Terminal Velocity Derivation

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Prove

According to the differential equation, we could simply get the formula:

$$\frac{1}{m}dt = \frac{1}{mg - bv^2}dv$$

Integrating both sides

$$\int \frac{1}{m}dt = \int \frac{1}{mg - bv^2}dv$$

Simplify the right integrand

$$\int \frac{1}{mg - bv^2} = \frac{1}{b}\int \frac{1}{a^2 - v^2}dv, where\ a = \sqrt{\frac{mg}{b}}$$

To solve the integral, we could change the form of it to

$$\int \frac{1}{a^2 - v^2}dv = \int \frac{(a+v)+(a-v)}{(a+v)(a-v)} \cdot \frac{1}{2a}dv$$

Simplify, and we got

$$\frac{1}{2a} \cdot \left[\int \frac{1}{a-v}dv + \int \frac{1}{a+v}dv\right]$$

Then, set this back to the original equation, with the limit that time from 0 to t and velocity from 0 to v(t) (in which the initial velocity is 0, I will discuss the case where the initial velocity is v_0 later if possible)

$$\int_0^t \frac{1}{m}dt = \int_0^{v(t)} \frac{1}{mg - bv^2}dv$$

$$\int_0^t \frac{1}{m}dt = \frac{1}{b}\int_0^{v(t)} \frac{1}{a^2 - v^2}dv,\ \ a = \sqrt{\frac{mg}{b}}$$

$$\int_0^t \frac{1}{m}dt = \frac{1}{b} \cdot \frac{1}{2a} \cdot \left[\int_0^{v(t)} \frac{1}{a-v}dv + \int_0^{v(t)} \frac{1}{a+v}dv\right]$$

With simple substitution and basic integral rules

$$\int \frac{1}{a-v}dv = -\ln(a-v)+C, \int \frac{1}{a+v}dv = \ln(a+v)+C$$

Plug in back to the definite integral and use some stuff that I have not learned in Calculus AB yet, and with some algebraic simplification, we could get

$$\frac{t}{m} = \frac{1}{2ab} \cdot \{-\ln[a-v(t)] + \ln[a+v(t)]\}$$

$$\frac{t}{m} = \frac{1}{2ab} \cdot \ln\left[\frac{a+v(t)}{a-v(t)}\right]$$

Keep simplify

$$\frac{2abt}{m} = \ln\left[\frac{a+v(t)}{a-v(t)}\right]$$

$$e^{\frac{2abt}{m}} = \frac{a+v(t)}{a-v(t)}$$

Then we get

$$v(t) = a \cdot \frac{e^{\frac{2abt}{m}}+1}{e^{\frac{2abt}{m}}-1},\ where\ a = \sqrt{\frac{mg}{b}}$$

Ultimately and finally:

$$v(t) = \sqrt{\frac{mg}{b}} \cdot \frac{e^{\frac{2\sqrt{\frac{mg}{b}}\cdot bt}{m}}+1}{e^{\frac{2\sqrt{\frac{mg}{b}}\cdot bt}{m}}-1}$$

With final simplification, remarkably, we substitute a and get:

$$v(t) = \sqrt{\frac{mg}{b}} \cdot \frac{e^{2t\sqrt{\frac{bg}{m}}}+1}{e^{2t\sqrt{\frac{bg}{m}}}-1}$$

Verify

To verify, we could simply use the fancy hyperbolic trigonometric function we get and simplify that with the definition of the hyperbolic trigonometric function.

According to the handout with the mysteriously occult hyperbolic trigonometric substitution, we get that

$$v(t) = \sqrt{\frac{mg}{b}} tanh\left(t\sqrt{\frac{bg}{m}} + arctanh\left(v_0\sqrt{\frac{b}{mg}}\right)\right)$$

We simplify step by step; first is arctanh, according to the definition:

$$arctanh(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

With the prerequisite that we assume $v_0=0$, easily we get

$$arctanh\left(v_0\sqrt{\frac{bg}{m}}\right) = \frac{1}{2}\ln\left(\frac{1+0\sqrt{\frac{b}{mg}}}{1-0\sqrt{\frac{b}{mg}}}\right) = 0$$

Then, with the definition of tanh:

$$tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{e^{2x}+1}{e^{2x}-1}$$

Substitute x with $t\sqrt{\frac{bg}{m}}$, profoundly, we get

$$v(t) = \sqrt{\frac{mg}{b}} \cdot \frac{e^{2t\sqrt{\frac{bg}{m}}}+1}{e^{2t\sqrt{\frac{bg}{m}}}-1}$$

$\square$