1. Consider the noiseless, scalar, continuous-time dynamical system

$$\dot{x}(t) = 0$$

that is accompanied by scalar, discrete-time measurements of the form

$$z_k = x_k + v_k\,,$$

where v_k is a zero-mean measurement noise with constant variance P_{vv}. If the initial mean and variance of the state are known to be $m_{x,0}$ and $P_{xx,0}$, respectively, derive closed-form expressions for the Kalman gain and posterior variance at the ℓ^{th} update time, where $k = 1, 2, \ldots, \ell$. Your results should be expressed entirely in terms of $P_{xx,0}$, P_{vv}, and ℓ.

Given :

→ $\mathbb{E}\{\underline{v}_k\} = 0$

→ $P_{vv,k} = P_{vv} \quad \forall k$

→ $\mathbb{E}\{\underline{x}(t_0)\} = \underline{m}_{x,0}$

$\dot{\underline{x}}(t) = 0$, thus $\underline{x}(t) = \alpha$ where α is a constant, $\alpha \in \mathbb{R}$

Thus, $x_k = \alpha \quad \forall k$

Starting with the definition of the Kalman gain, given on page 205 of the notes :

$$K_k = (P_{xz,k}^-)(P_{zz,k}^-)^{-1}$$

where $\underline{z}_k = \underline{H}_{x,k}\underline{x}_k + H_{v,k}\underline{v}_k$

$$P_{xz,k}^- = P_{xx,k}^- \cancelto{1}{H_{x,k}^T}$$

$$P_{zz,k}^- = \cancelto{1}{H_{x,k}} P_{xx,k}^- \cancelto{1}{H_{x,k}^T} + \cancelto{1}{H_{v,k}} \cancelto{P_{vv}}{P_{vv,k}} \cancelto{1}{H_{v,k}^T}$$

Need to find $P_{xx,k}^-$ ....

$P_{xx,k}^-$ is found by propagating the covariance forward, under the propagation model described by:

$$\dot{P}_{xx}(t) = F_x(t)P_{xx}(t) + P_{xx}(t)F_x^T(x) + \cancelto{0}{F_w(t)Q_{ww}(t)F_w^T(t)}$$

(no modelled process noise)

Since $\dot{x}(t) = 0$, $F_x(t) = 0$, $\therefore$ $\dot{P}_{xx}(t) = 0$

To find the results at an arbitrary update time, ℓ lets find the pattern by starting with $k = 1$ ...

For $k=1$, $P_{xx,1}^- = P_{xx,0}$ because $\dot{P}_{xx}(t) = 0$

Then, $$\left.\begin{aligned} P_{xz,1}^- &= P_{xx,0} \\ P_{zz,1} &= P_{xx,0} + P_{vv} \end{aligned}\right\} \quad k_1 = \left(P_{xx,0}\right)\left(P_{xx,0} + P_{vv}\right)^{-1}$$

The state covariance is then updated according to the equation given on page 228 of the notes, which is valid for this case since we have linear measurements:

$$P_{xx,1}^+ = \left[I_2 - K_1 H\right] P_{xx,1}^-$$

$$= \left[I_2 - \left(P_{xx,0}\right)\left(P_{xx,0} + P_{vv}\right)^{-1}\right] P_{xx,0}$$

$$= \left[I - \frac{P_{xx,0}}{P_{xx,0} + P_{vv}}\right] P_{xx,0}$$

$$= \left[\frac{\cancel{P_{xx,0}} + P_{vv}}{P_{xx,0} + P_{vv}} - \frac{\cancel{P_{xx,0}}}{P_{xx,0} + P_{vv}}\right] P_{xx,0}$$

$$P_{xx,1}^+ = \frac{P_{vv} P_{xx,0}}{P_{xx,0} + P_{vv}}$$

The same procedure can be done with $k=2$, using $P_{xx,1}^+$ as $P_{xx,2}^-$

$$P_{xz,2}^- = P_{xx,2}^-$$

$$P_{zz,2}^- = P_{xx,2}^- + P_{vv}$$

Thus, $$k_2 = \left(P_{xx,2}^-\right)\left(P_{xx,2}^- + P_{vv}\right)^{-1} = \left(\frac{P_{vv} P_{xx,0}}{P_{xx,0} + P_{vv}}\right)\left(\frac{P_{vv} P_{xx,0}}{P_{xx,0} + P_{vv}} + P_{vv}\right)^{-1}$$

$$= \left(\frac{P_{vv} P_{xx,0}}{P_{xx,0} + P_{vv}}\right)\left(\frac{P_{vv} P_{xx,0}}{P_{xx,0} + P_{vv}} + \frac{P_{vv}\left(P_{xx,0} + P_{vv}\right)}{P_{xx,0} + P_{vv}}\right)^{-1}$$

$$= \left(\frac{P_{vv} P_{xx,0}}{P_{xx,0} + P_{vv}}\right)\left(\frac{P_{vv} P_{xx,0} + P_{vv}\left(P_{xx,0} + P_{vv}\right)}{P_{xx,0} + P_{vv}}\right)^{-1}$$

$$= \left(\frac{P_{vv} P_{xx,0}}{P_{xx,0} + P_{vv}}\right)\left(\frac{P_{xx,0} + P_{vv}}{P_{vv} P_{xx,0} + P_{vv}(P_{xx,0} + P_{vv})}\right)$$

$$= \frac{P_{vv} P_{xx,0}}{2 P_{xx,0} P_{vv} + P_{vv}^2}$$

$$K_2 = \frac{P_{xx,0}}{2 P_{xx,0} + P_{vv}}$$

And again, we can solve for $P_{xx,2}^{+}$ which will be $P_{xx,3}^{-}$ in the next iteration using

$$P_{xx,2}^{+} = \left[I_2 - K_2 H\right] P_{xx,2}^{-}$$

$$= \left[\frac{2 P_{xx,0} + P_{vv} - P_{xx,0}}{2 P_{xx,0} + P_{vv}}\right]\left[\frac{P_{vv} P_{xx,0}}{P_{xx,0} + P_{vv}}\right]$$

$$= \left[\frac{P_{xx,0} + P_{vv}}{2 P_{xx,0} + P_{vv}}\right]\left[\frac{P_{vv} P_{xx,0}}{P_{xx,0} + P_{vv}}\right]$$

$$P_{xx,2}^{+} = \left[\frac{P_{vv} P_{xx,0}}{2 P_{xx,0} + P_{vv}}\right]$$

Doing the same for $k=3$ will result in $K_3 = \dfrac{P_{xx,0}}{3 P_{xx,0} + P_{vv}}$, & $P_{xx,3}^{+} = \dfrac{P_{vv} P_{xx,0}}{3 P_{xx,0} + P_{vv}}$

This pattern holds for all $k = 1, 2, 3, \ldots$ and so for an arbitrary $k = \ell$ the kalman gain, and the posterior variance will be:

$$\boxed{K_\ell = \frac{P_{xx,0}}{\ell P_{xx,0} + P_{vv}}} \quad \text{and} \quad \boxed{P_{xx,\ell}^{+} = \frac{P_{vv} P_{xx,0}}{\ell P_{xx,0} + P_{vv}}}$$