Intro about Cross Product

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Introduction

Here we will derive that given two vectors:

$$\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$$

$$\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$$

Their cross product is:

$$\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}$$

Properties

Recall the properties for the cross product (they might be useful):

$$\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u} \tag{1}$$
$$\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w}) \tag{2}$$
$$c(\mathbf{u} \times \mathbf{v}) = (c\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (c\mathbf{v}) \tag{3}$$
$$\mathbf{u} \times 0 = 0 \times \mathbf{u} = 0 \tag{4}$$
$$\mathbf{u} \times \mathbf{u} = 0 \tag{5}$$
$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} \tag{6}$$
$$\mathbf{i} \times \mathbf{j} = \mathbf{k} \quad \mathbf{j} \times \mathbf{k} = \mathbf{i} \quad \mathbf{k} \times \mathbf{i} = \mathbf{j} \tag{7}$$

Proof for the cross product

Proof. Let $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$

$$
\begin{aligned}
\mathbf{u} \times \mathbf{v} =& (u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}) \times (v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}) \\
=& (u_1\mathbf{i} \times v_1\mathbf{i}) + (u_1\mathbf{i} \times v_2\mathbf{j}) + (u_1\mathbf{i} \times v_3\mathbf{k}) + \\
& (u_2\mathbf{j} \times v_1\mathbf{i}) + (u_2\mathbf{j} \times v_2\mathbf{j}) + (u_2\mathbf{j} \times v_3\mathbf{k}) + && \text{(Property 2)} \\
& (u_3\mathbf{k} \times v_1\mathbf{i}) + (u_3\mathbf{k} \times v_2\mathbf{j}) + (u_3\mathbf{k} \times v_3\mathbf{k}) \\
=& u_1v_1(\mathbf{i} \times \mathbf{i}) + u_1v_2(\mathbf{i} \times \mathbf{j}) + u_1v_3(\mathbf{i} \times \mathbf{k}) + \\
& u_2v_1(\mathbf{j} \times \mathbf{i}) + u_2v_2(\mathbf{j} \times \mathbf{j}) + u_2v_3(\mathbf{j} \times \mathbf{k}) + && \text{(Property 3)} \\
& u_3v_1(\mathbf{k} \times \mathbf{i}) + u_3v_2(\mathbf{k} \times \mathbf{j}) + u_3v_3(\mathbf{k} \times \mathbf{k}) \\
=& u_1v_2(\mathbf{i} \times \mathbf{j}) + u_1v_3(\mathbf{i} \times \mathbf{k}) + u_2v_1(\mathbf{j} \times \mathbf{i}) + \\
& u_2v_3(\mathbf{j} \times \mathbf{k}) + u_3v_1(\mathbf{k} \times \mathbf{i}) + u_3v_2(\mathbf{k} \times \mathbf{j}) && \text{(Property 5)} \\
=& u_1v_2(\mathbf{i} \times \mathbf{j}) - u_2v_1(\mathbf{i} \times \mathbf{j}) + \\
& u_2v_3(\mathbf{j} \times \mathbf{k}) - u_3v_2(\mathbf{j} \times \mathbf{k}) + && \text{(Property 1)} \\
& u_3v_1(\mathbf{k} \times \mathbf{i}) - u_1v_3(\mathbf{k} \times \mathbf{i}) \\
=& (u_1v_2 - u_2v_1)\mathbf{k} + (u_2v_3 - u_3v_2)\mathbf{i} + (u_3v_1 - u_1v_3)\mathbf{j} && \text{(Property 7)} \\
=& (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k} && \text{(Rearrange)}
\end{aligned}
$$

□

End

I might doubt whether there is circular reasoning about Property 7, as we use the conclusion of this project to prove Property 7. However, we could get Property 7 through the original definition of the cross product. Thus, the proof should be valid.
Also, the relationship between the cross product and the parallelogram intrigued me; I might prove that in the future (lack of motivation due to no extra credit for that).