Integral Formula Derivation

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Intro

This project is the derivation of the advanced integral formula that you could find in the Calculus test book, and the one I found is written by Tongji University, where there are 147 integral formulas in total. The intention of this project stems from another project that I created, "Terminal Velocity Derivation." I have noticed that even math/physics teachers are unsure how those integral formulas are derived, and I think the derivation process is omitted extensively. In this project, I will derive those formulas step by step, ensuring that even a student with an AP Calculus BC background can understand how they are derived, and unveil those "secrets" beyond the general textbook.

Integral with $ax + b$

1. $\int \frac{1}{ax+b}dx = \int \frac{1}{a(x+\frac{b}{a})}dx = \frac{1}{a}\int \frac{1}{ax+b}d(ax+b) = \frac{1}{a}\ln|ax+b| + C$

$$\blacksquare \int \frac{1}{ax+b}dx = \frac{1}{a}\ln|ax+b| + C$$

2. $\int (ax+b)^{\mu}dx \xrightarrow[dx=\frac{du}{a}]{u=ax+b} \frac{1}{a}\int u^{\mu}du = \frac{1}{a}\cdot\frac{u^{\mu+1}}{\mu+1} + C = \frac{1}{a(\mu+1)}\cdot(ax+b)^{\mu+1} + C$

$$\blacksquare \int (ax+b)^{\mu}dx = \frac{1}{a(\mu+1)}(ax+b)^{\mu+1} + C \ (\mu \neq -1)$$

3. $\int \frac{x}{ax+b}dx = \int \frac{1}{a}\cdot\frac{ax+b-b}{ax+b}dx = \frac{1}{a}(\int \frac{ax+b}{ax+b}dx - \int \frac{b}{ax+b}dx)$
$\implies \frac{1}{a}(x - \frac{b}{a}\cdot\ln|ax+b|) + C = \frac{1}{a^2}(ax - b\ln|ax+b|) + C$

$$\blacksquare \int \frac{x}{ax+b}dx = \frac{1}{a^2}\left(ax - b\ln|ax+b|\right) + C$$

4. $\int \frac{x^2}{ax+b}dx = \int \frac{1}{a^2}\cdot\frac{a^2x^2-b^2+b^2}{ax+b}dx = \frac{1}{a^2}(\int \frac{(ax+b)(ax-b)}{ax+b}dx + \int \frac{b^2}{ax+b}dx)$
$\implies \frac{1}{a^2}(\int (ax-b)dx + b^2\int \frac{1}{ax+b}dx) = \frac{1}{a^3}[\frac{1}{2}(ax-b)^2 + b^2\ln|ax+b|] + C$

$$\blacksquare \int \frac{x^2}{ax+b} = \frac{1}{a^3}\left[\frac{1}{2}(ax-b)^2 + b^2\ln|ax+b|\right] + C$$

5. $\int \frac{1}{x(ax+b)}dx \implies \int(\frac{A}{x}+\frac{B}{ax+b})dx = \int \frac{A(ax+b)+Bx}{x(ax+b)}dx = \int \frac{Aax+Ab+Bx}{x(ax+b)} \xrightarrow{Aax+Ab+Bx=1}$
$Aa+B=0, Ab=1 \to A=\frac{1}{b}, B=-\frac{a}{b} \implies \int(\frac{\frac{1}{b}}{x}+\frac{-\frac{a}{b}}{ax+b})dx = \int[\frac{1}{bx}-\frac{a}{b(ax+b)}]dx$
$\implies \frac{1}{b}\int \frac{1}{x}dx - \frac{a}{b}\int \frac{1}{ax+b}dx = \frac{1}{b}(\ln|x|-\ln|ax+b|)+C = \frac{1}{b}\ln|\frac{x}{ax+b}|+C$

$$\blacksquare \int \frac{1}{x(ax+b)}dx = \frac{1}{b}\ln\left|\frac{x}{ax+b}\right| + C$$

6. $\int \frac{1}{x^2(ax+b)}dx \implies \int(\frac{A}{x}+\frac{B}{x^2}+\frac{C}{ax+b})dx = \int[\frac{Ax(ax+b)+B(ax+b)+Cx^2}{x^2(ax+b)}]dx \implies$
$\int \frac{Aax^2+Abx+Bax+Bb+Cx^2}{x^2(ax+b)}dx \xrightarrow{Aax^2+Abx+Bax+Bb+Cx^2=1} Bb=1, Ab+Ba=0,$
$Aa+C=0 \to A=-\frac{a}{b^2}, B=\frac{1}{b}, C=\frac{a^2}{b^2} \implies \int(\frac{-\frac{a}{b^2}}{x}+\frac{\frac{1}{b}}{x^2}+\frac{\frac{a^2}{b^2}}{ax+b})dx \implies$
$\int[-\frac{a}{bx}+\frac{1}{bx^2}+\frac{a^2}{b^2(ax+b)}]dx \xrightarrow[\int \frac{a^2}{b^2(ax+b)}-\frac{a}{bx}dx=\frac{a}{b^2}\ln|\frac{ax+b}{z}|+C]{\int \frac{1}{bx^2}dx=-\frac{1}{bx}+C} -\frac{1}{bx}+\frac{a}{b^2}\ln|\frac{ax+b}{x}|+C$

$$\blacksquare \int \frac{1}{x^2(ax+b)}dx = -\frac{1}{bx}+\frac{a}{b^2}\ln\left|\frac{ax+b}{x}\right| + C$$

7. $\int \frac{x}{(ax+b)^2}dx = \int \frac{1}{a}\cdot\frac{ax+b-b}{(ax+b)^2}dx = \frac{1}{a}\int[\frac{ax+b}{(ax+b)^2}-\frac{b}{(ax+b)^2}]dx$
$\implies \frac{1}{a}[\int \frac{1}{ax+b}dx - b\int \frac{1}{(ax+b)^2}]dx = \frac{1}{a}(\frac{1}{a}\ln|ax+b|+\frac{b}{a}\cdot\frac{1}{ax+b})+C$
$\implies \frac{1}{a^2}(\ln|ax+b|+\frac{b}{ax+b})+C$

$$\blacksquare \int \frac{x}{(ax+b)^2}dx = \frac{1}{a^2}\left(\ln|ax+b|+\frac{b}{ax+b}\right)+C$$

8. $\int \frac{x^2}{(ax+b)^2}dx = \frac{1}{a^2}\int \frac{a^2x^2-b^2+b^2}{ax+b}dx = \frac{1}{a^2}\int \frac{(ax+b)(ax-b)}{(ax+b)^2}+\frac{b^2}{(ax+b)^2}dx \implies$
$\frac{1}{a^2}\int \frac{ax-b}{ax+b}+\frac{b^2}{(ax+b)^2}dx = \frac{1}{a^2}\int \frac{ax+b-2b}{ax+b}+\frac{b^2}{(ax+b)^2}dx = \frac{1}{a^2}\int 1-\frac{2b}{ax+b}+\frac{b^2}{(ax+b)^2}dx$
$\implies \frac{1}{a^2}\left(x-\frac{2b}{a}\ln|ax+b|-\frac{b^2}{a(ax+b)}\right)+C = \frac{1}{a^3}\left(ax-2b\ln|ax+b|-\frac{b^2}{ax+b}\right)+C$

$$\blacksquare \int \frac{x^2}{(ax+b)^2}dx = \frac{1}{a^3}\left(ax-2b\ln|ax+b|-\frac{b^2}{ax+b}\right)+C$$

9. $\int \frac{1}{x(ax+b)^2}dx \implies \int[\frac{A}{x}+\frac{B}{(ax+b)}+\frac{C}{(ax+b)^2}]dx = \int \frac{A(ax+b)^2+Bx(ax+b)+Cx}{x(ax+b)^2}dx$
$\implies \int \frac{Aa^2x^2+2Aabx+Ab^2+Bax^2+Cx}{x(ax+b)^2}dx \xrightarrow{Aa^2x^2+2Aabx+Ab^2+Bax^2+Cx=1} Ab^2=1,$
$Aa^2+Ba=0, 2Aab+C=0 \to A=\frac{1}{b^2}, B=-\frac{a}{b^2}, C=-\frac{a}{b} \implies$
$\int \frac{\frac{1}{b^2}}{x}+\frac{-\frac{a}{b^2}}{(ax+b)}+\frac{-\frac{a}{b}}{(ax+b)^2}dx = \int \frac{1}{b^2x}-\frac{a}{b^2(ax+b)}-\frac{a}{b(ax+b)^2}dx \xrightarrow[\int \frac{a}{b(ax+b)^2}dx=\frac{1}{b(ax+b)}+C]{\int \frac{1}{b^2x}-\frac{a}{b^2(ax+b)}dx=\frac{1}{b^2}\ln|\frac{x}{ax+b}|+C}$
$\frac{1}{b^2}\ln|x|-\frac{1}{b^2}\ln|ax+b|+\frac{1}{b(ax+b)}+C = \frac{1}{b(ax+b)}+\frac{1}{b^2}\ln|\frac{x}{ax+b}|+C$

$$\blacksquare \int \frac{1}{x(ax+b)^2}dx = \frac{1}{b(ax+b)}+\frac{1}{b^2}\ln\left|\frac{x}{ax+b}\right|+C$$

Integral with $\sqrt{ax+b}$

10. $\int \sqrt{ax+b}\,dx \xrightarrow[du=adx]{u=ax+b} \int \frac{\sqrt{u}}{a}du = \frac{u^{\frac{1}{2}+1}}{a(\frac{1}{2}+1)}+C = \frac{2\sqrt{u^3}}{3a}+C = \frac{2}{3a}\sqrt{(ax+b)^3}+C$

$$\blacksquare \int \sqrt{ax+b} = \frac{2}{3a}\sqrt{(ax+b)^3}+C$$

11. $\int x\sqrt{ax+b}\ dx \xrightarrow[du=\frac{a}{2\sqrt{ax+b}}dx=\frac{a}{2u}dx]{u=\sqrt{ax+b},x=\frac{u^2-b}{a}} \int \frac{u^2-b}{a}\cdot u \cdot \frac{2u}{a}du = \frac{1}{a^2}\int(2u^4-2bu^2)du$

$\Longrightarrow \frac{1}{a^2}(\frac{2}{5}u^5-\frac{2b}{3}u^3)+C = \frac{1}{a^2}(\frac{6u^5-10bu^3}{15})+C = \frac{1}{a^2}[\frac{2u^3(3u^2-5b)}{15}]+C$

$\xrightarrow{u=\sqrt{ax+b}} \frac{2}{15a^2}\sqrt{(ax+b)^3}\cdot[3(ax+b)-5b]+C = \frac{2}{15a^2}(3ax-2b)\sqrt{(ax+b)^3}+C$

$$\blacksquare \int x\sqrt{ax+b}\ dx = \frac{2}{15a^2}(3ax-2b)\sqrt{(ax+b)^3}+C$$

12. $\int x^2\sqrt{ax+b}\,dx \xrightarrow[du=\frac{a}{2\sqrt{ax+b}}dx=\frac{a}{2u}dx]{u=\sqrt{ax+b},x^2=(\frac{u^2-b}{a})^2} \int(\frac{u^2-b}{a})^2\cdot u\cdot\frac{2u}{a}du = \int\frac{(u^4-2bu^2+b^2)\cdot(2u^2)}{a^2\cdot a}du$

$\Longrightarrow \frac{2}{a^3}\int(u^6-2bu^4+b^2u^2)du = \frac{2}{a^3}(\frac{u^7}{7}-\frac{2bu^5}{5}+\frac{b^2u^3}{3})+C = \frac{2u^3}{105a^3}(\frac{15u^4-42bu^2+35b^2}{105})+C$

$\xrightarrow{u=\sqrt{ax+b}} \frac{2}{a^3}[15(\sqrt{ax+b})^4-42b(\sqrt{ax+b})^2+35b^2](\sqrt{(ax+b)^3})+C$

$\Longrightarrow \frac{2}{105a^3}(15a^2x^2-12abx+8b^2)\sqrt{(ax+b)^3}+C$

$$\blacksquare \int x^2\sqrt{ax+b}\ dx = \frac{2}{105a^3}\left(15a^2x^2-12abx+8b^2\right)\sqrt{(ax+b)^3}+C$$

13. $\int \frac{x}{\sqrt{ax+b}}dx \xrightarrow[du=\frac{a}{2\sqrt{ax+b}}dx=\frac{a}{2u}dx]{u=\sqrt{ax+b},x=\frac{u^2-b}{a}} \int\frac{\frac{u^2-b}{a}}{u}\cdot\frac{2u}{a}du = \int\frac{2(u^2-b)}{a^2}du = \frac{2}{a^2}\int(u^2-b)du$

$\Longrightarrow \frac{2}{a^2}\cdot(\frac{1}{3}u^3-bu)+C = \frac{2}{3a^2}\cdot u\cdot(u^2-3b)+C$

$\xrightarrow{u=\sqrt{ax+b}} \frac{2}{3a^2}\cdot\sqrt{ax+b}\cdot(ax+b-3b)+C = \frac{2}{3a^2}\cdot(ax-2b)\cdot\sqrt{ax+b}+C$

$$\blacksquare \int \frac{x}{\sqrt{ax+b}}dx = \frac{2}{3a^2}\,(ax-2b)\,\sqrt{ax+b}+C$$

14. $\int \frac{x^2}{\sqrt{ax+b}}dx \xrightarrow[du=\frac{a}{2\sqrt{ax+b}}dx=\frac{a}{2u}dx]{u=\sqrt{ax+b}} \int\frac{(\frac{u^2-b}{a})^2}{u}\cdot\frac{2u}{a}du = \frac{2}{a^3}\int(u^2-b)^2du \Longrightarrow$

$\frac{2}{a^3}\int(u^4-2bu^2+b^2)du = \frac{2}{a^3}(\frac{1}{5}u^5-\frac{2bu^3}{3}+b^2u)+C = \frac{2}{a^3}\cdot u\cdot(\frac{3u^4-10bu^2+15b^2}{15})+C$

$\xrightarrow{u=\sqrt{ax+b}} \frac{2}{15a^3}\cdot\sqrt{ax+b}\cdot(3(ax+b)^2-10b(ax+b)+15b^2)+C$

$\Longrightarrow \frac{2}{15a^3}\cdot(3a^2x^2-4abx+8b^2)\cdot\sqrt{ax+b}+C$

$$\blacksquare \int \frac{x^2}{\sqrt{ax+b}}dx = \frac{2}{15a^3}\left(3a^2x^2-4abx+8b^2\right)\sqrt{ax+b}+C$$

15. $\int \frac{1}{x\sqrt{ax+b}}dx \xrightarrow[du=\frac{a}{2\sqrt{ax+b}}dx=\frac{a}{2u}dx]{u=\sqrt{ax+b},x=\frac{u^2-b}{a}} \int \frac{1}{\frac{u^2-b}{a}\cdot u}\cdot\frac{2u}{a}du = 2\int\frac{1}{u^2-b}du \implies$

① $b>0 \to 2\int\frac{1}{u^2-b}du = 2\int\frac{1}{u^2-(\sqrt{b})^2}du = 2\int\frac{1}{(u+\sqrt{b})\cdot(u-\sqrt{b})}du = 2\int\frac{(u+\sqrt{b})-(u-\sqrt{b})}{(u+\sqrt{b})\cdot(u-\sqrt{b})}\cdot\frac{1}{2\sqrt{b}}du$

$\implies \frac{1}{\sqrt{b}}\int(\frac{1}{u-\sqrt{b}}-\frac{1}{u+\sqrt{b}})du = \frac{1}{\sqrt{b}}\cdot(\ln|u-\sqrt{b}|-\ln|u+\sqrt{b}|)+C = \frac{1}{\sqrt{b}}\ln\left|\frac{u-\sqrt{b}}{u+\sqrt{b}}\right|+C$

$\xrightarrow{u=\sqrt{ax+b}} \frac{1}{\sqrt{b}}\ln\left|\frac{\sqrt{ax+b}-\sqrt{b}}{\sqrt{ax+b}+\sqrt{b}}\right|+C$

② $b<0 \to 2\int\frac{1}{u^2-b}du = 2\int\frac{1}{u^2+(\sqrt{-b})^2}du = 2\cdot\frac{1}{\sqrt{-b}}\arctan(\frac{u}{\sqrt{-b}})+C$

$\xrightarrow{u=\sqrt{ax+b}} \frac{2}{\sqrt{-b}}\arctan\sqrt{\frac{ax+b}{-b}}+C$

$$\blacksquare \int\frac{1}{x\sqrt{ax+b}}\,dx = \begin{cases}\frac{1}{\sqrt{b}}\ln\left|\frac{\sqrt{ax+b}-\sqrt{b}}{\sqrt{ax+b}+\sqrt{b}}\right|+C\ (b>0)\\ \frac{2}{\sqrt{-b}}\arctan\sqrt{\frac{ax+b}{-b}}+C\ (b<0)\end{cases}$$

16. $\int\frac{\sqrt{ax+b}}{x^2}dx = \int\sqrt{ax+b}\,d(-\frac{1}{x}) = -\frac{\sqrt{ax+b}}{x}-\int-\frac{1}{x}d(\sqrt{ax+b}) = -\frac{\sqrt{ax+b}}{x}+\frac{a}{2}\int\frac{1}{x\sqrt{ax+b}}dx$

$$\blacksquare \int\frac{\sqrt{ax+b}}{x^2}dx = -\frac{\sqrt{ax+b}}{x}+\frac{a}{2}\int\frac{1}{x\sqrt{ax+b}}dx$$

17. $\int\frac{\sqrt{ax+b}}{x}dx \xrightarrow[du=\frac{a}{2\sqrt{ax+b}}dx=\frac{a}{2u}dx]{u=\sqrt{ax+b},x=\frac{u^2-b}{a}} \int\frac{u}{\frac{u^2-b}{a}}\cdot\frac{2u}{a}du = \int\frac{2u^2}{u^2-b}du = 2\int\frac{u^2-b+b}{u^2-b}du$

$\implies 2\int(1+\frac{b}{u^2-b})du = 2\sqrt{ax+b}+2b\int\frac{1}{ax+b-b}d(\sqrt{ax+b}) = 2\sqrt{ax+b}+b\int\frac{1}{x\sqrt{ax+b}}dx$

$$\blacksquare \int\frac{\sqrt{ax+b}}{x} = 2\sqrt{ax+b}+b\int\frac{1}{x\sqrt{ax+b}}dx$$

18. $\int\frac{1}{x^2\sqrt{ax+b}}dx = \int(\frac{A}{x\sqrt{ax+b}}+\frac{B\sqrt{ax+b}}{x^2})dx = \int\frac{Ax+B(ax+b)}{x^2\sqrt{ax+b}}dx = \int\frac{Ax+Bax+Bb}{x^2\sqrt{ax+b}}dx$

$\xrightarrow{Ax+Bax+Bb=1} Bb=1, Ax+Bax=0 \to B=\frac{1}{b}, A=-\frac{a}{b} \implies \int(\frac{-\frac{a}{b}}{x\sqrt{ax+b}}+\frac{\frac{1}{b}\sqrt{ax+b}}{x^2})dx$

$\implies -\frac{a}{b}\int\frac{1}{x\sqrt{ax+b}}dx+\frac{1}{b}\int\frac{\sqrt{ax+b}}{x^2}dx = -\frac{a}{b}\int\frac{1}{x\sqrt{ax+b}}dx+\frac{1}{b}(-\frac{\sqrt{ax+b}}{x}+\frac{a}{2}\int\frac{1}{x\sqrt{ax+b}}dx)$

$\implies -\frac{\sqrt{ax+b}}{bx}+(-\frac{a}{b}+\frac{a}{2b})\int\frac{1}{x\sqrt{ax+b}}dx = -\frac{\sqrt{ax+b}}{bx}-\frac{a}{2b}\int\frac{1}{x\sqrt{ax+b}}dx$

$$\blacksquare \int\frac{1}{x^2\sqrt{ax+b}}dx = -\frac{\sqrt{ax+b}}{bx}-\frac{a}{2b}\int\frac{1}{x\sqrt{ax+b}}dx$$

Integral with $x^2 \pm a^2$

19. $\int \frac{1}{x^2+a^2}dx = \int \frac{1}{a^2(\frac{x^2}{a^2}+1)}dx = \int \frac{1}{a^2((\frac{x}{a})^2+1)^2} \cdot d(\frac{x}{a}) \cdot a = \frac{1}{a}\arctan(\frac{x}{a}) + C$

$$\blacksquare \int \frac{1}{x^2+a^2}dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C$$

20. $\int \frac{1}{(x^2+a^2)^n}dx = \frac{x}{(x^2+a^2)^n} - \int x \cdot d(\frac{1}{(x^2+a^2)^n}) = \frac{x}{(x^2+a^2)^n} - \int x \cdot \frac{(-n)\cdot 2x}{(x^2+a^2)^{n+1}}dx$

$\vdash 2n\int \frac{x^2}{(x^2+a^2)^{n+1}}dx = 2n\int \frac{x^2+a^2-a^2}{(x^2+a^2)^{n+1}}dx = 2n[\int \frac{1}{(x^2+a^2)^n}dx - \int \frac{a^2}{(x^2+a^2)^{n+1}}dx] \dashv$

$\implies 2na^2\int \frac{1}{(x^2+a^2)^{n+1}}dx = \frac{x}{(x^2+a^2)^n} + (2n-1)\int \frac{1}{(x^2+a^2)^n}dx \xrightarrow[n \implies n-1]{n+1 \implies n}$

$2(n-1)a^2 \cdot \int \frac{1}{(x^2+a^2)^n}dx = \frac{x}{(x^2+a^2)^{n-1}} + [2(n-1)-1]\int \frac{1}{(x^2+a^2)^{n-1}}dx$

$\xrightarrow[\text{both side}]{\div 2(n-1)a^2} \int \frac{1}{(x^2+a^2)^n}dx = \frac{x}{2(n-1)a^2(x^2+a^2)^{n-1}} + \frac{2n-3}{2(n-1)a^2}\int \frac{1}{(x^2+a^2)^{n-1}}dx$

$$\blacksquare \int \frac{1}{(x^2+a^2)^n}dx = \frac{x}{2(n-1)a^2(x^2+a^2)^{n-1}} + \frac{2n-3}{2(n-1)a^2}\int \frac{1}{(x^2+a^2)^{n-1}}dx$$

21. $\int \frac{1}{x^2-a^2}dx = \int \frac{1}{2a} \cdot \frac{x+a-(x-a)}{(x+a)\cdot(x-a)}dx = \frac{1}{2a}(\int \frac{1}{x-a}dx - \int \frac{1}{x+a}dx)$

$\implies \frac{1}{2a} \cdot (\ln|x-a| - \ln|x+a|) + C = \frac{1}{2a}\ln|\frac{x-a}{x+a}| + C$

$$\blacksquare \int \frac{1}{x^2-a^2}dx = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C$$

Integral with $ax^2 + b$

22. $\int \frac{1}{ax^2+b}dx = \int \frac{1}{(\sqrt{a}x)^2+b}dx$

① $b > 0 \to \int \frac{1}{(\sqrt{a}x)^2+b}dx = \int \frac{1}{(\sqrt{a}x)^2+(\sqrt{b})^2}dx = \int \frac{1}{(\sqrt{a}x)^2+(\sqrt{b})^2} \cdot \frac{d(\sqrt{a}x)}{\sqrt{a}}$

$\implies \frac{1}{\sqrt{a}} \cdot \int \frac{1}{(\sqrt{a}x)^2+(\sqrt{b})^2}d(\sqrt{a}x) \overset{\boxed{19}}{=} \frac{1}{\sqrt{ab}}\arctan\sqrt{\frac{a}{b}}x + C$

② $b < 0 \to \int \frac{1}{(\sqrt{a}x)^2+b}dx = \int \frac{1}{(\sqrt{a}x)^2-(\sqrt{-b})^2}dx = \int \frac{1}{(\sqrt{a}x)^2-(\sqrt{-b})^2} \cdot \frac{d(\sqrt{a}x)}{\sqrt{a}}$

$\implies \frac{1}{\sqrt{a}} \cdot \int \frac{1}{(\sqrt{a}x)^2-(\sqrt{-b})^2}d(\sqrt{a}x) \overset{\boxed{21}}{=} \frac{1}{2\sqrt{-ab}}\ln|\frac{\sqrt{a}x-\sqrt{-b}}{\sqrt{a}x+\sqrt{-b}}| + C$

$$\blacksquare \int \frac{1}{ax^2+b}dx = \begin{cases} \frac{1}{\sqrt{ab}}\arctan\sqrt{\frac{a}{b}}x + C(b>0) \\ \frac{1}{2\sqrt{-ab}}\ln\left|\frac{\sqrt{a}x-\sqrt{-b}}{\sqrt{a}x+\sqrt{-b}}\right| + C(b<0) \end{cases}$$

23. $\int \frac{x}{ax^2+b}dx = \int \frac{x}{ax^2+b} \cdot \frac{d(ax^2+b)}{2ax} = \frac{1}{2a} \cdot \int \frac{1}{ax^2+b}d(ax^2+b) = \frac{1}{2a}\ln|ax^2+b| + C$

$$\blacksquare \int \frac{x}{ax^2+b}dx = \frac{1}{2a}\ln\left|ax^2+b\right| + C$$

24. $\int \frac{x^2}{ax^2+b}dx = \int \frac{1}{a} \cdot \frac{ax^2+b-b}{ax^2+b}dx = \frac{1}{a}(\int \frac{ax^2+b}{ax^2+b}dx - \frac{b}{ax^2+b}dx) = \frac{x}{a} - \frac{b}{a}\int \frac{1}{ax^2+b}dx$

$$\blacksquare \int \frac{x^2}{ax^2+b}dx = \frac{x}{a} - \frac{b}{a}\int \frac{1}{ax^2+b}dx$$

25. $\int \frac{1}{x(ax^2+b)}dx = \int \frac{x}{x^2(ax^2+b)}dx = \int \frac{1}{x^2(ax^2+b)}d(\frac{1}{2}x^2) = \frac{1}{2}\int \frac{1}{x^2(ax^2+b)}d(x^2)$
$\implies \frac{1}{2}\int(\frac{A}{x^2} + \frac{B}{ax^2+b})d(x^2) = \frac{1}{2}\int \frac{A(ax^2+b)+Bx^2}{x^2(ax^2+b)}d(x^2) = \frac{1}{2}\int \frac{Aax^2+Bx^2+Ab}{x^2(ax^2+b)}d(x^2)$
$\xrightarrow{Aax^2+Bx^2+Ab=1} Ab = 1, Aa+B = 0 \to B = -\frac{a}{b}, A = \frac{1}{b} \implies \frac{1}{2}\int(\frac{\frac{1}{b}}{x^2}+\frac{-\frac{a}{b}}{ax^2+b})d(x^2)$
$\implies \frac{1}{2}[\frac{1}{b}\int \frac{1}{x^2}d(x^2) - \frac{a}{b}\int \frac{1}{ax^2+b}d(x^2)] = \frac{1}{2b}(\ln|x^2| - \ln|ax^2+b|) + C = \frac{1}{2b}\ln\frac{x^2}{|ax^2+b|} + C$

$$\blacksquare \int \frac{1}{x(ax^2+b)}dx = \frac{1}{2b}\ln\frac{x^2}{|ax^2+b|} + C$$

26. $\int \frac{1}{x^2(ax^2+b)}dx \implies \int(\frac{A}{x^2}+\frac{B}{ax^2+b})dx = \int \frac{A(ax^2+b)+Bx^2}{x^2(ax^2+b)}dx = \int \frac{Aax^2+Bx^2+Ab}{x^2(ax^2+b)}dx$
$\xrightarrow{Aax^2+Bx^2+Ab=1} Ab = 1, Aa + B = 0 \to B = -\frac{a}{b}, A = \frac{1}{b} \implies \int(\frac{\frac{1}{b}}{x^2} + \frac{-\frac{a}{b}}{ax^2+b})dx$
$\implies \frac{1}{b}\int \frac{1}{x^2}dx - \frac{a}{b}\int \frac{1}{ax^2+b}dx = -\frac{1}{bx} - \frac{a}{b}\int \frac{1}{ax^2+b}dx$

$$\blacksquare \int \frac{1}{x^2(ax^2+b)}dx = -\frac{1}{bx} - \frac{a}{b}\int \frac{1}{ax^2+b}dx$$

27. $\int \frac{1}{x^3(ax^2+b)}dx = \int \frac{x}{x^4(ax^2+b)}dx = \int \frac{1}{x^4(ax^2+b)}d(\frac{1}{2}x^2) = \frac{1}{2}\int \frac{1}{x^4(ax^2+b)}d(x^2)$
$\implies \frac{1}{2}[\int(\frac{A}{x^2} + \frac{B}{x^4} + \frac{C}{ax^2+b})]d(x^2) = \frac{1}{2}[\int \frac{Ax^2(ax^2+b)+B(ax^2+b)+Cx^4}{x^4(ax^2+b)}d(x^2)]$
$\implies \frac{1}{2}\int \frac{Aax^4+Cx^4+Abx^2+Bax^2+Bb}{x^4(ax^2+b)}d(x^2) \xrightarrow{Aax^4+Cx^4+Abx^2+Bax^2+Bb=1}$
$\implies Bb = 1, Ab + Ba = 0, Aa + C = 0 \to B = \frac{1}{b}, A = -\frac{a}{b^2}, C = \frac{a^2}{b^2}$
$\implies \frac{1}{2}\int(\frac{-\frac{a}{b^2}}{x^2}+\frac{\frac{1}{b}}{x^4}+\frac{\frac{a^2}{b^2}}{ax^2+b})d(x^2) = \frac{1}{2}[-\frac{a}{b^2}\int \frac{1}{x^2}d(x^2)+\frac{1}{b}\int \frac{1}{x^4}d(x^2)+\frac{a^2}{b^2}\int \frac{1}{ax^2+b}d(x^2)]$
$\implies \frac{1}{2}(-\frac{a}{b^2}\ln|x^2| + \frac{1}{b}\cdot\frac{(x^2)^{-1}}{-1} + \frac{a}{b^2}\ln|ax^2+b|) + C = \frac{a}{2b^2}\ln\frac{|ax^2+b|}{x^2} - \frac{1}{2bx^2} + C$

$$\blacksquare \int \frac{1}{x^3(ax^2+b)}dx = \frac{a}{2b^2}\ln\frac{|ax^2+b|}{x^2} - \frac{1}{2bx^2} + C$$

28. $\int \frac{1}{(ax^2+b)^2}dx = \int \frac{1}{(ax^2+b)^2}\cdot\frac{d(ax^2+b)}{2ax} = \int \frac{1}{2ax}d(-\frac{1}{ax^2+b}) = -\frac{1}{2ax(ax^2+b)} - \int -\frac{1}{ax^2+b}d(\frac{1}{2ax})$

$\vdash \int -\frac{1}{ax^2+b}d(\frac{1}{2ax}) = \frac{1}{2a}\int \frac{1}{x^2(ax^2+b)}dx \overset{\boxed{26}}{=} \frac{1}{2a}\cdot[-\frac{1}{bx} - \frac{a}{b}\int \frac{1}{ax^2+b}dx] = -\frac{1}{2abx} - \frac{1}{2b}\int \frac{1}{ax^2+b}dx \dashv$

$\implies -\frac{1}{2ax(ax^2+b)} - (-\frac{1}{2abx} - \frac{1}{2b}\int \frac{1}{ax^2+b}dx) = \frac{-b+ax^2+b}{2abx(ax^2+b)} + \frac{1}{2b}\int \frac{1}{ax^2+b}dx$
$\implies \frac{ax^2}{2abx(ax^2+b)} + \frac{1}{2b}\int \frac{1}{ax^2+b}dx = \frac{x}{2b(ax^2+b)} + \frac{1}{2b}\int \frac{1}{ax^2+b}dx$

$$\blacksquare \int \frac{1}{(ax^2+b)^2}dx = \frac{x}{2b(ax^2+b)} + \frac{1}{2b}\int \frac{1}{ax^2+b}dx$$

Integral with ax^2+bx+c $(a>0)$

29. $\int \frac{1}{ax^2+bx+c}dx \xrightarrow[\text{in vertex form: } y=a(x-h)+k]{ax^2+bx+c}$ with the vertex $(-\frac{b}{2a}, \frac{4ac-b^2}{4a})$

$\vdash ax^2+bx+v = a(x+\frac{b}{2a})^2+\frac{4ac-b^2}{4a} = \frac{4a^2(x+\frac{b}{2a})^2}{4a}+\frac{4ac-b^2}{4a} = \frac{1}{4a}[(2ax+b)^2+(4ac-b^2)] \dashv$

$\Longrightarrow \int \frac{1}{ax^2+bx+c}dx = 4a\int \frac{1}{(2ax+b)^2+(4ac-b^2)}\cdot\frac{d(2ax+b)}{2a} = 2\int \frac{1}{(2ax+b)^2+(4ac-b^2)}d(2ax+b)$

① $4ac > b^2 \to 2\int \frac{1}{(2ax+b)^2+(\sqrt{4ac-b^2})^2}d(2ax+b) \overset{\boxed{19}}{=} \frac{2}{\sqrt{4ac-b^2}}\arctan\frac{2ax+b}{\sqrt{4ac-b^2}}+C$

② $4ac < b^2 \to 2\int \frac{1}{(2ax+b)^2-(\sqrt{b^2-4ac})^2}d(2ax+b) \overset{\boxed{21}}{=} \frac{1}{\sqrt{b^2-4ac}}\ln|\frac{2ax+b-\sqrt{b^2-4ac}}{2ax+b+\sqrt{b^2-4ac}}|+C$

$$\blacksquare \int \frac{1}{ax^2+bx+c}dx = \begin{cases} \dfrac{2}{\sqrt{4ac-b^2}}\arctan\dfrac{2ax+b}{\sqrt{4ac-b^2}}+C(4ac>b^2) \\ \dfrac{1}{\sqrt{b^2-4ac}}\ln\left|\dfrac{2ax+b-\sqrt{b^2-4ac}}{2ax+b+\sqrt{b^2-4ac}}\right|+C(4ac<b^2) \end{cases}$$

30. $\int \frac{x}{ax^2+bx+c}dx = \int \frac{1}{2a}\cdot\frac{2ax+b-b}{ax^2+bx+c}dx = \frac{1}{2a}(\int \frac{2ax+b}{ax^2+bx+c}dx - \int \frac{b}{ax^2+bx+c}dx)$

$\Longrightarrow \frac{1}{2a}\int \frac{2ax+b}{ax^2+bx+c}\cdot\frac{d(ax^2+bx+c)}{2ax+b} - \frac{1}{2a}\int \frac{b}{ax^2+bx+c}dx = \frac{1}{2a}\ln|ax^2+bx+c| - \frac{b}{2a}\int \frac{1}{ax^2+bx+c}dx$

$$\blacksquare \int \frac{x}{ax^2+bx+c}dx = \frac{1}{2a}\ln\left|ax^2+bx+c\right| - \frac{b}{2a}\int \frac{1}{ax^2+bx+c}dx$$