Intro to Bernoulli Number

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Intro

Let us start with this question: $1 + 2 + 3 + 4 + \cdots + 99 + 100 =?$
The great mathematician Carl Friedrich Gauss gave a solution when he was in elementary school:

$$\begin{aligned}
S &= 1 + 2 + \cdots + 99 + 100 \\
S &= 100 + 99 + \cdots + 2 + 1 \\
2S &= (1 + 100) + (2 + 99) + \cdots + (99 + 2) + (100 + 1) \\
&= 101 \times 100 = 10100 \\
S &= 2S \div 2 = 10100 \div 2 = 5500
\end{aligned}$$

That is! $1 + 2 + 3 + 4 + \cdots + 99 + 100 = 5500$! We could use a similar method to find $1 + 2 + \cdots + n$, where n is any positive integer greater than one.

$$\begin{aligned}
S &= 1 + 2 + \cdots + (n - 1) + n \\
S &= n + (n - 1)) + \cdots + 2 + 1 \\
2S &= (1 + n) + (2 + n - 1) + \cdots + (n - 1 + 2) + (n + 1) \\
&= (n + 1) \times n \\
S &= 2S \div 2 = n \times (n + 1) \div 2 = \frac{n(n + 1)}{2}
\end{aligned}$$

With the fancy summation notation:

$$\sum_{k=1}^{n} k = \frac{n(n + 1)}{2} = \frac{n^2}{2} + \frac{n}{2}$$

Power Up!...?

After solving the problem of $1+2+\cdots+n$, or the sum of all the positive integers from 1 to n, with the power of 1, we naturally would like to raise that one to the power of 2 or even higher.
For example, we want to know $1^2 + 2^2 + 3^2 + 4^2 + \cdots + n^2 =?$

If we apply a similar way to what we did to the power of 1, it looks like:

$$\begin{aligned}
S &= 1^2 + 2^2 + \cdots + (n-1)^2 + n^2 \\
S &= n^2 + (n-1)^2 + \cdots + 2^2 + 1^2 \\
2S &= (1^2 + n^2) + (2^2 + (n-1)^2) + \cdots + ((n-1)^2 + 2^2) + (n^2 + 1^2) \\
&= ???(\text{What is this? How we know the square sum})???
\end{aligned}$$

Yes, this does not work, indicating that we could not use the way we treat the power of 1 to higher powers. But is there any other way to do that?

Jacob Bernoulli

In the 18th century, Jacob Bernoulli dabbled in the series. He tried to find a general solution for $1^p + 2^p + 3^p + \cdots + k^p$, which:

$$\sum_{k=1}^{n} k^p = ?$$

He started with the power of one. Instead of using the solution mentioned previously, he did something else.
First, I believe we all agree that:

$$\sum_{k=1}^{n} k^0 = \sum_{k=1}^{n} 1 = \underbrace{1 + 1 + 1 + \cdots + 1}_{n \text{ times}} = n$$

As we all acknowledge $(k+1)^2 = k^2 + 2k + 1$, the equation is still valid even when we add a summation notation in front of both sides of it:

$$\sum_{k=1}^{n} (k+1)^2 = \sum_{k=1}^{n} (k^2 + 2k + 1)$$

With some simplification of the property of summation:

$$\sum_{k=1}^{n} (k+1)^2 = \sum_{k=1}^{n} k^2 + 2\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1$$

Move the square together:

$$\sum_{k=1}^{n} ((k+1)^2 - k^2) = 2\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1$$

Now we try to simplify this. We start from the left. If we write a direct expansion of this, it will be like:

$$\begin{aligned}
\sum_{k=1}^{n} ((k+1)^2 - k^2) &= \cancel{2^2} - 1^2 + \cancel{3^2} - \cancel{2^2} + \cdots + \cancel{n^2} - \cancel{(n-1)^2} + (n+1)^2 - \cancel{n^2} \\
&= (n+1)^2 - 1
\end{aligned}$$

This is called the Telescoping series, which means that you can expand the series and cancel the terms through the similarity between each term.

Now we try to simplify the right part. $\sum_{k=1}^{n} k$ is what we want to know, and $\sum_{k=1}^{n} 1 = n$. Plugging those known values back into our equation, simply:

$$\begin{aligned}
\sum_{k=1}^{n} ((k+1)^2 - k^2) &= 2\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1 \\
(n+1)^2 - 1 &= 2\sum_{k=1}^{n} k + n \\
\sum_{k=1}^{n} k &= \frac{(n+1)^2 - (n+1)}{2} = \frac{(n+1)(n+1-1)}{2} \\
\sum_{k=1}^{n} k &= \frac{n(n+1)}{2} = \frac{n^2}{2} + \frac{n}{2}
\end{aligned}$$

This solution is... solid. It is a really good approach to reflect on a higher power. Notice that Bernoulli creates $(k+1)^2$ to find $\sum_{k=1}^{n} k^1$, maybe similarly we could create $(k+1)^3$ to find $\sum_{k=1}^{n} k^2$?

Power! Up!!!!!

As we already know how to do it, you could try it on yourself. I will simply show my process (less explanation).

$$\begin{aligned}
(k+1)^3 &= k^3 + 3k^2 + 3k + 1 \\
\sum_{k=1}^{n} (k+1)^3 &= \sum_{k=1}^{n} k^3 + 3k^2 + 3k + 1 \\
\sum_{k=1}^{n} ((k+1)^3 - k^3) &= 3\sum_{k=1}^{n} k^2 + 3\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1
\end{aligned}$$

Given that $\sum_{k=1}^{n} k$ and $\sum_{k=1}^{n} 1$ are known, and with Telescope series on the left:

$$\begin{aligned}
\sum_{k=1}^{n} ((k+1)^3 - k^3) &= 3\sum_{k=1}^{n} k^2 + 3\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1 \\
(n+1)^3 - 1 &= 3\sum_{k=1}^{n} k^2 + 3 \cdot \frac{n(n+1)}{2} + n \\
3\sum_{k=1}^{n} k^2 &= (n+1)^3 - \frac{3n(n+1)}{2} - (n+1) \\
\sum_{k=1}^{n} k^2 &= \frac{n(n+1)(2n+1)}{6} = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}
\end{aligned}$$

Now find the power to 3 in a similar way:

$$
\begin{aligned}
(k+1)^4 &= k^4+4k^3+6k^2+4k+1 \\
\sum_{k=1}^{n}(k+1)^4 &= \sum_{k=1}^{n} k^4+4k^3+6k^2+4k+1 \\
\sum_{k=1}^{n}((k+1)^4-k^4) &= 4\sum_{k=1}^{n} k^3+6\sum_{k=1}^{n} k^2+4\sum_{k=1}^{n} k+\sum_{k=1}^{n} 1 \\
(n+1)^4-1 &= 4\sum_{k=1}^{n} k^3+6\cdot\frac{n(n+1)(2n+1)}{6}+4\cdot\frac{n(n+1)}{2}+n \\
4\sum_{k=1}^{n} k^3 &= (n+1)^4-n(n+1)(2n+1)-(2n)(n+1)-(n+1) \\
\sum_{k=1}^{n} k^3 &= \frac{n^2(n+1)^2}{4}=\frac{n^4}{4}+\frac{n^3}{2}+\frac{n^2}{4}
\end{aligned}
$$

Notice that if we want to find $\sum_{k=1}^{n} k^p$, we need to know $\sum_{k=1}^{n} k^{p-1}$, $\sum_{k=1}^{n} k^{p-2}$, $\cdots$, $\sum_{k=1}^{n} k^0$. But eventually we could find the summation of k to the power of p step by step from the power of 0, 1, 2... In mathematics, this is called **Recursion**.
I used this to find the summation to the power of 8 with the assistance of my TI-Nspire CX II CAS (I have a video explanation), which Jacob Bernoulli found in his essay *Summae Potestatum*, published in 1713.
(Due to the limited space, I will start a new page for this.)

Jacob Bernoulli

$$\sum_{k=1}^{n} k^1 = \frac{1}{2}n^2 + \frac{1}{2}n \tag{1}$$

$$\sum_{k=1}^{n} k^2 = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n \tag{2}$$

$$\sum_{k=1}^{n} k^3 = \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2 \tag{3}$$

$$\sum_{k=1}^{n} k^4 = \frac{1}{5}n^5 + \frac{1}{2}n^4 + \frac{1}{3}n^3 - \frac{1}{30}n \tag{4}$$

$$\sum_{k=1}^{n} k^5 = \frac{1}{6}n^6 + \frac{1}{2}n^5 + \frac{5}{12}n^4 - \frac{1}{12}n^2 \tag{5}$$

$$\sum_{k=1}^{n} k^6 = \frac{1}{7}n^7 + \frac{1}{2}n^6 + \frac{1}{2}n^5 - \frac{1}{6}n^3 + \frac{1}{42}n \tag{6}$$

$$\sum_{k=1}^{n} k^7 = \frac{1}{8}n^8 + \frac{1}{2}n^7 + \frac{7}{12}n^6 - \frac{7}{24}n^4 + \frac{1}{12}n^2 \tag{7}$$

$$\sum_{k=1}^{n} k^8 = \frac{1}{9}n^9 + \frac{1}{2}n^8 + \frac{2}{3}n^7 - \frac{7}{15}n^5 + \frac{2}{9}n^3 - \frac{1}{30}n \tag{8}$$

$$\sum_{k=1}^{n} k^9 = \frac{1}{10}n^{10} + \frac{1}{2}n^9 + \frac{3}{4}n^8 - \frac{7}{10}n^6 + \frac{1}{2}n^4 - \frac{1}{12}n^2 \tag{9}$$

$$\sum_{k=1}^{n} k^{10} = \frac{1}{11}n^{11} + \frac{1}{2}n^{10} + \frac{5}{6}n^9 - n^7 + n^5 - \frac{1}{2}n^3 + \frac{5}{66} \tag{10}$$

Reflection

You might wonder what this is associated with the Bernoulli number. I will reveal this in this section.
After finding those summations, Bernoulli tries to find the pattern and uses that to reflect on the higher power: the power of p.
Obviously, the coefficient of the first term $\sum_{k=1}^{n} k^p$ will be $\frac{1}{k+1}$ and the coefficient of the second term would be $\frac{1}{2}$. For the coefficient of the third term, we could notice from (5) that it is $\frac{p}{12}$. However, the rest coefficient seems disordered. What Jacob Bernoulli did was that he extracted the coefficient of the first term and rewrote the previous equation like this:

$$\sum_{k=1}^{n} k^1 = \frac{1}{2}(n^2 + 2 \cdot \frac{1}{2}n)$$
$$\sum_{k=1}^{n} k^2 = \frac{1}{3}(n^3 + 3 \cdot \frac{1}{2}n^2 + 3 \cdot \frac{1}{6}n)$$
$$\sum_{k=1}^{n} k^3 = \frac{1}{4}(n^4 + 4 \cdot \frac{1}{2}n^3 + 6 \cdot \frac{1}{6}n^2 + 4 \cdot 0 \cdot n)$$
$$\sum_{k=1}^{n} k^4 = \frac{1}{5}(n^5 + 5 \cdot \frac{1}{2}n^4 + 10 \cdot \frac{1}{6}n^3 + 10 \cdot 0 \cdot n^2 - 5 \cdot \frac{1}{30}n)$$
$$\sum_{k=1}^{n} k^5 = \frac{1}{6}(n^6 + 6 \cdot \frac{1}{2}n^5 + 15 \cdot \frac{1}{6}n^4 + 20 \cdot 0 \cdot n^3 - 15 \cdot \frac{1}{30}n^2 + 6 \cdot 0 \cdot n)$$
$$\sum_{k=1}^{n} k^6 = \frac{1}{7}(n^7 + 7 \cdot \frac{1}{2}n^6 + 21 \cdot \frac{1}{6}n^5 + 35 \cdot 0 \cdot n^4 - 35 \cdot \frac{1}{30}n^3 + 21 \cdot 0 \cdot n + 7 \cdot \frac{1}{42}n)$$
$$\sum_{k=1}^{n} k^7 = \frac{1}{8}(n^8 + 8 \cdot \frac{1}{2}n^7 + 28 \cdot \frac{1}{6}n^6 + 56 \cdot 0 \cdot n^5 - 70 \cdot \frac{1}{30}n^4 + 56 \cdot 0 \cdot n^3 + 28 \cdot \frac{1}{42}n^2 + 8 \cdot 0 \cdot n)$$

Give an example for $\sum_{k=1}^{n} k^7$, the coefficient it has after in the parabola is 1, 8, 28, 56, 70, 56, 28, 8, which is exactly the binomial coefficient for $(x+1)^8$. If we use that to reflect on any power p, the coefficient will be the binomial coefficient for $(x+1)^{p+1}$.
Magically, notice that the vertical column, from left to right, is: $\frac{1}{2}$, $\frac{1}{6}$, 0, $-\frac{1}{30}$, 0, $\frac{1}{42}$...The number itself does not have too much pattern, but since it is vertically identical, Jacob Bernoulli named this series of numbers **Bernoulli number**. For the n-th term of the Bernoulli number, we write it as B_n
By this, we could write an expression the summation for any given non-negative integer power p as:

$$\sum_{k=1}^{n} k^p = \frac{1}{p+1} \sum_{k=0}^{p} \binom{p+1}{k} B_k \cdot n^{p+1-k}$$

This formula is also known as **Faulhaber's formula**. Problem solved for all non-negative powers' summation, but what if it is a negative integer like this:

$$\sum_{k=1}^{\infty} k^{-2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots =?$$

Through the analysis of the series, this converges. But what exact number does it converge to? Jacob Bernoulli could not answer, and he said, "Whoever solves this, I will greatly appreciate it."
This question is called **Basel problem**, which I will address in the future. I can tell you this problem was solved by another legendary mathematician **Leonhard Euler**.

Conclusion

Bernoulli's number is pivotal in number theory and occurs frequently in Mathematical Analysis. For instance, the summation we just discussed, the Taylor expansion of $\tan x$, the Euler–Maclaurin formula, the expression for specific values of the Riemann zeta function, and so on.

$$\begin{aligned}\tan x &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}2^{2n}(2^{2n}-1)B_{2n}}{(2n)!}x^{2n-1} \quad \text{for} |x| < \frac{\pi}{2} \\ &= x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \frac{62x^9}{2835} + \frac{1382x^{11}}{155925} + \cdots\end{aligned}$$

Back in my young age, the age when I learn the simple formula to find the sum of an Arithmetic sequence. How can I imagine that there is a mysterious Bernoulli number behind this simple, innocuous formula $S = \frac{n(n+1)}{2}$.

The beauty of mathematics lies in that knowing smile it gives you—cryptic at first, until years later, when the meaning suddenly dawns on you. That moment of revelation is where its true magic resides.
And I love it for all.