

Approximating open quantum systems

Using Stinespring dilation

BFP Applied Physics, Applied Mathematics

David Chen, 1742477

My project in one minute

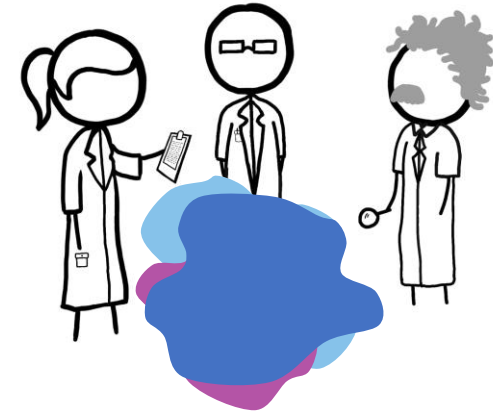
2

Create a model



(Most scientists)

Compare to experiment



My project in one minute

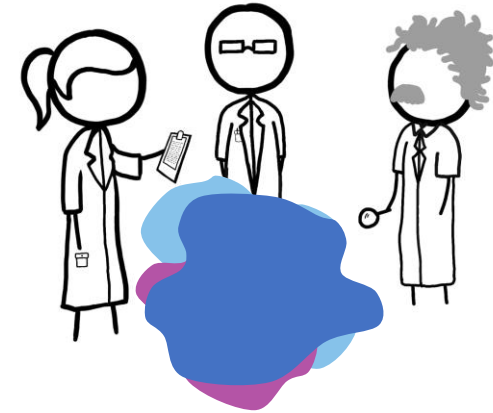
3

Create a model



(Most scientists)

Compare to experiment



Infer model



(Me)

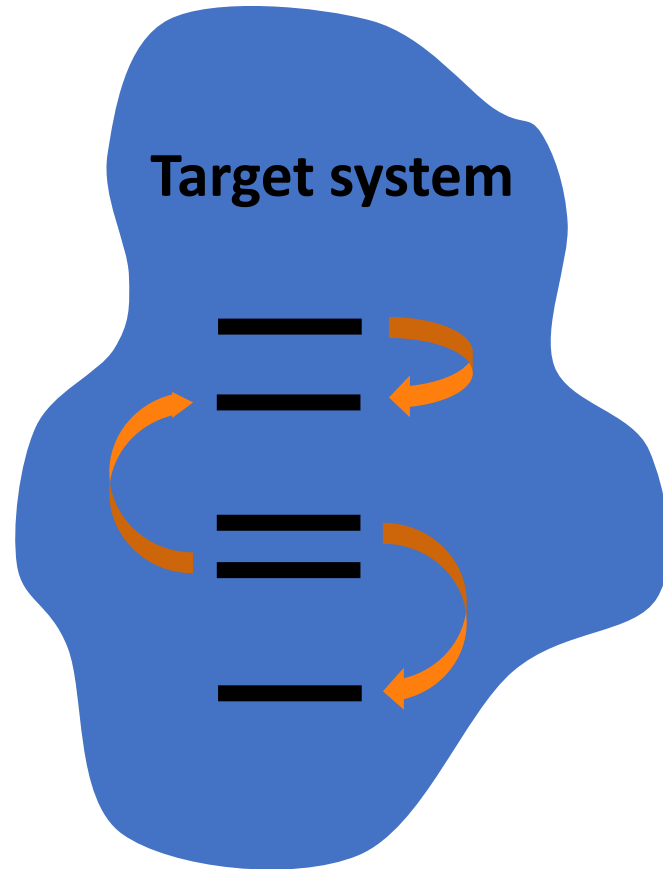
Make measurements

Background

- **Quantum systems**
- **Density matrix formalism**
- **Evolution equations**
- **Quantum channels**
- **Combining qubits**
- **Stinespring's dilation theorem**

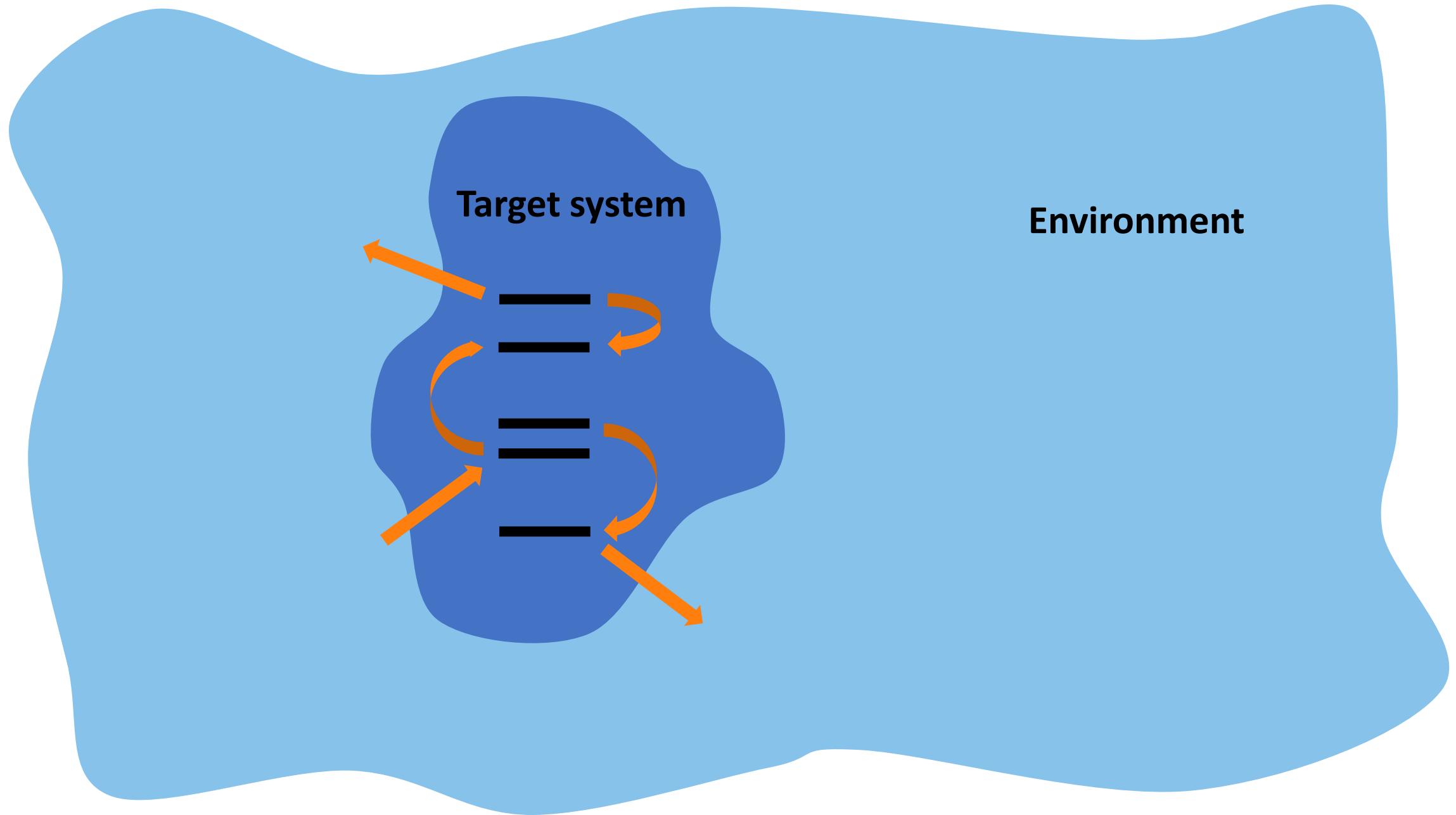
My project

What is a quantum system?



What is an open quantum system?

6



Density matrix formalism

7

Pure state

$$|\psi\rangle = \alpha_n |e^n\rangle$$
$$= \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_N \end{bmatrix}$$

$$\rho = |\psi\rangle\langle\psi|$$
$$= \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_N \end{bmatrix} \begin{bmatrix} \bar{\alpha}_1 & \dots & \bar{\alpha}_N \end{bmatrix} = \begin{bmatrix} \alpha_1 \bar{\alpha}_1 & \dots & \alpha_1 \bar{\alpha}_N \\ \vdots & \ddots & \vdots \\ \alpha_N \bar{\alpha}_1 & \dots & \alpha_N \bar{\alpha}_N \end{bmatrix}$$

Mixed state

???

$$\rho = \sum_k p_k |\psi^k\rangle\langle\psi^k|, \quad \sum_k p_k = 1$$

Density matrix formalism

8

Pure state

$$|\psi\rangle = \alpha_n |e^n\rangle$$
$$= \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_N \end{bmatrix}$$



$$\varrho = |\psi\rangle \langle \psi|$$
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Mixed state

???



$$\rho = \sum_k p_k |\psi^k\rangle \langle \psi^k|, \quad \sum_k p_k = 1$$

Evolution equations

9

Closed system (Schrödinger equation)

$$\frac{d}{dt} |\psi\rangle = -\frac{i}{\hbar} H |\psi\rangle$$

Density matrix formalism



(Von Neumann equation)

$$\frac{d}{dt} \rho = -\frac{i}{\hbar} [H, \rho]$$

Open system (Lindblad equation)

$$\frac{d}{dt} \rho = -\frac{i}{\hbar} [H, \rho] + \sum_k \gamma_k \left(A_k \rho A_k^\dagger - \frac{1}{2} \{ A_k^\dagger A_k, \rho \} \right)$$

Closed system (Schrödinger equation)

$$\frac{d}{dt} |\psi\rangle = -\frac{i}{\hbar} H |\psi\rangle$$

Density matrix formalism



(Von Neumann equation)

$$\frac{d}{dt} \rho = -\frac{i}{\hbar} [H, \rho]$$

“Hamiltonian”

Open system (Lindblad equation)

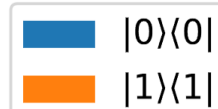
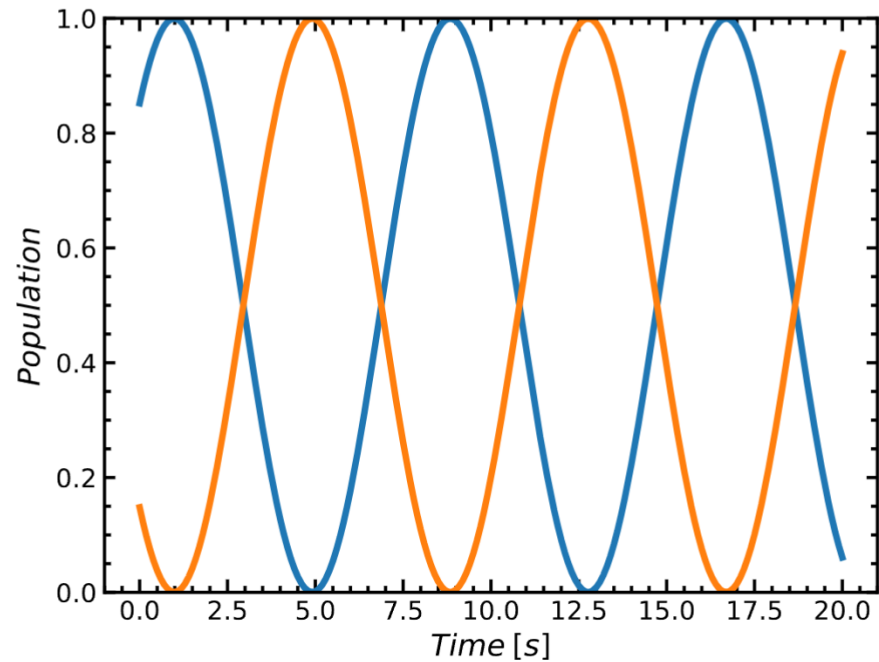
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“Hamiltonian”

“Interactions”

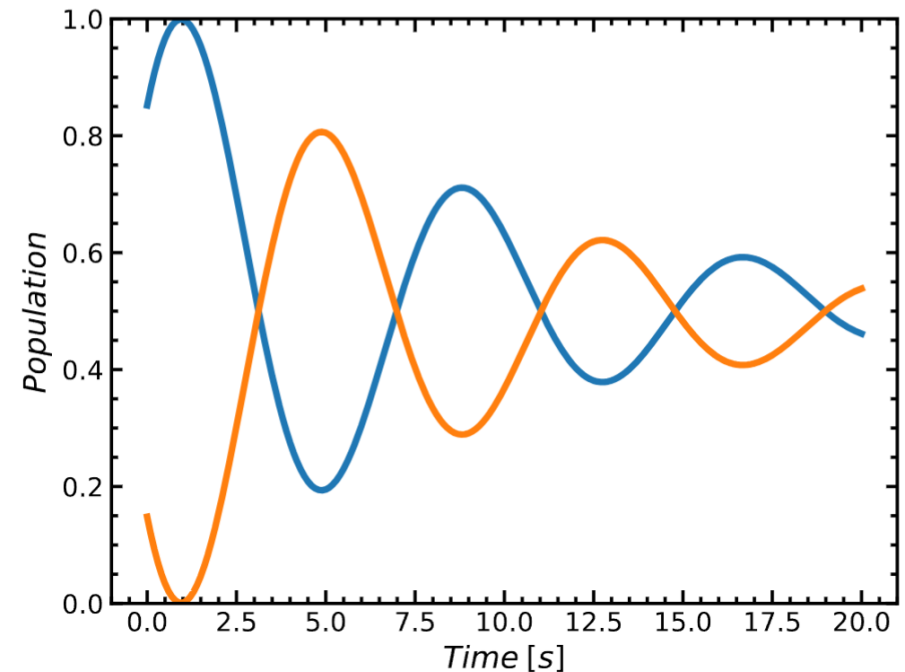
Closed system (Von Neumann)

$$\frac{d}{dt}\rho = -\frac{i}{\hbar}[H, \rho]$$



Open system (Lindblad)

$$\frac{d}{dt}\rho = -\frac{i}{\hbar}[H, \rho] + \sum_k \gamma_k \left(A_k \rho A_k^\dagger - \frac{1}{2} \{A_k^\dagger A_k, \rho\} \right)$$



Closed system (Von Neumann)

$$\frac{d}{dt}\rho = -\frac{i}{\hbar}[H, \rho]$$

$$U_t \rho_0 U_t^\dagger = \rho_t,$$

$$U_t = \exp\left(-\frac{i}{\hbar}Ht\right)$$

Open system (Lindblad)

$$\frac{d}{dt}\rho = -\frac{i}{\hbar}[H, \rho] + \sum_k \gamma_k \left(A_k \rho A_k^\dagger - \frac{1}{2} \{A_k^\dagger A_k, \rho\} \right)$$

Goal: approximate $\Phi_t(\rho_0) = \rho_t$

Closed system (Von Neumann)

$$\frac{d}{dt}\rho = -\frac{i}{\hbar}[H, \rho]$$

$$U_t \rho_0 U_t^\dagger = \rho_t,$$

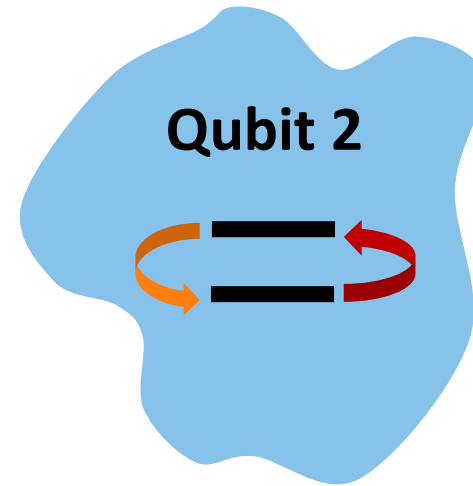
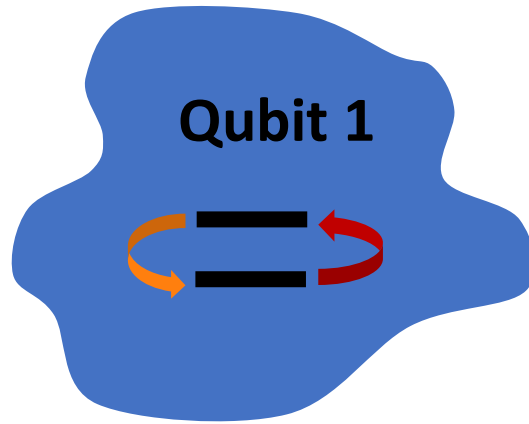
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Open system (Lindblad)

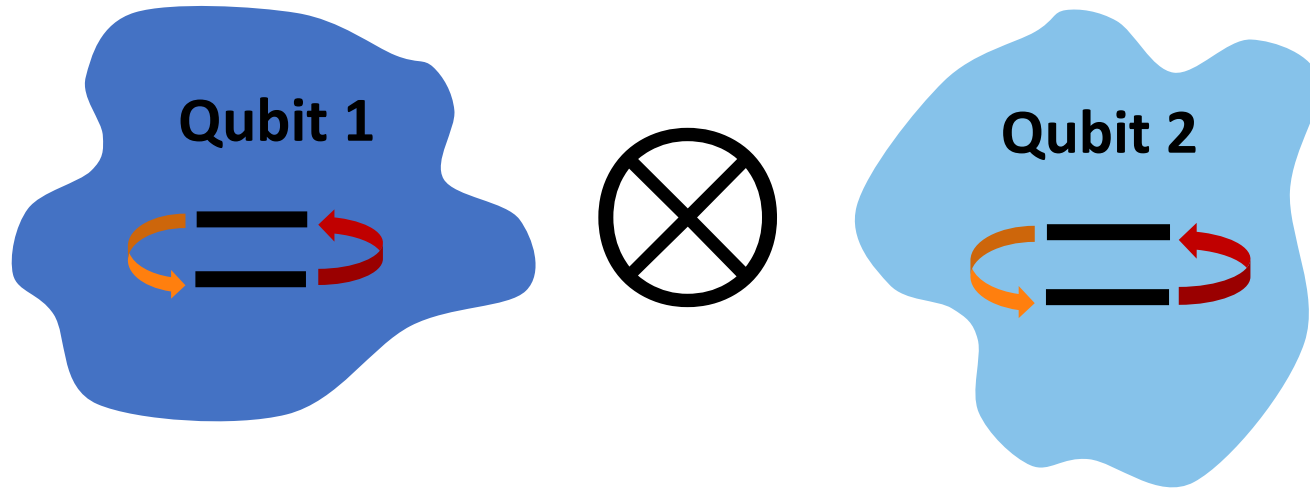
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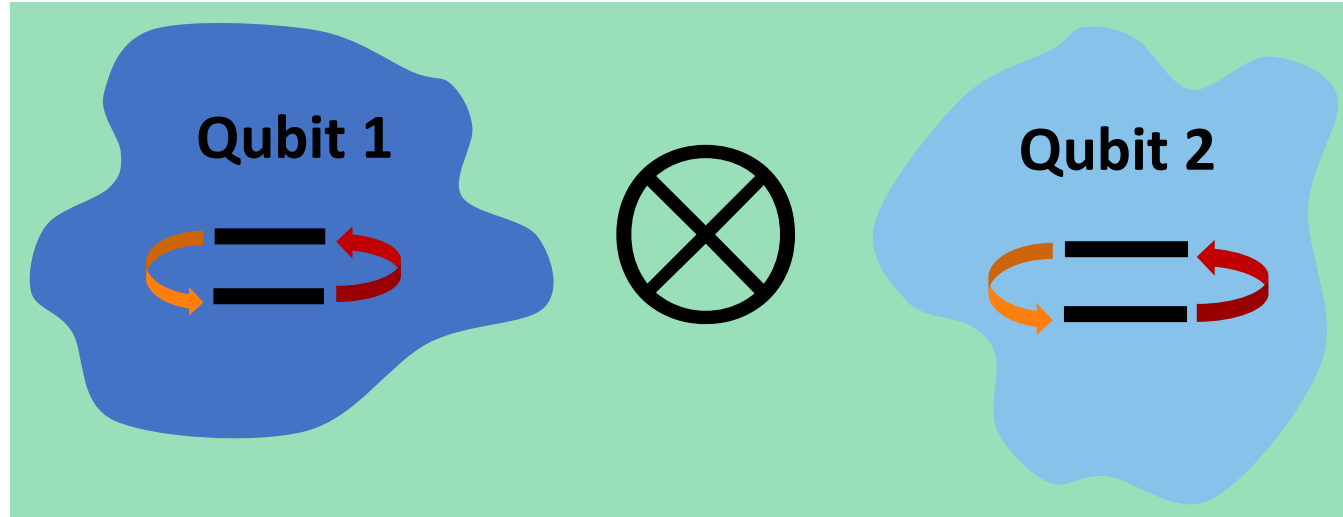
Combining qubits



Combining qubits

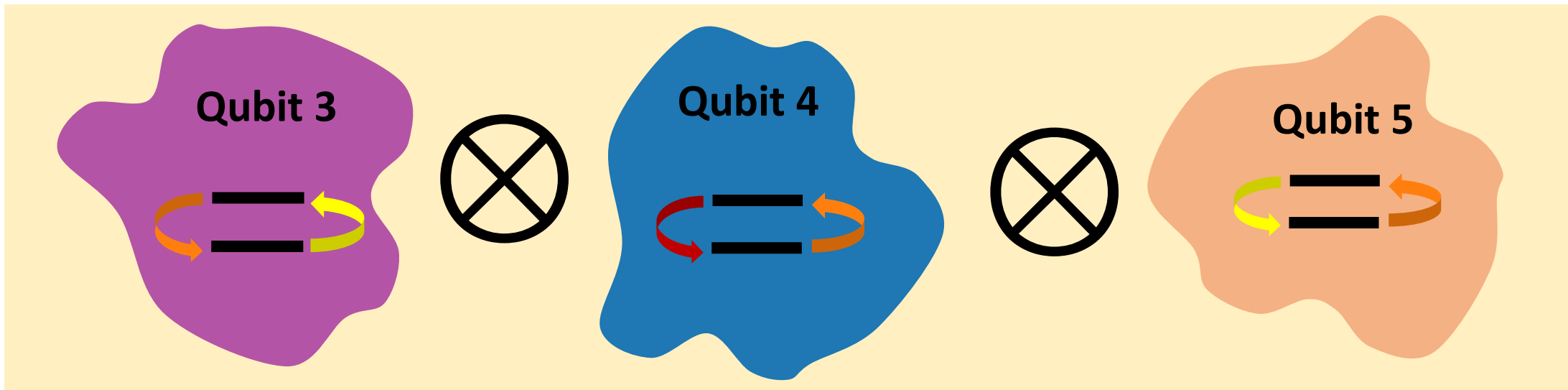
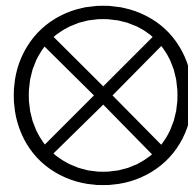


Combining qubits



Computational
qubits

"System A"



Ancilla
qubits

"System B"

Stinespring's dilation theorem

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There exists U_t on dilated space (computational qubits) \otimes (ancilla qubits) such that:

$$\Phi_t(\rho_0) = \text{Tr}_B[U_t (\rho_0 \otimes |0\rangle_B \langle 0|_B) U_t^\dagger]$$

Stinespring's dilation theorem

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There exists U_t on dilated space (computational qubits) \otimes (ancilla qubits) such that:

$$\Phi_t(\rho_0) = \text{Tr}_B [U_t (\rho_0 \otimes |0\rangle_B \langle 0|_B) U_t^\dagger]$$

“Stinespring unitary”

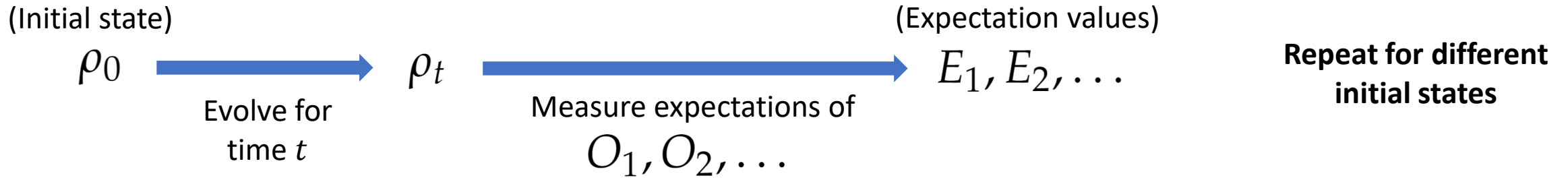
My project

- **Approximation method**
- **Parametrization of Stinespring unitary**
- **Results**

Approximation method

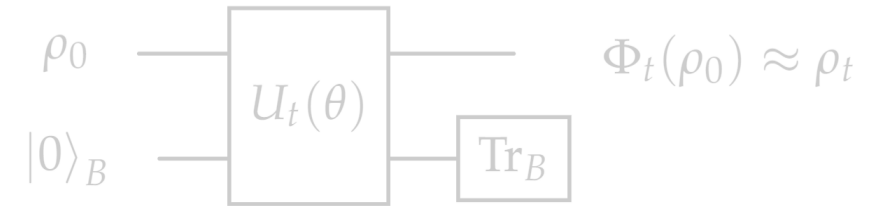
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Gather training data



Approximate quantum channel

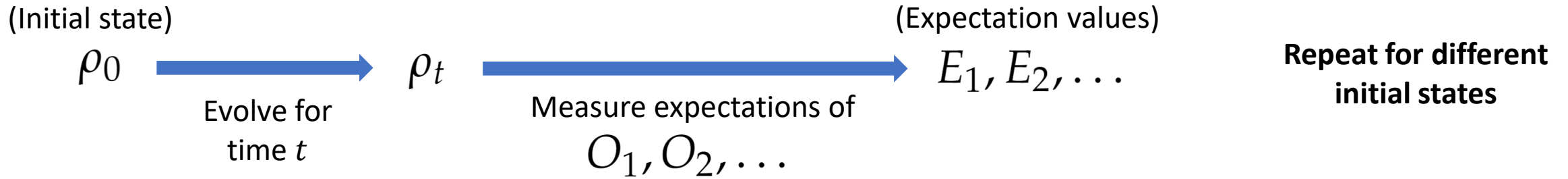
$$\Phi_t(\rho_0) = \text{Tr}_B[U_t(\rho_0 \otimes |0\rangle_B\langle 0|_B)U_t^\dagger]$$



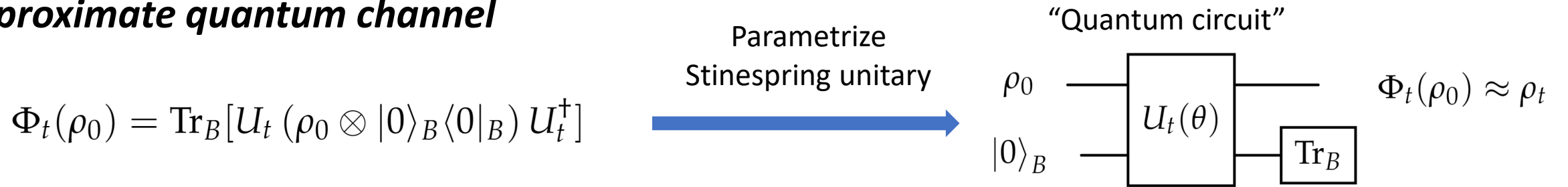
Optimize parameters

$$J(\theta) = \sum_{\ell} (E_{\ell} - \hat{E}_{\ell})^2$$

Gather training data



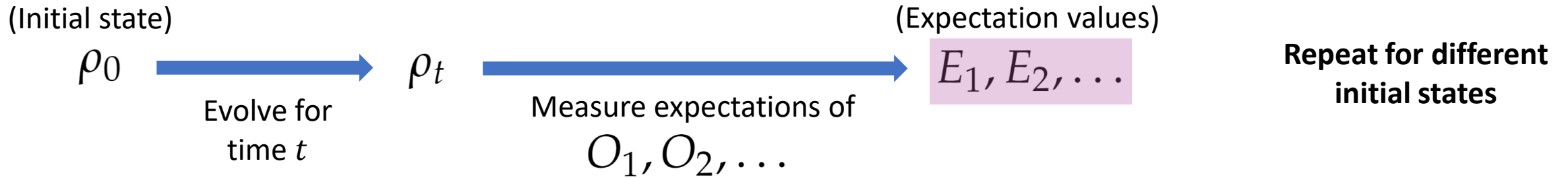
Approximate quantum channel



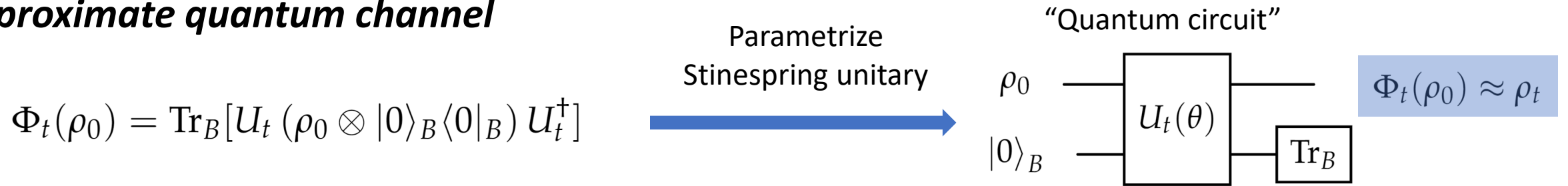
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Gather training data



Approximate quantum channel



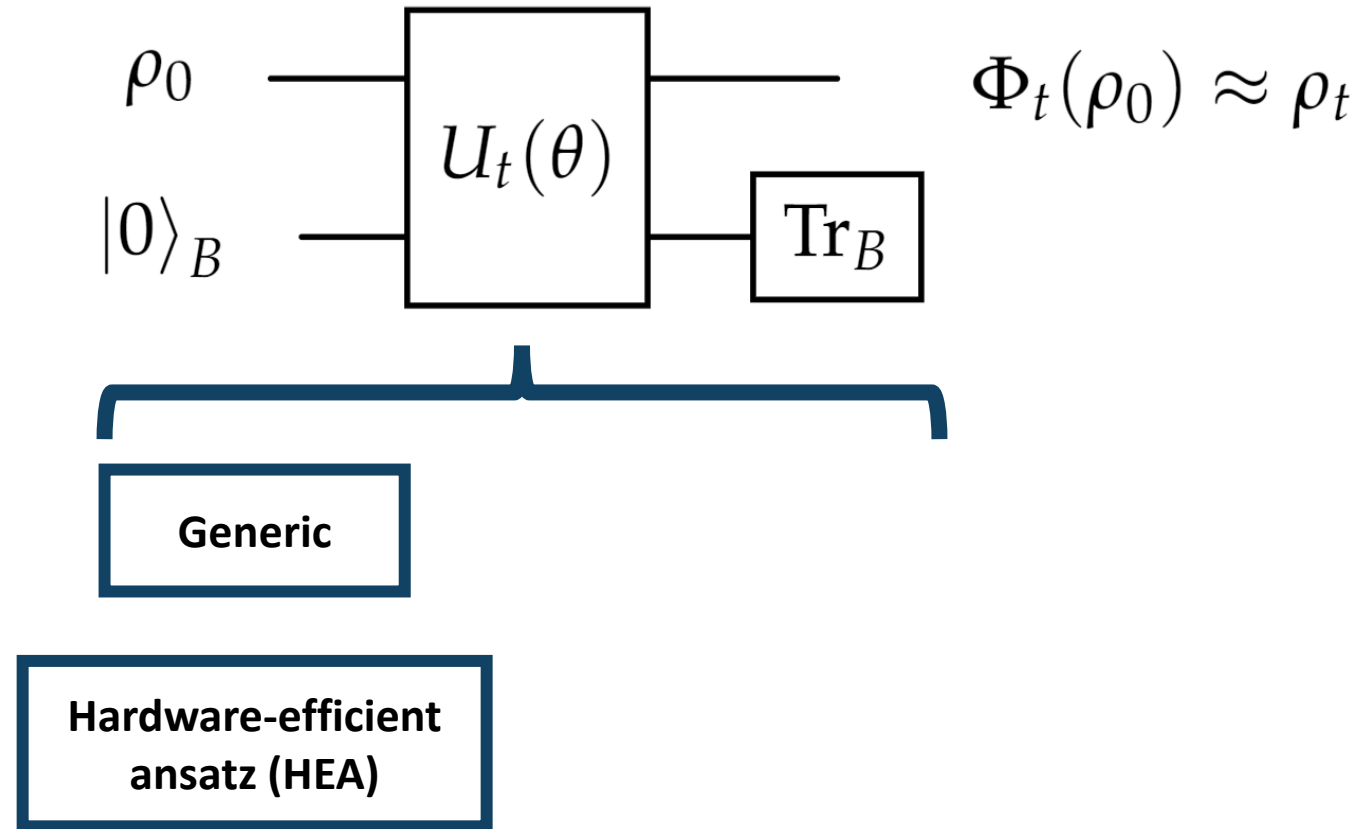
Optimize parameters

$$J(\theta) = \sum_{\ell} (E_{\ell} - \hat{E}_{\ell})^2$$

Measured Approximated

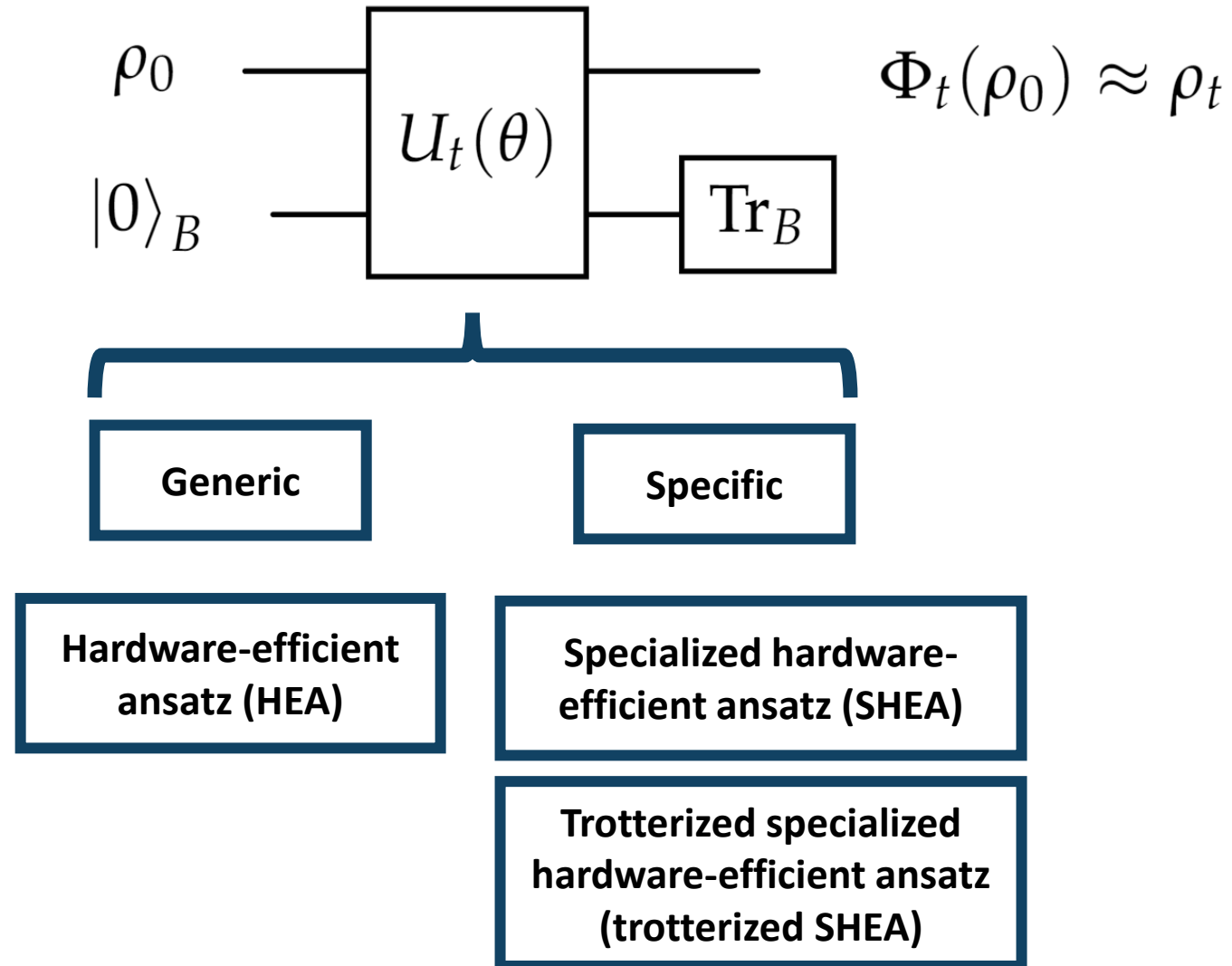
Parametrization of Stinespring unitary

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Parametrization of Stinespring unitary

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Hardware-efficient ansatz (HEA)

$$\begin{array}{c} \rho_0 \\ |0\rangle_B \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} \boxed{U_t(\theta)} \\ \boxed{U_t(\theta)} \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} = \left(\begin{array}{c} \text{---} \boxed{ZXZ} \text{---} \\ \text{---} \boxed{ZXZ} \text{---} \end{array} \begin{array}{c} \boxed{\text{Ent.}} \\ \boxed{\text{Ent.}} \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} \right)_d$$

Specialized hardware-efficient ansatz (SHEA)

$$\begin{array}{c} \rho_0 \\ |0\rangle_B \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} \boxed{U_t(\theta)} \\ \boxed{U_t(\theta)} \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} = \begin{array}{c} \text{---} \boxed{\exp(-iHt)} \text{---} \\ \text{---} \end{array} \begin{array}{c} \boxed{\text{HEA}_d} \\ \boxed{\text{HEA}_d} \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array}$$

Interlude: trotterization

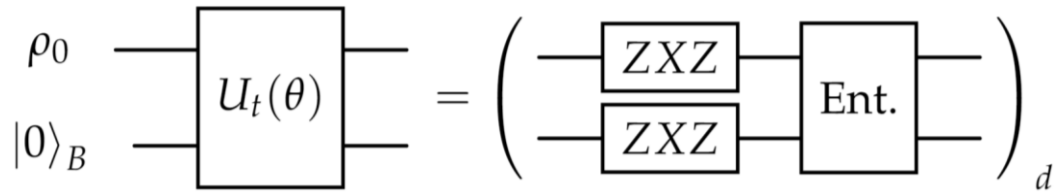
For Hermitian operators A and B we have:

$$\lim_{n \rightarrow \infty} (e^{-iA \frac{t}{n}} e^{-iB \frac{t}{n}})^n = e^{-i(A+B)t}$$

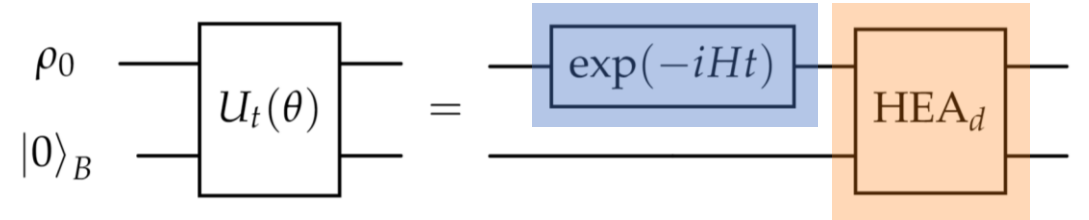
Trotterized specialized hardware-efficient ansatz (trotterized SHEA)

$$\begin{array}{c} \rho_0 \\ |0\rangle_B \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} \boxed{U_t(\theta)} \\ \boxed{U_t(\theta)} \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} = \left(\begin{array}{c} \text{---} \boxed{\exp(-iH_d \frac{t}{d})} \text{---} \\ \text{---} \end{array} \begin{array}{c} \boxed{\text{HEA}_q} \\ \boxed{\text{HEA}_q} \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} \right)_d$$

Hardware-efficient ansatz (HEA)



Specialized hardware-efficient ansatz (SHEA)



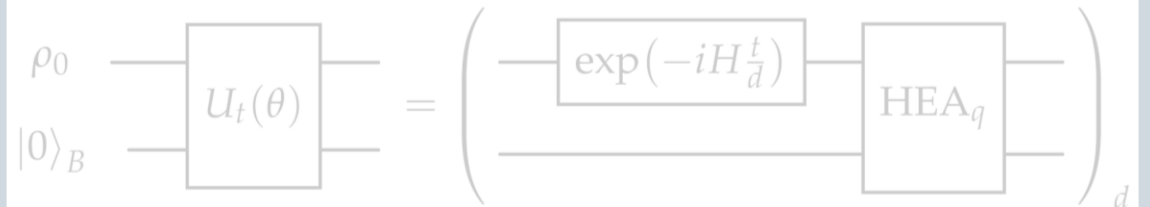
$$\frac{d}{dt}\rho = -\frac{i}{\hbar}[H, \rho] + \sum_k \gamma_k \left(A_k \rho A_k^\dagger - \frac{1}{2} \{A_k^\dagger A_k, \rho\} \right)$$

Interlude: trotterization

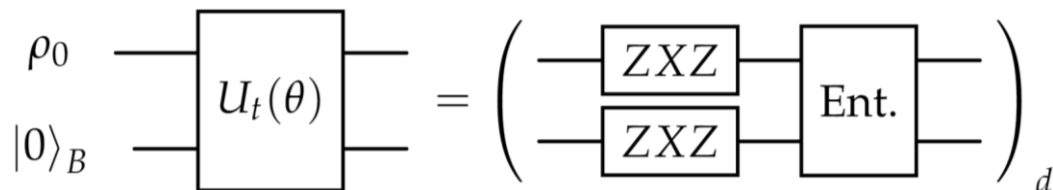
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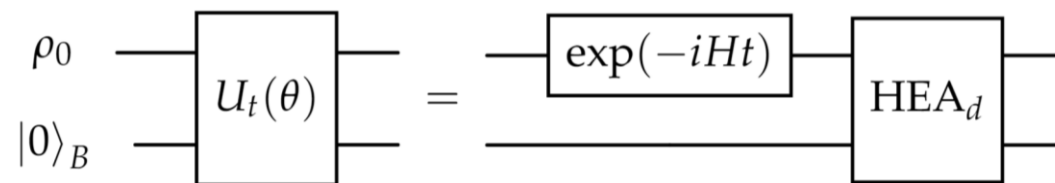
Trotterized specialized hardware-efficient ansatz (trotterized SHEA)



Hardware-efficient ansatz (HEA)



Specialized hardware-efficient ansatz (SHEA)



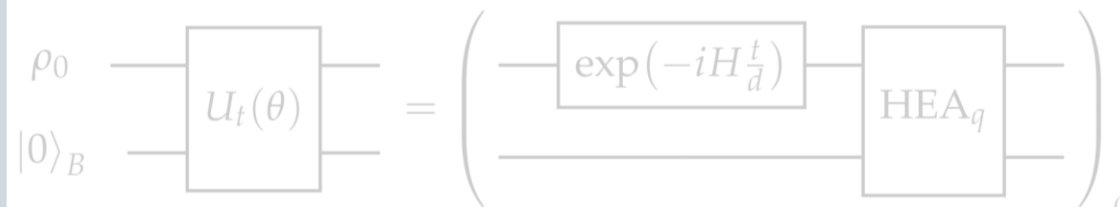
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Trotterized specialized hardware-efficient ansatz (trotterized SHEA)



Hardware-efficient ansatz (HEA)

$$\rho_0 \text{ --- } U_t(\theta) \text{ --- } \left(\begin{array}{c} \text{--- } \boxed{\text{ZXZ}} \text{ ---} \\ \text{--- } \boxed{\text{ZXZ}} \text{ ---} \end{array} \text{Ent.} \right)_d$$

Specialized hardware-efficient ansatz (SHEA)

$$\rho_0 \text{ --- } U_t(\theta) \text{ --- } \exp(-iHt) \text{ --- } \text{HEA}_d$$

$$\frac{d}{dt}\rho = -\frac{i}{\hbar}[H, \rho] + \sum_k \gamma_k \left(A_k \rho A_k^\dagger - \frac{1}{2} \{A_k^\dagger A_k, \rho\} \right)$$

Interlude: trotterization

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Trotterized specialized hardware-efficient ansatz (trotterized SHEA)

$$\rho_0 \text{ --- } U_t(\theta) \text{ --- } \left(\begin{array}{c} \text{--- } \boxed{\exp(-iH_d \frac{t}{d})} \text{ ---} \\ \text{--- } \boxed{\text{HEA}_q} \text{ ---} \end{array} \right)_d$$

Hardware-efficient ansatz (HEA)

$$\rho_0 \text{ --- } U_t(\theta) \text{ --- } \left(\begin{array}{c} \text{ZXZ} \\ \text{ZXZ} \end{array} \text{ --- Ent. } \right)_d$$

Specialized hardware-efficient ansatz (SHEA)

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Interlude: trotterization

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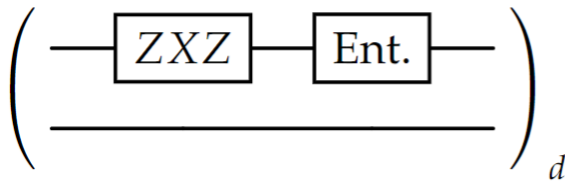
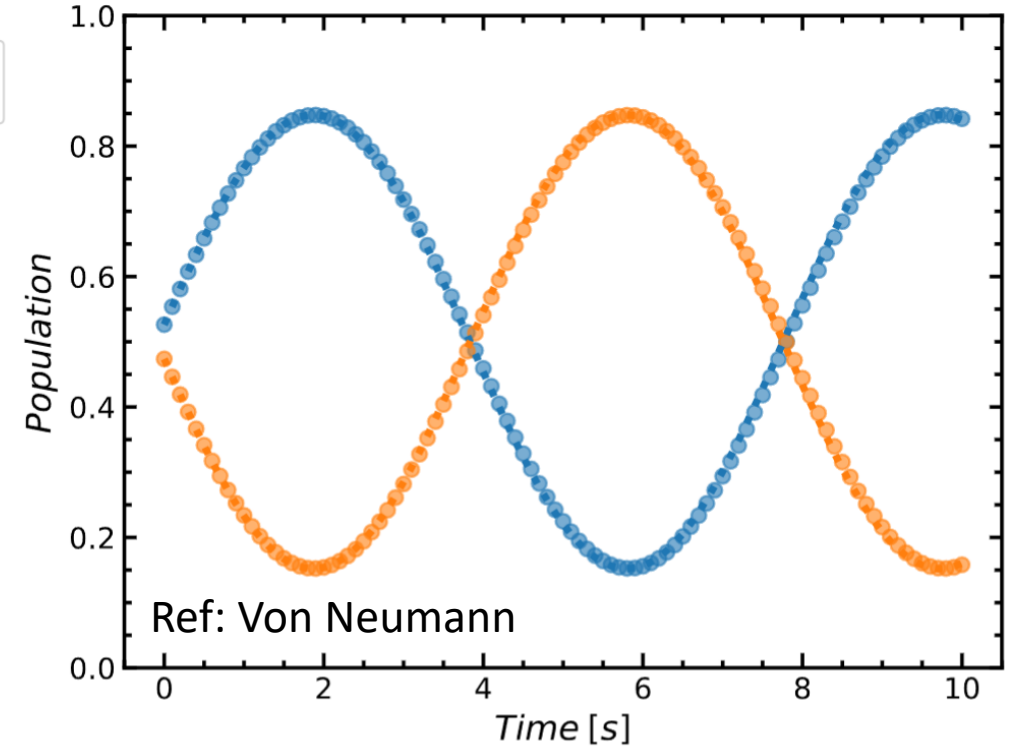
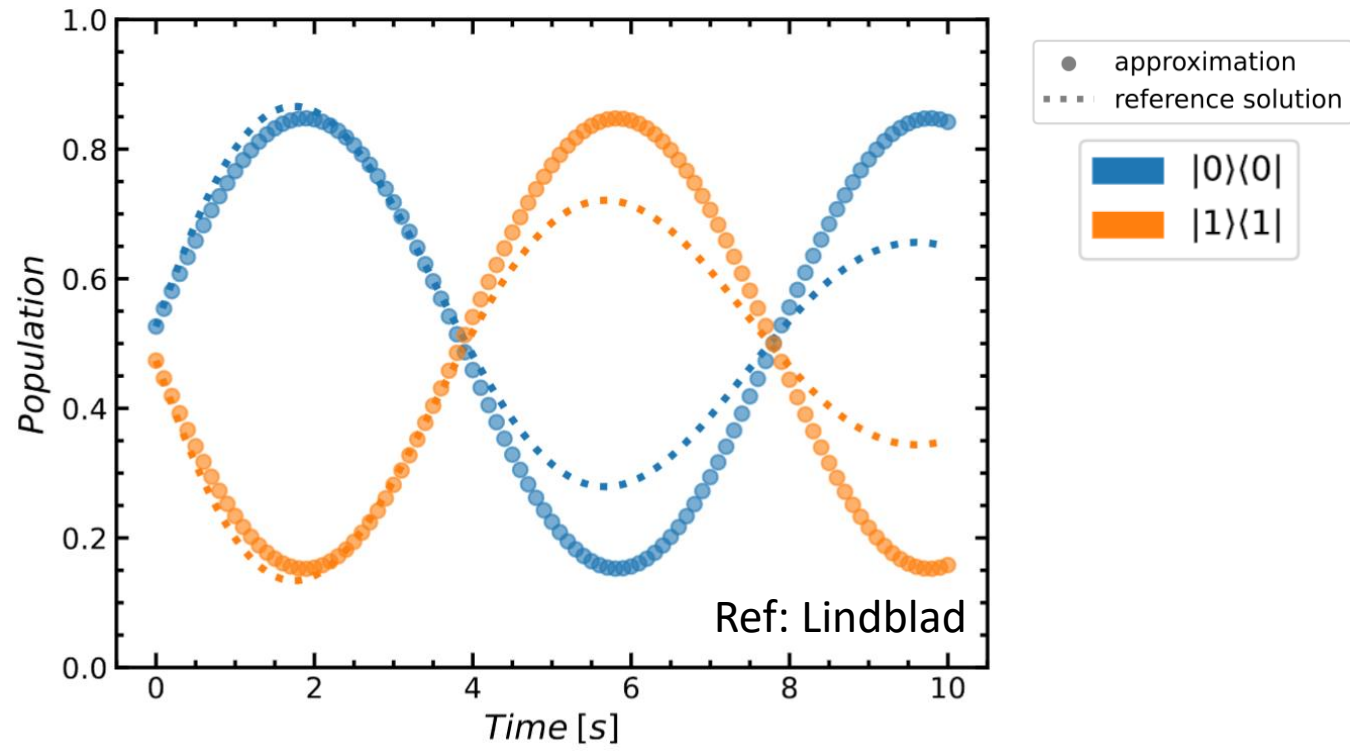
Trotterized specialized hardware-efficient ansatz (trotterized SHEA)

$$\rho_0 \text{ --- } U_t(\theta) \text{ --- } \left(\exp(-iH_d \frac{t}{d}) \text{ --- HEA}_q \right)_d$$

“Same parameters”

Results: approximating Rabi oscillation (one qubit)

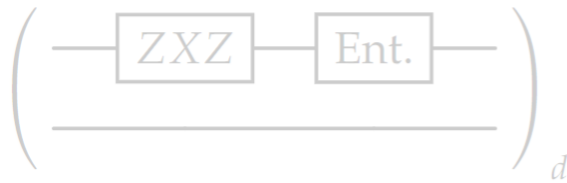
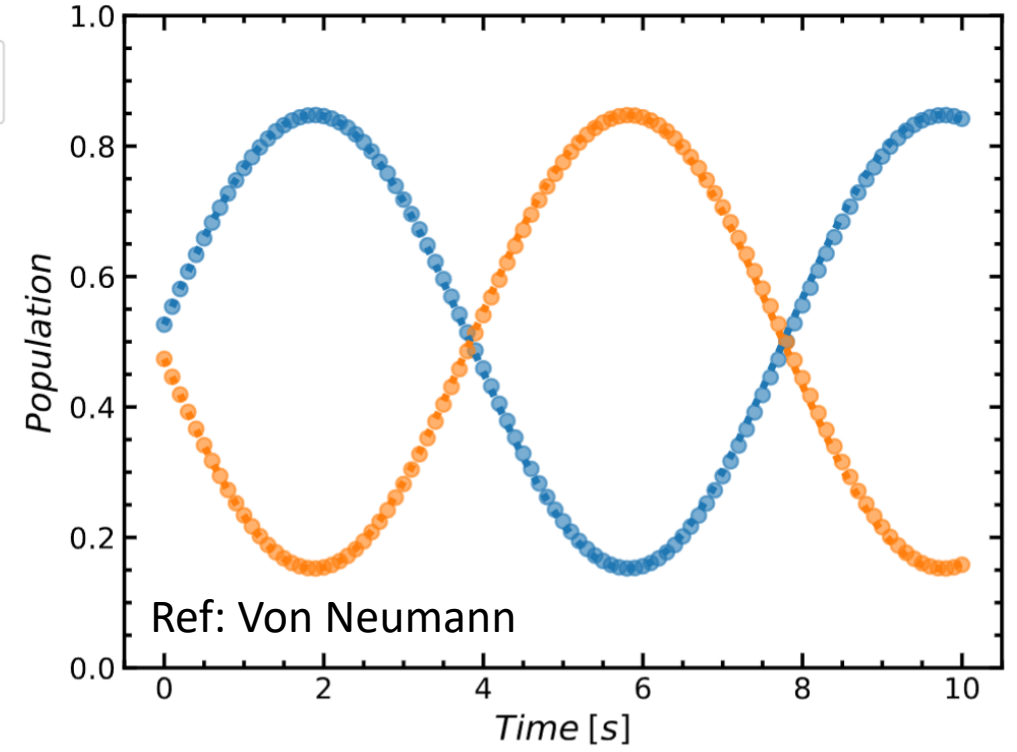
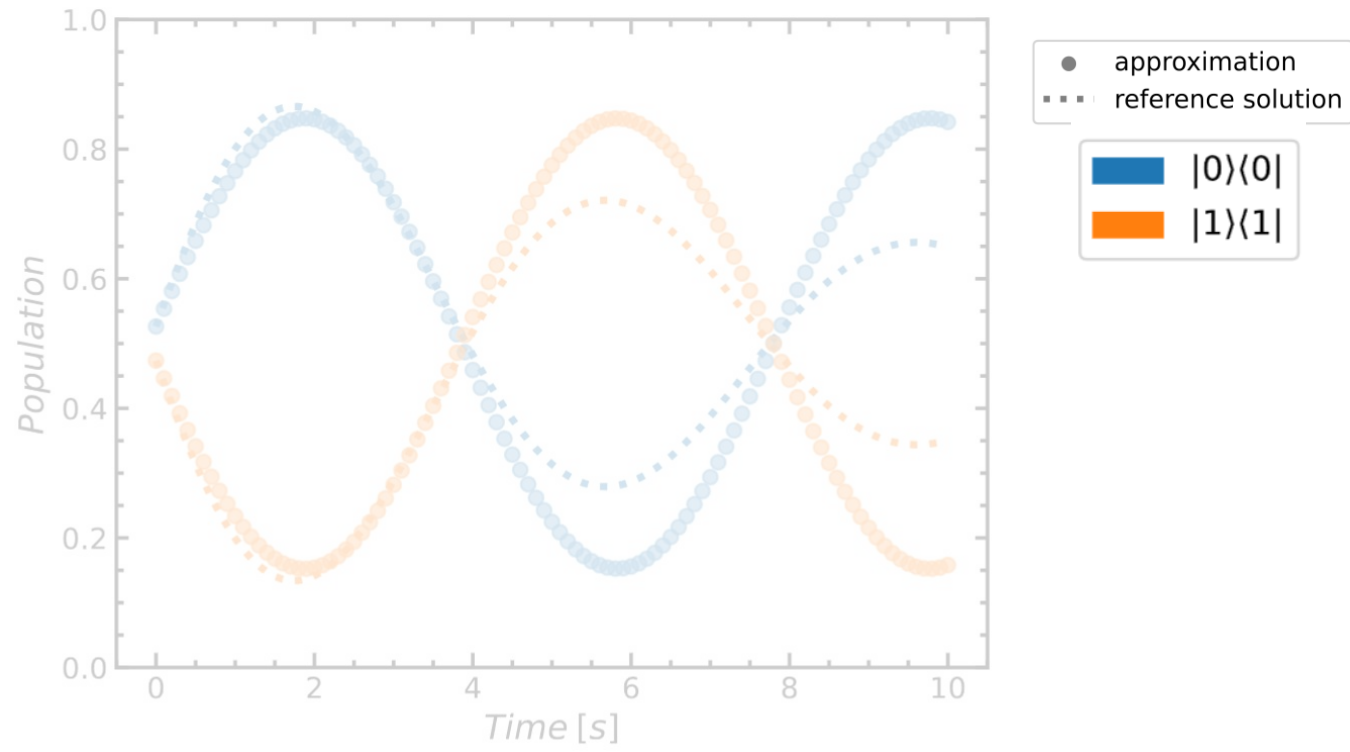
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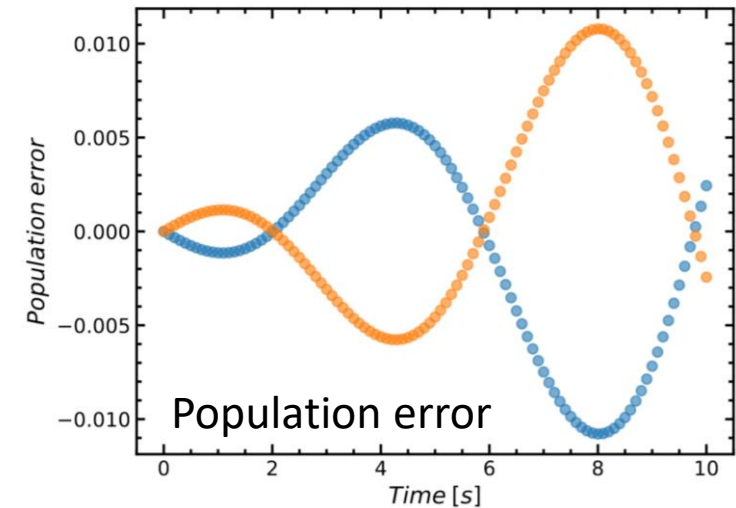
HEA

Results: approximating Rabi oscillation (one qubit)

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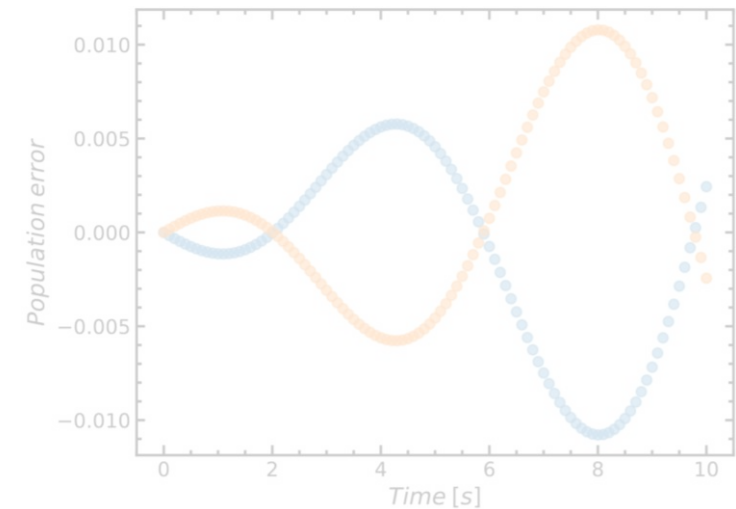
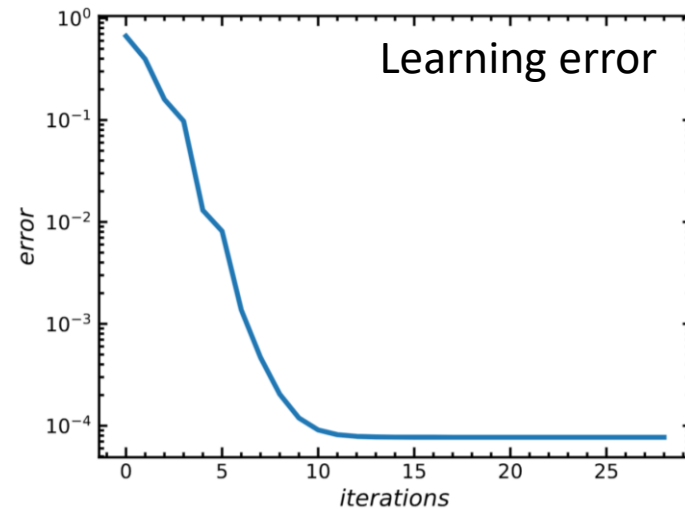
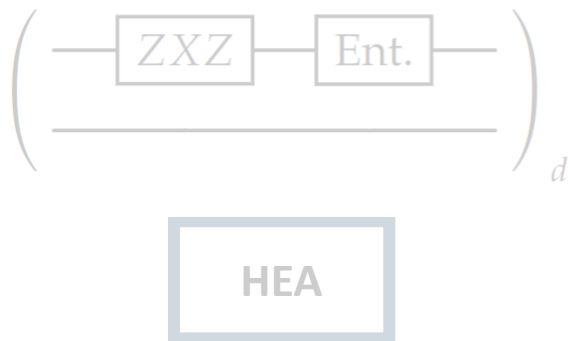
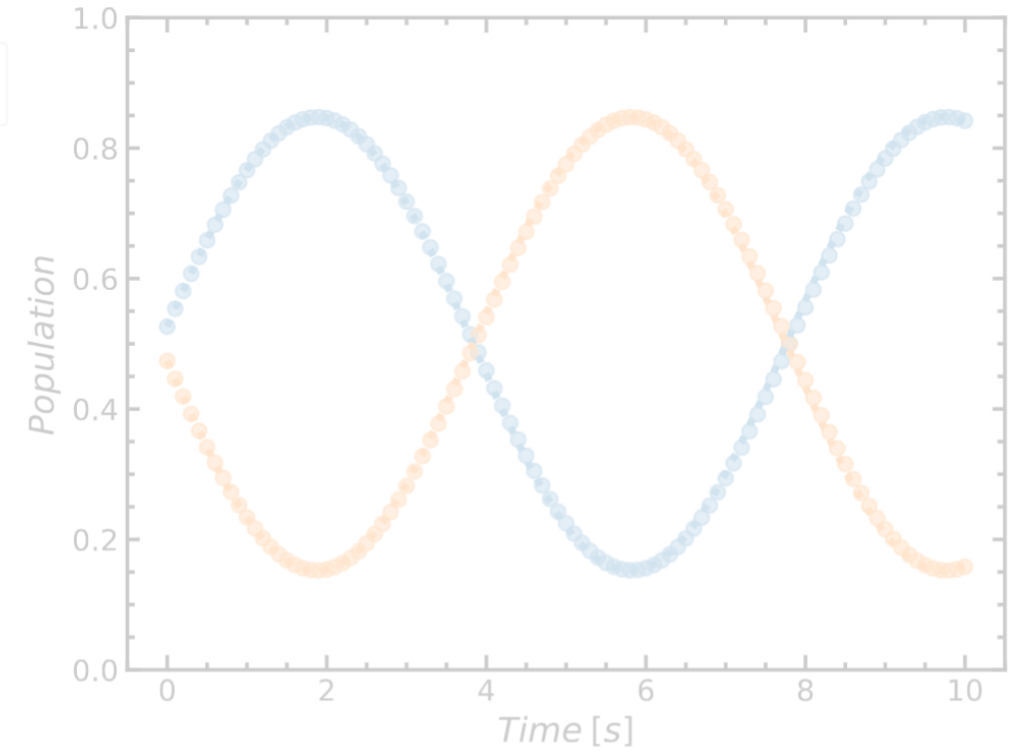
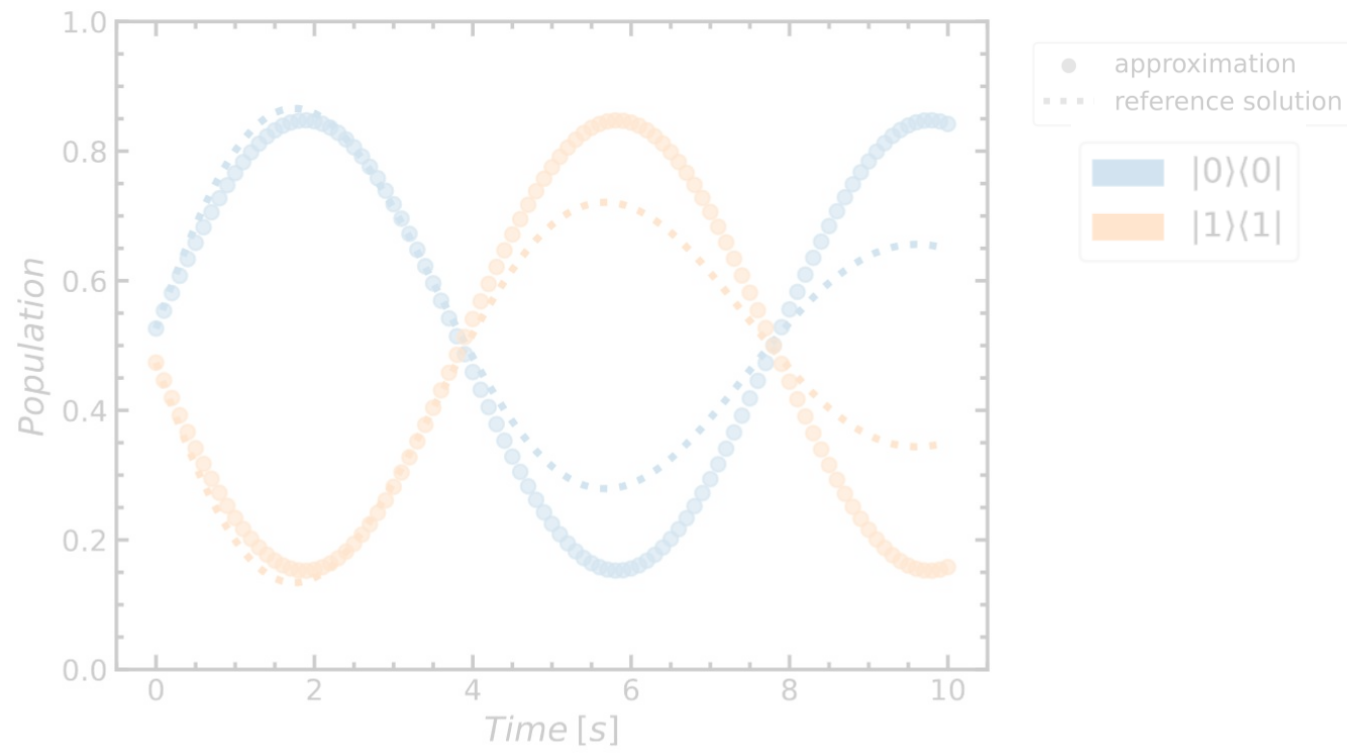


HEA

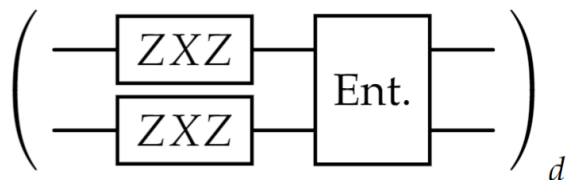
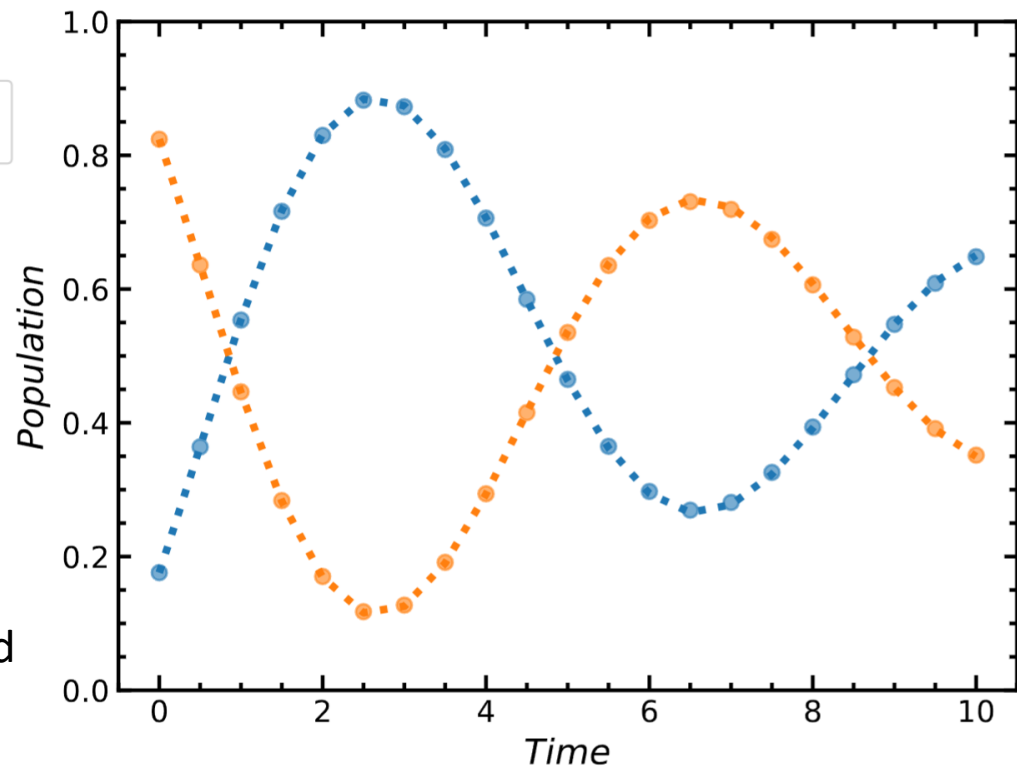
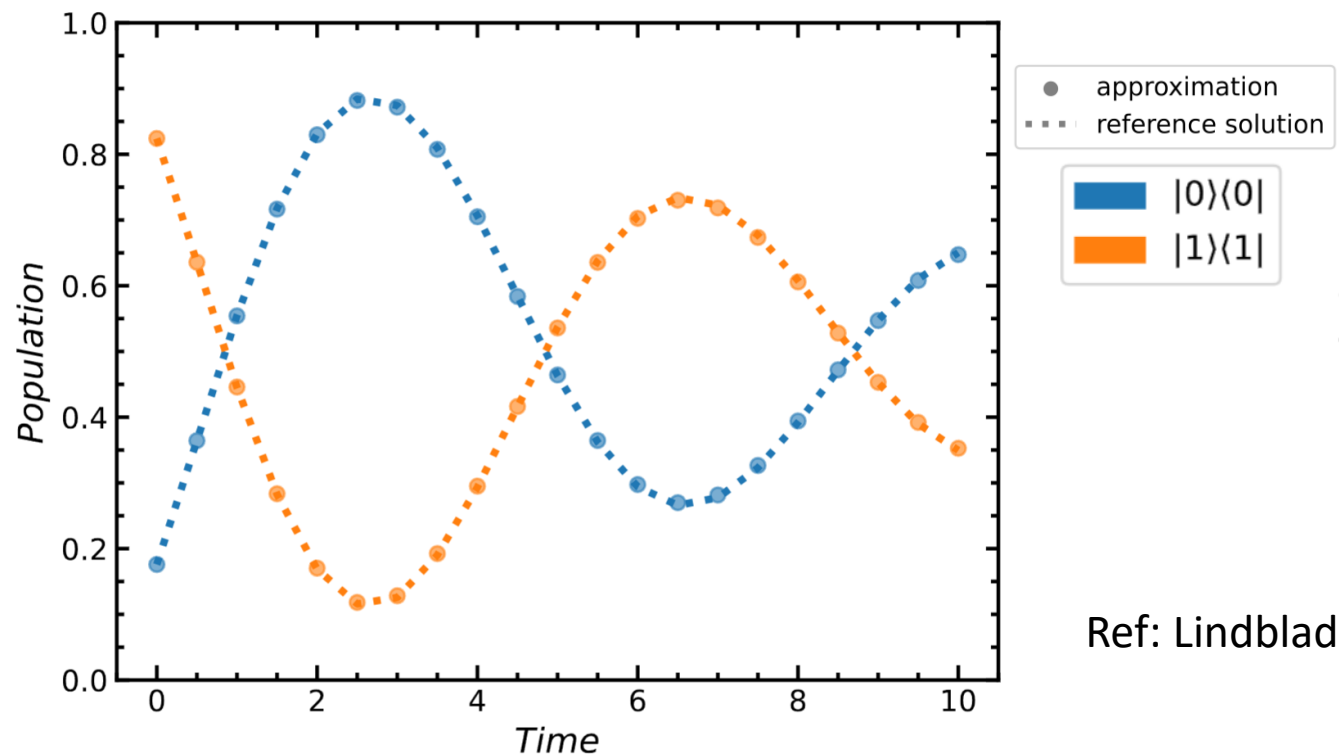


Results: approximating Rabi oscillation (one qubit)

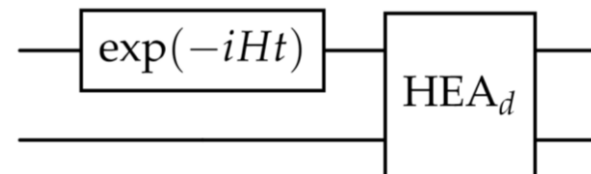
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Results: approximating Rabi oscillation (one qubit)



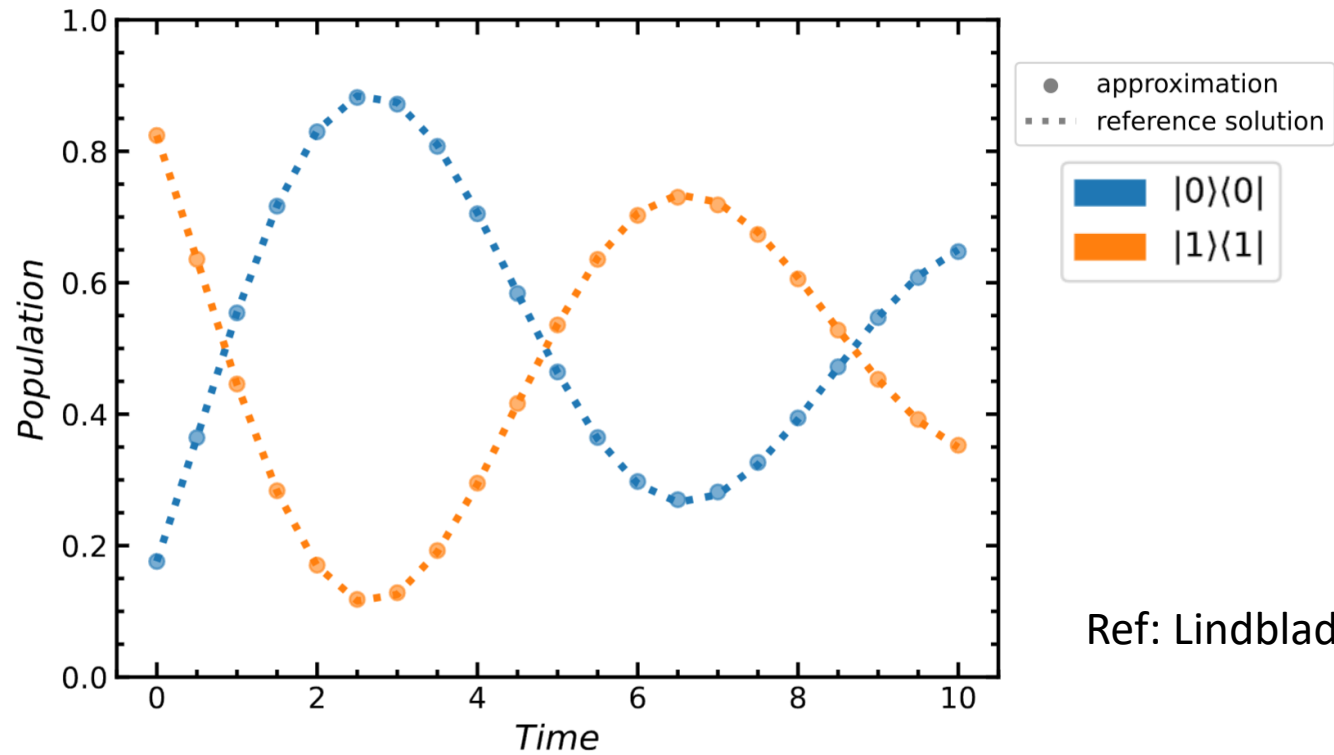
HEA



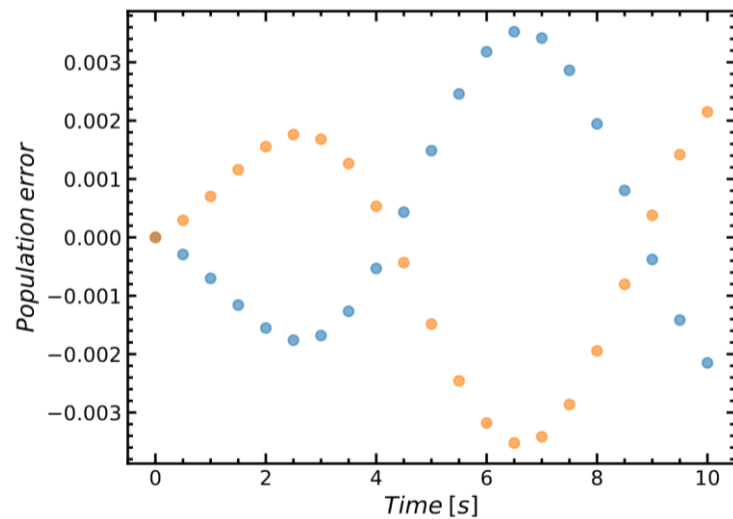
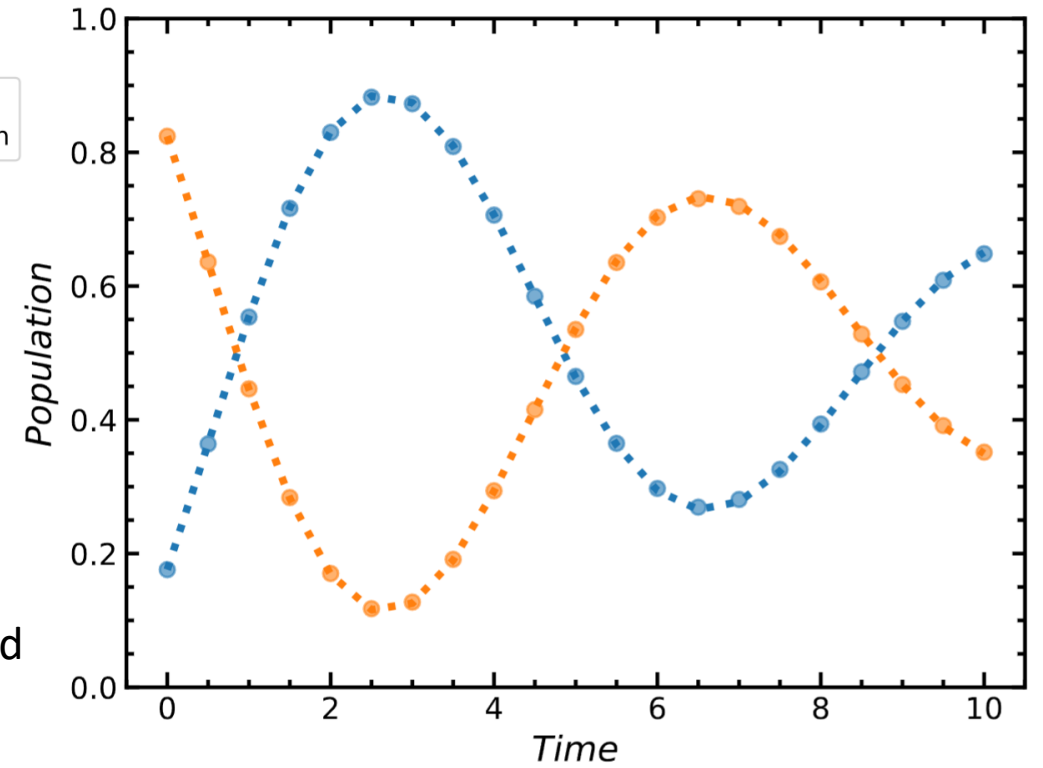
SHEA

Results: approximating Rabi oscillation (one qubit)

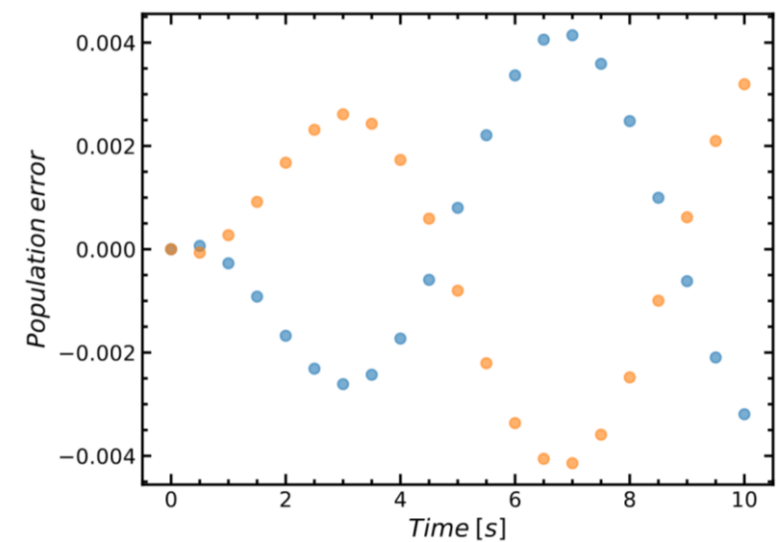
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Ref: Lindblad

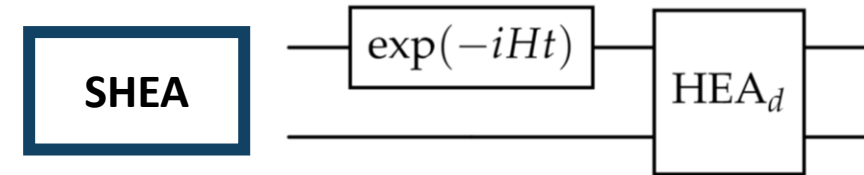
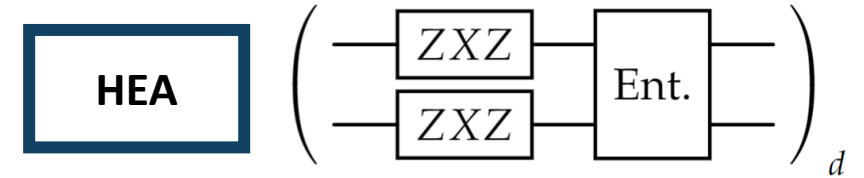
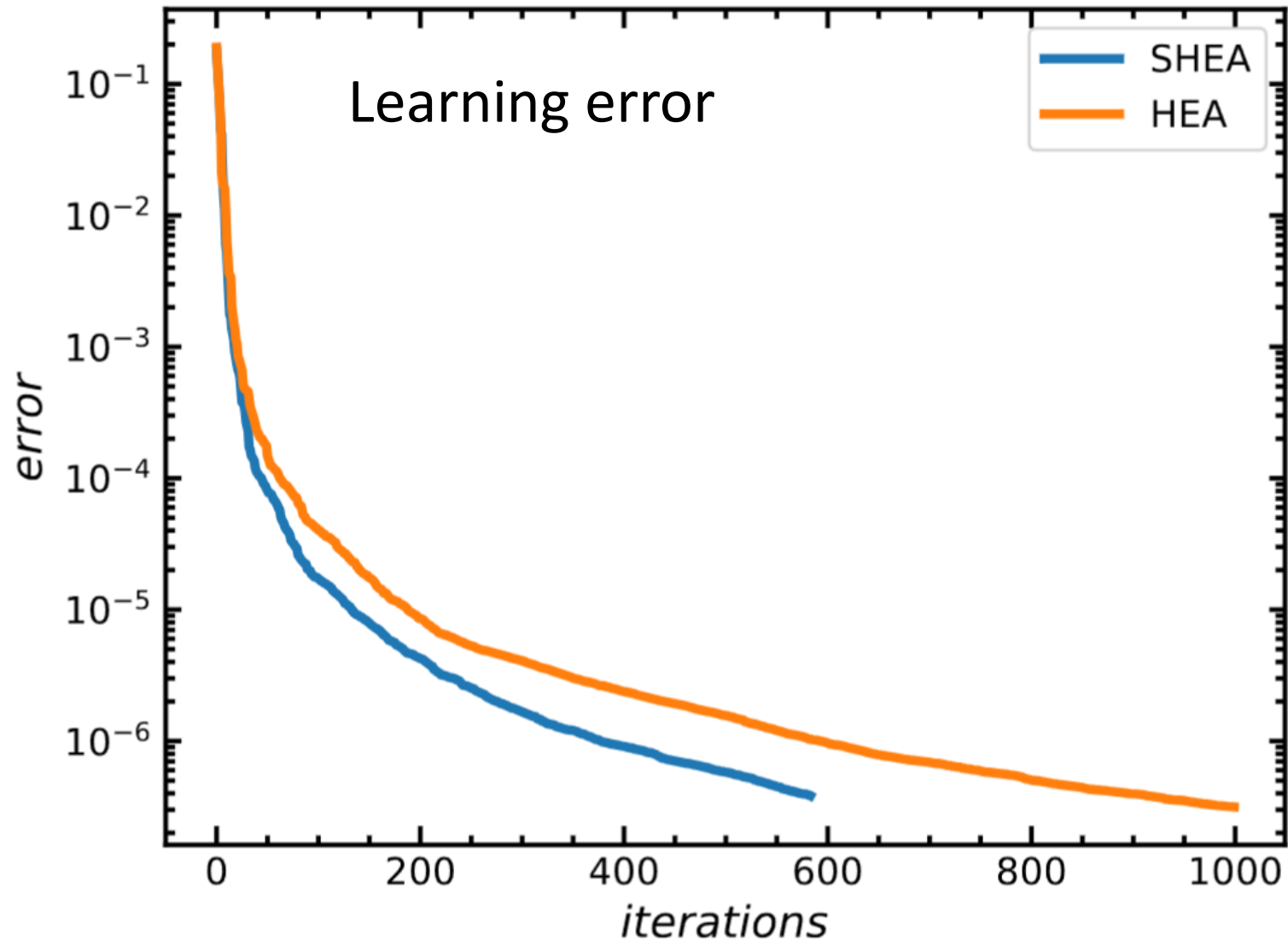


Population error



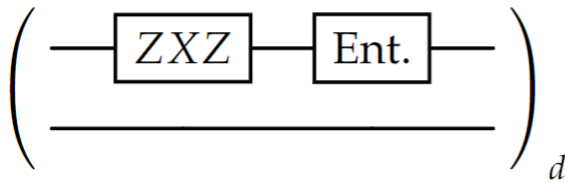
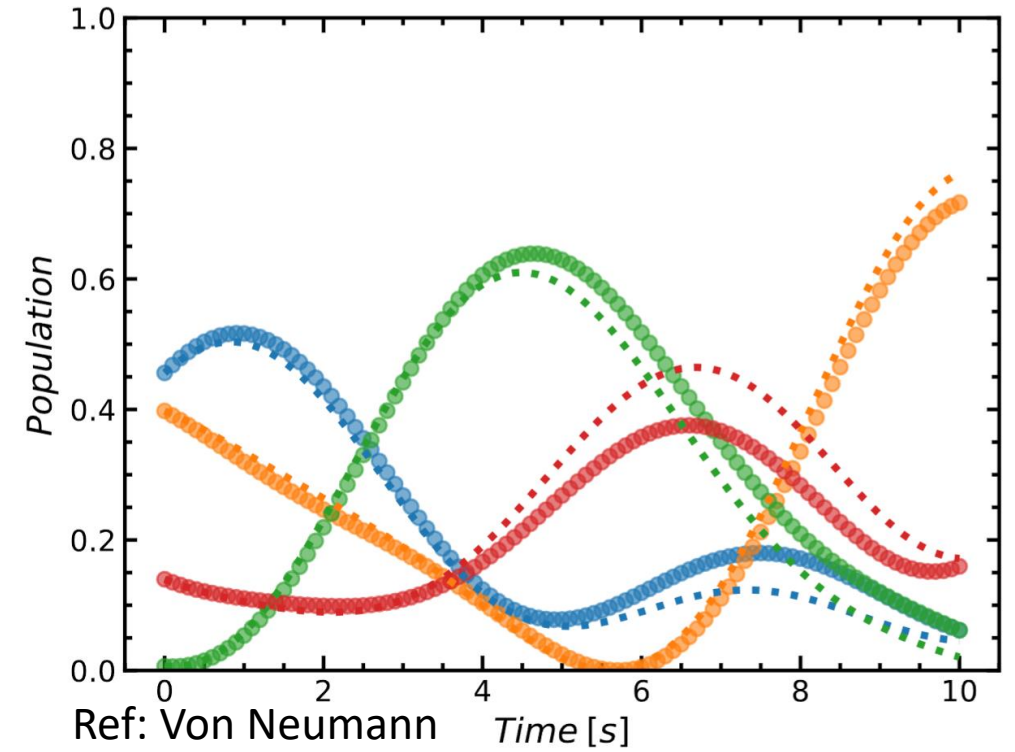
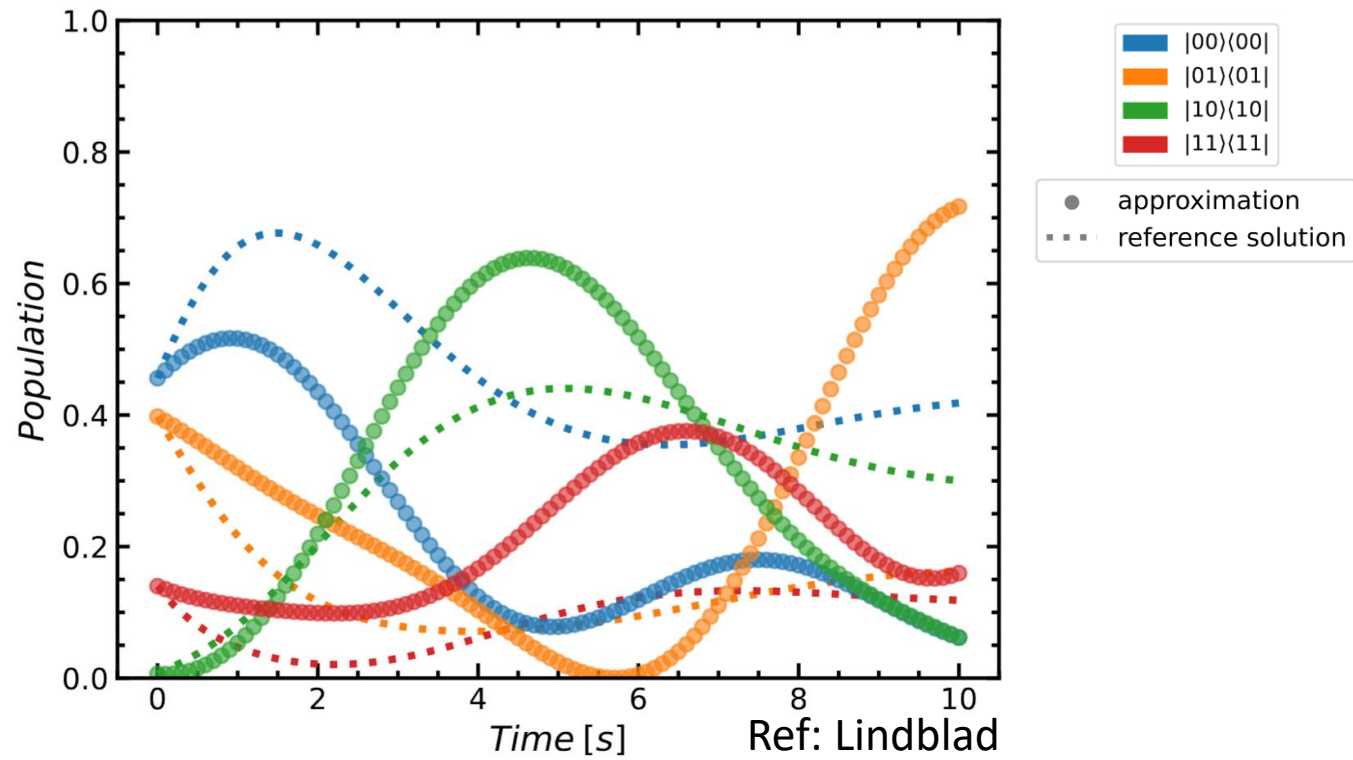
Results: approximating Rabi oscillation (one qubit)

35



Results: approximating Rabi oscillation (two qubits)

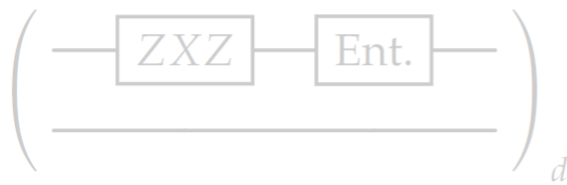
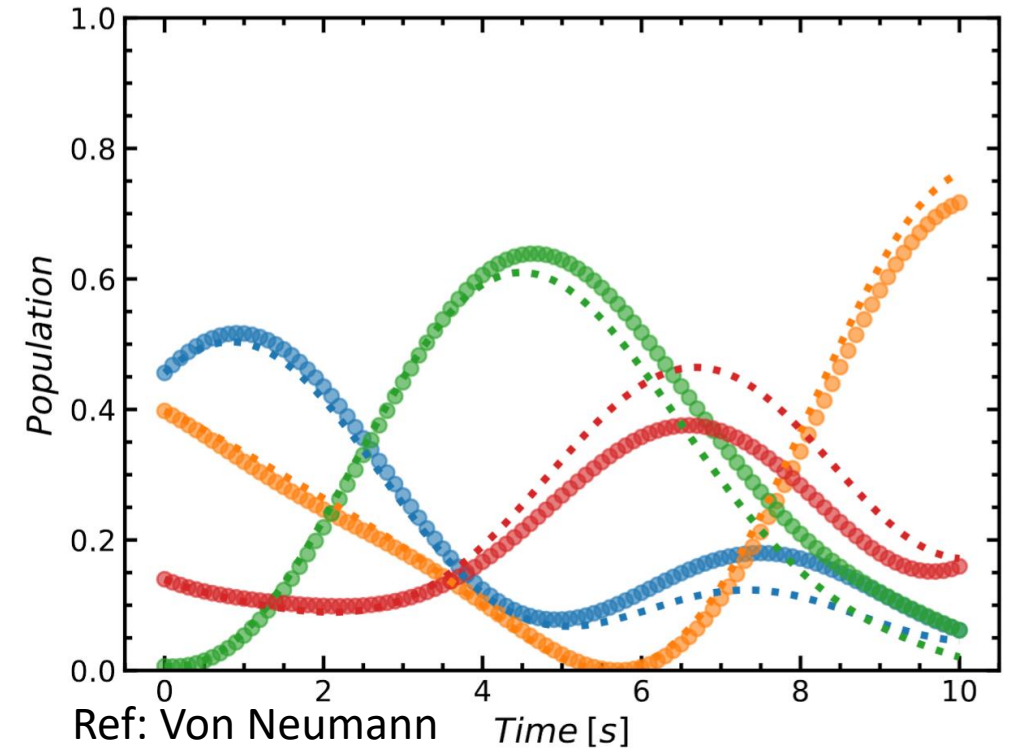
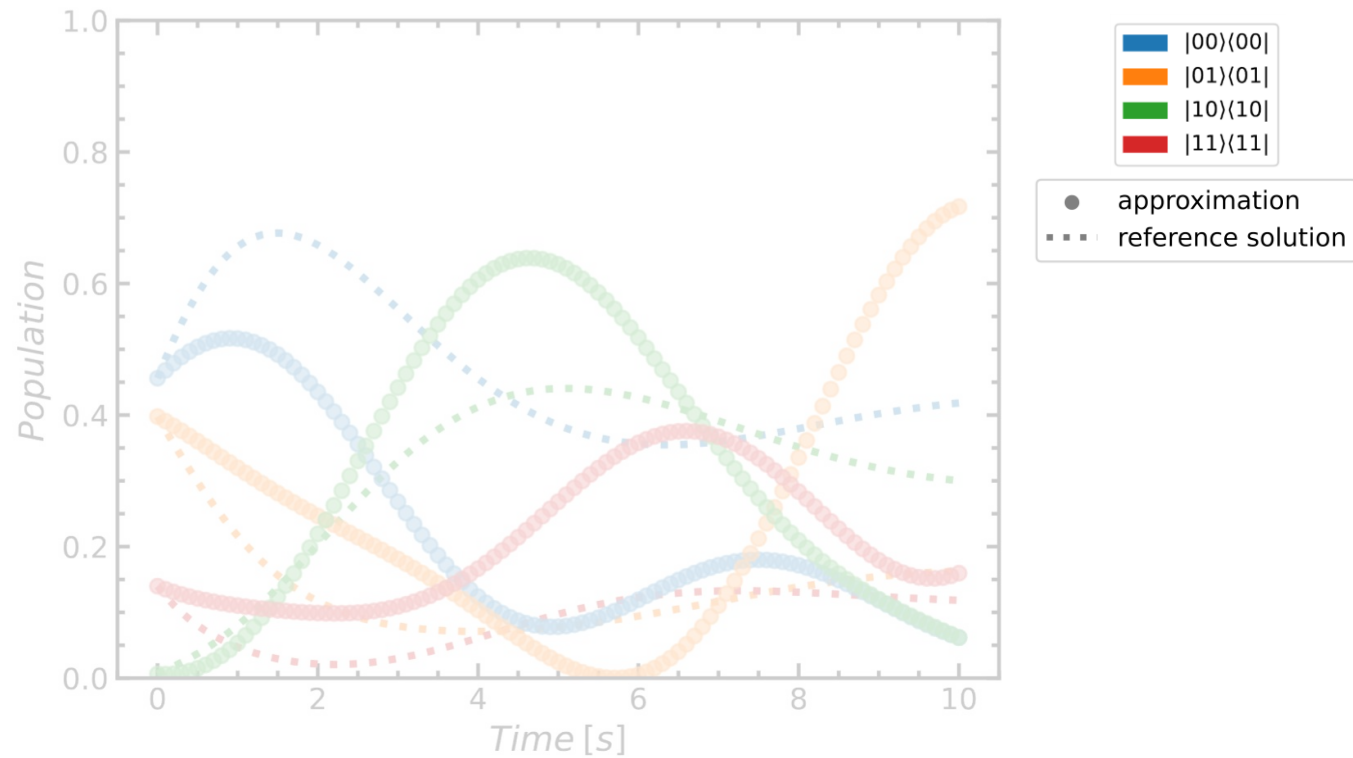
36



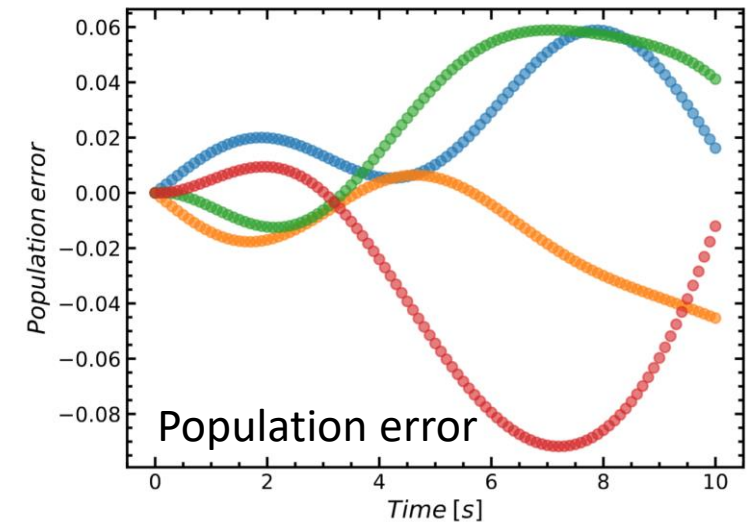
HEA

Results: approximating Rabi oscillation (two qubits)

37

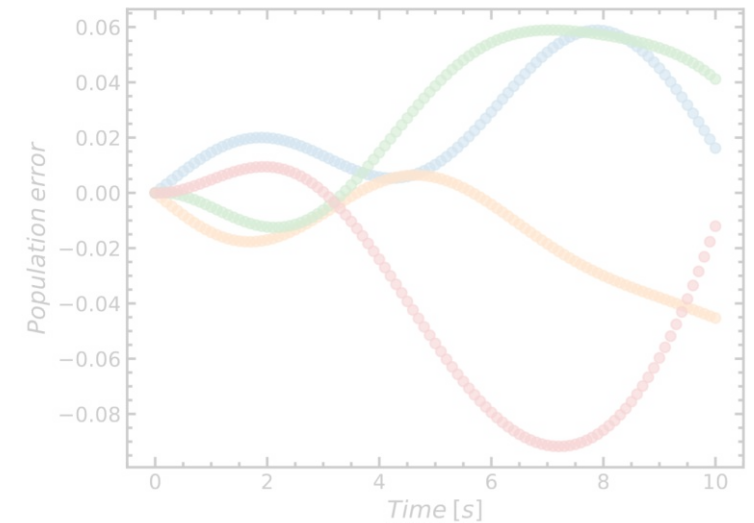
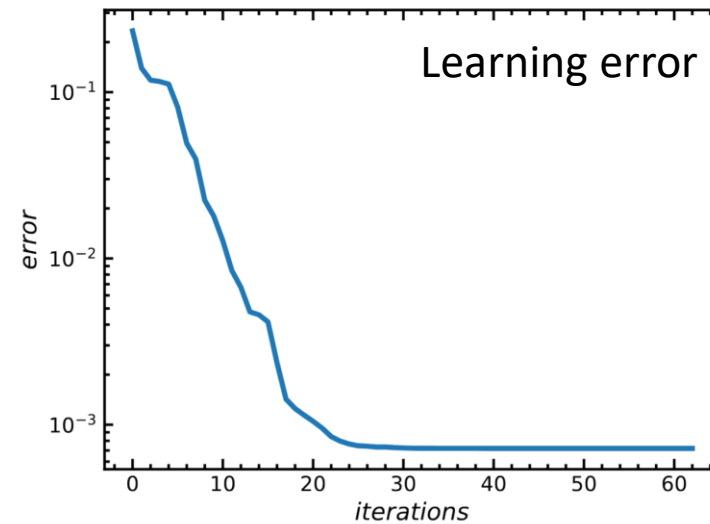
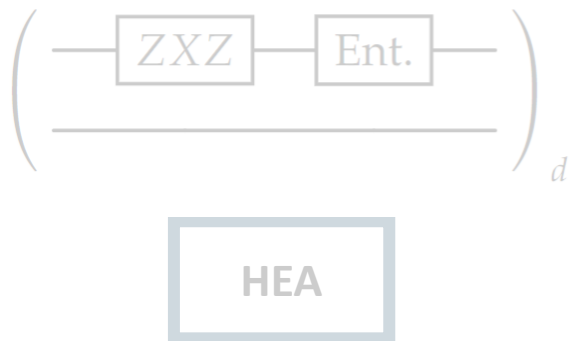
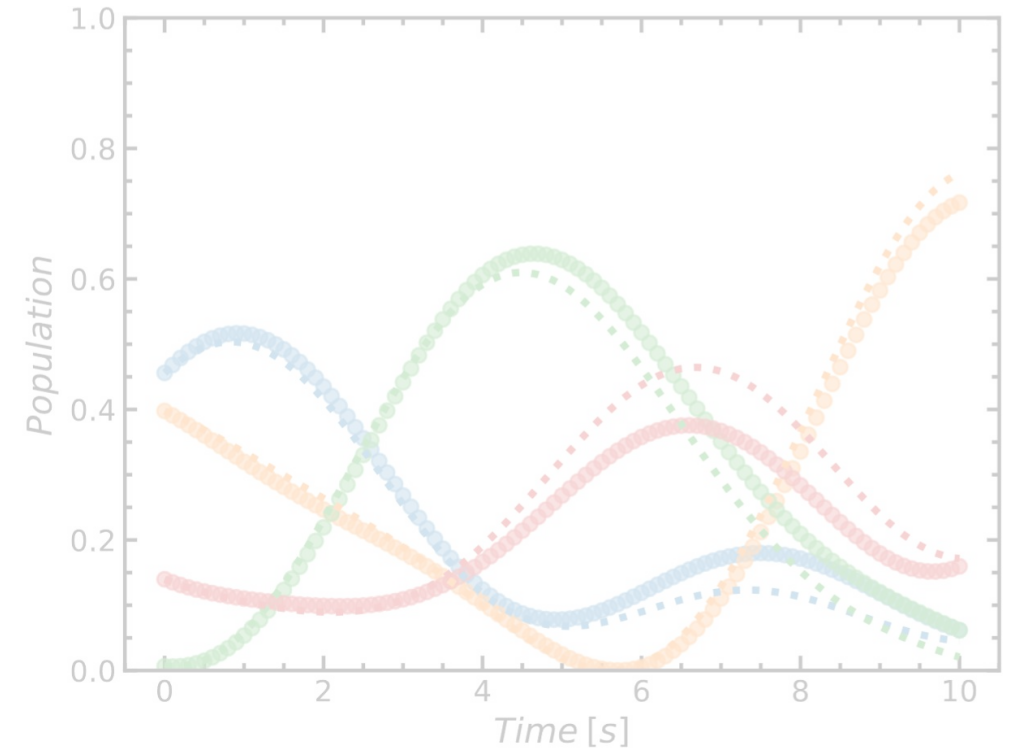
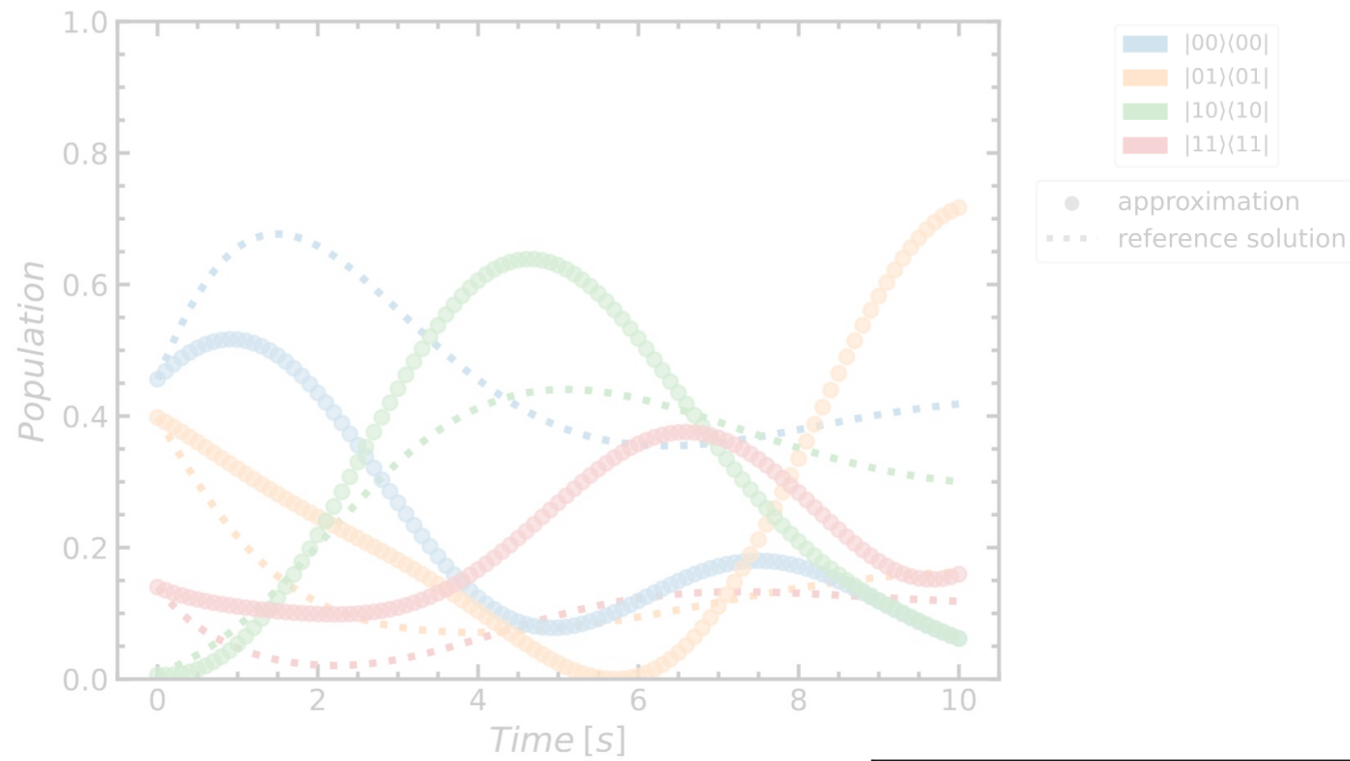


HEA



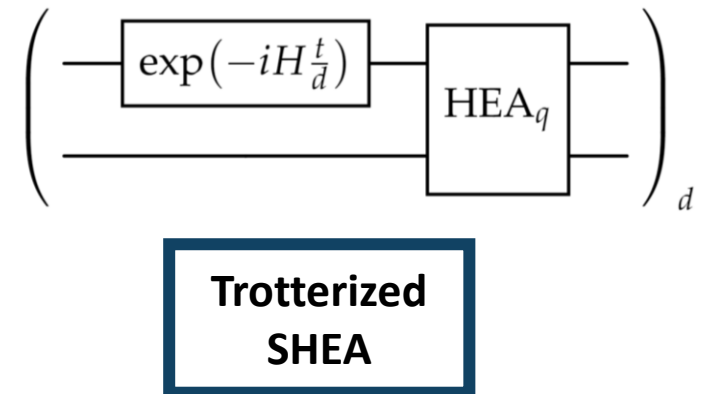
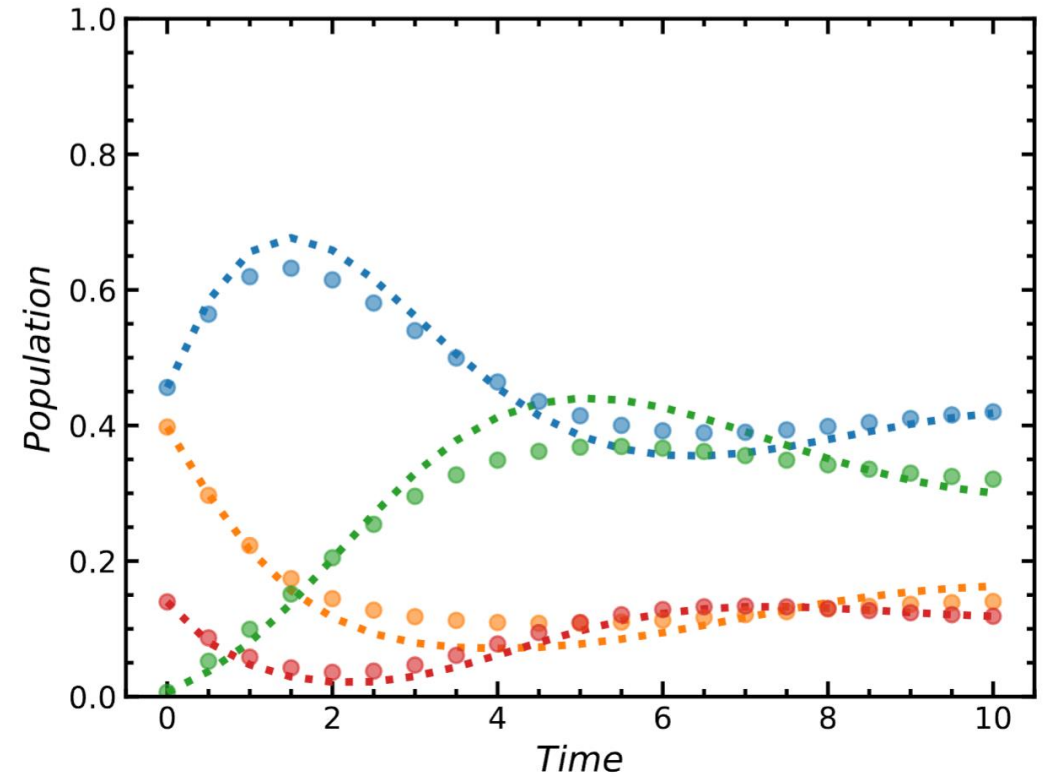
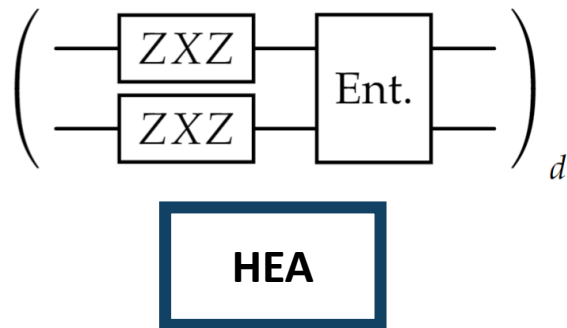
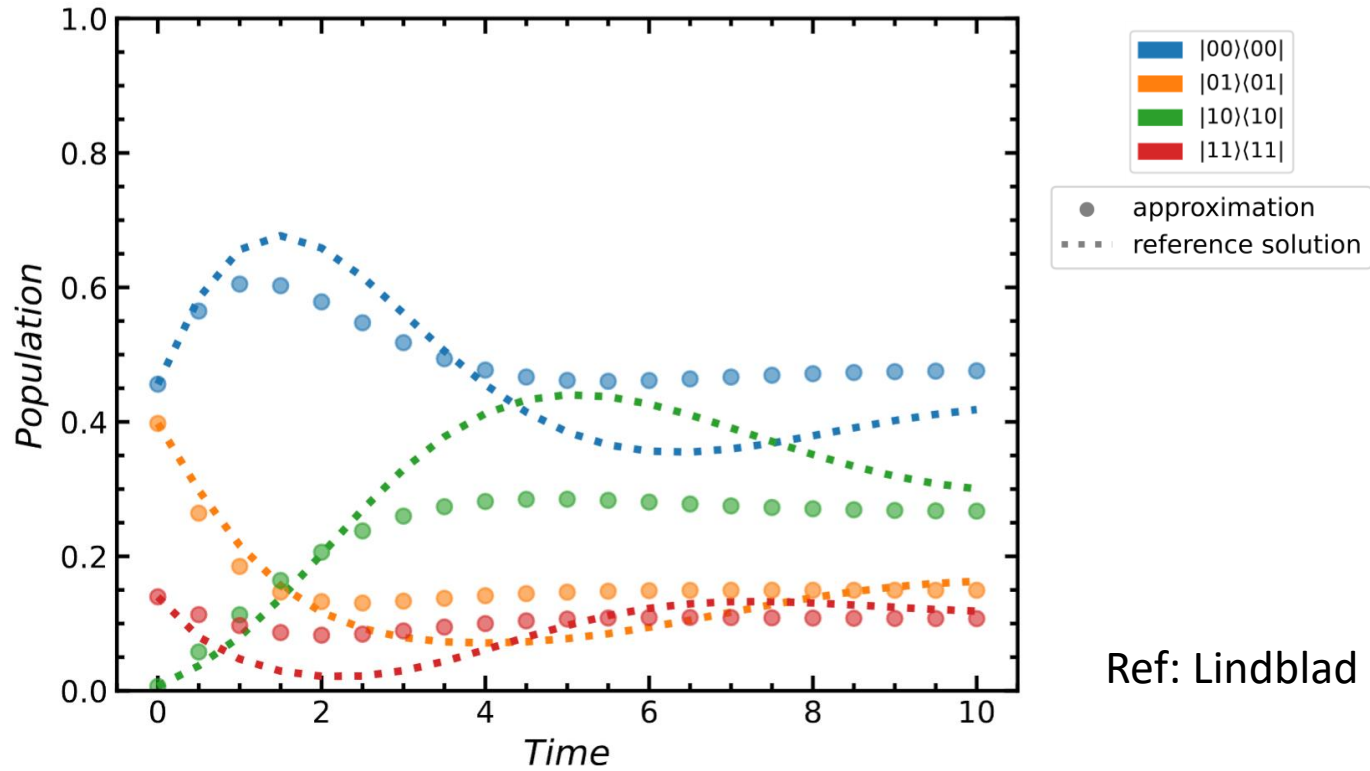
Results: approximating Rabi oscillation (two qubits)

38



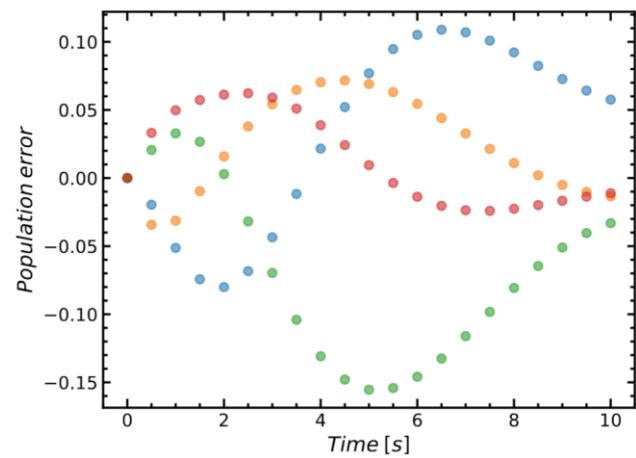
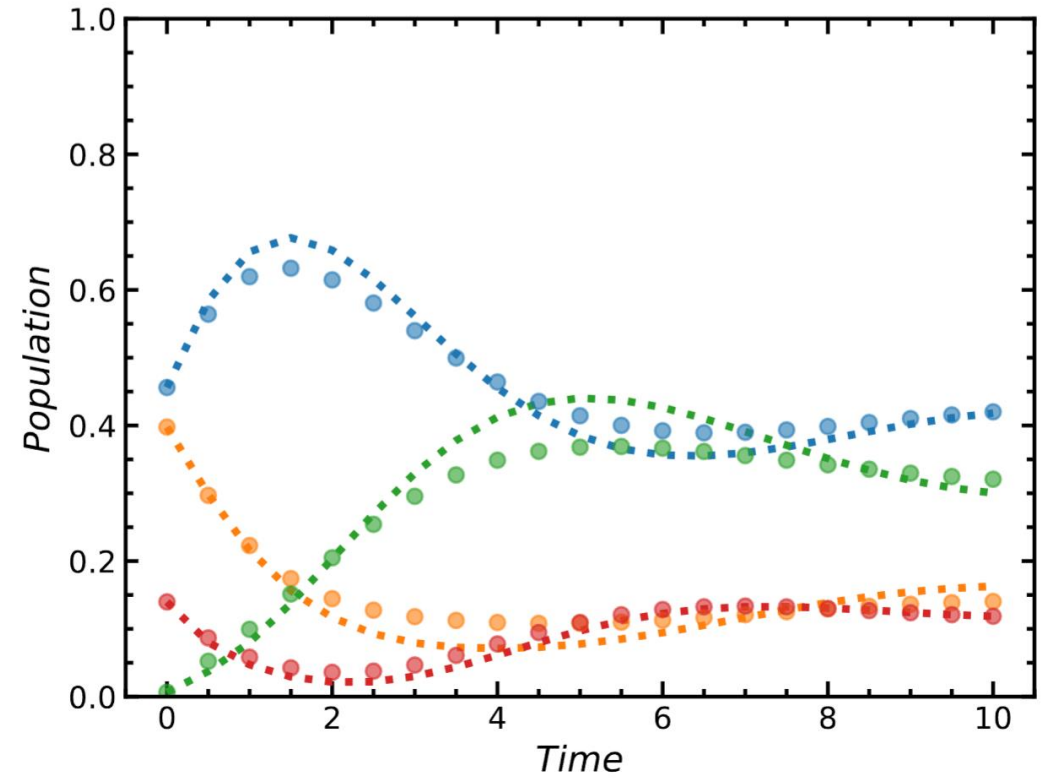
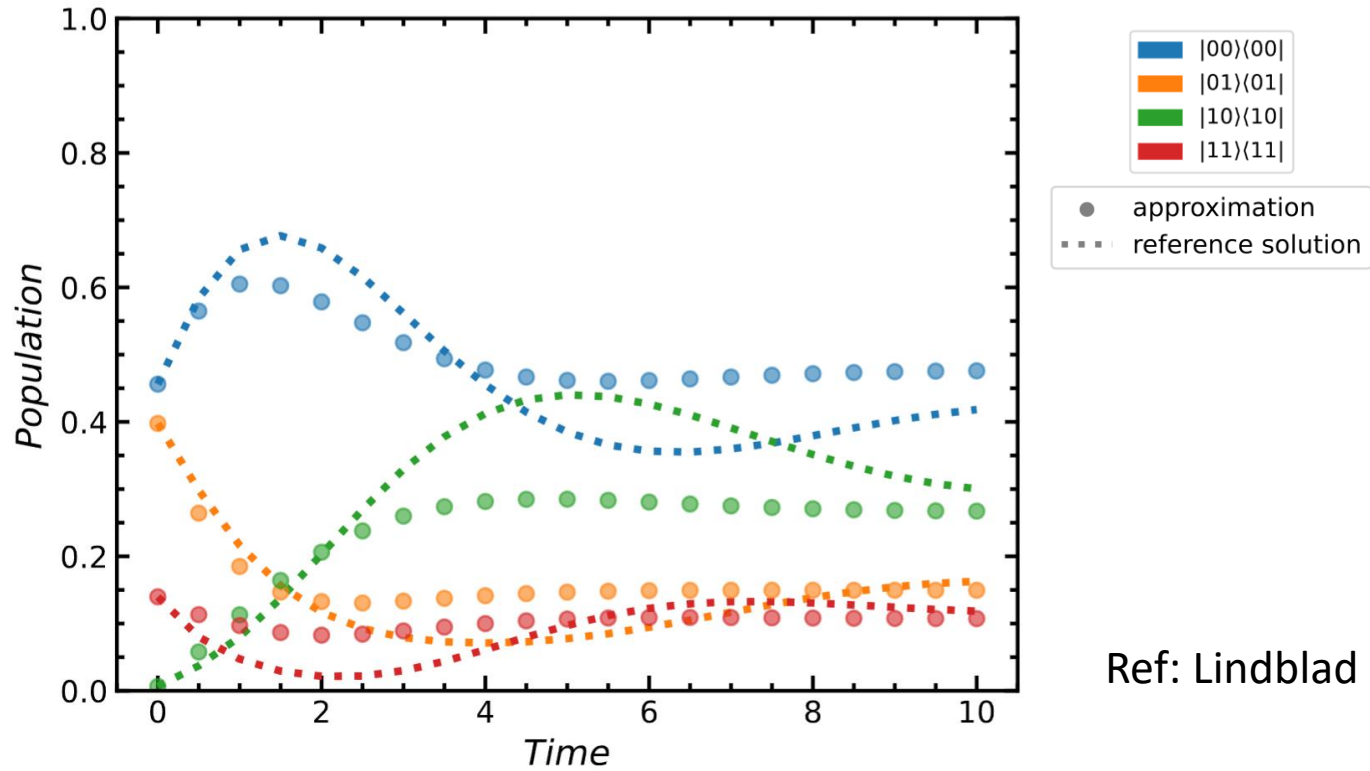
Results: approximating Rabi oscillation (two qubits)

39

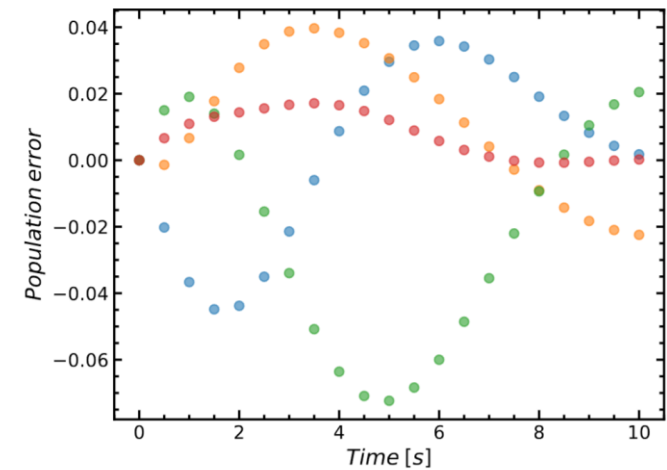


Results: approximating Rabi oscillation (two qubits)

40

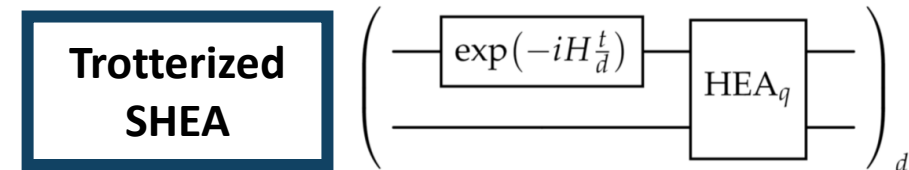
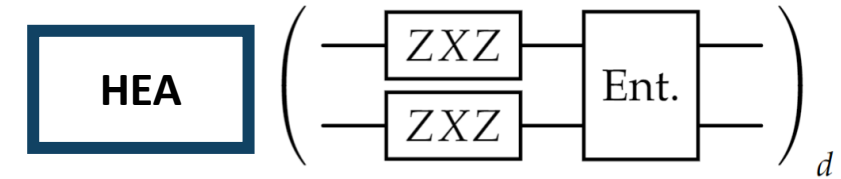
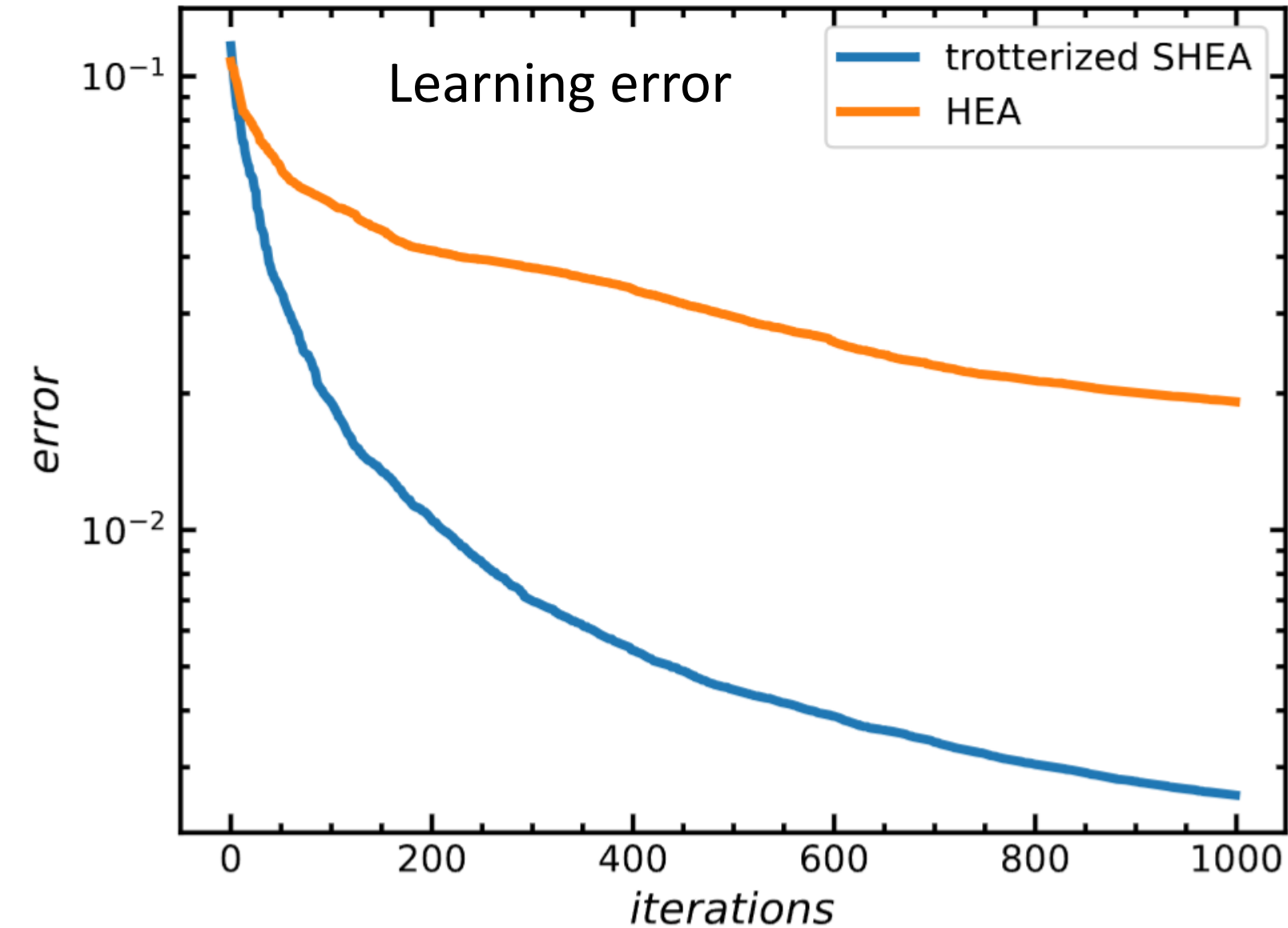


Population error



Results: approximating Rabi oscillation (two qubits)

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Future research

- **How much training data is needed?**
- **What is the relation between depth and expressibility?**
- **Implement on quantum computer**