AERO 626 HOMEWORK #1

(50 points)

1. Using the integral definition for the mean and variance of a probability density function (pdf), i.e.,

$$m = \int_{-\infty}^{\infty} x p(x) \mathrm{d}x \quad \text{and} \quad P = \int_{-\infty}^{\infty} (x - m)^2 p(x) \mathrm{d}x \,,$$

determine the mean and variance of a uniform pdf of the form

$$p(x) = \begin{cases} \dfrac{1}{b-a} \,, & a < x < b \\ 0 \,, & \text{otherwise} \end{cases} .$$

2. In MATLAB, a "standard" Gaussian random number can be generated according to

```
x = randn;
```

This can be generalized to the case of generating a Gaussian random number with mean m and variance P via

```
x = m + sqrt(P)*randn;
```

Similarly, a "standard" uniform random number (with $a = 0$ and $b = 1$) can be generated according to

```
y = rand;
```

and the generalized version (for $a < b$) is given by

```
y = a + (b - a)*rand;
```

Both `randn` and `rand` can be used to generate n random numbers in a single call by supplying optional arguments, i.e.,

```
x = m + sqrt(P)*randn(n,1);
y = a + (b - a)*rand(n,1);
```

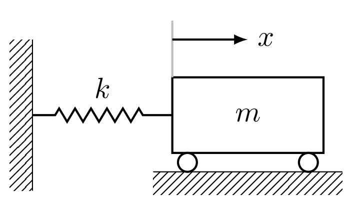
Generate 1×10^6 uniform random numbers with $a = 0$ and $b = 1$. Using the MATLAB functions `mean` and `cov`, compute the "sample mean" and "sample (co)variance" of the uniform random numbers. Repeat the process for 1×10^6 Gaussian random numbers with $m = \frac{1}{2}$ and $P = \frac{1}{12}$. What do you notice?

3. For the line-fitting problem discussed in class, it was found that $m = 2$ measurements given by z_1 and z_2, taken at times t_1 and t_2, respectively, leads to the least-squares solution

$$\hat{a} = \frac{z_1 t_2 - z_2 t_1}{t_2 - t_1} \quad \text{and} \quad \hat{b} = \frac{z_2 - z_1}{t_2 - t_1} \,.$$

Show that the residuals, $\epsilon_1 = z_1 - h_1$ and $\epsilon_2 = z_2 - h_2$, where $h_i = \hat{a} + \hat{b} t_i$ are both zero for the case of fitting a line through two measurements.

4. Consider the unforced spring-mass system



which has the equation of motion

$$\ddot{x}(t) = -\omega_n^2 x(t) \quad \text{where} \quad \omega_n^2 = k/m\,.$$

Assume that $\omega_n = 1$ for this problem.

(a) Define two states to be $x_1(t) = x(t)$ and $x_2(t) = \dot{x}(t)$, and determine the matrix $\boldsymbol{F}(t)$ for the linear dynamic system $\dot{\boldsymbol{x}}(t) = \boldsymbol{F}(t)\boldsymbol{x}(t)$, where $\boldsymbol{x}(t) = [x_1(t) \;\; x_2(t)]^T$.

(b) Determine an analytic representation for the state transition matrix $\boldsymbol{\Phi}(t_i, t_0)$. *Hint: use the fact that this is a harmonic oscillator to find analytic solutions for $x_1(t)$ and $x_2(t)$ in terms of $x_1(t_0)$, $x_2(t_0)$, and ω_n and then deduce the state transition matrix from these solutions. You may assume that $t_0 = 0$.*

(c) Assuming that the position at time t_i, $x(t_i)$, can be observed, determine $\tilde{\boldsymbol{H}}_i$, where the measurement model is $\boldsymbol{h}_i = \tilde{\boldsymbol{H}}_i \boldsymbol{x}_i$.