SUPPLEMENTARY MATERIAL - PROOFS

This section is concerned with proving that **Error! Reference source not found.** and **Error! Reference source not found.** are equivalent i.e., given that $\tilde{Z} \sim N(0,1)$ and $\omega_t \sim IG\left(\frac{\nu}{2}, \frac{\nu}{2}\right)$, we wish to show that $\tilde{Z}_t\left(\frac{\nu-2}{\nu}\omega_t\right)^{\frac{1}{2}}$ is t-distributed with mean 0, and unit variance. This is reformulated to form the following theorem:

Result:

If Y and W are independent random variables where $Y \sim N(0,1)$ and $W \sim IG\left(\frac{\nu}{2}, \frac{\nu}{2}\right)$, then the distribution of

$$X = Y\left(\sqrt{\frac{\nu-2}{\nu}W}\right)$$

is the scaled Student t-distribution with mean 0, and variance 1.

Proof:

Given that $W \sim IG\left(\frac{\nu}{2}, \frac{\nu}{2}\right)$, then for any constant $k \in \mathbb{R}$ $kW \sim IG\left(\frac{\nu}{2}, \frac{k\nu}{2}\right)$. Therefore, if we let $M = \left(\frac{\nu-2}{\nu}\right)W$, then $M \sim IG\left(\frac{\nu}{2}, \frac{\nu-2}{2}\right)$. Next, we let $Z = \sqrt{M}$, such that the cumulative density function (CDF) of Z, denoted by G, can be written as:

$$\begin{aligned} G(z) &= P[Z \leq z] \\ &= P\left[\sqrt{M} \leq z\right] \\ &= P[M \leq z^2] \end{aligned}$$

Given a random variable which follows an $IG(\alpha, \beta)$ distribution, the CDF is defined as:

$$\frac{\Gamma\left(\alpha, \frac{\beta}{x}\right)}{\Gamma(\alpha)}$$

where $\Gamma(\cdot,\cdot)$ refers to the incomplete Gamma function, and $\Gamma(\cdot)$ refers to the Gamma function. Since M is Inverse Gamma distributed, we can use this result to obtain $G(z)$:

$$G(z) = \frac{\Gamma\left(\frac{\nu}{2}, \frac{\nu-2}{2z^2}\right)}{\Gamma\left(\frac{\nu}{2}\right)}$$

To obtain the probability density function (PDF), we differentiate $G(z)$ with respect to z:

$$g(z) = G'(z) = \frac{2^{\frac{2-\nu}{2}} e^{-\frac{\nu-2}{2z^2}} \left(\frac{\nu-2}{z^2}\right)^{\frac{\nu}{2}}}{z\Gamma\left(\frac{\nu}{2}\right)} = \frac{2^{1-\frac{\nu}{2}} e^{-\frac{\nu-2}{2z^2}} (\nu-2)^{\frac{\nu}{2}} z^{-\nu-1}}{\Gamma\left(\frac{\nu}{2}\right)}$$

Since Y and Z are independent, we can multiply the distributions to generate their joint distribution, denoted by m:

$$m(y,z) = \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}} \times \frac{2^{1-\frac{\nu}{2}} e^{-\frac{\nu-2}{2z^2}} (\nu-2)^{\frac{\nu}{2}} z^{-\nu-1}}{\Gamma\left(\frac{\nu}{2}\right)}$$

$$= \frac{2^{\frac{1-\nu}{2}} e^{-\frac{y^2}{2}-\frac{\nu-2}{2z^2}} (\nu-2)^{\frac{\nu}{2}} z^{-\nu-1}}{\sqrt{\pi}\,\Gamma\left(\frac{\nu}{2}\right)}$$

To obtain the joint distribution of Y and X, denoted by h, we will perform a change of variable. Since $x = yz$, then $z = \frac{x}{y}$ where $\frac{dz}{dx} = \frac{1}{y}$. Thus,

$$\begin{aligned}
h(y,x) &= m(y,z)\left|\frac{dz}{dx}\right| \\
&= \frac{2^{\frac{1-\nu}{2}} e^{-\frac{y^2}{2}} (\nu-2)^{\frac{\nu}{2}}}{\sqrt{\pi}\,\Gamma\left(\frac{\nu}{2}\right)}\left(e^{-\frac{\nu-2}{2\left(\frac{x}{y}\right)^2}}\right)\left(\frac{x}{y}\right)^{-\nu-1}\left(\frac{1}{y}\right) \\
&= \frac{2^{\frac{1-\nu}{2}} e^{-\frac{y^2}{2}} (\nu-2)^{\frac{\nu}{2}}\; x^{-\nu-1}\, y^{\nu}}{\sqrt{\pi}\,\Gamma\left(\frac{\nu}{2}\right)} e^{-\frac{y^2(\nu-2)}{2x^2}} \\
&= \frac{2^{\frac{1-\nu}{2}}\; (\nu-2)^{\frac{\nu}{2}}\; x^{-\nu-1}\, y^{\nu}}{\sqrt{\pi}\,\Gamma\left(\frac{\nu}{2}\right)} \exp\left[-\frac{y^2}{2}\left(1+\frac{\nu-2}{x^2}\right)\right]
\end{aligned}$$

The PDF of X, denoted by r, can be found using $r(x) = \int_0^\infty h(y,x)\,dx$. To work this integral we will use another change of variable. Let $w = \frac{y^2}{2}\left(1+\frac{\nu-2}{x^2}\right)$, then $y = \frac{\sqrt{2w}\,x}{\sqrt{x^2+\nu-2}}$. Thus,

$$\begin{aligned}
\frac{dy}{dw} &= \frac{x}{\sqrt{2w(x^2+\nu-2\,)}} \\
dy &= \frac{x}{\sqrt{2w(x^2+\nu-2\,)}}\, dw
\end{aligned}$$

Therefore, we can transform

$$r(x) = \int_0^\infty \frac{2^{\frac{1-\nu}{2}}\; (\nu-2)^{\frac{\nu}{2}}\; x^{-\nu-1}\, y^{\nu}}{\sqrt{\pi}\,\Gamma\left(\frac{\nu}{2}\right)} \exp\left[-\frac{y^2}{2}\left(1+\frac{\nu-2}{x^2}\right)\right] dx$$

into

$$\begin{aligned}
r(x) &= \int_0^\infty \frac{2^{\frac{1-\nu}{2}}\; (\nu-2)^{\frac{\nu}{2}}\; x^{-\nu-1}}{\sqrt{\pi}\,\Gamma\left(\frac{\nu}{2}\right)}\left(\frac{\sqrt{2w}\,x}{\sqrt{x^2+\nu-2}}\right)^{\nu} e^{-w}\,\frac{x}{\sqrt{2w(x^2+\nu-2)}}\, dw \\
r(x) &= \frac{2^{\frac{1-\nu}{2}}\; (\nu-2)^{\frac{\nu}{2}}\; x^{-\nu-1}}{\sqrt{\pi}\,\Gamma\left(\frac{\nu}{2}\right)}\int_0^\infty \frac{2^{\frac{\nu}{2}}\, w^{\frac{\nu}{2}}\, x^{\nu}}{(x^2+\nu-2)^{\frac{\nu}{2}}}\, e^{-w}\,\frac{x}{2^{\frac{1}{2}}\, w^{\frac{1}{2}}\,(x^2+\nu-2)^{\frac{1}{2}}}\, dw \\
&= \frac{2^{\frac{1-\nu+\nu-1}{2}}\; x^{-\nu-1+1+\nu}}{\sqrt{\pi}\,\Gamma\left(\frac{\nu}{2}\right)\;(x^2+\nu-2)^{\frac{\nu+1}{2}}}\int_0^\infty w^{\frac{\nu-1}{2}}\, e^{-w}\, dw \\
&= \frac{(\nu-2)^{\frac{\nu}{2}}}{\sqrt{\pi}\,\Gamma\left(\frac{\nu}{2}\right)}\,(x^2+\nu-2)^{-\left(\frac{\nu+1}{2}\right)}\int_0^\infty w^{\frac{\nu-1}{2}}\, e^{-w}\, dw
\end{aligned}$$

To compute the integral, we let $\frac{\nu-1}{2} = \left(\frac{\nu+1}{2}\right) - 1$ and we compare it to:

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-\frac{x}{\beta}}\, dx$$

Thus, we can conclude that $\int_0^\infty w^{\left(\frac{\nu+1}{2}\right)-1}\, e^{-w}\, dw = \Gamma\left(\frac{\nu+1}{2}\right)$.

$$\therefore r(x) = \frac{(\nu-2)^{\frac{\nu}{2}}\, \Gamma\left(\frac{\nu+1}{2}\right)\, (x^2+\nu-2)^{-\left(\frac{\nu+1}{2}\right)}}{\sqrt{\pi}\, \Gamma\left(\frac{\nu}{2}\right)}$$

Now, a random variable P which is t-distributed with ν degrees of freedom, location μ and scale σ i.e. $Y \sim t_\nu(\mu, \sigma^2)$ has the following PDF:

$$t(p) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{\nu\pi}\,\sigma}\left(1 + \frac{1}{\nu}\left[\frac{y-\mu}{\sigma}\right]^2\right)^{-\frac{\nu+1}{2}}$$

Thus, we arrange $r(x)$ to follow the shape of the generalized Student t-distribution:

$$r(x) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{\nu\pi}\,\sqrt{\frac{\nu-2}{\nu}}}\left(1 + \frac{1}{\nu}\left[\frac{x}{\sqrt{\frac{\nu-2}{2}}}\right]^2\right)^{-\frac{\nu+1}{2}}$$

We can conclude that X follows a Student t-distribution with ν degrees of freedom, location 0, and scale $\sqrt{\frac{\nu-2}{2}}$ i.e. $X \sim t_\nu\left(0, \sqrt{\frac{\nu-2}{2}}\right)$

Thus, $E[X] = \mu = 0$ and $Var[X] = \frac{\nu}{\nu-2}\sigma^2 = \frac{\nu}{\nu-2}\left(\frac{\nu-2}{\nu}\right) = 1$

□