Relationship between Schneider's and Jarvis's Shear-Shear Correlation Function

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From Jarvis (Page 3):

$$\xi_+ = \left\langle \gamma_i \gamma_j^* \right\rangle = \texttt{xip} + i(\texttt{xip\_im}) \tag{1}$$
$$\xi_- = \left\langle \gamma_i \gamma_j e^{-4i\alpha} \right\rangle = \texttt{xim} + i(\texttt{xim\_im}) \tag{2}$$

where α is the angle between the two objects i, j and each γ is given by $\gamma_n = |\gamma_n| e^{2i\theta_n}$ in polar form.

From Schneider (Page 92):

$$\xi_\pm = \left\langle \gamma_{i_t} \gamma_{j_t} \right\rangle \pm \left\langle \gamma_{i_\times} \gamma_{j_\times} \right\rangle \tag{3}$$
$$\xi_\times = \left\langle \gamma_{i_t} \gamma_{j_\times} \right\rangle \tag{4}$$

where $\gamma_{n_t} = -\operatorname{Re}\left(\gamma_n e^{-2i\alpha}\right)$ and $\gamma_{n_\times} = -\operatorname{Im}\left(\gamma_n e^{-2i\alpha}\right)$.

Schneider's ξ_+ to Jarvis's `xip`

Starting with Schneider's definition, observe that

$$\begin{aligned}
\xi_+ &= \left\langle \gamma_{i_t} \gamma_{j_t} \right\rangle + \left\langle \gamma_{i_\times} \gamma_{j_\times} \right\rangle \\
&= \left\langle \operatorname{Re}\left(\gamma_i e^{-2i\alpha}\right) \cdot \operatorname{Re}\left(\gamma_j e^{-2i\alpha}\right) \right\rangle + \left\langle \operatorname{Im}\left(\gamma_i e^{-2i\alpha}\right) \cdot \operatorname{Im}\left(\gamma_j e^{-2i\alpha}\right) \right\rangle \\
&= \left\langle \operatorname{Re}\left(|\gamma_i| e^{2i(\theta_i - \alpha)}\right) \cdot \operatorname{Re}\left(|\gamma_j| e^{2i(\theta_j - \alpha)}\right) \right\rangle + \left\langle \operatorname{Im}\left(|\gamma_i| e^{2i(\theta_i - \alpha)}\right) \cdot \operatorname{Im}\left(|\gamma_j| e^{2i(\theta_j - \alpha)}\right) \right\rangle \\
&= \left\langle |\gamma_i| \cdot |\gamma_j| \cos\left(2(\theta_i - \alpha)\right) \cos\left(2(\theta_j - \alpha)\right) \right\rangle + \left\langle |\gamma_i| \cdot |\gamma_j| \sin\left(2(\theta_i - \alpha)\right) \sin\left(2(\theta_j - \alpha)\right) \right\rangle .
\end{aligned}$$

Using the trig identities

$$\cos(u)\cos(v) = \frac{1}{2}\left[\cos(u - v) + \cos(u + v)\right], \tag{5}$$
$$\sin(u)\sin(v) = \frac{1}{2}\left[\cos(u - v) - \cos(u + v)\right], \tag{6}$$

and the linearity of the expectation operator, it is straightforward to show that

$$\xi_+ = \left\langle |\gamma_i| \cdot |\gamma_j| \cos\left(2(\theta_i - \theta_j)\right) \right\rangle .$$

Now using the identity

$$\cos(u \pm v) = \cos(u)\cos(v) \mp \sin(u)\sin(v), \tag{7}$$

the previous equation can be written as

$$\begin{aligned}
\xi_+ &= \left\langle |\gamma_i| \cdot |\gamma_j| \cdot \left[\cos(2\theta_i)\cos(2\theta_j) + \sin(2\theta_i)\sin(2\theta_j)\right] \right\rangle \\
&= \left\langle \operatorname{Re}(\gamma_i) \cdot \operatorname{Re}(\gamma_j) + \operatorname{Im}(\gamma_i) \cdot \operatorname{Im}(\gamma_j) \right\rangle .
\end{aligned}$$

Using the complex number identities

$$\operatorname{Re}(z) = \frac{1}{2}\left[z + z^*\right], \tag{8}$$
$$\operatorname{Im}(z) = \frac{1}{2i}\left[z - z^*\right], \tag{9}$$

it follows that

$$\begin{aligned}
\xi_+ &= \left\langle \frac{1}{4}\left[(\gamma_i+\gamma_i^*)(\gamma_j+\gamma_j^*)-(\gamma_i-\gamma_i^*)(\gamma_j-\gamma_j^*)\right]\right\rangle \\
&= \left\langle \frac{1}{2}\left[\gamma_i\gamma_j^* + (\gamma_i\gamma_j^*)^*\right]\right\rangle \\
&= \left\langle \mathrm{Re}(\gamma_i\gamma_j^*)\right\rangle = \texttt{xip}.
\end{aligned}$$

Schneider's ξ_- to Jarvis's `xim`

The first few steps are identical to the previous section besides the negative in the definition of ξ_-, giving

$$\begin{aligned}
\xi_- &= \langle \gamma_{i_t}\gamma_{j_t}\rangle - \langle \gamma_{i_\times}\gamma_{j_\times}\rangle \\
&= \langle |\gamma_i|\cdot|\gamma_j|\cos\left(2(\theta_i-\alpha)\right)\cos\left(2(\theta_j-\alpha)\right)\rangle - \langle |\gamma_i|\cdot|\gamma_j|\sin\left(2(\theta_i-\alpha)\right)\sin\left(2(\theta_j-\alpha)\right)\rangle \\
&= \langle |\gamma_i|\cdot|\gamma_j|\cos\left(2(\theta_i+\theta_j-2\alpha)\right)\rangle .
\end{aligned}$$

Now using Equation (7) twice, first letting $u = 2(\theta_i+\theta_j)$ and $v=-4\alpha$, and then letting $u=2\theta_i$ and $v=2\theta_j$, this equation becomes

$$\begin{aligned}
\xi_- &= \Big\langle |\gamma_i|\cdot|\gamma_j|\Big([\cos(2\theta_i)\cos(2\theta_j)-\sin(2\theta_i)\sin(2\theta_j)]\cos(4\alpha) + [\sin(2\theta_i)\cos(2\theta_j)+\cos(2\theta_i)\sin(2\theta_j)]\sin(4\alpha)\Big)\Big\rangle \\
&= \Big\langle \big[\mathrm{Re}(\gamma_i)\cdot\mathrm{Re}(\gamma_j)-\mathrm{Im}(\gamma_i)\cdot\mathrm{Im}(\gamma_j)\big]\cos(4\alpha) + \big[\mathrm{Im}(\gamma_i)\cdot\mathrm{Re}(\gamma_j)+\mathrm{Re}(\gamma_i)\cdot\mathrm{Im}(\gamma_j)\big]\sin(4\alpha)\Big\rangle.
\end{aligned}$$

Using the identities (8) and (9), and simplifying the leftover terms, this equation can be shown to equal

$$\xi_- = \big\langle\, \mathrm{Re}(\gamma_i\gamma_j)\cos(4\alpha) + \mathrm{Im}(\gamma_i\gamma_j)\sin(4\alpha)\big\rangle.$$

Now observe that for two complex numbers a and b, it is true that

$$\mathrm{Re}(a\cdot b^*) = \mathrm{Re}(a)\cdot\mathrm{Re}(b) + \mathrm{Im}(a)\cdot\mathrm{Im}(b).$$

Then setting $a=\gamma_i\gamma_j$ and $b=e^{4i\alpha}$, it must be true that

$$\mathrm{Re}\left(\gamma_i\gamma_j e^{-4i\alpha}\right) = \mathrm{Re}(\gamma_i\gamma_j)\cos(4\alpha) + \mathrm{Im}(\gamma_i\gamma_j)\sin(4\alpha).$$

Combining this with our previous result, this means that

$$\xi_- = \big\langle\, \mathrm{Re}(\gamma_i\gamma_j)\cos(4\alpha) + \mathrm{Im}(\gamma_i\gamma_j)\sin(4\alpha)\big\rangle = \big\langle \mathrm{Re}\left(\gamma_i\gamma_j e^{-4i\alpha}\right)\big\rangle = \texttt{xim}.$$

Schneider's $\xi_\times$ to Jarvis's $\frac{1}{2}\left(\texttt{xim\_im} - \texttt{xip\_im}\right)$

Starting from Schneider's definition of $\xi_\times$,

$$\begin{aligned}
\xi_\times &= \langle \gamma_{i_t}\gamma_{j_\times}\rangle \\
&= \left\langle \mathrm{Re}\left(|\gamma_i|e^{2i(\theta_i-\alpha)}\right)\cdot\mathrm{Im}\left(|\gamma_j|e^{2i(\theta_j-\alpha)}\right)\right\rangle \\
&= \langle |\gamma_i|\cdot|\gamma_j|\cos\left(2(\theta_i-\alpha)\right)\sin\left(2(\theta_j-\alpha)\right)\rangle.
\end{aligned}$$

Using the trig identity

$$\cos(u)\sin(v) = \frac{1}{2}\left[\sin(u+v)-\sin(u-v)\right], \tag{10}$$

the previous equation becomes

$$\xi_\times = \left\langle |\gamma_i| \cdot |\gamma_j| \cdot \frac{1}{2} \Big[\sin\left(2(\theta_i + \theta_j - 2\alpha)\right) - \sin\left(2(\theta_i - \theta_j)\right) \Big] \right\rangle .$$

Next, applying the identities

$$\sin(u \pm v) = \sin(u)\cos(v) \pm \cos(u)\sin(v) \tag{11}$$

along with (7) iteratively until each trig function only has a single term, the equation becomes

$$\begin{aligned}
\xi_- = \Big\langle |\gamma_i| \cdot |\gamma_j| \cdot \frac{1}{2} \Big(& [\sin(2\theta_i)\cos(2\theta_j) + \cos(2\theta_i)\sin(2\theta_j)]\cos(4\alpha) \\
& - [\cos(2\theta_i)\cos(2\theta_j) - \sin(2\theta_i)\sin(2\theta_j)]\sin(4\alpha) \\
& - \sin(2\theta_i)\cos(2\theta_j) + \cos(2\theta_i)\sin(2\theta_j) \Big) \Big\rangle \\
= \Big\langle \frac{1}{2} \Big(& [\mathrm{Im}(\gamma_i) \cdot \mathrm{Re}(\gamma_j) + \mathrm{Re}(\gamma_i) \cdot \mathrm{Im}(\gamma_j)]\cos(4\alpha) \\
& - [\mathrm{Re}(\gamma_i) \cdot \mathrm{Re}(\gamma_j) - \mathrm{Im}(\gamma_i) \cdot \mathrm{Im}(\gamma_j)]\sin(4\alpha) \\
& - \mathrm{Im}(\gamma_i)\,\mathrm{Re}(\gamma_j) + \mathrm{Re}(\gamma_i) \cdot \mathrm{Im}(\gamma_j) \Big) \Big\rangle .
\end{aligned}$$

Using the identities (8) and (9), this can be simplified to

$$\xi_- = \left\langle \frac{1}{2} \Big[\mathrm{Im}(\gamma_i\gamma_j)\cos(4\alpha) - \mathrm{Re}(\gamma_i\gamma_j)\sin(4\alpha) - \mathrm{Im}(\gamma_i\gamma_j^*) \Big] \right\rangle .$$

Consider again two complex numbers a and b. Observe that

$$\mathrm{Im}(a \cdot b^*) = \mathrm{Im}(a) \cdot \mathrm{Re}(b) - \mathrm{Re}(a) \cdot \mathrm{Im}(b).$$

Then setting $a = \gamma_i\gamma_j$ and $b = e^{4i\alpha}$, it must be true that

$$\mathrm{Im}\left(\gamma_i\gamma_j e^{-4i\alpha}\right) = \mathrm{Im}(\gamma_i\gamma_j)\cos(4\alpha) - \mathrm{Re}(\gamma_i\gamma_j)\sin(4\alpha).$$

Combing these results give

$$\begin{aligned}
\xi_- &= \left\langle \frac{1}{2} \Big[\mathrm{Im}(\gamma_i\gamma_j)\cos(4\alpha) - \mathrm{Re}(\gamma_i\gamma_j)\sin(4\alpha) - \mathrm{Im}(\gamma_i\gamma_j^*) \Big] \right\rangle \\
&= \left\langle \frac{1}{2} \Big[\mathrm{Im}\left(\gamma_i\gamma_j e^{-4i\alpha}\right) - \mathrm{Re}(\gamma_i\gamma_j^*) \Big] \right\rangle \\
&= \left\langle \frac{1}{2} \Big[\texttt{xim_im} - \texttt{xip_im} \Big] \right\rangle .
\end{aligned}$$