Fun Algebra Question

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There is a fun question on today's Math League, and I solve it with a neat formula, and here is the question:

Given that:

$$a + b + c = 4$$

$$a^2 + b^2 + c^2 = 10$$

$$a^3 + b^3 + c^3 = 22$$

Find:

$$a^4 + b^4 + c^4$$

Before starting it, I would like to introduce the equations I used in the question.

$$(a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + ac + bc) ①$$

$$a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - ac - bc) ②$$

Then we could start solving it!

plug in $a + b + c = 4$ and $a^2 + b^2 + c^2 = 10$ to ①, we get:

$$4^2 = 10 + 2(ab + ac + bc)$$

$$2(ab + ac + bc) = 6$$

$$ab + ac + bc = 3$$

plug in $a + b + c = 4$, $a^2 + b^2 + c^2 = 10$, $a^3 + b^3 + c^3 = 22$, and $ab + ac + bc = 3$ to ②, we get:

$$22 - 3abc = 4 \cdot (10 - 3))$$

$$-3abc = 6$$

$$abc = -2$$

For $ab + ac + bc = 3$, we could square it as ① and get:

$$(ab + ac + bc)^2 = a^2b^2 + a^2c^2 + b^2c^2 + 2(ab \cdot ac + ab \cdot bc + ac \cdot bc)$$

$$(ab + ac + bc)^2 = a^2b^2 + a^2c^2 + b^2c^2 + 2[abc(a + b + c)]$$

Plug in $a + b + c = 4$, $ab + ac + bc = 3$, and $abc = -2$, we get:

$$3^2 = a^2b^2 + a^2c^2 + b^2c^2 + 2 \cdot (-2) \cdot 4$$

$$a^2b^2 + a^2c^2 + b^2c^2 = 25$$

Replace to a, b, c at ① to a^2, b^2, c^2, we get:

$$(a^2 + b^2 + c^2)^2 = a^4 + b^4 + c^4 + 2(a^2b^2 + a^2c^2 + b^2c^2)$$

Plug in $a^2 + b^2 + c^2 = 10$, $a^2b^2 + a^2c^2 + b^2c^2 = 25$, easily(I don't even want to write it) we get:

$$a^4 + b^4 + c^4 = 50$$

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Also, we could try not to use that fancy equation to find abc by simply cube the $(a + b + c)$, and we got:

$$(a + b + c)^3 = a^3 + b^3 + c^3 + 3(a^2b + a^2c + b^2a + b^2c + c^2a + c^2b) + 6abc$$

$$(a + b + c)^3 + 3abc = a^3 + b^3 + c^3 + 3(a^2b + a^2c + b^2a + b^2c + c^2a + c^2b) + 9abc$$

Matching abc with the things in the parabola and re-express right side:

$$a^3 + b^3 + c^3 + 3(a^2b + a^2c + b^2a + b^2c + c^2a + c^2b) + 9abc$$

$$= a^3 + b^3 + c^3 + (3a^2b + 3abc + 3a^2c) + (3b^2a + 3abc + 3b^2c) + (3c^2a + 3abc + 3c^2b)$$

$$= a^3 + b^3 + c^3 + 3a(ab + ac + bc) + 3b(ab + ac + bc) + 3c(ab + ac + bc)$$

$$= a^3 + b^3 + c^3 + 3(a + b + c)(ab + ac + bc)$$

and with the thing we know on the left sides, we could find abc:

$$(a + b + c)^3 + 3abc = a^3 + b^3 + c^3 + 3(a + b + c)(ab + ac + bc)$$

in which $a + b + c = 4$, $a^3 + b^3 + c^3 = 22$, $ab + ac + bc = 3$, easily we get:

$$abc = -2$$

Then start over the step we did at the beginning of this page, we also get:

$$a^4 + b^4 + c^4 = 50$$