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HW 5

Math 3124
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1. TODO.
2. *Proof.* Let f be entire. Recall that in the power series expansion $f(z) = \sum C_k z^k$, $C_k = \frac{f^{(k)}(0)}{k!}$. From the proof of Theorem 5.5, we found that $C_k = \frac{1}{2\pi i}\int_C \frac{f(w)}{w^{k+1}}dw$. where C is a circle centered at the origin of radius $|w|$ containing z. Setting the two equal, we have $f^{(k)}(0) = \frac{k!}{2\pi i}\int_C \frac{f(w)}{w^{k+1}}dw$ for $k = 1, 2, \ldots$. Define $g(z) := f(z+a)$. Then

$$\begin{aligned} f^{(k)}(a) = g^{(k)}(0) &= \frac{k!}{2\pi i}\int_C \frac{g(\omega)}{\omega^{k+1}}d\omega \\ &= \frac{k!}{2\pi i}\int_C \frac{f(\omega + a)}{\omega^{k+1}}d\omega \\ &= \frac{k!}{2\pi i}\int_C \frac{f(w)}{(w-a)^{k+1}}dw \end{aligned}$$

   using the parameterization $\omega = w - a$. □

3. (a) *Proof.* Suppose f is entire with $|f| \leq M$ along $|z| = R$. From above, we have that the coefficients of the power series expansion of f about 0 is given by $C_k = \frac{1}{2\pi i}\int_C \frac{f(w)}{w^{k+1}}dw$, where C is a circle centered at the origin of radius $|w|$ containing z. Then, the integrand is bounded above by $\frac{M}{R^{k+1}}$. The length of C is $2\pi R$, so by the M-L Theorem, $|C_k| \leq |\frac{1}{2\pi i}\left(\frac{M}{R^{k+1}}\right)(2\pi R)| = \frac{M}{R^k}$. □

   (b) *Proof.* Polynomial functions are entire, so by above, with $M = 1$ and $R = 1$, $|C_k| \leq \frac{M}{R^k} = 1$. □

4. *Proof.* Let f be entire with $|f(z)| \leq A + B|z|^k$. Then, along $|z| = R$, $|f| \leq A + BR^k$. From above, $|C_j| \leq \frac{A+BR^k}{R^j}$. Now suppose $j > k$. Taking circles of radius R of arbitrary size, $\lim_{R\to\infty} |C_j| = 0$. □
5. *Proof.* Let f be entire with $|f(z)| \leq A + B|z|^{3/2}$. From above, $|C_k| = 0$ for $k \geq 3/2$. That is, $|C_k| \neq 0$ only for $k = 0, 1$. So $f(z) = C_0 + C_1 z$. □
6. *Proof.* Suppose f is entire. Since it is entire, it is continuous, and thus bounded on the compact set $0 \leq x, y \leq 1$. By periodicity, it is also bounded on all 1-by-1 squares $a \leq x, y \leq a + 1$ where $a \in \mathbb{Z}$. So, the function is bounded on the entire complex plane, so it is constant, by Liouville's Theorem. □
7. *Proof.* Suppose $P(z) = (z - \alpha)^k Q(z)$ where Q is analytic and $Q(\alpha) \neq 0$. □