DSPotential Theory Document

David Moeller Sztajnbok

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1 Brief Vector Calculus Review

Before diving into the differential equations that govern fluid flow, particularly the flow we are interested in - irrotational, incompressible flow - a brief review of vector calculus is useful for the math that is to follow. This document will be void of complicated derivations, but will still include basic vector calculus concepts and notation. Below is a brief review of these concepts.

1.1 Vector Notation

This document will use standardized notation throughout, some of which is presented below.

Vectors are denoted by an arrow above them and are also bolded. A velocity vector, for instance, is denoted $\vec{\boldsymbol{u}}$ while a scalar quantity, density, is denoted ρ.

Vectors are presented in matrix form to more easily visualize operations like dot products, cross products, and vector multiplication. They are presented as an $nx1$ matrix, that is, a column vector. The example of velocity vector is shown below:

$$\vec{\boldsymbol{u}} = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

Where u, v, and w are the x, y, and z components of vector $\vec{\boldsymbol{u}}$.

1.2 Vector Operations

It is important to remember the different vector operations. Addition and subtraction are ommited from the discussion below. The focus is given to the scalar, vectorial, and simple multiplication of vectors, also known as the dot, cross, and simple products.

1.2.1 The Dot Product

Consider two vectors $\vec{\boldsymbol{a}}$ and $\vec{\boldsymbol{b}}$ such that:

$$\vec{\boldsymbol{a}} = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}$$

And

$$\vec{\boldsymbol{b}} = \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix}$$

Then the dot product between them is:

$$\vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{b}} = a_x b_x + a_y b_y + a_z b_z$$

With the geometric meaning that the dot product is the magnitude of the projection of $\vec{\boldsymbol{a}}$ onto $\vec{\boldsymbol{b}}$, also given by:

$$\vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{b}} = \bar{\vec{\boldsymbol{a}}}\bar{\vec{\boldsymbol{b}}} \cos\theta$$

$$\vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{b}} = \left|\vec{\boldsymbol{a}}\right| \left|\vec{\boldsymbol{b}}\right| \cos\theta$$

Where θ is the angle formed between $\vec{\boldsymbol{a}}$ and $\vec{\boldsymbol{b}}$.

Note: the dot product is an operation between two vectors and it results in a scalar. It is not to be confused with the multiplication of a scalar by a vector:

$$c\vec{\boldsymbol{a}} = \begin{bmatrix} ca_x \\ ca_y \\ ca_z \end{bmatrix}$$

Where c is a constant. The operation above is the product between a scalar and a vector, and results in a vector! So be weary of the dots in the text below.

1.2.2 The Cross Product

Consider the same vectors as before. The cross product between them is:

$$\vec{\boldsymbol{a}} \times \vec{\boldsymbol{b}} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = \hat{i} \begin{vmatrix} a_y & a_z \\ b_y & b_z \end{vmatrix} - \hat{j} \begin{vmatrix} a_x & a_z \\ b_x & b_z \end{vmatrix} + \hat{k} \begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix} = \hat{i}(a_y b_z - a_z b_y) + \hat{j}(a_z b_x - a_x b_z) + \hat{k}(a_x b_y - a_y b_x)$$

With the geometric meaning that the cross product is a vector normal to both $\vec{\boldsymbol{a}}$ and $\vec{\boldsymbol{b}}$ and its magnitude is the area formed by the parallellepiped between $\vec{\boldsymbol{a}}$ and $\vec{\boldsymbol{b}}$. This magnitude is:

$$\vec{\boldsymbol{a}} \times \vec{\boldsymbol{b}} = \left|\vec{\boldsymbol{a}}\right| \left|\vec{\boldsymbol{b}}\right| \sin\theta$$

Where θ is the angle between the vectors. Note that the cross product is an operation between two vectors that yields another vector.

1.3 Scalar vs. Vector Fields

It is important to make a subtle yet important distinction before speaking of derivatives. In fluid flow, we are interested in calculating different properties at different points in space and time. Therefore, quantities will be functions of space ($x, y,$ and z) and time (t.) *However, these quantities can be scalar and/or vectors.*

A vector field is a function of space and time wherein each point in space and time defines a vector. The best example and perhaps the most relevant here is the velocity vector, $\vec{\boldsymbol{u}}$:

$$\vec{\boldsymbol{u}} = u\hat{i} + v\hat{j} + w\hat{k}$$

Wherein each component of the velocity is a function of space and time:

$$\begin{aligned} u &= u(x, y, z, t) \\ v &= v(x, y, z, t) \\ w &= w(x, y, z, t) \end{aligned}$$

Therefore at each point in space at a given time, each component of the velocity vector at that point in space and time is defined, and a vector is therefore defined at that point.

Scalar fields, on the other hand, are scalars defined as functions of space and time. Thus, instead of a point in space and time defining three components that form a vector, only a scalar is defined. The best examples are pressure and temperature:

$$\begin{aligned} P &= P(x, y, z, t) \\ T &= T(x, y, z, t) \end{aligned}$$

1.4 The Gradient of a Scalar Field

Consider a scalar field $p(x, y, z)$. It defines a pressure (scalar) at every point in space. We are often interested in the direction at which the pressure increases the most, and the rate at which it increases. This is given by the gradient of the scalar field, ∇p.

The gradient of a scalar field ∇p is defined as the vector with the following properties:

1. Its direction is towards the greatest increase in the scalar field and;
2. Its magnitude is the rate of increase of the scalar field in that direction.

The gradient vector, ∇ is defined as:

$$\nabla = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix}$$

Such that the gradient of the scalar field is simply:

$$\nabla p = \begin{bmatrix} \frac{\partial p}{\partial x} \\ \frac{\partial p}{\partial y} \\ \frac{\partial p}{\partial z} \end{bmatrix}$$

1.5 Curl and Divergence

Another important set of concepts important in vector calculus that involve the gradient operator, ∇, are the curl and divergence of a vector field. The curl of a vector field $\vec{\boldsymbol{A}}$ is given by:

$$\text{curl}(\vec{\boldsymbol{A}}) = \nabla \times \vec{\boldsymbol{A}}$$

While the divergence of a vector field $\vec{\boldsymbol{A}}$ is given by:

$$\text{div}(\vec{\boldsymbol{A}}) = \nabla \cdot \vec{\boldsymbol{A}}$$

Note that both the curl and divergence operations are performed on vectors. However, the curl produces a vector while the divergence produces a scalar.

This will show itself useful when we define the concepts of rotationality, as well as the fundamental condition for the conservation of mass.

1.6 Total Derivative

In single-variable calculus, functions are dependent on one variable only. Therefore, some function $f(x)$ has as its derivative $\frac{df}{dx}$ and will be a function of x and x only (or some constant.)

In multivariable calculus, however, functions are dependent on more than one variable. It is typically the case in fluid flow that quantities are functions of space (x, y, and z) as well as time (t) which can be typically written as $f(x, y, z, t)$. However, the one of the variables in the function can be itself a function of another variable. It is imperative in that case to make use of the chain rule when differentiating the function, just as was the case in single-variable calculus when differentiating compound functions.

Let us consider a practical example. The acceleration of a fluid particle is given by:

$$\vec{\boldsymbol{a}} = \frac{d}{dt}[\vec{\boldsymbol{u}}(\vec{\boldsymbol{r}}, t)]$$

Where $\vec{\boldsymbol{r}}$ is the position vector given by:

$$\vec{\boldsymbol{r}} = \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix}$$

And the components of the velocity vector $\vec{\boldsymbol{u}}$ are given by:

$$\vec{\boldsymbol{u}} = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

The total derivative $\frac{d}{dt}[\vec{\boldsymbol{u}}(\vec{\boldsymbol{r}}, t)]$ must account for changes in position as functions of time, as well as changes in the velocity components as functions of position and time. The chain rule must be applied The total derivative is from now on denoted $\frac{D}{Dt}$ and is given by:

$$\frac{D\vec{\boldsymbol{u}}}{Dt} = \frac{\partial \vec{\boldsymbol{u}}}{\partial t} + \frac{\partial \vec{\boldsymbol{u}}}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial \vec{\boldsymbol{u}}}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial \vec{\boldsymbol{u}}}{\partial z}\frac{\partial z}{\partial t}$$

Note that in the equation above, r_x, r_y and r_z are simply x, y and z since we will have the velocity not as a function of the particle's position (Lagrangian approach to fluid flow analysis) but rather as a function of the space coordinates themselves (Eulerian approach to fluid flow analysis.)

The expression can be simplified. Note that $\frac{\partial x}{\partial t}$, $\frac{\partial y}{\partial t}$, and $\frac{\partial z}{\partial t}$ are, respectively, the x, y, and z components of the velocity vector, i.e., u, v, and w. Additionally, recall the gradient operator ∇:

$$\nabla = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix}$$

Such that:

$$\vec{\boldsymbol{u}} \cdot \nabla = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$$

And the above times the velocity vector itself is:

$$(\vec{\boldsymbol{u}} \cdot \nabla)\vec{\boldsymbol{u}} = u\frac{\partial \vec{\boldsymbol{u}}}{\partial x} + v\frac{\partial \vec{\boldsymbol{u}}}{\partial y} + w\frac{\partial \vec{\boldsymbol{u}}}{\partial z}$$

Note, therefore, that the total derivative can be simplified to:

$$\frac{D\vec{\boldsymbol{u}}}{Dt} = \frac{\partial \vec{\boldsymbol{u}}}{\partial t} + (\vec{\boldsymbol{u}} \cdot \nabla)\vec{\boldsymbol{u}}$$

In fact, the total derivative of any flow field quantity ϕ can be written as:

$$\frac{D\phi}{Dt} = \frac{\partial \phi}{\partial t} + (\vec{\boldsymbol{u}} \cdot \nabla)\phi$$

And it applies to any variable, be it scalar-valued or vector-valued.

1.7 Theorems of Vector Calculus

Several theorems are studied in vector calculus, but the ones that relate the different types of integrals and their bounds are of particular interest to us. These are of great aid in the derivation of the differential equations that govern fluid flow.

The reader is encouraged to review the fundamental background behind these theorems - namely the definition and physical meaning of line, surface, and volume integrals - since they are beyond the scope of this document. However, the theorems themselves are presented below.

1.7.1 Stokes' Theorem

Consider an open surface S bounded by a closed curve C and a vector field $\vec{\boldsymbol{A}}$. Stokes' theorem relates the line integral of $\vec{\boldsymbol{A}}$ over the closed curve C to the surface integral of the curl of $\vec{\boldsymbol{A}}$ over the surface S. Mathematically we write:

$$\oint_C \vec{\boldsymbol{A}} \cdot \vec{\boldsymbol{dr}} = \iint_S (\nabla \times \vec{\boldsymbol{A}}) \cdot \vec{\boldsymbol{dS}}$$

Which is a powerful tool, since it relates line integrals to surface integrals.

1.7.2 Divergence Theorem

Consider instead a volume ν bounded by a closed surface S and a vector field $\vec{\boldsymbol{A}}$. The Divergence Theorem relates the surface integral of $\vec{\boldsymbol{A}}$ over the bounding surface to the volume integral of the divergence of $\vec{\boldsymbol{A}}$ over the volume ν. Mathematically we write:

$$\iint_S \vec{\boldsymbol{A}} \cdot \vec{\boldsymbol{dS}} = \iiint_\nu (\nabla \cdot \vec{\boldsymbol{A}}) d\nu$$

Which is a powerful tool, since it relates surface integrals to volume integrals.

1.7.3 Gradient Theorem

An analogous theorem exists for the gradient of a scalar field. Consider again a volume ν bounded by a closed surface S and instead a scalar field $p(x, y, z)$. The Gradient Theorem relates the surface integral of p over S to the volume integral of the gradient of p over ν. Mathematically we write:

$$\iint_S p\vec{\boldsymbol{dS}} = \iiint_\nu \nabla p \, d\nu$$

2 Differential Form of Conservation Laws

We are now ready to introduce the conservation laws that govern fluid flow. Although I will not go into the full derivation for the laws below, a brief introduction into how they come to be is presented below. The analysis below is conducted in the frame of an arbitrary control volume. A control volume is a closed region in space in which we analyze the change in given quantities - in our case, we are particularly interested in mass, momentum, and energy. Therefore, in the text below, when the phrase "the change of quantity B" is said, it really means "the change of quantity B within the control volume chosen for our analysis." This is for the sake of physical rigidity.

Consider a differential ammount (given a small control volume) of quantity B, that is, dB. The quantity per unit mass, $\frac{dB}{dm}$, is said to be an *intensive* quantity, while dB is said to be an *extensive* quantity. For the laws below, we are interested in expressing the conservation of B within a control volume. It can be stated that the change of B comes from two sources: changes of B *within* the control volume and changes of B *through* the control volume. An example is the amount of elephants in a zoo: the total change in number of elephants in the zoo is within the zoo, that is, through the birth and death of elephants, as well as through the zoo, that is, elephants leaving and entering the zoo.

This general statement of conservation is known as *Reynold's Transport Theorem.* The change within the control volume is given by the time rate of change of the extensive quantity B within the control volume, while the change through the control volume is given by the flux of the intensive quantity $\frac{dB}{dm}$ through the control volume. The theorem is given by:

$$\frac{dB}{dt} = \frac{d}{dt}\iiint_{\nu} \rho\beta d\nu + \iint_{S} \rho\beta(\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{n}})dS$$

Where the first term is the time rate of change of B within the control volume and the second term is the flux of B through the control surface (the surface within which the volume is bounded.)

This can be written for various quantities B. Most notably:

1. MASS: $B = m \implies \beta = 1$

2. MOMENTUM: $B = m\vec{\boldsymbol{u}} \implies \beta = \vec{\boldsymbol{u}}$

3. ENERGY: $B = E \implies \beta = e = E/m$

The idea is thus to write out Reynold's Transport Theorem for each quantity and, using the theorems of vector calculus (Section 1.7), to write the entire expression in terms of volume integrals. Using a small enough control volume ensures that the integrand is equal to the result of the conservation law. For example, if:

$$\iiint_{\nu} (\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho\vec{\boldsymbol{u}}))d\nu = 0$$

This ensures that:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{\boldsymbol{u}}) = 0$$

For an arbitrarily small control volume.

This is enough for an introduction, and the reader is encouraged to seek literature for the full derivations of the conservation laws. They are simply presented below, and will be used for the derivations in potential flow.

2.1 Conservation of Mass