Exam 2 - Problem 1

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a If we consider the following partitions of A and L, where A is a SPD matrix and L is the Cholesky factorization of A:

$$A = \left(\begin{array}{c|cc} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ A_{20} & a_{21} & A_{22} \end{array}\right), L = \left(\begin{array}{c|cc} L_{00} & 0 & 0 \\ \hline l_{10}^T & \lambda_{11} & 0 \\ L_{20} & l_{21} & L_{22} \end{array}\right)$$

We know that $A = LL^T$. This means, based on our partition:

$$A_{00} = L_{00}L_{00}^T$$

$$a_{10}^T = l_{10}^T L_{00}^T$$

$$a_{01} = L_{00}l_{10}$$

$$\alpha_{11} = l_{10}^T l_{10} + \lambda^2$$

Furthermore, because A is a SPD matrix, we also know that:

$$a_{01}^T = a_{10}^T$$

This means we can use either equation to solve for l_{10}^T. Assuming L_{00} is known, then we can find l_{10}^T by solving $a_{01} = L_{00}l_{10}$ for l_{10} and transposing the solution. Then, we can find λ_{11} by solving:

$$\alpha_{11} = l_{10}^T l_{10} + \lambda^2$$

$$\lambda^2 = \alpha_{11} - l_{10}^T l_{10}$$

$$\lambda = \sqrt{\alpha_{11} - l_{10}^T l_{10}}$$

Using all of these, we can create the following algorithm to find the Cholesky factorization of A:

$A \rightarrow \left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right)$
A_{TL} is 0×0
while $n(A_{TL}) < n(A)$ **do**

$\left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) \rightarrow \left(\begin{array}{c|cc} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ A_{20} & a_{21} & A_{22} \end{array}\right)$

Solve $L_{00} l_{10} = a_{01}$ for l_{10} and transpose the results, overwriting a_{10}^T

$\alpha_{11} := \lambda_{11} = \sqrt{\alpha_{11} - a_{10}^T a_{10}}$ (a_{10}^T has been overwritten with l_{10}^T)

$a_{01} := 0$

$\left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) \leftarrow \left(\begin{array}{cc|c} A_{00} & a_{01} & A_{02} \\ a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array}\right)$

end while

At the end of this algorithm, A will have been overwritten with L.

b Assume $n = 1$. Then It is trivial to prove that any SPD matrix has a Cholesky Factorization:

$$A = (\alpha), L = (\sqrt{\alpha})$$

$$x^T A x = x^2 \alpha > 0 \text{ for all } x \in \mathbb{R}$$

$$\alpha > 0$$

For any other matrix, we can partition as follows:

$$A = \left(\begin{array}{c|c} A_{00} & a_{01} \\ \hline a_{10}^T & \alpha_{11} \end{array}\right), L = \left(\begin{array}{c|c} L_{00} & 0 \\ \hline l_{10}^T & \lambda_{11} \end{array}\right)$$

Where L_{00} is the previously computed Cholesky factorization of A_{00}. l_{10}^T can be found by solving:

$$l_{10}^T = a_{10}^T L_{00}^{-T}$$

Because we know that A_{00} is a SPD matrix, we know the diagonal of $L_{00} > 0$ and L can therefore be inverted, making l_{10}^T well defined and unique. We can find λ_{11} with the following formula:

$$\lambda_{11} = \sqrt{\alpha_{11} - l_{10}^T l_{10}}$$

In order for λ_{11} to be a real number, $\alpha_{11} > l_{10}^T l_{10}$ must be true. This will be true for all SPD matrices.