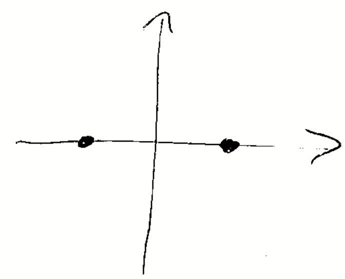
Roots of unity

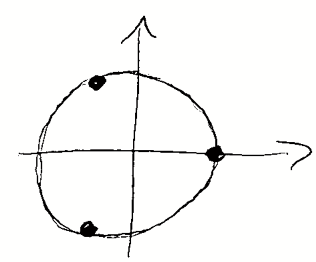
Def. A complex number $w \in \mathbb{C}$ is an N:th root of unity if $\boxed{w^N = 1}$

Ex $N=2$ $\quad w^2 = 1 \quad \Rightarrow \quad w = \pm 1$



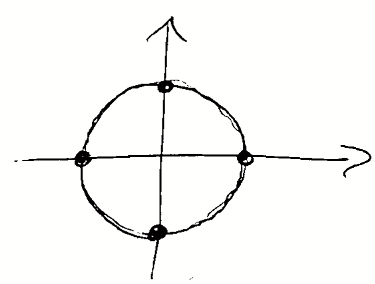
$N=3$ $\quad w^3 = 1 \iff (w-1)(w^2+w+1) = 0$

$$\Rightarrow \quad w = 1 \quad \text{or} \quad w = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$

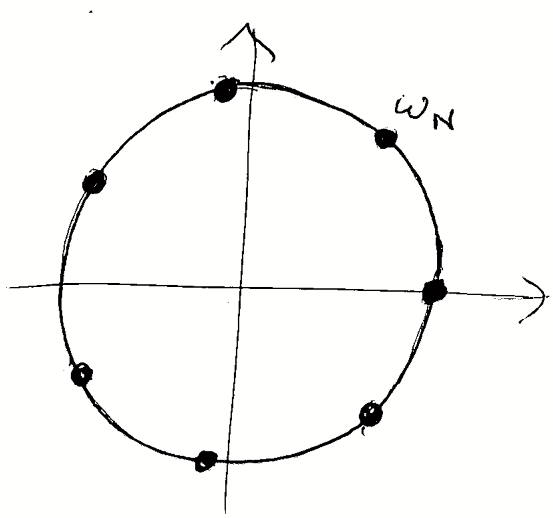


$N=4$ $\quad w^4 = 1 \quad \Rightarrow \quad w^2 = \pm 1$

$$\Rightarrow \quad w = 1, -1, i, -i$$



The Nth roots of unity are evenly spread along the unit circle in $\mathbb{C}$.



$$\omega_N = e^{i\frac{2\pi}{N}}$$

$$\Rightarrow \quad 1, \omega_N, \omega_N^2, \omega_N^3, \ldots, \omega_N^{N-1}$$

are all the Nth roots of unity

Properties of $\omega_N = e^{i\frac{2\pi}{N}}$ (Nth roots of unity)

(1) $\omega_N^N = 1$

(2) $\omega_N^{N-1} = \overline{\omega_N}$

(3) $\overline{\omega_N} = \omega_N^{-1}$ $\left(\omega_N^{-1} = \frac{\overline{\omega_N}}{|\omega_N|^2} = \overline{\omega_N} \right)$

Theorem Let $0<k<N$. Then $\sum_{i=0}^{N-1} (\omega_N^k)^i = 0$

Proof Recall that for a geometric series,

$$1 + q + q^2 + \dots + q^{N-1} = \frac{1-q^N}{1-q}$$

$$\Rightarrow \sum_{i=0}^{N-1} (\omega_N^k)^i = \frac{1-(\omega_N^k)^N}{1-\omega_N^k} = \frac{1-(\omega_N^N)^k}{1-\omega_N^k} = 0$$

(Note that $1-\omega_N^k \neq 0$ as $0<k<N$)

Fourier basis

Note : Indexing starts from 0

The standard basis of $\mathbb{C}^N$ is $\underline{e}_0, \underline{e}_1, \ldots, \underline{e}_{N-1}$

where

$$\underline{e}_k = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow \text{entry } k$$

Def. The <u>Fourier basis</u> of $\mathbb{C}^N$ is $\underline{f}_0, \underline{f}_1, \underline{f}_2, \ldots, \underline{f}_{N-1}$

where

$$\underline{f}_k = \begin{bmatrix} 1 \\ \omega_N^{k} \\ \omega_N^{2k} \\ \vdots \\ \omega_N^{(N-1)k} \end{bmatrix} \quad \text{where } \omega_N = e^{i\frac{2\pi}{N}}$$

e.g. $N=4$ $\quad \omega_4 = e^{i\frac{2\pi}{4}} = i$

$$\underline{f}_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad \underline{f}_1 = \begin{bmatrix} 1 \\ i \\ -1 \\ -i \end{bmatrix} \quad \underline{f}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} \quad \underline{f}_4 = \begin{bmatrix} 1 \\ -i \\ -1 \\ i \end{bmatrix}$$

Theorem The Fourier basis $\underline{f_0}, \underline{f_1}, \ldots, \underline{f_{N-1}}$ is an orthogonal basis of $\mathbb{C}^N$, with

$$\| \underline{f_k} \| = \sqrt{N}.$$

That is,

$$\langle \underline{f_k}, \underline{f_m} \rangle = \begin{cases} N, & \text{if } k = m \\ 0, & \text{if } k \neq m \end{cases}$$

Proof

$$\langle \underline{f_k}, \underline{f_m} \rangle = \underline{f_k}^T \overline{\underline{f_m}}$$

$$= \begin{bmatrix} 1 & \omega_N^k & \omega_N^{2k} & \cdots & \omega_N^{(N-1)k} \end{bmatrix} \begin{bmatrix} 1 \\ \overline{\omega_N}^m \\ \overline{\omega_N}^{2m} \\ \vdots \\ \overline{\omega_N}^{(N-1)m} \end{bmatrix}$$

$$= \sum_{j=0}^{N-1} \omega_N^{jk} \, \overline{\omega_N}^{jm}$$

$$= \sum_{j=0}^{N-1} \omega_N^{jk} \, \omega_N^{-jm}$$

$$= \sum_{j=0}^{N-1} \omega_N^{j(k-m)} = \begin{cases} N, & k = m \\ 0, & k \neq m \end{cases}$$

(see previous theorem)

Theorem $\quad \overline{\underline{f}_k} = \underline{f}_{N-k} \quad$ for $0 < k < N$

Proof $\quad$ Note that $\omega_N^N = 1$

$$\overline{\underline{f}_k} = \begin{bmatrix} 1 \\ \overline{\omega_N}^{k} \\ \overline{\omega_N}^{2k} \\ \vdots \\ \overline{\omega_N}^{(N-1)k} \end{bmatrix} = \begin{bmatrix} 1 \\ \omega_N^{-k} \\ \omega_N^{-2k} \\ \vdots \\ \omega_N^{-(N-1)k} \end{bmatrix} = \begin{bmatrix} 1 \\ \omega_N^{N-k} \\ \omega_N^{2(N-k)} \\ \vdots \\ \omega_N^{(N-1)(N-k)} \end{bmatrix} = \underline{f}_{N-k}$$

For instance, $N=8$:

$$\underline{f}_0, \ \underline{f}_1, \ \underline{f}_2, \ \underline{f}_3, \ \underline{f}_4, \ \underline{f}_5, \ \underline{f}_6, \ \underline{f}_7$$

real
(only 1s)

conjugate vectors

Note that if N is even, $\underline{f}_{\frac{N}{2}} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ \vdots \\ -1 \end{bmatrix}$ is also real

Discrete Fourier Transform (DFT)

The DFT is the vector of coefficients of a vector $\underline{x} \in \mathbb{C}^N$ with respect to the Fourier basis (except a factor $\frac{1}{N}$)

Let $\underline{x} \in \mathbb{C}^N$ and project $\underline{x}$ onto the Fourier basis:

$$\underline{x} = \frac{\langle \underline{x}, \underline{f_0} \rangle}{\langle \underline{f_0}, \underline{f_0} \rangle} \underline{f_0} + \cdots + \frac{\langle \underline{x}, \underline{f_{N-1}} \rangle}{\langle \underline{f_{N-1}}, \underline{f_{N-1}} \rangle} \underline{f_{N-1}}$$

$$= \frac{1}{N} \begin{bmatrix} \underline{f_0} & \cdots & \underline{f_{N-1}} \end{bmatrix} \begin{bmatrix} \langle \underline{x}, \underline{f_0} \rangle \\ \vdots \\ \langle \underline{x}, \underline{f_{N-1}} \rangle \end{bmatrix}$$

$$\left(\langle \underline{x}, \underline{f_k} \rangle = \overline{\langle \underline{f_k}, \underline{x} \rangle} = \overline{\underline{f_k}}^T \underline{x} \right)$$

$$= \frac{1}{N} \begin{bmatrix} \underline{f_0} & \cdots & \underline{f_{N-1}} \end{bmatrix} \underbrace{\begin{bmatrix} \overline{f_0}^T \\ \overline{f_1}^T \\ \vdots \\ \overline{f_{N-1}}^T \end{bmatrix} \underline{x}}_{= \mathrm{DFT}(\underline{x})}$$

<u>Definition</u> The discrete Fourier transform of $\underline{x} \in \mathbb{C}^N$ is

$$DFT(\underline{x}) = F_N \underline{x}$$

where
$$F_N = \begin{bmatrix} \bar{f}_0^T \\ \bar{f}_1^T \\ \vdots \\ \bar{f}_{N-1}^T \end{bmatrix} = \begin{bmatrix} f_0^T \\ f_{N-1}^T \\ \vdots \\ f_1^T \end{bmatrix}$$

is the Fourier matrix.

Note: $F_N^T = F_N$, so we can also write

$$F_N = \begin{bmatrix} \underline{f_0} & \underline{f_{N-1}} & \underline{f_{N-2}} & \cdots & \underline{f_1} \end{bmatrix}$$

Example Compute $\mathrm{DFT}(\underline{x})$ for $\underline{x} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}$

Solution $N = 4$

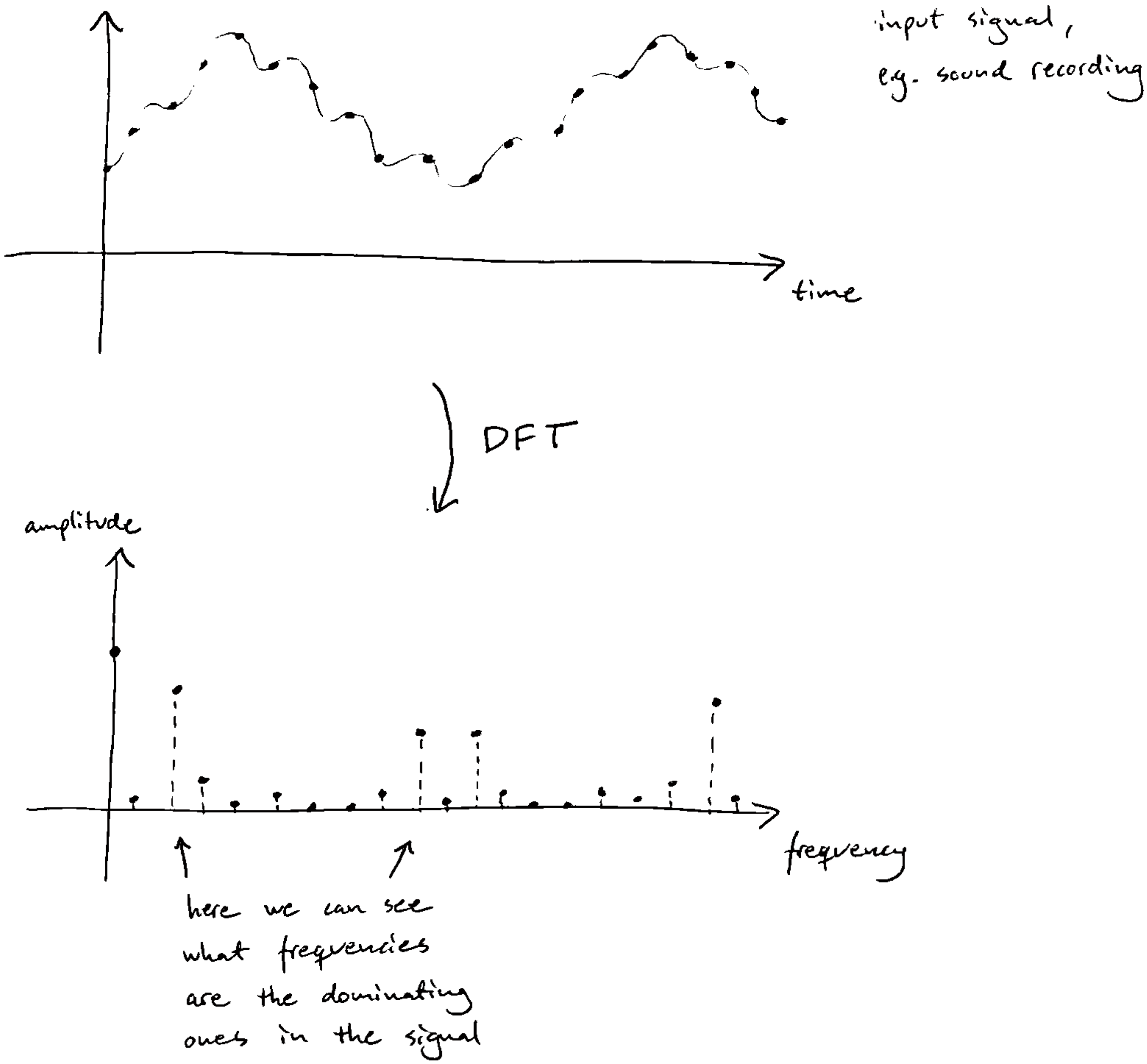
We saw already that

$$\underline{f_0} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad \underline{f_1} = \begin{bmatrix} 1 \\ i \\ -1 \\ -i \end{bmatrix} \quad \underline{f_2} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} \quad \underline{f_3} = \begin{bmatrix} 1 \\ -i \\ -1 \\ i \end{bmatrix}$$

$$\Rightarrow \quad F_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix}$$

$$\Rightarrow \mathrm{DFT}(\underline{x}) = F_4 \underline{x} = \begin{bmatrix} 4 \\ 1-i \\ -2 \\ 1+i \end{bmatrix}$$

Basic idea of the DFT:



- Fourier basis vectors $\to$ correspond to different frequencies
- DFT = coefficients when writing the input signal in the Fourier basis

$\Rightarrow$ The DFT tells us how much of each frequency is part of the signal.