Turing Machine III

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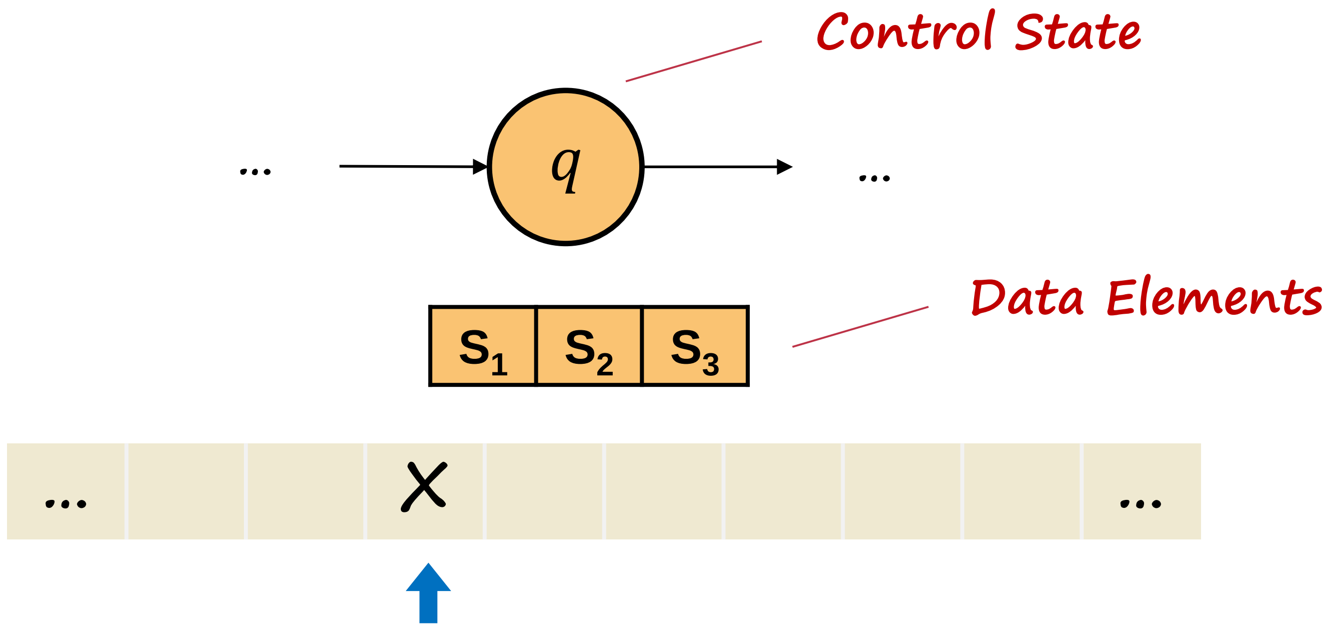
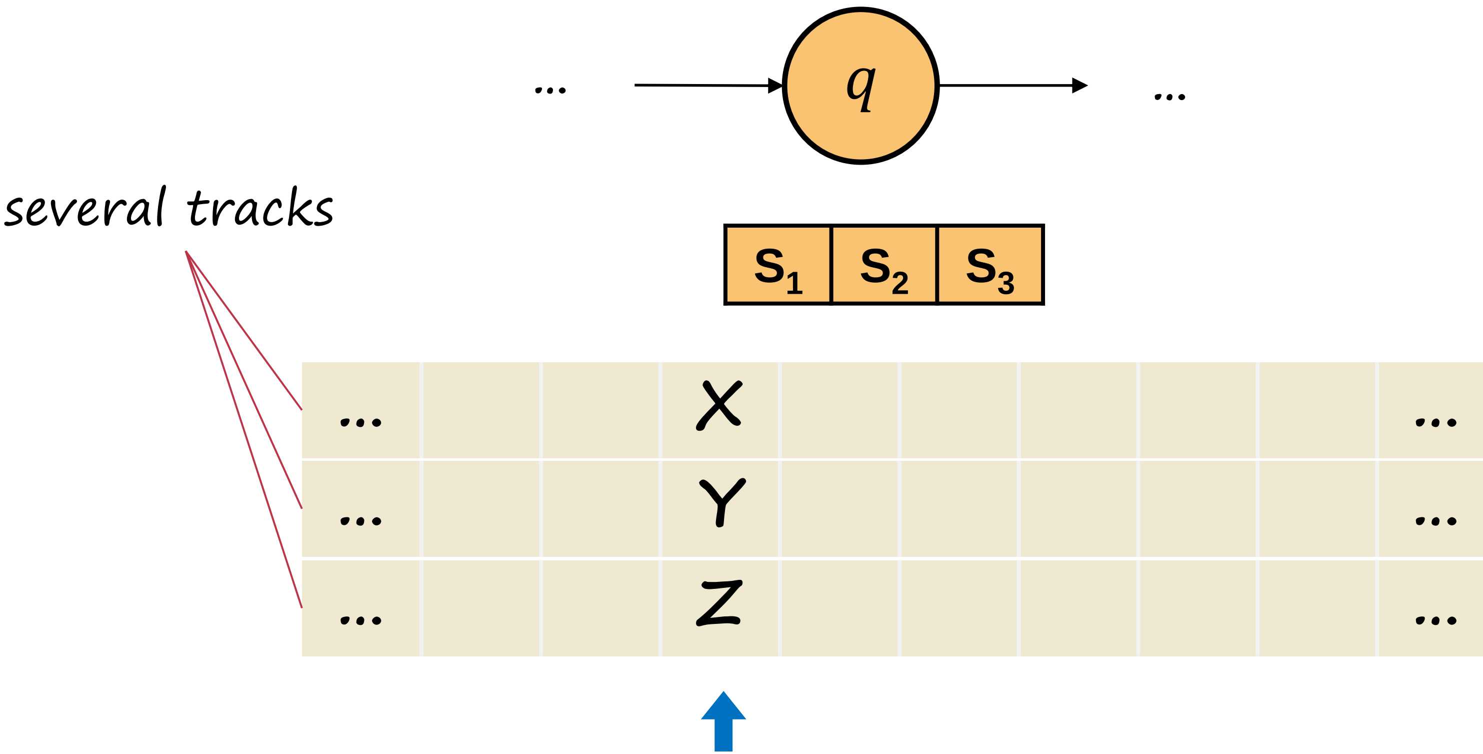
2019-12-9

Adapted From:
IALC 9.1.2, 9.2.1, 9.2.2, 9.2.3
Stanford CS103 Turing Machine

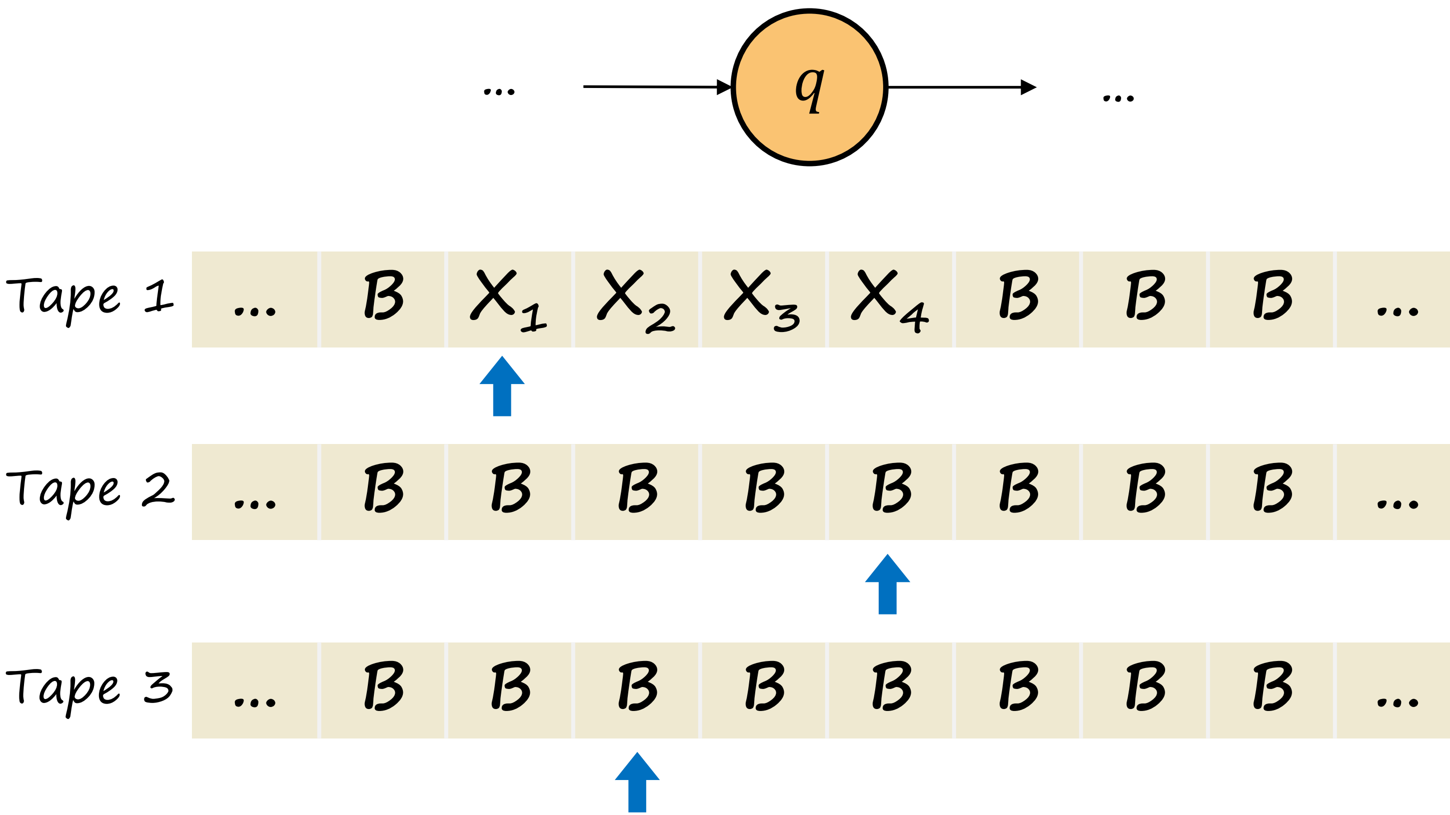
Review: TM with Storage



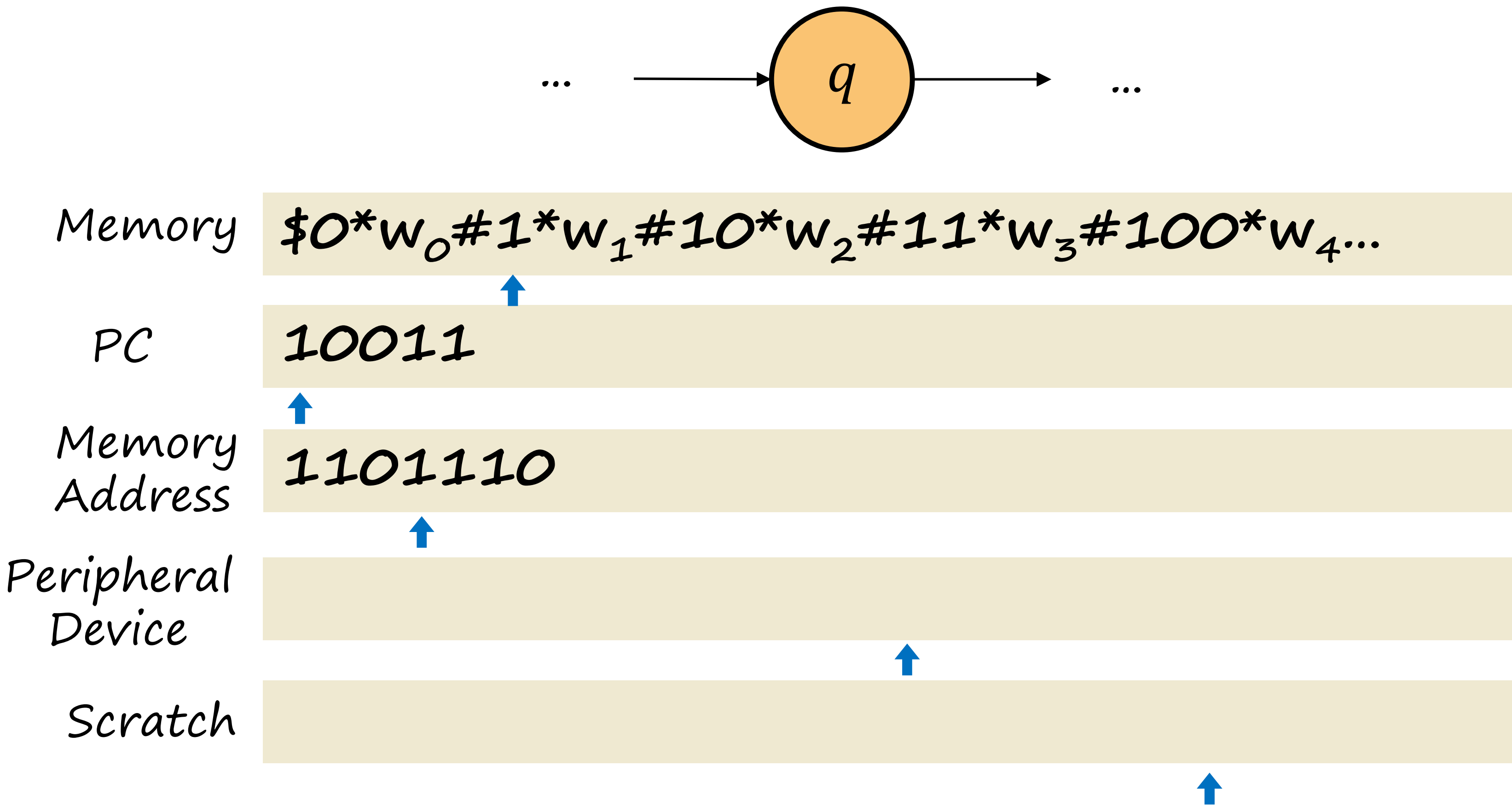
Review: Multitrack TM



Review: Multi-tape TM



Review: TM ≈ Idealized Computer



Effective Computation

- An ***effective method of computation*** is a form of computation with the following properties:
  - The computation consists of a set of steps.
  - There are fixed rules governing how one step leads to the next.
  - Any computation that yields an answer does so in finitely many steps.
  - Any computation that yields an answer always yields the correct answer.
- This is not a formal definition. Rather, it's a set of properties we expect out of a computational system.

Church-Turing Thesis

> Every effective method of computation is either equivalent to or weaker than a Turing machine.

- This is not a **theorem** but a falsifiable scientific hypo**thesis**. But it has been thoroughly tested! So we have strong faith in its correctness.
- This means Turing Machine can model any “computation”.

How we investigate computability?

2.1 Discussion on Problems

Part1. Intro & Set theory(I) : Basics & Formal Language.

Part2. Set Theory(II): Axiom system & Cardinality.

Part3. Capture Structures: Binary Relation & Function

2.2 Discussion on Computation

Part1. Turing Machine Basics.

Part2. Variants of Turing Machine. Church-Turing Thesis.

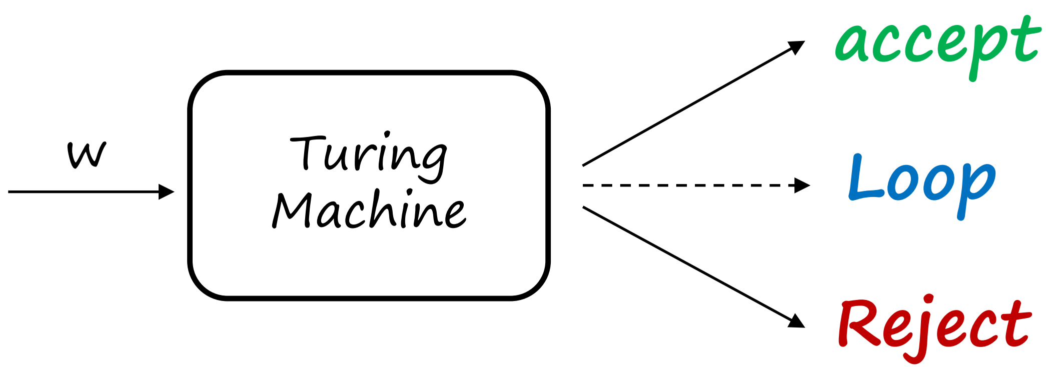
2.3 Discussion on computability

Part1. The Language of Turing Machine. R & RE

Part2. Undecidability.

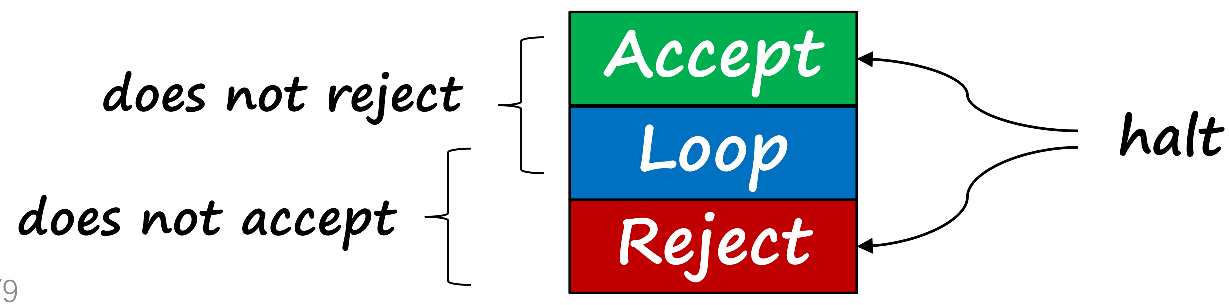
Output of TM

- For a certain input string, the output of a TM can be one in three kinds:
  - The TM **accepts** this string.
  - The TM **rejects** this string.
  - Rather than accepting or rejecting, the TM **loops** forever on the input string and never stops.



Very Important Terminology

- Let M be a Turing machine and let s be a string.
  - M **accepts** s if it enters an accepting state when running on s
  - M **rejects** s if it enters a rejecting state when running on s.
  - M **loops infinitely** on s when running on s if it enters neither an accepting nor a rejecting state.
  - M **does not accept** s if it either rejects s or loops on s.
  - M **does not reject** s if it either accepts s or loops on s.
  - M **halts** on s if it accepts w or rejects s.



More Details of RE

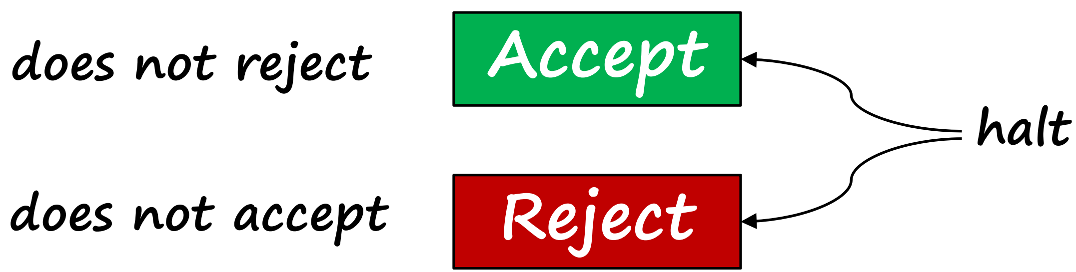
- We've known that for a certain language L, if there exists a TM $\boldsymbol{M}$ such that $L = L(\boldsymbol{M})$, then we call L is a **recursively enumerable language(递归可枚举语言)**, or ***RE***.
- So for any RE $\boldsymbol{L}$, a TM $\boldsymbol{M}$ that $L(M) = L$, and any string $\boldsymbol{w}$
  - If $w \in L, M$ accepts w in finite steps.
  - If $w \notin L, M$ does not accept w, it means M rejects w or loops forever
- We also call the TM $\boldsymbol{M}$ a **recognizer** for the language $\boldsymbol{L}$, and $\boldsymbol{L}$ is **Turing-recognizable(图灵可识别的).**

More Details of RE

- In other words, as an recognizer, M will tell you correct on every correct input, but M is not necessary to tell you wrong on every wrong input.
  - You can not determine the input is correct or wrong until M halts.
- But a problem is solvable(computable) if the computation process finishes in finite steps.

Decider

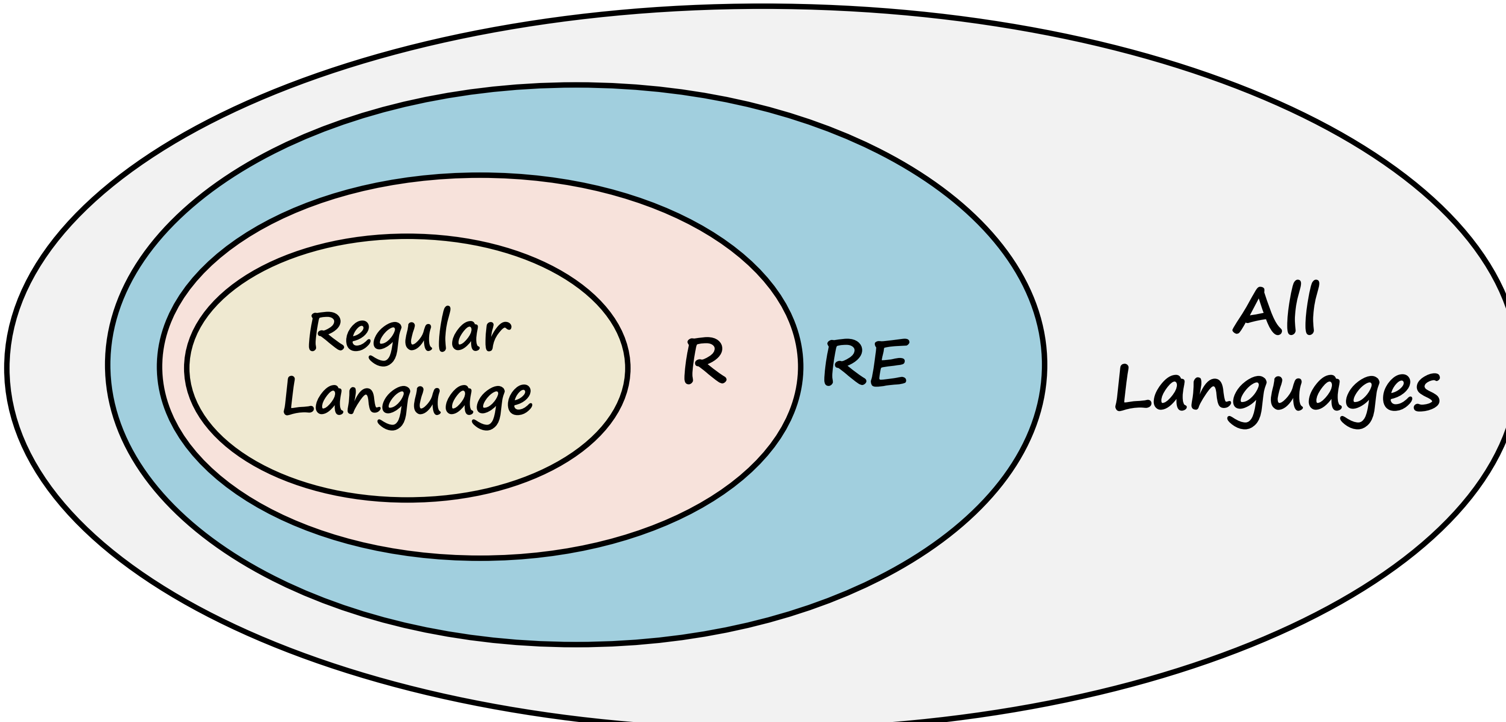
- Some Turing machines always halt; they never go into an infinite loop.
- If $\boldsymbol{M}$ is a TM and $\boldsymbol{M}$ halts on every possible input, then we say that $\boldsymbol{M}$ is a decider.
- For deciders, accepting is the same as not rejecting and rejecting is the same as not accepting.



Recursive Language

- A language $\boldsymbol{L}$ is **recursive(递归的)** if there's a TM $\boldsymbol{M}$ such that:
  - If $w \in L, M$ accepts w in finite steps.
  - If $w \notin L, M$ rejects w in finite steps.
- We also call the TM $\boldsymbol{M}$ a **decider** for the language $\boldsymbol{L}$, and $\boldsymbol{L}$ is **Turing-decidable(图灵可判定的).**
- Decidable problems, in some sense, are that can definitely be "solved" by a computer.
- Recursive Language is a subset of RE language

Language Hierarchy

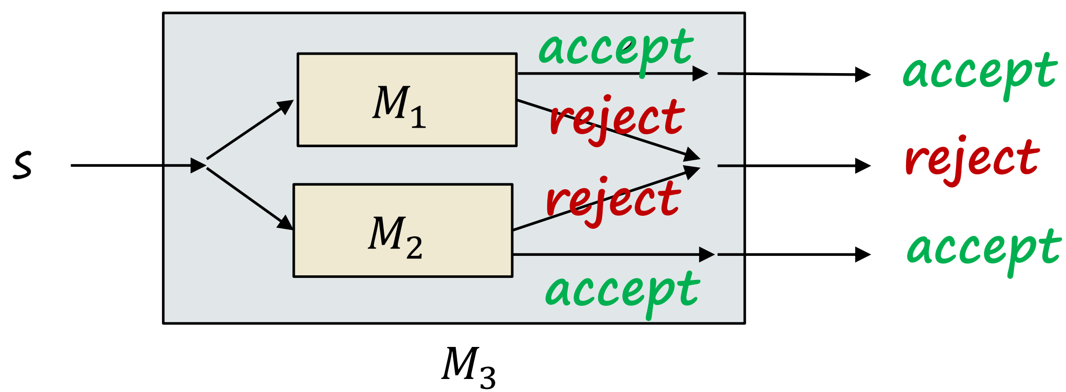


Properties of R & RE

- Union/Intersection of two **RE** language is **RE**.
- Union/Intersection of two **R** language is **R**.

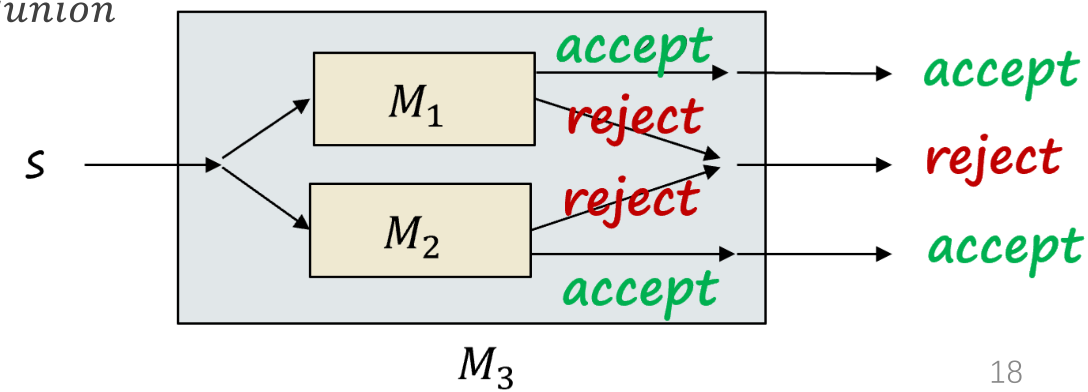
Union of Two RE

- Let L_1, L_2 be two RE language, $L_{union} = L_1 \cup L_2$
- According to definition, there are two TMs M_1, M_2 accept L_1, L_2
- We can design a new TM M_3 based on M_1 and M_2
- M_3 concurrently simulates M_1 and M_2, if any one accepts, then M_3 accepts. (How to simulate this process?)



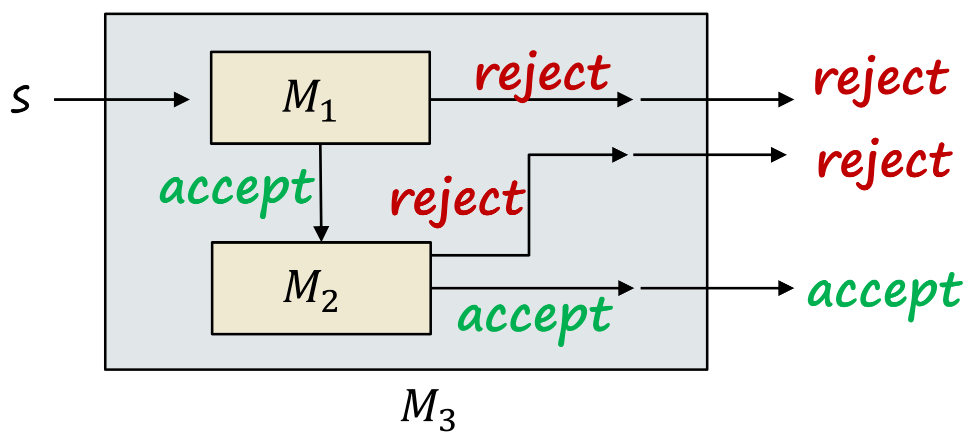
Union of Two RE

- If $s \in L_{union}$
  $\Rightarrow s \in L_1 \vee s \in L_2$
  $\Rightarrow M_1 \; or \; M_2 \; accepts \; s \Rightarrow M_3 \; accepts \; s$
- If $s \notin L_{union}$
  $\Rightarrow s \notin L_1 \wedge s \notin L_2$
  $\Rightarrow M_1 \; and \; M_2 \; do \; not \; accept \; s \Rightarrow M_3 \; does \; not \; accept \; s$
- M_3 is a TM recognizing L_{union}
- L_{union} is RE



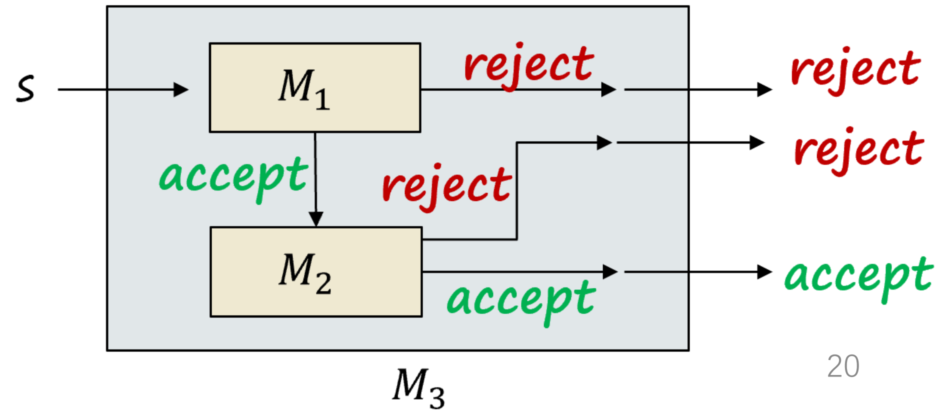
Intersection of Two RE

- Let L_1, L_2 be two RE language, $L_{intersection} = L_1 \cap L_2$
- According to definition, there are two TMs M_1, M_2 accept L_1, L_2
- We can design a new TM M_3 based on M_1 and M_2
- M_3 first simulates M_1, if M_1 accepts then simulates M_2. If M_2 still accepts, then M_3 accepts. (How to simulate this process?)



Intersection of Two RE

- If $s \in L_{intersection}$
  $\Rightarrow s \in L_1 \wedge \ s \in L_2$
  $\Rightarrow M_1 \ and \ M_2 \ accepts \ s \Rightarrow M_3 \ accepts \ s$
- If $s \notin L_{intersection}$
  $\Rightarrow s \notin L_1 \vee \ s \notin L_2$
  $\Rightarrow M_1 \ or \ M_2 \ does \ not \ accept \ s \Rightarrow M_3 \ does \ not \ accept \ s$
- M_3 is a TM recognizing $L_{intersection}$
- $L_{intersection}$ is RE



Properties of R & RE

- Union/Intersection of two **RE** language is **RE**.
- Union/Intersection of two **R** language is **R**.

It can be similarly proved.

Complements of Language

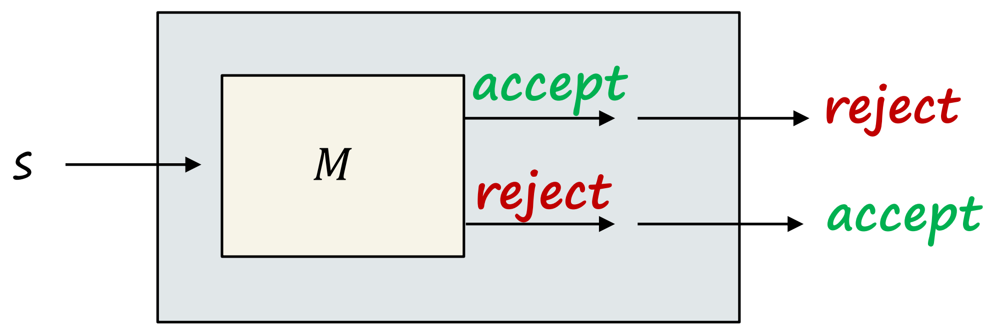
- A language $\boldsymbol{L}$'s complement $\overline{\boldsymbol{L}}$ is the set containing all valid strings not in $\boldsymbol{L}$
  - $\overline{\boldsymbol{L}} = \boldsymbol{\Sigma}^* - \boldsymbol{L}$, recall that $\boldsymbol{\Sigma}^*$ is total available string set
- Properties:
  - If $\boldsymbol{L}$ is recursive, so is $\overline{\boldsymbol{L}}$
  - If both $\boldsymbol{L}$ and $\overline{\boldsymbol{L}}$ is RE, then $\boldsymbol{L}$ is recursive

Complements of Language

If L is recursive, so is $\overline{L}$.

Proof:

- Let $L = L(M)$ for some TM M that always halts. We construct a TM $\overline{M}$ such that $\overline{L} = L(\overline{M})$ by "flipping" the result of M.

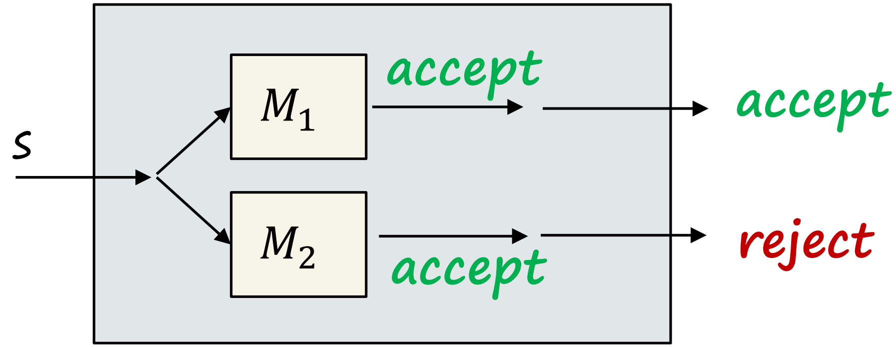


Complements of Language

If both $\boldsymbol{L}$ and $\overline{\boldsymbol{L}}$ is RE, then $\boldsymbol{L}$ is recursive

Proof:

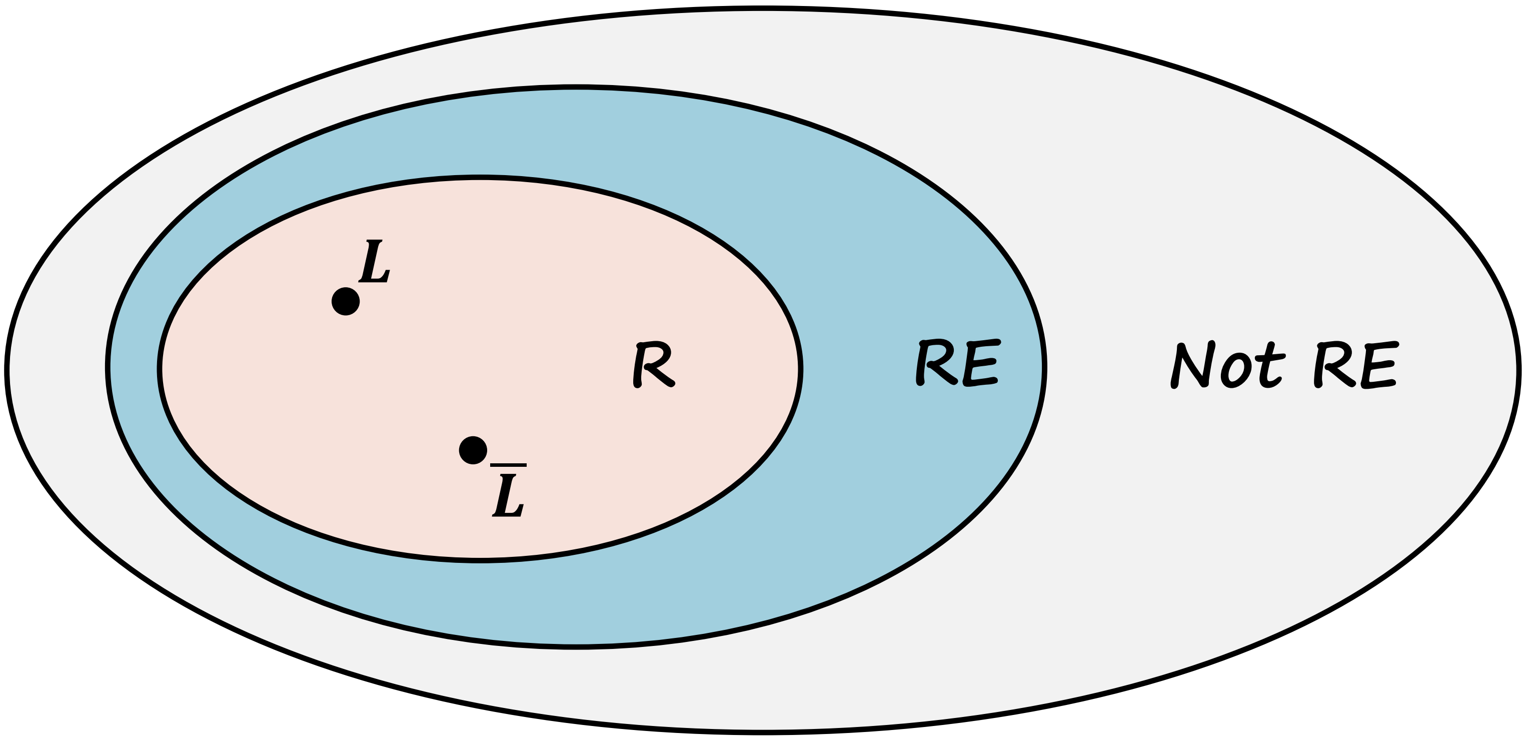
- Let $L = L(M_1)$ and $\bar{L} = L(M_2)$ for some TM M_1 and M_2. We construct a TM M_3 that always halts such that $L = L(M_3)$ by combining M_1 and M_2



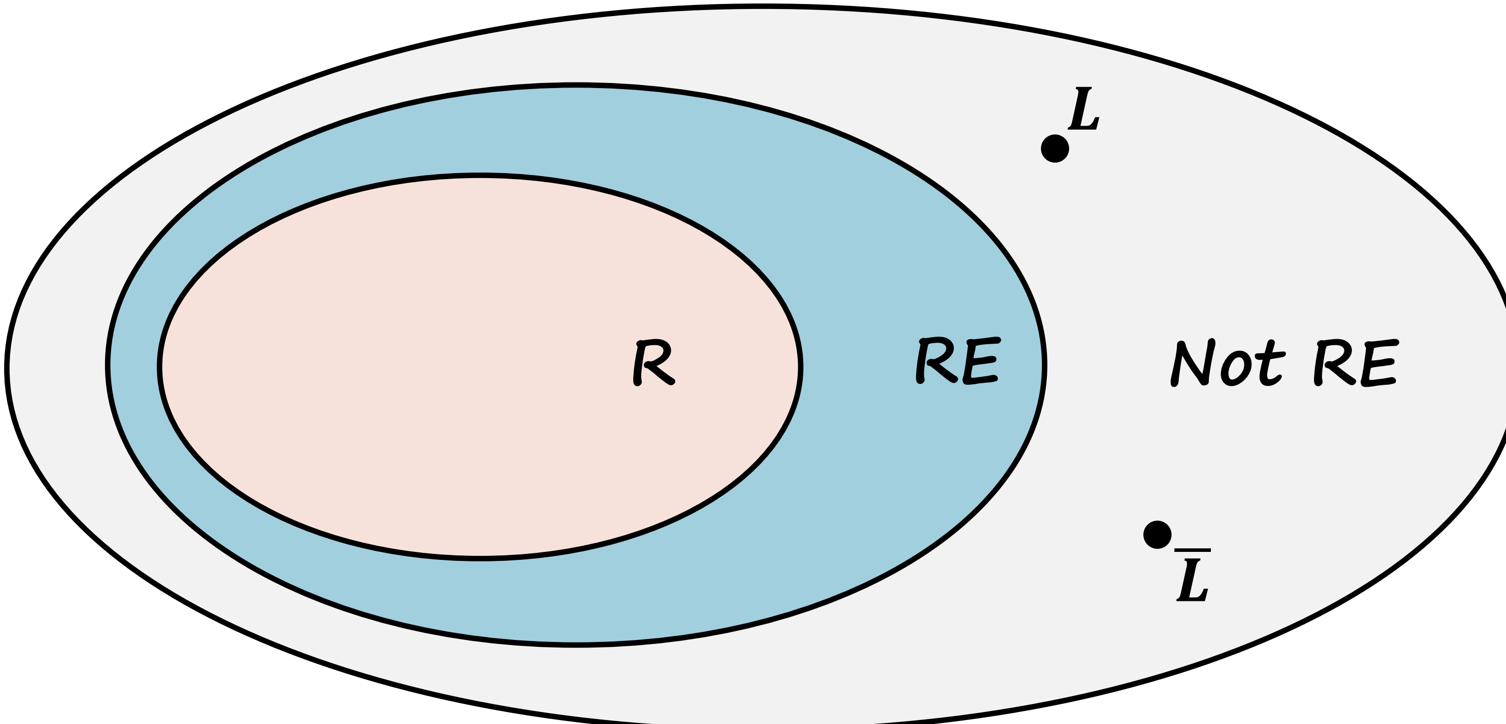
Complements of Language

- For any language L, L can be one of three kinds types: **R**, **RE** but not **R** (If any), not **RE.**
- But there are only four possible types combination of L and $\overline{L}$
  - Both L and $\overline{L}$ are R
  - Both L and $\overline{L}$ are not RE
  - One of L and $\overline{L}$ is RE, the other one is not RE

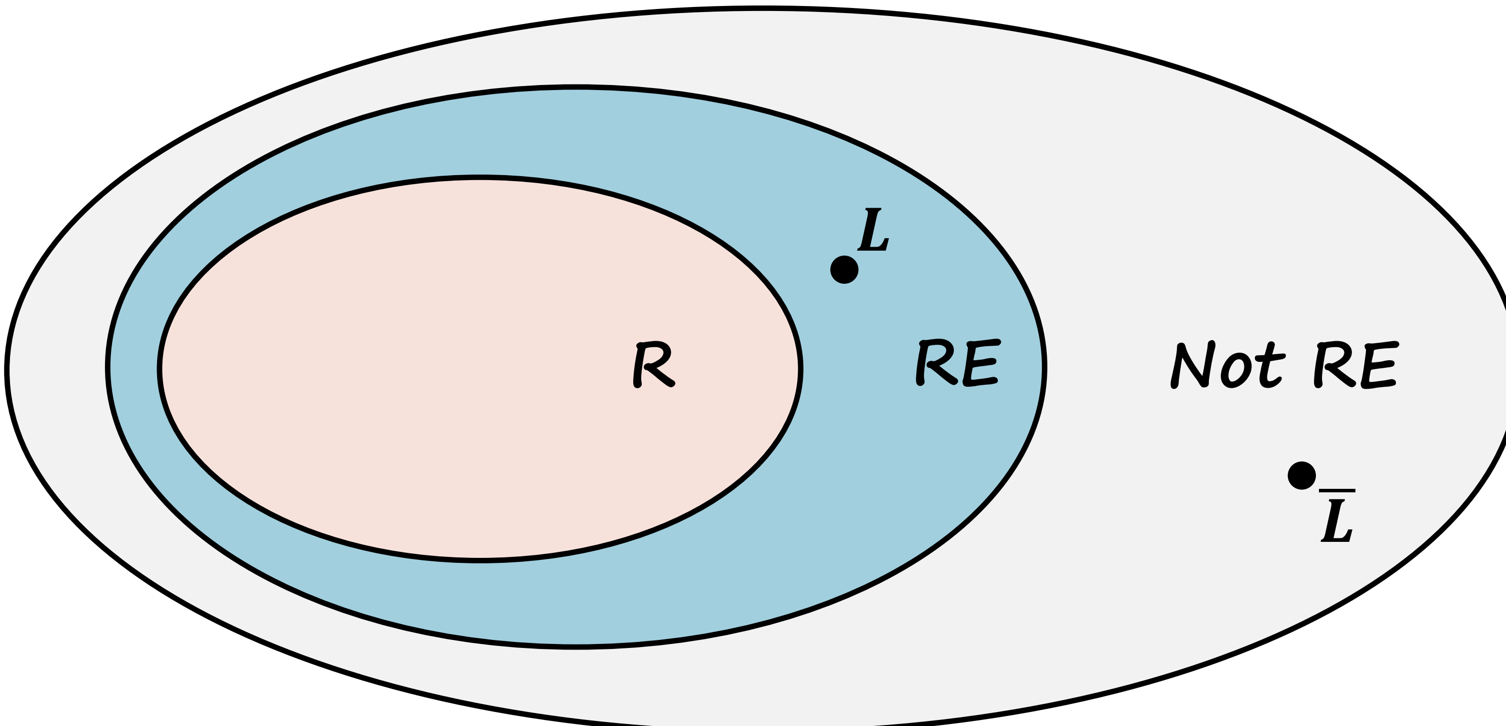
Both in Recursive



Both in Not RE



Both in Not RE

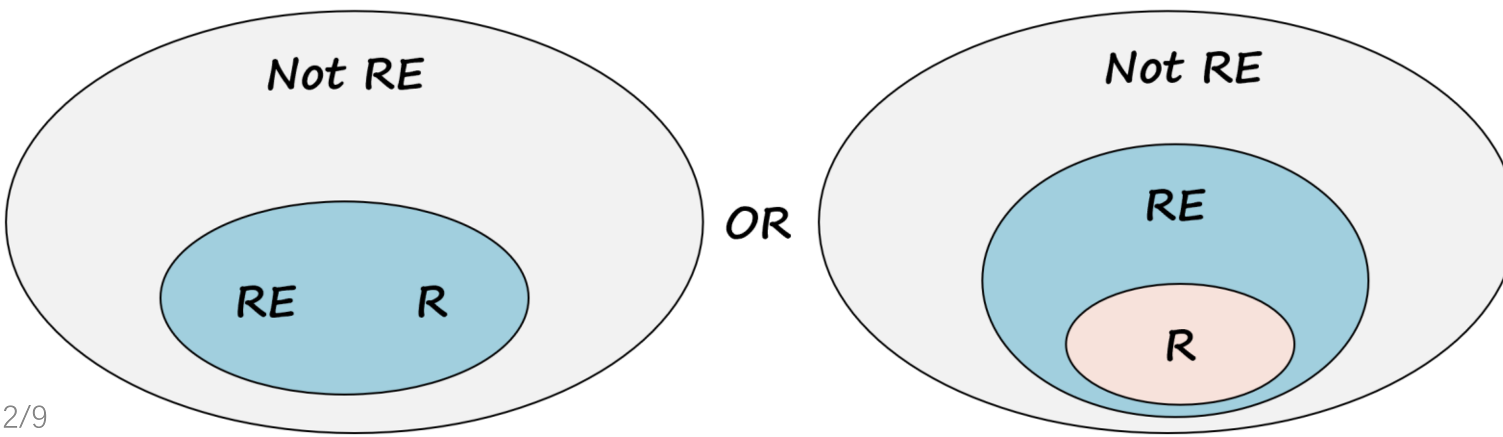


R & RE

- We've know $\mathbf{R} \subseteq \mathbf{RE}$, but more important question is that:

$$\mathbf{R} = \mathbf{RE}?$$

- That is, if you can just confirm "yes" answers to a problem, can you necessarily solve that problem?



Universal Turing Machine

通用图灵机

An Observation

- When we've been discussing Turing machines, we've talked about designing specific TMs to solve specific problems.
  - TM for $0^n 1^n$
- Does this match your real-world experiences? Do you have one computing device for each task you need to perform?
- Or can we make a “**reprogrammable** Turing machine?”

A TM Simulator

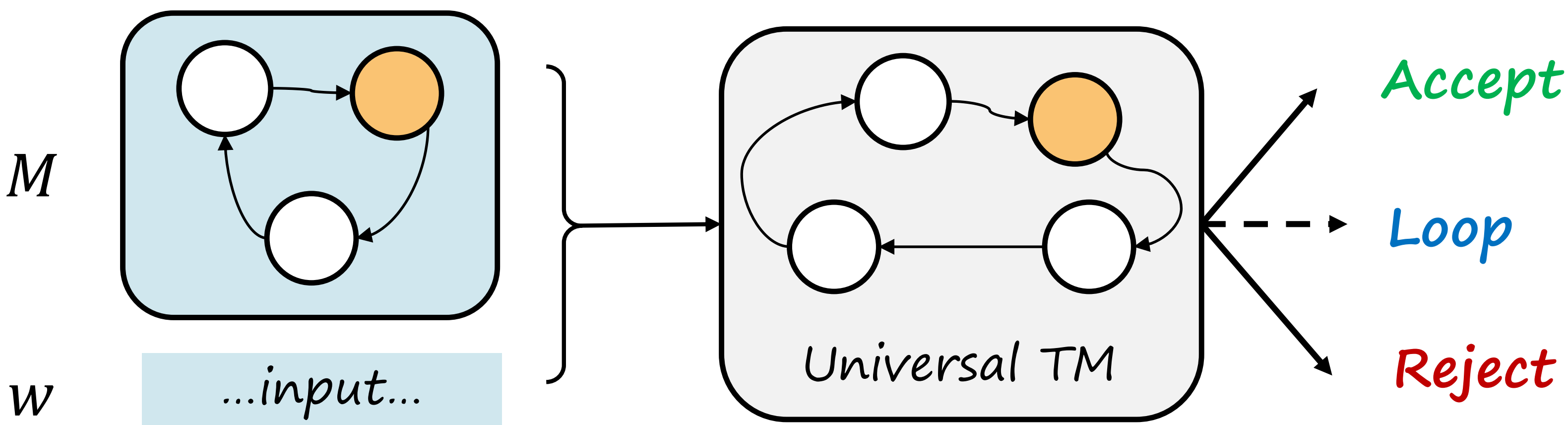
- We've known it is possible to program a TM simulator on an unbounded-memory computer.
- We could imagine it as a method

$$\text{boolean simulateTM(TM } M, \text{ string } w)$$

- with the following behavior:
  – If M accepts w, then $simulateTM(M, w)$ returns true.
  – If M rejects w, then $simulateTM(M, w)$ returns false.
  – If M loops on w, then $simulateTM(M, w)$ loops infinitely.

A TM Simulator

- It is also known that anything that can be done with an unbounded-memory computer can be done with a TM
- This means that there must be some TM that has the behavior of this *simulateTM* method.



The Universal Turing Machine

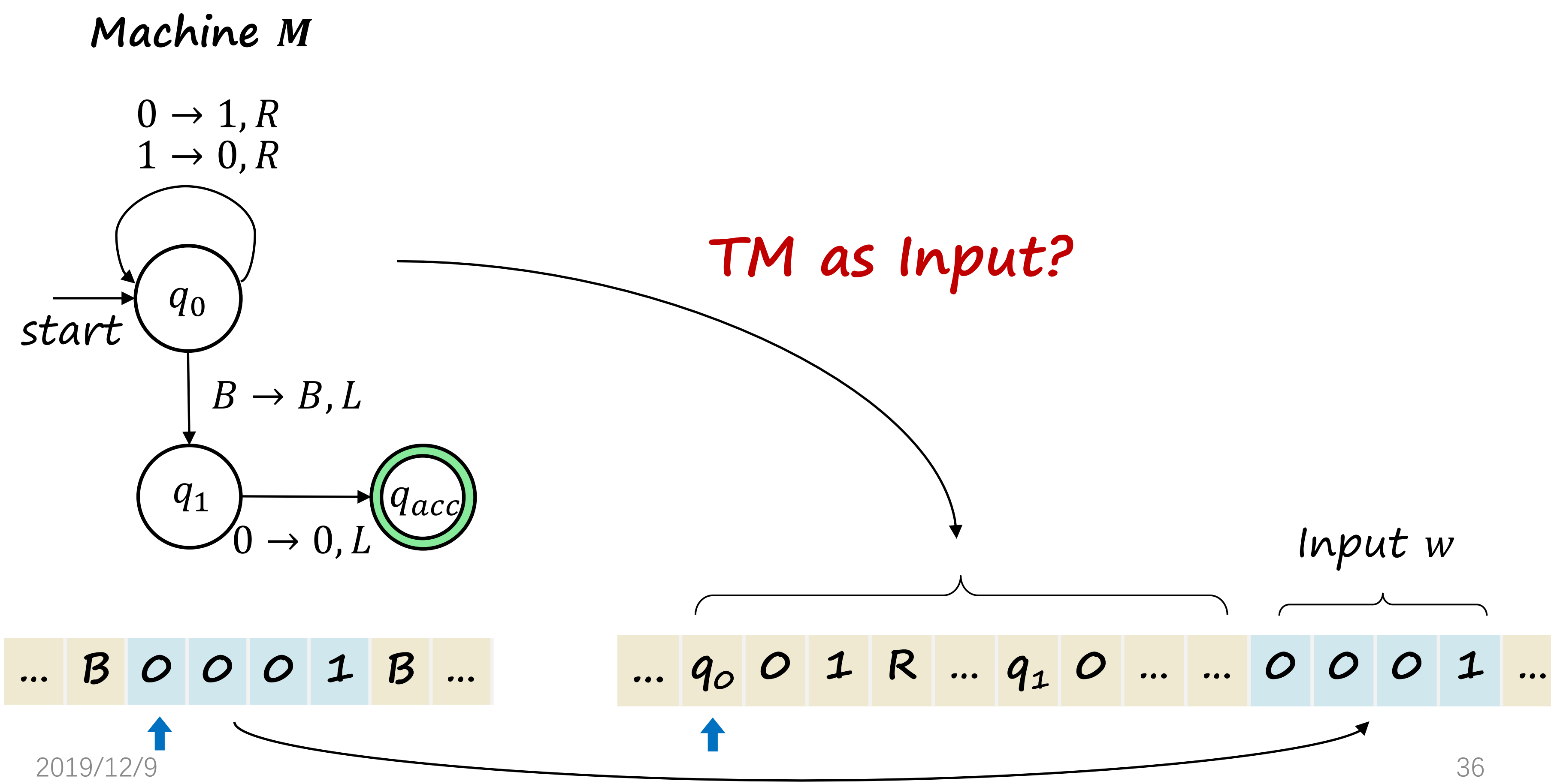
- **Theorem (Turing, 1936):** There is a Turing machine $\boldsymbol{U_{TM}}$ called **the Universal Turing Machine** that, when run on an input of the form $\langle M, w \rangle$, where M is a Turing machine and w is a string, simulates M running on w and **does whatever** M does on w (accepts, rejects, or loops).

- U_{TM} behaves as follows**:**
  - If M accepts w, then U_{TM} accepts $\langle M, w \rangle$
  - If M rejects w, then U_{TM} rejects $\langle M, w \rangle$
  - If M loops on w, then U_{TM} loops on $\langle M, w \rangle$

Encoding Input with binary

- Input string may contains any possible character in the input alphabet of M
- But we know everything on your computer is a string over $\{0, 1\}$
- We can let the input alphabet to be $\{0,1\}$
- It not necessary to limit the alphabet as $\{0,1\}$, but only for simplicity.

The Universal Turing Machine



Encoding TM

- In order to take a Turing Machine as an input, we need to encoding the TM. Similarly, we shall encode TM with binary.
- We first assign integers to the states, tape symbols and directions
  - We assume the states are $q_1, q_2, \ldots, q_k$ for some k. $\boldsymbol{q_1}$ is the **input state** and $\boldsymbol{q_2}$ is the **only accept state**.(Is it right?)
  - The tape symbols are $X_1, X_2, \ldots, X_m$ for some m. $\boldsymbol{X_1, X_2, X_3}$ **are** $\boldsymbol{0, 1, B}$ respectively.
  - Refer to direction $\boldsymbol{L}$ **as** $\boldsymbol{D_1}$, $\boldsymbol{R}$ **as** $\boldsymbol{D_2}$

Encoding TM

- After assigning each state, symbol and direction an integer, we can encode the transition function $\boldsymbol{\delta}$.
- Suppose one transition rule is $\boldsymbol{\delta}(\boldsymbol{q}_{\boldsymbol{i}}, \boldsymbol{X}_{\boldsymbol{j}}) = (\boldsymbol{q}_{\boldsymbol{k}}, \boldsymbol{X}_{\boldsymbol{l}}, \boldsymbol{D}_{\boldsymbol{m}})$, we shall code this rule by the string $\mathbf{0}^{\boldsymbol{i}}\mathbf{10}^{\boldsymbol{j}}\mathbf{10}^{\boldsymbol{k}}\mathbf{10}^{\boldsymbol{l}}\mathbf{10}^{\boldsymbol{m}}$.
  - Notice $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}, \boldsymbol{l}, \boldsymbol{m}$ are at least one, so there're no occurrences of two or more consecutive 1's with in the code for a transition
- So let $\boldsymbol{C}_{\boldsymbol{i}}$ donate the code for the ith transition rule, we can encode the whole TM as:
  - $C_1 11 C_2 11 C_3 11 \ldots 11 C_n$

Example of Code for TM

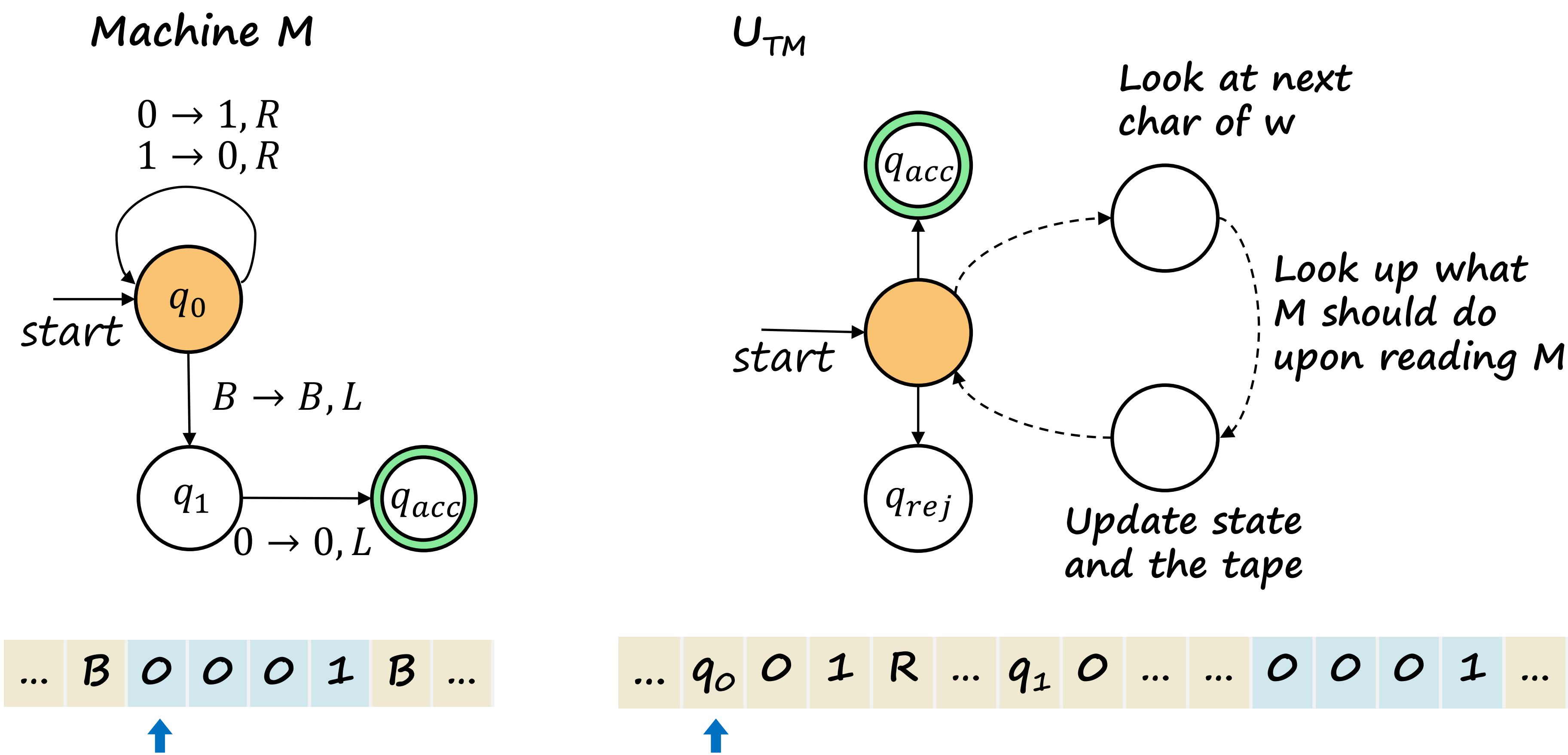
- Let a TM: $M = (\{q_1, q_2, q_3\}, \{0,1\}, \{0,1, B\}, \{\delta\}, q_1, B, q_2)$
  - $X_1 = 0, X_2 = 1, X_3 = B, D_1 = L, D_2 = R$
- Transition Function δ**:**
  - $\boldsymbol{\delta(q_1, 1) = (q_3, 0, R)}$
  - $\boldsymbol{\delta(q_3, 0) = (q_1, 1, R)}$
  - $\boldsymbol{\delta(q_3, 1) = (q_2, 0, R)}$
  - $\boldsymbol{\delta(q_3, B) = (q_3, 1, L)}$

Example of Code for TM

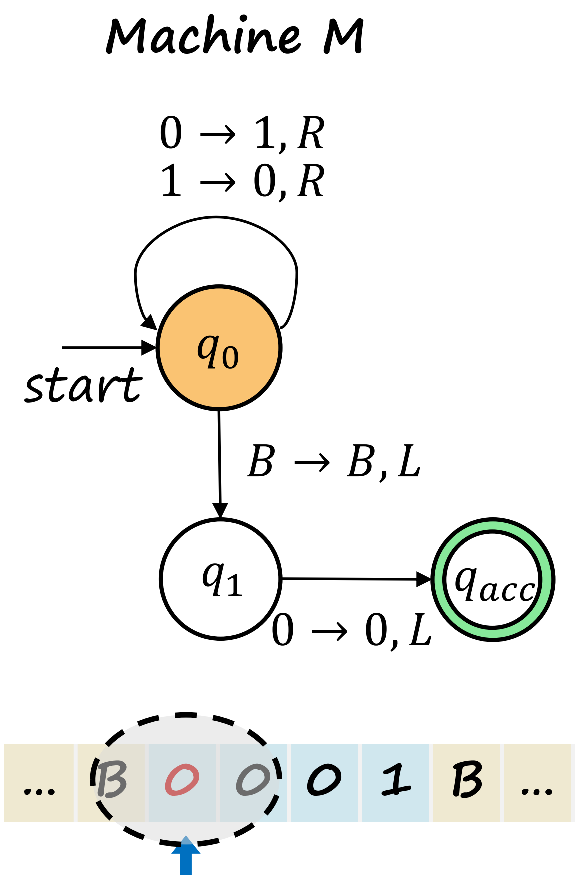
- Let a TM: $M = (\{q_1, q_2, q_3\}, \{0,1\}, \{0,1, B\}, \{\delta\}, q_1, B, q_2)$
  - $X_1 = 0, X_2 = 1, X_3 = B, D_1 = L, D_2 = R$
- Transition Function δ:
  - $\boldsymbol{\delta(q_1, 1) = (q_3, 0, R) \Rightarrow \delta(q_1, X_2) = (q_3, X_1, D_2) \Rightarrow 0100100010100}$
  - $\boldsymbol{\delta(q_3, 0) = (q_1, 1, R) \Rightarrow \delta(q_3, X_1) = (q_1, X_2, D_2) \Rightarrow 0001010100100}$
  - $\boldsymbol{\delta(q_3, 1) = (q_2, 0, R) \Rightarrow \delta(q_3, X_2) = (q_2, X_1, D_2) \Rightarrow 00010010010100}$
  - $\boldsymbol{\delta(q_3, B) = (q_3, 1, L) \Rightarrow \delta(q_3, X_3) = (q_3, X_2, D_1) \Rightarrow 0001000100010010}$
- **Code for this TM:**

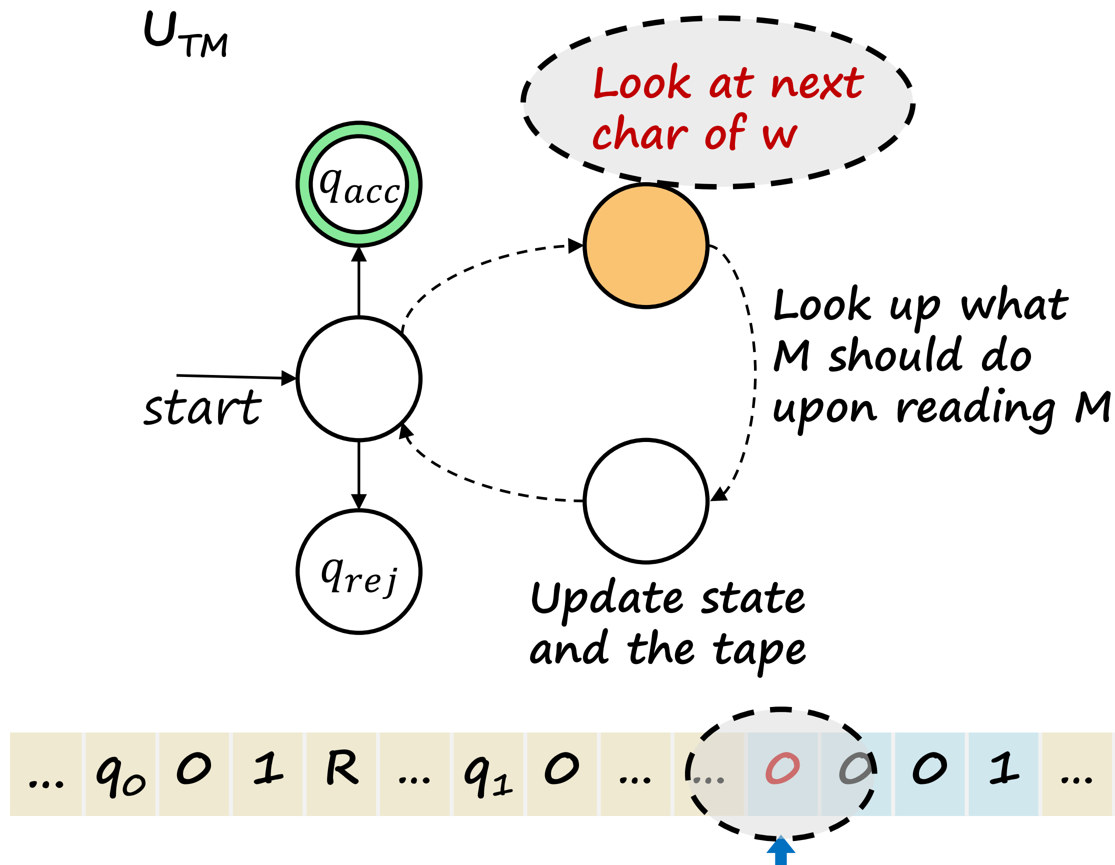
$$\mathbf{0100100010100110001010100100110001001001010011 0001000100010010}$$

The Universal Turing Machine



The Universal Turing Machine





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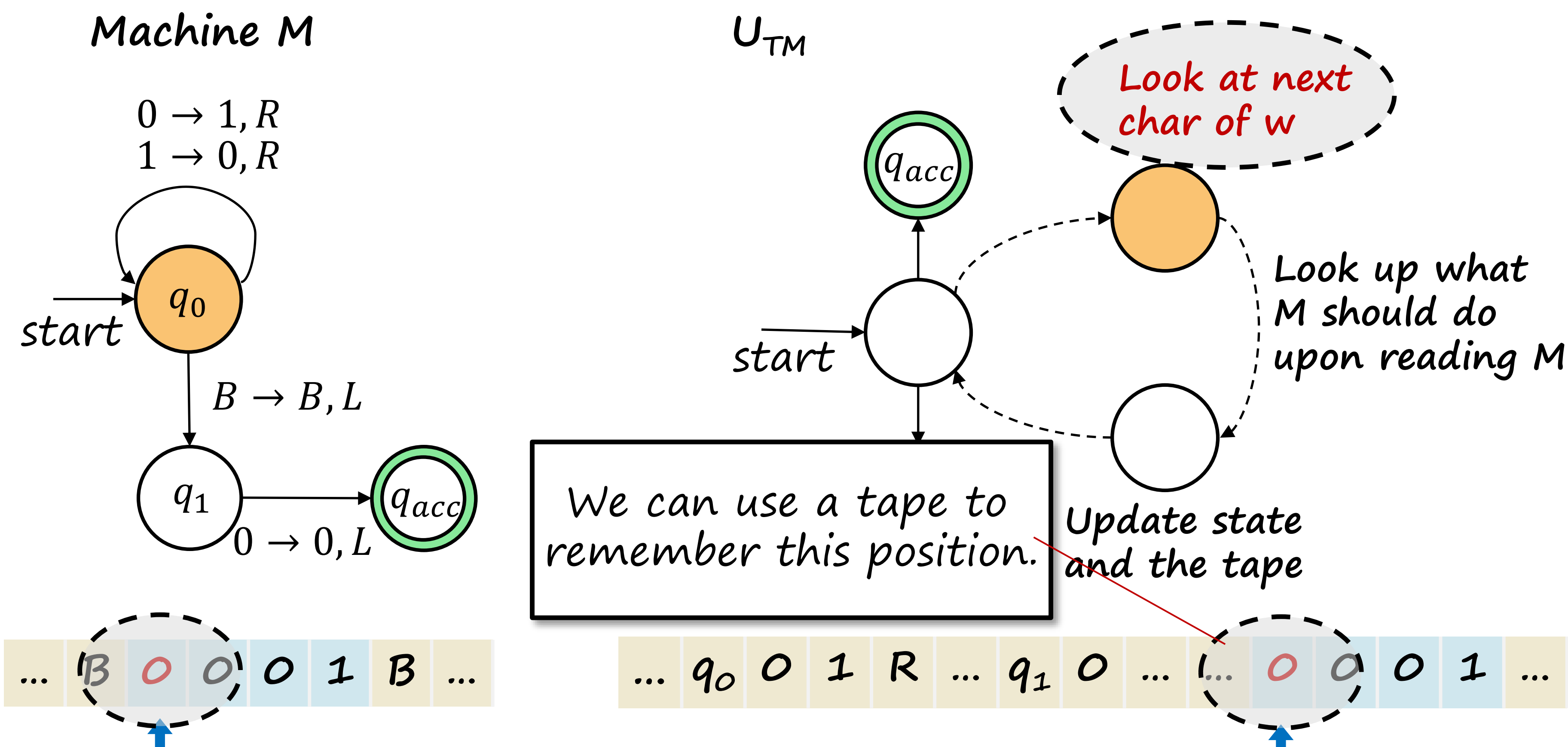

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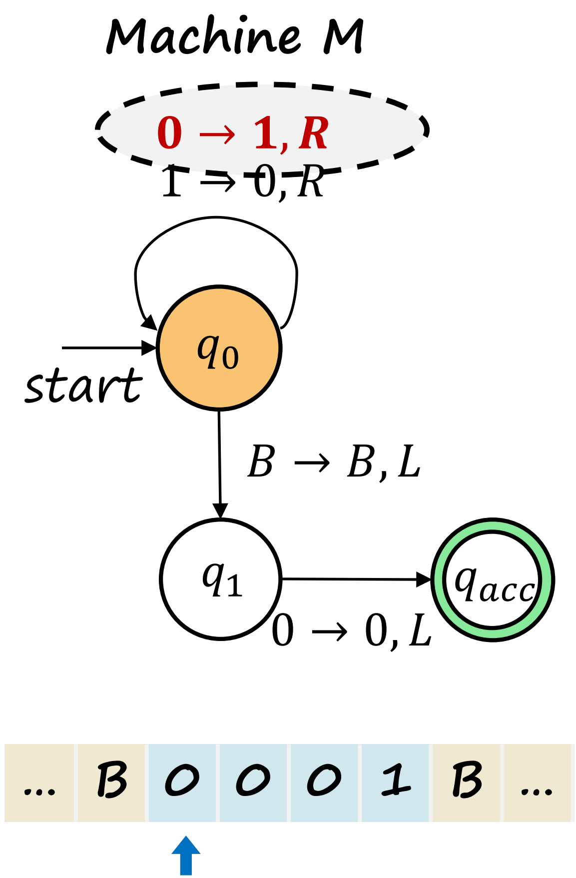

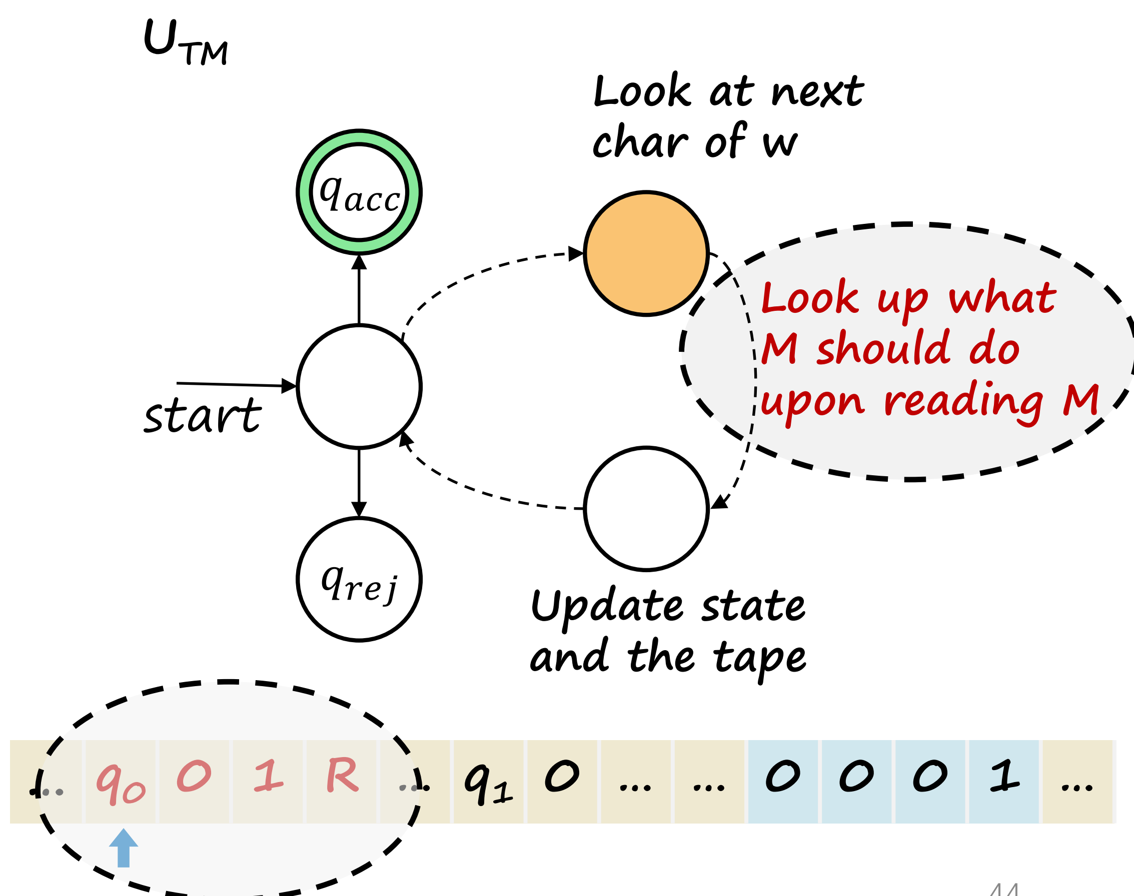

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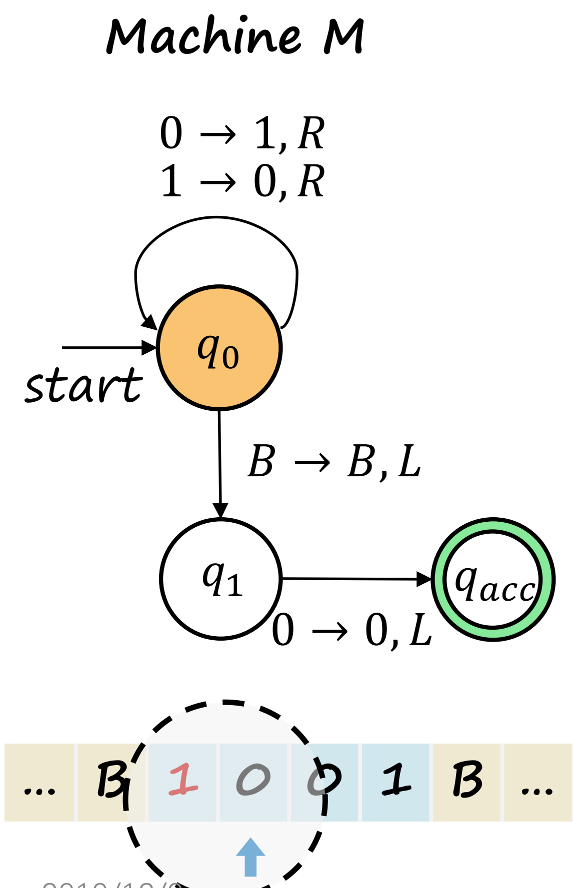

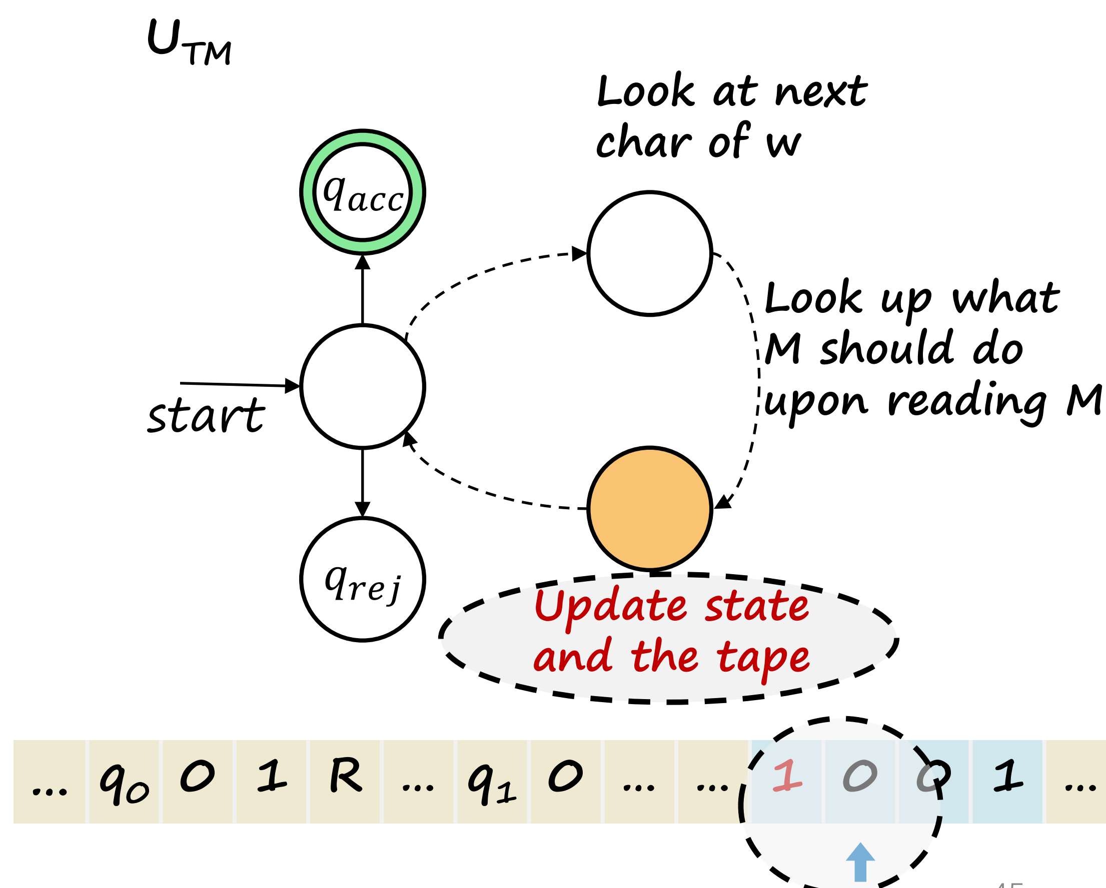

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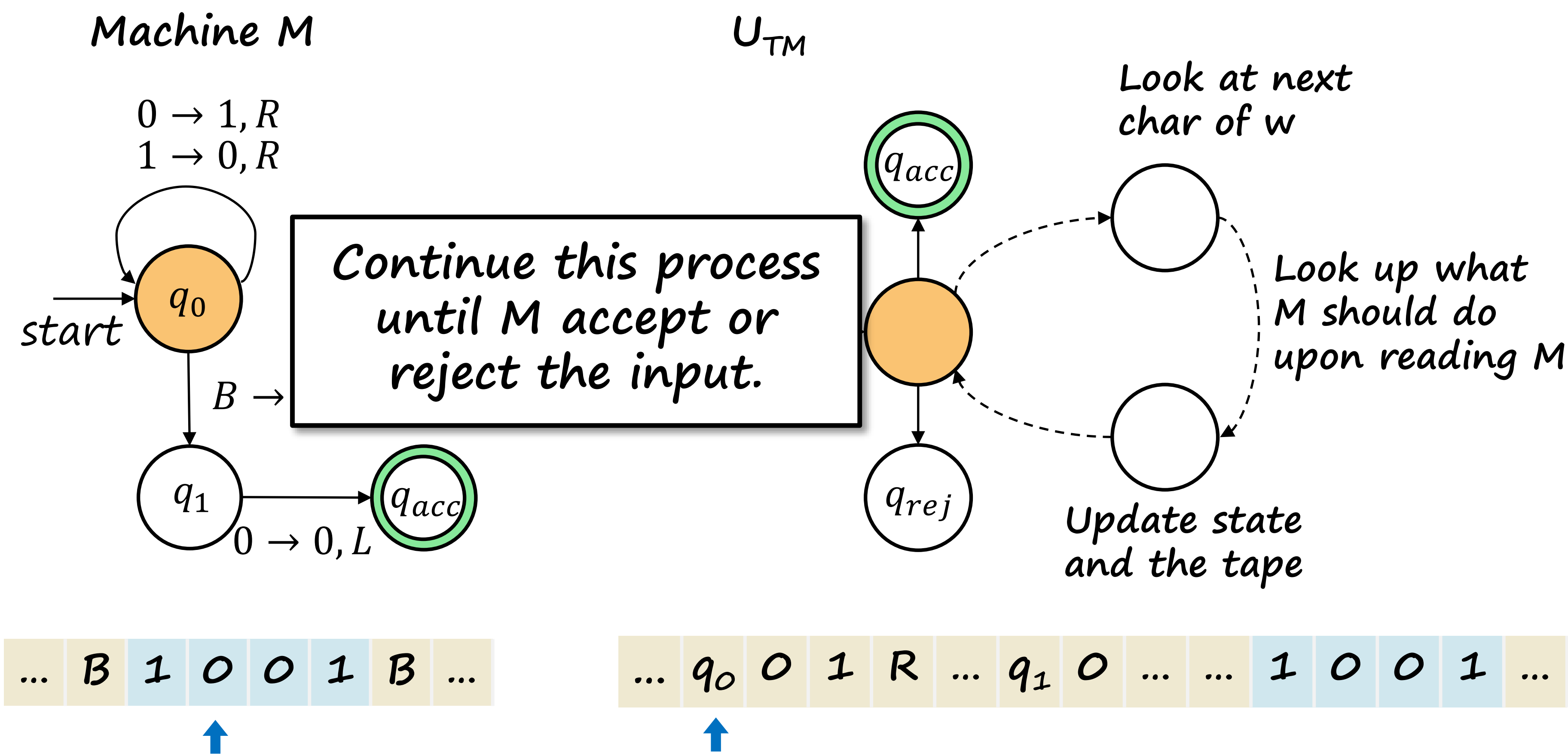
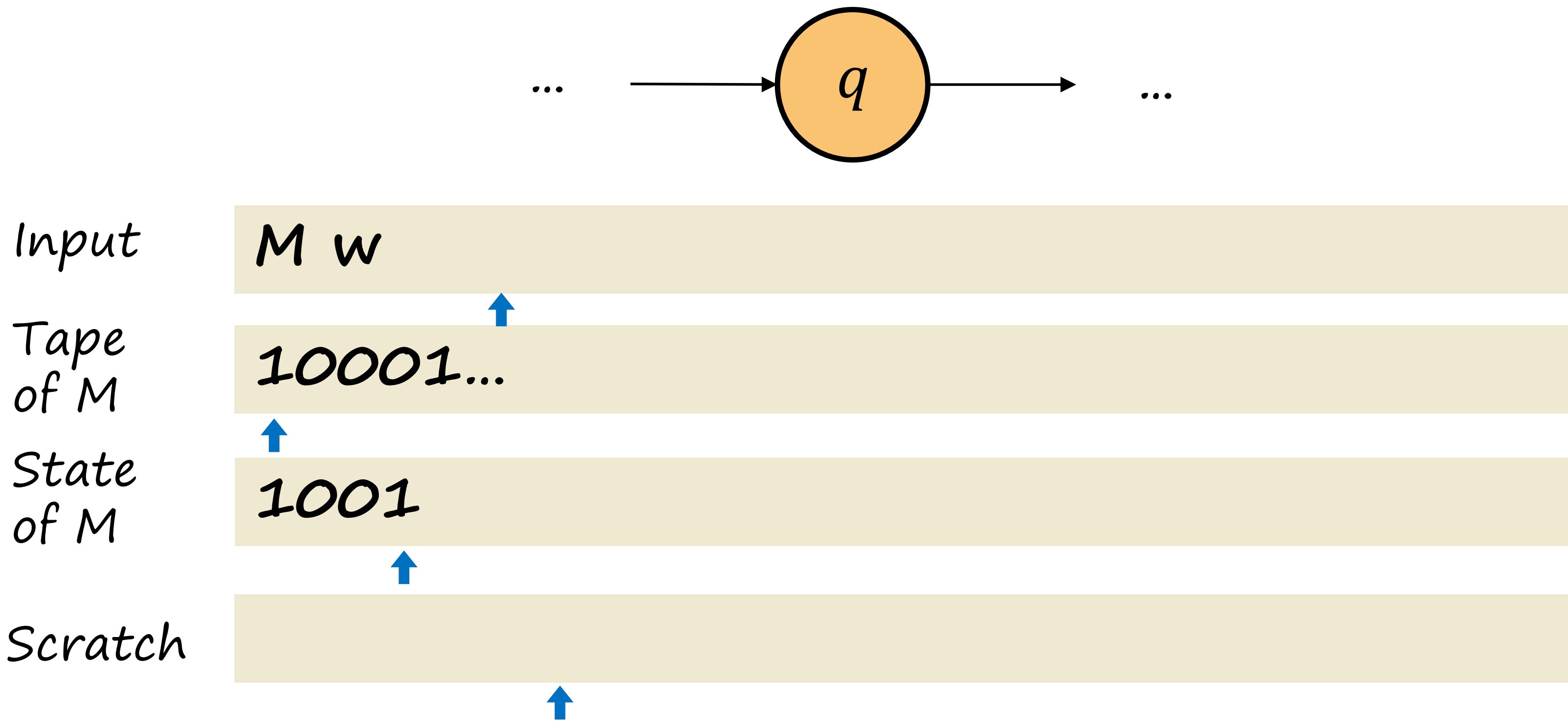


The Universal Turing Machine



The Language of U_{TM}

- Recall that the language of a TM is the set of all strings that TM accepts.
- $\boldsymbol{U_{TM}}$ when run on a string $\langle \boldsymbol{M}, \boldsymbol{w} \rangle$, where $\boldsymbol{M}$ is a TM and $\boldsymbol{w}$ is a string, will
  - Accept $\langle M, w \rangle$ if M accepts w
  - Reject $\langle M, w \rangle$ if M rejects w
  - Loop on $\langle M, w \rangle$ if M loops on w.

The Language of U_{TM}

- The universal language, donated $\boldsymbol{L_u}$, is the language of the U_{TM}

$$L_u = L(U_{TM}) = \{\langle M, w\rangle | M \text{ } is \text{ } a \text{ } TM \text{ } and \text{ } M \text{ } accepts \text{ } w\}$$

$$= \{\langle M, w\rangle | M \text{ } is \text{ } a \text{ } TM \text{ } and \text{ } w \in L(M)\}$$

- Useful fact:

$$\langle \boldsymbol{M}, \boldsymbol{w}\rangle \in \boldsymbol{L_u} \Leftrightarrow \boldsymbol{M} \text{ } accepts \text{ } \boldsymbol{w}$$

- Because $L_u = L(U_{TM})$, we know that $L_u \in RE$

The Language of U_{TM}

- If M accepts w, then we have:
  - U_{TM} accepts $\langle M, w\rangle$
  - U_{TM} accepts $\left\langle U_{TM}, \langle M, w\rangle\right\rangle$
  - U_{TM} accepts $\left\langle U_{TM}, \left\langle U_{TM}, \langle M, w\rangle\right\rangle\right\rangle$
  - ….

Next

Self-Reference

- Turing machines that compute on themselves!

Undecidable Problems

- Problems truly beyond the limits of algorithmic problem-solving!


THERE'S STILL
HOPE

测试范围

两部分：

- 形式化验证 （70%）
- 图灵机（30%）

测试范围

形式化验证部分：70%

- 命题逻辑：程序转化，DPLL算法
- 谓词逻辑：程序转化，lazy SMT techniques，EUF solver，Nelson-Oppen method，trigger matching
- 霍尔逻辑：公理和推导规则，循环不变量，最弱前置条件

不考的部分：

Switch variable，GSAT，eager SMT techniques，对lazy SMT techniques的优化(incremental T-solver，theory propagation)，e-matching，公理系统，最强后置条件

测试范围

图灵机部分：30%

- 图灵机基础：DFA；图灵机定义、表示、计算；设计图灵机解决某一问题。RE&R的定义，性质（交并补）
- 不可判定性：不可判定问题（通用图灵机语言、停机问题）；了解构造证明和规约。

期末考试范围

课程组统一出卷阅卷

- 命题范围: 逻辑1,2,4,5章
- 集合论9,10,11章