2. Consider the linear system described by the state-space model

$$\boldsymbol{x}_k = \boldsymbol{F}_{x,k-1}\boldsymbol{x}_{k-1} + \boldsymbol{w}_{k-1}$$
$$\boldsymbol{z}_k = \boldsymbol{H}_{x,k}\boldsymbol{x}_k + \boldsymbol{v}_k \,,$$

where the process noise is taken to be zero-mean, Gaussian with covariance $\boldsymbol{P}_{ww,k-1}$ and the measurement noise is taken to be zero-mean, Gaussian with covariance $\boldsymbol{P}_{vv,k}$. Given the posterior density at t_{k-1} to be

$$p(\boldsymbol{x}_{k-1} \,|\, \boldsymbol{z}_{1:k-1}) = \sum_{\ell=1}^{L^+_{x,k-1}} w^{(\ell)+}_{x,k-1} p_g(\boldsymbol{x}_{k-1}; \boldsymbol{m}^{(\ell)+}_{x,k-1}, \boldsymbol{P}^{(\ell)+}_{xx,k-1}) \,,$$

determine a closed-form expression for the *a priori* mean and covariance of $\boldsymbol{x}_k$, i.e., $\boldsymbol{m}^-_{x,k}$ and $\boldsymbol{P}^-_{xx,k}$. Express your answers solely in terms of $\boldsymbol{F}_{x,k-1}$, $\boldsymbol{P}_{ww,k-1}$, and the mean and covariance of the posterior density at t_{k-1}, i.e., $\boldsymbol{m}^+_{x,k-1}$ and $\boldsymbol{P}^+_{xx,k-1}$. Comment on your findings, especially with respect to other uncertainty propagation schemes.

Given : $p(\underline{x}_{k-1} \,|\, \underline{z}_{1:k-1}) = \sum_{\ell=1}^{L^+_{x,k-1}} \omega^+_{x,k-1} \, p_g(\underline{x}_{k-1} ; \underline{m}^+_{x,k-1} , P^+_{xx,k-1}) \longrightarrow p^+(\underline{x}_{k-1})$

$\Rightarrow$ We want to propagate via Chapman-Kolmogorov $\longrightarrow p^-(\underline{x}_k)$

System Model (zero-mean Gaussian Noise)

$$\left.\begin{aligned} \underline{x}_k &= F_{x,k-1} \underline{x}_{k-1} + \underline{\omega}_{k-1} \\ \underline{z}_k &= H_{x,k} \underline{x}_k + \underline{v}_k \end{aligned}\right\} \quad \begin{aligned} p(\underline{x}_k \,|\, \underline{x}_{k-1}) &= p_g(\underline{x}_k ; F_{x,k-1} \underline{x}_{k-1} , P_{ww,k-1}) \\ p(\underline{z}_k \,|\, \underline{x}_k) &= p_g(\underline{z}_k ; H_{x,k} \underline{x}_k , P_{vv,k}) \end{aligned}$$

Applying the Chapman-Kolmogorov Equation, we can simplify the expression...

$$p(\underline{x}_k \,|\, \underline{z}_{1:k-1}) = \int p_g(\underline{x}_k ; F_{x,k-1} \underline{x}_{k-1} , P_{ww,k-1}) \sum_{\ell=1}^{L^+_{x,k-1}} \omega^{(\ell)+}_{x,k-1} \, p_g(\underline{x}_{k-1} ; \underline{m}^{(\ell)+}_{x,k-1} , P^{(\ell)+}_{xx,k-1}) \, d\underline{x}_{k-1}$$

$$= \sum_{\ell=1}^{L^+_{x,k-1}} \omega^{(\ell)+}_{x,k-1} \int p_g(\underline{x}_k ; F_{x,k-1} \underline{x}_{k-1} , P_{ww,k-1}) \, p_g(\underline{x}_{k-1} ; \underline{m}^{(\ell)+}_{x,k-1} , P^{(\ell)+}_{xx,k-1}) \, d\underline{x}_{k-1}$$

$$= \sum_{\ell=1}^{L^+_{x,k-1}} \omega^{(\ell)+}_{x,k-1} \, p_g(\underline{x}_k ; F_{x,k-1} \underline{m}^{(\ell)+}_{x,k-1} , F_{x,k-1} P^{(\ell)+}_{xx,k} F^T_{x,k-1} + P_{ww,k-1})$$

Finally, we can determine the a priori mean and covariance of $\underline{x}_k$ by using the individual components of the GM.

$$\underline{m}^-_{x,k} = \int \underline{x} \, p(\underline{x}_k \,|\, \underline{z}_{1:k-1}) \, d\underline{x}$$

$$= \sum_{\ell=1}^{L^+_{x,k-1}} \omega^{(\ell)+}_{x,k-1} \int \underline{x}\, \mathcal{N}_g\left(\underline{x}_k ;\ F_{x,k-1}\, \underline{m}^{(\ell)+}_{x,k-1},\ F_{x,k-1} P^{(\ell)+}_{xx,k} F^T_{x,k-1} + P_{ww,k-1}\right) d\underline{x}$$

$$= \sum_{\ell=1}^{L^+_{x,k-1}} \omega^{(\ell)+}_{x,k-1} \left[F_{x,k-1}\, \underline{m}^{(\ell)+}_{x,k-1} \right]$$

$$\boxed{\underline{m}^-_{x,k} = F_{x,k-1}\, \underline{m}_{x,k-1}}$$

Similarly, for the a priori covariance,

$$P^-_{xx,k} = \mathbb{E}\left\{ (\underline{x}_k - \underline{m}^-_{x,k})(\underline{x}_k - \underline{m}^-_{x,k})^T \right\}$$

$$= \mathbb{E}\left\{ \underline{x}_k \underline{x}_k^T \right\} - \underline{m}^-_{x,k}\, \underline{m}^{-T}_{x,k}$$

$$= \int \underline{x}_k \underline{x}_k^T\, p(\underline{x}_k \mid \underline{z}_{1:k-1})\, d\underline{x} - \underline{m}^-_{x,k}\, \underline{m}^{-T}_{x,k}$$

$$= \int \underline{x}_k \underline{x}_k^T \left[\sum_{\ell=1}^{L^+_{x,k-1}} \omega^{(\ell)+}_{x,k-1}\, \mathcal{N}_g\left(\underline{x}_k ;\ F_{x,k-1}\, \underline{m}^{(\ell)+}_{x,k-1},\ F_{x,k-1} P^{(\ell)+}_{xx,k} F^T_{x,k-1} + P_{ww,k-1}\right) \right] d\underline{x} - \underline{m}^-_{x,k}\, \underline{m}^{-T}_{x,k}$$

$$= \sum_{\ell=1}^{L^+_{x,k-1}} \omega^{(\ell)+}_{x,k-1} \left[\left(F_{x,k-1} P^{(\ell)+}_{xx,k} F^T_{x,k-1} + P_{ww,k-1} \right) + \left(F_{x,k-1}\, \underline{m}^{(\ell)+}_{x,k-1} \right)\left(F_{x,k-1}\, \underline{m}^{(\ell)+}_{x,k-1} \right)^T \right] - \underline{m}^-_{x,k}\, \underline{m}^{-T}_{x,k}$$

Substituting from above...

$$= \sum_{\ell=1}^{L^+_{x,k-1}} \omega^{(\ell)+}_{x,k-1} \left[\left(F_{x,k-1} P^{(\ell)+}_{xx,k} F^T_{x,k-1} + P_{ww,k-1} \right) + \left(F_{x,k-1}\, \underline{m}^{(\ell)+}_{x,k-1} \right)\left(F_{x,k-1}\, \underline{m}^{(\ell)+}_{x,k-1} \right)^T - \left(F_{x,k-1}\, \underline{m}_{x,k-1} \right)\left(F_{x,k-1}\, \underline{m}_{x,k-1} \right)^T \right]$$

We can pull out $P^+_{xx,k-1}$ using the definition given on page 541, and the result is simply:

$$\boxed{P^-_{xx,k} = F_{x,k-1} P^+_{xx,k-1} F^T_{x,k-1} + P_{ww,k-1}}$$

The results for the a priori mean and covariance of $\underline{x}_k$ are no surprise. Since this system is linear and all the PDF's are Gaussian, we expect the mean and covariance to propagate exactly like a discrete Kalman filter, which is in fact exactly the result we obtain in the above solution.