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HW 6

Math 3124
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1. Using the fact that $\sum_{k=0}^{\infty}(-1)^n r^k = \frac{1}{1+r}$ when $|r| < 1$,

$$\begin{aligned}\frac{1}{z} &= \frac{1}{1+i+(z-1-i)} \\ &= \frac{1}{(1+i)(1+\frac{z-1-i}{1+i})} \\ &= \frac{1}{1+i}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z-(1+i)}{1+i}\right)^n\end{aligned}$$

when $|z| < 1+i$.

2. Factoring and applying partial fraction decomposition,

$$\frac{1}{1-z-2z^2} = \frac{1}{-(2z-1)(z+1)} = -\frac{A}{2z-1} - \frac{B}{z+1}$$

and solving for A, B in $-A(z+1) - B(2z-1) = 1$ for all z gives $A = -2/3$ and $B = 1/3$. So,

$$\begin{aligned}f(z) &= \frac{2}{3}\left(\frac{1}{1-2z}\right) + \frac{1}{3}\left(\frac{1}{1+z}\right) \\ &= \frac{2}{3}\sum_{n=0}^{\infty}(2z)^n + \frac{1}{3}\sum_{n=0}^{\infty}(-1)^n z^n \\ &= \sum_{n=0}^{\infty}\frac{2^{n+1}+(-1)^n}{3}z^n\end{aligned}$$

when $|z| < 1/2$.

3. *Proof.* Suppose for contradiction that for some function f analytic in $|z| \leq 1$, $f(\frac{1}{n}) = \frac{1}{n+1}$ for all $n \in \mathbb{N}$. Consider the sequence $z_n = \frac{1}{n}$, $n \in \mathbb{N}$. Then $z_n \to 0$ as $n \to \infty$, so by the Uniqueness Theorem, $f(z) = f(z_n) = \frac{z_n}{1+z_n}$ for $|z| \leq 1$. But then, f is discontinuous at $z = -1$, contradicting the fact that it is analyatic in $|z| \leq 1$. $\square$

4. *Proof.* Let $z_n = \frac{1}{n}$ for $n \in \mathbb{N}$. Then $z_n \subseteq \mathbb{R}$, so for some fixed $x \in \mathbb{R}$, $f(z) = \sin(x+z)$ and $g(z) = \sin x \cos z + \cos x \sin z$ coincide for all z_n (by the trigonometric identity for the reals). Since sin and cos are entire and $z_n \to 0$ as $n \to \infty$, by the Uniqueness theorem, $f(z) = g(z)$ for all $z \in \mathbb{C}$. Now consider $f^\star(z_1, z_2) = \sin(z_1+z_2)$ and $g^\star(z_1, z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2$. Since $f(z) = g(z)$, $f^\star(z_1, z_2) = g^\star(z_1, z_2)$ for all $z_1 \subseteq (z_n)$ and $z_2 \in \mathbb{C}$. Then because z_n converges to 0 and $f^\star$ and $g^\star$ are entire, $f^\star(z_1, z_2) = g^\star(z_1, z_2)$ for all $z_1, z_2 \in \mathbb{C}$. $\square$

5. Let $z = x + iy$. Then

$$\begin{aligned}|z^2 - z|^2 &= |z(z-1)|^2 \\ &= |(x+iy)(x+iy-1)|^2 \\ &= (x^2-x-y^2)^2 + (2xy-y)^2\end{aligned}$$

By the Maximum-Modulus theorem, the maximum occurs on the boundary of the disk, that is, when $|z| = 1$. Constraining to $x^2 + y^2 = 1$ by substituting $y^2 = 1 - x^2$ above, we have

$$\begin{aligned}|z^2 - z|^2 &= (x^2 - x - (1 - x^2))^2 + (2x(\sqrt{1 - x^2}) - \sqrt{1 - x^2})^2 \\ &= 2 - 2x\end{aligned}$$

So the modulus is monotonically decreasing with respect to x. Since the domain is $|z| \leq 1$, the maximum occurs at $x = -1$, that is $z = -1$.

The minimum modulus occurs at $z = 0$ at $z = 1$, where $|z^2 - z| = 0$.

6. *Proof.* By the Cauchy Integral Formula,

$$\begin{aligned}|f(z_0)^n| &= \left|\frac{1}{2\pi i}\int_C \frac{f(z)^n}{z - z_0} dz\right| \\ &\leq \frac{1}{2\pi} \cdot \frac{M^n}{K'} \cdot 2\pi R \\ &= KM^n\end{aligned}$$

by the M-L formula, where $K = R/K'$ for some constant K' dependent on the distance between z_0 and C. Note that K' is constant and bounded since z_0 is inside C. Taking the n-th root of both sides, we have $|f(z_0)| \leq K^{\frac{1}{n}} M$, so as $n \to \infty$, $|f(z_0)| \leq M$. $\square$