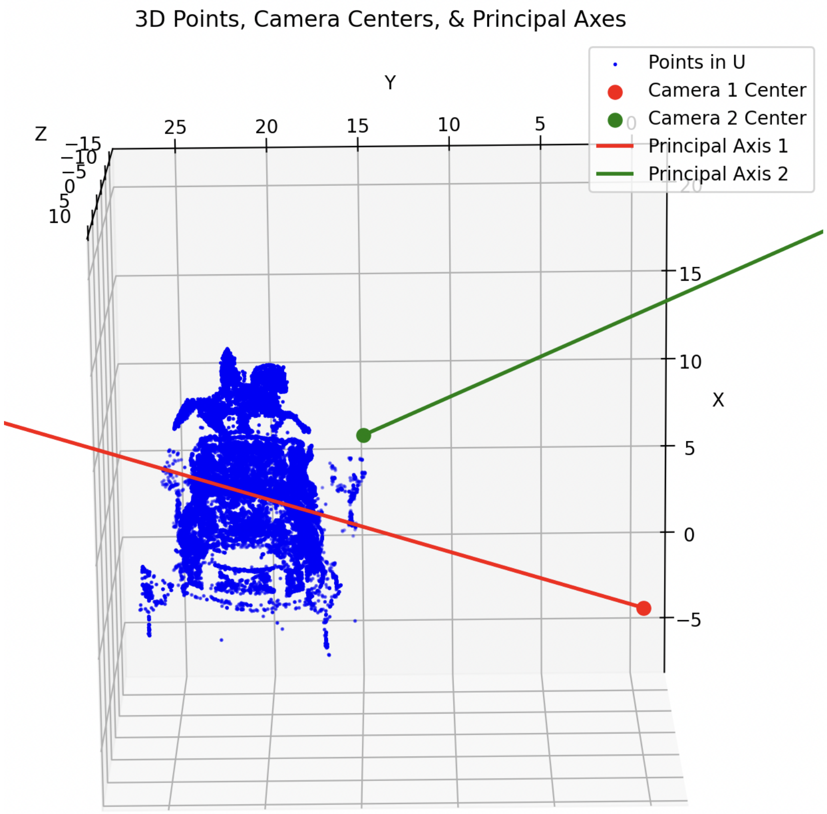

Figure 5: A plot of the points in U, the camera centers, and the principal axes.

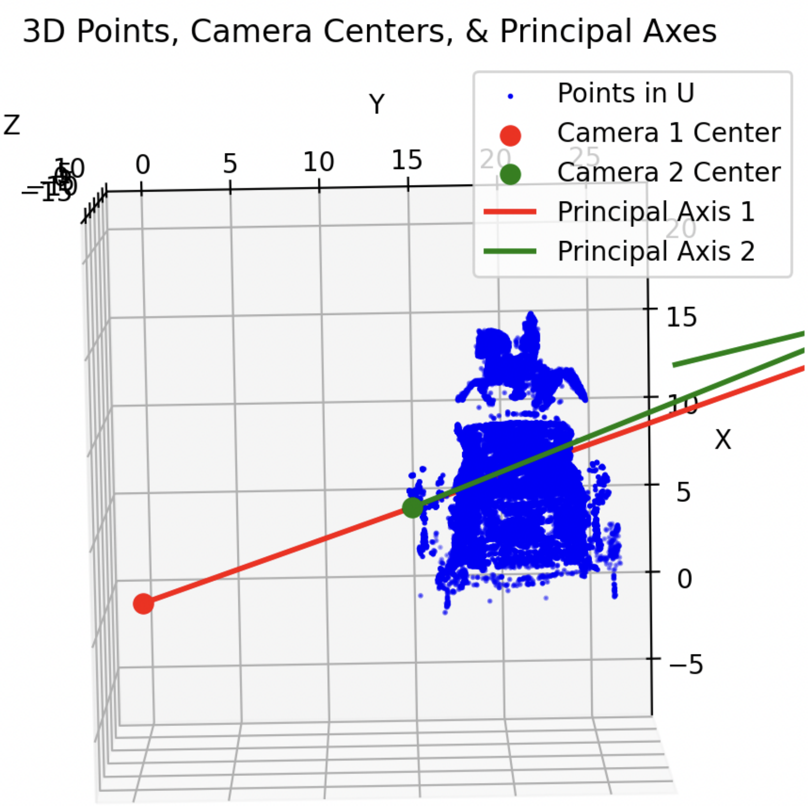

Figure 6: Another plot from a different angle of the points in U, the camera centers, and the principal axes.

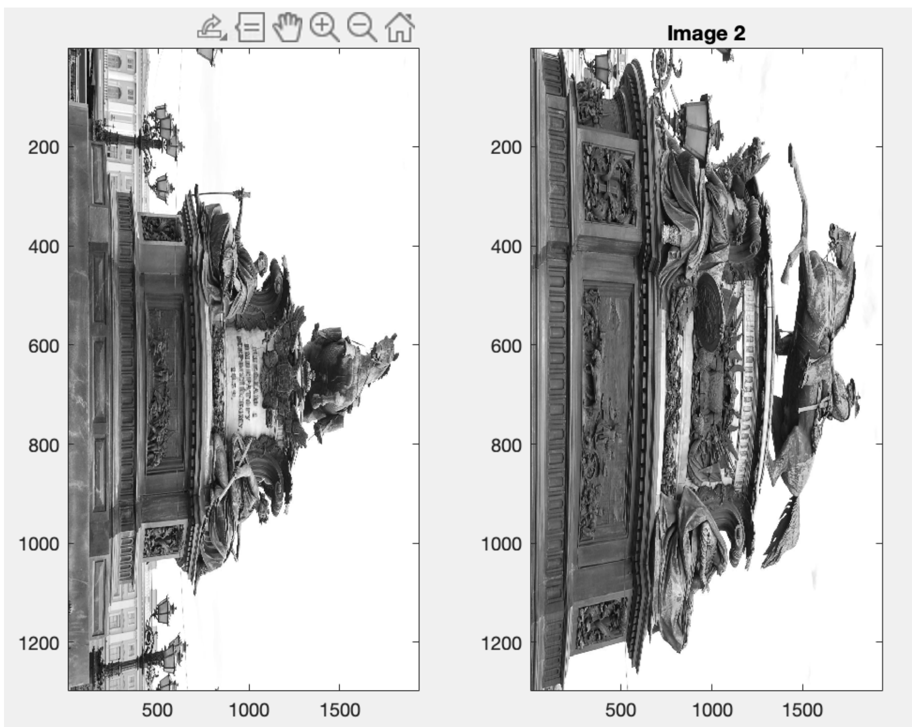

Figure 7: Original horse picture

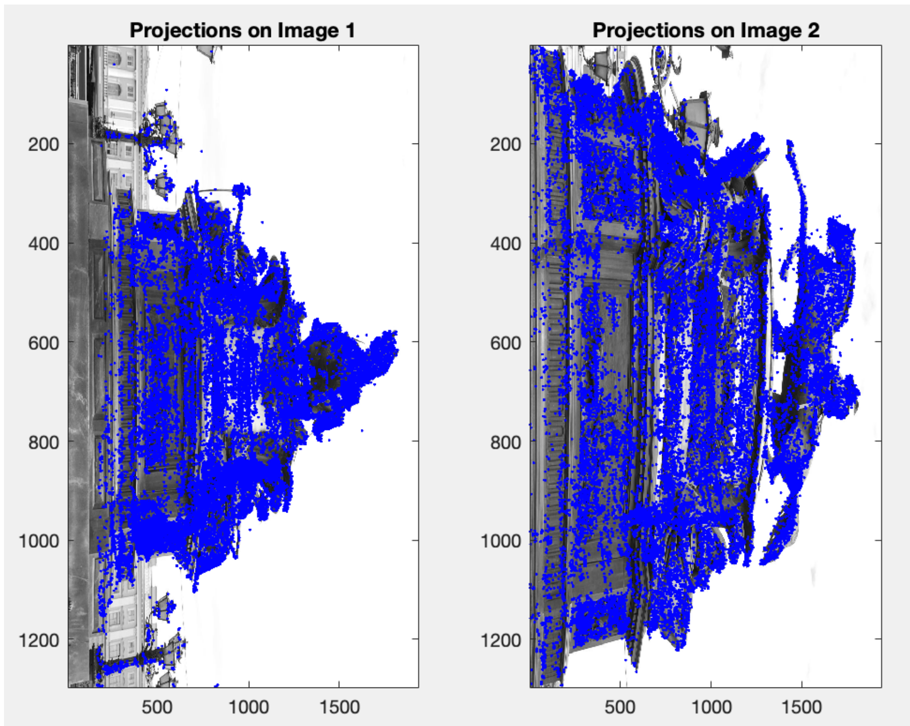

Figure 8: Horse picture with the points projected onto it.

I think the projections onto the images look reasonable since almost all points get mapped onto points of relevance, only a few outliers that get misplaced in the sky/white background. By doing so it captures the most important information of the image, in case you would want to restore it from the points. Since we have a quite symmetric motive with both its front and side being similar, apart from the horse, we can also note how it on both projections has captured and distributed the points similarly on the similar part which validates each other. Principal axis 1 (red color) is $\begin{pmatrix} 0.313 \\ 0.946 \\ 0.084 \end{pmatrix}$, principal axis 2 (green color) is $\begin{pmatrix} 0.032 \\ 0.340 \\ 0.939 \end{pmatrix}$. Camera center 1 (red color) is $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, camera center 2 (green color) is $\begin{pmatrix} 6.6 \\ 14.8 \\ -15.0 \end{pmatrix}$.

Exercise 6

Exercise 6.1

With $P_1 = [I\ 0]$ and $U \sim \begin{pmatrix} X \\ s \end{pmatrix}$, we have:

$$x \sim P_1 U = \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} X \\ s \end{bmatrix} = X$$

Therefore, $X = x$, and any point on $U(s) = \begin{bmatrix} x \\ s \end{bmatrix}$ projects to the same x. Since this is independent of s, we can therefore neither determine s from P_1.

This means that $U(s)$ representing a line of points in 3D all maps onto a single point no matter if U per se is a vanishing point or not because of P_1's inherit structure.

Exercise 6.2

Assuming U belongs to a plane Π defined by:

$$\Pi = \begin{bmatrix} \pi \\ 1 \end{bmatrix}, \quad \pi \in \mathbb{R}^3,$$

we find s such that $\Pi^T U = 0$ through:

$$\begin{bmatrix} \pi^T & 1 \end{bmatrix} \begin{bmatrix} x \\ s \end{bmatrix} = 0$$

$$\iff \pi^T x + s = 0$$

$$\iff s = -\pi^T x.$$

To verify a homography considering $x \sim P_1 U$, $y \sim P_2 U$, and $\Pi^T U = 0$. The homography is defined as:

$$H = R - t\pi^T.$$

Computing $P_2 U(s)$:

$$P_2 U = \begin{bmatrix} R & t \end{bmatrix} \begin{bmatrix} x \\ s \end{bmatrix} = Rx + ts.$$

Substituting $s = -\pi^T x$:

$$P_2 U = Rx + t(-\pi^T x) = (R - t\pi^T)x.$$

Since $y \sim P_2 U$ and $H = R - t\pi^T$,

$$y \sim Hx,$$

confirming that the homography H maps x to y as required.

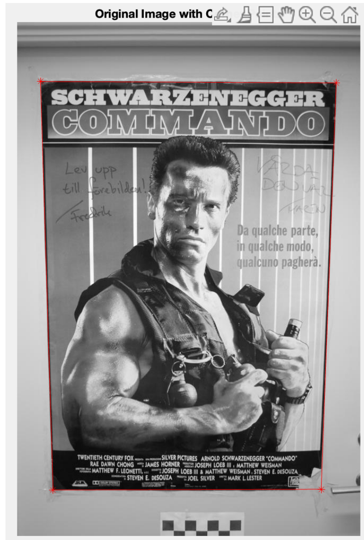

Figure 9: Arnold with Corner Points.
In the original image coordinate system, the origin (0,0) is located at the top-left corner of the image.

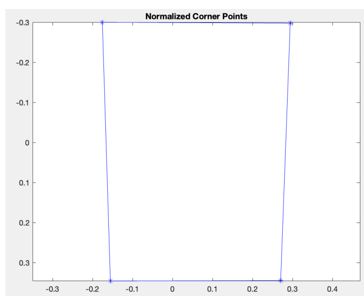

Figure 10: Normalized Corner Points.
After normalization, the origin corresponds to the principal point being at the center of the image coordinate system.

2 Computer Exercise 5

The new camera matrix:

$$P_{\text{new}} = 10^3 \cdot \begin{bmatrix} 0.9023 & 0.0000 & 0.8134 & 2.3762 \\ -0.2349 & 1.1913 & 0.4069 & 0 \\ -0.0005 & 0 & 0.0009 & 0 \end{bmatrix}$$

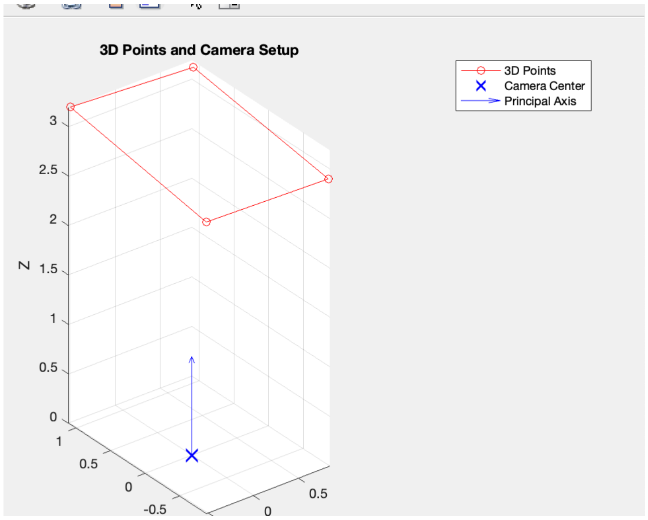

Figure 11: The new Camera Setup with our principal point located in the center.
It looks reasonable with the camera center being in the xy-center of the image and the principal axis being perpendicular pointing towards the projcetion image.

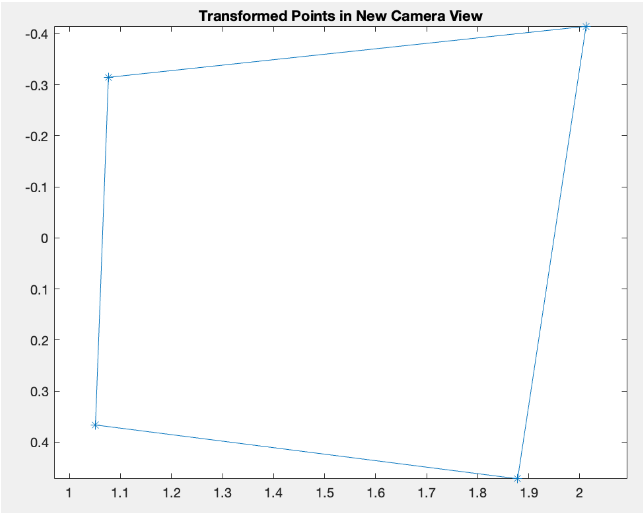

Figure 12: New Camera View after shifted 2m and rotated 30 degrees.

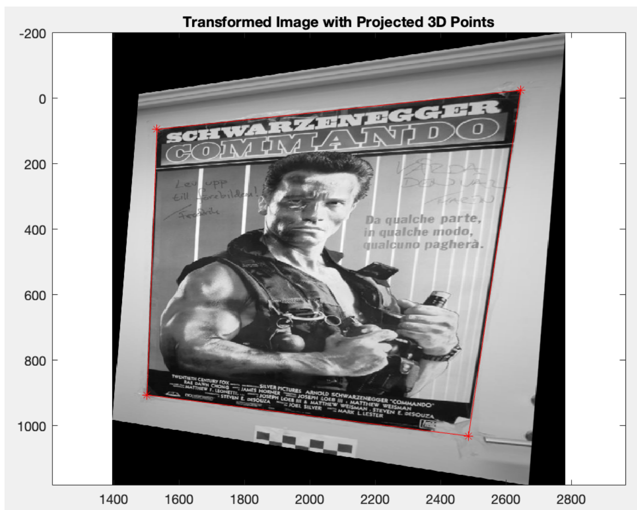

Figure 13: Transformed Arnold using the New Camera Position and projected 3D points plotted in red.

The result looks kind of like expected since the poster appears to be shifted 2 meters and rotated according to the camera's new position and -30° rotation. However, the image does look a little bit distored where it has went from a rectangular shape that's about 2:1 in height:width to be roughly a square with proporitons 1:1.

We also note that the points we got from the homography (blue shape in figure 11) and the projected red 3D points looks almost identical, which they should, yielding the same result.