Basel problem

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Intro

In 1650, Italian mathematician Pietro Mengoli tried to solve a problem like this:

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots = ?$$

Back then, people had really limited methods for an infinite series like this, so this is Mengoli's approach:

$$\begin{aligned}
\frac{1}{1^2} + \frac{1}{2^2} &= 1.25000\cdots \\
\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} &= 1.36111\cdots \\
&\cdots\cdots \\
&\cdots\cdots \\
\frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{998^2} &= 1.643932\cdots \\
\frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{999^2} &= 1.643933\cdots \\
\frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{1000^2} &= 1.643934\cdots
\end{aligned}$$

With thousands of calculations, Mengoli could only conclude that the answer should be $1.643934\cdots$. However, this number provides pivotal evidence for future proof.

(Di/Con)verge?

Of course, according to the p-series test, the Basel problem converges. But can we prove that? (I simply know an elegant way to prove the p-series test, and I would like to show off)

We again start with a series with lower power like this:

$$\sum_{k=1}^{\infty} \frac{1}{k} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \cdots = ?$$

Behind the veil of ignorance about the p-series test, how to find whether this series converges or diverges?
Notice that $\frac{1}{3}$ is greater than $\frac{1}{4}$, and $\frac{1}{5}$, $\frac{1}{6}$, $\frac{1}{7}$ is greater than $\frac{1}{8}$...... Then, if we replace those terms, we rewrite the series like this:

$$\sum_{k=1}^{\infty} \frac{1}{k} > \frac{1}{1} + \frac{1}{2} + (\frac{1}{4} + \frac{1}{4}) + (\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}) + \cdots = 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots = \infty$$

Thus, we prove that $\sum_{k=1}^{\infty} \frac{1}{k}$ is divergent. We could apply a similar method to determine $\sum_{k=1}^{\infty} \frac{1}{k^2}$. For all positive integer n:

$$\frac{1}{n^2} < \frac{1}{n^2 - n} = \frac{1}{n(n-1)} = \frac{1}{n-1} - \frac{1}{n}$$

$$\frac{1}{2^2} < \frac{1}{1} - \frac{1}{2}, \ \frac{1}{3^2} < \frac{1}{2} - \frac{1}{3}, \ \frac{1}{4^2} < \frac{1}{3} - \frac{1}{4}, \cdots$$

Thus:

$$\sum_{k=1}^{\infty} \frac{1}{k^2} < \frac{1}{1^2} + (\frac{1}{1} - \cancel{\frac{1}{2}}) + (\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}) + (\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}) + \cdots = \lim_{k \to \infty} (\frac{1}{1^2} + \frac{1}{1} - \frac{1}{k}) = 2$$

$$1 < \sum_{k=1}^{\infty} \frac{1}{k^2} < 2$$

We know that this series is convergent. According to Mengoli, it converges to $1.643934\cdots$. But can we find that exact value? Since 1650, numerous mathematicians, including Sir Isaac Newton, Gottfried Wilhelm Leibniz, Jacob Bernoulli, and Johann Bernoulli (Jacob's younger brother), have attempted to solve this problem, but they have all failed to do so.

Genius?

In 1735, 85 years after the problem was posed, Johann Bernoulli introduced this problem to his student Leonhard Euler. There is a fun anecdote (rumor): When Euler first saw the number $1.643934\cdots$, he blurted out: "it's $\frac{\pi^2}{6}$."
Does that mean the sum of infinite rational numbers equals the square of an irrational number over a rational number? It seemed unsound when Euler first gave this answer. However, based on this answer, Euler gave his proof.

Euler's approach

Since there is a π^2 in the answer, why don't we start with $\sin x$? We know that $\sin x$ could be written in the form of a sum through the Taylor series:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

We have a form of sum, and need another form of product. We know that if a polynomial has roots x_1, x_2, x_3; it could be written as:

$$P(x) = (x - x_1)(x - x_2)(x - x_3)$$

As we know $\sin x$ has roots at $x = 0,\ \pm\pi,\ \pm 2\pi,\ \pm 3\pi, \cdots$, we could rewrite $\sin x$ as:

$$\sin x = \cdots (x + 3\pi)(x + 2\pi)(x + \pi)x(x - \pi)(x - 2\pi)(x - 3\pi)\cdots$$

With the property of the difference of two squares:

$$\sin x = x(x^2 - \pi^2)(x^2 - (2\pi)^2)(x^2 - (3\pi)^2)\cdots$$

However, regardless of the value of x, the right side will diverge. Perhaps we need a constant C to ensure convergence, as:

$$\sin x = Cx(x^2 - \pi^2)(x^2 - 2^2\pi^2)(x^2 - 3^2\pi^2)\cdots$$

Divide by x on both sides and find the limit to 0:

$$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} C(x^2 - \pi^2)(x^2 - 2^2\pi^2)(x^2 - 3^2\pi^2)\cdots$$

The left side is probably the first rule you should recognize in AP Calculus AB, in which $\lim_{x\to 0} \frac{\sin x}{x} = 1$; for the right side, we simply plug in $x = 0$ to the series:

$$1 = C(-\pi^2)(-2^2\pi^2)(-3^2\pi^2)\cdots$$
$$C = \frac{1}{(-\pi^2)(-2^2\pi^2)(-3^2\pi^2)\cdots}$$

Plug in this C back to $\sin x$:

$$\sin x = \frac{1}{(-\pi^2)(-2^2\pi^2)(-3^2\pi^2)\cdots} \cdot x(x^2 - \pi^2)(x^2 - 2^2\pi^2)(x^2 - 3^2\pi^2)\cdots$$

$$\sin x = x(1 - \frac{x^2}{\pi^2})(1 - \frac{x^2}{2^2\pi^2})(1 - \frac{x^2}{3^2\pi^2})\cdots$$

Now, we have both the sum form and the product form of $\sin x$; we could make an equation about them:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = x(1 - \frac{x^2}{\pi^2})(1 - \frac{x^2}{2^2\pi^2})(1 - \frac{x^2}{3^2\pi^2})\cdots$$

Divide by x on all sides:

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots = (1 - \frac{x^2}{\pi^2})(1 - \frac{x^2}{2^2\pi^2})(1 - \frac{x^2}{3^2\pi^2})\cdots$$

Notice that the rightmost side is still difficult to simplify. However, in fact, we only need to find the coefficient of x^2. To find how to simplify, we could start with a simple form:

$$(1-a^2x^2)(1-b^2x^2)(1-c^2x^2) = -a^2b^2c^2 \cdot x^6 + (a^2b^2+a^2c^2+b^2c^2) \cdot x^4 - (a^2+b^2+c^2) \cdot x^2 + 1$$

Notice the coefficient of x^2 is $-(a^2+b^2+c^2)$. Reflect on $\frac{\sin x}{x}$, the right side's coefficient for x^2 is $-(\frac{1}{\pi^2}+\frac{1}{2^2\pi^2}+\frac{1}{3^2\pi^2}+\cdots)$, and the left side's coefficient for x^2 is $-\frac{1}{3!}$. According to the identity of a polynomial, the coefficient of the corresponding exponential has to be equal. Thus:

$$-(\frac{1}{\pi^2}+\frac{1}{2^2\pi^2}+\frac{1}{3^2\pi^2}+\cdots) = -\frac{1}{3!}$$

Now it is the most miraculous moment: time $-\pi^2$ on both sides:

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots = \frac{\pi^2}{3!} = \frac{\pi^2}{6}$$

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Reflection

Euler's proof seems flawless. With this elegant proof, 28-year-old Euler represents nothing but a newborn math genius. However, there is a critical mistake Euler made: he applied the property of a polynomial function to an infinite series—the definition of a polynomial states that all polynomial functions have a finite number of terms. Because of this, you cannot use the root of $\sin x$ to find the product form of $\sin x$

In the 19th century, German mathematician Karl Weierstrass discovered that all Entire functions, which are holomorphic (complex differentiable) at every point in the complex plane, can be expressed in the form of products, known as the **Weierstrass factorization theorem**. According to this theorem, using the root of an infinite series to express a product form as a polynomial function is generally incorrect, but it coincidentally works for $\sin x$; this is why Euler obtained the correct answer.

But with Euler's insight, many other mathematician found their own correct solution. Here, I will introduce the easiest proof by French mathematician Augustin-Louis Cauchy.

Cauchy's proof

We start with a formula about complex numbers introduced in my Precalculus Honors class: **De Moivre's formula**

$$(\cos x + i \sin x)^n = \cos nx + i \sin nx$$

We now replace n to $2m+1$:

$$(\cos x + i\sin x)^{2m+1} = \cos[(2m+1)x] + i\sin[(2m+1)x]$$

Notice that the imaginary part on the right side is $\sin[(2m+1)x]$, just keep that in mind.
Then, we use **Binomial Theorem** to expand left side:

$$\begin{aligned}(\cos x + i\sin x)^{2m+1} = &\binom{2m+1}{0}\cos^{2m+1}x+\\ &\binom{2m+1}{1}\cos^{2m}x\cdot i\sin x+\\ &\binom{2m+1}{2}\cos^{2m-1}x\cdot i^2\sin^2 x+\\ &\binom{2m+1}{3}\cos^{2m-2}x\cdot i^3\sin^3 x+\\ &\binom{2m+1}{4}\cos^{2m-3}x\cdot i^4\sin^4 x+\cdots\end{aligned}$$

We just want the imaginary part; with the property of $i^2=-1$, the imaginary part of $(\cos x + i\sin x)^{2m+1}$ is:

$$\binom{2m+1}{1}\cos^{2m}x\sin x - \binom{2m+1}{3}\cos^{2m-2}x\sin^3 x + \binom{2m+1}{5}\cos^{2m-4}x\sin^5 x - \cdots$$

Now is the time to show some magic. If we factor out $\sin^{2m+1}x$, the first term will leave $\frac{\cos^{2m}x}{\sin^{2m}x} = \cot^{2m}x$; similarly, the second term will leave $\frac{\cos^{2m-2}x}{\sin^{2m-2}x} = \cot^{2x-2}x$. The following term could all get simplified in this form, as:

$$\sin^{2m+1}x\left[\binom{2m+1}{1}\cot^{2m}x - \binom{2m+1}{3}\cot^{2m-2}x + \binom{2m+1}{5}\cot^{2m-4}x - \cdots\right]$$

As we know, the imaginary part is also equal to $(\cos x + i\sin x)^{2m+1}$; thus, we make an equation:

$$\sin[(2m+1)x] = \sin^{2m+1}x\left[\binom{2m+1}{1}\cot^{2m}x - \binom{2m+1}{3}\cot^{2m-2}x + \cdots\right]$$

Now things are getting interesting. In order to make it neater, let $\cot^2 x = t$; rewrite this equation:

$$\sin[(2m+1)x] = \sin^{2m+1}x\left[\binom{2m+1}{1}t^m - \binom{2m+1}{3}t^{m-1} + \binom{2m+1}{5}t^{m-2} - \cdots\right]$$

When $x = \frac{n\pi}{2m+1}$, where n is an integer, the left side is equal to 0. For the right side, $\sin^{2m+1}x$ does not equal 0 when $x = \frac{n\pi}{2m+1}$. Thus, the huge thing in parentheses has to equal 0, and we have a polynomial:

$$\binom{2m+1}{1}t^m - \binom{2m+1}{3}t^{m-1} + \binom{2m+1}{5}t^{m-2} - \cdots = 0$$

Of course, according to **Fundamental Theorem of Algebra**, the maximum power is m, which means this polynomial has m roots, and we note them from r_1 to r_m. And according to **Vieta's formulas**, the sum of all roots $t_1 + t_2 + \cdots + t_m = -\frac{a_{n-1}}{a_n}$, where a_n is the coefficient of the highest power and a_{n-1} is the coefficient of the second highest power. Plug in those coefficients from the polynomial we have:

$$t_1 + t_2 + \cdots + t_m = \cancel{-}\frac{\cancel{-}\binom{2m+1}{3}}{\binom{2m+1}{1}} = \frac{\frac{\cancel{(2m+1)}(\cancel{2}m)(2m-1)}{3\cdot\cancel{2}\cdot 1}}{\cancel{2m+1}} = \frac{m(2m-1)}{3}$$

Remember that t is $\cot^2 x$, which means that for those root $t_1, t_2, \cdots, t_m$, the will have a corresponding $\cot^2 x_1, \cot^2 x_2, \cdots, \cot^2 x_m$ as the root for our polynomial. In addition, $x = \frac{n\pi}{2m+1}$ is also the root for our polynomial; thus, we could say that $x_1, x_2, \cdots, x_m$ are $\frac{(1\to m)\pi}{2m+1}$. Replacing t in Vieta's formulas:

$$\cot^2 \frac{\pi}{2m+1} + \cot^2 \frac{2\pi}{2m+1} + \cdots + \cot^2 \frac{m\pi}{2m+1} = \frac{m(2m-1)}{3}$$

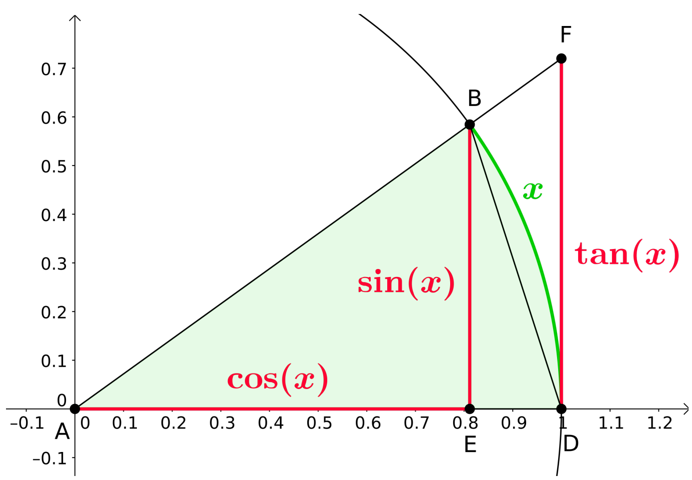
Now we add m to both sides. For the left side, we could distribute that m to m ones; for the right side, we could simply add $\frac{3m}{3}$:

$$(\cot^2 \frac{\pi}{2m+1} + 1) + (\cot^2 \frac{2\pi}{2m+1} + 1) + \cdots + (\cot^2 \frac{m\pi}{2m+1} + 1) = \frac{m(2m-1)}{3} + \frac{3m}{3}$$

Remember the Trigonometric Identity that $\cot^2 x + 1 = \csc^2 x$, we keep simplify to:

$$\csc^2 \frac{\pi}{2m+1} + \csc^2 \frac{2\pi}{2m+1} + \cdots + \csc^2 \frac{m\pi}{2m+1} = \frac{m(2m+2)}{3}$$

You might not feel it, but we are actually really close to the end. Now, I will introduce another inequality through this graph:



It is really obvious that in this unit circle, $\tan x \geq x \geq \sin x$. We take the reciprocal of it and invert the inequality, we get:

$$\frac{1}{\tan x} \leq \frac{1}{x} \leq \frac{1}{\sin x}$$

Square, and simplify:

$$\cot^2 x \leq \frac{1}{x^2} \leq \csc^2 x$$

I'm not sure if you're ready for this final moment, but I'm incredibly excited right now.
For this final inequality, we replace $\frac{(1 \to m)\pi}{2m+1}$ with each x, so that we will have m inequalities like this:

$$\cot^2 \frac{\pi}{2m+1} \leq \frac{(2m+1)^2}{\pi^2} \leq \csc^2 \frac{\pi}{2m+1}, \cdots, \cot^2 \frac{m\pi}{2m+1} \leq \frac{(2m+1)^2}{(m\pi)^2} \leq \csc^2 \frac{m\pi}{2m+1}$$

Now we sum up all the inequalities. Remember what we got previously:

$$\cot^2 \frac{\pi}{2m+1} + \cot^2 \frac{2\pi}{2m+1} + \cdots + \cot^2 \frac{m\pi}{2m+1} = \frac{m(2m-1)}{3}$$

$$\csc^2 \frac{\pi}{2m+1} + \csc^2 \frac{2\pi}{2m+1} + \cdots + \csc^2 \frac{m\pi}{2m+1} = \frac{m(2m+2)}{3}$$

So the side part is known, and we write:

$$\frac{m(2m-1)}{3} \leq \frac{(2m+1)^2}{\pi^2} + \frac{(2m+1)^2}{(2\pi)^2} + \cdots + \frac{(2m+1)^2}{(m\pi)^2} \leq \frac{m(2m+2)}{3}$$

Times $\frac{\pi^2}{(2m+1)^2}$ through all sides:

$$\frac{\pi^2 \cdot m \cdot (2m+1)}{3 \cdot (2m+1)^2} \leq \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{m^2} \leq \frac{\pi^2 \cdot m \cdot (2m+2)}{3 \cdot (2m+1)^2}$$

Notice that:

$$\lim_{m \to \infty} \left(\frac{\pi^2 \cdot m \cdot (2m+1)}{3 \cdot (2m+1)^2} \right) = \frac{\pi^2}{6}, \lim_{m \to \infty} \left(\frac{\pi^2 \cdot m \cdot (2m+2)}{3 \cdot (2m+1)^2} \right) = \frac{\pi^2}{6}$$

Thus, according to something introduced in my AP Calculus AB course but never applied—**Squeeze theorem**:

$$\lim_{m \to \infty} \left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{m^2} \right) = \frac{\pi^2}{6}$$

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Conclusion

Euler's proof is flawed, but the interesting coincidence made his track work (accidentally). His intuition on distinguishing $1.643934 \approx \frac{\pi^2}{6}$ is the mountain that none of the other mathematicians could realize; and that makes genius a real Genius.
The entire track of Cauchy's proof contains all the knowledge within the range of AP Calculus AB. It is so elegant that all students in my Calculus class, with certain assistance, can understand it (I should create a test to verify this). The previous day, I tried not to make too many connections to higher levels of math using elementary mathematics, because I had heard that a smart Math Olympiad winner had attempted to use elementary math to solve everything and ended up unable to prove a simple theorem in Calculus. But Cauchy changed my mind with his proof.
Of course, there are many other proofs of the Basel Question, such as using Euler's formula, the Fourier transform, the double integral, etc (I will add more in the future once I know more).

$$\int_0^1 \int_0^1 \frac{1}{1 - xy} \, dx \, dy = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

Because there are so many continuous discoveries on this sum, German mathematician Bernhard Riemann created the **Riemann zeta function** as:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots$$

This function was introduced by Euler and later extended to complex analysis by Riemann; it can be considered one of the most essential concepts in analytic number theory, as it is relevant to the distribution of all prime numbers.

The Basel problem is just a beginning; the mystery of math is still ongoing...