TOPOLOGY MATH4121A/9021
LECTURE 1

Definition. Suppose X is a set. Then $\mathscr{T}$ is a **topology** on X if and only if $\mathscr{T}$ is a collection of subsets of X such that

(1) $\emptyset \in \mathscr{T}$,
(2) $X \in \mathscr{T}$,
(3) if $U \in \mathscr{T}$ and $V \in \mathscr{T}$, then $U \cap V \in \mathscr{T}$, and
(4) if $\{U_\alpha\}_{\alpha \in \lambda}$ is any collection of sets of $\mathscr{T}$, then $\bigcup_{\alpha \in \lambda} U_\alpha \in \mathscr{T}$.

Theorem 1. *Let $\{U_i\}_{i=1}^n$ be a* finite *collection of open sets in a topological space $(X, \mathscr{T})$. Then $\bigcap_{i=1}^n U_i$ is open.*

Proof.

Exercise 2. Why does your proof not prove the false statement that the infinite intersection of open sets is necessarily open?

Solution.

Theorem 3. *A set U is open in a topological space $(X, \mathscr{T})$ if and only if for every point $x \in U$, there exists an open set U_x such that $x \in U_x \subset U$.*

Proof.

Definition. An open set containing x is called a **neighbourhood** of x.

This means that a set is open if and only if every point has an open neighborhood that lies within U.

Definition. The standard topology on $\mathbb{R}$, denoted $\mathscr{T}_{std}$ is defined as follows: a subset U of $\mathbb{R}$ belongs to $\mathscr{T}_{std}$ if and only if for every point p of U there is some $\epsilon_p > 0$ such that the interval $(p - \epsilon_p, p + \epsilon_p)$ is contained in U. We write $\mathbb{R}_{std}$ for $(\mathbb{R}, \mathscr{T}_{std})$, but if we every see $\mathbb{R}$ without some toplogy mentioned, we should assume it has the standard topology.

Exercise 4. Illustrate an open set on $\mathbb{R}$. Generalize the standard topology to $\mathbb{R}^2$. Can you generalize the standard topology to $\mathbb{R}^n$?

Solution.

Exercise 5. Verify that $\mathscr{T}_{\text{std}}$ is a topology on $\mathbb{R}^n$; in other words, it satisfies the four conditions of the definition of a topology.

Solution.

Definition. Let $(X, \mathscr{T})$ be a topological space, A be a subset of X, and p be a point in X. Then p is a **limit point** of A if and only if for each open set U containing p, $(U - \{p\}) \cap A \neq \emptyset$. Notice that p may or may not belong to A.

Exercise 6. Illustrate the definition of a limit point. The *indiscrete topology* is the topology such that $\mathscr{T} = \{\emptyset, X\}$. Let $X = \mathbb{R}$ and $A = (1, 2)$. Verify that 0 is a limit point of A in the indiscrete topology but not in the standard topology on $\mathbb{R}$.

Solution.