Implicit Partial Differentiation

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Introduction

So, here we will try to derive the implicit (partial) differentiation.

Variable with x and y

The implicit form of the function that has variable with x and y could be expressed as:

$$z = f(x, y) = 0$$

Using the **Chain Rule** that we have learned previously to differentiate all based on x, we get:

$$\frac{dz}{dx} = \frac{\partial f(x, y)}{\partial x}\frac{dx}{dx} + \frac{\partial f(x, y)}{\partial y}\frac{dy}{dx} = 0$$

We know $\frac{dx}{dx} = 1$, so once we move the terms, we get:

$$\frac{dy}{dx} = \frac{-\frac{\partial f(x,y)}{\partial x}}{\frac{\partial f(x,y)}{\partial y}} = -\frac{F_x}{F_y}$$

Similarly, with partial differentiation:

$$\frac{\partial z}{\partial x} = \frac{\partial f(x, y)}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial f(x, y)}{\partial y}\frac{\partial y}{\partial x} = 0$$

$$\frac{\partial y}{\partial x} = -\frac{\frac{\partial f(x,y)}{\partial x}}{\frac{\partial f(x,y)}{\partial y}} = -\frac{F_x}{F_y}$$

Variable with x, y, and z

The implicit form of the function that has variables with x, y, and z could be expressed as:

$$w = f(x, y, z) = 0$$

Again, according to the **Chain Rule**, we get:

$$\frac{dw}{dx} = \frac{\partial f(x,y,z)}{\partial x}\frac{dx}{dx} + \frac{\partial f(x,y,z)}{\partial y}\frac{dy}{dx} + \frac{\partial f(x,y,z)}{\partial z}\frac{dz}{dx} = 0$$

With simplification, we get the following:

$$\frac{dy}{dx} = -\frac{\frac{\partial f(x,y,z)}{\partial x} + \frac{\partial f(x,y,z)}{\partial z}\frac{dz}{dx}}{\frac{\partial f(x,y,z)}{\partial y}} = -\frac{F_x + F_z\frac{dz}{dx}}{F_y}$$

Again, with partial differentiation, because z in this case becomes a "constant", so $F_z = 0$, and everything else remains the same, as:

$$\frac{\partial y}{\partial x} = -\frac{F_x}{F_y}$$