TOPOLOGY MATH4121A/9021
LECTURE 2

Definition. Let $(X, \mathscr{T})$ be a topological space, A be a subset of X, and p be a point in X. Then p is a **limit point** of A if and only if for each open set U containing p, $(U - \{p\}) \cap A \neq \emptyset$. Notice that p may or may not belong to A.

Theorem 1. *Suppose $p \notin A$ in a topological space $(X, \mathscr{T})$. Then p is* not *a limit point of A if and only if there exists a neighborhood U of p such that $U \cap A = \emptyset$.*

Proof.

Definition. Let $(X, \mathscr{T})$ be a topological space, A be a subset of X, and p be a point in X. If $p \in A$ but p is not a limit point of A, then p is an **isolated point** of A.

Theorem 2. *If p is an isolated point of a set A in a topological space X, then there exists an open set U such that $U \cap A = \{p\}$.*

Proof.

Exercise 3. Give examples of sets A in topological spaces $(X, \mathscr{T})$ with

(1) a limit point of A that is an element of A;
(2) a limit point of A that is not an element of A;
(3) an isolated point of A;
(4) a point not in A that is not a limit point of A.

Solution.

Definition. Let $(X, \mathscr{T})$ be a topological space, and $A \subset X$. Then the **closure** of A in X, denoted $\bar{A}$ or $\mathrm{Cl}(A)$ or $\mathrm{Cl}_X(A)$, is the set A together with all its limit points in X.
Let $(X, \mathscr{T})$ be a topological space and $A \subset X$. The subset A is **closed** if and only if $\bar{A} = A$, in other words, if A contains all its limit points.

The following theorem tells us that closed sets and open sets are complements of one another.

Theorem 4. *Let $(X, \mathscr{T})$ be a topological space. Then the set A is closed if and only if $X - A$ is open.*

Solution.

Similarly, one can show that removing a closed set from an open set leaves an open set. A theorem which may be helpful in proving this theorem is DeMorgans Law:

Theorem. (DeMorgan's Laws) Let X be a set, and let $\{A_k\}_{k=1}^N$ be a finite collection of sets such that $A_k \subset X$ for each $k = 1, 2, \ldots, N$. Then

$$X - \left(\bigcup_{k=1}^{N} A_k\right) = \bigcap_{k=1}^{N}(X - A_k)$$

and

$$X - \left(\bigcap_{k=1}^{N} A_k\right) = \bigcup_{k=1}^{N}(X - A_k).$$

Theorem 5. *Let $(X, \mathscr{T})$ be a topological space, and let U be an open set and A be a closed subset of X. Then the set $U - A$ is open and the set $A - U$ is closed.*

Proof.

Theorem 6. *$y \in \bar{A}$ iff for each open set U containing x, $U \cap A \neq \emptyset$.*

Proof.

Theorem 7. *For any topological space* $(X, \mathscr{T})$ *and* $A \subset X$, $\bar{A}$ *is closed. That is, for any set* A *in a topological space,* $\overline{\overline{A}} = \overline{A}$*. Hint: The previous theorem may be useful in the "*$\subset$*" direction.*

Proof.

Theorem 8. *For any set A in a topological space X, the closure of A equals the intersection of all closed sets containing A, that is,*

$$\overline{A} = \bigcap_{B \supset A, B \in \mathscr{C}} B$$

where $\mathscr{C}$ is the collection of all closed sets in X.