Menelaus's Theorem

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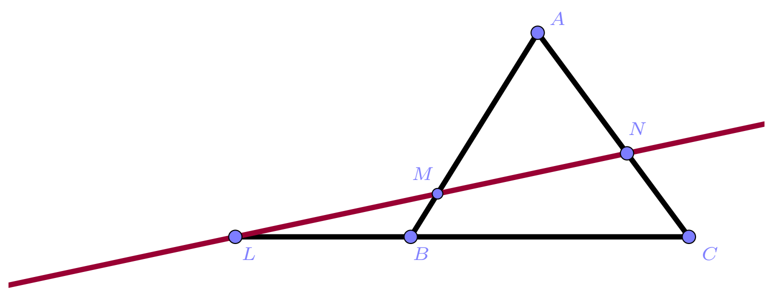
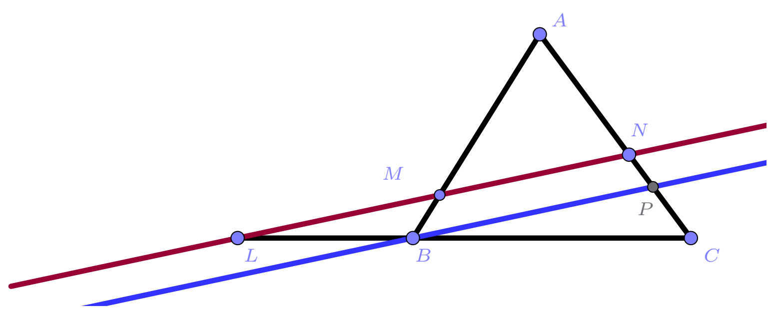


Menelaus's Theorem states that given a figure shown above, we have:

$$\frac{AM}{MB} \cdot \frac{BL}{LC} \cdot \frac{CN}{NA} = 1$$

Proof 1



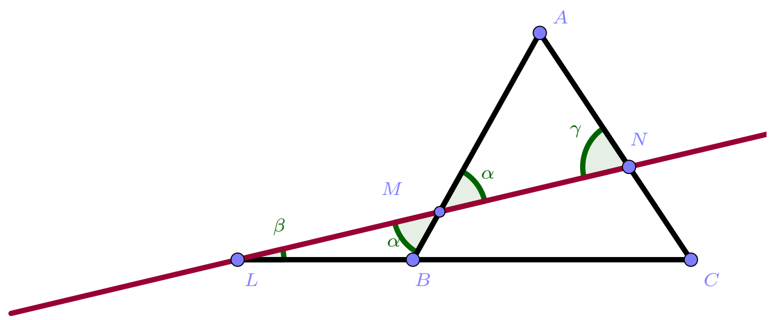
Create a line $BP \parallel LN$, with the Properties of Parallel Lines, we easily get:

$$\frac{AM}{MB} = \frac{AN}{NP}, \ \frac{BL}{LC} = \frac{PN}{NC}, \ \frac{CN}{NA} = \frac{CN}{NA}$$

Then we could rewrite the original equation directly to get the result:

$$\frac{AM}{MB} \cdot \frac{BL}{LC} \cdot \frac{CN}{NA} = \frac{AN}{NP} \cdot \frac{PN}{NC} \cdot \frac{CN}{NA} = 1$$

Proof 2



Apply the Law of Sines to $\triangle AMN$, $\triangle MLB$, and $\triangle NLC$, we get:

$$\frac{AM}{\sin\gamma} = \frac{AN}{\sin\alpha},\ \frac{BL}{sin\alpha} = \frac{MB}{\sin\beta},\ \frac{CN}{\sin\beta} = \frac{LC}{\sin(180^\circ - \gamma)}$$

According to trigonometric identities, we know that $\sin(180^\circ - \gamma) = \sin(\gamma)$, then we could simplify to:

$$\frac{AM}{AN} = \frac{\sin\gamma}{\sin\alpha},\ \frac{BL}{MB} = \frac{\sin\alpha}{\sin\beta},\ \frac{CN}{LC} = \frac{\sin\beta}{\sin(180^\circ - \gamma)} = \frac{\sin\beta}{\sin\gamma}$$

Thus:

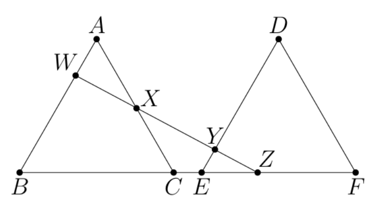
$$\frac{AM}{MB} \cdot \frac{BL}{LC} \cdot \frac{CN}{NA} = \frac{AM}{AN} \cdot \frac{BL}{MB} \cdot \frac{CN}{LC} = \frac{\sin\gamma}{\sin\alpha} \cdot \frac{\sin\alpha}{\sin\beta} \cdot \frac{\sin\beta}{\sin\gamma} = 1$$

There are numerous other proofs. But now, try a real competition question.

Purple Comet Math Meet 2020 Problem 21

Problem 21

Two congruent equilateral triangles $\triangle ABC$ and $\triangle DEF$ lie on the same side of line BC so that B, C, E, and F are collinear as shown. A line intersects $\overline{AB}$, $\overline{AC}$, $\overline{DE}$, and $\overline{EF}$ at W, X, Y, and Z, respectively, such that $\frac{AW}{BW} = \frac{2}{9}$, $\frac{AX}{CX} = \frac{5}{6}$, and $\frac{DY}{EY} = \frac{9}{2}$. The ratio $\frac{EZ}{FZ}$ can then be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.



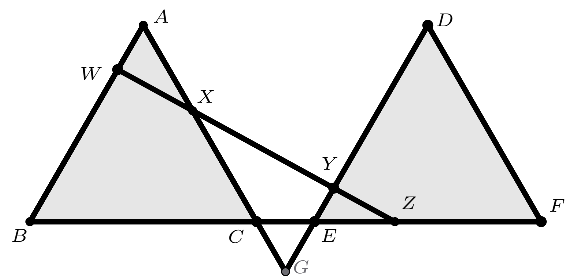
Answer

Because the ratio of the sides of equilateral triangles sums up to 11, we assume the triangle has side length of 11.

According to Menelaus's Theorem, we could know that:

$$\frac{AX}{XC} \cdot \frac{CZ}{ZB} \cdot \frac{BW}{WA} = 1$$

With $\frac{AW}{BW} = \frac{2}{9}$, $\frac{AX}{CX} = \frac{5}{6}$, we get: $\frac{CZ}{ZB} = \frac{4}{15}$, in which $BC = 11,\ CZ = 4$



Here, we extend AE and DE to point G, easily we get equilateral $\triangle CEG$, and we assume $CE = EG = CG = x$; it is also obvious that $\triangle AWX \sim \triangle GYX$ and:

$$\frac{AX}{GX} = \frac{AM}{GY} \implies \frac{5}{6+x} = \frac{2}{2+x} \implies x = \frac{2}{3}$$

With $CZ = 4$, $CE = \frac{2}{3}$, we get $EZ = 4 - \frac{2}{3}$ and $FZ = 11 - (4 - \frac{2}{3})$. Thus:

$$\frac{EZ}{FZ} = \frac{4 - \frac{2}{3}}{11 - (4 - \frac{2}{3})} = \frac{\frac{10}{3}}{\frac{23}{3}} = \frac{10}{23}$$

$$m + n = EZ + FZ = 10 + 23 = 33$$

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