EI 1

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Suppose that $\boldsymbol{W}_1$ and $\boldsymbol{W}_2$ are, respectively, $n \times l_1$ and $n \times l_2$ matrices of instruments, and that $\boldsymbol{W}_2$ consists of $\boldsymbol{W}_1$ plus $l_2 - l_1$ additional columns. Prove that the generalized IV estimator using $\boldsymbol{W}_2$ is asymptotically more efficient than the generalized IV estimator using $\boldsymbol{W}_1$. To do this you need to show that the matrix $(\boldsymbol{X}'\boldsymbol{P}_{\boldsymbol{W}_1}\boldsymbol{X})^{-1} - (\boldsymbol{X}'\boldsymbol{P}_{\boldsymbol{W}_2}\boldsymbol{X})^{-1}$ is positive semidefinite. Prove the result by first proving $\boldsymbol{P}_{\boldsymbol{W}_2} = \boldsymbol{P}_{\boldsymbol{W}_1} + \boldsymbol{P}_{\{(\boldsymbol{I}-\boldsymbol{P}_{\boldsymbol{W}_1})\boldsymbol{W}_3\}}$, where $\boldsymbol{W}_3$ contains the columns of $\boldsymbol{W}_2$ not contained in $\boldsymbol{W}_1$.

Solution Exercise 18

By Lemma 3 of the syllabus we have that $\boldsymbol{P}_{\boldsymbol{W}_2} = \boldsymbol{P}_{\boldsymbol{W}_1} + \boldsymbol{P}_{\{(\boldsymbol{I}-\boldsymbol{P}_{\boldsymbol{W}_1})\boldsymbol{W}_3\}}$.

Since by result 37 of the Matrix algebra syllabus for any projection matrix $\boldsymbol{P}_{\boldsymbol{A}}$, where the arbitrary matrix $\boldsymbol{A}$ has full column rank, the following holds $\boldsymbol{0} \leq \boldsymbol{P}_{\boldsymbol{A}} \leq \boldsymbol{I}$. This implies

$$\boldsymbol{P}_{\boldsymbol{W}_2} = \boldsymbol{P}_{\boldsymbol{W}_1} + \boldsymbol{P}_{\{(\boldsymbol{I}-\boldsymbol{P}_{\boldsymbol{W}_1})\boldsymbol{W}_3\}} \geq \boldsymbol{P}_{\boldsymbol{W}_1} \geq \boldsymbol{O}.$$

If we apply assumption 10 and 11 of the syllabus we have that

$$\frac{\boldsymbol{W}'\boldsymbol{X}}{n} \xrightarrow{p} \boldsymbol{S}_{\boldsymbol{W}'\boldsymbol{X}}, \quad \text{f.c.r.}$$

and

$$\frac{\boldsymbol{W}'\boldsymbol{W}}{n} \xrightarrow{p} \boldsymbol{S}_{\boldsymbol{W}'\boldsymbol{W}} > \boldsymbol{O},$$

with $\boldsymbol{W}$ a matrix of instruments.
So, by result 26 of the matrix algebra syllabus we get

$$\boldsymbol{X}'\boldsymbol{P}_{\boldsymbol{W}}\boldsymbol{X} = n\left(\frac{\boldsymbol{W}'\boldsymbol{X}}{n}\right)'\left(\frac{\boldsymbol{W}'\boldsymbol{W}}{n}\right)^{-1}\left(\frac{\boldsymbol{W}'\boldsymbol{X}}{n}\right) > \boldsymbol{O}.$$

Subsequently, if we apply result 29 of the matrix algebra syllabus we can write

$$\boldsymbol{X}'\boldsymbol{P}_{\boldsymbol{W}_2}\boldsymbol{X} \geq \boldsymbol{X}'\boldsymbol{P}_{\boldsymbol{W}_1}\boldsymbol{X} > \boldsymbol{O} \Leftrightarrow (\boldsymbol{X}'\boldsymbol{P}_{\boldsymbol{W}_1}\boldsymbol{X})^{-1} \geq (\boldsymbol{X}'\boldsymbol{P}_{\boldsymbol{W}_2}\boldsymbol{X})^{-1} > \boldsymbol{O}.$$

As a result we get

$$(\boldsymbol{X}'\boldsymbol{P}_{\boldsymbol{W}_1}\boldsymbol{X})^{-1} - (\boldsymbol{X}'\boldsymbol{P}_{\boldsymbol{W}_2}\boldsymbol{X})^{-1} \geq \boldsymbol{O} \tag{1}$$

is positive semidefinite.

Since the asymptotic variance of the generalized IV estimator is given by

$$\sigma^2 \text{plim} \left(\frac{\boldsymbol{X}'\boldsymbol{P}_{\boldsymbol{W}}\boldsymbol{X}}{n} \right)^{-1} \quad \text{as} \quad n \to \infty,$$

with σ^2 the population variance, and by using (1), we can conclude that the generalized IV estimator using $\boldsymbol{W}_2$ is asymptotically more efficient than the generalized IV estimator using $\boldsymbol{W}_1$.