Contents

Topological Spaces

Open Sets and the Definition of a Topological Space

Definition: Suppose X is a set. Then $\mathcal{T}$ is a **topology** on X if and only if $\mathcal{T}$ is a collection of subsets of X such that 1. $\emptyset \in \mathcal{T}$ 2. $X \in \mathcal{T}$ 3. If $U \in \mathcal{T}$ and $V \in \mathcal{T}$, then $U \cap V \in \mathcal{T}$. 4. The union of any collection of sets of $\mathcal{T}$ is contained in $\mathcal{T}$.

Theorem 2.1: Let $\mathcal{U}$ be a collection of open sets in a topology T. Then $\bigcap \mathcal{U}$ is open.

Proof:

> By induction. If $\mathcal{U}$ contains only one open set $\mathcal{U}$ is open. Suppose the intersection of n open sets is open and suppose $\mathcal{U}$ contains $n+1$ open sets. Then $\bigcap_{i=1}^{n+1} U_i$ is open since $\bigcap_{i=1}^{n+1} U_i = \bigcap_{i=1}^{n} U_i \cap U_{n+1}$ is open, by the inductive hypothesis and Axiom 3. Thus, by induction, the intersection of any finite number of open sets is open.

Remark: The above prove does not prove that the infinite intersection of open sets is necessarily open. This is because the proof is by induction and induction only works for finite sets. For example, consider the open sets $U_n = (-1/n, 1/n)$ for $n \in \mathbb{N}$ in the standard topology of $\mathbb{R}$. Then $\bigcap_{n=1}^{\infty} U_n = \{0\}$ which is not open.

Theorem 2.3: A set U is open in a toplogical space $(X, \mathcal{T})$ if and only if for every $x \in U$, there exists an open set U_x such that $x \in U_x \subseteq U$. We call U_x a **neighborhood** of x.

Proof:

> Suppose U is open in $(X, \mathcal{T})$ and let $x \in U$. Then x is contained in an open set contained in U, namely U itself. Conversely, suppose every $x \in U$ is contained in some open set within U. Then $U = \bigcup_{x \in U} U_x$ is open since the union of open sets is open.

Example: Consider the following open interval $(a, b) \subseteq \mathbb{R}$ with the standard topology. $x \in (a, b)$ is contained in the open interval $(x - \epsilon, x + \epsilon)$ for some $\epsilon > 0$, which itself is contained in (a, b).

Definition: A subset $U \subseteq \mathbb{R}^n$ belongs to the **standard topology** $\mathcal{T}_{std}$ on $\mathbb{R}^n$ if and only if every point $x \in U$ is contained in an open ball $B(x, \epsilon) \subseteq U$ for some $\epsilon > 0$. Note that the open ball $B(x, \epsilon)$ is defined as the set of all points

$y \in \mathbb{R}^n$ such that $\|x - y\| < \epsilon$, where $\| \cdot \|$ is the standard Euclidean norm (the square root of the sum of the squares of the components).

Proposition: The standard topology is a topology.

Proof:

(1) $\emptyset \in \mathcal{T}_{std}$ vacuously.

(2) $\mathbb{R}^n \in \mathcal{T}_{std}$ because every point $x \in \mathbb{R}^n$ is contained in the open ball $B(x, 1) \subseteq \mathbb{R}^n$.

(3) If $U_1, U_2 \in \mathcal{T}_{std}$, then for every $x \in U_1 \cap U_2$, there exist $\epsilon_1, \epsilon_2 > 0$ such that $B(x, \epsilon_1) \subseteq U_1$ and $B(x, \epsilon_2) \subseteq U_2$. Let $\epsilon = \min(\epsilon_1, \epsilon_2)$. Then $B(x, \epsilon) \subseteq B(x, \epsilon_1) \subseteq U_1$ and $B(x, \epsilon) \subseteq B(x, \epsilon_2) \subseteq U_2$, so $B(x, \epsilon) \subseteq U_1 \cap U_2$.

(4) If $\{U_\alpha\}_{\alpha \in A} \subseteq \mathcal{T}_{std}$, then for every $x \in \bigcup_{\alpha \in A} U_\alpha$, there exists $\alpha \in A$ such that $x \in U_\alpha$, so there exists $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U_\alpha \subseteq \bigcup_{\alpha \in A} U_\alpha$. Therefore, $\bigcup_{\alpha \in A} U_\alpha \in \mathcal{T}_{std}$.

Definition: The **discrete topology** on a set X is the power set $\mathcal{P}(X)$.

Definition: The **indiscrete topology** on a set X is $\{\emptyset, X\}$.

Definition: The **finite complement (cofinite) topology** on X is the set of all subsets $U \subseteq X$ such that either $U = \emptyset$ or $X \quad U$ is finite.

Definition: The **countable complement (cocountable) topology** on X is the set of all subsets $U \subseteq X$ such that either $U = \emptyset$ or $X \quad U$ is countable.

Limit Points and Closed Sets

Definition: Let $(X, \mathcal{T})$ be a topological space and $A \subseteq X$. A point $x \in X$ is a **limit point** of A if every open set $U \in \mathcal{T}$ containing x intersects A at some point other than x.

Example 2.8: Let $X = \mathbb{R}$ and $A = (1, 2)$. Then 0 is a limit point of A in the indiscrete topology and the finite complement topology, but not the standard topology nor discrete topology on $\mathbb{R}$.

> The only open sets in the indiscrete topology which contain 0 is $\mathbb{R}$ itself, which contains A. In the finite complement topology, every open set containing 0 contains all but finitely many points of A. In the standard topology, the open set $(-1, 1)$ contains 0 but does not intersect A. In the discrete topology, the open set $\{0\}$ contains 0 but does not intersect A.

Theorem 2.9: Suppose $p \notin A$ in a topological space $(X, \mathcal{T})$. Then p is not a limit point of A if and only if there exists a neighborhood U of p such that $U \cap A = \emptyset$.

Proof:

> ($\Rightarrow$) Suppose p is not a limit point of A. Then there exists an open set $U \in \mathcal{T}$ containing p such that $U \cap A = \{p\}$. Since $p \notin A$, $U \cap A = \emptyset$.
>
> ($\Leftarrow$) Suppose there exists a neighborhood U of p such that $U \cap A = \emptyset$. Then U is an open set containing p which does not intersect A, so p is not a limit point of A.

Definition: Let $(X, \mathcal{T})$ be a topological space. A point $p \in A \subseteq X$ is an **isolated point** of A if p is not a limit point of A.

Proposition 2.10: If p is an isolated point of A, then there exists an open set $U \in \mathcal{T}$ such that $U \cap A = \{p\}$.

Proof:

> Since p is an isolated point of A, there exists a neighborhood U of p such that $U \cap A = \emptyset$. Since $p \in A$, and $A \subseteq U$, no other elements are in $U \cap A$, that is, $U \cap A = \{p\}$.

Example 2.11: Consider the standard topology on $\mathbb{R}$. Then 0 is a limit point of $A := (-1, 1)$ contained in A, and -1 is a limit point of A not contained in A. 0 is an isolated point of A in the discrete topology on $\mathbb{R}$, and an isolated point of $B := \{0\}$ in the standard topology on $\mathbb{R}$. The point 2 is not a limit point of A.

Definition: Let $(X, \mathcal{T})$ be a topological space. The **closure** of A in X, denoted $\overline{A}$, is the set A together with all its limit points in X.

Definition: A subset $A \subseteq X$ is **closed** if and only if $A = \overline{A}$.