AERO 626 Homework #1

(50 points)

1. Using the integral definition for the mean and variance of a probability density function (pdf), i.e.,

$$m = \int_{-\infty}^{\infty} x p(x) \mathrm{d}x \quad \text{and} \quad P = \int_{-\infty}^{\infty} (x-m)^2 p(x) \mathrm{d}x \,,$$

determine the mean and variance of a uniform pdf of the form

$$p(x) = \begin{cases} \dfrac{1}{b-a}\,, & a < x < b \\ 0\,, & \text{otherwise} \end{cases} .$$

First, let's look at the mean:

$$\begin{aligned} m &= \int_{-\infty}^{\infty} x p(x) \mathrm{d}x = \frac{1}{b-a} \int_a^b x \mathrm{d}x = \frac{1}{b-a} \left[\frac{1}{2} x^2 \right]_{x=a}^{x=b} \\ &= \frac{1}{2} \cdot \frac{1}{b-a} (b^2 - a^2) = \frac{1}{2} \cdot \frac{(b+a)(b-a)}{b-a} = \frac{b+a}{2} \end{aligned}$$

Now, for the variance. First, note that

$$P = \int_{-\infty}^{\infty} (x-m)^2 p(x) dx = \int_{-\infty}^{\infty} x^2 p(x) \mathrm{d}x - m^2$$

Then, substituting for the form of the uniform pdf and the previously determined mean, it follows that

$$\begin{aligned} P &= \frac{1}{b-a} \int_a^b x^2 p(x) \mathrm{d}x - \frac{(b+a)^2}{4} = \frac{1}{b-a} \left[\frac{1}{3} x^3 \right]_{x=a}^{x=b} - \frac{(b+a)^2}{4} = \frac{1}{3} \cdot \frac{b^3 - a^3}{b-a} - \frac{(b+a)^2}{4} \\ &= \frac{1}{12} \cdot \frac{4b^3 - 4a^3 - 3(b+a)^2(b-a)}{b-a} = \frac{1}{12} \cdot \frac{b^3 - 3ab^2 + 3a^2 b - a^3}{b-a} = \frac{1}{12} \cdot \frac{(b-a)^3}{b-a} \\ &= \frac{(b-a)^2}{12} \end{aligned}$$

2. In Matlab, a "standard" Gaussian random number can be generated according to

```
x = randn;
```

This can be generalized to the case of generating a Gaussian random number with mean m and variance P via

```
x = m + sqrt(P)*randn;
```

Similarly, a "standard" uniform random number (with $a = 0$ and $b = 1$) can be generated according to

```
y = rand;
```

and the generalized version (for $a < b$) is given by

```
y = a + (b - a)*rand;
```

Both randn and rand can be used to generate n random numbers in a single call by supplying optional arguments, i.e.,

```
x = m + sqrt(P)*randn(n,1);
y = a + (b - a)*rand(n,1);
```

Generate 1×10^6 uniform random numbers with $a = 0$ and $b = 1$. Using the MATLAB functions mean and cov, compute the "sample mean" and "sample (co)variance" of the uniform random numbers. Repeat the process for 1×10^6 Gaussian random numbers with $m = \frac{1}{2}$ and $P = \frac{1}{12}$. What do you notice?

The MATLAB code should look something like

```
n = 1e6;
a = 0.0;
b = 1.0;
m = 0.5;
P = 1.0/12.0;

x = m + sqrt(P)*randn(n,1);
y = a + (b - a)*rand(n,1);

mx = mean(x);
Px = cov(x);

my = mean(y);
Py = cov(y);
```

Depending upon the random number seed, results will vary, but they should be something along the lines of

$$m_x = 0.5000\,, \quad P_x = 0.0832\,, \quad m_y = 0.5000\,, \quad \text{and} \quad P_y = 0.0835\,.$$

You should notice that the mean and variance of the uniform random numbers are in very close agreement to the mean and variance of the Gaussian random numbers, even though the distributions look very different. Mean and variance are *not* unique to the Gaussian pdf.

3. For the line-fitting problem discussed in class, it was found that $m = 2$ measurements given by z_1 and z_2, taken at times t_1 and t_2, respectively, leads to the least-squares solution

$$\hat{a} = \frac{z_1 t_2 - z_2 t_1}{t_2 - t_1} \quad \text{and} \quad \hat{b} = \frac{z_2 - z_1}{t_2 - t_1}\,.$$

Show that the residuals, $\epsilon_1 = z_1 - h_1$ and $\epsilon_2 = z_2 - h_2$, where $h_i = \hat{a} + \hat{b}t_i$ are both zero for the case of fitting a line through two measurements.

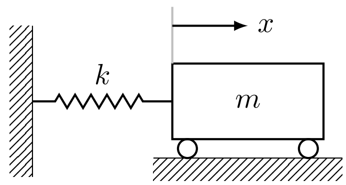
Using the definition of the first residual and substituting for the model evaluated at t_1 using the least-squares estimate of the parameters, it follows that

$$\begin{aligned}
\epsilon_1 = z_1 - [\hat{a} + \hat{b}t_1] &= z_1 - \frac{z_1 t_2 - z_2 t_1}{t_2 - t_1} - \frac{z_2 - z_1}{t_2 - t_1} \cdot t_1 = z_1 - \frac{z_1 t_2 - z_2 t_1}{t_2 - t_1} - \frac{z_2 t_1 - z_1 t_1}{t_2 - t_1} \\
&= z_1 - \frac{z_1 t_2 - z_2 t_1 + z_2 t_1 - z_1 t_1}{t_2 - t_1} = z_1 - \frac{z_1 t_2 - z_1 t_1}{t_2 - t_1} = z_1 - z_1 \cdot \frac{t_2 - t_1}{t_2 - t_1} = z_1 - z_1 = 0
\end{aligned}$$

A similar procedure applied to the second residual yields

$$\begin{aligned}\epsilon_2 &= z_2 - [\hat{a} + \hat{b}t_2] = z_2 - \frac{z_1 t_2 - z_2 t_1}{t_2 - t_1} - \frac{z_2 - z_1}{t_2 - t_1} \cdot t_2 = z_2 - \frac{z_1 t_2 - z_2 t_1}{t_2 - t_1} - \frac{z_2 t_2 - z_1 t_2}{t_2 - t_1} \\ &= z_2 - \frac{z_1 t_2 - z_2 t_1 + z_2 t_2 - z_1 t_2}{t_2 - t_1} = z_2 - \frac{z_2 t_2 - z_2 t_1}{t_2 - t_1} = z_2 - z_2 \cdot \frac{t_2 - t_1}{t_2 - t_1} = z_2 - z_2 = 0\end{aligned}$$

4. Consider the unforced spring-mass system



which has the equation of motion

$$\ddot{x}(t) = -\omega_n^2 x(t) \quad \text{where} \quad \omega_n^2 = k/m\,.$$

Assume that $\omega_n = 1$ for this problem.

(a) Define two states to be $x_1(t) = x(t)$ and $x_2(t) = \dot{x}(t)$, and determine the matrix $\boldsymbol{F}(t)$ for the linear dynamic system $\dot{\boldsymbol{x}}(t) = \boldsymbol{F}(t)\boldsymbol{x}(t)$, where $\boldsymbol{x}(t) = [x_1(t) \;\; x_2(t)]^T$.

$$\dot{\boldsymbol{x}}(t) = \begin{bmatrix}\dot{x}_1(t) \\ \dot{x}_2(t)\end{bmatrix} = \begin{bmatrix}x_2(t) \\ -\omega_n^2 x_1(t)\end{bmatrix} = \begin{bmatrix}0 & 1 \\ -\omega_n^2 & 0\end{bmatrix}\begin{bmatrix}x_1(t) \\ x_2(t)\end{bmatrix} \quad \rightarrow \quad \boldsymbol{F}(t) = \boldsymbol{F} = \begin{bmatrix}0 & 1 \\ -\omega_n^2 & 0\end{bmatrix}$$

(b) Determine an analytic representation for the state transition matrix $\boldsymbol{\Phi}(t_i, t_0)$. *Hint: use the fact that this is a harmonic oscillator to find analytic solutions for $x_1(t)$ and $x_2(t)$ in terms of $x_1(t_0)$, $x_2(t_0)$, and ω_n and then deduce the state transition matrix from these solutions. You may assume that $t_0 = 0$.*
Since this is a harmonic oscillator, we expect the solution at time t_i to take the form

$$x(t_i) = A\cos\omega_n t_i + B\sin\omega_n t_i\,.$$

Therefore, we see that the solutions for the two states $x_1(t_i) = x(t_i)$ and $x_2(t_i) = \dot{x}(t_i)$ take the forms

$$\begin{aligned}x_1(t_i) &= A\cos\omega_n t_i + B\sin\omega_n t_i \\ x_2(t_i) &= B\omega_n\cos\omega_n t_i - A\omega_n\sin\omega_n t_i\,.\end{aligned}$$

Given initial conditions as $x_1(t_0)$ and $x_2(t_0)$ for $t_0 = 0$, it follows that $A = x_1(t_0)$ and $B = x_2(t_0)/\omega_n$, which gives the solutions as

$$\begin{aligned}\begin{bmatrix}x_1(t_i) \\ x_2(t_i)\end{bmatrix} &= \begin{bmatrix}x_1(t_0)\cos\omega_n t_i + \frac{1}{\omega_n}x_2(t_0)\sin\omega_n t_i \\ x_2(t_0)\cos\omega_n t_i - x_1(t_0)\omega_n\sin\omega_n t_i\end{bmatrix} \\ &= \underbrace{\begin{bmatrix}\cos\omega_n t_i & \frac{1}{\omega_n}\sin\omega_n t_i \\ -\omega_n\sin\omega_n t_i & \cos\omega_n t_i\end{bmatrix}}_{\boldsymbol{\Phi}(t_i, t_0)}\begin{bmatrix}x_1(t_0) \\ x_2(t_0)\end{bmatrix}\end{aligned}$$

Let's try another method for determining the state transition matrix. First, we recognize that $\boldsymbol{F}(t) = \boldsymbol{F}$, i.e., it is a constant matrix, so we know that the state transition matrix can be generated according to the matrix exponential:

$$\boldsymbol{\Phi}(t_i, t_0) = \sum_{k=0}^{\infty} \frac{1}{k!}(\boldsymbol{F}t_i)^k = \boldsymbol{I} + \boldsymbol{F}t_i + \frac{1}{2}\boldsymbol{F}^2 t_i^2 + \frac{1}{3!}\boldsymbol{F}^3 t_i^3 + \frac{1}{4!}\boldsymbol{F}^4 t_i^4 + \cdots,$$

where we have made use of $t_0 = 0$ to simplify the preceding expression. Let's look at powers of $\boldsymbol{F}$:

$$\begin{aligned}
\boldsymbol{F} &= \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & 0 \end{bmatrix} \\
\boldsymbol{F}^2 = \boldsymbol{F}\boldsymbol{F} &= \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -\omega_n^2 & 0 \end{bmatrix} = \begin{bmatrix} -\omega_n^2 & 0 \\ 0 & -\omega_n^2 \end{bmatrix} \\
\boldsymbol{F}^3 = \boldsymbol{F}\boldsymbol{F}^2 &= \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & 0 \end{bmatrix}\begin{bmatrix} -\omega_n^2 & 0 \\ 0 & -\omega_n^2 \end{bmatrix} = \begin{bmatrix} 0 & -\omega_n^2 \\ \omega_n^4 & 0 \end{bmatrix} \\
\boldsymbol{F}^4 = \boldsymbol{F}\boldsymbol{F}^3 &= \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & 0 \end{bmatrix}\begin{bmatrix} 0 & -\omega_n^2 \\ \omega_n^4 & 0 \end{bmatrix} = \begin{bmatrix} \omega_n^4 & 0 \\ 0 & \omega_n^4 \end{bmatrix} \\
\boldsymbol{F}^5 = \boldsymbol{F}\boldsymbol{F}^4 &= \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & 0 \end{bmatrix}\begin{bmatrix} \omega_n^4 & 0 \\ 0 & \omega_n^4 \end{bmatrix} = \begin{bmatrix} 0 & \omega_n^4 \\ -\omega_n^6 & 0 \end{bmatrix} \\
\boldsymbol{F}^6 = \boldsymbol{F}\boldsymbol{F}^5 &= \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & 0 \end{bmatrix}\begin{bmatrix} 0 & \omega_n^4 \\ -\omega_n^6 & 0 \end{bmatrix} = \begin{bmatrix} -\omega_n^6 & 0 \\ 0 & -\omega_n^6 \end{bmatrix}
\end{aligned}$$

We could keep going, but we have enough terms to deduce the pattern that's coming out of the powers of $\boldsymbol{F}$. Let's look at the individual elements of the state transition matrix:

$$\begin{aligned}
\Phi_{(1,1)}(t_i, t_0) &= 1 - \frac{1}{2}\omega_n^2 t_i^2 + \frac{1}{4!}\omega_n^4 t_i^4 - \frac{1}{6!}\omega_n^6 t_i^6 + \cdots \\
\Phi_{(1,2)}(t_i, t_0) &= t_i - \frac{1}{3!}\omega_n^2 t_i^3 + \frac{1}{5!}\omega_n^4 t_i^5 + \cdots \\
\Phi_{(2,1)}(t_i, t_0) &= -\omega_n^2 t_i + \frac{1}{3!}\omega_n^4 t_i^3 - \frac{1}{5!}\omega_n^6 t_i^5 + \cdots \\
\Phi_{(2,2)}(t_i, t_0) &= 1 - \frac{1}{2}\omega_n^2 t_i^2 + \frac{1}{4!}\omega_n^4 t_i^4 - \frac{1}{6!}\omega_n^6 t_i^6
\end{aligned}$$

Let's take these expressions and modify the middle two slightly to make the powers of ω_n match the powers of t_i, such that

$$\begin{aligned}
\Phi_{(1,1)}(t_i, t_0) &= 1 - \frac{1}{2}\omega_n^2 t_i^2 + \frac{1}{4!}\omega_n^4 t_i^4 - \frac{1}{6!}\omega_n^6 t_i^6 + \cdots \\
\Phi_{(1,2)}(t_i, t_0) &= \frac{1}{\omega_n}\Big[\omega_n t_i - \frac{1}{3!}\omega_n^3 t_i^3 + \frac{1}{5!}\omega_n^5 t_i^5 - \cdots\Big] \\
\Phi_{(2,1)}(t_i, t_0) &= -\omega_n\Big[\omega_n t_i - \frac{1}{3!}\omega_n^3 t_i^3 + \frac{1}{5!}\omega_n^5 t_i^5 - \cdots\Big] \\
\Phi_{(2,2)}(t_i, t_0) &= 1 - \frac{1}{2}\omega_n^2 t_i^2 + \frac{1}{4!}\omega_n^4 t_i^4 - \frac{1}{6!}\omega_n^6 t_i^6
\end{aligned}$$

At this point, we note that the series expansions for sine and cosine are given by

$$\cos x = 1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots$$
$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots$$

Making the substitution $x = \omega_n t_i$, it is clearly seen that the elements of the state transition matrix are

$$\Phi_{(1,1)}(t_i, t_0) = \cos \omega_n t_i$$
$$\Phi_{(1,2)}(t_i, t_0) = \frac{1}{\omega_n} \sin \omega_n t_i$$
$$\Phi_{(2,1)}(t_i, t_0) = -\omega_n \sin \omega_n t_i$$
$$\Phi_{(2,2)}(t_i, t_0) = \cos \omega_n t_i$$

These are clearly the same expressions found using the first approach.

(c) Assuming that the position at time t_i, $x(t_i)$, can be observed, determine $\tilde{\boldsymbol{H}}_i$, where the measurement model is $\boldsymbol{h}_i = \tilde{\boldsymbol{H}}_i \boldsymbol{x}_i$.
The measurements are simply of the position, which is the first state, so we can write

$$h_i = x(t_i) = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{\tilde{\boldsymbol{H}}_i} \begin{bmatrix} x_1(t_i) \\ x_2(t_i) \end{bmatrix}$$