Heteroskedastic Matrix Factorisation (HMF) with Alternating Least Squares (ALS)

NOTE: converted to LaTeX via LLM.

Let f_{ij} be observed values for observations $i \in \{1, \ldots, N\}$ at coordinates (pixels) $j \in \{1, \ldots, M\}$ with known variance σ_{ij}^2.

We seek coefficients a_{ik} and basis g_{kj} for $k \in \{1, \ldots, K\}$ such that

$$f_{ij} \approx \sum_{k=1}^{K} a_{ik} g_{kj} + e_{ij}, \quad e_{ij} \sim \mathcal{N}(0, \sigma_{ij}^2).$$

Therefore, equivalently, HMF minimises the objective

$$\chi^2(a, g) = \sum_{i=1}^{N} \sum_{j=1}^{M} \frac{\left(f_{ij} - \sum_{k=1}^{K} a_{ik} g_{kj}\right)^2}{\sigma_{ij}^2},$$

which is proportional to the negative log-likelihood.

Non-convexity

The problem is bi-linear and not convex. ALS will reduce χ^2 monotonically to a stationary point, which will not necessarily be the global minimum.

ALS Updates

With $G = \{g_{kj}\}$ fixed, each row of $A = \{a_{ik}\}$ solves a WLS. With A fixed, each column of G solves WLS.

a-step

$$A_i \leftarrow G_i^{-1} F_i,$$
$$\left[G_i\right]_{kk'} = \sum_{j=1}^{M} \frac{g_{kj} g_{k'j}}{\sigma_{ij}^2},$$
$$\left[F_i\right]_k = \sum_{j=1}^{M} \frac{g_{kj} f_{ij}}{\sigma_{ij}^2}.$$

This is WLS for a_i at fixed G.

g-step

$$
\begin{aligned}
G_j &\leftarrow A_j^{-1} F_j, \\
\left[A_j\right]_{kk'} &= \sum_{i=1}^{N} \frac{a_{ik} a_{ik'}}{\sigma_{ij}^2}, \\
\left[F_j\right]_k &= \sum_{i=1}^{N} \frac{a_{ik} f_{ij}}{\sigma_{ij}^2}.
\end{aligned}
$$

This is WLS for g_j at fixed A.