Cross Product and Parallelogram

David Zhang

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Introduction

Here we will derive that given two vectors:

$$\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$$

$$\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$$

We have equation that connects cross product to the area of a parallelogram:

$$\|\mathbf{u}\|\|\mathbf{v}\|\sin\theta = \|\mathbf{u} \times \mathbf{v}\|$$

Properties

As we normally do, we will introduce some properties that could help the proof.

$$\|\mathbf{u}\| = \sqrt{u_1^2 + u_2^2 + u_3^2} \tag{1}$$

$$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3 \tag{2}$$

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\|\cos\theta \tag{3}$$

$$\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k} \tag{4}$$

$$\cos^2\theta + \sin^2\theta = 1 \tag{5}$$

$$(a-b)^2 = a^2 - 2ab + b^2 \tag{6}$$

$$(a+b+c)^2 = a^2 + b^2 + c^2 + 2(ab + ac + bc) \tag{7}$$

Proof

Since we want $\|\mathbf{u}\|\|\mathbf{v}\|\sin\theta = \|\mathbf{u} \times \mathbf{v}\|$, and both side is positive because it is magnitude. Thus, we could alternatively prove the squared version:

$$\|\mathbf{u}\|^2\|\mathbf{v}\|^2\sin^2\theta = \|\mathbf{u} \times \mathbf{v}\|^2$$

Then we start:

Proof. Let $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$

$$
\begin{aligned}
\|\mathbf{u}\|^2\|\mathbf{v}\|^2\sin^2\theta =& \|\mathbf{u}\|^2\|\mathbf{v}\|^2(1-\cos^2\theta) && \text{(Property 5)}\\
=& \|\mathbf{u}\|^2\|\mathbf{v}\|^2 - (\|\mathbf{u}\|^2\|\mathbf{v}\|^2\cos^2\theta)\\
=& \|\mathbf{u}\|^2\|\mathbf{v}\|^2 - (\mathbf{u}\cdot\mathbf{v})^2 && \text{(Property 3)}\\
=& \|\mathbf{u}\|^2\|\mathbf{v}\|^2 - (u_1v_1 + u_2v_2 + u_3v_3)^2 && \text{(Property 2)}\\
=& (u_1^2+u_2^2+u_3^2)(v_1^2+v_2^2+v_3^2) - && \text{(Property 1)}\\
& [u_1^2v_1^2 + u_2^2v_2^2 + u_3^2v_3^2 + 2(u_1v_1u_2v_2 + u_1v_1u_3v_3 + u_2v_2u_3v_3)] && \text{(Property 7)}\\
=& \cancel{u_1^2v_1^2} + u_1^2v_2^2 + u_1^2v_3^2 +\\
& u_2^2v_1^2 + \cancel{u_2^2v_2^2} + u_2^2v_3^2 +\\
& u_3^2v_1^2 + u_3^2v_2^2 + \cancel{u_3^2v_3^2} - && \text{(Cancel terms)}\\
& \cancel{u_1^2v_1^2} - \cancel{u_2^2v_2^2} - \cancel{u_3^2v_3^2} -\\
& 2u_1v_1u_2v_2 - 2u_1v_1u_3v_3 - 2u_2v_2u_3v_3\\
=& (u_2^2v_3^2 - 2u_2v_2u_3v_3 + u_3^2v_2^2) +\\
& (u_1^2v_3^2 - 2u_1v_1u_3v_3 + u_3^2v_1^2) + && \text{(Rearrange)}\\
& (u_1^2v_2^2 - 2u_1v_1u_2v_2 + u_2^2v_1^2)\\
=& (u_2v_3 - u_3v_2)^2 + (u_1v_3 - u_3v_1)^2 + (u_1v_2 - u_2v_1)^2 && \text{(Property 6)}\\
=& \|\mathbf{u}\times\mathbf{v}\|^2 && \text{(Property 4 and 1)}
\end{aligned}
$$

□

Remorse

I could definitely prove this in the quiz if I have more time; I am already in the step of attempting to square the dot product. It is a fun journey, though. I used less time on this project than on the previous cross product project.