Optimal Decision Trees

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Before we start

Without loss of generality, throughout this presentation, we will simplify the decision trees (DT for short) as follows.

- We will always talk about the classification problem rather than the regression problem, since there is no fundamental difference between them.
- We will not distinguish the binary DT from n-ary DT and will use both interchangeably, since any n-ary DT is equivalent to a binary DT of larger heights.

Overview I

Overview II

Current section

1. A quick review on DT
    - Tree construction algorithm: use ID3 as an example.
    - Prevent overfitting: make our DT “less suboptimal”.
    - Pros & cons of DT

A quick review on DT

Tree construction algorithm: use ID3 as an example.

```
1: $A \leftarrow$ the best attribute for splitting {examples}
2: Decision attribute for this node $\leftarrow A$
3: for value in $A$ do
4:    Create a new child node.
5: end for
6: Split {examples} into child nodes.
7: for all [child_node, {subset}] do
8:    if {subset} is pure then
9:       Stop
10:   else
11:      Split(child_node, {Subset})
12:   end if
13: end for
```

Algorithm 1: Split(node, {examples})

A quick review on DT

Tree construction algorithm: use ID3 as an example.

- We can use *Information Gain (IG)* to measure the purity of a node: suppose we have a set of examples S and we want to evaluate the effect of splitting the dataset using attribute A.

$$IG(S, A) := H(S) - \sum_{v \in \{-1,1\}} \frac{|S_v|}{|S|} H(S_v)$$

  where S_v is the subset of S with $A = v$, and the *entropy* is defined as

$$H(S) = - \sum_{v \in \{-1,1\}} p_v \log p_v$$

- The entropy, just like in physics, measures the degree of chaos in our probability distribution.

A quick review on DT

Prevent overfitting: make our DT "less suboptimal".

- Direction 1: optimize a single tree.
  - Stop splitting when it is not statistically significant.
  - Grow a full DT, and then post-prune it on the CV set.
  - If we use IG as our metric, the DT will by nature favoring the attributes which can divide the training set into a lot of small subsets (think of the index of each observation as an extreme example). In this case, we can use *Gain Ratio (GR)* instead of IG during the splits.

$$SplitEntropy(S, A) = -\sum_{v\in\{-1,1\}} \frac{|S_v|}{|S|}\log\frac{|S_v|}{|S|}$$

$$GR(S, A) := \frac{IG(S, A)}{SplitEntropy(S, A)}$$

  - Modify the DT to make the decision boundaries oblique.
  - Run an overall optimization instead of greedy algorithm.
- Direction 2: use law of large numbers.
  - Random forest

A quick review on DT

Pros & cons of DT

- Pros
  - Interpretable: humans can understand the decision process.
  - Can easily handle the noise attributes (since their $IG \approx 0$).
  - Can handle missing data.
  - Compact: After pruning, number of nodes $\ll$ size of the dataset.
  - Very fast in prediction: $O(h)$ time complexity.
- Cons
  - Only axis-aligned splits of data.
  - It is a greedy algorithm, and therefore cannot guarantee the optimality.

Current section

OC1: the big picture

- Just like ID3, we shall still use the greedy algorithm to recursively construct the tree.
- Therefore, the general framework of ID3 needs no change:
    1. Find the best "multivariate split".
    2. Divide our data into two subsets using the split function we found.
    3. For each child node, if it is not pure, repeat step 1 & 2.
    4. Post-prune the DT if needed.
- The only step which needs further discussion is step 1.

Hyperplane search strategy of OC1: perturbation

- The strategy of searching through the space of possible hyperplanes is that we consistently perturb the current hyperplane into a new location.
- Denote our sample space as P, which contains n examples, each with d attributes. Each example belongs to a particular category.
- The equation of the current hyperplane H can be written as $\sum_{i=1}^{d}(a_i x_i) + a_{d+1} = 0$.
- Let $P_j := (x_{j1}, x_{j2}, ..., x_{jd})$ be the jth observation from P. If we substitute P_j into the equation for H, we get $V_j = \sum_{i=1}^{d}(a_i x_{ij}) + a_{d+1}$.
- $\mathrm{sgn}(V_j)$ tells us whether P_j is above or below the hyperplane H.

Hyperplane search strategy of OC1: perturbation

- OC1 perturbs the coefficients of H one at a time.
- Denote the current attribute index we are perturbing is m $(1 \leq m \leq d)$.
- Since we perturb only one attribute at a time, all other coefficients can be seen as constants, and V_j can be viewed as a function of a_m.
- Define
$$U_j = \frac{a_m x_{jm} - V_j}{x_{jm}} \tag{1}$$
Then the point P_j is above H if $a_m > U_j$ and vice versa.
- Thus, by fixing the value of $a_1, a_2, ..., a_{d+1}$ except a_m, we can obtain n constraints on a_m, using n data points in P.
- The problem then is to find a value for a_m that satisfies as many of these constraints as possible.

Hyperplane search strategy of OC1: perturbation

```
1: for $j = 1$ to $n$ do
2:     Use (1) to compute $U_j$.
3: end for
4: Sort $U_1, U_2, ..., U_m$ in the increasing order. Then possible splits are chosen as
   $\frac{U_1+U_2}{2}, \frac{U_2+U_3}{2}, ..., \frac{U_{m-1}+U_m}{2}$.
5: $a_{m1}$ = best univariate split among all the candidates above
6: Let $H_1$ be the result of substituting $a_{m1}$ for $a_m$ in H.
7: if impurity$(H) >$ impurity$(H_1)$ then
8:     $a_m = a_{m1}$
9:     stagnant $= 1$
10: else if impurity$(H) =$ impurity$(H_1)$ then
11:     $a_m = a_{m1}$ with probability $= 1 - \exp(-stagnant)$
12:     stagnant $=$ stagnant $+ 1$
13: else
14:     Do not update
15: end if
```

Algorithm 2: Perturb(H, m)

Hyperplane search strategy of OC1: perturbation

In terms of which coefficient among $a_1, ..., a_{d+1}$ we shall choose, we use the following method called *R-50*.

```
1: for $i = 1$ to 50 do
2:     Generate a random integer $m$ ranging in $[1, d+1]$.
3:     Perturb($H$, $m$)
4: end for
```

Algorithm 3: R-50

Hyperplane search strategy of OC1: jump out of local minima.

- Since algorithm 2 is similar to SGD, one of its big drawbacks is that we may be frequently stuck in the local minima.
- We can deal with the problem in two ways:
  1. perturbing the hyperplane in a random direction;
  2. re-running the perturbation algorithm with additional intial hyperplaces.
- The second method is rather trivial, and we will only focus on the first one.

Hyperplane search strategy of OC1: jump out of local minima.

When a hyperplane $H = \sum_{i=1}^{d} a_i x_i + a_{d+1}$ cannot be improved by the deterministic perturbation, we do the following.

- Let $R = (r_1, r_2, ..., r_{d+1})$ be a random vector. Let α be the amount by which we want to perturb H in the direction R.
- That is to say, after the random perturbation, the new hyperplane is $H_1 = \sum_{i=1}^{d}(a_i + \alpha r_i)x_i + (a_{d+1} + \alpha r_{d+1})$.
- Notice that the only variable here is α. Therefore, just like before, the n examples in P impose n constraints on α, depending on their categories.
- We use the perturbation algorithm 2 to compute the best value of α.
- If H_1 improves (i.e., lowers) the impurity, we adopt H_1 and continue with the deterministic coefficient perturbation procedure; otherwise stop and output H as the best possible split.

OC1: impurity measure, stopping criteria and pruning strategy

- Impurity metric: Gini impurity
- Stopping criteria: stop splitting when all the leaves are pure.
- Pruning strategy: the only pruning done by the paper consists of cutting off subtrees at nodes whose impurity measures are less than a certain threshold.

OC1: my implementation

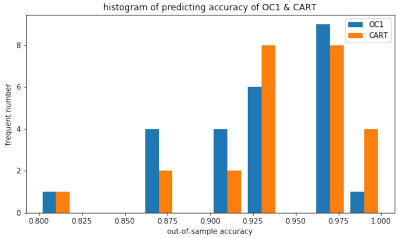
- I have implemented OC1 in Python in *oblique_decision_tree.py* and *gini.py*
- Compared with OC1 introduced above, my implementation is a bit simplified in the sense that I do not implement the random perturbation (in order to escape from local minima) and the post-pruning.
- For the pruning process, it is ok not to implement it because we can use “OC1 forest” in practice.

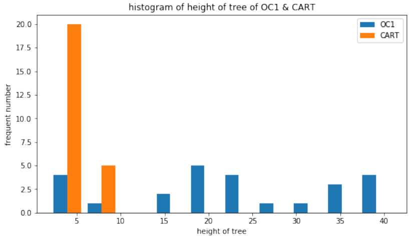
OC1: results and discussion

- We compare my simplified OC1 with *sklearn.tree.DecisionTreeClassifier*, which I believe is a modification of CART.
- We compare the performance in both single-tree and forest level.
- The evaluation metric is out-of-sample accuracy.
- The data set used here is iris dataset with 150 data points and 4 features.

OC1: results and discussion

- We use both OC1 and CART to fit a single tree 25 times. Train : test = 4 : 1.

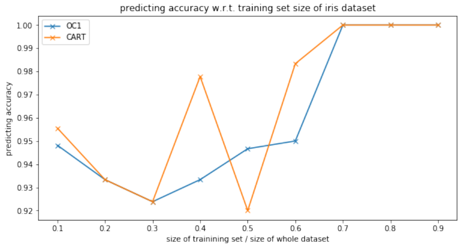




- In terms of accuracy, simplified OC1 achieves a similar level with modified CART.
- OC1 has much larger heights than CART, which in turn proves that *sklearn.tree.DecisionTreeClassifier* has been modified in some aspects in order to control the height and prevent overfitting.

OC1: results and discussion

- In terms of random forest, we set training set : whole dataset = 0.1, 0.2, ..., 0.9. In each case, we fit a forest of 50 trees using OC1 and CART respectively.



- Both forests do pretty good jobs, probably because iris is just a toy dataset and not noisy or complicated at all.
- But we do notice that as the size of training set grows, the accuracy of OC1 forest increases much more stably than CART forests.
- This probably means that in the forest level, OC1 forest better captures the data and thus has less overfitting.

Current section

Finding the optimal DT: the big picture

Before getting into any detail, let's first try to answer a fundamental question:

Fundamental question

Regardless of which methodology we adopt, what has to be done in order to find the optimal DT?

Finding the optimal DT: the big picture

The answer to the question above consists of the following steps.

- Model the current state of a DT, including both internal and leaf nodes.
- Define a loss function, since the big picture is to minimize the loss so that we can get the "optimal tree". The loss function should be a function of the prediction results and the true labels.
- Tackle a fundamental problem: the status of internal nodes has to be decided (i.e., fixed) in order to get our prediction results. So there is a time order between the internal nodes and leaves. How do we take gradient of both of them simultaneously?

Finding the optimal DT: the big picture

- If all these problems can be solved, there is nothing to keep us from find the so-called “optimal DT”.
- We will solve them one by one.

Finding the optimal DT: some notations

- m: number of leaf nodes. Since we will only talk about full binary trees, the number of leaves is thus $(m+1)$.
- x: $(p \times 1)$ vector representing a data point. Here p is the number of attributes.
- k: number of categories.
- i: index of internal nodes, $i = 1 : m$.
- j: index of leaves, $j = 1 : m + 1$.
- l: index of categories, $l = 1 : k$.
- $\mathcal{D} := \{x_z, y_z\}_{z=1:n}$: training set, where n is the size of training set, and $y_z \in \{1, 2, ..., k\}$ is the categorical label we want to predict.
- $v[i]$: the ith element of a column vector v. All indices start from 1 in this section.
- $\mathbb{I}_n$: the set of all the column vectors of length n, s.t. only one of the elements is 1 and all the others are 0.
- $\mathbb{H}^m := \{-1, 1\}^m$: the linear space of length-m vectors whose elements are either -1 or $+1$.

Finding the optimal DT: model internal nodes.

- We use the splitting function $s_i(x) : \mathbb{R}^p \mapsto \{-1, 1\}$ to represent an internal node.
- If $s_i(x) = -1$, it means that if we put observation x on node i, it will be classified into the left subtree, and vice versa.
- Just like OC1, we shall assume that $s_i(x) = \operatorname{sgn}(w_i^\top x)$, where $w_i : (p \times 1)$ is a column vector describing the decision process inside node i.
- Note that by introducing w_i, the decision boundaries can be oblique.
- There is no need to introduce an "intercept term" such as $s_i(x) = \operatorname{sgn}(w_i^\top x + b_i)$, since this can be achieved by adding a column of 1 to our X matrix.
- Therefore, based on the discussion above, we can use

  $$W_{(m \times p)} := (w_1^\top, w_2^\top, ..., w_m^\top)^\top$$

  to represent the current status of all internal nodes.

Finding the optimal DT: model leaf nodes.

- For a leaf node j, we use the predictive log-probability $\theta_j[l] := \log \Pr(y = l|j)$ to describe its prediction on class l.
- Apparently θ_j is a $(k \times 1)$ column vector.
- Similar with internal nodes, we can use

$$\Theta_{(m+1)\times k} := (\theta_1^\top, \theta_2^\top, ..., \theta_{m+1}^\top)^\top$$

to describe our current status of leaves in the DT.

Finding the optimal DT: link internal nodes to leaves.

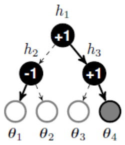
- The key to linking internal nodes to leaves is to define a latent vector

$$h := \operatorname{sgn}(\mathrm{Wx}) \in \{-1, 1\}^m := \mathbb{H}^m$$

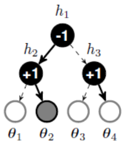
- Intuitively speaking, for a single input x, if we put it on each of the m internal nodes (no matter whether x will be directed to that node in reality), and combine all the results to a column vector, we will get h.
- Such a vector fully determines the leaf to which a data point is directed.

Finding the optimal DT: link internal nodes to leaves

- After having the latent vector h, let's further define a *tree navigation function* $f : \mathbb{H}^m \mapsto \mathbb{I}_{m+1}$ that maps an m-bit sequence of decisions (i.e., $h = \text{sgn}(Wx)$) to a one-hot vector specifying the selected leaf.
- Some examples are as below.



$$f([+1, -1, +1]^\top) = [0, 0, 0, 1]^\top = \mathbb{1}_4$$
$$\theta = \Theta^\top f(\mathrm{h}) = \theta_4$$



$$f([-1, +1, +1]^\top) = [0, 1, 0, 0]^\top = \mathbb{1}_2$$
$$\theta = \Theta^\top f(\mathrm{h}) = \theta_2$$

- Note that f is fully determined by Wx.
- Therefore, using the notation above, $\theta := \Theta^\top f(\text{sgn}(Wx))$ is the prediction of our DT when the input is x.

Finding the optimal DT: define the loss function.

- Clearly a generic loss function should be of the form $l = l(\theta, y)$ where θ is our predicted probability mass function and y is the true ground label.
- For a regression problem, y and θ are just scalars, and we can simply define $l(\theta, y) := (\theta - y)^2$
- For a multi-classification problem, we can use the *log loss*

$$l(\theta, y) := -\theta[y] + \log\big(\sum_{\beta=1}^{k} \exp\theta[\beta]\big)$$

- The goal of learning is just to find $\{W, \Theta\}$ that minimize the empirical loss given a training set $\mathcal{D}$

$$\mathcal{L}(W, \Theta; \mathcal{D}) := \sum_{(x,y)\in\mathcal{D}} l(\theta, y) = \sum_{(x,y)\in\mathcal{D}} l(\Theta^\top f(\operatorname{sgn}(Wx)), y) \tag{2}$$

Finding the optimal DT: is there something magical going on?

- Obviously by introducing W, Θ and f, we completed the task of modelling a DT.
- Moreover, it seems that we de-coupled the before-and-after order between internal and child nodes too!
- But how did we achieve that?
- We de-coupled W and Θ by incorporating some "useless information": for an observation x, from the root to the leaf it belongs to, it will only go through $\sim \log(m)$ nodes, but we incorporated the decision of every internal node in W.
- But that "redundant information", in turn, gives us the freedom to de-couple the dependency in the time axis!
- This is the real magical part of this paper.

Finding the optimal DT: make the loss function nice and smooth.

- Note that there is no way for us to compute $\frac{\partial \mathcal{L}(W,\Theta;\mathcal{D})}{\partial W}$ during the optimization process.
- The problem can be solved by the following observation

$$\operatorname{sgn}(Wx) = \arg\max_{h \in \mathbb{H}^m}(h^\top Wx)$$

- Using the equation and (2), we can re-express our loss function as

$$\mathcal{L}(W,\Theta;\mathcal{D}) = \sum_{(x,y)\in\mathcal{D}} l(\Theta^\top f(\hat{h}(x)), y) \\ \text{where } \hat{h}(x) = \arg\max_{h\in\mathbb{H}^m}(h^\top Wx) \tag{3}$$

- This is the form of loss function we shall use in the next step.

Finding the optimal DT: make the loss function nice and smooth.

- But the loss function (3) is still not easy to take gradient and thus needs further modification.
- Here we develop an upper bound for $l(\Theta^\top f(\mathrm{sgn}(Wx)), y)$

$$\begin{aligned} l(\Theta^\top f(\mathrm{sgn}(Wx)), y) &\leq \max_{g \in \mathbb{H}^m} (g^\top Wx + l(\Theta^\top f(g), y)) - \max_{h \in \mathbb{H}^m} (h^\top Wx) \\ &:= \mathrm{I} - \mathrm{II} \end{aligned} \tag{4}$$

- The proof of (4) can be found in reference 3.
- One important observation is that LHS can be achieved when $g = h = \mathrm{sgn}(Wx)$ in RHS.
- Term II (and also the first part in term I) is called the *inference problem*, as it infers the latent variable h.
- Term I is called the *loss-augmented inference*, as it augments the inference problem by an additional loss term.

Finding the optimal DT: make the loss function nice and smooth.

- Notice that in (4), the LHS does not depend on the scale of W, whereas the RHS does.
- To dig a little deeper, let's divert a bit and discuss the effect of $||W||$ on the upper bound. For $a \geq b \geq 0$,

$$\max_{g \in \mathbb{H}^m} (ag^\top Wx + l(\Theta^\top f(g), y)) - \max_{h \in \mathbb{H}^m} (ah^\top Wx) \leq$$
$$\max_{g \in \mathbb{H}^m} (bg^\top Wx + l(\Theta^\top f(g), y)) - \max_{h \in \mathbb{H}^m} (bh^\top Wx)$$

- That is to say, the larger the scale, the tighter the upper bound.

Finding the optimal DT: make the loss function nice and smooth.

- Therefore, we want the scale of W to be large.
- But on the other hand, when $||W||$ approaches $+\infty$, the loss term $l(\Theta^\top f(g), y)$ in term I of (4) becomes negligible compared with $g^\top Wx$. Therefore, the solutions to our loss-augmented inference and the original inference problem will be almost identical.
- This is not good, because even though a larger $||W||$ yields a tighter bound, it actually makes the bound approach the loss itself, and therefore becomes nearly piecewise constant, which is hard to optimize.
- In conclusion, we do not want $||W||$ to be either too large or too small. A reasonable approach would be to set a hard constraint on $||W||$.
- Let's come back to our optimization problem.

Finding the optimal DT: the optimization problem

- Finally, we formulate our optimization problem as

$$\text{minimize } \mathcal{L}'(W, \Theta; \mathcal{D}) = \sum_{(x,y)\in\mathcal{D}} \{\max_{g\in\mathbb{H}^m}(g^\top Wx + l(\Theta^\top f(g), y)) - \max_{h\in\mathbb{H}^m}(h^\top Wx)\}$$
$$s.t. \;\; ||w_i|| \leq \nu, \; \forall i \in \{1, ..., m\} \tag{5}$$

- Here w_i is the ith row of matrix W.
- $\mathcal{L}'(W, \Theta; \mathcal{D})$ is called the *surrogate objective.* This is the optimization problem we want to solve.
- We can apply constraints on each w_i separately because each node (represented by w_i) acts independently in the DT in terms of the splitting decision.

Finding the optimal DT: algorithm to solve the optimization problem

- We will definitely use the gradient descent algorithm family to solve the optimization problem.
- But during that process, we also have to evaluate our target function given a pair of (W, Θ) from time to time.
- That is to say, we need to solve the maximization sub-problem of term I and II.
- As discussed before, the solution to term II is just $\hat{h} = \operatorname{sgn}(Wx)$.
- If we can solve term I, then we can build the optimal DT following the algorithm on the next page.

Finding the optimal DT: algorithm to solve the optimization problem

```
1: Initialize $W^{(0)}, \Theta^{(0)}$ using OC1.
2: **for** $t = 0$ to $\tau$ **do**
3:     Sample a pair $(x, y)$ randomly from $\mathcal{D}$.
4:     $\hat{h} \leftarrow \mathrm{sgn}(W^{(t)}x)$ (solution to term II)
5:     $g \leftarrow \arg\max_{g \in \mathbb{H}^m} \{g^\top W^{(t)} x + l(\Theta^\top f(g), y)\}$ (solution to term I)
6:     $W^{(tmp)} \leftarrow W^{(t)} - \eta \hat{g} x^\top + \eta \hat{h} x^\top$ (SGD for $W$, $\eta$ is learning rate)
7:     **for** $i = 1$ to $m$ **do**
8:         $W_{i,:}^{(t+1)} \leftarrow \min\{1, \frac{\sqrt{\nu}}{||W_{i,:}^{(tmp)}||_2}\} \cdot W_{i,:}^{(tmp)}$ (projection operation)
9:     **end for**
10:    $\Theta^{(t+1)} \leftarrow \Theta^{(t)} - \eta \frac{\partial}{\partial \Theta} l(\Theta^\top f(\hat{g}), y)\Big|_{\Theta = \Theta^{(t)}}$ (SGD for $\Theta$)
11: **end for**
```

Algorithm 4: Algorithm to construct the optimal DT

Finding the optimal DT: algorithm to solve the optimization problem

- So far, there is only one problem left: how to solve the maximization problem of term I

$$\hat{g}(x) = \arg\max_{g \in \mathbb{H}^m} \{ g^\top W x + l(\Theta^\top f(g), y) \}$$

- This is actually not as scary as it looks like.
- An observation that makes it tractable is that $f(g)$ can only take on $m+1$ distinct values, which correspond to terminating at one of the $m+1$ leaves and selecting a distribution from $\{\theta_j\}_{j=1}^{m+1}$.

Finding the optimal DT: algorithm to solve the optimization problem

- That is to say, we actually just need to solve the following problem

$$\hat{g}(x) = \arg\max_{g \in \mathbb{H}^m} \{g^\top W^x + l(\theta_j, y)\}, \;\; s.t. \;\; f(g) = \mathbb{I}_j = (0, .., 0, 1 \; (j\text{th index}), 0, ..., 0)^\top$$

- Now it becomes clear about how to solve the problem:
    1. For any given j, set all the binary bits in g corresponding to the path from the root to leaf j to be consistent with the path direction towards leaf j in our DT, according to W.
    2. Bits of g that do not appear on this path have no effect on the output of $f(g)$ and should be set based on $g[i] = \text{sgn}(w_i^\top x)$ to obtain the maximum $g^\top Wx$.
    3. Repeat step 1 & 2 for each j, and find the $j^\star$ that maximizes the target function. Then $\hat{g}(x) = f^{-1}(\mathbb{I}_{j^\star})$.

Finding the optimal DT: time complexity analysis

- Let's analysis the time complexity of the optimization problem above.
- Denote the height (i.e., depth) of our DT as $d \sim \log_2(m) \sim \log(m)$.
- For each data point, we search for each $j = 1 : m + 1$, and the within each internal node i, the time complexity of computing inner product $w_i^\top x$ is $\mathcal{O}(p)$.
- Thus the time complexity of each round of SGD is $\mathcal{O}(2^d p) \sim \mathcal{O}(mp)$.
- The complexity is too high to let us train a deep tree.

Finding the optimal DT: improve the time complexity.

- We can improve the time complexity by using a slightly different upper bound on the loss

$$l(\Theta^\top f(\text{sgn}(Wx)), y) \leq \max_{g \in \mathbb{B}_1(\text{sgn}(Wx))} (g^\top Wx + l(\Theta^\top f(g), y)) - \max_{h \in \mathbb{H}^m} (h^\top Wx) \quad (6)$$

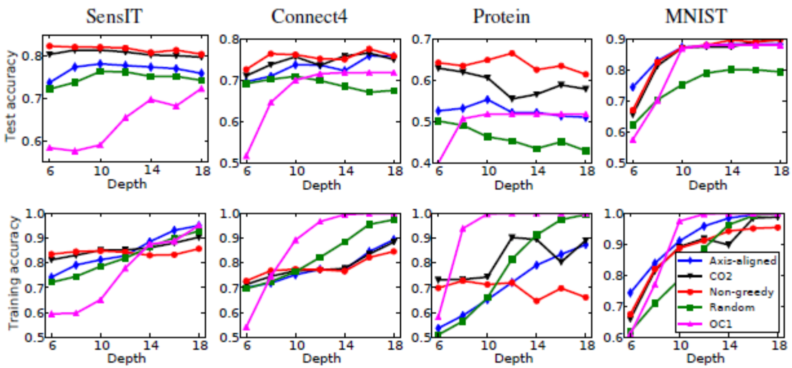
  where $\mathbb{B}_1(\text{sgn}(Wx))$ is the set of all vectors which differ from $\text{sgn}(Wx)$ by at most one bit.
- This set is called the *Hamming ball* of radius 1 around $\text{sgn}(Wx)$.
- The proof can be found in reference 3.
-
  - Since $\hat{g}$ and $\text{sgn}(Wx)$ can only differ in at most one bit;
  - and among all the bits in $\hat{g}$, we care about only d of them,
- $\hat{g}$ can only take $\sim d$ different values, and $f(\hat{g})$ can also take $\sim d$ different values, instead of $m = 2^d$ in the prior case.
- Therefore, after the improvement, the time complexity of each round of gradient descent is $\mathcal{O}(dp) = \mathcal{O}(p \log m)$.[1]

[1] It confuses me that time complexity here is $\mathcal{O}(d^2 p)$, according to the paper.

Finding the optimal DT

- We have now finished all the technical parts of finding the optimal DT.
- Let's see the results provided by the paper.

Finding the optimal DT: results and discussion



- In terms of test accuracy, non-greedy tree is the best among almost all the four datasets.
- Moreover, in terms of the shape similarity between in-sample and out-of-sample accuracy curves, the non-greedy tree is the best. This indicates that non-greedy tree does have less overfitting.

Current section

Take-aways

- OC1
    - OC1 is a promising extension on the regular DT.
    - OC1 is almost guaranteed to be better than the regular DT, since the latter is a subset of OC1.
- Non-greedy optimal DT
    - Finding the “optimal DT” is a complicated problem, which is still under active research now.
    - The difficulties mostly come from the before-and-after order between the internal nodes and leaves. This order makes it difficult to apply optimization techniques.
    - The problem above can be solved by incorporating some information which may be redundant for now but useful later in the evolution of our DT.

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