Magnetic Damping

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April 14, 2021

Derivation

As a function of time, the magnetic field in the x-direction $H_x(t)$ may be written as

$$H_x(t) = \int_{0^-}^{t} \left[\beta_{xx}^T + \left(\frac{1}{\mu_0} - \beta_{xx}^T\right) e^{-\mathrm{a}(t-\tau)}\right] \frac{\partial B_x}{\partial \tau} d\tau + \int_{0^-}^{t} \beta_{xy}^T \left(1 - e^{-\mathrm{a}(t-\tau)}\right) \frac{\partial B_y}{\partial \tau} d\tau \tag{1}$$

where the constants are given in the following table:

M_s	$4.85E5$	A/m
$\|B\|$	$\mu_0 M_s/50$	N/A/m
γ	$1.759/(2\pi)E11$	rad · A · m/N/s
β_{xx}^T	$1/2.1152/\mu_0$	A^2/N
α	0.045	Dimensionless
ω	$2\pi \times 400E6$	rad/s
μ_0	$4\pi E-7$	$\mathrm{N/A}^2$
a	$\frac{\gamma\mu_0 M_s}{\alpha}$	1/s

The analytical solution to this convolution integral is

$$H_x(t) = |B|\beta_{xx}^T \sin(\omega t) + |B| \left(\frac{1}{\mu_0} - \beta_{xx}^T\right) \frac{\omega/\mathrm{a}}{(1 + (\omega/\mathrm{a})^2)} \left\{\cos(\omega t) - e^{-\mathrm{at}} + \frac{\omega}{\mathrm{a}} \sin(\omega t)\right\} \tag{2}$$

We first observe that $B_y = 0$, which cancels out the 2nd integral from equation (1), resulting in

$$H_x(t) = \int_{0^-}^{t} \left[\beta_{xx}^T + \left(\frac{1}{\mu_0} - \beta_{xx}^T\right) e^{-\mathrm{a}(t-\tau)}\right] \frac{\partial B_x}{\partial \tau} d\tau \tag{3}$$

We also note that $B_x(\tau) = |B| \sin(\omega\tau)$, so $\dfrac{\partial B_x}{\partial \tau} = |B|\omega \cos(\omega\tau)$. So,

$$H_x(t) = |B|\omega \int_{0^-}^{t} \left[\beta_{xx}^T + \left(\frac{1}{\mu_0} - \beta_{xx}^T\right) e^{-\mathrm{a}(t-\tau)}\right] \cos(\omega\tau) d\tau$$

To prove that equation (2) is true, we appeal to integration by parts which states

$$\int u dv = uv - \int v du$$

We use

$$\int_{0^-}^{t} e^{-a(t-\tau)} \cos(\omega\tau) d\tau$$

$$\begin{array}{ll} u = \cos(\omega\tau) & v = \frac{1}{a}e^{-\mathrm{a}(t-\tau)} \\ du = -\omega\sin(\omega\tau)d\tau & dv = e^{-\mathrm{a}(t-\tau)}d\tau \end{array}$$

Performing integration by parts once again,

$$\begin{aligned} \int_{0^-}^{t} e^{-a(t-\tau)}\cos(\omega\tau)d\tau &= \left[\cos(\omega\tau)\left(\frac{1}{a}e^{-\mathrm{a}(t-\tau)}\right)\right]_{0^-}^{t} + \int_{0^-}^{t}\frac{1}{a}\omega\sin(\omega\tau)e^{-\mathrm{a}(t-\tau)}d\tau \\ &= \frac{1}{a}\left[\cos(\omega t) - e^{-at}\right] + \frac{\omega}{a}\left\{\left[\frac{1}{a}\sin(\omega\tau)e^{-a(t-\tau)}\right]_{0^-}^{t}\right. \\ &\left. - \frac{\omega}{a}\int_{0^-}^{t} e^{-a(t-\tau)}\cos(\omega\tau)d\tau\right\} \end{aligned}$$

Solving for $\int_{0^-}^{t} e^{-a(t-\tau)}\cos(\omega\tau)d\tau$, we obtain

$$\int_{0^-}^{t} e^{-a(t-\tau)}\cos(\omega\tau)d\tau = \left(\frac{1}{1+(\omega/a)^2}\right)\left\{\frac{1}{a}\left[\cos(\omega t) - e^{-at}\right] + \frac{\omega}{a^2}\sin(\omega t)\right\}$$

Plugging this result back into equation (3) and then subsequently factoring out $\frac{1}{a}$ yields equation (2). It is also important to not neglect the presence of an ω term that is pulled out after the evaluation of $\dfrac{\partial B_x}{\partial \tau}$.

$$H_x(t) = |B|\beta_{xx}^{T}\sin(\omega t) + |B|\left(\frac{1}{\mu_0} - \beta_{xx}^{T}\right)\frac{\omega/\mathrm{a}}{(1+(\omega/\mathrm{a})^2)}\left\{\cos(\omega t) - e^{-\mathrm{a}t} + \frac{\omega}{\mathrm{a}}\sin(\omega t)\right\}$$