2. The equations of motion for the two-body problem in inertial, Cartesian coordinates under the assumption of Keplerian motion are

$$\ddot{\boldsymbol{r}} = -\frac{\mu}{\|\boldsymbol{r}\|^3}\boldsymbol{r}.$$

Define two (vector) states to be the position, $\boldsymbol{r}$, and velocity, $\boldsymbol{v} = \dot{\boldsymbol{r}}$, of the object, and formulate a first-order nonlinear differential equation of the form $\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}(t))$.

$$x(t) = \begin{bmatrix} \underline{r} \\ \underline{v} \end{bmatrix} \qquad \text{where } \underline{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \underline{v} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix}$$

$$\boxed{\dot{x}(t) = \begin{bmatrix} \dot{\underline{r}} \\ \dot{\underline{v}} \end{bmatrix} = \begin{bmatrix} \underline{v} \\ -\frac{\mu}{\|\underline{r}\|^3}\underline{r} \end{bmatrix}} \qquad (1)$$

3. Determine the Jacobian of $\boldsymbol{f}(\boldsymbol{x}(t))$, given by $\boldsymbol{F}(\boldsymbol{x}(t))$.

The Jacobian of $\underline{f}(\underline{x}(t))$ is obtained by taking the partials of $\underline{f}(\underline{x}(t))$ with respect to the vector states. Using vector calculus we can obtain $\underline{F}(\underline{x}(t))$

$$\underline{F}(\underline{x}(t)) = \begin{bmatrix} \frac{\partial \dot{\underline{r}}}{\partial \underline{r}} & \frac{\partial \dot{\underline{r}}}{\partial \underline{v}} \\ \frac{\partial \dot{\underline{v}}}{\partial \underline{r}} & \frac{\partial \dot{\underline{v}}}{\partial \underline{v}} \end{bmatrix} = \begin{bmatrix} \frac{\partial \underline{v}}{\partial \underline{r}} & \frac{\partial \underline{v}}{\partial \underline{v}} \\ \frac{\partial \dot{\underline{v}}}{\partial \underline{r}} & \frac{\partial \dot{\underline{v}}}{\partial \underline{v}} \end{bmatrix}$$

Clearly from (1), $\underline{v}$ has no dependence on $\underline{r}$ and thus $\frac{\partial \underline{v}}{\partial \underline{r}} = \underline{0}_{3,3}$. The same can be said for $\frac{\partial \dot{\underline{v}}}{\partial \underline{v}}$, and thus $\frac{\partial \dot{\underline{v}}}{\partial \underline{r}} = \underline{0}_{3,3}$. Also, $\frac{\partial \underline{v}}{\partial \underline{v}}$ is clearly identity; $\frac{\partial \underline{v}}{\partial \underline{v}} = I_3$

Thus the only term that we must labor to compute is the matrix $\frac{\partial \dot{\underline{v}}}{\partial \underline{r}}$.

$$\frac{\partial \dot{\underline{v}}}{\partial \underline{r}} = \frac{\partial}{\partial \underline{r}}\left[-\frac{\mu}{\|\underline{r}\|^3}\underline{r}\right] = -\frac{\mu}{\|\underline{r}\|^3}\frac{\partial}{\partial \underline{r}}\left[\underline{r}\right] + \underline{r}\frac{\partial}{\partial \underline{r}}\left[-\frac{\mu}{\|\underline{r}\|^3}\right]$$

$$= -\frac{\mu}{\|\underline{r}\|^3} I_3 + \underline{r}\left[\frac{3\mu}{\|\underline{r}\|^4}\right] \cdot \underbrace{\frac{\partial}{\partial \underline{r}}\left[\|\underline{r}\|\right]}_{\frac{\underline{r}^T}{\|\underline{r}\|}}$$

$$= -\frac{\mu}{\|\underline{r}\|^3} I_3 + \frac{3\mu\,\underline{r}\,\underline{r}^T}{\|\underline{r}\|^5}$$

$$\frac{\partial \dot{\underline{v}}}{\partial \underline{r}} = \underline{A}(t) = \mu\left[\frac{3\underline{r}\,\underline{r}^T}{\|\underline{r}\|^5} - \frac{I_3}{\|\underline{r}\|^3}\right]$$

$$\boxed{\underline{F}(\underline{x}(t)) = \begin{bmatrix} \underline{0}_{3,3} & I_3 \\ \underline{A}(t) & \underline{0}_{3,3} \end{bmatrix}}$$

4. Given the position of an observer, $\boldsymbol{r}_{\text{obs}}$, and the position of the object, $\boldsymbol{r}$, the right-ascension and declination angles may be computed as

$$\alpha = \tan^{-1}\frac{y}{x}$$
$$\delta = \tan^{-1}\frac{z}{\sqrt{x^2+y^2}},$$

where x, y, and z are the components of the relative position vector $\boldsymbol{r}-\boldsymbol{r}_{\text{obs}}$. If $\boldsymbol{h}(\boldsymbol{x}) = [\alpha \quad \delta]^T$, determine the Jacobian of $\boldsymbol{h}(\boldsymbol{x})$, given by $\tilde{\boldsymbol{H}}(\boldsymbol{x})$.

$$\tilde{H}(x) = \left[\frac{\partial \underline{h}(x)}{\partial \underline{x}}\right] \quad \text{where} \quad \underline{x} = \begin{bmatrix} x \\ y \\ z \\ \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} \quad \text{and} \quad \underline{h}(x) = [\alpha \;\; \delta]^T$$

$$\text{and } \alpha = \tan^{-1}\left(\frac{y}{x}\right)$$
$$\delta = \tan^{-1}\left(\frac{z}{\sqrt{x^2+y^2}}\right)$$

$$\tilde{H}(x) = \begin{bmatrix} \frac{\partial h_1}{\partial x_1} & \frac{\partial h_1}{\partial x_2} & \cancel{\frac{\partial h_1}{\partial x_3}} \, 0 & \cancel{\frac{\partial h_1}{\partial x_4}} \, 0 & \cancel{\frac{\partial h_1}{\partial x_5}} \, 0 & \cancel{\frac{\partial h_1}{\partial x_6}} \, 0 \\ \frac{\partial h_2}{\partial x_1} & \frac{\partial h_2}{\partial x_2} & \frac{\partial h_2}{\partial x_3} & \cancel{\frac{\partial h_2}{\partial x_4}} \, 0 & \cancel{\frac{\partial h_2}{\partial x_5}} \, 0 & \cancel{\frac{\partial h_2}{\partial x_6}} \, 0 \end{bmatrix}$$

$$\frac{\partial h_1}{\partial x_1} = \frac{\partial \alpha}{\partial x} = \frac{\partial}{\partial x}\left(\tan^{-1}\left(\frac{y}{x}\right)\right) = \frac{1}{\left(\frac{y}{x}\right)^2+1}\frac{\partial}{\partial x}\left(\frac{y}{x}\right) = -\frac{y}{y^2+x^2}$$

$$\frac{\partial h_1}{\partial x_2} = \frac{\partial \alpha}{\partial y} = \frac{\partial}{\partial y}\left(\tan^{-1}\left(\frac{y}{x}\right)\right) = \frac{1}{\left(\frac{y}{x}\right)^2+1}\frac{\partial}{\partial y}\left(\frac{y}{x}\right) = \frac{x}{y^2+x^2}$$

$$\frac{\partial h_2}{\partial x_1} = \frac{\partial \delta}{\partial x} = \frac{\partial}{\partial x}\left(\tan^{-1}\left(\frac{z}{\sqrt{x^2+y^2}}\right)\right) = \frac{1}{\left(\frac{z}{\sqrt{x^2+y^2}}\right)^2+1} \cdot \underbrace{\frac{\partial}{\partial x}\left[\frac{z}{\sqrt{x^2+y^2}}\right]}_{-\frac{1}{\cancel{2}}\frac{\cancel{2}xz}{(x^2+y^2)^{3/2}}}$$

$$= \frac{1}{\left(\frac{z}{\sqrt{x^2+y^2}}\right)^2+1} \cdot \frac{-xz}{(x^2+y^2)^{3/2}}$$

$$= \frac{1}{\frac{z^2}{x^2+y^2}+1} \cdot \frac{-xz}{(x^2+y^2)^{3/2}} = \frac{x^2+y^2}{z^2+x^2+y^2}\,\frac{-xz}{(x^2+y^2)^{3/2}}$$

$$= \frac{-xz}{(z^2+x^2+y^2)\sqrt{x^2+y^2}}$$

Similarly, $\dfrac{\partial h_2}{\partial x_2} = \dfrac{-yz}{(z^2+x^2+y^2)\sqrt{x^2+y^2}}$

$$\frac{\partial h_2}{\partial x_3} = \frac{\partial}{\partial z}\left[\tan^{-1}\left(\frac{z}{\sqrt{x^2+y^2}}\right)\right] = \frac{1}{\left(\frac{z}{\sqrt{x^2+y^2}}\right)^2+1}\cdot(x^2+y^2)^{-1/2} = \frac{\sqrt{x^2+y^2}}{z^2+x^2+y^2}$$

Collecting our partial terms...

$$\boxed{\tilde{H}(x) = \begin{bmatrix} -\dfrac{y}{y^2+x^2} & \dfrac{x}{y^2+x^2} & 0 & 0 & 0 & 0 \\ \dfrac{-xz}{(z^2+x^2+y^2)\sqrt{x^2+y^2}} & \dfrac{-yz}{(z^2+x^2+y^2)\sqrt{x^2+y^2}} & \dfrac{\sqrt{x^2+y^2}}{z^2+x^2+y^2} & 0 & 0 & 0 \end{bmatrix}}$$