MATH 307 Lecture: DFT

★ Info : · final Python assignment
(deadline next week Sat.)

· office hour

★ Today's theme : DFT

— (2-3 min)

★ First : roots of unity

· Def. $\omega^N = 1$

maybe guide
$\omega^2 = \pm 1$
↓

· A few examples, $N = 2, 3, 4$

→ students find roots for $N=2$ and $N=4$

? Q: What is the pattern here?

→ spread evenly around unit circle

· write all roots in terms of $\omega_N = e^{i \frac{2\pi}{N}}$

= first root going counterclockwise from 1

→ our "building block" for the Fourier basis

★ Properties of ω_N

- $\omega_N^N = 1$
- $\omega_N^{N-1} = \overline{\omega_N}$ (show graphically)
- $\overline{\omega_N} = \omega_N^{-1}$ (can use $z^{-1} = \dfrac{\overline{z}}{\underbrace{|z|^2}_{=1}}$)

<u>Thm</u> $0 < k < N$: $\displaystyle\sum_{i=0}^{N-1} (\omega_N^k)^i = 0$

(?) Formula for geometric series?

- first infinite $\dfrac{1}{1-q}$
- then finite $\dfrac{1-q^{N}}{1-q}$ get exponent right!

- Use this to prove thm.
- Point out that $1 - \omega_N^k \neq 0$ since $0 < k < N$

— (~20 min)

★ Fourier basis

- indexing from 0 !
- standard basis of $\mathbb{C}^N$ → same as for $\mathbb{R}^N$
- Fourier basis of $\mathbb{C}^N$

$$\underline{f_k} = \begin{bmatrix} 1 \\ \omega_N^k \\ \omega_N^{2k} \\ \vdots \end{bmatrix}$$ → powers of ω_N^k

★ Demonstration ! (GarageBand)

- Filter out frequency to improve sound
- EQ "=" DFT

——— (~ 30 min)

★ Example of Fourier basis <u>N=4</u>

(?) Recall : 4th roots of unity ?

★ Orthagonality of Fourier basis

<u>Thm</u> $\langle f_{\underline{k}}, f_{\underline{m}} \rangle = \begin{cases} N, & k=m \\ 0, & k \neq m \end{cases}$

Proof : $\langle f_{\underline{k}}, f_{\underline{m}} \rangle = \begin{bmatrix} \cdots & \omega_N^{jk} & \cdots \end{bmatrix} \begin{bmatrix} \vdots \\ \overline{\omega_N^{jm}} \\ \vdots \end{bmatrix}$

[conjugated !]

$= \sum \omega_N^{jk} \, \omega_N^{-jm}$

$= \sum \omega_N^{(k-m)j} = 0 \, , \; k \neq m$

using previous result.

★ Conjugate symmetry

$$\overline{\underline{f}_k} = \begin{bmatrix} \vdots \\ \overline{\omega_N^{jk}} \\ \vdots \end{bmatrix} = \begin{bmatrix} \vdots \\ \omega_N^{-jk} \\ \vdots \end{bmatrix} = \begin{bmatrix} \vdots \\ \omega_N^{j(N-k)} \\ \vdots \end{bmatrix} = \underline{f}_{N-k}$$

$\omega_N^N = 1$

e.g. $N=8$

→ Note: $\underline{f}_4$ is its own conjugate ⇒ real

? What is $\underline{f}_{\frac{N}{2}}$ for even N ?

$$\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ \vdots \end{bmatrix}$$

— (45 min)

★ DFT

→ basically the coefficients when writing a vector in the Fourier basis

(extra factor $\frac{1}{N}$)

$$\underline{x} = \frac{\langle \underline{x}, \underline{f}_0 \rangle}{\langle \underline{f}_0, \underline{f}_0 \rangle} \underline{f}_0 + \dots + \frac{\langle \underline{x}, \underline{f}_{N-1} \rangle}{\langle \underline{f}_{N-1}, \underline{f}_{N-1} \rangle} \underline{f}_{N-1}$$

$= N$

$$\cdots \qquad \underline{x} = \frac{1}{N} \underbrace{\left[\underline{f}_0 \ \underline{f}_1 \ \cdots \ \underline{f}_{N-1} \right]}_{\text{lin. comb.}} \underbrace{\begin{bmatrix} \langle x, f_0 \rangle \\ \vdots \\ \langle x, f_{N-1} \rangle \end{bmatrix}}_{\text{coeff.}}$$

- Note: $\langle \underline{x}, \underline{f}_k \rangle = \overline{\langle \underline{f}_k, \underline{x} \rangle} = \bar{f}_k^T \underline{x}$

$$\Rightarrow \quad \underline{x} = \frac{1}{N} \left[\underline{f}_0 \ \cdots \ \underline{f}_{N-1} \right] \underbrace{\begin{bmatrix} \bar{f}_0^T \\ \vdots \\ \bar{f}_{N-1}^T \end{bmatrix} \underline{x}}_{\text{DFT !}}$$

Def. $\mathrm{DFT}(\underline{x}) = F_N \underline{x}$

where $F_N = \begin{bmatrix} \bar{f}_0^T \\ \vdots \\ \bar{f}_{N-1}^T \end{bmatrix} \underset{\text{conj. symm.}}{=} \begin{bmatrix} f_0^T \\ f_{N-1}^T \\ \vdots \\ f_1^T \end{bmatrix}$

$\rightarrow$ Fourier matrix

- Note: F_N symmetric! (basis vectors as columns)

— (60 min)

★ Example : DFT of $\underline{x} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}$

- $N = 4$
- $F_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix}$

(?) Check if they remember to conjugate!
e.g. i or $-i$ here?

$\Rightarrow \text{DFT}(\underline{x}) = F_4 \underline{x} = \begin{bmatrix} 4 \\ 1-i \\ -2 \\ 1+i \end{bmatrix}$ ← sum of entries

conj.

(?) This could be in-class exercise, if enough time

★ Python example

- DFT of $\underline{x} = [1, 3, 7, \ldots]$
- np.fft.fft
- np.fft.ifft for inverse (next time...)
- Briefly : example with filtering