Consider $f : [0,1] \to [0,1]$. Define

$$f(x) = \begin{cases} 1, & x \neq 1 \\ 0, & x = 1 \end{cases}$$

Then $f(x) \neq x$ for all $x \in [0,1]$, but $\lim_{x \to 1} d(x, f(x)) = 0$. So there is no $\delta > 0$ such that $d(x, f(x)) \geq \delta$ for all $x \in X$?