ERDŐS–PÓSA DUALITIES IN GRAPH THEORY

by

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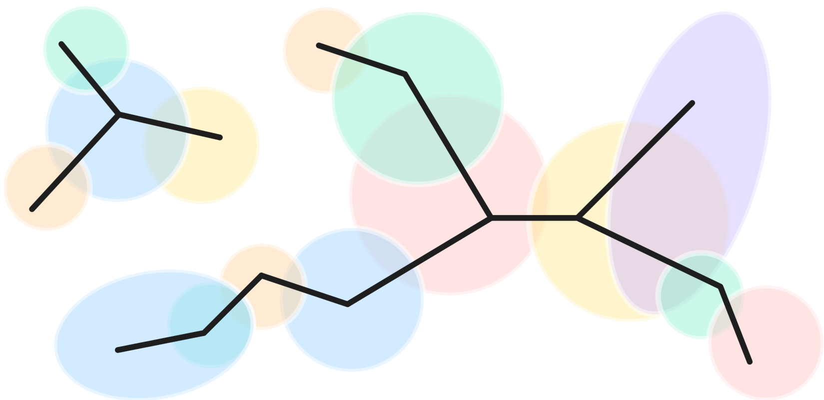

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Abstract

In 1965, Erdős and Pósa showed that there exists a function $f(k) = O(k \log k)$ such that for every graph G and all integers $k \geqslant 1$, G contains k pairwise vertex-disjoint cycles, or there exists a subset $X \subseteq V(G)$ with $|X| \leqslant f(k)$ and $G - X$ is a forest. This sparked extensive research into so-called "Erdős–Pósa type problems". In essence, these problems take the following form: Given graph classes $\mathcal{G}$ and $\mathcal{F}$, the pair $(G, \mathcal{F})$ is said to have the Erdős–Pósa property if there exists a function $f : \mathbb{N} \to \mathbb{N}$ such that for every graph $G \in \mathcal{G}$ and all integers $k \geqslant 1$, G contains k pairwise vertex-disjoint subgraphs in $\mathcal{F}$, or there exists a subset $X \subseteq V(G)$ with $|X| \leqslant f(k)$ and $G - X$ has no subgraph in $\mathcal{F}$. Such a function f is called a bounding function for $(\mathcal{G}, \mathcal{F})$. If $\mathcal{G}$ is the class of all graphs, then the pair $(\mathcal{G}, \mathcal{F})$ is abbreviated to $\mathcal{F}$. For graphs H, let $\mathcal{F}(H)$ denote the class of all graphs containing H as a minor.

This thesis surveys various Erdős–Pósa type results in graph theory, including the classic Erdős–Pósa Theorem, Robertson and Seymour's extension of the Erdős–Pósa Theorem to planar graph minors, and the Coarse Erdős–Pósa Conjecture. The necessary background required for the content is also provided, including graph minor theory, chordal graphs, tree-width, and coarse graph theory. In particular, the background on tree-width presents Leaf and Seymour's proof of the Grid Minor Theorem, since it is the key tool in the proof of the Erdős–Pósa property of $\mathcal{F}(H)$ when H is a planar graph. Once a bounding function is established for a pair $(\mathcal{G}, \mathcal{F})$, the question of optimality comes into the picture. The Erdős–Pósa Theorem gives a $O(k \log k)$ bounding function for $\mathcal{F}(K^3)$, which is optimal. More generally, Cames van Batenburg, Huynh, Joret and Raymond showed that a $O(k \log k)$ bounding function can be achieved for $\mathcal{F}(H)$ when H is a fixed planar graph. Interestingly, more can be said when restricting to forest minors. Namely, a $O(k)$ bounding function can be achieved for $\mathcal{F}(F)$ when F is a fixed forest. Dujmović, Joret, Micek and Morin showed that for any fixed t-vertex tree T, $f(k) = t(k-1)$ is the absolutely tight bounding function for $\mathcal{F}(T)$. Inspired by the formula $t(k-1)$, we conjecture that the same formula is a bounding function for $\mathcal{F}(F)$ when F is a t-vertex forest. We also verify the conjecture for some forests.

The last of the three main Erdős–Pósa dualities examined in this thesis is the Coarse Erdős–Pósa Conjecture. Coarse graph theory is an emerging field that studies the "large-scale" geometry of graphs. This often involves finding coarse analogues of graph-theoretic notions by replacing disjointness with being far apart. In the direction of the Coarse Erdős–Pósa Conjecture, the material of the latter part of this thesis expounds the following distance variant of the classic Erdős–Pósa Theorem that was proven by Dujmović, Joret, Micek and Morin for $f(k) = O(k^{18} \text{polylog}\, k)$ and $g(d) = 19d$: There exists functions $f, g : \mathbb{N} \to \mathbb{N}$ such that for every graph G and all integers $k, d \geqslant 1$, G contains k cycles at pairwise distance greater than d, or there exists a subset $X \subseteq V(G)$ with $|X| \leqslant f(k)$ and $G - B_G(X, g(d))$ is a forest. Related to this result, an original contribution of this thesis is proving the necessity $g(d) \geqslant d$. Another contribution is pointing out that $f(k)$ can be brought down to $O(k^3 \log k)$ by applying a result of Alon. The final topic of this thesis discusses the need for coarse analogues of tools from structural graph theory, and follows a proof by Hickingbotham characterising the graphs that are quasi-isometric to graphs of bounded tree-width.

CHAPTER 1

Introduction

Packing is a topic in mathematics in which one aims to cram as many objects into a container as possible. Sometimes there is an obvious obstruction for the existence of a large packing, and results asserting that this obvious necessary condition is also sufficient are highly sought after. A prime example of this phenomenon comes from matchings in bipartite graphs. Given a bipartite graph G with bipartition $\{A, B\}$, suppose that there exists a subset $S \subseteq A$ such that $|S| > |N_G(S)|$. Since there are not enough vertices available in $N_G(S)$ to match with S, G has no matching for S, hence no matching for A. Thus, if G has a matching for A, every subset $S \subseteq A$ satisfies $|S| \leqslant |N_G(S)|$. Hall's Marriage Theorem asserts that this necessary condition for G to have a matching for A is sufficient.

Theorem 1.0.1 (Marriage Theorem. Hall [64])**.** *Let G be a bipartite graph with bipartition $\{A, B\}$. Then G has a matching for A if and only if $|S| \leqslant |N_G(S)|$ for all $S \subseteq A$.*

Another example comes from perfect matchings in arbitrary graphs. Let G be a graph with a perfect matching M, and let $S \subseteq V(G)$. Since there are no edges between different components of $G - S$ and each odd component C of $G - S$ has no perfect matching, some vertex of C is matched by M to some vertex of S. Consequently, if G has a perfect matching, then for every subset $S \subseteq V(G)$, $G - S$ has at most $|S|$ odd components. Tutte's Perfect Matching Theorem asserts that this necessary condition is sufficient.

Theorem 1.0.2 (Perfect Matching Theorem. Tutte [133])**.** *A graph G has a perfect matching if and only if $G - S$ has at most $|S|$ odd components for all $S \subseteq V(G)$.*

Moving away from matchings, another celebrated example of this phenomenon comes from tree packing. Let G be a graph, and suppose T is a spanning tree of G. Let $\mathcal{P}$ be a partition of $V(G)$ and define H to be the auxiliary graph with vertex set $\mathcal{P}$, and an edge $XY \in E(H)$ if and only if there exists an X-Y-edge in T. As T is spanning and connected in G, H is connected, and so H has at least $|\mathcal{P}| - 1$ edges. That is, at least $|\mathcal{P}| - 1$ edges of T are *cross-edges* for $\mathcal{P}$, edges whose ends lie in different parts of $\mathcal{P}$. See Figure 1.1.

It follows that for any integer $k \geqslant 0$, if G contains k pairwise edge-disjoint spanning trees, then for every partition $\mathcal{P}$ of $V(G)$, G has at least $k(|\mathcal{P}| - 1)$ cross-edges for $\mathcal{P}$. As with the previous examples, this necessary condition is also sufficient.

Theorem 1.0.3 (Nash-Williams [98]; Tutte [132])**.** *For every graph G and all integers $k \geqslant 0$, G contains k pairwise edge-disjoint spanning trees if and only if for every partition $\mathcal{P}$ of $V(G)$, G has at least $k(|\mathcal{P}| - 1)$ cross-edges for $\mathcal{P}$.*

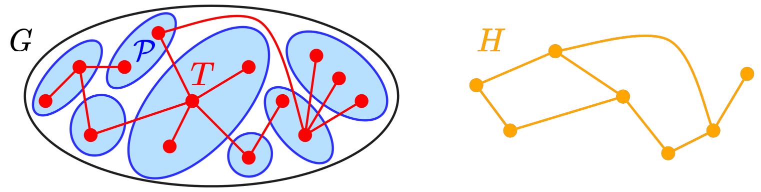

FIGURE 1.1. Left: G with spanning tree T and $V(G)$ partitioned by $\mathcal{P}$. Right: Auxiliary graph H.

Hall and Tutte say that a desired matching exists in G as long as some obvious obstruction does not occur. In packing however, there is often a dual covering problem which leads to a so-called min-max result. An archetypal min-max result is a theorem of Menger, which states that for every graph G and all subsets $A, B \subseteq V(G)$, the maximum size of a collection of pairwise vertex-disjoint A-B-paths in G equals the minimum size of a set $X \subseteq V(G)$ meeting every A-B-path in G. Menger's Theorem can be restated in the following way:

Theorem 1.0.4 (Menger [93])**.** *For every graph G, all integers $k \geqslant 1$, and all subsets $A, B \subseteq V(G)$, G has k pairwise vertex-disjoint A-B-paths, or there exists $X \subseteq V(G)$ with $|X| \leqslant k-1$ and $G - X$ has no A-B-paths.*

More generally in packing and covering, given graph classes $\mathcal{G}$ and $\mathcal{F}$, one may ask the following question: Does there exists a function $f : \mathbb{N} \to \mathbb{N}$ such that for every graph $G \in \mathcal{G}$ and all integers $k \geqslant 1$, G contains k pairwise vertex-disjoint subgraphs in $\mathcal{F}$, or there exists a subset $X \subseteq V(G)$ with $|X| \leqslant f(k)$ and $G - X$ has no subgraph in $\mathcal{F}$? We call such a function f a *bounding function* for the pair $(\mathcal{G}, \mathcal{F})$. In the context of $\mathcal{G}$ being the class of all graphs, we abbreviate the pair $(\mathcal{G}, \mathcal{F})$ to $\mathcal{F}$. Bounding functions capture the notion of approximate dualities in packing and covering. Once the existence of a bounding function is established, a secondary question is: What is the asymptotically smallest bounding function? Restated in this framework, Menger's Theorem reads: for every graph G and all subsets $A, B \subseteq V(G)$, $f(k) = k - 1$ is a bounding function for the pair $(\{G\}, \{P : P \text{ is an } A\text{-}B\text{-path in } G\})$. This thesis hopes to elucidate the many relationships in graph theory that fit into this framework.

Prolific twentieth century Hungarian mathematician Paul Erdős had a knack for finding the simplest non-trivial examples of interesting problems which would, and still do, spark extensive research related to said problem. Indeed, Erdős wrote around 1500 articles [59] and posed around 1000 open problems, some offering cash prizes. The prizes depended on the importance of the problem, and ranged from \$25 all the way to \$10,000. See [15] for a list of those problems. Central to this thesis is the following influential theorem named after Erdős and his protégé Pósa:

Theorem 1.0.5 (Erdős–Pósa Theorem [46])**.** *There exists a $O(k \log k)$ bounding function for the class of all cycles. Moreover, any bounding function for the class of all cycles is in $\Omega(k \log k)$.*

The Erdős–Pósa Theorem is interesting because it says that for every graph G, every cycle is covered by a set of vertices whose size depends only on the maximum size of a packing of cycles in G. The Erdős–Pósa Theorem inspired research into so-called

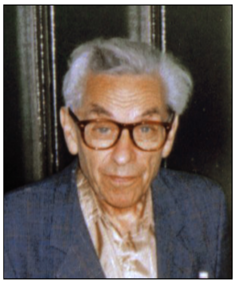

FIGURE 1.2. Erdős.

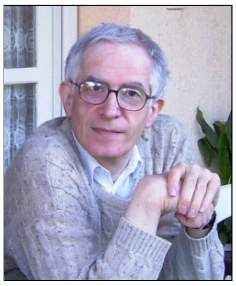

FIGURE 1.3. Pósa.

"Erdős–Pósa type problems", which study packing-covering dualities of combinatorial structures, particularly graphs. If a bounding function exists for a pair $(\mathcal{G}, \mathcal{F})$, then $(\mathcal{G}, \mathcal{F})$ is said to have the *Erdős–Pósa property*. The term *Erdős–Pósa function* is synonymous with bounding function.

A graph H is a *minor* of a graph G if H can be obtained from a subgraph of G by contracting edges. Then G contains a cycle if and only if G contains K^3 as a minor. Let $\mathcal{F}(H)$ be the class of all graphs containing H as a minor. Then the Erdős–Pósa Theorem states that $\mathcal{F}(K^3)$ has the Erdős–Pósa property. In 1986, as a part of their Graph Minors programme, Robertson and Seymour [118] proved the following striking extension of the Erdős–Pósa Theorem:

Theorem 1.0.6 (Erdős–Pósa Property of Planar Graph Minors. Robertson, Seymour [118])**.** *For graphs H, $\mathcal{F}(H)$ has the Erdős–Pósa property if and only if H is a planar.*

FIGURE 1.4. Robertson.

FIGURE 1.5. Seymour.

The Erdős–Pósa Theorem says that any bounding function for $\mathcal{F}(K^3)$ is in $\Omega(k \log k)$. More generally, it can be shown that for every planar graph H that contains a cycle, every bounding function for $\mathcal{F}(H)$ is in $\Omega(k \log k)$. On the other hand, the Erdős–Pósa Theorem also says that there is a bounding function for $\mathcal{F}(K^3)$ in $O(k \log k)$. And remarkably, Cames van Batenburg, Huynh, Joret and Raymond [20] showed that for every planar graph H, there is a bounding function for $\mathcal{F}(H)$ in $O(k \log k)$. Hence, for all planar graphs H that contain a cycle, the optimal bounding function for $\mathcal{F}(H)$ is in $\Theta(k \log k)$. In contrast, if H has no cycle, then H is a forest. For this case, Fiorini, Joret and Wood [49] showed that the optimal bounding function for $\mathcal{F}(H)$ is in $\Theta(k)$.

The discussion so far has paid no attention to the "scale at which a packing is realised". We explain. Consider the space obtained from $\mathbb{R} \times \mathbb{R}$ by placing at each point of $\mathbb{Z} \times \mathbb{Z}$, a ball from $\mathbb{R}^{100}$. This space, as depicted in Figure 1.6, looks to be 100-dimensional at "small-scales", however, when "zoomed out" the space looks like a plane.

This sort of "large-scale dimension" of a space is captured by Gromov's [58] asymptotic dimension. It is also a key concept in the emerging field of coarse graph theory. More broadly, coarse graph theory studies the large-scale geometry of graphs and properties preserved under some controlled "distortion" of the space. This includes finding "coarse analogues" of graph-theoretic notions by replacing disjointness with being far apart.

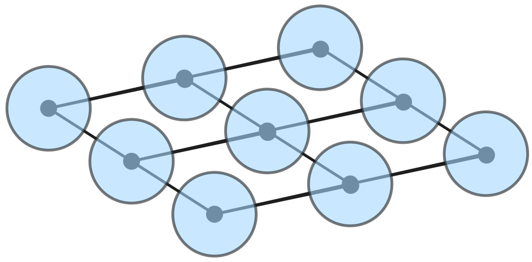

FIGURE 1.6. High dimension at small-scale and low dimension at large-scale.

Now back to packing. In light of Figure 1.6, the presence of certain features of a graph may depend on what "scale" the graph is being observed. Intuitively, a minor that is "insufficiently spread out" amounts to noise when "zoomed out". Fat minors are similar to minors except that they are forced to be "spread out" for the purpose of being "visible" at a desired "scale". Fat minors are defined as follows: Let $d \geqslant 1$ be an integer, and G and H be graphs. Suppose $(X_v : v \in V(H)) \cup (P_e : e \in E(H))$ is a collection of connected subgraphs of G such that the following hold:

- if v is an end of e then $V(X_v) \cap V(P_e) \neq \emptyset$, and
- for all distinct $A, B \in (X_v : v \in V(H)) \cup (P_e : e \in E(H))$ which are not covered by the first condition, $\text{dist}_G(A, B) \geqslant d$.

Then G is said to contain H as a *d-fat minor*. Georgakopoulos and Papasoglu [55] conjectured the following "Erdős–Pósa type of result" in the coarse graph theory setting:

Conjecture 1.0.7 (Coarse Erdős–Pósa Conjecture. Georgakopoulos, Papasoglu [55]). *There exists a function $f : \mathbb{N} \to \mathbb{N}$ and a constant $c > 0$, such that for every graph G and all integers $n, d \geqslant 1$, G contains the disjoint union of n copies of K^3 as a d-fat minor, or there exists a subset $X \subseteq V(G)$ with $|X| \leqslant f(n)$ and $G - B_G(X, cd)$ contains no K^3 as a d-fat minor.*

1.1. Purpose and Structure of Thesis

At a high level, the purpose of this thesis is to survey various "Erdős–Pósa type" results in graph theory, to show how they fit into the rest of the theory by building them up from basic definitions and theorems, and to tell a story. The main Erdős–Pósa dualities this thesis focuses on are the Erdős–Pósa Theorem (Theorem 1.0.5), the Erdős–Pósa Property of Planar Graph Minors (Theorem 1.0.6), and the Coarse Erdős–Pósa Conjecture (Conjecture 1.0.7). See Figure 1.7 for the high-level connections between the main concepts of this thesis.

The classic Erdős–Pósa Theorem is where the story of this thesis begins. As such, Chapter 3 gives two proofs for the existence of a $O(k \log k)$ bounding function for the

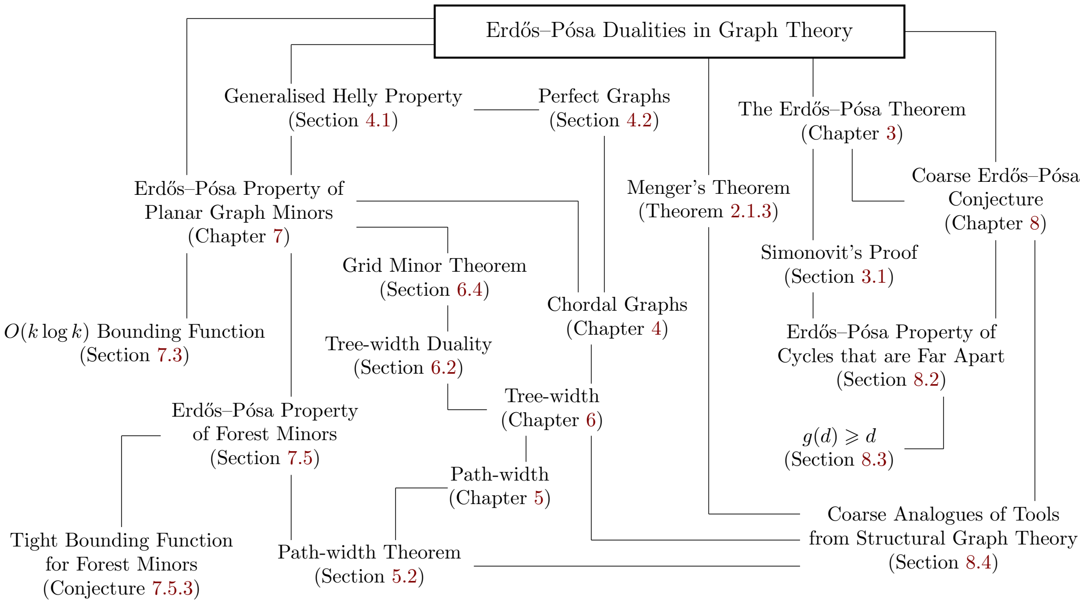

FIGURE 1.7. High-level structure of this thesis.

class of all cycles, each showcasing vastly different techniques. The first is by Simonovits [127] and the second is by Cames van Batenburg, Joret and Ulmer [21]. The latter part of Chapter 3 surveys an assortment of generalisations of the Erdős–Pósa Theorem, including long cycles and modularity constrained cycles. Prior to the main content, Chapter 2 provides all the graph theory background required to understand this thesis, including basic definitions and important concepts in graph minor theory. After tackling cycles, the goal of this thesis turns to proving Robertson and Seymour's extension of the Erdős–Pósa Theorem, which says that $\mathcal{F}(H)$ has the Erdős–Pósa property if and only if H is planar (Theorem 1.0.6). In its proof, the notion of what it means for a graph to be "tree-like" is vital. The standard measure for how similar a graph is to a tree is its tree-width, where a lower tree-width means more "tree-like". Intuitively, grids are not tree-like, and indeed the $k \times k$-grid has tree-width k. On the other hand, the Grid Minor Theorem states that graphs with sufficiently large tree-width contain the $k \times k$-grid as a minor. The Grid Minor Theorem is *the* main tool in the proof of Theorem 1.0.6, and for this reason, Chapter 6 is dedicated to introducing tree-width and proving the Grid Minor Theorem. In this thesis, the main uses for tree-width and tree-decompositions are related to how they describe the structure of graphs. However, there are also algorithmic motivations for studying these concepts, some of which are briefly discussed in Section 6.1.

When working with tree-decompositions, it is helpful to study chordal graphs since they describe how a collection of subtrees of a tree intersect. Chordal graphs also give an equivalent formulation of tree-width. Another useful tool for analysing subtrees of a tree is the generalised Helly property, which states that for any collection $\mathcal{F}$ of subtrees of a tree T, $(\{T\}, \mathcal{F})$ has the Erdős–Pósa property with bounding function $f(k) = k - 1$. Properties of chordal graphs and the generalised Helly property are used regularly in Chapter 6 and Chapter 7. This dependence justifies Chapter 4 being allocated to proving these useful properties. Keeping to the theme of dualities in graph theory, Chapter 4 also includes a brief discussion of perfect graphs, a topic in which dualities are inescapable and chordal graphs are an important example.

Next, this thesis looks at the asymptotics of the optimal bounding function for $\mathcal{F}(H)$ where H is a fixed planar graph. One may deduce a $O(k(2k|H|)^{8(2k|H|)^2})$ bounding function for $\mathcal{F}(H)$ by combining the results up to the end of Section 7.1. In the special case of $H = K^3$, the Erdős–Pósa Theorem gives a $O(k \log k)$ bounding function for $\mathcal{F}(K^3)$. Trivially, bounding functions are always in $\Omega(k)$, since any set of vertices meeting every member in a vertex-disjoint packing must contain at least one vertex from each member in a maximum such packing. This lower bound may be sharpened under certain assumptions, such as in Section 7.2, which gives a probabilistic proof showing that every bounding function for $\mathcal{F}(K^3)$ is in $\Omega(k \log k)$. This implies the optimality of the bounding function in the Erdős–Pósa Theorem. Interestingly, in the general setting of H being a planar graph, $O(k(2k|H|)^{8(2k|H|)^2})$ may be brought all the way down to $O(k \log k)$, matching the order of the optimal bounding function in the Erdős–Pósa Theorem. This optimal result for the general setting was proven by Cames van Batenburg, Huynh, Joret and Raymond, and is the topic of Section 7.3.

The aforementioned lower bound of $\Omega(k \log k)$ does not apply when H is a forest. But the lower bound of $\Omega(k)$ is still valid. It turns out that the optimal bounding function for $\mathcal{F}(H)$ when H is a forest is in $\Theta(k)$. This is explored in Section 7.5. Moreover, the proof of this fact relies on the notion of how "path-like" a graph is, which is measured by a special case of tree-width called path-width. The path-width analogue of the Grid Minor Theorem, called the Path-width Theorem, is a key tool in proving the existence of a $O(k)$ bounding function for $\mathcal{F}(F)$ where F is a forest. Consequently, Chapter 5 is devoted to introducing path-width and proving the Path-width Theorem. For trees T, Dujmović, Joret, Micek and Morin proved that $|T|(k-1)$ is the absolutely tight bounding function for $\mathcal{F}(T)$. Motivated by this, one of the contributions of this thesis is a conjecture, which, if proven true implies $|F|(k-1)$ is a bounding function for $\mathcal{F}(F)$. We prove some specific cases of our conjecture in Section 7.5.

The final Erdős–Pósa duality of this thesis is the Coarse Erdős–Pósa Conjecture, discussed in Chapter 8. To appreciate the Coarse Erdős–Pósa Conjecture, one must understand the basic definitions in the area of coarse graph theory. As such, Chapter 8 begins by laying out the main definitions and motivations of coarse graph theory. A recent step in the direction of the Coarse Erdős–Pósa Conjecture was the following distance variant of the classic Erdős–Pósa Theorem, which was proven for $f(k) = O(k^{18} \operatorname{polylog} k)$ and $g(d) = 19d$:

Theorem 1.1.1 (Dujmović, Joret, Micek, Morin [41])**.** *There exists functions $f, g : \mathbb{N} \to \mathbb{N}$ such that for every graph G and all integers $k, d \geqslant 1$, G contains k cycles at pairwise distance greater than d, or there exists $X \subseteq V(G)$ with $|X| \leqslant f(k)$ and $G - B_G(X, g(d))$ is a forest.*

In the proof of Theorem 1.1.1, and more generally, many complications could arise when trying to produce a packing whose members are at pairwise distance d, for example, deleting vertices or edges usually changes the distance function. Thus induction on the size of the graph is often not possible. This makes the proof of Theorem 1.1.1 intriguing. Its proof also has a connection to the main calculation in Simonovits' proof of the Erdős–Pósa Theorem. To demonstrate this connection, Section 8.2 proves Theorem 1.1.1 by combining the main lemmas from Dujmović, Joret, Micek and Morin's paper [41]. Furthermore, Dujmović, Joret, Micek and Morin [41] ask what is the smallest possible g satisfying Theorem 1.1.1 (with no restriction on f)? An original contribution of this thesis is the necessity $g(d) \geqslant d$ for all $d \geqslant 1$, and is found in Section 8.3. Another contribution is pointing out that the bound on $f(k)$ can be brought down to $O(k^3 \log k)$ by applying a result of Alon.

FIGURE 1.8. Left to Right: Dujmović, Joret, Micek, Morin.

Coarse graph theory poses questions that combine graph-theoretic notions with coarse geometry. Many of those questions, such as the Coarse Erdős–Pósa Conjecture, Coarse Hadwiger, and Coarse Kuratowski–Wagner would become more approachable if one had coarse versions of standard tools from structural graph theory, including Menger's theorem, tree-width, and the Graph Minor Structure Theorem. As a finale to the chapter on the Coarse Erdős–Pósa Conjecture, Section 8.4 discusses the need for coarse analogues of tools from structural graph theory. In that direction, Section 8.4 also proves a characterisation of the graphs that are quasi-isometric to graphs with bounded tree-width. The final chapter, Chapter 9, briefly catalogues some topics for future research.

1.2. Original Contribution

This thesis includes the following original contributions:

- Conjectured (Conjecture 7.5.3) that $f(k) = t(k-1)$ is the absolutely tight bounding function for $\mathcal{F}(F)$ when F is a fixed t-vertex forest.
- Proved that Conjecture 7.5.3 holds for certain forests (Theorem 7.5.4, Theorem 7.5.9).
- Spotted some typos in [41] and found that Claim 8.2.9 (in the proof of Theorem 8.2.1) had some missing details. As such, we notified the authors who have since made the corrections.
- Pointing out that in the proof of Theorem 8.2.1, one may replace Theorem 8.2.2 by a result of Alon (Theorem 8.2.3) to show that Theorem 8.2.1 holds with $f(k) = O(k^3 \log k)$ and $g(d) = 19d$.
- Showing the necessity of $g(d) \geqslant d$ in Theorem 8.2.1.
- Spotted some typos in [67] and found that an incorrect version of Lemma 8.4.16 was used. As such, we notified the author who has since made the corrections.

CHAPTER 2

Graph Theory Background

This section provides all the graph theory background required to understand this thesis, including basic definitions and important concepts in graph minor theory. We refer the reader to Diestel's book [34] for any terms that go undefined.

2.1. Basic Definitions

A *graph* is a pair (V, E) consisting of a finite set of *vertices* V and a set of *edges* $E \subseteq \binom{V}{2}$. The sets V and $\binom{V}{2}$ are assumed to be disjoint. The vertex and edge sets of G are denoted by $V(G)$ and $E(G)$ respectively. Define $|G| := |V(G)|$ and $||G|| := |E(G)|$. A graph is *null* if it has no vertices. If $uv \in E(G)$, then u and v are *adjacent*, and u and v are *neighbours*. Vertices u and v are the *ends* of uv, and uv is *incident* to u and v. The *$k \times m$-grid* is the graph with vertex set $[k] \times [m]$, where (x_1, y_1) and (x_2, y_2) are adjacent if and only if $|x_1 - x_2| + |y_1 - y_2| = 1$. An illustration of the 4×4-grid is shown in Figure 2.1.

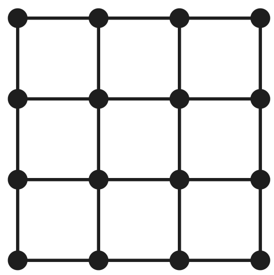

FIGURE 2.1. 4×4-grid.

An *isomorphism* from a graph G to a graph H is a bijection $\varphi : V(G) \to V(H)$ with the property that $uv \in E(G)$ if and only if $\varphi(u)\varphi(v) \in E(H)$. If such a φ exists, G and H are said to be *isomorphic*, which may be written as $G \cong H$. This thesis predominantly discusses graphs up to isomorphism. In this spirit, what is usually meant by a graph G is the isomorphism type of G.

For integers $n \geqslant 1$, any graph G with $|G| = n$ and $||G|| = \binom{n}{2}$ is said to be a K^n, and is called a *complete graph*. For integers $m, n \geqslant 1$, other examples of isomorphism types include *paths P_n*, *cycles C_n*, and *complete bipartite graphs $K_{m,n}$*. A graph G is a P_n if $|G| = n$ and there is an ordering of the vertices $(v_1, \dots, v_n)$ such that $E(G) = \{v_iv_{i+1} : i \in [n-1]\}$. It is common to designate a first and last vertex of a path. A graph G is a C_n if $|G| = n$ and there is an ordering of the vertices $(v_1, \dots, v_n)$ such that $E(G) = \{v_iv_{i+1} : i \in [n-1]\} \cup \{v_nv_1\}$. A graph G is a $K_{m,n}$ if $V(G)$ can be partitioned into two sets A and B of size m and n respectively, such that $E(G) = \{ab : (a, b) \in A \times B\}$. Figure 2.2 illustrates $K^5, P_7, C_4, K_{3,3}$.

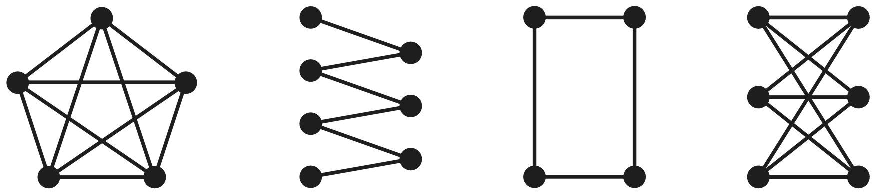

FIGURE 2.2. Left to right: $K^5, P_7, C_4, K_{3,3}$.

The *neighbourhood* of a vertex v in a graph G is the set of neighbours of v, and is denoted by $N_G(v)$. For subsets $S \subseteq V(G)$ define $N_G(S) := \bigcup_{v \in S} N_G(v) \setminus S$. The *degree* of a vertex v in G is number of edges incident to v, denote by $\deg_G(v)$. $\delta(G)$ is the minimum degree of a vertex of G. For integers $r \geqslant 0$, a graph G is *r-regular* if $\deg_G(v) = r$ for all $v \in V(G)$. The following fact is ceremonial in graph theory:

Lemma 2.1.1 (Handshaking Lemma)**.** *For every graph G,*

$$2||G|| = \sum_{v \in V(G)} \deg_G(v).$$

PROOF. Since $\deg_G(v)$ counts the number of edges incident to the vertex v and each edge of G is incident to exactly two vertices, $\sum_{v \in V(G)} \deg_G(v)$ counts each edge of G twice. ■

A set of pairwise adjacent vertices in a graph G is a *clique* in G. The *clique number* of G, denoted by $\omega(G)$, is the maximum size of a clique in G. A set of pairwise non-adjacent vertices in a graph G is an *independent set* in G. The *independence number* of G, denoted by $\alpha(G)$, is the maximum size of an independent set in G.

A graph H is a *subgraph* of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. $H \subseteq G$ means that H is a subgraph of G. A subgraph H of G is *spanning* for G if $V(H) = V(G)$. Let $U \subseteq V(G)$, then the *subgraph of G induced on U* is the graph $G[U] := (U, \binom{U}{2} \cap E(G))$. The *complement* of a graph G is the graph $\overline{G} := (V(G), \binom{V(G)}{2} \setminus E(G))$. For subsets $X \subseteq V(G)$, define $G - X := G[V(G) \setminus X]$. For subsets $M \subseteq \binom{V(G)}{2}$, define $G - M := (V(G), E(G) \setminus M)$ and $G + M := (V(G), E(G) \cup M)$. For graphs G and H, define the graphs $G \cup H := (V(G) \cup V(H), E(G) \cup E(H))$ and $G \cap H := (V(G) \cap V(H), E(G) \cap E(H))$. A graph is a $G \sqcup H$, and is called a *disjoint union* of G and H, if it equals the union of two vertex-disjoint graphs, one isomorphic to G and the other isomorphic to H. If $\mathcal{F}$ is a finite collection of graphs, then $\bigcup \mathcal{F}$ and $\bigcap \mathcal{F}$ is the union and intersection respectively of all graphs in $\mathcal{F}$. A graph G is a $\bigsqcup \mathcal{F}$, if there exists a collection $\mathcal{D} := (G_F : F \in \mathcal{F})$ of vertex-disjoint graphs such that $G_F \cong F$ for all $F \in \mathcal{F}$, and $G = \bigcup \mathcal{D}$.

Let $A, B \subseteq V(G)$. An *A-B-path* in G is a path P in G whose first vertex is in A and last vertex is in B, and has no internal vertex in $A \cup B$. An *A-B-edge* in G is an edge ab of G such that $a \in A$ and $b \in B$. For subgraphs H of G, an *H-path* in G is a $V(H)$-$V(H)$-path P such that $||P|| \geqslant 1$, and the edges of P lie outside of H. A graph G is *connected* if $V(G) \neq \emptyset$ and for all distinct vertices $u, v \in V(G)$ there exists a u-v-path in G. A set $U \subseteq V(G)$ is *connected* if $G[U]$ is connected. A subgraph-maximal connected subgraph of G is a *component* of G. For integers $k \geqslant 0$, a graph G is *k-connected* if $|G| \geqslant k + 1$ and $G - X$ is connected for every subset $X \subseteq V(G)$ with

$|X| \leqslant k-1$. A graph F that has no cycle as a subgraph is a *forest*, and a connected forest is a *tree*. The following proof shows a useful technique which may be applied to trees:

Proposition 2.1.2. *Every tree T has a vertex $v \in V(T)$ such that every component of $T-v$ has at most $\frac{1}{2}|T|$ vertices.*

PROOF. The case of $||T|| = 0$ is trivial, so it may be assumed that $||T|| \geqslant 1$. For each $e \in E(T)$, $T-e$ has exactly two components, each containing exactly one end of e. Choose a component C of $T-e$ such that $|C| \leqslant \frac{1}{2}|T|$, and orient e away from its end lying in C. This defines an orientation of the edges of T. Consider the following process: Starting at any vertex of T, whenever possible, arbitrarily follow edges of T along their orientations, arriving at another vertex at each step. If it is not possible to take a step, then the current vertex has out-degree 0, so stop the process. Since T has no cycle, each step arrives at a vertex that is different from all previous vertices. Then, the finiteness of $|T|$ implies that the process stops at some final vertex, call it v. According to the stopping condition, every edge incident to v is oriented towards v. By definition of the orientation, every component of $T-v$ has at most $\frac{1}{2}|T|$ vertices. ■

A *separation* of a graph G is a pair (A, B) such that A and B are subsets of $V(G)$ with $A \cup B = V(G)$ and there is no $(A \setminus B)$-$(B \setminus A)$-edge in G. The *order* of a separation (A, B) is the number $|A \cap B|$, and (A, B) is said to be *proper* if $A \setminus B$ and $B \setminus A$ are non-empty. A set $X \subseteq V(G)$ *separates* two subsets $A, B \subseteq V(G)$ if X meets every A-B-path in G. Such an X is called an *A-B-separator* of G. The following theorem of Menger is foundational and will see much use in this thesis:

Theorem 2.1.3 (Menger's [93])**.** *For all graphs G and all subsets $A, B \subseteq V(G)$, the minimum size of an A-B-separator of G equals the maximum size of a collection of pairwise vertex-disjoint A-B-paths in G.*

The *length* of a path or a cycle is its number of edges. The length of a shortest cycle in a graph G is the *girth* of G, denoted by $g(G)$. If G is a forest, then $g(G) = +\infty$. The *distance* between two vertices u, v in a graph G is the length of a shortest u-v-path in G, and is denoted by $\mathrm{dist}_G(u, v)$. If G has no u-v-path, $\mathrm{dist}_G(u, v) = +\infty$. It is easy to check that $(V(G), \mathrm{dist}_G)$ defines a metric-space. For subsets $X, Y \subseteq V(G)$, define $\mathrm{dist}_G(X, Y) := \min\{\mathrm{dist}_G(x, y) : (x, y) \in X \times Y\}$. For integers $r \geqslant 0$ and subsets $X \subseteq V(G)$, the ball of radius r around X is $B_G(X, r) := \{v \in V(G) : \mathrm{dist}_G(X, v) \leqslant r\}$. Thus $B_G(X, r) = \bigcup_{x \in X} B_G(x, r)$.

Let G be a graph, $e = xy$ be an edge of G, and $n \geqslant 1$ be an integer. Let H be the graph obtained from $G-e$ by adding a path $(z_1, \ldots, z_n)$ made of new vertices, and adding new edges xz_1 and yz_n. Then H is the graph obtained from G by *subdividing* e exactly n times. A *subdivision* of G is a graph obtained from G by subdividing each edge any number of times. The *n-subdivision* of G is the graph obtained from G by subdividing every edge of G exactly n times.

A *pseudograph* $(V, E, \mathcal{E})$ is a tuple consisting of a set of vertices V, a set of edges E, and a map $\mathcal{E} : E \to V \cup \binom{V}{2}$ that specifies the ends of edges. The sets V and $\binom{V}{2}$ are assumed to be disjoint. Where a graph has at most one edge between any two distinct vertices, a pseudograph may have multiple edges between any two (not necessarily distinct) vertices. The definitions stated up till now for graphs are analogous for pseudographs. The members of $\mathcal{E}^{-1}(V)$ are called *loops*, and each loop incident to a vertex v contributes $+2$ to the degree of v. If $e_1, e_2 \in E$ are distinct and $\mathcal{E}(e_1) = \mathcal{E}(e_2)$, then e_1 and e_2 are said to be *parallel*.

Let v be a degree-2 vertex of a pseudograph G. If a loop is incident to v put $H := G - v$. If not, then two distinct edges e_1 and e_2, each having an end different from v, are incident to v. Let u_1 and u_2 be the ends of e_1 and e_2 respectively that are different from v, and let H be the pseudograph obtained from $G - v$ by adding an edge with ends $\{u_1, u_2\}$. Then H is called the graph obtained from G by *suppressing* the vertex v, see Figure 2.3 for an example.

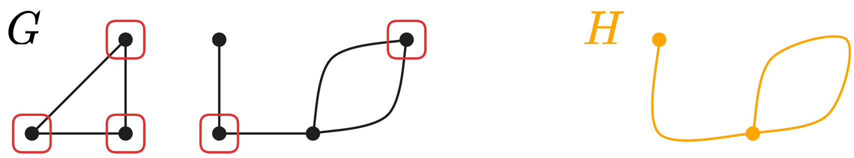

FIGURE 2.3. Suppressing the marked vertices of G to produce H.

A *colouring* of a graph G is a function $f : V(G) \to \mathbb{N}$. f is a *proper-colouring* if $f(u) \neq f(v)$ for all $uv \in E(G)$. Then every colouring of G uses at most $|V(G)|$ colours. For integers $k \geqslant 1$, G is k-colourable if G has a proper-colouring f with $|f(V(G))| = k$. 2-colourable graphs are *bipartite*. The *chromatic number* of G, denoted by $\chi(G)$, is the minimum integer k such that G is k-colourable.

For integers $m, n \geqslant 0$ with $\binom{n}{2} \geqslant m$, $\mathbf{G}(n, m)$ is the uniform probability space on the set of all graphs G such that $V(G) = [n]$ and $||G|| = m$.

2.2. Graph Minors

Given a graph G and an edge $e = xy \in E(G)$, let G/e be the graph obtained from G by adding a new vertex v_e, adding new edges such that v_e is adjacent to every vertex in $(N_G(x) \cup N_G(y)) \setminus \{x, y\}$, followed by deleting the vertices x and y. G/e is said to be the graph obtained from G by *contracting* the edge e. See Figure 2.4.

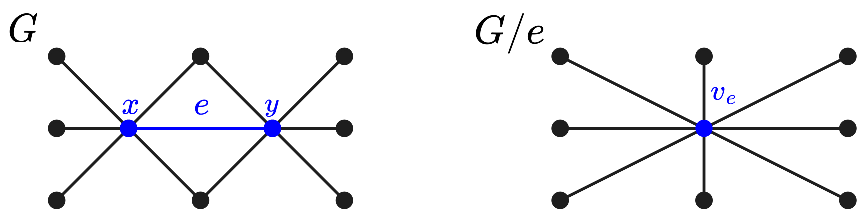

FIGURE 2.4. Contracting the edge $e = xy$ to a vertex v_e.

A graph H is a *minor* of a graph G if H is isomorphic to a graph that can be obtained from a subgraph of G by a sequence of edge contractions. $H \preccurlyeq G$ means that H is a minor of G. If in addition $H \ncong G$, H is a *proper-minor* of G. A *model* of a graph H in a graph G is a collection $\mathbf{X} = (X_v : v \in V(H))$ of pairwise disjoint connected sets of vertices in G, called *branch sets*, such that for all $uv \in E(G)$ there exists an

X_u-X_v-edge in G. Thus H is a minor of G if and only if there exists a model of H in G.

A *surface* is a compact 2-manifold. Let X be a topological space, $V \subseteq X$ a finite set of points, and E a collection of homeomorphic images of $[0,1]$ in X, called arcs. Suppose that for each arc in E, its endpoints are different points of V, and its interior is disjoint from V and all other arcs in E. Suppose also that each pair of distinct points in V form the endpoints of at most one arc in E. Let G be the graph with vertex set V, where distinct points x, y of V are adjacent if and only if x and y are the ends of a common arc in E. Then $V \cup E$ is said to be an *embedding* of G in X. A graph G is *embeddable* in X if there exists an embedding of a G in X. Every graph is embeddable on a surface, say on the boundary of a thickening of the graph. The *Euler genus* of the surface obtained by the addition of h handles and c crosscaps to a sphere is $2h+c$. The *Euler genus* of a graph G is the minimum Euler genus of a surface $\mathcal{S}$ such that G is embeddable in $\mathcal{S}$. Intuitively, G "uses up" the handles and crosscaps in such a surface $\mathcal{S}$, and so many disjoint copies of G cannot be embedded in $\mathcal{S}$. Indeed this is the case as shown by the following result:

Theorem 2.2.1 (Additivity of the Euler-genus. Miller [94]; Richter [113])**.** *Let G_1 and G_2 be graphs meeting on at most one vertex. Then the Euler genus of $G_1 \cup G_2$ is the sum of the Euler geni of G_1 and G_2.*

On the other hand, if a graph G is embeddable in a surface $\mathcal{S}$, then so is every minor of G. These graph classes associated to surfaces are key examples of so-called minor-closed classes. As the name suggests, a class of graphs $\mathcal{F}$ is *minor-closed* if for every $G \in \mathcal{F}$, $\mathcal{F}$ contains every minor of G. A class of graphs is *proper* if some graph is not in the class.

Other examples of minor-closed classes include the class of *knotlessly embeddable* graphs, and *linklessly embeddable* graphs. The former consists of all graphs which are embeddable in $\mathbb{R}^3$ such that none of its cycles are knotted. The latter consists of all graphs which are embeddable in $\mathbb{R}^3$ such that no two disjoint cycles form a link.

One can check that the $\preccurlyeq$ is a partial ordering on the class of all graphs. As such, every minor-closed graph class $\mathcal{F}$ is characterised by the set $\mathcal{M}$ of minor-minimal graphs not in $\mathcal{F}$ in the following way:

$$G \in \mathcal{F} \text{ if and only if } H \not\preccurlyeq G \text{ for all } H \in \mathcal{M}.$$

$\mathcal{M}$ is called the *set of minimally excluded minors* of $\mathcal{F}$. A graph class that is central to this thesis is the class of *planar graphs*, the graphs that are embeddable in $\mathbb{R}^2$. The Kuratowski–Wagner Theorem characterises the planar graphs by exhibiting its minimally excluded minors.

Theorem 2.2.2 (Kuratowski [82]; Wagner [134])**.** *A graph H is planar if and only if H is K^5-minor-free and $K_{3,3}$-minor-free.*

The linklessly embeddable graphs can also be characterised by a finite set of minimally excluded minors as shown by the following theorem:

Theorem 2.2.3. *A graph G is linklessly embeddable if and only if G does not contain any graph from the Petersen family as a minor (see Figure 2.6).*

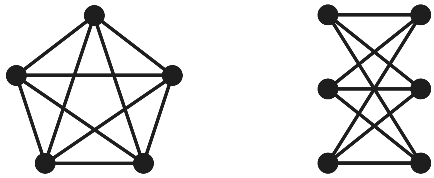

FIGURE 2.5. The minor-minimal non-planar graphs. Left: K^5. Right: $K_{3,3}$.

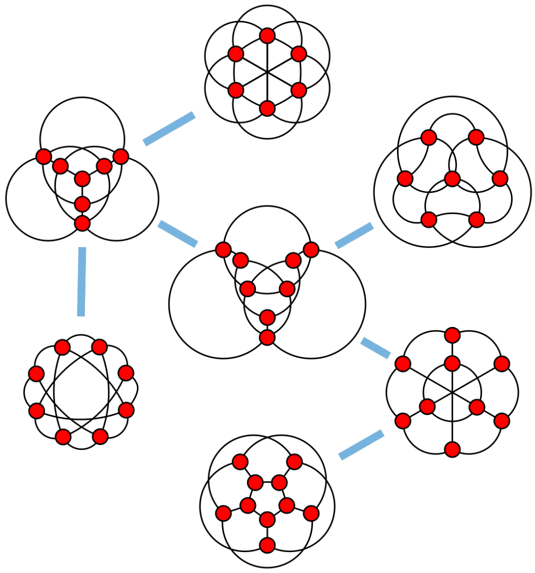

FIGURE 2.6. Petersen family.

The following is a landmark achievement in graph theory:

Theorem 2.2.4 (Graph Minor Theorem. Robertson, Seymour [121])**.** *In every infinite sequence of graphs $G_1, G_2, \dots$ there exists two G_i and G_j with $i < j$, such that $G_i \preccurlyeq G_j$.*

The Graph Minor Theorem implies that every minor-closed class is characterised by a finite set of minimally excluded minors, since any infinite set of excluded minors would not be minimal. This massive generalisation of the Kuratowski–Wagner Theorem is considered to be "among one of the deepest theorems mathematics has to offer"[34]. One of the interesting consequences of the Graph Minor Theorem is that the question of whether an n-vertex graph is embeddable in a fixed surface, knotlessly embeddable, or linklessly embeddable, all have polynomial time algorithms. Indeed, Robertson and Seymour [115] proved an algorithm that decides in $O(n^3)$ time whether a fixed graph H is a minor of an n-vertex graph. Kawarabayashi, Kobayashi and Reed [77] showed that the $O(n^3)$ can be improved to $O(n^2)$. This along with the Graph Minor Theorem implies that membership in a minor-closed class can be tested in $O(n^2)$ time. The algorithm is as follows: Let $\mathcal{F}$ be a minor-closed class and let $\mathcal{M}$ be its set of minimally excluded minors, which is finite by the Graph Minor Theorem. Since the size of $\mathcal{M}$ does not depend on the input G, one may test whether G contains a minor in $\mathcal{M}$ in $O(n^2)$ time. If G contains no minor in $\mathcal{M}$, then $G \in \mathcal{F}$. Otherwise, $G \notin \mathcal{F}$. The "big O" hides a complicated dependence on $\mathcal{F}$. The Graph Minor Theorem guarantees a finite set of minimally excluded minors but gives no way of finding such a set. In general, the set of minimally excluded minors is hard to find and can be huge.

To prove the Graph Minor Theorem, Robertson and Seymour first proved a structure theorem for graphs not containing a fixed graph as a minor called the Graph Minor Structure Theorem [119]. At a high level, Robertson and Seymour's [121] method for proving the Graph Minor Theorem is roughly as follows: Suppose $G_1, G_2, \ldots$ is an infinite sequence of graphs such that G_i is not a minor G_j for all $i < j$. Then $G_2, G_3, \ldots,$ is an infinite sequence of G_1-minor-free graphs, and so Robertson and Seymour use the structure of G_1-minor-free graphs, offered by the Graph Minor Structure Theorem, to derive a contradiction.

The Graph Minor Structure Theorem lays the ground work for a lot of structural graph theory. The statement of the Graph Minor Structure Theorem is technical and would require a sizeable departure from the main topic of this thesis in order to explain the reasons for why each part of the theorem is the way it is. For this reason, we only say that the structure of H-minor-free graphs is well studied, resembling Figure 2.7. See [88, 104] for surveys on the Graph Minor Structure Theorem.

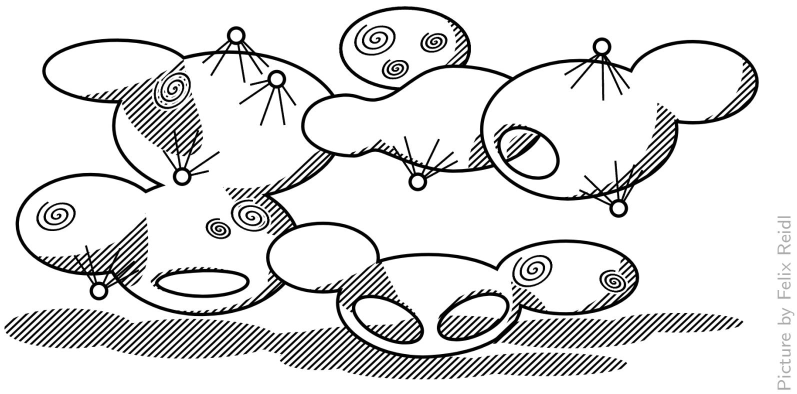

FIGURE 2.7. Structure of H-minor-free graphs.

The Graph Minor Structure Theorem is a common avenue for one to go through when studying graph minor theory. Chapter 8 discusses some of the questions raised by the emerging field of coarse graph theory. Roughly, coarse graph theory studies the "large-scale" features present in graphs, including fat minors, a notion similar to minors except with the additional property that they are realised at "large-scales". In the coarse graph theory setting, the ability to access tools from structural graph theory, would be hugely beneficial. Section 8.4 continues this discussion by surveying some tools from structural graph theory that have, in some sense, been translated to the coarse setting.

CHAPTER 3

The Erdős–Pósa Theorem

Recall the following theorem of Erdős and Pósa that is central to this thesis:

Theorem 3.0.1 (Erdős–Pósa Theorem [46]). *There exists a function $f : \mathbb{N} \to \mathbb{N}$, $f(k) = O(k \log k)$ such that for every graph G and all integers $k \geqslant 1$, G contains k pairwise vertex-disjoint cycles, or there exists $X \subseteq V(G)$ with $|X| \leqslant f(k)$ and $G - X$ is a forest. Moreover, $f(k) = \Omega(k \log k)$.*

Section 3.1 gives two proofs of the existence of a $O(k \log k)$ bounding function in the Erdős–Pósa Theorem (Theorem 3.0.1), and Section 3.2 explores various Erdős–Pósa type results. The proof of the $\Omega(k \log k)$ lower bound in the Erdős–Pósa Theorem is delayed till Section 7.2.

3.1. Existence of Bounding Function

This section showcases two proofs of the existence of a $O(k \log k)$ bounding function in the classic Erdős–Pósa Theorem, each exhibiting vastly different techniques. The first is based on Simonovits' proof [127] for which we follow Diestel's book [34], and the second is by Cames van Batenburg, Joret and Ulmer [21]. A third proof, albeit with a worse bounding function, is implied by Theorem 7.1.3.

Define the following[1] for all integers $k \geqslant 1$:

$$s(k) := \begin{cases} 4k(\log k + \log \log k + 4) & \text{if } k \geqslant 2, \\ 2 & \text{if } k = 1. \end{cases}$$

In pseudographs, loops and pairs of parallel edges are considered to be cycles.

Lemma 3.1.1 (Erdős, Pósa [47]). *For all integers $n \geqslant 2$, every n-vertex pseudograph G with $\delta(G) \geqslant 3$ contains a cycle C of length at most $2 \log n$.*

Proof. Holds trivially if G has a loop or parallel edges. Hence, it may be assumed that G has no loops and no parallel edges. Let x be any vertex of G. $\delta(G) \geqslant 3$ implies that G has a cycle, so let $r \geqslant 1$ be the maximum integer so that $2r < g(G)$. Then $n \geqslant |B_G(x, r)| \geqslant 1 + 3\sum_{i=0}^{r-1} 2^i = 1 + 3(2^r - 1) \geqslant 2^{r+1}$. Therefore $\log n \geqslant r + 1$. By choice of r, $2(r + 1) \geqslant g(G)$, hence $2 \log n \geqslant g(G)$. ■

Lemma 3.1.2 (Simonovits [127]). *For all integers $k \geqslant 1$, every 3-regular pseudograph G with at least $s(k)$ vertices contains k pairwise vertex-disjoint cycles.*

Proof. Proceed by induction on $k \geqslant 1$ with the hypothesis that every 3-regular pseudograph G with at least $s(k)$ vertices contains k pairwise vertex-disjoint cycles. In the base case, since G is 3-regular, G contains a cycle. Now suppose $k \geqslant 2$ and the

[1] The convention $\log = \log_2$ is used throughout this thesis.

hypothesis holds for smaller values of k. Let C the cycle promised by Lemma 3.1.1. Let $X_1, \dots, X_p$ be a maximal sequence of subsets $V(G)$ such that $X_1 = V(C)$ and for each $i \in [p-1]$, $X_{i+1} = X_i \cup \{v\}$ for some $v \in V(G - X_i)$ with $\deg_{G-X_i}(v) \leqslant 1$. Let m_i be the number of edges in G with exactly one end in X_i. Observe that since G is 3-regular, $m_{i+1} \leqslant m_i - 1$ for all $i \in [p-1]$. Moreover, since C is 2-regular, $m_1 \leqslant |C|$. Thus $m_p + p - 1 \leqslant m_1 \leqslant |C|$. By maximality of p, the vertices of $G - X_p$ have degree 2 or 3. Hence, in G, each vertex in $V(G - X_p)$ sends at most one edge to a vertex in X_p, thus $G - X_p$ has exactly m_p vertices of degree 2. Let H be the 3-regular pseudograph obtained from $G - X_p$ by suppressing all vertices of degree 2. Then $|H| = |G| - X_p - m_p = |G| - (|C| + p - 1) - m_p \geqslant |G| - 2|C| \geqslant |G| - 4\log|G|$. Taking a derivative shows $x \mapsto x - 4\log x$ is increasing for $x \geqslant \frac{4}{\ln 2}$, thus $|G| \geqslant s(k) \geqslant s(2) = 40 \geqslant \frac{4}{\ln 2}$ implies that $|H| \geqslant s(k) - 4\log s(k)$. Hence, it suffices to show $s(k) - 4\log s(k) \geqslant s(k-1)$. The desired inequality holds for $k = 2$ since $s(2) - 4\log s(2) \geqslant 40 - 4\log 64 \geqslant s(1)$. Now suppose $k \geqslant 3$.

Claim 3.1.3. $\log k + \log\log k + 4 \leqslant 4\log k$ *for all* $k \geqslant 3$.

Subproof. If $k \geqslant 4$, then $4 \leqslant 2\log k$, and so $\log k + \log\log k + 4 \leqslant \log k + \log k + 2\log k = 4\log k$. For $k = 3$, one may verify that $3^{16} \leqslant 2^{27}$. Then taking $\log\log$ of both sides shows $\log\log 3 + 4 \leqslant 3\log 3$, hence $\log 3 + \log\log 3 + 4 \leqslant 4\log 3$. ■

Finally, observe that:

$$s(k) - 4\log s(k) = s(k) - 8 - 4\log k - 4\log(\log k + \log\log k + 4).$$

By Claim 3.1.3:

$$\begin{aligned} s(k) - 4\log s(k) &\geqslant s(k) - 8 - 4\log k - 4\log(4\log k) \\ &= s(k) - 4(\log k + \log\log k + 4) \\ &= 4(k-1)(\log k + \log\log k + 4) \\ &\geqslant s(k-1). \end{aligned}$$

The result follows by induction. ■

1st proof of existence in Theorem 3.0.1. The result is shown for $f(k) = s(k) + k - 1$. Let G be a graph and H be a maximal subgraph such that $\deg_H(v) \in \{2, 3\}$ for all $v \in V(H)$. Let S be the set of all degree 3 vertices in H. If $|S| \geqslant s(k)$, let H' be the 3-regular pseudograph obtained from H by suppressing all vertices of degree 2, then H' has k pairwise vertex-disjoint cycles by Lemma 3.1.2. Appropriately subdividing edges of those cycles produces k pairwise vertex-disjoint cycles in G. Hence, it may be assume that $|S| < s(k)$.

Let $\mathcal{C}$ be the collection of all cycles in G that avoid S and meet H at exactly one vertex. Let Z be the set of those vertices and for each $z \in Z$ choose a cycle $C_z \in \mathcal{C}$ meeting H at z. Put $\mathcal{C}' := \{C_z : z \in Z\}$. Note that $Z \subseteq V(H) \setminus S$, so if two cycles of $\mathcal{C}'$ meet, there is an H-path in G whose ends have degree 2 in H, contradicting the maximality of H. Hence, it may be assumed that $\mathcal{C}'$ is a collection of pairwise vertex-disjoint cycles. Let $\mathcal{D}$ be the set of all 2-regular components of H that avoid Z, then $\mathcal{C}' \cup \mathcal{D}$ is a collection of pairwise vertex-disjoint cycles in G. If $|\mathcal{C}' \cup \mathcal{D}| \geqslant k$, then the desired packing of cycles has been found. Now suppose $|\mathcal{C}' \cup \mathcal{D}| \leqslant k-1$. Let A be the set obtained from Z by adding one vertex from each member of $\mathcal{D}$, then $|A| \leqslant k-1$. The following demonstrates that $X := S \cup A$ meets every cycle of G. Let C be a cycle in G, then C meets H by maximality of H. It may be assumed that C avoids S. Hence, if $C \subseteq H$, then C is a component of H, and so $C \in \mathcal{D}$, which implies C meets A. Hence, it may be assumed that there exists $e \in E(C) \setminus E(H)$. If C meets H at exactly one vertex, C meets Z, then C meets A. Otherwise, C has an H-path P through e such that both ends of P have degree 2 in H. Then $H \cup P$ contradicts the maximality of H. Finally, verify $|X| \leqslant s(k)+k-1 = f(k)$, as required. ■

Now to the second proof.

Lemma 3.1.4 (Mader [90]). *For all integers $d \geqslant 3$ and $k \geqslant 1$, every graph G with $\delta(G) \geqslant d$ and $g(G) \geqslant 8k+3$ has a minor with minimum degree at least $d(d-1)^k$.*

Proof. Let $X \subseteq V(G)$ be maximal such that $\mathrm{dist}_G(u,v) \geqslant 2k+1$ for all distinct $u,v \in X$. Consider the following iterative process that produces a family of pairwise-vertex disjoint trees $(T_x : x \in X)$ with $V(G) = V(\bigcup_{x\in X} T_x)$ and $V(T_x) \subseteq B_G(x, 2k)$. For each $x \in X$ initialise a tree $T_x := G[\{x\}]$. While $V(G) \setminus V(\bigcup_{x\in X} T_x) \neq \emptyset$, choose $u \in V(G) \setminus V(\bigcup_{x\in X} T_x)$ to minimise $m := \mathrm{dist}_G(X,u)$. Since $V(G) = B_G(X,2k)$, $1 \leqslant m \leqslant 2k$. So there exists $v \in V(G)$ with $\mathrm{dist}_G(X,v) = m-1$. Then by choice of u, $v \in V(T_x)$ for some $x \in X$. Update T_x by adding the vertex u and the edge uv. This completes the description of the process.

Since any two vertices of X are at distance at least $2k+1$, for every $x \in X$, every $u \in V(G)$ with $\mathrm{dist}_G(u,x) \leqslant k$ satisfies $\mathrm{dist}_G(u, X \setminus \{x\}) \geqslant k+1$, and so $u \in V(T_x)$. Therefore $B_G(x,k) \subseteq V(T_x)$. Furthermore, for each $x \in X$, since $V(T_x) \subseteq B_G(x,2k)$ and $g(G) \geqslant 8k+3$, T_x is induced in G. Then $\delta(G) \geqslant d$ implies T_x contains at least $d(d-1)^{k-1}$ vertices at distance k from x, thus T_x has at least $d(d-1)^{k-1}$ leaves. Again $\delta(G) \geqslant d$ implies each leaf of T_x has at least $d-1$ neighbours in other trees (i.e. T_y's with $y \in X \setminus \{x\}$). However, there is at most one T_x-T_y-edge for distinct $x,y \in X$, otherwise there is a cycle of length at most $8k+2$ in G, which contradicts $g(G) \geqslant 8k+3$. Hence contracting each T_x to a single vertex produces a graph with minimum degree at least $d(d-1)^k$. ■

Kühn and Osthus [81] showed a qualitative strengthening of Mader's lemma (Lemma 3.1.4). We roughly sketch their proof. Notice that by only assuming $g(G) \geqslant 4k+3$, one can still show as in the above proof that each $B_G(x,k)$ induces a tree. By choosing the set of roots X randomly, with positive probability there is an outcome X for which there are relatively few edges with an end outside of $B_G(X,k)$, so-called bad edges. By a slight adjustment to the process described in the above proof, produce a family of pairwise vertex-disjoint trees $(T_x : x \in X)$ with $B_G(X,k) = V(\bigcup_{x\in X} T_x)$ and $V(T_x) \subseteq B_G(x,k)$. The $g(G) \geqslant 4k+3$ condition implies each non-bad edge not in $E(\bigcup_{x\in X} T_x)$ is a T_x-T_y-edge for some distinct $x,y \in X$, and there is at most one T_x-T_y-edge for distinct $x,y \in X$. Since there are few bad edges and $\delta(G) \geqslant d$, one can show that $(T_x : x \in X)$ are the branch sets of a model in G of a graph with large average degree, implying G has a minor with large minimum degree. More precisely, Kühn and Osthus showed that for all integers $d \geqslant 3$ and $k \geqslant 1$, every graph G with $\delta(G) \geqslant d$ and $g(G) \geqslant 4k+3$ has a minor with minimum degree at least $\frac{1}{48}(d-1)^{k+1}$.

Lemma 3.1.5. *For every integer $k \geqslant 1$, every graph G with $\delta(G) \geqslant 3k-1$ contains k pairwise vertex-disjoint cycles.*

Proof. Proceed by induction on $k \geqslant 1$ with the hypothesis that every graph G with $\delta(G) \geqslant 3k-1$ contains k pairwise vertex-disjoint cycles. In the base case $\delta(G) \geqslant 2$, so G is not a forest, implying G contains a cycle. Now suppose $k \geqslant 2$ and the hypothesis holds for smaller values of k. Let C be a shortest cycle in G. Then every vertex of G not in $V(C)$ has at most three of its neighbours in $V(C)$, otherwise there exists a cycle shorter than C. It follows that $\delta(G - V(C)) \geqslant \delta(G) - 3 \geqslant 3(k-1) - 1$. By induction $G - V(C)$ has $k-1$ pairwise vertex-disjoint cycles. This along with C shows that G contains k pairwise vertex-disjoint cycles. The result follows by induction. ∎

2nd proof of existence in Theorem 3.0.1. Proceed by induction on $n \geqslant 1$ with the hypothesis that for every n-vertex graph G and every integer $k \geqslant 1$, G contains k pairwise vertex-disjoint cycles or there exists $X \subseteq V(G)$ with $|X| \leqslant 8k\lceil \log k \rceil + 2k$ and $G - X$ is a forest. In the base case of $n = 1$, G has no cycle, so $G - X$ is a forest where $X := \emptyset$. Now suppose $n \geqslant 2$ and the hypothesis holds for smaller values of n. Notice that the result is trivial if $k = 1$, hence it may be assumed that $k \geqslant 2$. Suppose G has a vertex v with $\deg_G(v) \leqslant 1$ and consider $H := G - v$. If induction gives k pairwise vertex-disjoint cycles in H, then these cycles suffice for G. Otherwise induction gives $X \subseteq V(H)$ with $|X| \leqslant 8k\lceil \log k \rceil + 2k$ and $H - X$ is a forest. Since no cycle of G uses v, $G - X$ is a forest. Hence, it may be assumed that $\delta(G) \geqslant 2$. Suppose G contains a cycle C of length at most $8\lceil \log k \rceil + 2$ and consider $H' := G - V(C)$. If induction gives $k-1$ pairwise vertex-disjoint cycles in H', then along with C, G has k pairwise vertex-disjoint cycles. Otherwise induction gives $X' \subseteq V(H')$ with $|X'| \leqslant 8(k-1)\lceil \log(k-1) \rceil + 2(k-1)$ and $H' - X'$ is a forest. Put $X := X' \cup V(C)$, then $|X| \leqslant 8k\lceil \log k \rceil + 2k$ and $G - X$ is a forest. Hence, it may be assumed that $g(G) \geqslant 8\lceil \log k \rceil + 3$. Suppose G has a vertex v with $\deg_G(v) = 2$ and write $N_G(v) = \{x, y\}$. Since $k \geqslant 2$, $g(G) > 4$, thus $xy \notin E(G)$. Let H'' be the graph obtained from G by suppressing v. If induction gives k pairwise vertex-disjoint cycles in H'', then by possibly subdividing an edge of one of those cycles, one finds k pairwise vertex-disjoint cycles in G. Otherwise induction gives $X'' \subseteq V(H'')$ with $|X''| \leqslant 8k\lceil \log k \rceil + 2k$ and $H'' - X''$ is a forest. Assume for a contradiction that $G - X''$ has a cycle C'. Since $G - v \subseteq H''$, $G - v - X''$ is a forest, hence C' uses v. Since $xy \notin E(C')$, suppressing v in C' gives a cycle in $H'' - X''$, a contradiction. Hence, it may be assumed that $\delta(G) \geqslant 3$ and $g(G) \geqslant 8\lceil \log k \rceil + 3$. By Lemma 3.1.4 G has a minor M with minimum degree at least $3 \cdot 2^{\lceil \log k \rceil} \geqslant 3k$. By Lemma 3.1.5 M has k pairwise vertex-disjoint cycles, hence so does G. The result follows by induction. ∎

3.2. Close Relatives

This section surveys some close relatives of the classic Erdős–Pósa Theorem to highlight the variety one can find in this topic. Recall the following definitions: Given graph classes $\mathcal{G}$ and $\mathcal{F}$, a *bounding function* $f : \mathbb{N} \to \mathbb{N}$ for the pair $(\mathcal{G}, \mathcal{F})$, is a function such that for every graph $G \in \mathcal{G}$ and all integers $k \geqslant 1$, G contains k pairwise vertex-disjoint subgraphs in $\mathcal{F}$, or there exists a subset $X \subseteq V(G)$ with $|X| \leqslant f(k)$ and $G - X$ has no subgraph in $\mathcal{F}$. If a bounding function exists for the pair $(\mathcal{G}, \mathcal{F})$, then $(\mathcal{G}, \mathcal{F})$ is said to have the *Erdős–Pósa property*. For convenience, $\mathcal{G}$ and $\mathcal{F}$ are often referred to as the *host* and *guest* classes respectively. If $\mathcal{G}$ is the class of all graphs, we write more succinctly that $\mathcal{F}$ has the Erdős–Pósa property. Unfortunately, this definition does not quite encompass the diversity of Erdős–Pósa type results found in the literature. In fact, to the author's knowledge, there is no known definition of "Erdős–Pósa property" that covers every case that one would want it to. By the end of this section, one should be able to distinguish Erdős–Pósa type results.

3.2.1. Packing Minors

Recall that for graphs H, $\mathcal{F}(H)$ is the class of all graphs containing H as a minor. Then the Erdős–Pósa Theorem says that $\mathcal{F}(K^3)$ has the Erdős–Pósa property with bounding function $O(k \log k)$. A key result by Robertson and Seymour [118] characterises the graphs H for which $\mathcal{F}(H)$ has the Erdős–Pósa property: For graphs H, $\mathcal{F}(H)$ has the Erdős–Pósa property if and only if H is planar. Furthermore, Cames van Batenburg, Huynh, Joret and Raymond [20] showed that for any planar graph H, there exists a $O(k \log k)$ bounding function for $\mathcal{F}(H)$, truly generalising the Erdős–Pósa Theorem.

So far, the host class has been the class of all graphs, and the guest class has varied. By adjusting the host class and keeping the guest class fixed, one would expect different bounding functions. Indeed, in the Erdős–Pósa Theorem, if one restricts the host class to planar graphs, one obtains a linear bounding function as shown by the following theorem:

Theorem 3.2.1 (Kloks, Lee, Liu [79])**.** *Let $\mathcal{P}$ be the class of all planar graphs. Then $f(k) = 5k$ is a bounding function for $(\mathcal{P}, \mathcal{F}(K^3))$.*

The fact that $f(k)$ is linear in Theorem 3.2.1 was originally shown by Bienstock and Dean [12]. Famously, Kloks, Lee and Liu conjectured that Theorem 3.2.1 holds when the "5" is replaced by "2". Called "Jones' Conjecture", it is tight for wheels and holds for outer-planar graphs [79]. Furthermore, Bonamy, Dross, Masařík, Nadara, Pilipczuk and Pilipczuk [18] showed that Jones' Conjecture holds when the host class is subcubic planar pseudographs.

Strikingly, Theorem 3.2.1 generalises in the following way to any proper minor-closed class:

Theorem 3.2.2 (Fomin, Saurabh, Thilikos [51])**.** *For every planar graph H and every proper minor-closed class $\mathcal{G}$, there exists a $O(k)$ bounding function for $(\mathcal{G}, \mathcal{F}(H))$.*

Chapter 7 studies Erdős–Pósa properties of planar graph minors in more detail.

3.2.2. Constrained Cycles

There is a large variety of Erdős–Pósa type results that try generalise the Erdős–Pósa Theorem by adding constraints to the cycles. An example of such a generalisation is the following Erdős–Pósa property of long cycles:

Theorem 3.2.3 (Mousset, Noever, Škorić, Weissenberger [97])**.** *There exists a function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, $f(k, \ell) = O(k\ell + k \log k)$ such that for every graph G and all integers*

$k \geqslant 1$ and $\ell \geqslant 3$, G contains k pairwise vertex-disjoint cycles of length at least ℓ, or there exists a subset $X \subseteq V(G)$ with $|X| \leqslant f(k,\ell)$ and $G - X$ has no cycle of length at least ℓ.

The first bound on the function f in Theorem 3.2.3 was an exponential tower implied by Robertson and Seymour's [118] extension of the Erdős–Pósa Theorem to planar graphs. A subsequent paper by Thomassen [129] implies $f(k,\ell) = 2^{\ell^{O(k)}}$. Later, a bound of $f(k,\ell) = O(\ell k^2)$ was produced by Birmelé, Bondy and Reed [14], which was followed by an improvement to $f(k,\ell) = O(\ell k \log k)$ by Fiorini and Herinckx [48]. The bound on f by Mousset, Noever, Škorić and Weissenberger [97] (as shown in Theorem 3.2.3) has the optimal order.

The following theorem gives another generalisation of the Erdős–Pósa Theorem in terms of cycles through a prescribed set of vertices:

Theorem 3.2.4 (Pontecorvi, Wollan [106])**.** *There exists a function $f : \mathbb{N} \to \mathbb{N}$, $f(k) = O(k \log k)$ such that for every graph G, every subset $S \subseteq V(G)$, and every integer $k \geqslant 1$, G contains k pairwise vertex-disjoint cycles each containing at least one vertex in S, or there exists a subset $X \subseteq V(G)$ with $|X| \leqslant f(k)$ and $G - X$ has no such cycle.*

Notice that taking $S := V(G)$ in Theorem 3.2.4 yields the Erdős–Pósa Theorem. Consequently, $f(k) = O(k \log k)$ is optimal for Theorem 3.2.4. The first result extending the Erdős–Pósa Theorem to cycles through a prescribed set of vertices was by Kakimura, Kawarabayashi and Marx [75], who obtained a bound of $f(k) = O(k^2 \log k)$.

Interestingly, Bruhn, Joos and Schaudt [19] showed that the constraints in Theorem 3.2.3 and in Theorem 3.2.4 can be combined:

Theorem 3.2.5 (Bruhn, Joos, Schaudt [19])**.** *There exists a function $f : \mathbb{N} \to \mathbb{N}$, $f(k,\ell) = O(\ell k \log k)$ such that for every graph G, every subset $S \subseteq V(G)$, and all integers $k \geqslant 1$ and $\ell \geqslant 3$, G contains k pairwise vertex-disjoint cycles of length at least ℓ and each containing at least one vertex in S, or there exists a subset $X \subseteq V(G)$ with $|X| \leqslant f(k,\ell)$ and $G - X$ has no such cycle.*

At this point, it is worth mentioning that Simonovits's calculation [127] (as in Lemma 3.1.2) was utilised by [48, 106, 19].

Cycles with modularity constraints have also been studied extensively. For example, Thomassen [129] showed that for every integer $m \geqslant 1$, the class of all cycles with length $0 \mod m$ has the Erdős–Pósa property with bounding function $f(k) = 2^{m^{O(k)}}$. Cames van Batenburg, Huynh, Joret and Raymond's tight bounding function for planar graph minors discussed in Section 7.3 implies that Thomassen's result holds with $f(k) = O(k \log k)$ (m is fixed). Thomassen's result was extended to cycles with length $0 \mod m$ through prescribed vertices by Kakimura and Kawarabayashi [74]. On the other hand, according to Reed [112], Lovász and Schrijver pointed out that Erdős–Pósa property does not hold for odd cycles. Reed [112] elucidates on this by showing that there is a family of so-called Escher walls that have no disjoint odd cycles and no small set of vertices meeting every odd cycle. The proceeding defines Escher walls.

The elementary wall of height 8 is depicted in Figure 3.1. In general, the *elementary wall* of height $h \geqslant 3$ is the graph obtained from the $(h+1) \times 2(h+1)$-grid by deleting

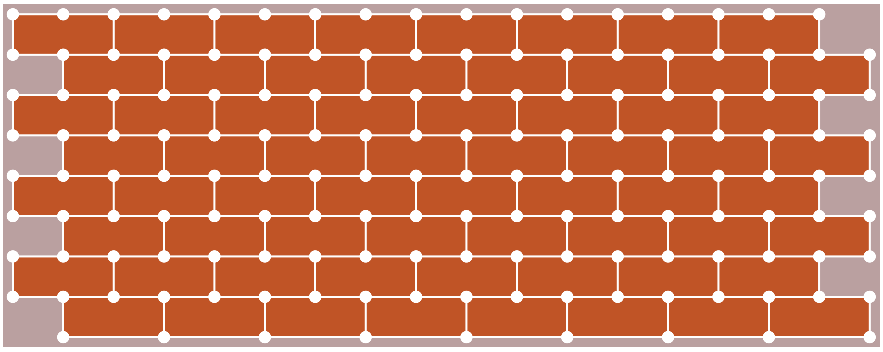

FIGURE 3.1. Elementary wall of height 8.

the following edges:

$$\{(2i,2j-1)(2i+1,2j-1) : i \in [\lfloor \tfrac{1}{2}h \rfloor], j \in [h+1]\} \\ \cup \{(2i-1,2j)(2i,2j) : i \in [\lfloor \tfrac{1}{2}h \rfloor], j \in [h+1]\}.$$

A *wall* of height $h \geqslant 3$ is any graph obtained from the elementary wall of height h by subdividing its edges. Walls have a natural embedding in the plane (see Figure 3.1). A *brick* of a wall W is a cycle that forms the boundary of an inner face in a natural embedding of W in the plane. Informally, a wall of height h consists of h layers of bricks, each containing h bricks. An *Escher wall* of height $h \geqslant 3$ is a graph consisting of a wall W of height h along with h pairwise vertex-disjoint W-paths $P_1, \ldots, P_h$ such that:

(i) W is bipartite.
(ii) For each $i \in [h]$, each endpoint of P_i lies in only one brick of W, one endpoint of P_i is in the ith brick of the top layer of W, and its other endpoint is in the $(h+1-i)$th brick of the bottom layer of W.
(iii) For each $i \in [h]$, $W \cup P_i$ contains an odd cycle.

See Figure 3.2 for an example.

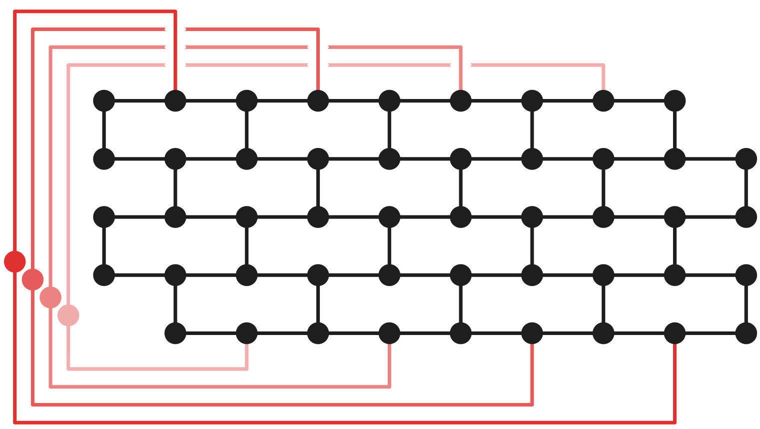

FIGURE 3.2. Example of Escher wall of height 4.

Dejter and Neumann-Lara [31] found infinitely many *pairs* (ℓ, m) such that the Erdős–Pósa property does not hold for cycles of length $\ell \mod m$, and they asked to

($\star$) determine all pairs (ℓ, m) such that the Erdős–Pósa property does not hold.

The class of all cycles of non-zero length $\mod m$ for a fixed even integer m does not have the Erdős–Pósa property (see [135]).

In his paper titled *"Mangoes and Blueberries"* [112], Reed proved the following conjecture of Erdős and Hajnal, giving sufficient conditions for a graph to have a hitting set for odd cycles:

Theorem 3.2.6 (Reed [112])**.** *There exists a function $f : \mathbb{N} \to \mathbb{N}$ such that for every graph G and all integers $k \geqslant 0$, if every subgraph H of G has an independent set of size at least $\frac{1}{2}(V(H) - k)$, then there exists a subset $X \subseteq V(G)$ with $|X| \leqslant f(k)$ and $G - X$ is bipartite.*

The title *"Mangoes and Blueberries"* is in relation to Gyárfás' *"Fruit Salad"* [60]. However, this was no ordinary fruit salad. Instead it was mixed from various ingredients that Gyárfás obtained while working on Erdős' problems, including the $k = 0$ case of Theorem 3.2.6.

FIGURE 3.3. Bruce Reed with a bowl of mangoes and blueberries.

Reed's approach to proving Theorem 3.2.6 implies a version of the Erdős–Pósa property for odd cycles when vertex-disjointness is relaxed in the following way: For integers $k \geqslant 1$ and graphs G, a *half integral packing* in G is a collection $\mathcal{C}$ of subgraphs in G such that each vertex of G lies in at most two members of $\mathcal{C}$.

Theorem 3.2.7 (Reed [112])**.** *There exists a function $f : \mathbb{N} \to \mathbb{N}$ such that for every graph G and all integers $k \geqslant 1$, G has a half integral packing of k odd cycles, or there exists a subset $X \subseteq V(G)$ with $|X| \leqslant f(k)$ and $G - X$ is bipartite.*

In his method, Reed showed essentially showed that Escher walls are the canonical obstructions to the Erdős–Pósa property for odd cycles. That is, a graph with neither k pairwise vertex-disjoint odd cycles nor a small hitting set for odd cycles, has a large Escher wall. As a consequence, this ensures that odd cycles have the Erdős–Pósa property when the host class consists of graphs embeddable on any fixed orientable surface.[2]

[2]This is because Escher walls are not embeddable in orientable surfaces.

As mentioned earlier, the Erdős–Pósa property does not hold for odd cycles, and so it does not hold for odd cycles through prescribed vertices. The following theorem of Kakimura and Kawarabayashi [73] shows that odd cycles through prescribed vertices have the Erdős–Pósa property when vertex-disjointness is relaxed to half integral packing, generalising Reed's Theorem 3.2.7:

Theorem 3.2.8 (Kakimura, Kawarabayashi [73]). *There exists a function $f : \mathbb{N} \to \mathbb{N}$ such that for every graph G, every subset $S \subseteq V(G)$, and every integer $k \geqslant 1$, G has a half integral packing of k odd cycles each containing at least one vertex in S, or there exists a subset $X \subseteq V(G)$ with $|X| \leqslant f(k)$ and $G - X$ has no odd cycle that meets S.*

One recovers Reed's Theorem 3.2.7 from Theorem 3.2.8 with $S := V(G)$.

Recall that Robertson and Seymour showed that the planarity of H is necessary and sufficient for $\mathcal{F}(H)$ to have the Erdős–Pósa property. If instead one packs half integrally, the planarity condition may be dropped:

Theorem 3.2.9 (Liu [84]). *For every graph H, there exists a function $f : \mathbb{N} \to \mathbb{N}$ such that for every graph G and every integer $k \geqslant 1$, G has a half integral packing of k subgraphs in $\mathcal{F}(H)$, or there exists a subset $X \subseteq V(G)$ with $|X| \leqslant f(k)$ and $G - X$ has no subgraph in $\mathcal{F}(H)$.*

Liu showed more strongly that the above theorem holds when $\mathcal{F}(H)$ is replaced by the class of graphs that contain a subdivision of H. According to Liu [84], Theorem 3.2.9 was conjectured by Thomas, and Norin announced a proof of Theorem 3.2.9 that was not published.

It turns out that odd cycles have the Erdős–Pósa property when packing into highly connected graphs, as shown by the following result:

Theorem 3.2.10 (Rautenbach, Reed [109]). *There exists a function $g : \mathbb{N} \to \mathbb{N}$, $g(k) = O(k)$ such that for every integer $k \geqslant 1$, and every $g(k)$-connected graph G, G has k pairwise vertex-disjoint odd cycles, or there exists a subset $X \subseteq V(G)$ with $|X| \leqslant 2k - 2$ and $G - X$ is bipartite.*

Thomassen [130] proved a result similar to Theorem 3.2.10 except g was doubly-exponential. The best known function g satisfying Theorem 3.2.10 is $g(k) = \frac{31}{2}k$ by Kawarabayashi and Wollan [78].

Linear connectivity also ensures the Erdős–Pósa property for odd cycles through prescribed vertices, as shown by the following theorem that parallels Kakimura and Kawarabayashi's Theorem 3.2.8:

Theorem 3.2.11 (Joos [71]). *For any integer $k \geqslant 1$, any $50k$-connected graph G, and any subset $S \subseteq V(G)$, G has k pairwise vertex-disjoint odd cycles each containing at least one vertex in S, or there exists a subset $X \subseteq V(G)$ with $|X| \leqslant 2k - 2$ and $G - X$ has no such cycle.*

Wollan considered a different type of modularity constraint as shown by the following theorem:

Theorem 3.2.12 (Wollan [135]). *For odd integers $m \geqslant 1$, the class of all cycles of non-zero length $\bmod m$ has the Erdős–Pósa property. Moreover, there is a bounding function that does not depend on m.*

To prove Theorem 3.2.12, Wollan proved a more general statement regarding group labelled graphs. For abelian groups Γ, a *Γ-labelling* of a graph G is a function $\gamma : E(G) \to \Gamma$. For each subgraph $H \subseteq G$, its *γ-value* is defined as $\gamma(H) := \sum_{e \in E(H)} \gamma(e)$. A *non-zero cycle* C in G is one whose γ-value is non-zero.

Theorem 3.2.13 (Wollan [135])**.** *There exists constants c and c' such that the following holds. Let Γ be an abelian group that does not have any elements of order two, and let G be a Γ-labelled graph. Then for all integers $k \geqslant 1$, G contains k pairwise vertex-disjoint non-zero cycles, or there exists a subset $X \subseteq V(G)$ with $|X| \leqslant c^{k^{c'}}$ and X meets every such cycle.*

Then Theorem 3.2.12 follows from Theorem 3.2.13 by labelling every edge $1 \in \mathbb{Z}_m$. Later, Huynh, Joos and Wollan [70] produced a unified framework for proving Erdős–Pósa type results for constrained cycles. Their method studied *(Γ_1, Γ_2)-direct-labelled* graphs $(\overrightarrow{G}, \gamma)$, where $\overrightarrow{G}$ is a digraph and $\gamma : E(\overrightarrow{G}) \to \Gamma_1 \oplus \Gamma_2$ is an edge labelling of $\overrightarrow{G}$ by elements of a direct sum of groups $\Gamma_1 \oplus \Gamma_2$. The *group-value* of a walk $W = v_0 e_1 v_1 e_2 v_2 \dots e_\ell v_\ell$ is defined to be $\gamma(W) := \gamma'(e_1, v_1)\gamma'(e_2, v_2)\cdots\gamma'(e_1, v_1)\gamma'(e_\ell, v_\ell)$ where $\gamma'(e_i, v_i) := \gamma(e_i)$ if v_i is the head of e_i, otherwise $\gamma'(e_i, v_i) := \gamma(e_i)^{-1}$. Much like how Reed showed that Escher walls are the canonical obstructions for the Erdős–Pósa property for odd cycles, Huynh, Joos and Wollan determined all canonical obstructions to the Erdős–Pósa property for the non-zero cycles in (Γ_1, Γ_2)-direct-labelled graphs, and used this to show the following "half integral Erdős–Pósa type theorem" for non-zero cycles in (Γ_1, Γ_2)-direct-labelled graphs:

Theorem 3.2.14 (Huynh, Joos, Wollan [70])**.** *There exists a function $f : \mathbb{N} \to \mathbb{N}$ such that for all groups Γ_1 and Γ_2, all (Γ_1, Γ_2)-direct-labelled graphs $(\overrightarrow{G}, \gamma)$, and all integers $k \geqslant 1$, $(\overrightarrow{G}, \gamma)$ has a half integral packing of k non-zero cycles, or there exists a subset $X \subseteq V(\overrightarrow{G})$ with $|X| \leqslant f(k)$ and X meets every such cycle.*

Using their results one may derive (with worse bounding functions) the Erdős–Pósa property of long cycles (Theorem 3.2.3), cycles through prescribed vertices (Theorem 3.2.4), long cycles through prescribed vertices (Theorem 3.2.5) and much more. The half integral Erdős–Pósa property for odd cycles (Theorem 3.2.7) and odd cycles through prescribed vertices are also recovered (Theorem 3.2.8).

Now looking at (undirected)labellings of graphs, Gollin, Hendrey, Kawarabayashi, Kwon and Oum [57] proved a far reaching generalisation of Theorem 3.2.7, they essentially showed that in any graph whose edges are labelled by multiple abelian groups, the cycles whose value in each group avoids some fixed bounded set have the half integral Erdős–Pósa property. The following is their main theorem:

Theorem 3.2.15 (Gollin, Hendrey, Kawarabayashi, Kwon, Oum [57])**.** *For all integers $m, \omega \geqslant 1$, there exists a function $f_{m,\omega} : \mathbb{N} \to \mathbb{N}$ such that, for all abelian groups $\Gamma_1, \dots, \Gamma_m$, for all sets $\Omega_1, \dots, \Omega_m$ with $\Omega_i \subseteq \Gamma_i$ and $|\Omega_i| \leqslant \omega$ for all $i \in [m]$, for all graphs G, for all $\Gamma_1 \oplus \dots \oplus \Gamma_m$-labellings $\gamma : E(G) \to \Gamma_1 \oplus \dots \oplus \Gamma_m$ of G, and all integers $k \geqslant 1$, G has a half integral packing of k cycles whose γ-values lie in $(\Gamma_1 \setminus \Omega_1) \times \dots \times (\Gamma_m \setminus \Omega_m)$, or there exists a subset $X \subseteq V(G)$ with $|X| \leqslant f_{m,\omega}(k)$ and X meets every such cycle.*

Suppose $m = \omega = 1$, $\Gamma_1 = \mathbb{Z}_2$, $\Omega_1 = \{0\}$, and labelling every edge 1. Then the odd cycles are precisely those whose value is 1. Consequently, one obtains Theorem 3.2.7 from Theorem 3.2.15. One may recover Theorem 3.2.8 from Theorem 3.2.15 by $\mathbb{Z}_2 \oplus \mathbb{Z}$-labelling G in the following way: Assign a weight of $(1,1)$ to the edges that are incident to a vertex in S, and assign a weight of $(1,0)$ to every other edge. Then the cycles with value in $\{1\} \times (\mathbb{Z} \setminus \{0\})$ are odd and contain at least one vertex in S. Therefore Theorem 3.2.8 follows by applying Theorem 3.2.15 with this $\mathbb{Z}_2 \oplus \mathbb{Z}$-labelling, $m = 2$, $\omega = 1$, $\Gamma_1 = \mathbb{Z}_2$, $\Gamma_2 = \mathbb{Z}$, $\Omega_1 = \{0\}$ and $\Omega_2 = \{0\}$. Analogously, one may prove the following half integral Erdős–Pósa result for cycles through m prescribed sets:

Corollary 3.2.16. *Let $m \geqslant 1$ be an integer. There exists a function $f : \mathbb{N} \to \mathbb{N}$ such that for every graph G, all subset $S_1, \dots, S_m \subseteq V(G)$, and every integer $k \geqslant 1$, G has a half integral packing of k cycles each containing at least one vertex in each of $S_1, \dots, S_m$, or there exists a subset $X \subseteq V(G)$ with $|X| \leqslant f(k)$ and $G - X$ has no such cycle.*

This corollary is obtained from Theorem 3.2.15 by taking $\omega = 1$, $\Gamma_1 = \cdots = \Gamma_m = \mathbb{Z}$, $\Omega_1 = \cdots = \Omega_m = \{0\}$, and defining a labelling $\gamma : E(G) \to \Gamma_1 \oplus \cdots \oplus \Gamma_m$ by setting the ith entry of $\gamma(e)$ to be

$$(\gamma(e))_i = \begin{cases} 1 & \text{if } e \text{ is incident to a vertex in } S_i, \\ 0 & \text{otherwise.} \end{cases}$$

For more examples of applications of Theorem 3.2.15, see [57].

This section concludes with the following result that relates to the question of Dejter and Neumann-Lara ($\star$). Gollin, Hendrey, Kwon, Oum and Yoo [56] characterised all pairs (ℓ, m) for which the Erdős–Pósa property holds for cycles of length $\ell \mod m$, thus solving Dejter and Neumann-Lara's problem ($\star$):

Theorem 3.2.17 (Gollin, Hendrey, Kwon, Oum, Yoo [56])**.** *Let ℓ and z be integers with $z \geqslant 2$, and let $p_1^{a_1} \cdots p_n^{a_n}$ be the prime factorisation of z with $p_i < p_{i+1}$ for all $i \in [n-1]$. Then the following bullets are equivalent:*

- *The class of all cycles of length $\ell \mod m$ has the Erdős–Pósa property.*
- *Both of the following hold:*
  - *(1) If $p_1 = 2$, then $\ell = 0 \mod p_1^{a_1}$, and*
  - *(2) There do not exist distinct $i_1, i_2, i_3 \in [n]$ such that $\ell \neq 0 \mod p_{i_j}^{a_{i_j}}$ for each $j \in [3]$.*

There are many more Erdős–Pósa dualities that were not stated in this section, too many to list here. Thus, we refer the reader to the survey by Raymond and Thilikos [110].

CHAPTER 4

Chordal and Perfect Graphs

A graph G is *chordal* if every cycle $C \subseteq G$ of length at least 4 has a *chord*, an edge $e \in E(G) \setminus E(C)$ joining two vertices of C. Equivalently, G is chordal if G has no induced cycle of length at least 4. Thus, induced subgraphs of a chordal graph are chordal. The main take away from this chapter comes from Section 4.1, which describes how a collection of subtrees of a tree intersect. Such a description is especially useful in Chapter 6 to prove results about tree-width, and in Chapter 7 in the proof of Robertson and Seymour's extension of the Erdős–Pósa Theorem. Intuitively, there ought to be a good reason for why a collection of subtrees $\mathcal{F}$ of a tree T does not have many pairwise vertex-disjoint members. Indeed, it turns out the reason why is because there exists a small set of vertices of T which meet every graph in $\mathcal{F}$. This property of subtrees of a tree is known as the generalised Helly property. More precisely, the generalised Helly property states that for any collection of subtrees $\mathcal{F}$ of a tree T, the pair $(\{T\}, \mathcal{F})$ has the Erdős–Pósa property with bounding function $f(k) = k-1$. Another way to analyse how subtrees of a tree intersect is to study an auxiliary graph, one whose vertices are the subtrees of interest and two vertices are adjacent if and only if the corresponding subtrees intersect. These types of auxiliary graphs are suitably named intersection graphs. Section 4.1 proves that the intersection graphs of subtrees of a tree are the chordal graphs. In order to build up to that, this chapter begins with the following characterisation of chordal graphs by Dirac:

Proposition 4.0.1 (Dirac [37])**.** *A graph is chordal if and only it can be obtained recursively from complete graphs, by pasting along cliques.*

According to Proposition 4.0.1, the components of chordal graphs decompose into a "tree-like" structure as illustrated in Figure 4.1.

Tree-width is a standard measure of how "tree-like" a graph is. One way of defining the tree-width of a graph G is in the following way:

$$\mathrm{tw}(G) = \min\{\omega(H) - 1 : H \text{ is a chordal graph and } G \subseteq H\}.$$

An equivalent definition is given later in Chapter 6. Moreover, in Chapter 6 it will be shown that the underlying tree-like structure of graphs allows one to transfer techniques used for trees, such as Proposition 2.1.2, to other classes of graphs, for instance Theorem 6.0.5. If a graph has low tree-width, there is intuitively a low cost associated with this "transfer".

Proof of Proposition 4.0.1. ($\implies$) Proceed by induction on $n \geqslant 1$ with the hypothesis that every n-vertex chordal graph G can be obtained from complete graphs by pasting along cliques. The base case of $n = 1$ is trivial, so assume $n \geqslant 2$ and the hypothesis holds for smaller values of n. It may be assumed that G is not a complete

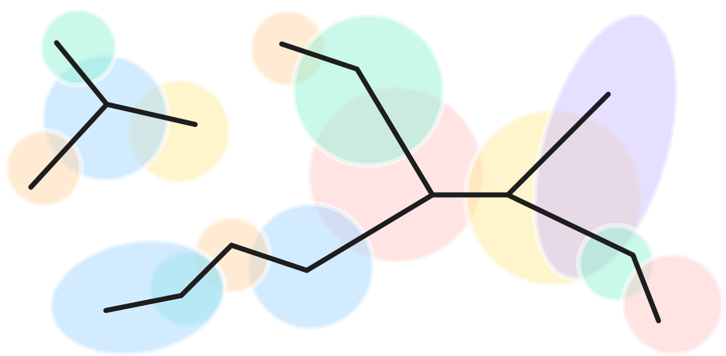

FIGURE 4.1. Illustration of the structure of chordal graphs implied by Proposition 4.0.1: The blobs are complete graphs, and the intersections of different blobs are cliques. Notice the “tree-like” structure suggested by the black lines.

graph. Then consider any two non-adjacent vertices $a, b \in V(G)$. Let $K \subseteq V(G) \setminus \{a, b\}$ with $|K|$ minimum possible and K meets every a-b-path in G. Let H be the component of $G - K$ containing a, and put $G_1 := G[V(H) \cup K]$ and $G_2 := G - V(H)$. Since each of G_1 and G_2 are induced subgraphs of G, they are chordal, hence by induction G_1 and G_2 can be obtained from complete graphs by pasting along cliques. It suffices to show K is a clique in G. This is true if $|K| \leqslant 1$, so assume $|K| \geqslant 2$. Assume for a contradiction that there exists distinct non-adjacent vertices $p, q \in K$. Then by choice of K, Menger's Theorem (Theorem 2.1.3) promises two internally disjoint a-b-paths P, Q through p and q respectively. Since $P \cup Q$ contains an induced cycle of G with length at least 4, G is not chordal, a contradiction. Hence G is obtained from G_1 and G_2 by pasting along the clique K. The result follows by induction.

($\Longleftarrow$) Proceed by induction on $n \geqslant 1$ with the hypothesis that every n-vertex graph G that can be obtained from complete graphs by pasting along cliques is chordal. The base case of $n = 1$ is trivial, so assume $n \geqslant 2$ and the hypothesis holds for smaller values of n. It may be assumed that G is not a complete graph. Then there exists graphs G_1 and G_2 such that $|G_1| \leqslant n - 1$, G_1 can be obtained from complete graphs by pasting along cliques, G_2 is a complete graph, and G is obtained from G_1 and G_2 by pasting along a clique K. By induction G_1 and G_2 are chordal.

Consider any cycle C of G with $|C| \geqslant 4$. Then C does not lie entirely in G_1 or entirely in G_2. Then C meets $V(G_1) \setminus V(G_2)$ and $V(G_2) \setminus V(G_1)$. Consequently, $(V(G_1) \cap V(C), V(G_2) \cap V(C))$ is a proper separation of C. Thus at least two distinct vertices p, q of C, that are not adjacent in C, lie in $V(G_1) \cap V(G_2)$. However, since $V(G_1) \cap V(G_2) = K$ is a clique in G, pq is a chord of C. It follows that every cycle in G of length at least 4 has a chord, and so G is chordal. The result follows by induction. ∎

The following useful corollary is obtained by choosing any vertex in the last clique pasted in Proposition 4.0.1:

Corollary 4.0.2. *Every chordal graph has a vertex whose neighbourhood is a clique.*

4.1. Helly Property and Intersection Graphs

The following theorem from discrete geometry gave rise to the notion of Helly properties in combinatorics:

Theorem 4.1.1 (Helly [66]). *Let $\mathcal{F}$ be a finite collection of at least $n+1$ convex subsets of $\mathbb{R}^n$ such that every sub-collection $\mathcal{G}$ of size $n+1$ satisfies $\bigcap \mathcal{G} \neq \emptyset$. Then $\bigcap \mathcal{F} \neq \emptyset$.*

Roughly speaking, Helly properties describe how objects in a family intersect, making them interesting and useful for proving other things. This section describes how subtrees of a tree intersect by studying a Helly property, and by studying intersection graphs of families of subtrees of a tree. The results in this section are used in Chapter 6 to prove facts about tree-width, and in Chapter 7 in the proof of Robertson and Seymour's extension of the Erdős–Pósa Theorem. The first proof of this section is the following Helly property for subtrees of a tree, called the generalised Helly property:

Lemma 4.1.2 (Generalised Helly Property). *For every tree T, every collection $\mathcal{F}$ of subtrees of T, and every integer $k \geqslant 1$, there exists k pairwise vertex-disjoint members of $\mathcal{F}$, or there exists $X \subseteq V(T)$ with $|X| \leqslant k-1$ and X meets every member of $\mathcal{F}$.*

As mentioned before, Helly properties describe how objects in a family intersect. This parallels the topic of this thesis; Erdős–Pósa dualities. In fact, one may state Lemma 4.1.2 as an Erdős–Pósa property as follows: Let $\mathcal{F}$ be a family of subtrees of a fixed tree T, then $(\{T\}, \mathcal{F})$ has the Erdős–Pósa property with bounding function $f(k) = k-1$.

Proof of Lemma 4.1.2. Proceed by induction on $n \geqslant 1$ with the hypothesis that for every n-vertex tree T, every collection $\mathcal{F}$ of subtrees of T, and every integer $k \geqslant 1$, there exists k pairwise vertex-disjoint members of $\mathcal{F}$, or there exists $X \subseteq V(T)$ with $|X| \leqslant k-1$ and X meets every member of $\mathcal{F}$. In the base case of $n = 1$, every member of $\mathcal{F}$ equals T. Then the result is trivial in the case of $k = 1$. If $k \geqslant 2$, then $X := V(T)$ has size at most $k-1$ and meets every member of $\mathcal{F}$. Now suppose $n \geqslant 2$ and the hypothesis holds for smaller values of n. As in the base case, the result is trivial if $k = 1$, so assume $k \geqslant 2$. Let x be a leaf of T and y its neighbour, and put $T' := T - x$. Consider the two cases $T[\{x\}] \notin \mathcal{F}$ or $T[\{x\}] \in \mathcal{F}$. In the former case, consider the collection $\mathcal{F}' := (H - x : H \in \mathcal{F})$ of subtrees of T'. If induction gives k pairwise vertex-disjoint members of $\mathcal{F}'$, then the k corresponding members of $\mathcal{F}$ are pairwise vertex-disjoint since at most one of them contains y. Otherwise induction gives $X \subseteq V(T')$ with $|X| \leqslant k-1$ and X meets every member of $\mathcal{F}'$. Thus X meets every member of $\mathcal{F}$. Now consider the latter case of $T[\{x\}] \in \mathcal{F}$. Let $\mathcal{F}''$ be the sub-collection of $\mathcal{F}$ consisting of all trees avoiding x. Then $\mathcal{F}''$ is a collection of subtrees of T'. If induction gives $k-1$ pairwise vertex-disjoint members of $\mathcal{F}''$, then along with $T[\{x\}]$, $\mathcal{F}$ has k pairwise vertex-disjoint members. Otherwise induction gives $X' \subseteq V(T')$ with $|X'| \leqslant k-2$ and X' meets every member of $\mathcal{F}''$, thus $X := X' \cup \{x\}$ has size at most $k-1$ and meets every member of $\mathcal{F}$. The result follows by induction. ∎

By setting $k = 2$ in the Generalised Helly Property (Lemma 4.1.2), one obtains the following corollary that graph theorists usually call the *Helly property*:

Corollary 4.1.3 (Helly Property)**.** *For every tree T and every collection $\mathcal{F}$ of pairwise intersecting subtrees of T, there exists a vertex $x \in V(T)$ such that $x \in V(F)$ for all $F \in \mathcal{F}$.*

The remainder of this section proves a characterisation of chordal graphs as the intersection graphs of subtrees of a tree. This, along with Proposition 4.0.1 gives the structure of a family of subtrees of a tree. Let $\mathcal{F} := (H_i : i \in \Lambda)$ be a family of subgraphs of a graph H. The *intersection graph* of $\mathcal{F}$ is the graph with vertex set Λ, and the edge ij is present if and only if $V(H_i \cap H_j) \neq \emptyset$.

Theorem 4.1.4 (Gavril [53]; Surányi (see [61]))**.** *A graph is chordal if and only if it is isomorphic to the intersection graph of a family of subtrees of a tree.*

Proof. ($\Longrightarrow$) Proceed by induction on $n \geqslant 1$ with the hypothesis that every n-vertex chordal graph G is isomorphic to the intersection graph of a family of subtrees of a tree. The base case of $n = 1$ is trivial. Now suppose $n \geqslant 2$ and the hypothesis holds for smaller values of n. By Corollary 4.0.2 there exists $v \in V(G)$ such that $N_G(v)$ is a clique. Since $G - v$ is chordal, by induction $G - v$ is isomorphic to the intersection graph of a family of subtrees $\mathcal{F}$ of a tree T. Since the members $T_1, \dots, T_d$ of $\mathcal{F}$ corresponding to the neighbours of v are pairwise intersecting subtrees of T, the Helly property (Corollary 4.1.3) implies that there exists $t \in V(T)$ such that $t \in V(\bigcap_{i=1}^{d} T_i)$. Let T' be the tree obtained from T by adding a new vertex t' adjacent to t. Let $\mathcal{F}'$ be obtained from $\mathcal{F}$ by replacing $T_1, \dots, T_d$ by $T'[V(T_1) \cup \{t'\}], \dots, T'[V(T_d) \cup \{t'\}]$ respectively, and adding $T'[\{t'\}]$. Then G is isomorphic to the intersection graph of $\mathcal{F}'$. The result follows by induction.

($\Longleftarrow$) Let G be the intersection graph of a family subtrees $\mathcal{F}$ of a tree T. Assume for a contradiction G has an induced cycle $C = (v_1, \dots, v_n)$ of length at least 4. For each $i \in [n]$ let T_i be the member of $\mathcal{F}$ corresponding to v_i. Notice that $V(T_i \cap T_{i+1}) \neq \emptyset$ for all $i \in [n-1]$ and $V(T_n \cap T_1) \neq \emptyset$, so let P' be a shortest T_3-T_1-path in $T' := \bigcup_{i=4}^{n} T_i$. Furthermore, let P be a shortest T_1-T_3-path in T_2. Since C has no chords, T_2 and T' are vertex-disjoint, and so P and P' are vertex-disjoint. For each $i \in \{1, 3\}$, let P_i be the path in T_i linking the two vertices of $T_i \cap (P \cup P')$. Since T_1 and T_3 are vertex-disjoint, so are P_1 and P_3. Hence $P_1 \cup P \cup P_3 \cup P'$ is a cycle in T, a contradiction. ∎

4.2. Perfection

The following question is classical in graph theory: How few colours are needed to properly-colour the vertices of a given graph? Some of the most famous theorems and conjectures in graph theory are in the area of graph colouring, including the Four Colour Theorem [6, 116], which states that planar graphs are 4-colourable, and Hadwiger's Conjecture [62], which states that every K^t-minor-free graph is $(t-1)$-colourable. The Four Colour Theorem and Hadwiger's Conjecture are interesting because they relate two completely different concepts in graph theory, graph minors and colouring. They imply upper bounds for the chromatic number when particular graphs are excluded as minors. On the other hand, when it comes to lower bounds, since every vertex in a clique receives a different colour, one obtains $\chi(G) \geqslant \omega(G)$ for every graph G. Large cliques present a "local reason" for large chromatic number. However, there could be other reasons for large chromatic since there exists triangle-free graphs with arbitrarily large chromatic number as shown by Blanche Descartes [32], and independently by Zykov [136]. Hence, the gap in the lower bound $\chi(G) \geqslant \omega(G)$ can be arbitrarily large. Worse yet, Erdős' Theorem [45] states that there exists graphs with arbitrarily large girth and chromatic number. This is surprising because graphs with large girth locally look like a tree, and trees are 2-colourable. Erdős' Theorem is famous for being one of the first uses of the powerful probabilistic method. Ideas from the proof of Erdős' Theorem [45] are used in Section 7.2 to prove the $\Omega(k \log k)$ lower bound in the Erdős–Pósa Theorem. Erdős' Theorem suggests that the chromatic number of a graph can be a "global property". Nevertheless, graph theorists ask, when is the chromatic determined by a local property? This leads to the topic of this section, perfect graphs. Formulated by Berge [9], a graph G is *perfect* if $\chi(H) = \omega(H)$ for every induced subgraph H of G. Simple examples of perfect graphs are the bipartite graphs. On the other hand, any odd cycle of length at least 5 is an example of an imperfect graph since its chromatic number is 3 and clique number is 2.

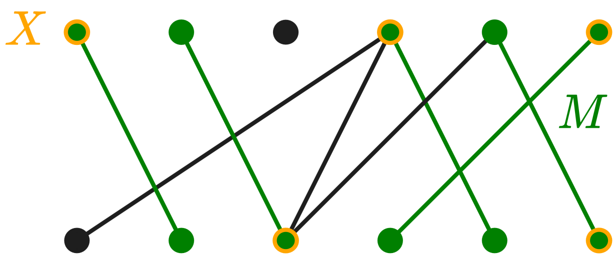

FIGURE 4.2. Example of a minimum sized vertex cover X having the same size as a maximum sized matching M.

A *vertex cover* of a graph G is a set $X \subseteq V(G)$ such that every edge of G has an end in X. The next example of perfect graphs is related to the following duality of König:

Theorem 4.2.1 (König [80]). *For every bipartite graph G, the maximum size of a matching in G equals the minimum size of a vertex cover of G.*

One way to prove König's Theorem (Theorem 4.2.1) is by application of Menger's Theorem (Theorem 2.1.3), where A and B are the parts of a bipartition of G. See Figure 4.2 for an example of König's Theorem in action. As an aside, notice that König's Theorem is the statement that the pair (Bipartite Graphs, K^2) has the Erdős–Pósa property with bounding function $f(k) = k - 1$.

The following argument shows that the complements of bipartite graphs are perfect, giving another example of perfect graphs: Let G be a bipartite graph. Since the induced subgraphs of bipartite graphs are bipartite, it suffices to show that $\chi(\overline{G}) = \omega(\overline{G})$. Let M be a maximum matching in G. Since $\chi(\overline{G})$ equals the minimum size of a collection $\mathcal{I}$ of pairwise disjoint independents sets in $\overline{G}$ with $\bigcup \mathcal{I} = V(\overline{G})$, $\chi(\overline{G})$ equals the minimum size of a collection $\mathcal{C}$ of pairwise disjoint cliques in G with $\bigcup \mathcal{C} = V(G)$. Since G is triangle-free, the members of $\mathcal{C}$ correspond to cliques of size 1 or 2. As such, $|\mathcal{C}| = |M| + (|G| - 2|M|) = |G| - |M|$, and so $\chi(\overline{G}) = |G| - |M|$. Let $X \subseteq V(G)$

be a minimum sized vertex cover of G. Then by König's Theorem, $\chi(\overline{G}) = |G - X|$. Since a maximum independent set is obtained by deleting a minimum set of vertices that meet all the edges, $V(G - X)$ is an independent set in G of size $\alpha(G)$. Hence $\chi(\overline{G}) = \alpha(G) = \omega(\overline{G})$, as required.

It turns out that many dualities in combinatorics are connected to perfect graphs, for instance, the previous example connects König's Theorem to perfect graphs. Another example of such a connection is related to the following duality theorems from order theory:

Theorem 4.2.2 (Dilworth [36]). *For all finite partial orders $(P, \leqslant)$, the maximum size of an antichain in P equals the minimum size of a collection $\mathcal{C}$ of chains in P with $\bigcup \mathcal{C} = P$.*

Theorem 4.2.3 (Mirsky [95]). *For all finite partial orders $(P, \leqslant)$, the maximum size of a chain in P equals the minimum size of a collection $\mathcal{A}$ of antichains in P with $\bigcup \mathcal{A} = P$.*

The *comparability graph* of a partial order $(P, \leqslant)$, is the graph with vertex set P, where two vertices are adjacent if and only if they are comparable. The following argument shows that comparability graphs are perfect: Let G be the comparability graph of a finite partial order $(P, \leqslant)$, and let H be an induced subgraph of G. It suffices to show that $\chi(H) \leqslant \omega(H)$. Notice that H is the comparability graph of the partial order $(V(H), \leqslant)$, obtained by restricting $\leqslant$ to the subset $V(H)$ of P. Let C be a maximum sized chain in $V(H)$, and let $\mathcal{A}$ be a minimum sized collection of antichains in $V(H)$ with $\bigcup \mathcal{A} = V(H)$. It may be assumed that $\mathcal{A}$ is a pairwise disjoint collection. Since chains and antichains in $V(H)$ are cliques and independent sets in H, C is a clique in H and $\mathcal{A}$ are the colour classes of a proper-colouring of H. As such, $|C| \leqslant \omega(H)$ and $\chi(H) \leqslant |\mathcal{A}|$. Then by Mirsky's Theorem, $\chi(H) \leqslant \omega(H)$, as required.

The following argument shows that the complements of comparability graphs are perfect, and is nearly identical to the one above, with the major difference being that Mirsky's Theorem is replaced by Dilworth's Theorem: Let G be the comparability graph of a finite partial order $(P, \leqslant)$, and let H be an induced subgraph of $\overline{G}$. It suffices to show that $\chi(H) \leqslant \omega(H)$. Notice that $\overline{H}$ is an induced subgraph of G, and so $\overline{H}$ is the comparability graph of the partial order $(V(\overline{H}), \leqslant)$, obtained by restricting $\leqslant$ to the subset $V(\overline{H})$ of P. Let A be a maximum sized antichain in $V(\overline{H})$, and let $\mathcal{C}$ be a minimum sized collection of chains in $V(\overline{H})$ with $\bigcup \mathcal{C} = V(\overline{H})$. It may be assumed that $\mathcal{C}$ is pairwise disjoint collection. Since chains and antichains in $V(\overline{H})$ are independent sets and cliques in H, A is a clique in H and $\mathcal{C}$ are the colour classes of a proper-colouring of H. As such, $|A| \leqslant \omega(H)$ and $\chi(H) \leqslant |\mathcal{C}|$. Then by Dilworth's Theorem, $\chi(H) \leqslant \omega(H)$, as required.

So far, we have seen that bipartite graphs and comparability graphs are perfect, and the complements of bipartite graphs and comparability graphs are perfect. It is a surprising fact that this pattern generalises to all perfect graphs as shown by the following theorem of Lovász:

Theorem 4.2.4 (Perfect Graph Theorem. Lovász [87]). *A graph is perfect if and only if its complement is perfect.*

Using the Perfect Graph Theorem (Theorem 4.2.4) and the generalised Helly property (Lemma 4.1.2), one obtains the following proposition:

Proposition 4.2.5. *Chordal graphs and their complements are perfect.*

Proof. Let G be a chordal graph. By the Perfect Graph Theorem, it suffices to show that $\overline{G}$ is perfect. Let H be an induced subgraph of G. Notice that the induced subgraphs of $\overline{G}$ are precisely the complements of induced subgraphs of G. Thus, it suffices to show that $\chi(\overline{H}) \leqslant \omega(\overline{H})$. Since H is chordal, Theorem 4.1.4 implies that H is isomorphic to the intersection graph of a collection of subtrees $\mathcal{F}$ of a tree T. Since $\mathcal{F}$ does not have $\alpha(H) + 1$ pairwise vertex-disjoint members, the generalised Helly property implies that there exists a set $X \subseteq V(T)$ with $|X| \leqslant \alpha(H)$ and X meets every member of $\mathcal{F}$. For each $x \in X$, the set of all $F \in \mathcal{F}$ that contain x is a clique in H. Hence, there is a collection $\mathcal{I}$ of independent sets in $\overline{H}$ with $|\mathcal{I}| = |X|$ and $\bigcup \mathcal{I} = V(\overline{H})$. Consequently, $\chi(\overline{H}) \leqslant |\mathcal{I}| = |X| \leqslant \alpha(H) = \omega(\overline{H})$, as required. ■

The perfection of the complements of chordal graphs was first shown by Hajnal and Suranyi [63], and the perfection of chordal graphs was shown by Berge [9], both prior to the proof of the Perfect Graph Theorem. One way to show the perfection of chordal graphs without using the Perfect Graph Theorem is by using Proposition 4.0.1 and the fact that if G is obtained by pasting G_1 and G_2 along a clique, then $\chi(G) = \max\{\chi(G_1), \chi(G_2)\}$ and $\omega(G) = \max\{\omega(G_1), \omega(G_2)\}$. The purpose of the proof given for Proposition 4.2.5 is to connect the two main concepts from the start of this chapter, the generalised Helly property and chordal graphs.

4.2.1. Proof of the Perfect Graph Theorem

This part of the section presents a proof of the Perfect Graph Theorem that is based on Trotignon's proof [131], which is essentially Preissmann and Sebő's proof [107]. The following lemma gives a basic intuition for what perfect graphs look like:

Lemma 4.2.6. *For graphs G, the following are equivalent:*

(1) G is perfect.
(2) For every induced subgraph H of G and every vertex $v \in V(H)$, there exists an independent set X in H such that $v \in X$, and X meets every maximum sized clique in H.
(3) For every induced subgraph H of G, there exists an independent set X in H such that X meets every maximum sized clique in H.

Proof. (1) $\implies$ (2) : Note that H is perfect. Let $\mathcal{C}$ be the colour classes of an optimal colouring of H. Then there exists an $X \in \mathcal{C}$ with $v \in X$. If X misses a maximum sized clique in H, then $\chi(H-X) = \omega(H-X) = \omega(H) = \chi(H) > \chi(H-X)$, a contradiction. Consequently, X meets every maximum sized clique in H.

(2) $\implies$ (3) : Holds trivially.

(3) $\implies$ (1) : Let H be an induced subgraph of G. According to (3), consider a maximal sequence of pairs $(H_0, X_0), (H_1, X_1), \ldots, (H_k, X_k)$ such that $H_0 = H$, $H_{i+1} = H_i - X_i$, and each X_i is a non-empty independent set in H_i that meets every maximum sized clique in H_i. Consequently, $\omega(H_i) \geqslant \omega(H_{i+1}) + 1$, and so $\omega(H) \geqslant k$. On the

other hand, $(X_0, \dots, X_k)$ defines a proper-colouring of H, and so $k \geqslant \chi(H)$. Hence G is perfect. ■

The following definition and lemma are key to the proof of the Perfect Graph Theorem: Let G be a graph and v a vertex of G. The graph obtained from G by *replicating* v is the graph G' obtained from G by adding a new vertex v' that is adjacent to v and its neighbours as in Figure 4.3.

FIGURE 4.3. Replicating v.

Lemma 4.2.7 (Replication Lemma; Lovász [87]). *Replication preserves perfection.*

PROOF. Let G be a perfect graph and G' be the graph obtained from G be replicating a vertex v. It suffices to show that G' satisfies (3) in Lemma 4.2.6. Let H be an induced subgraph of G'. If H contains at most one of v or v', then H is isomorphic to an induced subgraph of G, and so H is perfect. Now suppose that H contains both v and v'. Since $H - v'$ is an induced subgraph of G, $H - v'$ is perfect. Thus, by (2) in Lemma 4.2.6, there exists an independent set X in $H - v'$ such that $v \in X$, and X meets every maximum sized clique in $H - v'$. To complete the proof, the following demonstrates that X meets every maximum sized clique in H: Let C be a maximum sized clique in H. If C does not contain v', then C is a maximum sized clique of $H - v'$, and so X meets C. Now suppose C contains v'. Since v and v' are adjacent and have the same neighbourhood, any clique containing v' remains a clique after adding v to it. Then, since C is maximum, C contains v, and so X meets C. ■

PROOF OF THE PERFECT GRAPH THEOREM (THEOREM 4.2.4). It is sufficient to prove the forward implication. Let G be a perfect graph. By (3) in Lemma 4.2.6, to prove that $\overline{G}$ is perfect, it suffices to show that for every induced subgraph H of G, H has a clique that meets every maximum sized independent set in H.

Let H be an induced subgraph of G, and let A be the graph obtained from H by deleting all vertices not appearing in any maximum sized independent set of H. Let $\mathcal{I}$ be the set of all maximum sized independent sets in A. Let $\mathcal{I}'$ be obtained from $\mathcal{I}$ by making its members disjoint. For each $v \in V(A)$, let C_v be the set of all clones of v in $\bigcup \mathcal{I}'$. Let H' be the graph with vertex set $\bigcup \mathcal{I}'$, and edge set defined in the following way: For each $v \in V(A)$, have all edges between vertices of C_v to make C_v a clique, and have all possible C_u-C_v-edges whenever $uv \in E(A)$.

The following observation is crucial: Let α_v be the number of maximum sized independent sets in H that contain v. Then H' is isomorphic to the graph obtained from H by replicating each vertex v exactly $\alpha_v - 1$ times, where $\alpha_v - 1 = -1$ means deleting v. Therefore, since H is perfect, the Replication Lemma (Lemma 4.2.7) implies that H' is perfect.

Since each C_v is a clique in H', every independent set X in H' meets each C_v at most once. Thus X corresponds to an independent set in A, hence in H. Consequently, $\alpha(H') \leqslant \alpha(H)$. On the other hand, any member of $\mathcal{I}'$ is an independent set in H' of size $\alpha(H)$, and so $\alpha(H') = \alpha(H)$. Therefore, $\mathcal{I}'$ partitions $V(H')$ into maximum sized independent sets of H', which implies that $\chi(H') = |\mathcal{I}'|$. Since H' is perfect, it has a clique K of size $\chi(H')$. Consequently, K meets each member of $\mathcal{I}'$ exactly once. Thus K corresponds to a clique in A that meets every maximum sized independent set in A. It follows that H has a clique that meets every maximum sized independent set, as required. This concludes the proof of the Perfect Graph Theorem. ■

4.2.2. Origins of Perfect Graph Theory

An article by Berge [10], and another by Berge and Ramírez Alfonsín [108], outline Berge's motivations for studying perfect graphs as coming from Shannon's zero-error capacity [126], a concept that studies how one may restrict the set of input messages through a noisy channel so that no errors are introduced during transmission. At a high level, Shannon considered an alphabet Σ that makes up the input strings, and a matrix P that encodes the probability that each letter in Σ is transmitted as another. Here P is thought of as a noisy channel. Two letters are said to be *confoundable* if it is possible for them to result in the same letter during transmission. More generally, two length m strings are confoundable if for each $1 \leqslant i \leqslant m$, their ith letters are identical or confoundable. A set of length m pairwise non-confoundable input strings is said to be *m-admissible*. Then the members of an m-admissible set can not be confused with each other during transmission.

If one chose a maximum sized subset of Σ of pairwise non-confoundable letters, and formed all length m strings from these letters, then one would would obtain an m-admissible set. If α_P is the maximum size of a subset of Σ of pairwise non-confoundable letters, then this method produces an m-admissible set of size α_P^m. Interestingly, one may find significantly larger m-admissible sets. For example [108], suppose that $\Sigma = \{a, b, c, d, e\}$ and the pairs ab, bc, cd, de, ea are confoundable. The confoundable pairs can be visualised by C_5 as in Figure 4.4. Then $\alpha_P = 2$, and so $\alpha_P^m = 2^m$. However, $\{ab, bd, ca, dc, ee\}$ is a set of five input strings, no two of which are confoundable. Therefore, for even m, one may construct an m-admissible set of size at least $5^{m/2} = 2^{(m/2)\log 5}$.

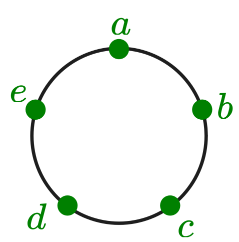

FIGURE 4.4. Confoundable pairs as per the example.

In general, for $n \leqslant m$, if C is a set of length n pairwise non-confoundable input strings, then one may construct an n-admissible set of size at least $2^{\lfloor m/n \rfloor \log |C|}$. This motivated Shannon to define the *zero-error capacity* of a transmission channel P as

$C_0 := \sup_n \frac{1}{n} \log M_0(n)$, where $M_0(n)$ is the maximum size of an n-admissible set (depending on Σ and P). Now consider the graph G_P on Σ where two vertices are adjacent if and only if they are confoundable according to P. Then it is clear that $M_0(1) = \alpha(G_P)$. Similarly, one finds that $M_0(n)$ depends only on the graph of P in the following way: For graph G and H, the *strong-product* of G and H is defined to be the graph $G \boxtimes H$ whose vertex set is $V(G) \times V(H)$, and where distinct pairs $(a, b), (c, d) \in V(G) \times V(H)$ are adjacent if and only if $a = c$ or $ac \in E(G)$, and $b = d$ or $bd \in E(H)$. Notice that $G_P \boxtimes G_P$ models the length 2 input strings that are confoundable, and so $M_0(2) = \alpha(G_P \boxtimes G_P)$. More generally, for graphs G, define $G^{\boxtimes k}$ to be the strong-product of k copies of G. Then $M_0(n) = \alpha(G_P^{\boxtimes n})$. The *Shannon capacity* of a graph G is defined to be

$$\phi(G) := \sup_n \sqrt[n]{\alpha(G^{\boxtimes n})}$$

Notice that for any graph G, $\log \phi(G)$ is the zero-error capacity of a suitable P. $\phi(G)$ has resisted being known even for small graphs, for example, the value of $\phi(C_7)$ is still unknown. Above, we saw that $\log \phi(C_5) \geqslant \frac{1}{2} \log 5$, and so $\phi(C_5) \geqslant \sqrt{5}$, which Shannon knew [126]. Twenty years later, Lovász [86] showed $\phi(C_5) = \sqrt{5}$ by calculating a different graph invariant called the Lovász number.

FIGURE 4.5. Left to Right: Shannon, Berge, Lovász.

It turns out that if $\alpha(G) = \chi(\overline{G})$, then $\phi(G) = \alpha(G)$ as shown by the following: It is obvious that $\alpha(G)^n \leqslant \alpha(G^{\boxtimes n})$ and $\chi(G^{\boxtimes n}) \leqslant \chi(G)^n$. Furthermore, $\alpha(G) \leqslant \chi(\overline{G})$, and so $\alpha(G) \leqslant \phi(G) \leqslant \chi(\overline{G})$. According to Berge and Ramírez Alfonsín [108], this was the motivation for Berge to study the graphs satisfying $\alpha(G) = \chi(\overline{G})$, which led to the birth of perfect graph theory. Berge [10] explains that he investigated the following four classes (verbatim from [10])[1]:

---

(1) a graph G is in *Class 1* if $\phi(G) = \alpha(G)$;
(2) a graph G is in *Class 2* if $\alpha(G') = \theta(G')$ (the 'Beautiful Property') for all induced subgraphs G' of G;
(3) a graph G is in *Class 3* if $\gamma(G') = \omega(G')$ (the chromatic number is equal to the clique number) for all induced subgraphs G' of G;
(4) a graph G is in *Class 4* if G contains no induced C_{2k+1}, the odd cycle of length $2k+1 \geqslant 5$ (called *odd hole*) and no induced complement of an odd hole (called *odd antihole*).

The equivalence of Classes 2 and 3 is precisely the Perfect Graph Theorem (Theorem 4.2.4). Since perfection is closed under taking induced subgraphs, a graph is perfect

[1] Using our notation, $\theta(G) = \chi(\overline{G})$ and $\gamma(G) = \chi(G)$.

if and only if it does not contain an induced-subgraph-minimal imperfect graph as an induced subgraph. The start of Section 4.2 showed that the odd cycles of length at least 5 are imperfect, and so their complements are imperfect by Theorem 4.2.4. Thus, if a graph G contains an odd hole or an odd antihole, then G is imperfect. Remarkably, the so-called Strong Perfect Graph Theorem says that the converse is also true, and so Classes 2, 3 and 4 are are all equivalent:

Theorem 4.2.8 (Strong Perfect Graph Theorem. Chudnovsky, Robertson, Seymour, Thomas [25])**.** *A graph is perfect if and only if it contains no odd hole and no odd antihole.*

Berge conjectured [11] the Perfect Graph Theorem and the Strong Perfect Graph Theorem. The original proof of the Strong Perfect Graph Theorem spanned 150 pages [25], and awarded its authors the 2009 Fulkerson Prize. Attempts to prove Berge's conjectures contributed to the development of perfect graph theory and graph theory as a whole. For surveys, see [108, 69, 131].

CHAPTER 5

Path-width

This chapter introduces path-width, a standard measure of how "path-like" a graphs is. This concept plays a key role in Section 7.5 when obtaining a tight bounding function for the Erdős–Pósa property of forest minors.

A *path-decomposition* of a graph G is a sequence $(B_1, \ldots, B_q)$ of subsets of $V(G)$ called *bags*, such that following hold:

- *Vertex-property.* For every $v \in V(G)$, $\{i \in [q] : v \in B_i\}$ is a non-empty interval of integers.
- *Edge-property.* For every $uv \in E(G)$ there exists $i \in [q]$ such that $\{u, v\} \subseteq B_i$.

See Figure 5.1 for an example.

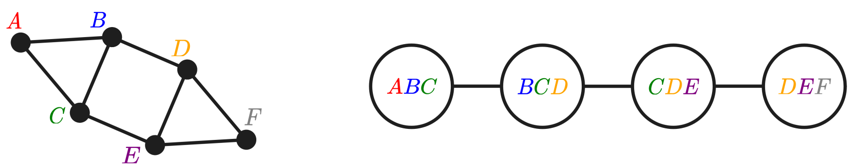

FIGURE 5.1. Example of a path-decomposition. Notice that each vertex $v \in V(G)$ is assigned a subinterval of $[q]$ such that the subintervals corresponding to adjacent vertices of G overlap.

The *width* of $(B_1, \ldots, B_q)$ is the number $\max_{i\in[q]} |B_i| - 1$. The minimum width of a path-decompositions of G is called the *path-width* of G, and is denoted by $\operatorname{pw}(G)$. Note that path-width is well-defined since for every graph G, $(V(G))$ is a path-decomposition of G.

Proposition 5.0.1 (Monotonicity of Path-width)**.** *If $H \preccurlyeq G$, then $\operatorname{pw}(H) \leqslant \operatorname{pw}(G)$.*

PROOF. Since H is obtained from a subgraph of G by contracting edges, it suffices to prove the result for the following cases: (a) $H = G - v$ for some $v \in V(G)$, (b) $H = G - e$ for some $e \in E(G)$, or (c) $H = G/e$ for some $e \in E(G)$. Let $(B_1, \ldots, B_q)$ be a path-decomposition of G of minimum width. In case (a), deleting v from each bag yields a path-decomposition of H. In case (b), $(B_1, \ldots, B_q)$ is a path-decomposition of H. In case (c), write $e = xy$ and suppose contracting e in G forms a vertex v_e in H. Then for each bag, replacing every instance of x or y with v_e yields a path-decomposition of H. Since all cases produce a path-decomposition of H of width at most $\operatorname{pw}(G)$, $\operatorname{pw}(H) \leqslant \operatorname{pw}(G)$. ■

5.1. Forests

This section proves that the class of all forests has unbounded path-width. Combining this with Proposition 5.0.1 shows the following important concept: For every integer

$k \geqslant 0$, there exists a forest F such that every graph G that contains F as a minor has path-width more than k. This section begins by proving a formula (Theorem 5.1.2) for the path-width of a tree.

Lemma 5.1.1 (Ellis, Sudborough, Turner [44]; Suderman [128])**.** *Every non-empty connected graph G has a path P such that* $\operatorname{pw}(G - V(P)) \leqslant \operatorname{pw}(G) - 1$*.*

Proof. Let $(B_1, \ldots, B_q)$ be a path-decomposition of G of minimum width. Let $x, y \in [q]$ be the first and last indices respectively such that $B_x \neq \emptyset$ and $B_y \neq \emptyset$. Since G is connected, there exists a B_x-B_y-path $P = (v_1, \ldots, v_k)$ in G. By the vertex-property, for each vertex $v_j \in V(P)$ the set $I(v_j) := \{i \in [q] : v_j \in B_i\}$ is a subinterval of $[q]$. By the edge-property, subintervals corresponding to adjacent vertices of P overlap. That is, $I(v_j) \cap I(v_{j+1}) \neq \emptyset$ for all $j \in [k-1]$. Then, since $x \in I(v_1)$ and $y \in I(v_k)$, $\{x, x+1, \ldots, y\} \subseteq \bigcup_{j=1}^{k-1} I(v_j) = \{i \in [q] : B_i \cap V(P) \neq \emptyset\}$. Thus P meets B_i for all $i \in \{x, x+1, \ldots, y\}$. Then $(B_x \setminus V(P), B_{x+1} \setminus V(P), \ldots, B_y \setminus V(P))$ is a path-decomposition of $G - V(P)$ of width at most $\operatorname{pw}(G) - 1$. ■

Theorem 5.1.2 (Ellis, Sudborough, Turner [44])**.** *For every tree T,*

$$\operatorname{pw}(T) = \min_P \operatorname{pw}(T - V(P)) + 1,$$

where the minimum is taken over all subpaths of T.

Proof. Lemma 5.1.1 implies $\operatorname{pw}(T) \geqslant \min_P \operatorname{pw}(T - V(P)) + 1$. On the other hand, let $P := (v_1, \ldots, v_n)$ be a subpath of T realising the expression on the right-hand-side. For each $i \in [n]$, let $\mathcal{C}_i$ be the set of all components C of $T - V(P)$ for which there exists a $V(C)$-v_i-edge in T. Since T has no cycles and is connected, each component of $T - V(P)$ is a member of exactly one of the $\mathcal{C}_i$. By Proposition 5.0.1, for each $i \in [n]$, $\bigcup \mathcal{C}_i$ has a path-decomposition $(B^i_1, \ldots, B^i_{q_i})$ of width at most $\operatorname{pw}(T - V(P))$. Then

$$\begin{aligned} &B^1_1 \cup \{v_1\}, B^1_2 \cup \{v_1\}, \ldots, B^1_{q_1} \cup \{v_1\}, \{v_1, v_2\}, B^2_1 \cup \{v_2\}, B^2_2 \cup \{v_2\}, \ldots \\ &\qquad \ldots, B^{n-1}_{q_{n-1}} \cup \{v_{n-1}\}, \{v_{n-1}, v_n\}, B^n_1 \cup \{v_n\}, B^n_2 \cup \{v_n\}, \ldots, B^n_{q_n} \cup \{v_n\} \end{aligned}$$

defines a path-decomposition of T of width at most $\operatorname{pw}(T - V(P)) + 1$. ■

The formula in Theorem 5.1.2 is used in the following theorem, which implies that the class of all forests has unbounded path-width:

Theorem 5.1.3. *Let $h \geqslant 0$ be an integer and T_h be a complete ternary tree of height h. Then $\operatorname{pw}(T_h) = h$. Hence the class of all forests has unbounded path-width.*

Proof. Proceed by induction on $h \geqslant 0$ with the hypothesis that $\operatorname{pw}(T_h) = h$. In the base case, $T_h = K^1$ and the path-decomposition $(V(T_h))$ shows that $\operatorname{pw}(T_h) = 0$. Now suppose $h \geqslant 1$ and the hypothesis holds for smaller values of h. Let r be the unique degree-3 vertex of T_h. Observe that every subpath P of T avoids at least one of the three component of $T_h - r$. Furthermore, the components of $T_h - r$ are isomorphic to T_{h-1}, thus Proposition 5.0.1 implies the inequality $\operatorname{pw}(T_{h-1}) \leqslant \operatorname{pw}(T - V(P))$ for all subpaths P of T. The following shows that equality is achieved if $P = (r)$: Since $T_h - r \cong T_{h-1} \sqcup T_{h-1} \sqcup T_{h-1}$, concatenating three optimal path-decompositions of T_{h-1} in any order produces a path-decomposition of $T_h - r$ of width $\operatorname{pw}(T_{h-1})$, and so $\operatorname{pw}(T_h - r) \leqslant \operatorname{pw}(T_{h-1})$. Hence $\operatorname{pw}(T_{h-1}) = \min_P \operatorname{pw}(T - V(P))$, where the minimum is taken over all subpaths P of T. By induction $\operatorname{pw}(T_{h-1}) = h - 1$, and by Theorem 5.1.2 $\operatorname{pw}(T_h) = \min_P \operatorname{pw}(T - V(P)) + 1$. Hence $\operatorname{pw}(T_h) = h$. The result follows by induction. ■

5.2. Path-width Theorem

Section 5.1 showed that for every integer $k \geqslant 0$, there exists a forest F such that every graph G that contains F as a minor has path-width more than k. On the other hand, this section shows that for every forest F, there exists an integer $k \geqslant 0$ such that every graph G with path-width greater than k contains an F minor. Initially, Robertson and Seymour [120] proved that there exists a function $f : \mathbb{N} \to \mathbb{N}$ such that for every forest F, every graph G with $\operatorname{pw}(G) \geqslant f(|F|)$ contains an F minor. A subsequent paper by Bienstock, Robertson, Seymour and Thomas [13] produced a simpler proof that brought the bound on f to its best possible value:

Theorem 5.2.1 (Path-width Theorem. Bienstock, Robertson, Seymour, Thomas [13]). *For every forest F, every graph G with $\operatorname{pw}(G) \geqslant |F| - 1$ contains an F minor.*

Theorem 5.2.1 is best possible for every forest F, since $K^{|F|-1}$ has path-width $|F|-2$ and is F-minor-free. The story does not end there. Diestel [33] produced a short and simpler proof while achieving the same optimal bound bound on f. And most recently, Seymour [124] produced a shorter proof with the optimal bound on f.

This section follows Seymour's [124] proof of the Path-width Theorem, which is used in Section 7.5 to obtain a tight bounding function for the Erdős–Pósa property of forest minors. The proof relies on the following preliminary definitions: If (A, B) and (A', B') are separations of a graph G with $A \subseteq A'$ and $B' \subseteq B$, then write $(A, B) \leqslant (A', B')$. For integers $k \geqslant 0$, a separation (A, B) of a graph G is *k-friendly* if $G[A]$ has a path-decomposition of width at most k such that the last bag equals $A \cap B$.

Lemma 5.2.2. *Let $k \geqslant 0$ be an integer and (A', B'), (P, Q) be separations of a graph G with $(P, Q) \leqslant (A', B')$. If (A', B') is k-friendly and there exists a collection $\mathcal{R}$ of $|P \cap Q|$ pairwise vertex-disjoint P-B'-paths in G, then (P, Q) is k-friendly.*

Proof. Note that the set of P-ends of the paths in $\mathcal{R}$ equals $P \cap Q$. Now consider the subgraph $H := G[P] \cup \bigcup \mathcal{R}$ of $G[A']$. Since (A', B') is k-friendly, H has a path-decomposition of width at most k with last bag equal to the set of B'-ends of the paths in $\mathcal{R}$. Let $\mathcal{P}$ be the path-decomposition of $G[P]$ inherited from H, as in Proposition 5.0.1, by contracting each path in $\mathcal{R}$ to a vertex. Then $\mathcal{P}$ has width at most k and last bag equal to $P \cap Q$, as required. ■

If (A, B) and (A', B') are separations of a graph G with $(A, B) \leqslant (A', B')$ and $|A \cap B| \geqslant |A' \cap B'|$, then write $(A, B) \ll (A', B')$. It is easy to see that $\ll$ defines a partial-ordering on the set of separations of G. Let $k \geqslant 0$ be an integer, and T be a tree or the null graph. A separation (A, B) of a graph G is *(k, T)-spanning* if the following hold:

- $|A \cap B| = |T|$,
- there exists a T model $\mathbf{X}$ in $G[A]$ such that $X \cap A \cap B \neq \emptyset$ for all $X \in \mathbf{X}$,

- if $|T| \leqslant k+1$, then (A,B) is $\ll$-maximal k-friendly, and
- ⋆ if $|T| = k+2$, then (A,B) is $(k+1)$-friendly.

It is possible to modify the proof of the Path-width Theorem so that definition of (k,T)-spanning does not include property $\star$. However, $\star$ is needed to obtain a useful corollary (Corollary 5.2.4).

Lemma 5.2.3. *Let $k \geqslant 0$ be an integer, G be a graph with $\operatorname{pw}(G) \geqslant k+1$, and T be a tree or the null graph, with $|T| \leqslant k+2$. Then there exists a (k,T)-spanning separation of G.*

Proof. Fix an integer $k \geqslant 0$ and proceed by induction on $n \in \{0,\dots,k+2\}$ with the hypothesis that for every graph G with $\operatorname{pw}(G) \geqslant k+1$ and every tree (or null graph) T with $|T| = n$, there exists a (k,T)-spanning separation of G.

In the base case of $|T| = 0$, since $(\emptyset, V(G))$ is k-friendly with $|\emptyset \cap V(G)| = 0$, G has a $\ll$-maximal k-friendly separation (A,B) with $|A \cap B| = 0$. Since T is the null graph, (A,B) is (k,T)-spanning. Now suppose $|T| \in \{1,\dots,k+2\}$ and the result holds for trees on fewer vertices. Choose $\ell \in V(T)$ with $\deg_T(\ell) \leqslant 1$. If $|T| \geqslant 2$, then choose $x \in N_T(\ell)$. By induction, there exists a $(k, T-\ell)$-spanning separation (A,B) of G. Since $|T-\ell| \leqslant k+1$, (A,B) is $\ll$-maximal k-friendly. Let $\mathbf{X}'$ be a $T-\ell$ model in $G[A]$ such that $X \cap A \cap B \neq \emptyset$ for all $X \in \mathbf{X}'$, and let $\mathcal{P}$ be a path-decomposition of $G[A]$ of width at most k such that the last bag equals $A \cap B$.

Next, choose a vertex $v \in B \setminus A$ as follows: Note that $\operatorname{pw}(G[A]) \leqslant k$ and $\operatorname{pw}(G) \geqslant k+1$ together imply that $B \setminus A \neq \emptyset$. If $|T| = 1$, then choose $v \in B \setminus A$ arbitrarily. Now suppose $|T| \geqslant 2$. Let $X' \in \mathbf{X}'$ be the branch set of x and choose $u \in X' \cap B$. Assume for a contradiction that $N_G(u) \cap (B \setminus A) = \emptyset$. Then concatenating a new bag $A \cap (B \setminus \{u\})$ to the end of $\mathcal{P}$ yields a path-decomposition of $G[A]$ that shows $(A, B\setminus\{u\})$ is k-friendly. Then $(A,B) \ll (A, B\setminus\{u\})$ and $(A,B) \neq (A, B\setminus\{u\})$ contradicts the maximality of (A,B). Hence one may choose $v \in N_G(u) \cap (B \setminus A)$. This completes the description of the vertex v.

Add the set $\{v\}$ to $\mathbf{X}'$.

If $|T| = k+2$, let $\mathcal{P}'$ be the path-decomposition obtained by concatenating a new bag $(A \cup \{v\}) \cap B$ to the end of $\mathcal{P}$. Then $\mathbf{X}'$ and $\mathcal{P}'$ together show that $(A \cup \{v\}, B)$ is (k,T)-spanning.

Hence, it may be assumed that $|T| \leqslant k+1$. Therefore $|A \cap B| = |T - \ell| \leqslant k$, and so concatenating a new bag $(A \cup \{v\}) \cap B$ to the end of $\mathcal{P}$ yields a path-decomposition of $G[A \cup \{v\}]$ that shows that $(A \cup \{v\}, B)$ is k-friendly. Then one may choose a $\ll$-maximal k-friendly separation (A', B') of G with $(A \cup \{v\}, B) \ll (A', B')$. Then $|A \cap B| + 1 \geqslant |A' \cap B'|$, and so $(A,B) \ll (A',B')$ and $(A,B) \neq (A',B')$. Then, since (A,B) is a $\ll$-maximal k-friendly separation of G, $|A \cap B| < |A' \cap B'|$ must hold. It follows that $|A' \cap B'| = |A \cap B| + 1 = |T|$.

Assume for a contradiction that there exists a separation (P,Q) of G of order at most $|T|-1$ and $(A \cup \{v\}, B) \leqslant (P,Q) \leqslant (A', B')$. Choose such a (P,Q) with minimum possible order. From $(A,B) \leqslant (A \cup \{v\}, B) \leqslant (P,Q)$ and $|A \cap B| = |T| - 1 \geqslant |P \cap Q|$, it follows that $(A,B) \ll (P,Q)$. Notice that $(A,B) \neq (P,Q)$. By choice of (P,Q), Menger's Theorem (Theorem 2.1.3) gives a collection $\mathcal{R}$ of $|P \cap Q|$ pairwise vertex-disjoint P-B'-paths in G. Then Lemma 5.2.2 implies (P,Q) is k-friendly, contradicting the maximality of (A, B). Hence there is no such separation (P, Q). By Menger's Theorem there exists a collection $\mathcal{L}$ of $|T|$ pairwise vertex-disjoint $(A \cup \{v\})$-B'-paths in G. Let $\mathbf{X}$ be the collection of all sets $X \cup V(L)$ where $X \in \mathbf{X}'$ and L is the unique member of $\mathcal{L}$ meeting X. Then $\mathbf{X}$ is a T model in $G[A']$ satisfying $X \cap A' \cap B' \neq \emptyset$ for all $X \in \mathbf{X}$. Hence (A', B') is (k, T)-spanning. The result follows by induction. ■

Proof of the Path-width Theorem (Theorem 5.2.1). Choose a tree T that contains F as a spanning subgraph. Since $\mathrm{pw}(G) \geqslant |T| - 1$, Lemma 5.2.3 with $k := |T| - 2$ implies there exists a (k, T)-spanning separation of G. Hence G contains an F minor. ■

For graphs G and subsets $Y \subseteq V(G)$, define

$$\partial_G Y := \{v \in Y : \exists u \in V(G - Y), uv \in E(G)\}.$$

The following corollary of Lemma 5.2.3 is key for obtaining an optimal Erdős–Pósa function for forest minors, and was originally implicit in Diestel's [33] proof of the Path-width Theorem:

Corollary 5.2.4 (Diestel [33])**.** *For every integer $t \geqslant 1$, every t-vertex forest F, and every graph G with $\mathrm{pw}(G) \geqslant t - 1$, there exists $Y \subseteq V(G)$ such that $G[Y]$ has a path-decomposition $(B_1, \dots, B_q)$ of width at most $t - 1$ with $\partial_G Y \subseteq B_q$, and $G[Y]$ contains an F minor.*

Proof. If $t = 1$, then choose $v \in V(G)$ arbitrarily ($V(G) \neq \emptyset$ since $\mathrm{pw}(G) \geqslant 0$) and put $Y := \{v\}$. Then (Y) is a path-decomposition of $G[Y]$ of width 0, and $\partial_G Y \subseteq Y$, and $G[Y]$ contains an F minor.

Now suppose $t \geqslant 2$. Let $k := t - 2$ and choose a tree T that contains F as a spanning subgraph. Then $k \geqslant 0$, $\mathrm{pw}(G) \geqslant k + 1$, and $|T| = k + 2$. By Lemma 5.2.3, there exists a (k, T)-spanning separation (A, B) of G. Then $G[A]$ contains an F minor. By property $\star$ in the definition of (k, T)-spanning, $G[A]$ has a path-decomposition $(B_1, \dots, B_q)$ of width at most $k + 1$ and last bag equal to $A \cap B$. It follows that $\partial_G A \subseteq B_q$. Hence $Y := A$ suffices. ■

CHAPTER 6

Tree-width

This chapter begins by introducing and proving some useful properties of tree-width, which is a measure of how "tree-like" a graphs is. In this thesis, tree-width is studied for its structural properties. However, there are also algorithmic motivations to study tree-width, some of which are highlighted in Section 6.1. Connected to the theme of dualities is Section 6.2, which proves the Tree-width Duality Theorem. The Tree-width Duality Theorem is used in Section 6.4 to prove the Grid Minor Theorem, which is *the* key tool in Section 7.1 for proving Robertson and Seymour's extension of the Erdős–Pósa Theorem to planar graph minors.

A *tree-decomposition* of a graph G is a tuple $\mathcal{T} = (T, \beta)$ consisting of a tree T and a map $\beta : V(T) \to 2^{V(G)}$, such that following hold:

- *Vertex-property.* For every $v \in V(G)$, $\{t \in V(T) : v \in \beta(t)\}$ is a non-empty connected set of vertices in T.
- *Edge-property.* For every $uv \in E(G)$ there exists $t \in V(T)$ such that $\{u, v\} \subseteq \beta(t)$.

The vertices of T are called *nodes* and the members of $(\beta(t) : t \in V(T))$ are the *bags* of $\mathcal{T}$. See Figure 6.1 for an example.

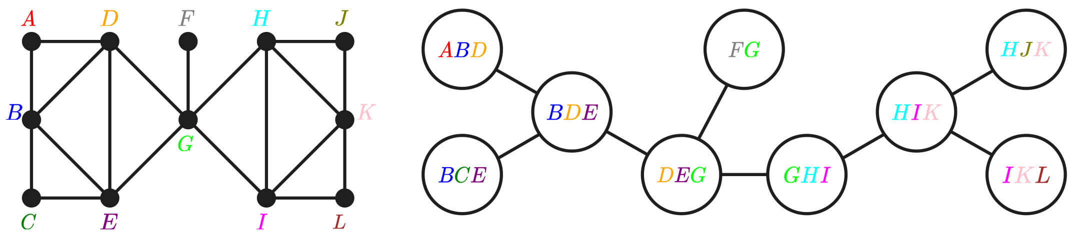

FIGURE 6.1. Example of a tree-decomposition.

For $xy \in E(T)$, the set $\beta(x) \cap \beta(y)$ is an *adhesion set* of $\mathcal{T}$. The *width* of $\mathcal{T}$ is the number $\max_{t \in V(T)} |\beta(t)| - 1$. The minimum width of a tree-decompositions of G is called the *tree-width* of G, denoted $\text{tw}(G)$. Note that tree-width is well-defined since every graph G has a tree-decomposition with $T = K^1$ and a single bag equal to $V(G)$. A tree-decomposition whose underlying tree is a path is a path-decomposition. In this way, tree-decompositions generalise path-decompositions.

The intuition one should have for tree-width is that it tells us how "tree-like" a graph is. Indeed, $\text{tw}(G) \leqslant 1$ if and only if G is a forest. It turns out that many properties of trees can be extended to tree-decompositions. For example, deleting any edge from a tree disconnects it, similarly for a tree-decomposition (T, β) of a graph G, every edge $xy \in E(T)$ induces a separation of G in the following way:

Proposition 6.0.1. *Let (T,β) be a tree-decomposition of a graph G. Let $xy \in E(T)$, and write T_x and T_y for the components of $T - xy$ such that $x \in V(T_x)$ and $y \in V(T_y)$. Put $S_{xy} := \beta(x) \cap \beta(y)$. Then S_{xy} separates $\bigcup \beta(V(T_x))$ and $\bigcup \beta(V(T_y))$ in G.*

A proof of the above proposition can be found in Diestel's book [34].

Just as path-width is monotonic with respect to the minor relation (Proposition 5.0.1), a nearly identical proof shows that the same is true for tree-width:

Proposition 6.0.2 (Monotonicity of Tree-width)**.** *If $H \preccurlyeq G$, then* $\operatorname{tw}(H) \leqslant \operatorname{tw}(G)$.

Chordal graphs from Chapter 4 provide another way for understanding tree-width. The characterisation given by Proposition 4.0.1 can be said in another way: A graph is chordal if and only if it has a tree-decomposition where every bag is a clique. This is especially clear in view of Figure 4.1. The following proposition connects chordal graphs and tree-width:

Proposition 6.0.3. *For every graph G,*

$$\operatorname{tw}(G) = \min\{\omega(H) - 1 : H \text{ is a chordal graph and } G \subseteq H\}.$$

Proof. By Proposition 4.0.1, every chordal graph H has a tree-decomposition of width $\omega(H) - 1$. Then, by Proposition 6.0.2, $\operatorname{tw}(G) \leqslant \omega(H) - 1$ for every chordal graph H with $G \subseteq H$. It remains to show that there exists a chordal graph H with $G \subseteq H$ and $\omega(H) - 1 \leqslant \operatorname{tw}(G)$. Let (T,β) be a tree-decomposition of G with minimum width. Define H to be the graph obtained from G by adding edges so that $\beta(t)$ is a clique for all $t \in V(T)$. Then (T,β) is a tree-decomposition of H with width $\operatorname{tw}(G)$, and every bag is a clique in H. Consequently, H is chordal. Moreover, any maximum sized clique in H equals some bag of (T,β), and so $\omega(H) - 1$ is at most the width of (T,β). Hence $\omega(H) - 1 \leqslant \operatorname{tw}(G)$, as required. ∎

Let (T,β) be a tree-decomposition of a graph G and $U \subseteq V(G)$. Define $(T,\beta)[U]$ to be the set of nodes $\{t \in V(T) : U \cap \beta(t) \neq \emptyset\}$. The following is an important property of tree-decompositions:

Proposition 6.0.4. *Let G be a graph, (T,β) be a tree-decomposition of G, and U be a non-empty connected set of vertices in G. Then $(T,\beta)[U]$ is a non-empty connected set of vertices in T.*

Proof. Apply induction on $|U|$. The base case of $|U| = 1$ is exactly the vertex-property. Suppose $|U| \geqslant 2$ and the result holds for smaller sets. Let x and y be different leaves of a spanning tree of $G[U]$, then $X := U \setminus \{x\}$ and $Y := U \setminus \{y\}$ are non-empty and connected sets of vertices in G. By induction $(T,\beta)[X]$ and $(T,\beta)[Y]$ are non-empty and connected in T. Since $X \cup Y = U$ and U is connected in G, there exists an X-Y-edge in G. By the edge-property some bag of (T,β) meets both X and Y, hence $(T,\beta)[X] \cup (T,\beta)[Y] = (T,\beta)[U]$ is non-empty and connected in T. The result follows by induction. ∎

The following result for tree-decompositions is analogous to Proposition 2.1.2:

Theorem 6.0.5 (Robertson, Seymour [117])**.** *For every graph G and every set $U \subseteq V(G)$, there exists $B \subseteq V(G)$ with $|B| \leqslant \operatorname{tw}(G) + 1$ such that every component of $G - B$ has at most $\frac{1}{2}|U|$ vertices in U.*

Proof. Let (T, β) be a tree-decomposition of G of minimum width. It may be assumed that $|V(G)| > \mathrm{tw}(G) + 1$, otherwise the result is trivial. Then (T, β) has at least two bags, and so $||T|| \geqslant 1$. For each edge $xy \in E(T)$, using the notation of Proposition 6.0.1, define $V_x^y := \bigcup \beta(T_x) \setminus S_{xy}$, and $V_y^x := \bigcup \beta(T_y) \setminus S_{xy}$. Then S_{xy} separates V_x^y and V_y^x in G. Since $\{V_x^y, V_y^x\}$ partitions $V(G - S_{xy})$, choose a part, say $P := V_x^y$, such that $|P \cap U| \leqslant \frac{1}{2}|U|$, and orient xy towards y. Orient every edge of T in this way. Since T has no cycles, there exists a sink node $t \in V(T)$. Put $B := \beta(t)$ and consider any component C of $G - B$. Since $V(C) \subseteq V_x^t$ for some $x \in N_T(t)$, the definition of the orientation implies $|V(C) \cap U| \leqslant |V_x^t \cap U| \leqslant \frac{1}{2}|U|$. ■

Corollary 6.0.6. *For every graph G and all $k \geqslant 0$, if there exists $U \subseteq V(G)$ such that for every $B \subseteq V(G)$ with $|B| \leqslant k + 1$ there exists a component C of $G - B$ with $|V(C) \cap U| > \frac{1}{2}|U|$, then $\mathrm{tw}(G) > k$.*

By monotonicity of tree-width, for every $k \geqslant 0$, the class of graphs with tree-width at most k is minor-closed. The Graph Minor Theorem (Theorem 2.2.4) implies that there exists a finite set $\mathcal{M}_k$ of minimally excluded minors for this class. It is easy to see $\mathcal{M}_1 = \{K^3\}$ and a standard exercise to show $\mathcal{M}_2 = \{K^4\}$. Interestingly, $\mathcal{M}_3$ consists of K^5 and three other graphs $V_8, K_{2,2,2}, C_5 \square K^2$ depicted in Figure 6.2.

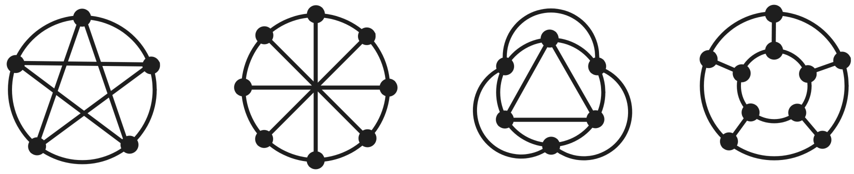

Figure 6.2. Left to right: $K^5, V_8, K_{2,2,2}, C_5 \square K^2$.

Sanders [122] showed $\mathcal{M}_4$ contains at least 76 graphs, and in general there is no proof of a complete list for $\mathcal{M}_k$ with $k \geqslant 4$. Section 6.3 shows that the $k \times k$-grid has tree-width k, therefore by monotonicity of tree-width, a large grid minor implies large tree-width. Robertson and Seymour proved a kind of converse:

Theorem 6.0.7 (Grid Minor Theorem. Robertson, Seymour [118])**.** *There exists a function $f : \mathbb{N} \to \mathbb{N}$ such that for every integer $k \geqslant 1$, every graph G with $\mathrm{tw}(G) > f(k)$ contains a $k \times k$-grid as a minor.*

By overlaying a fine grid on an embedding of a planar graph H, one can visualise "snapping" H to the grid. This visual argument inclines one to believe that every planar graph H is a minor of some large enough grid. This is true, in fact a result by Robertson, Seymour and Thomas [118] implies one only needs a $2|H| \times 2|H|$-grid. This, along with the Grid Minor Theorem gives the following corollary:

Corollary 6.0.8. *For every planar graph H, there exists an integer $w(H)$ such that every graph G with $\mathrm{tw}(G) > w(H)$ contains H as a minor.*

For more on tree-width, see surveys [16, 111, 65]. The next section briefly discusses algorithmic consequences of tree-width, making precise the notion that graphs with bounded tree-width are "simple".

6.1. Algorithmic Consequences

An approach for tackling computationally intractable problems is to find a parameter of the problem instance and classify the difficulty of the problem with respect to this parameter. A problem is *fixed-parameter tractable* with respect some parameter k if there exists an algorithm with time complexity $O(f(k)n^c)$, where $f(k)$ is some function depending only on k, and c is a constant. An intractable problem may become tractable when some parameter is small. The concept of fixed-parameter tractability allows one to classify the inherent difficulty of computational problems at a finer scale. One of the formative books on fixed-parameter tractability is by Downey and Fellows [40], and it shapes the material in this section. The purpose of this section is to make precise the notion that graphs with bounded tree-width are "simple". The theme of "complex" graphs being "similar" to "simple" graphs is returned to amidst the coarse graph theory discussion in Section 8.4.

The problem of finding a minimum sized vertex cover of a graph is a classic NP-hard problem and an example of parametrised tractability [39], since for each k, there is an algorithm running in $O(2^k|G|)$ time that determines whether G has a vertex cover of size at most k. It turns out that many graph problems become fixed-parameter tractable when parametrised by tree-width. A quintessential example of this is the following theorem of Courcelle:

Theorem 6.1.1 (Courcelle [28])**.** *Let F be a formula in the extended monadic second-order logic of graphs, and $k \geqslant 0$ be an integer. Then there exists a constant $c(F,k)$ and an algorithm that decides in $O(c(F,k)n)$ time whether an n-vertex graph with tree-width at most k satisfies F.*

The ingredients that make up formulas in the *extended monadic second-order logic of graphs*, or simply MSO_2, are logical connectives $\wedge, \vee, \neg, \Rightarrow, \Leftrightarrow$, variables for vertices $v_1, v_2, \ldots$, edges $e_1, e_2, \ldots$, sets of vertices $V_1, V_2, \ldots$, sets of edges $E_1, E_2, \ldots$, quantifiers $\forall, \exists$ which may be applied to those variables, and the binary predicates shown in Fig. 6.3.

Binary predicate	Interpretation
$v \in V$	The vertex variable v is a member of the vertex set variable V
$e \in E$	The edge variable e is a member of the edge set variable E
$\mathrm{in}(e,v)$	The edge variable e is incident to the vertex variable v
$\mathrm{ad}(u,v)$	The vertex variables u and v are adjacent
$=$	Equality for vertices, edges, sets of vertices and sets of edges

Table 6.3. Basic binary predicates in MSO_2.

The following formula expresses the property of having minimum degree at least 2:

$$\forall v_1 \exists v_2 \exists v_3 \Big(\neg(v_2 = v_3) \wedge \mathrm{ad}(v_1, v_2) \wedge \mathrm{ad}(v_1, v_3) \Big).$$

Since a graph has a cycle if and only if it has a subgraph with minimum degree at least 2, then the following formula expresses the property of containing a cycle:

$$\exists V_1 \Big(\exists v_1 (v_1 \in V_1) \wedge \forall v_2 \exists v_3 \exists v_4 \Big(\\ v_2 \in V_1 \wedge v_3 \in V_1 \wedge v_4 \in V_1 \wedge \neg(v_3 = v_4) \wedge \mathrm{ad}(v_2, v_3) \wedge \mathrm{ad}(v_2, v_4) \Big) \Big).$$

The next formula expresses the property of having a perfect matching:

$$\exists E_1 \forall v_1 \exists e_1 \Big(e_1 \in E_1 \wedge \mathrm{in}(e_1, v_1) \wedge \forall e_2 \Big((e_2 \in E_1 \wedge \mathrm{in}(e_2, v_1)) \Rightarrow e_1 = e_2 \Big) \Big)$$

In general, testing whether a graph admits a perfect matching can be done in polynomial time. Courcelles Theorem is interesting because many other graph properties may be expressed in MSO_2, including 3-colourability and Hamiltonicity. 3-colourability may be expressed as:

$$\exists V_1 \exists V_2 \exists V_3 \Big(\forall v_1 (v_1 \in V_1 \vee v_1 \in V_2 \vee v_1 \in V_3) \wedge \forall v_2 \forall v_3 \Big(\\ \Big((v_2 \in V_1 \wedge v_3 \in V_1) \Rightarrow \neg \mathrm{ad}(v_2, v_3) \Big) \\ \wedge \Big((v_2 \in V_2 \wedge v_3 \in V_2) \Rightarrow \neg \mathrm{ad}(v_2, v_3) \Big) \\ \wedge \Big((v_2 \in V_3 \wedge v_3 \in V_3) \Rightarrow \neg \mathrm{ad}(v_2, v_3) \Big) \Big) \Big).$$

Hence, by way of Courcelle's Theorem, 3-colourability can be decided in linear time when restricted to graphs with bounded tree-width. This is surprising because 3-colourability is an archetypal NP-complete problem.

6.2. Brambles and the Tree-width Duality Theorem

To show that a graph has small tree-width, it is enough to produce a tree-decomposition whose bags are small. On the other hand, to show that a graph has large tree-width, one must show that every tree-decomposition admits a large bag. An alternative strategy for proving large tree-width is instead to prove the existence of a large complete graph minor. Indeed, for any tree-decomposition (T, β) of a graph G, the branch sets of a K^t model induce connected subtrees of T by Proposition 6.0.4, which by the edge-property are pairwise intersecting. Then, the Helly property (Corollary 4.1.3) implies some bag meets every branch set of the K^t model. The maximum integer t such that $K^t \preccurlyeq G$ is the *Hadwiger number* of G, denote by $\mathrm{had}(G)$. Hence, this paragraph proves the following proposition:

Proposition 6.2.1. *For every graph G,*

$$\mathrm{had}(G) - 1 \leqslant \mathrm{tw}(G).$$

This lower bound is tight for complete graphs, however, as the following example shows, there exists graphs whose Hadwiger number is much smaller than its tree-width: Let $n \geqslant 3$ be an integer and G be the $n \times n$-grid. Since G is planar, the Kuratowski–Wagner Theorem (Theorem 2.2.2) implies that $\mathrm{had}(G) \leqslant 4$. On the other hand, $\mathrm{tw}(G) = \Omega(n)$. To see this, let U be the set of vertices of the first and last columns of G. By pigeonhole, any subset $B \subseteq V(G)$ with $|B| \leqslant (\frac{1}{2}n - 2) + 1$ avoids more than half of the rows and more than half of the columns of G. Then the subgraph C of $G - B$ induced by these rows and at least one of these columns is connected, and $|V(C) \cap U| > n = \frac{1}{2}|U|$. Hence $\mathrm{tw}(G) > \frac{1}{2}n - 2$ by Corollary 6.0.6.

Now instead consider the following generalisation of complete graph model, called a bramble: Firstly, two subgraphs A and B of a graph G are *touching* if $G[V(A \cup B)]$ is connected. A *bramble* $\mathcal{B}$ for a graph G is a collection of pairwise touching subgraphs of G. The *order* of $\mathcal{B}$ is the minimum size of a *hitting set* for $\mathcal{B}$, a set $S \subseteq V(G)$ meeting every member of $\mathcal{B}$. The *bramble number* of G, denoted by $\mathrm{bn}(G)$, is the maximum order of a bramble for G. Since the subgraphs of G induced on the branch sets of a K^t model form a bramble of order t, $\mathrm{had}(G) \leqslant \mathrm{bn}(G)$. Furthermore, a proof analogous to that of Proposition 6.2.1 shows the following:

Lemma 6.2.2. *For every graph G,*

$$\mathrm{bn}(G) - 1 \leqslant \mathrm{tw}(G).$$

The following duality theorem of Seymour and Thomas [123] asserts that Lemma 6.2.2 holds when the inequality is replaced by an equality:

Theorem 6.2.3 (Tree-width Duality Theorem. Seymour, Thomas [123])**.** *For every graph G,*

$$\mathrm{bn}(G) - 1 = \mathrm{tw}(G).$$

Proof. We follow Mazoit's proof [92]. Let G be given and put $k := \mathrm{bn}(G) - 1$. By Lemma 6.2.2, $\mathrm{tw}(G) \geqslant k$. A tree-decomposition (T, β) of G is *good* if the following hold:

- $\beta(t) \leqslant k$ for every non-leaf node $t \in V(T)$, and
- there exists $t \in V(T)$ such that $\beta(t) \leqslant k$.

Notice that the tree-decomposition of G with two bags $\emptyset$ and $V(G)$ is good. Since $\mathrm{tw}(G) \geqslant k$, every good tree-decomposition (T, β) of G has a leaf $t \in V(T)$ such that $\beta(t) \geqslant k + 1$. Let t' be the neighbour of t, then $\beta(t) \setminus \beta(t')$ is the *petal* of (T, β) *based* at t. Write $A \ll B$ if A and B are petals of tree-decompositions and $A \subseteq B$. Define $\mathcal{P}$ to be a minimal set of petals of good tree-decompositions of G such that the following hold:

(i) every good tree-decomposition of G has a petal in $\mathcal{P}$, and
(ii) if $A \in \mathcal{P}$ and $A \ll B$, then $B \in \mathcal{P}$.

Assume for a contradiction that there exists non-touching $X, Y \in \mathcal{P}$. Choose such X and Y to be $\ll$-minimal, that is, if $X' \subseteq X'$ and $Y' \subseteq Y$ with $X', Y' \in \mathcal{P}$ and X', Y' are non-touching, then $X' = X$ and $Y' = Y$. Since $\mathcal{P}_X := \mathcal{P} \setminus \{X\}$ is a set of petals of good tree-decompositions of G, at least one of (i) or (ii) does not hold for $\mathcal{P}_X$ by minimality of $\mathcal{P}$. Note that (ii) says $\mathcal{P}$ is closed up. Then, since $\mathcal{P}_X$ is obtained from $\mathcal{P}$ by deleting a minimal element, $\mathcal{P}_X$ satisfies (ii). Consequently $\mathcal{P}_X$ does not satisfy (i). Similarly, $\mathcal{P}_Y := \mathcal{P} \setminus \{Y\}$ satisfies (ii) but not (i). Hence there exists good tree-decompositions $\mathcal{T}_X := (T_X, \beta_X)$ and $\mathcal{T}_Y := (T_Y, \beta_Y)$ of G whose only petals in $\mathcal{P}$ are X and Y respectively. Let $x \in V(T_X)$ be a leaf such that X is the petal of $\mathcal{T}_X$ based at x, and let $y \in V(T_y)$ be a leaf such that Y is the petal of $\mathcal{T}_Y$ based at y. By Proposition 6.0.1 a petal of a tree-decomposition is based at exactly one leaf node of that tree-decomposition, hence such x and y are unique. Since X and Y are non-touching, $(X \cup N(X), V(G) \setminus (X \cup N(X)))$ is a separation of G. Hence, one may choose a separation (A, B) of G of minimum order such that $X \subseteq A \setminus B$ and $Y \subseteq B \setminus A$. Moreover, let $t \in V(T_X)$ be the neighbour of x, then $N(X) \subseteq \beta_X(t)$. Since $\mathcal{T}_X$ is good, $|N(X)| \leqslant |\beta_X(t)| \leqslant k$. Hence, the optimality of (A, B) implies $|A \cap B| \leqslant |N(X)| \leqslant k$. Put $S := A \cap B$. By Menger's Theorem there exists a collection of pairwise vertex-disjoint X-S-paths $(P_s : s \in S)$ in $G[A]$. Let H be the minor of G by deleting $A \setminus \bigcup_{s \in S} V(P_s)$, then for each $s \in S$, contracting P_s to s. Let $\mathcal{T}'_X := (T_X, \beta'_X)$ be the tree-decomposition of H inherited from G as in Proposition 6.0.2. Then for all $t \in V(T_X)$,

$$\beta'_X(t) := (\beta_X(t) \cap B) \cup \{s \in S : \beta_X(t) \cap V(P_s) \neq \emptyset\}. \tag{1}$$

From (1), $\beta'_X(t)$ can be seen as being obtained from $\beta_X(t)$ by first removing vertices not in B, then adding the vertices $\{s \in S : \beta_X(t) \cap V(P_s) \neq \emptyset\}$. Then for every vertex $s \in \beta'_X(t) \setminus \beta_X(t)$ at least one vertex of P_s is removed to form $\beta'_X(t)$, hence $|\beta'_X(t)| \leqslant |\beta_X(t)|$. Furthermore, since $X \subseteq A \setminus B$, $\beta'_X(x) = S$. Similarly, let $\mathcal{T}'_Y := (T_Y, \beta'_Y)$ be obtained in the same way for $G[A]$, then $|\beta'_Y(t)| \leqslant |\beta_Y(t)|$ for all $t \in V(T_Y)$ and $\beta'_Y(y) = S$. Let T be the tree obtained from $T_X \sqcup T_Y$ by identifying the leaves x and y, and for all $t \in V(T)$ put

$$\beta(t) := \begin{cases} \beta'_X(t) & \text{if } t \in V(T_X), \\ \beta'_Y(t) & \text{if } t \in V(T_Y). \end{cases}$$

Then $\mathcal{T} = (T, \beta)$ is a good tree-decomposition of G.

Claim 6.2.4. *Every petal of $\mathcal{T}$ is a subset of a petal of $\mathcal{T}_X$ or $\mathcal{T}_Y$, and both X and Y are not petals of $\mathcal{T}$.*

Subproof. Let Z be a petal of $\mathcal{T}$ based at some leaf $z \in V(T)$, and w be the neighbour of z. Without loss of generality z is a leaf of T_X different from x. Since the z-x-path of T starts with $(z, w, \dots)$, the properties of tree-decompositions imply $\beta(z) \cap \beta(x) \subseteq \beta(z) \cap \beta(w)$. This along with $\beta(x) = S$ and $Z = \beta(z) \setminus \beta(w)$ implies $Z \cap S = \emptyset$. Hence $Z \subseteq B \setminus A$. From (1), for all $t \in V(T_X)$, $\beta_X(t)$ and $\beta'_X(t)$ differ only in the vertices of A. Hence, $Z \subseteq \beta_X(z) \setminus \beta_X(w) =: Z_X$. Then $k + 1 \leqslant |\beta(z)| \leqslant |\beta_X(z)|$ implies Z_X is a petal of T_X containing Z. It remains to show $Z \neq X$ and $Z \neq Y$. The former is implied by $Z \subseteq B \setminus A$ and $X \subseteq A$. For the latter, assume for a contradiction that $Z = Y$. Then $Y \ll Z_X$. Since X and Y are non-touching, $Z_X \neq X$. Then by (ii), Y is a petal of $\mathcal{T}_X$ in $\mathcal{P}$ different from X, contradicting the definition of $\mathcal{T}_X$. Hence $Z \neq Y$. ∎

By (i), there exists $Z \in \mathcal{P}$ such that Z is a petal of $\mathcal{T}$. However, Claim 6.2.4 implies that Z is subset of a petal of $\mathcal{T}_X$ or $\mathcal{T}_Y$, and $Z \notin \{X, Y\}$. Without loss of generality Z is a subset of a petal Z' of $\mathcal{T}_X$. Then by (ii), $Z' \in \mathcal{P}$. By definition of $\mathcal{T}_X$, its only petal in $\mathcal{P}$ is X, hence $Z' = X$. Consequently, $Z \in \mathcal{P}$ and Z is a proper subset of X, contradicting the choice of X. Hence the assumption that there exists non-touching members of $\mathcal{P}$ was false.

Define $\mathcal{B}$ to be the set of all $G[P]$ where $P \in \mathcal{P}$ is connected in G. By the previous considerations, $\mathcal{B}$ is a bramble for G. It suffices to show that $\mathcal{B}$ has order at least $k+1$. Let $S \subseteq V(G)$ with $|S| = k$, and let $H_1, \dots, H_r$ be the components of components of $G - S$. Then G has a good tree-decomposition $\mathcal{T}$ with tree $K_{1,r}$, centre bag S, and leaf bags $V(H_1) \cup S, \dots, V(H_r) \cup S$. Hence every petal of $\mathcal{T}$ avoids S, implying S avoids a member of $\mathcal{B}$. Consequently, $\mathcal{B}$ has no hitting set of order k, as required. ■

6.3. Grids

Recall that the $n \times n$-grid is the graph with vertex set $[n] \times [n]$, where (x_1, y_1) and (x_2, y_2) are adjacent if and only if $|x_1 - x_2| + |y_1 - y_2| = 1$. Letting G_n be the $n \times n$-grid, this section shows that $\mathrm{tw}(G_n) = n$ for all integers $n \geqslant 2$.

First it is shown that $\mathrm{tw}(G_n) \leqslant n$. It suffices to produce a path-decomposition of G_n of width n. Enumerate the vertices of G_n in the following order:

$$(1,1), (1,2), \dots, (1,n), (2,1), (2,2), \dots, (2,n), (3,1), (3,2), \dots, \dots, (n,n). \tag{2}$$

For each $i \in [n^2 - n]$, let B_i be the set of vertices of G_n from the ith term to the $(i+n)$th term of (2). For example, $B_1 = \{(1,1), (1,2), \dots, (1,n), (2,1)\}$, and see Figure 6.4 for more examples in the case of $n = 4$.

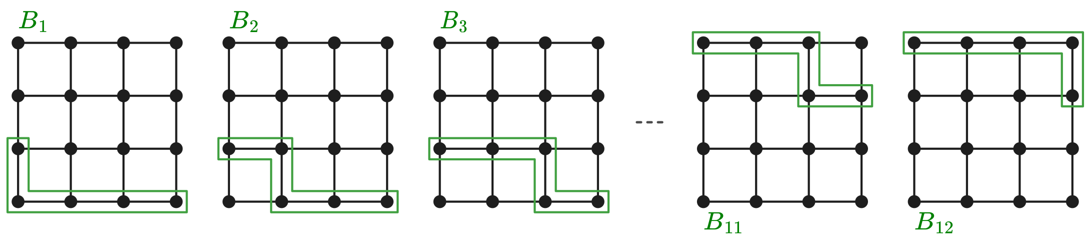

FIGURE 6.4. Illustration of the B_i's in the case of $n = 4$.

Then $(B_i : i \in [n^2 - n])$ consists of all sets of $n+1$ consecutive terms in (2). The following demonstrates that $(B_i : i \in [n^2 - n])$ is a path-decomposition of G_n: Since the kth term in (2) only appears in the sets $(B_i : i \in [n^2 - n], |i - k| \leqslant n)$, the vertex-property holds. For the edge-property, consider any edge $(x_1, y_1)(x_2, y_2) \in E(G_n)$. It remains to show that (x_1, y_1) and (x_2, y_2) lie in some common B_i. Since the edges of G_n are either along a row or a column of G_n, without loss of generality it suffices to consider the two cases (a) $x_1 = x_2 + 1$ and $y_1 = y_2$, or (b) $x_1 = x_2$ and $y_1 = y_2 + 1$. In case (a), there are $n+1$ terms in (2) from (x_1, y_1) to (x_2, y_2). By definition, these $n+1$ consecutive terms all lie in a common B_i, and so $\{(x_1, y_1), (x_2, y_2)\} \subseteq B_i$. In case (b), consider the last $i \in [n^2 - n]$ such that $(x_1, y_1) \in B_i$. Since (x_2, y_2) is the successor of (x_1, y_1) in (2), $(x_2, y_2) \in B_i$. It follows that $(B_i : i \in [n^2 - n])$ is a path-decomposition of G_n is a path-decomposition of G_n, and since each B_i has size $n+1$, $\mathrm{tw}(G_n) \leqslant \mathrm{pw}(G_n) \leqslant n$.

Next, it is shown that $\mathrm{tw}(G_n) \geqslant n$. By the Tree-width Duality Theorem (Theorem 6.2.3), it suffices to produce a bramble of G of order at least $n+1$. A cross of G_n is the union of row and a column subgraph of G_n. Define $\mathcal{B}$ to be the set consisting of

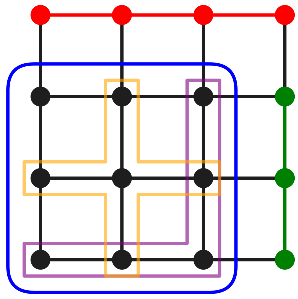

FIGURE 6.5. Illustration of some members of $\mathcal{B}$ in the case of $n = 4$. Orange and purple are examples of crosses of G_3.

the crosses of G_{n-1}, along with $G_n[\{n\} \times [n]]$ and $G_n[[n-1] \times \{n\}]$. See Figure 6.5 for examples of members of $\mathcal{B}$ when $n = 4$.

Firstly, note that any two crosses of G_{n-1} intersect. Secondly, every cross of G_{n-1} touches $G_n[\{n\}\times[n]]$ and $G_n[[n-1]\times\{n\}]$. Thirdly, $G_n[\{n\}\times[n]]$ and $G_n[[n-1]\times\{n\}]$ touch. Consequently, $\mathcal{B}$ is a bramble for G. The following demonstrates that $\mathcal{B}$ has order at least $n+1$: Assume for a contradiction that $\mathcal{B}$ has a hitting set S of size at most n. Since G_{n-1}, $G_n[\{n\} \times [n]]$, and $G_n[[n-1] \times \{n\}]$ are pairwise vertex-disjoint members of $\mathcal{B}$, all of the crosses of G_{n-1} are hit by $|S| - 2$ vertices in S. However $|S| - 2 \leqslant n - 2$, so S misses a row and a column of G_{n-1}, hence S misses a crosses of G_{n-1}, a contradiction. It follows that $\mathrm{bn}(G_n) \geqslant n + 1$, and so $\mathrm{tw}(G_n) \geqslant n$ by the Tree-width Duality Theorem. Hence $\mathrm{tw}(G_n) = n$, as announced.

6.4. Grid Minor Theorem

This section proves the following theorem:

Theorem 6.4.1 (Grid Minor Theorem. Robertson, Seymour [118]). *There exists a function $f : \mathbb{N} \to \mathbb{N}$ such that for every integer $g \geqslant 1$, every graph G with $\mathrm{tw}(G) > f(g)$ contains the $g \times g$-grid as a minor.*

The Grid Minor Theorem is used in Section 7.1 for proving Robertson and Seymour's extension of the Erdős-Pósa Theorem to planar graph minors. Initially, Robertson and Seymour proved their Grid Minor Theorem [118] with f being an exponential tower. A subsequent paper by Robertson, Seymour and Thomas [114] brought it down to $f(g) = 2^{O(g^5)}$, and they suspect a bound as low as $O(g^2 \log g)$. Since then, much work has been done [35, 76, 83] for the upper bound on f in Theorem 6.4.1, with the first polynomial bound of $O(g^{98} \mathrm{polylog}\, g)$ by Chekuri and Chuzhoy [23], a later improvement to $O(g^{36} \mathrm{polylog}\, g)$ by Chuzhoy [26], and the current best sitting at $O(g^9 \mathrm{polylog}\, g)$ by Chuzhoy and Tan [27]. This section follows Leaf and Seymour's proof [83] of the Grid Minor Theorem:

Theorem 6.4.2 (Leaf, Seymour [83]). *For every integer $g \geqslant 1$, every graph G with $\mathrm{tw}(G) > g^{8g^2}$ contains the $g \times g$-grid as a minor.*

Before launching into the proof, we roughly sketch Leaf and Seymour's method. Assuming G has large enough tree-width, one looks for a large grid minor H. Let T be a spanning tree of such H, and for each vertex v of T, add a large enough star whose centre is identified with v. Call this new tree T'. Since G has large tree-width, one may find a separation (A, B) of G such that, roughly, the following hold:

- $G[A]$ contains a T' model $\mathbf{X}$,
- every branch set of $\mathbf{X}$ meets $A \cap B$, and
- for any two subsets C and D of $A \cap B$, there exists a large collection of pairwise vertex-disjoint C-D-paths in $G[B]$.

Hence, one may obtain the desired H minor as follows: For each edge in $uv \in E(H) \setminus E(T)$, link the branch sets X_u and X_v of $\mathbf{X}$ via paths in $G[B]$. The addition of the large stars to form T', along with the third bullet point, yields many pairwise disjoint X_u-X_v-paths in $G[B]$. However, there is no guarantee that after linking two branch sets of $\mathbf{X}$ via some path P, there still exists enough usable paths in $G[B] - V(P)$ to finish the job. To remedy this, one assumes for a contradiction that G does not contain a large grid as a minor. Then, the so-called "Linkage Lemma" implies that useable paths can be chosen so that each successive choice decreases the number of useable paths only a little. In this way, the job can be finished. Much of the proof is pinning down how much is "large enough" when it comes to the tree-width of G and the size of the stars. Another detail is that instead of directly looking for a grid, the proof optimises by looking for a "grill".

6.4.1. Grill Lemma

A collection $\mathcal{P}$ of pairwise vertex-disjoint paths in a graph G is a *linkage* in G. For subsets $A, B \subseteq V(G)$, an *(A, B)-linkage* in G is a linkage $\mathcal{P}$ in G, every member of which has its first vertex in A and last vertex in B.

For integers $m, n \geqslant 1$, an *(m, n)-grill* is a graph G with $V(G) = \{v_{i,j} : (i, j) \in [m] \times [n]\}$, such that the following hold:

- for all $p \in [m]$, $P_p := (v_{p,1}, v_{p,2}, \ldots, v_{p,n})$ is a path in G, and
- for all $j \in [n]$, the subgraph $T_i := G[\{v_{i,j} : j \in [m]\}]$ is connected.

See Figure 6.6 for an example

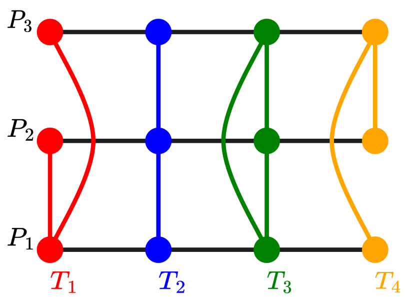

FIGURE 6.6. A $(3, 4)$-grill.

Lemma 6.4.3 (Grill Lemma)**.** *Let g, m, n be integers satisfying:*

$$g \geqslant 2,\ m \geqslant (2g + 1)(2g^2 - 5) + 2,\ \textit{and}\ n = g^2(2g + g^2 - 2).$$

Then every (m,n)-grill G contains a $g \times g$-grid minor.

The proof of Lemma 6.4.3 relies on the following lemmas:

Lemma 6.4.4. *Let h, m, n be integers satisfying $h \geqslant 2$, $m \geqslant h+1$, and $n = h^2$. Let G be an (m,n)-grill such that for every $j \in [h]$, $T_{(h+1)j-h}$ has a spanning tree with at least h leaves. Then G contains a K^h minor.*

Proof. For each $j \in [h]$ let $X_j \subseteq V(T_{(h+1)j-h})$ be a set of h leaves of some spanning tree of $T_{(h+1)j-h}$, then $T'_j := T_{(h+1)j-h} - X_j$ is non-null and connected. For each $j \in [h-1]$, put $H_j := G[X_j \cup X_{j+1} \bigcup (V(T_i) : (h+1)j-h < i < (h+1)(j+1)-h)]$. By pigeonhole, for every set $Z \subseteq V(H_j)$ with $|Z| < h$, there exists $p, q \in [m]$ and i in the range $(h+1)j-h < i < (h+1)(j+1)-h)$ such that $X_j \cap V(P_p) \neq \emptyset$, $X_{j+1} \cap V(P_q) \neq \emptyset$, $Z \cap V(P_p \cup P_q) = \emptyset$, and $Z \cap V(T_i) = \emptyset$. Then $H_j \cap (P_p \cup P_q \cup T_i)$ is a connected subgraph of H_j that avoids Z, and meets X_j and X_{j+1}. Hence H_j has no X_j-X_{j+1}-separator of cardinality less than h, so by Menger's Theorem there exists an (X_j, X_{j+1})-linkage $\mathcal{P}_j$ with cardinality h such that $V(\bigcup \mathcal{P}_j) \subseteq V(H_j)$. Finally, note that $A := \bigcup_{j \in [h-1]} \bigcup \mathcal{P}_j$ has h components, $B := \bigcup_{j \in [h]} T'_j$ has h components, and any two components of A and B are joined by an edge of G. Hence G contains a $K_{h,h}$ minor. Finally, contracting the edges of a perfect matching of $K_{h,h}$ produces a K^h. ∎

Lemma 6.4.5. *Let $r \geqslant 1$ and $h \geqslant 3$ be integers, and G be a connected graph with $|G| \geqslant (r+2)(2h-5)+2$. Then*

- *G has a spanning tree T with at least h leaves, or*
- *G has an r-vertex path P whose internal vertices have degree 2 in G.*

Proof. Let T be a spanning tree of G with as many leaves as possible. It may be assumed that T has at most $h-1$ leaves. Since $|G| \geqslant 2$, an easy induction proof on $h \geqslant 3$ shows that T has at most $2h-4$ vertices of degree different from 2. Hence, T is a subdivision of a tree with at most $2h-5$ edges, which implies that T is the union of at most $2h-5$ paths whose internal vertices have degree 2 in T. Consider a longest such path $(v_1, \dots, v_t)$. Then T has at most $(t-2)(2h-5)$ vertices of degree 2.

The following shows that every vertex v_i with $i \in \{4, \dots, t-3\}$ has $\deg_G(v_i) = 2$: Suppose not, then there exists a vertex $v \in N_G(v) \setminus \{v_{i-1}, v_{i+1}\}$. Without loss of generality the v_i-v-path in T contains the subpath (v_i, v_{i-1}, v_{i-2}). Then $T' := T - v_{i-1}v_{i-2} + v_i v$ is a spanning tree of G. If $v = v_{i-2}$, then T' has one more leaf v_{i-1} compared to T. If $v \neq v_{i-2}$, then v_{i-2} and v_{i-1} are leaves of T' but not of T, and v is not a leaf of T'. All in all, T' has more leaves than T, contradicting the choice of T.

It follows that every internal vertex of $P = (v_3, \dots, v_{t-2})$ has degree 2 in G. From

$$2h-4+(t-2)(2h-5) \geqslant |T| = |G| \geqslant (r+2)(2h-5)+2,$$

it follows that $(t-r-3)(2h-5) \geqslant 1$. Since $2h-5 \geqslant 1$, $t-r-3 \geqslant 1$. Thus $|P| = t-4 \geqslant r$. ∎

Proof of Lemma 6.4.3. Put $h := g^2$. It may be assumed that $n = h(2g+h-2)$. Let G be an (m,n)-grill and for each $i \in [n-2g+2]$ put $H_i := G[V(\bigcup_{j=i}^{i+2g-2} T_j)]$. Let M_i be the minor of H_i obtained by, for each $p \in [m]$, contracting $P_p \cap H_i$ to a vertex.

Suppose for all $i \in [n-2g+2]$, M_i has a spanning tree with at least h leaves. Put $d := 2g+h-1$, and perform the aforementioned edge-contractions of G taking $H_1, H_{d+1}, H_{2d+1}, \ldots$ to $M_1, M_{d+1}, M_{2d+1}, \ldots$ respectively. As depicted in Figure 6.7, this produces an (m, h^2)-grill which satisfies the premise of Lemma 6.4.4. Thus G contains a K^{g^2} minor, and so G contains a $g \times g$-grid minor.

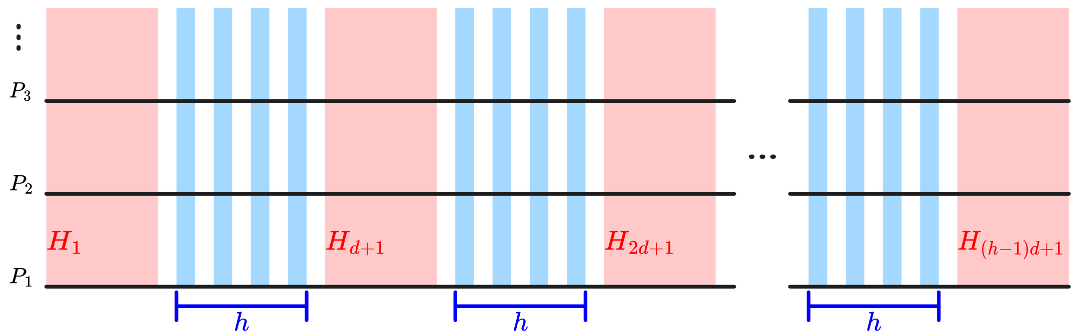

FIGURE 6.7. Layout of G prior to contractions.

Hence, it may be assumed that there exists $i \in [n-2g+2]$ such that M_i has no spanning tree with at least h leaves. Since $|M_i| = m \geqslant (2g+1)(2h-5)+2$, Lemma 6.4.5 with $r := 2g-1$ implies M_i has an r-vertex path P whose internal vertices have degree 2 in M_i. For each $p \in [m]$, let c_p be the vertex of M_i formed by contracting $P_p \cap H_i$. Re-index if necessary so that $P = (c_1, \ldots, c_r)$. Then G has no $V(P_{p-1} \cap H_i)$-$V(P_{p+1} \cap H_i)$-edge for all $p \in \{2, \ldots, r-1\}$, implying $N_{T_j}(v_{p,j}) \subseteq \{v_{p-1,j}, v_{p+1,j}\}$ for all $j \in \{i, \ldots, i+2g-2\}$. With this in mind, consider some $j \in \{i, \ldots, i+2g-2\}$. Since T_j is connected, $(v_{1,j}, \ldots, v_{g,j})$ or $(v_{g,j}, \ldots, v_{r,j})$ is a path in T_j. Without loss of generality $(v_{1,j}, \ldots, v_{g,j})$ is a path in T_j for at least half of the values of $j \in \{i, \ldots, i+2g-2\}$. Let H be the union of g such paths and $\bigcup_{p=1}^{r} P_p \cap H_i$. Then H contains a $g \times g$-grid minor. ∎

The Grill Lemma is not applied to any particular grill subgraph. Instead, the following corollary extends Lemma 6.4.3 to allow application on a well structured subgraph containing a grill minor; a pregrill:

For integers $m, n \geqslant 1$ and real $\varepsilon \in [0,1]$, an *(m, n, ε)-pregrill* is a graph G such that for some subsets $A, B \subseteq V(G)$, the following hold:

- there exists an (A, B)-linkage $\{P_1, \ldots, P_m\}$ in G,
- there exists pairwise vertex-disjoint connected subgraphs $T_1, \ldots, T_n$ of G,
- for all integers p, i, j satisfying $p \in [m]$ and $1 \leqslant i < j \leqslant n$, as P_p is traversed from A to B, *every* vertex of $P_p \cap T_i$ is seen before *any* vertex of $P_p \cap T_j$ is seen,
- for all $j \in [n]$, $P_p \cap T_j$ is null for at most εm values of $p \in [m]$.

See Figure 6.8 for an example.

Corollary 6.4.6. *Let g, m, n be integers satisfying*

$$g \geqslant 2,\ m \geqslant 2(2g+1)(2g^2-2),\ n = 2g^2(2g+g^2-2),\ \textit{and put}\ \varepsilon := \frac{1}{4(2g+1)(g^2-2)}.$$

Then every (m, n, ε)-pregrill G contains a $g \times g$-grid minor.

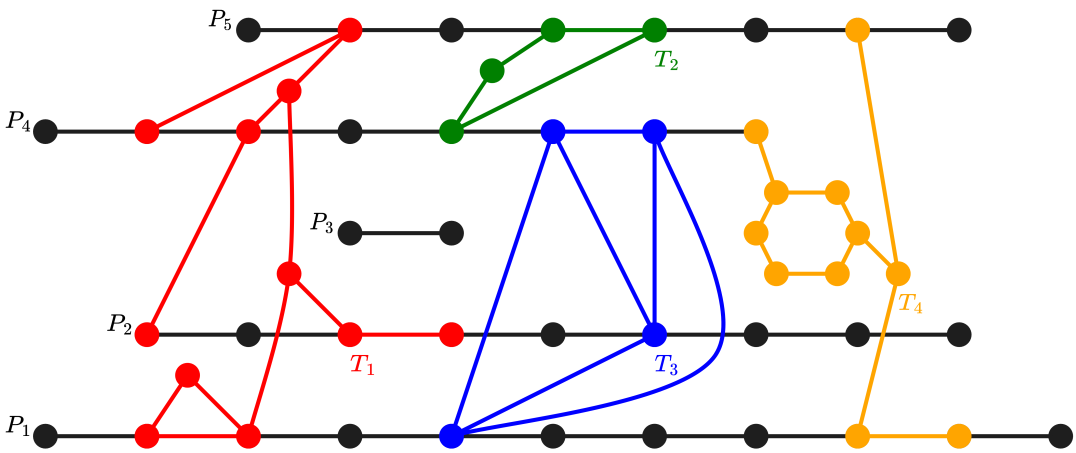

FIGURE 6.8. A $(5, 4, \frac{3}{5})$-pregrill.

PROOF. There exists at most εmn pairs $(p, j) \in [m] \times [n]$ for which $P_p \cap T_j$ is null. Hence on average, each P_p is disjoint from at most εn of $T_1, \dots, T_n$. Choose $k := (2g + 1)(2g^2 - 2)$ of the P_i which contributed least to the calculation of that average, say $P_1, \dots, P_k$. Then there exists at most $\varepsilon nk = \frac{1}{2}n$ pairs $(p, j) \in [k] \times [n]$ for which $P_p \cap T_j$ is null. Hence at least half of $T_1, \dots, T_n$ meet all of $P_1, \dots, P_k$, say $T_1, \dots, T_{\lceil n/2 \rceil}$. Then for each $(p, j) \in [k] \times [\lceil \frac{1}{2}n \rceil]$, contracting the edges of P_p between the first and last vertex of each T_j along with further contractions within each T_j leaves a $(k, \lceil \frac{1}{2}n \rceil)$-grill, which by Lemma 6.4.3 contains a $g \times g$-grid minor. ■

6.4.2. Linkage Lemma

Informally, this section shows that if a graph G has no $g \times g$-grid minor and has two linkages $\mathcal{P}, \mathcal{Q}$, then some subset $\mathcal{P}'$ of is disjoint from some member $Q \in \mathcal{Q}$. Moreover, $|\mathcal{P}'|$ is not much smaller than $|\mathcal{P}|$. More precisely, this section proves the following lemma:

Lemma 6.4.7 (Linkage Lemma). *Let g, m, n be integers satisfying*

$$g \geqslant 2,\ m \geqslant 2(2g + 1)(g^2 - 2),\ n = 2g^2(2g + g^2 - 2)m, \text{ and put } \varepsilon := \frac{1}{4(2g + 1)(g^2 - 2)}.$$

Let G be a graph with the following properties:

- *G has no $g \times g$-grid minor,*
- *there exists $A, B \subseteq V(G)$ and an (A, B)-linkage in G of cardinality m, and*
- *$\mathcal{Q}$ is a set of n pairwise vertex-disjoint connected subgraphs of G.*

Then there exists $Q \in \mathcal{Q}$ and an $(A \setminus V(Q), B \setminus V(Q))$-linkage in $G - V(Q)$ of cardinality greater than εm.

The proof of Lemma 6.4.7 relies on the following lemma:

Lemma 6.4.8. *Let m, p be integers satisfying $p \geqslant m \geqslant 1$. Let G be a p-vertex graph and $A, B \subseteq V(G)$ such that the following hold:*

- *there exists a unique (A, B)-linkage $\mathcal{P}$ of cardinality m in G, and*
- *$V(\bigcup \mathcal{P}) = V(G)$.*

Then there exists a sequence $B_1, \dots, B_{p-m+1}$ of subsets of $V(G)$ such that the following hold:

- $|B_i| = m$ *for all* $i \in [p-m+1]$,
- $B_1 = A$ *and* $B_{p-m+1} = B$,
- *for every connected set* $U \subseteq V(G)$, *the set* $\{i \in [p-m+1] : U \cap B_i \neq \emptyset\}$ *is an interval of integers.*

Proof. We say that the sequence in the conclusion of the lemma is good *suffices* for the tuple (m, p, G, A, B). Proceed by induction on $p \geqslant 1$. For the base case of $p = 1$, it follows that $m = 1$. Since G consists of only a single vertex, the sequence $V(G)$ suffices. Now suppose $p \geqslant 2$ and the result holds for smaller values of p. Suppose there exists a separation (C, D) of G of order m with $A \subseteq C$ and $B \subseteq D$, and both $|C| > m$ and $|D| > m$. Then, by induction there exist a sequence, $B_1, \ldots, B_r$ say, that suffices for $(m, |C|, G[C], A, C \cap D)$, and there exists a sequence $B'_1, \ldots, B'_t$ say, that suffices for $(m, |D|, G[D], C \cap D, B)$. Note that $B_r = B'_1$, then the sequence $B_1, \ldots, B_r, B'_2, \ldots, B'_t$ suffices for (m, p, G, A, B). Hence, it may be assumed that G has no such separation (C, D).

If $||\bigcup \mathcal{P}|| = 0$, then $|G| = |A| = |B| = m$, so the sequence $V(G)$ suffices. Now suppose that there exists an edge $e \in E(\bigcup \mathcal{P})$. By uniqueness of $\mathcal{P}$, the maximum cardinality of an (A, B)-linkage in $G - e$ is $m - 1$, then by Menger's Theorem $G - e$ has a separation (X, Y) of order $m - 1$ with $A \subseteq X$ and $B \subseteq Y$. Since G has an (A, B)-linkage of cardinality m, e is an $(X \setminus Y)$-$(Y \setminus X)$-edge of G. Write $e = xy$ where $x \in X \setminus Y$ and $y \in Y \setminus X$. Therefore $(X, Y \cup \{x\})$ is a separation of G of order m with $A \subseteq X$ and $B \subseteq Y \cup \{x\}$. By assumption, $|X| \leqslant m$ or $|Y \cup \{x\}| \leqslant m$. However, the latter is false since $|Y \cup \{x\}| > |Y| \geqslant |B| \geqslant m$. It follows that $m \geqslant |X| \geqslant |A| \geqslant m$, thus $|A| = |X| = m$ and $A = X$. Similarly, analysing $(X \cup \{y\}, Y)$ shows $|B| = |Y| = m$ and $B = Y$. Consequently, the sequence A, B suffices. The result follows by induction. ∎

Proof of Lemma 6.4.7. Fix g and m, and proceed by induction on $|G| + ||G|| \geqslant n$. Consider the base case of $|G| + ||G|| = n$. From the definition of $\mathcal{Q}$, $\bigcup \mathcal{Q} = G$ consists of n isolated vertices. Let $\mathcal{P}$ be an (A, B)-linkage in G of cardinality m. Since $n \geqslant 2m > m$, one may choose $Q \in \mathcal{Q}$ disjoint from $\bigcup \mathcal{P}$, thus $\mathcal{P}$ is an $(A \setminus V(Q), B \setminus V(Q))$-linkage in $G - V(Q)$ of cardinality $m \geqslant \varepsilon m$.

Now turning to the induction step, suppose $|G| + ||G|| \geqslant n + 1$. Let $\mathcal{P}$ be an (A, B)-linkage in G of cardinality m. If it is possible to choose $\mathcal{P}$ such that some vertex or edge of G does not belong to the subgraph $(\bigcup P) \cup (\bigcup Q)$, then let G' be the subgraph of G obtained by deleting said vertex or edge. Then $\mathcal{P}$ is an $(A \cap V(G'), B \cap V(G'))$-linkage in G' of cardinality m, and $\mathcal{Q}$ is a collection of n pairwise vertex-disjoint connected subgraphs of G'. Since $n \leqslant |G'| + ||G'|| < |G| + ||G||$, by induction there exists $Q \in \mathcal{Q}$ and an $((A \cap V(G')) \setminus V(Q), (B \cap V(G')) \setminus V(Q))$-linkage $\mathcal{L}$ in $G' - V(Q)$ of cardinality greater than εm. Consequently, $\mathcal{L}$ is a desired linkage in $G - V(Q)$. Hence, it may be assumed that $G = (\bigcup \mathcal{P}) \cup (\bigcup \mathcal{Q})$.

If it is possible to choose $\mathcal{P}$ such that some edge $e \in E(G)$ lies in both subgraphs $\bigcup \mathcal{P}$ and $\bigcup \mathcal{Q}$, then let G', A', B', $\mathcal{P}'$, $\mathcal{Q}'$ be the result of G, A, B, $\mathcal{P}$, $\mathcal{Q}$, respectively, after contracting e. Then $\mathcal{P}'$ is an (A', B')-linkage in G' of cardinality m, and $\mathcal{Q}'$ is a collection of n pairwise vertex-disjoint connected subgraphs of G'. Since $n \leqslant |G'| + ||G'|| < |G| + ||G||$, by induction there exists $Q' \in \mathcal{Q}'$ and an $(A' \setminus V(Q'), B' \setminus V(Q'))$-linkage $\mathcal{L}$ in $G' - V(Q')$ of cardinality greater than εm. By possibly subdividing an edge of some member of $\mathcal{L}$, one obtains a desired linkage in $G - V(Q)$. Hence, it may be assumed that $E(\bigcup \mathcal{P}) = E(G) \setminus E(\bigcup \mathcal{Q}) = \emptyset$ for every choice of $\mathcal{P}$.

Suppose there exists a vertex $v \in V(G) \setminus V(\bigcup \mathcal{P})$. If v is an isolated vertex of G, then $G[\{v\}] \in \mathcal{Q}$, thus one may take $\mathcal{P}$ as the desired linkage. On the other hand, there exists some edge $uv \in E(G)$, thus $uv \in E(\bigcup \mathcal{Q})$. Let G', A', B', $\mathcal{Q}'$ be the result of G, A, B, $\mathcal{Q}$, respectively, after contracting uv to u. Then $\mathcal{P}$ is an (A', B')-linkage in G' of cardinality m, and $\mathcal{Q}'$ is a collection of n pairwise vertex-disjoint connected subgraphs of G'. Since $n \leqslant |G'| + ||G'|| < |G| + ||G||$, by induction there exists $Q' \in \mathcal{Q}'$ and an $(A' \setminus V(Q'), B' \setminus V(Q'))$-linkage $\mathcal{L}$ in $G' - V(Q')$ of cardinality greater than εm. By possibly subdividing an edge of some member of $\mathcal{L}$, one obtains a desired linkage in $G - V(Q)$. Hence, it may be assumed that $V(\bigcup \mathcal{P}) = V(G)$ for every choice of $\mathcal{P}$.

It follows that $\mathcal{P}$ satisfies the premise of Lemma 6.4.8. Put $p := |G|$ and let $B_1, \ldots, B_{p-m+1}$ be as promised by Lemma 6.4.8. For each $Q \in \mathcal{Q}$ put $\mathcal{I}(Q) := \{i \in [p - m + 1] : V(Q) \cap B_i \neq \emptyset\}$. Since each Q is non-null and connected, each $\mathcal{I}(Q)$ is a non-empty interval of integers. Moreover, for each $i \in [p - m + 1]$, since $|B_i| = m$, at most m members of $(\mathcal{I}(Q) : Q \in \mathcal{Q})$ contain i. Then any subset $S \subseteq [p - m + 1]$ with $|S| < \frac{n}{m}$ meets less than n members of $(\mathcal{I}(Q) : Q \in \mathcal{Q})$, hence missing at least one. By the generalised Helly property (Lemma 4.1.2), there exists $\frac{n}{m}$ members of $\mathcal{Q}$, say $Q_1, \ldots, Q_{n/m}$, such that $(\mathcal{I}(Q_1), \ldots, \mathcal{I}(Q_{n/m}))$ is a pairwise disjoint collection. Re-index if necessary so that $\mathcal{I}(Q_1), \ldots, \mathcal{I}(Q_{n/m})$ are in increasing order. Write $\mathcal{P} = \{P_1, \ldots, P_m\}$, then for all integers p, i, j satisfying $p \in [m]$ and $1 \leqslant i < j \leqslant \frac{n}{m}$, as P_p is traversed from A to B, every vertex of $P_p \cap Q_i$ is seen before any vertex of $P_p \cap Q_j$ is seen. If for all $j \in [\frac{n}{m}]$, $P_p \cap Q_j$ is null for at most εm values of $p \in [m]$, then $\mathcal{P}$ along with $Q_1, \ldots, Q_{n/m}$ form an $(m, \frac{n}{m}, \varepsilon)$-pregrill and Corollary 6.4.6 implies G contains a $g \times g$-grid minor, a contradiction. Hence there exists $j \in [\frac{n}{m}]$, such that $P_p \cap Q_j$ is null for more than εm values of $p \in [m]$. Consequently, those members of $\mathcal{P}$ form a desired linkage in $G - V(Q_j)$. The result follows by induction. ■

For a graph G and a subset $Z \subseteq V(G)$, a path P in G is *Z-proper* if $||P|| \neq 1$ and only its end vertices are in Z. Note that a 1-vertex path is Z-proper if its unique vertex is in Z. A linkage is *Z-proper* if all its members are Z-proper. The Linkage Lemma is applied in the proof of the following lemma:

Lemma 6.4.9. *Let $g \geqslant 2$ and $n \geqslant 1$ be integers. Put*

$$d := 2g^2(2g + g^2 - 2), \ \varepsilon := \frac{1}{4(2g+1)(g^2-2)}, \ k_1 := \varepsilon^{1-n},$$
$$k_i := d(1+d)^{i-2}k_1 \text{ for } 2 \leqslant i \leqslant n.$$

Let G be a graph and $Z \subseteq V(G)$ such that the following hold:

- *G has no $g \times g$-grid minor, and*
- *for all $i \in [n]$ there exists subsets $A_i, B_i \subseteq Z$ and a Z-proper (A_i, B_i)-linkage in G of cardinality k_i.*

Then there exists pairwise vertex-disjoint Z-proper paths $P_1, \ldots, P_n$ in G, such that for all $i \in [n]$, the first vertex of P_i is in A_i and the last vertex is in B_i.

Lemma 6.4.9 follows quickly from the Lemma 6.4.10, so its proof is delayed till the end of the section:

Lemma 6.4.10. *Let g, m, n be integers satisfying*

$$g \geqslant 2,\ m \geqslant 2(2g+1)(g^2-2),\ n = 2g^2(2g+g^2-2)m,\ \textit{and put}\ \varepsilon := \frac{1}{4(2g+1)(g^2-2)}.$$

Let G be a graph and $Z \subseteq V(G)$ such that the following hold:

- *G has no $g \times g$-grid minor,*
- *there exists subsets $A, B \subseteq Z$ and a Z-proper (A, B)-linkage in G of cardinality m, and*
- *$\mathcal{Q}$ is a Z-proper linkage in G of cardinality n.*

Then there exists $Q \in \mathcal{Q}$ and a $Z \setminus V(Q)$-proper $(A \setminus V(Q), B \setminus V(Q))$-linkage in $G - V(Q)$ of cardinality greater than εm.

Proof. Let $\mathcal{P}$ be a Z-proper (A, B)-linkage in G of cardinality m and $G' := (\bigcup \mathcal{P}) \cup (\bigcup \mathcal{Q})$. By Lemma 6.4.7, there exists $Q \in \mathcal{Q}$ and an $(A \setminus V(Q), B \setminus V(Q))$-linkage $\mathcal{L}$ in $G' - V(Q)$ of cardinality greater than εm. Choose such an $\mathcal{L}$ so that $\bigcup \mathcal{L}$ is subgraph-minimal. The following demonstrates that $\mathcal{L}$ is a desired linkage in $G - V(Q)$: It suffices to show $\mathcal{L}$ is Z-proper. Since $\mathcal{P}$ and $\mathcal{Q}$ are Z-proper and $E(\bigcup \mathcal{L}) \subseteq E(\bigcup \mathcal{P}) \cup E(\bigcup \mathcal{Q})$, no edge of $\bigcup \mathcal{L}$ has both ends in Z. Now assume for a contradiction that some internal vertex v of some $L \in \mathcal{L}$ lies in Z. Then $\deg_{G'}(v) \geqslant 2$ and v is an end point of a member of $\mathcal{P} \cup \mathcal{Q}$. It follows that v is an endpoint of a member of $\mathcal{P}$ and an endpoint of a member of $\mathcal{Q}$. Thus $v \in A \cup B$, implying a proper subpath of L is a Z-proper path in G' from $A \setminus V(Q)$ to $B \setminus V(Q)$, contradicting the choice of $\mathcal{L}$. ∎

Corollary 6.4.11. *Let $g \geqslant 2$ and $n \geqslant 1$ be integers, $k_1, \ldots, k_n \geqslant 2(2g+1)(g^2-2)$ be integers satisfying $k_n \geqslant 2g^2(2g+g^2-2)\sum_{i=1}^{n-1} k_i$, and put $\varepsilon := (4(2g+1)(g^2-2))^{-1}$. Let G be a graph and $Z \subseteq V(G)$ such that the following hold:*

- *G has no $g \times g$-grid minor,*
- *for all $i \in [n]$ there exists subsets $A_i, B_i \subseteq Z$ and a Z-proper (A_i, B_i)-linkage in G of cardinality k_i.*

Then there exists a Z-proper path Q whose first vertex is in A_n and last vertex is in B_n, such that for all $i \in [n-1]$ there is a $Z \setminus V(Q)$-proper $(A_i \setminus V(Q), B_i \setminus V(Q))$-linkage in $G - V(Q)$ of cardinality greater than εk_i.

Proof. Let $\mathcal{Q}$ be a Z-proper (A_n, B_n)-linkage in G of cardinality k_n. For each $i \in [n-1]$, let $\mathcal{Q}_i$ be the set of all $Q \in \mathcal{Q}$ for which there is no $Z \setminus V(Q)$-proper $(A_i \setminus V(Q), B_i \setminus V(Q))$-linkage of cardinality greater than εk_i in $G - V(Q)$. By Lemma 6.4.10 with $m := k_i$, $|\mathcal{Q}_i| < 2g^2(2g+g^2-2)k_i$. Then $|\bigcup_{i=1}^{n-1} \mathcal{Q}_i| < 2g^2(2g+g^2-2)\sum_{i=1}^{n-1} k_i \leqslant k_n = |\mathcal{Q}|$, which implies that there exists $Q \in \mathcal{Q} \setminus \bigcup_{i=1}^{n-1} \mathcal{Q}_i$. It follows that for all $i \in [n-1]$ there is a $Z \setminus V(Q)$-proper $(A_i \setminus V(Q), B_i \setminus V(Q))$-linkage in $G - V(Q)$ of cardinality greater than εk_i. ∎

Proof of Lemma 6.4.9. Proceed by induction on $n \geqslant 1$. In the base case of $n = 1$, it follows that $k_1 = 1$, thus a desired path P_1 exists by hypothesis. Now suppose $n \geqslant 2$ and the result holds for smaller values of n. Then $k_i \geqslant \varepsilon^{1-n} \geqslant \varepsilon^{-1} \geqslant 2(2g+1)(g^2-2)$ for all $i \in [n]$, and $2g^2(2g+g^2-2)\sum_{i=1}^{n-1} k_i = d(1+d)\varepsilon^{1-n} = k_n$. By Corollary 6.4.11, there exists a Z-proper path Q whose first vertex is in A_n and last vertex is in B_n, such that for all $i \in [n-1]$ there is a $Z \setminus V(Q)$-proper $(A_i \setminus V(Q), B_i \setminus V(Q))$-linkage in $G - V(Q)$ of cardinality greater than εk_i. By induction there exists pairwise vertex-disjoint Z-proper paths $P_1, \dots, P_{n-1}$ in G, such that for all $i \in [n-1]$, the first vertex of P_i is in A_i and the last vertex is in B_i. Put $Q := P_n$. The result follows by induction. ■

6.4.3. Growing a Tree

For a graph G, a subset $Z \subseteq V(G)$ is *linked* in G if for all subsets $A, B \subseteq Z$ with $|A| = |B|$, there exists a Z-proper (A, B)-linkage in G of cardinality $|A|$. The final lemma grows a tree similar to Lemma 5.2.3, except with some additional properties.

Lemma 6.4.12. *For every tree T and every graph G with $\operatorname{tw}(G) \geqslant \frac{3}{2}|T| - 1$, there exists a separation (A, B) of G such that the following hold:*

- *(A, B) has order $|T|$,*
- *$G - A$ is connected,*
- *every vertex in $A \cap B$ has a neighbour in $B \setminus A$,*
- *$A \cap B$ is linked in $G[B]$, and*
- *there exists a T model $\mathbf{X}$ in $G[A]$ such that $X \cap A \cap B \neq \emptyset$ for all $X \in \mathbf{X}$.*

Proof. Put $w := |T|$ and choose an enumeration $t_1, \dots, t_w$ of the vertices of T such that $T_i := T[\{t_1, \dots, t_i\}]$ is connected for all $i \in [w]$. By the Tree-width Duality Theorem (Theorem 6.2.3), G has a bramble $\mathcal{B}$ of order at least $\frac{3}{2}w$. Choose a maximal such bramble. For each $S \subseteq V(G)$ with $|S| \leqslant \frac{3}{2}w - 1$, some member of $\mathcal{B}$ avoids S. By definition of bramble, there exists a unique component of $G - S$ containing all such members of $\mathcal{B}$. Define $\mathcal{B}(S)$ to be the vertex set of that component, then $G[\mathcal{B}(S)] \in \mathcal{B}$ by maximality of $\mathcal{B}$. A separation (A, B) of G is *good* if there exists an integer $k \in [w]$ such that the following hold:

(i) $|A \cap B| = k$,
(ii) there exists a T_k model $\mathbf{X}$ in $G[A]$ such that $X \cap A \cap B \neq \emptyset$ for all $X \in \mathbf{X}$,
(iii) $\mathcal{B}(A \cap B) \subseteq B$, and
(iv) every separation (C, D) of G with $A \subseteq C$, $D \subseteq B$, and $\mathcal{B}(C \cap D) \subseteq D$ satisfies $|C \cap D| \geqslant k$.

To see that a good separation exists, choose $v \in \mathcal{B}(\emptyset)$ and consider the separation $((V(G) \setminus \mathcal{B}(\emptyset)) \cup \{v\}, \mathcal{B}(\emptyset))$ of order 1. It is easy to see that properties (i-iii) hold with $k = 1$. For property (iv), more strongly, observe that for every separation (C, D) of G with $\{v\} \subseteq C$ and $\mathcal{B}(C \cap D) \subseteq D$. If $|C \cap D| = 0$, then $\mathcal{B}(\emptyset) \subseteq D$, which implies $v \in D$, contradicting $C \cap D = \emptyset$. Consequently, $|C \cap D| \geqslant 1$.

Choose a good separation (A, B) that maximises the number $|A| - |B|$. Put $k := |A \cap B|$ and let $\mathbf{X}$ be the T_k model promised by (ii). The following claims together prove Lemma 6.4.12:

Claim 6.4.13. *There is no separation (C, D) of G of order k satisfying*

$$A \subseteq C,\ D \subseteq B,\ (C, D) \neq (A, B),\text{ and } \mathcal{B}(C \cap D) \subseteq D.$$

Subproof. Assume for a contradiction that such a separation exists. We provide Figure 6.9 for the reader's convenience. If some separation (A', B') of G with $C \subseteq A'$,

$B' \subseteq D$, and $\mathcal{B}(A' \cap B') \subseteq B'$ satisfies $|A' \cap B'| < k$. Then (A', B') contradicts property (iv) of (A, B). Thus (C, D) has property (iv). Since $|C| - |D| > |A| - |B|$, (C, D) is not good, thus (C, D) does not have property (ii).

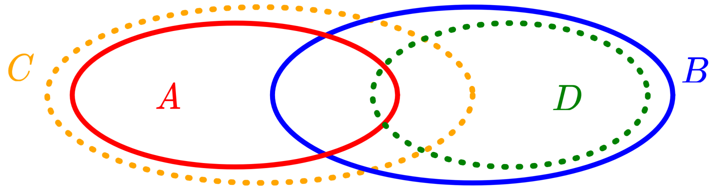

FIGURE 6.9. Layout of (A, B) and (A', B') in G.

If there exists an $(A \cap B)$-$(C \cap D)$-linkage $\mathcal{L}$ in $G[C \cap B]$ of cardinality k, then let $\mathbf{Y}$ be the collection of all sets $X \cup V(L)$, where $X \in \mathbf{X}$ and L is the unique member of $\mathcal{L}$ meeting X. Then $\mathbf{Y}$ is a T_k model in $G[C]$ such that $Y \cap C \cap D \neq \emptyset$ for all $Y \in \mathbf{Y}$, a contradiction. By Menger's Theorem, there exists a separation (P, Q) of G of order less than k, such that $A \subseteq P$ and $D \subseteq Q$. Since $\mathcal{B}(P \cap Q)$ and $\mathcal{B}(C \cap D)$ touch, $\mathcal{B}(C \cap D) \subseteq D \subseteq Q$ implies $\mathcal{B}(P \cap Q) \subseteq Q$, contradicting property (iv) of (A, B). ■

Claim 6.4.14. *$G - A$ is connected and every vertex in $A \cap B$ has a neighbour in $B \setminus A$.*

SUBPROOF. Put $C := V(G) \setminus \mathcal{B}(A \cap B)$ and $D := (A \cap B) \cup \mathcal{B}(A \cap B)$. Since $G[\mathcal{B}(A \cap B)]$ is a component of $G - A$, (C, D) is separation of G. Moreover, (C, D) is k-good. Since $A \subseteq C$ and $D \subseteq B$, the optimality of (A, B) implies $(A, B) = (C, D)$. Then $G - A = G[\mathcal{B}(A \cap B)]$, which implies $G - A$ is connected. ■

Claim 6.4.15. *Every vertex in $A \cap B$ has a neighbour in $B \setminus A$.*

SUBPROOF. Assume for a contradiction that there exists a vertex $v \in A \cap B$ such that v has no neighbour in $B \setminus A$. Consider the separation $(A, B \setminus \{v\})$. By definition of $\mathcal{B}$, $\mathcal{B}(A \cap (B \setminus \{v\}))$ and $\mathcal{B}(A \cap B)$ touch. Consequently, $\mathcal{B}(A \cap (B \setminus \{v\})) \subseteq B \setminus \{v\}$, contradicting property (iv) of (A, B). ■

Claim 6.4.16. *$k = w$.*

SUBPROOF. Assume for a contradiction that $k < w$. Let $X \in \mathbf{X}$ be the branch set of $t_k \in V(T_k)$ and v be the unique vertex of $X \cap A \cap B$. By Claim 6.4.15, v has a neighbour u in $B \setminus A$. Put $A' := A \cup \{u\}$. Then the separation (A', B) of G has order $k + 1 \in [w]$. Observe that $\mathbf{Y} := \mathbf{X} \cup \{\{u\}\}$ is a T_{k+1} model in $G[A']$ such that $Y \cap A' \cap B \neq \emptyset$ for all $Y \in \mathbf{Y}$. Since $\mathcal{B}(A' \cap B)$ and $\mathcal{B}(A \cap B)$ touch, $\mathcal{B}(A' \cap B) \subseteq B$. Moreover, if some separation (C, D) of G with $A' \subseteq C$, $D \subseteq B$, and $\mathcal{B}(C \cap D) \subseteq D$ satisfies $|C \cap D| < k + 1$. Then $|C \cap D| = k$ by property (iv) of (A, B), contradicting Claim 6.4.13. Consequently, (A', B) has property (iv). It follows that (A', B) is good. However, $|A'| - |B| > |A| - |B|$, contradicting the optimality of (A, B). ■

The following proof contains details that were missing in its original proof [83]:

Claim 6.4.17. *$A \cap B$ is linked in $G[B]$.*

Subproof. Assume for a contradiction that there exists subsets $X, Y \subseteq A \cap B$ with $|X| = |Y|$ and there is no $A\cap B$-proper (X, Y)-linkage in $G[B]$. One may choose X and Y to be disjoint. Let H be the subgraph of $G[B]$ obtained by deleting all X-Y-edges and the vertices $Z := (A \cap B) \setminus (X \cup Y)$. By assumption, Menger's Theorem implies that there exists a separation (C, D) of H of order less than $|X|$ such that $X \subseteq C$ and $Y \subseteq D$. Put $P := C \cup Z$ and $Q := D \cup Z$. Then (P, Q) is a separation of H of order less than $|X| + |Z|$. Then $(A \cup P, Q)$ is a separation of G of order $|P \cap Q| + |A \cap B| - |A \cap P|$, see Figure 6.10.

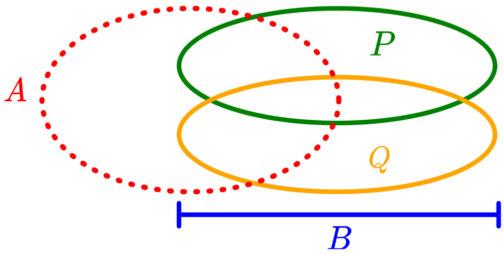

Figure 6.10. $|(A \cup P) \cap Q| = |P \cap Q| + |A \cap B| - |A \cap P|$ in the proof of Claim 8.2.10.

Since $|X| + |Z| \leqslant |A \cap P|$, $(A \cup P, Q)$ has order less than $|A \cap B|$, which equals w by Claim 6.4.16. Hence $\mathcal{B}((A \cup P) \cap Q)$ is well-defined. By property (iv) of (A, B), $\mathcal{B}((A\cup P)\cap Q) \nsubseteq Q$, and so $\mathcal{B}((A\cup P)\cap Q) \subseteq A\cup P$. Moreover, since $|X|+|Z| \leqslant |A\cap Q|$, a similar argument shows that $\mathcal{B}((A\cup Q)\cap P) \subseteq A\cup Q$. Put $S := (A\cap B)\cup(P\cap Q)$. Looking at Figure 6.10 again, one finds that $2|S| \leqslant |A\cap B|+|(A\cup P)\cap Q|+|(A\cup Q)\cap P| \leqslant 3w-2$. Hence $|S| \leqslant \frac{3}{2}w - 1$, which implies that $\mathcal{B}(S)$ is well-defined. Since $\mathcal{B}(S)$ and $\mathcal{B}(A\cap B)$ touch, $\mathcal{B}(S) \subseteq B$. Consequently, $\mathcal{B}(S) \subseteq P$ or $\mathcal{B}(S) \subseteq Q$. Without loss of generality assume $\mathcal{B}(S) \subseteq P$. Since $\mathcal{B}(S)$ and $\mathcal{B}((A \cup Q) \cup P)$ touch, $\mathcal{B}(S) \subseteq A \cup Q$. It follows that $\mathcal{B}(S) \subseteq (A \cup Q) \cap P \subseteq S$, a contradiction. ■

This concludes the proof of Lemma 6.4.12. ■

6.4.4. Proof of the Grid Minor Theorem

The following combines Lemma 6.4.9 and Lemma 6.4.12 to prove that for every integer $g \geqslant 1$, every graph G with $\operatorname{tw}(G) > g^{8g^2}$ contains the $g \times g$-grid as a minor:

Proof of Theorem 6.4.2. It may be assumed that $g \geqslant 2$. Put $h := g^2$, let H be the $g \times g$-grid. Put $m := |E(H)| - |V(H)| = 2g(g-1) - g^2$. Let T_0 be a spanning tree of H, and write $E(H) \setminus E(T_0) = \{a_1b_1, \ldots, a_{m+1}b_{m+1}\}$. Put

$$\varepsilon := \frac{1}{4(2g+1)(g^2-2)},\ d := 2g^2(2g+g^2-2),\ k_1 := \varepsilon^{-m},$$
$$k_i := d(1+d)^{i-2}k_1 \text{ for } 2 \leqslant i \leqslant m+1.$$

For each $i \in [m+1]$, add to T_0, two disjoint copies of K_{1,k_i-1} called A_i and B_i. Identify the centre vertex of A_i with a_i, and identify the centre vertex of B_i with b_i. Let T be the resulting tree, and observe the following calculation:

$$|T| = g^2 + 2\sum_{i=1}^{m+1}(k_i - 1) \leqslant g^2 + 2\sum_{i=1}^{m+1} k_i = g^2 + 2\left(\frac{1+d}{\varepsilon}\right)^m.$$

Since $(1+d)/\varepsilon \leqslant 8g^2(g^2-2)(2g+g^2)(2g+1)$ and $m = 2g(g-1) - g^2$, one finds:

$$|T| \leqslant g^2 + 2(8g^2(g^2-2)(2g+g^2)(2g+1))^{2g(g-1)-g^2} \leqslant \frac{2}{3}g^{8g^2}.$$

Put $w := |T|$, then $\mathrm{tw}(G) \geqslant \frac{3}{2}w - 1$. By Lemma 6.4.12, G has a separation (A, B) of order w such that $A \cap B$ is linked in $G[B]$, and G contains a T minor rooted at $A \cap B$. Put $Z := A \cap B$. There exists a T model $\mathbf{X}$ in $G[A]$ such that $X \cap Z \neq \emptyset$ for all $X \in \mathbf{X}$. Let G' be the graph obtained from G by producing T from $G[A]$ by deleting the vertices $A \setminus \bigcup \mathbf{X}$ and contract each $X \in \mathbf{X}$ to a vertex. Then, for all $i \in [m+1]$ there exists a Z-proper $(V(A_i), V(B_i))$-linkage in $G'[B]$ of cardinality k_i. Assume for a contradiction that G has no H minor. The following demonstrates that G has an H minor, giving a contradiction: Since G has no $g \times g$-grid minor, Lemma 6.4.9 implies that there exists pairwise vertex-disjoint Z-proper paths $P_1, \ldots, P_{m+1}$ in $G'[B]$, such that for all $i \in [m+1]$, the first vertex of P_i is in $V(A_i)$ and the last vertex is in $V(B_i)$. Then contracting $A_i \cup P_i \cup B_i$ to an edge for all $i \in [m+1]$ shows that H is a minor of G'. Hence G has an H minor, the desired contradiction. It follows that, G has an H minor. ∎

CHAPTER 7

Erdős–Pósa Property of Planar Graph Minors

Recall that a graph class $\mathcal{F}$ is said to have the Erdős–Pósa property if there exists a function $f : \mathbb{N} \to \mathbb{N}$ such that for every graph G and all integers $k \geqslant 1$, G contains k pairwise vertex-disjoint subgraphs from $\mathcal{F}$, or there exists $X \subseteq V(G)$ with $|X| \leqslant f(k)$ and $G - X$ has no subgraph from $\mathcal{F}$. Recall also that, for a graph H, $\mathcal{F}(H)$ is the class of all graphs that contain H as a minor. Section 7.1 uses the Grid Minor Theorem (Theorem 6.0.7) to prove that $\mathcal{F}(H)$ has the Erdős–Pósa property if and only if H is a planar graph. In particular, the proof relies on the following corollary of the Grid Minor Theorem, which was stated in Chapter 6:

Corollary 7.0.1. *For every planar graph H, there exists an integer $w(H)$ such that every graph G with $\operatorname{tw}(G) > w(H)$ contains H as a minor.*

Sections 7.2 to 7.3 discuss the asymptotics of the bounding function for $\mathcal{F}(H)$ when H is a planar graph. Section 7.5 discusses the asymptotically tight bounding function for $\mathcal{F}(H)$ when H is a fixed forest, and proves that $f(k) = |H|(k-1)$ is the absolutely tight bounding function for $\mathcal{F}(H)$ when H is a fixed tree.

7.1. Robertson and Seymour's Extension of the Erdős–Pósa Theorem

This section proves Robertson and Seymour's striking extension of Theorem 3.0.1, that $\mathcal{F}(H)$ has the Erdős–Pósa Property if and only if H is planar. The proof is based on Robertson and Seymour's proof [118], and begins with the following lemma for chordal graphs:

Lemma 7.1.1 (Robertson, Seymour [118])**.** *Let $m \geqslant 1$ and $\ell \geqslant 0$ be integers and G be a chordal graph. Let $Z_1, \dots, Z_m$ be pairwise disjoint independent sets in G of size ℓ. Let $x_1, \dots, x_m \geqslant 0$ be integers with $\sum_{i \in [m]} x_i = \ell$. Then there exists an independent set $X \subseteq V(G)$ such that $|X \cap Z_i| = x_i$ for all $i \in [m]$.*

Proof. We say that an independent set X as in the conclusion of the lemma is *good* for the tuple $(m, \ell, G, (Z_i)_{i\in[m]}, (x_i)_{i\in[m]})$. Proceed by induction on $|G|$. The result is trivial for $|G| = 1$, hence it may be assumed that $|G| \geqslant 2$ and the result holds for graphs on fewer vertices. The result is trivial if $m = 1$ or $\ell = 0$, hence it may be assumed that $m \geqslant 2$ and $\ell \geqslant 1$. If there exists $u \in V(G - \bigcup_{i\in[m]} Z_i)$, induction gives an $X \subseteq V(G - u)$ that is good for $(m, \ell, G - u, (Z_i)_{i\in[m]}, (x_i)_{i\in[m]})$. Such an X is good for $(m, \ell, G, (Z_i)_{i\in[m]}, (x_i)_{i\in[m]})$. Hence, it may be assumed that $V(G) = \bigcup_{i\in[m]} Z_i$. By Corollary 4.0.2, there exists $v \in V(G)$ such that $N_G(v)$ is a clique. Without loss of generality $v \in Z_m$. Since $Z_1, \dots, Z_m$ are independents sets, Z_m has no neighbour of v and each $Z_1, \dots, Z_{m-1}$ has at most one neighbour of v. For each $i \in [m-1]$, if possible choose $v_i \in Z_i$ to be a neighbour of v, otherwise choose $v_i \in Z_i$ arbitrarily. If $x_m = 0$, induction gives an $X \subseteq V(G - Z_m)$ that is good for $(m-1, \ell, G -$

$Z_m, (Z_i)_{i\in[m-1]}, (x_i)_{i\in[m-1]})$. Such an X is good for $(m, \ell, G, (Z_i)_{i\in[m]}, (x_i)_{i\in[m]})$. Hence, it may be assumed that $x_m \geqslant 1$. Put $\ell' := \ell - 1$ and $G' := G - \{v, v_1, \dots, v_{m-1}\}$. For each $i \in [m-1]$ put $Z'_i := Z_i \setminus \{v_i\}$ and $x'_i := x_i$. Put $Z'_m := Z_m \setminus \{v\}$ and $x'_m := x_m - 1$. Induction gives $X' \subseteq V(G')$ that is good for $(m, \ell', G', (Z'_i)_{i\in[m]}, (x'_i)_{i\in[m]})$. Since v has no neighbour in $\bigcup_{i\in[m]} Z'_i$, $X' \cup \{v\}$ is good for $(m, \ell, G, (Z_i)_{i\in[m]}, (x_i)_{i\in[m]})$. The result follows by induction. ■

Corollary 7.1.2 (Robertson, Seymour [118])**.** *Let $m \geqslant 1$ and $\ell \geqslant 0$ be integers and T be a tree. Let $\mathcal{F}_1, \dots, \mathcal{F}_m$ be families of subtrees of T. Let $x_1, \dots, x_m \geqslant 0$ be integers with $\sum_{i\in[m]} x_i = \ell$. Suppose that for each $i \in [m]$, there exists ℓ pairwise vertex-disjoint members of $\mathcal{F}_i$. Then for each $i \in [m]$ there exists x_i members $T^i_1, \dots, T^i_{x_i}$ of $\mathcal{F}_i$ such that $T^1_1, \dots, T^1_{x_1}, T^2_1, \dots, T^2_{x_2}, \dots, T^m_1, \dots, T^m_{x_m}$ are pairwise vertex-disjoint.*

Proof. For each $i \in [m]$ let $\mathcal{A}_i$ be a collection of ℓ pairwise vertex-disjoint members of $\mathcal{F}_i$. Let G be the intersection graph of $\bigcup_{i\in[m]} \mathcal{A}_i$. By Theorem 4.1.4, G is chordal and each $\mathcal{A}_i$ corresponds to an independent set of vertices Z_i in G of size ℓ. By Lemma 7.1.1, there exists an independent set X in G such that $|X \cap Z_i| = x_i$ for all $i \in [m]$. Without loss of generality it may be assumed that $X \subseteq \bigcup_{i\in[m]} Z_i$. Then the subtrees of T corresponding to the vertices of X form the desired family $T^1_1, \dots, T^1_{x_1}, T^2_1, \dots, T^2_{x_2}, \dots, T^m_1, \dots, T^m_{x_m}$. ■

For graphs H and integers $k \geqslant 1$, define $kH := \bigsqcup_{i=1}^k H$. Now turning to the main theorem of this section.

Theorem 7.1.3 (Robertson, Seymour [118])**.** *$\mathcal{F}(H)$ has the Erdős–Pósa Property if and only if H is planar. That is, for every planar graph H, there exists a function $f : \mathbb{N} \to \mathbb{N}$ such that for every graph G and all integers $k \geqslant 1$, G contains a kH minor or there exists $X \subseteq V(G)$ with $|X| \leqslant f(k)$ and $G - X$ is H-minor-free. Moreover, there is no such f for non-planar H.*

Proof. Suppose H is planar and write $H_1, \dots, H_m$ for the components of H. Since H is planar, kH is planar, so let $w := w(kH)$ be the integer given by Corollary 7.0.1. The result is shown for $f(k) = (mk - 1)(w + 1)$. Suppose G has no kH minor and let (T, β) be a tree-decomposition of G with width at most w. For each $i \in [m]$ put

$$\mathcal{F}_i := ((T,\beta)[U] : U \subseteq V(G), U \text{ is connected in } G, H_i \preccurlyeq G[U]),$$

where $(T, \beta)[U]$ is the subtree of T corresponding to U (see Proposition 6.0.4). Then each $\mathcal{F}_i$ is a family of subtrees of T. Put $\ell := mk$ and $x_i := k$ for all $i \in [m]$. Assume for a contradiction that for each $i \in [m]$ there exists ℓ pairwise vertex-disjoint members of $\mathcal{F}_i$. Then Corollary 7.1.2 implies for each $i \in [m]$ there exists k members $T^i_1, \dots, T^i_k$ of $\mathcal{F}_i$ such that $T^1_1, \dots, T^1_k, T^2_1, \dots, T^2_k, \dots, T^m_1, \dots, T^m_k$ are pairwise vertex-disjoint. Then for each $i \in [m]$ and $j \in [k]$ there exists a connected set $U^i_j \subseteq V(G)$ with $H_i \preccurlyeq G[U^i_j]$, such that $U^1_1, \dots, U^1_k, U^2_1, \dots, U^2_k, \dots, U^m_1, \dots, U^m_k$ are pairwise disjoint. Hence G contains kH as a minor, a contradiction. Hence, it may be assumed that $\mathcal{F}_i$ does not have ℓ pairwise vertex-disjoint members for some $i \in [m]$. By the generalised Helly property (Lemma 4.1.2) there exists $S \subseteq V(T)$ with $|S| \leqslant \ell - 1$ and S meets every member of $\mathcal{F}_i$. Put $X := \bigcup \beta(S)$, then $|X| \leqslant (mk-1)(w+1) = f(k)$ and X meets every connected set

$U \subseteq V(G)$ with $H_i \preccurlyeq G[U]$. Since H_i is connected, this implies $G - X$ is H_i-minor-free. Hence $G - X$ is H-minor-free.

The following demonstrates that the planarity of H is necessary: It suffices to show that for every non-planar graph H and every integer $\ell \geqslant 1$, there exists a graph G such that G has no $2H$ minor, and for every $X \subseteq V(G)$ with $|X| \leqslant \ell$, $G - X$ contains H as a minor.

Let g be the Euler genus of H, then H is embeddable on a surface $\mathcal{S}$ of Euler genus g. As H is non-planar, $g \geqslant 1$. Let $H_1, \ldots, H_{2\ell+1}$ be disjoint copies of H. Embed each H_i into $\mathcal{S}$ such that:

- for all $i \neq j$, $V(H_i)$ and $V(H_j)$ are disjoint sets of points in $\mathcal{S}$,
- each H_i has no edge-crossings,
- at each point of $\mathcal{S}$ at most two edges cross between different $H_1, \ldots, H_{2\ell+1}$, and these crossings occur transversally.

Draw a vertex at each point where there is an edge-crossing to obtain an embedding of a graph G in $\mathcal{S}$ without edge-crossings. If G contains $2H$ as a minor, then by the additive property of the Euler genus (Theorem 2.2.1) G has Euler genus at least $2g$. Since $g \geqslant 1$, $2g > g$, which is a contradiction since G has Euler genus at most g by construction. Hence G has no $2H$ minor. Finally, back in $\mathcal{S}$, since every point of G lies in at most two of $H_1, \ldots, H_{2\ell+1}$, any set of ℓ vertices of G avoids at least one of the H_i. ■

The growth of the bounding function in Theorem 7.1.3 depends on the Grid Minor Theorem. The version of the Grid Minor Theorem proven in this thesis (Theorem 6.4.2) produces a $O(k(2k|H|)^{8(2k|H|)^2})$ bounding function in Theorem 7.1.3. Running the same proof with the best known bound on the Grid Minor Theorem yields a $O(k^{10}\text{polylog}\,(k))$ bounding function. Once a bounding function is established, the question of optimality comes into the picture. Section 7.2 begins this discussion by examining a lower bound.

7.2. Lower Bound

This section gives a probabilistic proof that showing that every bounding function for $\mathcal{F}(K^3)$ is in $\Omega(k \log k)$. This shows the optimality of the bounding function in the Erdős–Pósa Theorem (Theorem 3.0.1). We remark that the literature often cites explicit constructions of so-called Ramanujan graphs [89, 91, 96] to derive the $\Omega(k \log k)$ lower bound. These constructions rely on algebraic concepts that are outside the scope of this thesis. For this reason, we follow Erdős and Pósa's [46] proof.

Theorem 7.2.1 (Erdős, Pósa [46])**.** *Every bounding function of $\mathcal{F}(K^3)$ is in $\Omega(k \log k)$.*

In the probability space $\mathbf{G}(n, m)$ with $m := 100n$, let A_n be the event consisting of all $G \in \mathbf{G}(n, m)$ satisfying $||G[X]|| \geqslant 2n$ for every $X \subseteq V(G)$ with $|X| = \lfloor n/2 \rfloor$. Let B_n be the event consisting of all $G \in \mathbf{G}(n, m)$ that have less than n cycles of length at most $\frac{1}{100} \log n$. The following lemmas are key to the proof of Theorem 7.2.1:

Lemma 7.2.2. $\mathbf{P}({A_n}^c) = o(1)$.

PROOF. ${A_n}^c$ is the set of all $G \in \mathbf{G}(n,m)$ for which there exists a set $X \subseteq V(G)$ with $|X| = \lfloor n/2 \rfloor$ and $||G[X]|| < 2n$. Therefore:

$$|{A_n}^c| = \binom{n}{\lfloor n/2 \rfloor} \sum_{0 \leqslant r < 2n} \binom{\binom{\lfloor n/2 \rfloor}{2}}{r} \binom{\binom{n}{2} - \binom{\lfloor n/2 \rfloor}{2}}{100n - r}. \tag{3}$$

Let n be large enough so that $r < \frac{1}{2}\binom{\lfloor n/2 \rfloor}{2}$ for all $r \in \{0, \dots, 2n\}$. Then for all $r \in \{0, \dots, 2n\}$,

$$\binom{\binom{\lfloor n/2 \rfloor}{2}}{r} \leqslant \binom{\binom{\lfloor n/2 \rfloor}{2}}{2n}. \tag{4}$$

Next, since $\binom{n}{2} = \frac{1}{2}n^2 - \frac{1}{2}n$ and $\binom{\lfloor n/2 \rfloor}{2} \leqslant \frac{1}{8}n^2$, $\binom{n}{2} - \binom{\lfloor n/2 \rfloor}{2} \geqslant \frac{3}{8}n^2 - \frac{1}{2}n$. So let n be large enough so that, $100n - r < \frac{1}{2}(\binom{n}{2} - \binom{\lfloor n/2 \rfloor}{2})$ for all $r \in \{0, \dots, 2n\}$. Then for all $r \in \{0, \dots, 2n\}$:

$$\binom{\binom{n}{2} - \binom{\lfloor n/2 \rfloor}{2}}{100n - r} \leqslant \binom{\binom{n}{2} - \binom{\lfloor n/2 \rfloor}{2}}{98n}. \tag{5}$$

Combining (3), (4), and (5) shows that the following holds for large enough n:

$$|{A_n}^c| \leqslant 2n2^n \binom{\binom{\lfloor n/2 \rfloor}{2}}{2n} \binom{\binom{n}{2} - \binom{\lfloor n/2 \rfloor}{2}}{98n}. \tag{6}$$

The standard bound $\binom{a}{b} < (\frac{ae}{b})^b$ along with $\binom{\lfloor n/2 \rfloor}{2} \leqslant \frac{1}{8}n^2$ implies that:

$$\binom{\binom{\lfloor n/2 \rfloor}{2}}{2n} < \Big(\frac{\binom{\lfloor n/2 \rfloor}{2} e}{2n}\Big)^{2n} \leqslant \Big(\frac{ne}{16}\Big)^{2n}. \tag{7}$$

Write $[a]_b = a(a-1)\cdots(a-(b-1))$ for the the falling factorial. Then:

$$\begin{aligned}
\binom{\binom{n}{2} - \binom{\lfloor n/2 \rfloor}{2}}{98n} &= \binom{\binom{n}{2}}{98n} \frac{[\binom{n}{2} - \binom{\lfloor n/2 \rfloor}{2}]_{98n}}{[\binom{n}{2}]_{98n}} \\
&\leqslant \binom{\binom{n}{2}}{98n} \left(1 - \binom{\lfloor n/2 \rfloor}{2} \Big/ \binom{n}{2}\right)^{98n} \\
&= (1 + o(1))^n \binom{\binom{n}{2}}{98n} \Big(\frac{3}{4}\Big)^{98n}. \qquad (8)
\end{aligned}$$

Next observe that:

$$\begin{aligned}
\binom{\binom{n}{2}}{98n} &= \binom{\binom{n}{2}}{100n} \frac{[100n]_{2n}}{[\binom{n}{2} - 98n]_{2n}} \\
&\leqslant \binom{\binom{n}{2}}{100n} \Big(\frac{100n}{\binom{n}{2} - 100n}\Big)^{2n} \\
&= (1 + o(1))^n \binom{\binom{n}{2}}{100n} \Big(\frac{100n}{n^2/2}\Big)^{2n} \\
&= (1 + o(1))^n \binom{\binom{n}{2}}{100n} \Big(\frac{200}{n}\Big)^{2n}. \qquad (9)
\end{aligned}$$

Combining (6), (7), (8) and (9) shows that the following holds:

$$\begin{aligned}|A_n{}^c| &< (1+o(1))^n 2n2^n\left(\frac{ne}{16}\right)^{2n}\binom{\binom{n}{2}}{100n}\left(\frac{200}{n}\right)^{2n}\left(\frac{3}{4}\right)^{98n}\\ &\leqslant (1+o(1))^n\binom{\binom{n}{2}}{100n}(200)^{2n}\left(\frac{3}{4}\right)^{98n}\\ &= o\left(\binom{\binom{n}{2}}{100n}\right).\end{aligned}$$

Dividing both sides by $\binom{\binom{n}{2}}{100n}$ shows that $\mathbf{P}(A_n{}^c) = o(1)$. ■

Lemma 7.2.3. $\mathbf{P}(B_n{}^c) = o(1)$.

Proof. Let X be the random variable that counts the number of cycles of length at at most $\frac{1}{100}\log n$. For each integer k in the range $3 \leqslant k\frac{1}{100}\log n$, there are at most n^k cycles of length k and each appears with the following probability:

$$\binom{\binom{n}{2}-k}{100n-k}\bigg/\binom{\binom{n}{2}}{100n}.$$

Hence:

$$\begin{aligned}\mathbf{E}(X) &\leqslant \sum_{3\leqslant k\leqslant\frac{1}{100}\log n} n^k\binom{\binom{n}{2}-k}{100n-k}\bigg/\binom{\binom{n}{2}}{100n}\\ &= \sum_{3\leqslant k\leqslant\frac{1}{100}\log n} n^k\frac{[100n]_k}{[\binom{n}{2}]_k}\\ &\leqslant (1+o(1))\sum_{3\leqslant k\leqslant\frac{1}{100}\log n} n^k\frac{(100n)^k}{\binom{n}{2}^k} = o(n).\end{aligned}$$

Then by Markov's inequality $\mathbf{P}(B_n{}^c) = \mathbf{P}(X \geqslant n) \leqslant \frac{1}{n}\mathbf{E}(X) = o(1)$. ■

Now turning to the proof of Theorem 7.2.1, which utilises Erdős' [45] famous trick for showing the existence of graphs with arbitrarily large girth and chromatic number:

Proof of Theorem 7.2.1. By Lemma 7.2.2 and Lemma 7.2.3, there exists an integer n_0, such that for all integers $n \geqslant n_0$, there exists a graph G with the following properties:

(i) $|G| = n$,
(ii) $||G|| = 100n$,
(iii) $||G[X]|| \geqslant 2n$ for every $X \subseteq V(G)$ with $|X| = \lfloor n/2\rfloor$, and
(iv) G has less than n cycles of length at most $\frac{1}{100}\log n$.

It suffices to show that for any integer $k \geqslant \frac{10{,}000n_0}{\log n_0}$, there exists a graph H such that H does not contain k vertex-disjoint cycles, and for any set $S \subseteq V(H)$ with $|S| < \frac{1}{20{,}000}k\log(\frac{1}{10{,}000}k) - \frac{1}{2}$, $H - S$ has a cycle.

Let n be the largest integer satisfying the following:

$$k \geqslant \frac{10,000n}{\log n}. \tag{10}$$

Since $\frac{10{,}000(n+1)}{\log(n+1)} > k$, $n+1 > \frac{1}{10{,}000}k$ and $n > \frac{1}{10{,}000}k\log(n+1) - 1$, which implies:

$$\frac{1}{2}n \geqslant \frac{1}{20,000}k\log\left(\frac{1}{10,000}k\right) - \frac{1}{2}. \tag{11}$$

Let G be graph satisfying (i), (ii), (iii), and (iv). Let H be a graph obtained from G by deleting an edge from each cycle of length at most $\frac{1}{100}\log n$. Let $\mathcal{C}$ be a maximum sized collection of pairwise vertex-disjoint cycles in H. Since $||H|| \leqslant 100n$ and $g(H) > \frac{1}{100}\log n$ and (10), one finds that:

$$|\mathcal{C}| < \frac{100n}{\frac{1}{100}\log n} \leqslant \frac{10,000n}{\log n} \leqslant k.$$

Thus H does not contain k pairwise vertex-disjoint cycles. Next, let X be a minimum sized subset of $V(H)$ such that $H-X$ is a forest. The following demonstrates that $|X| \geqslant \frac{1}{2}n$: Assume for a contradiction that $|X| < \frac{1}{2}n$. Then by property (iii), $||G-X|| \geqslant 2n$. Since $H-X$ is a forest, $||H-X|| \leqslant |H| - |X| - 1 < |H|$. On the other hand, one finds that:

$$||H-X|| \geqslant ||G-X|| - (||G|| - ||H||) > 2n - n = n,$$

a contradiction. It follows that $|X| \geqslant \frac{1}{2}n$. Then by (11),

$$|X| \geqslant \frac{1}{20,000}k\log\left(\frac{1}{10,000}k\right) - \frac{1}{2}.$$

Hence by choice of X, for any set $S \subseteq V(H)$ with $|S| < \frac{1}{20{,}000}k\log(\frac{1}{10{,}000}k) - \frac{1}{2}$, $H-S$ has a cycle. This completes the proof of Theorem 7.2.1. ■

7.3. Tight Erdős–Pósa Function for Planar Graph Minors

The topic of this section is the following asymptotically optimal result:

Theorem 7.3.1 (Cames van Batenburg, Huynh, Joret, Raymond [20])**.** *For every planar graph H, there exists an $f(k) = O(k\log k)$ bounding function for $\mathcal{F}(H)$.*

This is best possible since for $H = K^3$, the Erdős–Pósa Theorem says that every bounding function for $\mathcal{F}(K^3)$ is in $\Omega(k\log k)$. The above theorem improved the previous upper bound of $f(k) = O(k\log^d k)$ established by Chekuri and Chuzhoy [22], where, according to Cames van Batenburg, Huynh, Joret and Raymond [20], d was at least in the double digits.

The following notation is helpful in this section: For graphs G and H, define $\nu_H(G)$ to be the maximum integer $\nu \geqslant 0$ such that G contains ν pairwise vertex-disjoint subgraphs in $\mathcal{F}(H)$. Define $\tau_H(G)$ to be the minimum integer $\tau \geqslant 0$ such that there exists $X \subseteq V(G)$ with $|X| = \tau$ and $G - X$ has no subgraph in $\mathcal{F}(H)$. The *size of a model* $\mathbf{X}$ in G is the number $|\bigcup \mathbf{X}|$. The following theorem is at the heart of the proof of Theorem 7.3.1:

Theorem 7.3.2 (Cames van Batenburg, Huynh, Joret, Raymond [20])**.** *For every integer $p \geqslant 1$, every planar graph H, and every non-decreasing function $g : \mathbb{N} \to \mathbb{N}$ with $g(0) = 1$, there exists an integer $\sigma \geqslant 1$ such that for every graph G, at least one of the following holds:*

(i) G has an H model of size at most σ
(ii) G has a K^p model of size at most $\sigma\log|G|$, or

(iii) G has a separation (A, B) of order at most σ such that $G[A]$ is H-minor-free and $|A| \geqslant g(|A \cap B|)$.

The remainder of this section states technical theorems and combines them to prove Theorem 7.3.1.

Theorem 7.3.3 (Fomin, Lokshtanov, Misra, Saurabh [50]). *For every planar graph H, there exists a polynomial $P(x)$ such that for every integer $t \geqslant 1$, every graph G that is minor-minimal with $\tau_H(G) = t$ satisfies $|G| \leqslant P(t)$.*

Theorem 7.3.4 (Fiorini, Joret, Wood [49]). *For every connected planar graph H, there exists a computable function $h : \mathbb{N} \to \mathbb{N}$ such that for every graph G, if (A, B) is a separation of G such that $G[A]$ is H-minor-free and $|A| \geqslant h(|A \cap B|)$, then there exists a graph G' satisfying $\nu_H(G') = \nu_H(G)$, $\tau_H(G') = \tau_H(G)$, and $|G'| < |G|$.*

Observe that if (A, B) is a separation of G of order 0 such that $G[A]$ is H-minor-free and $|A| \geqslant 1$, then $G' := G[B]$ satisfies $\nu_H(G') = \nu_H(G)$, $\tau_H(G') = \tau_H(G)$, and $|G'| < |G|$. Hence it may be assumed that $h(0) \geqslant 1$ in Theorem 7.3.4. Furthermore, it may be assumed that the function h in Theorem 7.3.4 is non-decreasing, since the function $h'(x) := \max h(\{0, \dots, x\})$ works in place of h.

Lemma 7.3.5 (Aboulker, Fiorini, Huynh, Joret, Raymond, Sau [1]). *Let H' be a planar graph, $\mathcal{G}$ a graph class, f' be a bounding function for $(\mathcal{G}, \mathcal{F}(H'))$, and let $w := w(H')$ be the integer given by Corollary 7.0.1. If H is a minor of H with m components, then $f'(k) + (mk - 1)(w + 1)$ is a bounding function for $(\mathcal{G}, \mathcal{F}(H))$.*

The proof of Lemma 7.3.5, is essentially as follows: Suppose $G \in \mathcal{G}$ has no kH minor. Then G has no kH' minor, and so there exists $X \subseteq V(G)$ with $|X| \leqslant f'(k)$ and $G - X$ is H'-minor-free. Consequently, $\operatorname{tw}(G) \leqslant w$. Write $H_1, \dots, H_m$ for the components of H. Then a method analogous to the proof of Theorem 7.1.3 shows that there exists $Y \subseteq V(G - X')$ with $|Y| \leqslant (mk - 1)(w + 1)$ and $G - X \cup Y$ is H-minor-free. Finally, observe that $|X \cup Y| \leqslant f'(k) + (mk - 1)(w + 1)$, as required.

The following combines the technical theorems stated in this section and proves Theorem 7.3.1:

Proof of Theorem 7.3.1. We follow Cames van Batenburg, Huynh, Joret and Raymond's method [20]. It suffices to show that there exists a function $f(k) = O(k \log k)$ such that $\tau_H(G) \leqslant f(\nu_H(G))$ for every graph G.

Assume for now that H is connected. Let P be the polynomial for H given by Theorem 7.3.3. One may choose integers $\alpha, \beta \geqslant 1$ such that $P(x) \leqslant \alpha x^{\beta}$ for all $x \geqslant 1$. Let h be the function for H given by Theorem 7.3.4. As explained after the statement of Theorem 7.3.4, it may be assumed that $h(0) \geqslant 1$ and h is non-decreasing. Let $\sigma \geqslant 1$ be the integer promised by Theorem 7.3.2 when applied to $p := |H|$, H, and $g := h$. Then for every graph G, at least one the following hold:

(i) G has an H model of size at most σ,
(ii) G has a $K^{|H|}$ model of size at most $\sigma \log |G|$, or
(iii) G has a separation (A, B) of order at most σ such that $G[A]$ is H-minor-free and $|A| \geqslant h(|A \cap B|)$.

Choose a positive integer c such that $c - \sigma\beta \log c \geqslant \sigma(\log \alpha + 2\beta)$. The following demonstrates that $f(k) := ck\log(k+1)$ is a bounding function for $\mathcal{F}(H)$. Assume for a contradiction that there exists a graph G such that $\tau_H(G) > f(\nu_H(G))$. Choose such a G with $\nu_H(G)$ minimum possible. Among all such G, choose such one with $|G|$ minimum possible. Subject to the previous constraints, choose such a G with $||G||$ minimum possible. Put $k_0 := \nu_H(G)$. If (iii) holds for G, then Theorem 7.3.4 implies that there exists a graph G' satisfying $\nu_H(G') = \nu_H(G)$, $\tau_H(G') = \tau_H(G)$, and $|G'| < |G|$. However, this violates the minimality of G. Hence it may be assumed that outcome (i) or (ii) holds. In either case, G has an H model $\mathbf{X}$ of size at most $\sigma \log |G|$.

The following shows that G is minor-minimal with the property that $\tau_H(G) > f(\nu_H(G))$: If G'' is a proper-minor of G, then $|G''| < |G|$, or $|G''| = |G|$ and $||G''|| < ||G||$. Since $\nu_H(G'') \leqslant \nu_H(G)$, the choice of G implies that $\tau_H(G'') \leqslant f(\nu_H(G''))$. Additionally, since f is non-decreasing, then $\tau_H(G'') \leqslant f(\nu_H(G'')) \leqslant f(k_0)$, and so G is minor-minimal with the property that $\tau_H(G) > f(k_0)$. Next, observe that $\tau_H(G) \leqslant \tau_H(G-v)+1 \leqslant f(k_0)+1$, for any $v \in V(G)$. Consequently, $\tau_H(G) = f(k_0)+1$, and so G is minor-minimal with the property that $\tau_H(G) = f(k_0)+1$. Then applying Theorem 7.3.3 shows that $|G| \leqslant P(f(k_0)+1)$. Then:

$$\begin{aligned}
|\bigcup \mathbf{X}| \leqslant \sigma \log |G| = \sigma \log P(f(k_0)+1) &\leqslant \sigma \log(\alpha(ck_0\log(k_0+1)+1)^{\beta}) \\
&\leqslant \sigma \log(\alpha(c(k_0+1)^2)^{\beta}) \\
&= \sigma(\log \alpha + \beta \log c + 2\beta \log(k_0+1)).
\end{aligned}$$

If $\nu_H(G) = 0$, then $\tau_H(G) = 0$. Therefore, since $\tau_H(G) > f(\nu_H(G)) \geqslant 0$, $\nu_H(G) \geqslant 1$. This justifies that $|\bigcup \mathbf{X}| \leqslant \sigma(\log\alpha + \beta\log c + 2\beta)\log(k_0+1) \leqslant c\log(k_0+1)$.

Next, since $\nu_H(G - \bigcup \mathbf{X}) + 1 \leqslant \nu_H(G)$, $\nu_H(G - \bigcup \mathbf{X}) \leqslant k_0 - 1$. Since f is non-decreasing, $f(\nu_H(G - \bigcup \mathbf{X})) \leqslant f(k_0-1)$. It follows that:

$$\tau_H(G) \leqslant \tau_H(G - \bigcup \mathbf{X}) + |\bigcup \mathbf{X}| \leqslant f(k_0-1) + c\log(k_0+1) \leqslant f(k_0) = f(\nu_H(G)).$$

A contradiction.

Now suppose that H has $m \geqslant 1$ components. Let H' be a connected planar graph obtained from H by adding edges. By the previous consideration, there exists an $f'(k) = O(k\log k)$ bounding function for $\mathcal{F}(H')$. Let $w := w(H')$ be the integer given by Corollary 7.0.1. Then by Lemma 7.3.5, $f'(k) + (mk-1)(w+1) = O(k\log k)$ is a bounding function for $\mathcal{F}(H)$. ■

7.4. Proper Minor-closed Host Classes

Section 3.2 stated the following theorem:

Theorem 7.4.1 (Fomin, Saurabh, Thilikos [51])**.** *For every planar graph H and every proper minor-closed class $\mathcal{G}$, there exists a $O(k)$ bounding function for $(\mathcal{G}, \mathcal{F}(H))$.*

This section assumes the technical theorems stated in Section 7.3, and combines them to prove Theorem 7.4.1. The following proof follows Cames van Batenburg, Huynh, Joret and Raymond's method [20], and shares many steps with the proof of Theorem 7.3.1.

Proof of Theorem 7.4.1. Assume for now that H is connected. It suffices to show that there exists a function $f(k) = O(k)$ such that $\tau_H(G) \leqslant f(\nu_H(G))$ for every graph $G \in \mathcal{G}$.

Let F be a graph not in $\mathcal{G}$, then every graph in $\mathcal{G}$ is F-minor-free. Let h be the function for H given by Theorem 7.3.4. As explained after the statement of Theorem 7.3.4, it may be assumed that $h(0) \geqslant 1$ and h is non-decreasing. Let $\sigma \geqslant 1$ be the integer promised by Theorem 7.3.2 when applied to $p := |F|$, H, and $g := h$. Then for every graph G, at least one the following hold:

(i) G has an H model of size at most σ,
(ii) G has a $K^{|F|}$ model of size at most $\sigma \log |G|$, or
(iii) G has a separation (A, B) of order at most σ such that $G[A]$ is H-minor-free and $|A| \geqslant h(|A \cap B|)$.

The following demonstrates that $f(k) := \sigma k$ is a bounding function for $(\mathcal{F}(H), \mathcal{G})$. Assume for a contradiction that there exists a graph $G \in \mathcal{G}$ such that $\tau_H(G) > f(\nu_H(G))$. Choose such a G with $\nu_H(G)$ minimum possible. Among all such G, choose such one with $|G|$ minimum possible. Subject to the previous constraints, choose such a G with $||G||$ minimum possible. If (iii) holds for G, then Theorem 7.3.4 implies that there exists a graph G' satisfying $\nu_H(G') = \nu_H(G)$, $\tau_H(G') = \tau_H(G)$, and $|G'| < |G|$. However, this violates the minimality of G. Hence it may be assumed that outcome (i) or (ii) holds. However, (ii) does not hold as G is F-minor-free. Hence, G has an H model $\mathbf{X}$ of size at most σ.

Just as in the proof of Theorem 7.3.1, G is minor-minimal with the property that $\tau_H(G) > f(\nu_H(G))$: If G'' is a proper-minor of G, then $|G''| < |G|$, or $|G''| = |G|$ and $||G''|| < ||G||$. Since $\nu_H(G'') \leqslant \nu_H(G)$, the choice of G implies that $\tau_H(G'') \leqslant f(\nu_H(G''))$.

Since $\nu_H(G - \bigcup \mathbf{X}) + 1 \leqslant \nu_H(G)$ and f is non-decreasing, $f(\nu_H(G - \bigcup \mathbf{X})) \leqslant f(\nu_H(G) - 1)$. It follows that:

$$\tau_H(G) \leqslant \tau_H(G - \bigcup \mathbf{X}) + |\bigcup \mathbf{X}| \leqslant f(\nu_H(G - \bigcup \mathbf{X})) + \sigma \leqslant f(\nu_H(G) - 1) + \sigma = f(\nu_H(G)).$$

A contradiction.

Now suppose that H has $m \geqslant 1$ components. Let H' be a connected planar graph obtained from H by adding edges. By the previous consideration, there exists an $f'(k) = O(k)$ bounding function for $(\mathcal{G}, \mathcal{F}(H'))$. Let $w := w(H')$ be the integer given by Corollary 7.0.1. Then by Lemma 7.3.5, $f'(k) + (mk - 1)(w + 1) = O(k)$ is a bounding function for $(\mathcal{G}, \mathcal{F}(H))$. ■

7.5. Tight Erdős–Pósa Function for Forest Minors

The lower bound of $\Omega(k \log k)$ from Theorem 7.2.1 is not valid when H is a forest. However, bounding functions are always in $\Omega(k)$, since any set of vertices meeting every member of a vertex-disjoint packing must contain at least one vertex from each member in a maximum such packing. It turns out that the optimal bounding function for $\mathcal{F}(H)$ when H is a forest is in $\Theta(k)$. This was first shown by Fiorini, Joret, Wood [49]. A subsequent shorter proof with a better bounding function was produced by Dujmović, Joret, Micek and Morin [42].

Theorem 7.5.1 (Dujmović, Joret, Micek, Morin [42])**.** *Let F be a forest and t' be the maximum number of vertices in a component of F. Then $f(k) = |F|k - t'$ is a bounding function for $\mathcal{F}(F)$.*

Theorem 7.5.1 yields an optimal bounding function for $\mathcal{F}(T)$ when T is a tree, as shown by the following theorem:

Theorem 7.5.2 (Dujmović, Joret, Micek, Morin [42])**.** *Let T be a tree. For all integers $k \geqslant 1$ and every graph G, G contains a kT minor, or there exists $X \subseteq V(G)$ with $|X| \leqslant |T|(k-1)$ and $G - X$ is T-minor-free. Moreover, the bound on the size of X is absolutely tight for all trees T and all integers $k \geqslant 1$.*

Motivated by the formula for the bounding function in Theorem 7.5.2, we conjecture the following generalisation:

Conjecture 7.5.3. *Let F be a forest. For all integers $k \geqslant 1$ and every graph G, G contains a kF minor or there exists $X \subseteq V(G)$ with $|X| \leqslant |F|(k-1)$ and $G - X$ is F-minor-free. Moreover, the bound on the size of X is absolutely tight for all forests F and all integers $k \geqslant 1$.*

By a slight adjustment to the proof of Theorem 7.5.2 in [42], we found that Conjecture 7.5.3 holds for all forests of the form $F = cT$. More precisely, we prove the following:

Theorem 7.5.4. *Let $c \geqslant 1$ be an integer, T a tree, and $F := cT$. For every graph G and all integers $k \geqslant 1$, G contains a kF minor or there exists $X \subseteq V(G)$ with $|X| \leqslant |F|(k-1)$ and $G - X$ is F-minor-free. Moreover, the bound on the size of X is absolutely tight for every such forest F and all integers $k \geqslant 1$.*

Notice that Theorem 7.5.4 generalises Theorem 7.5.2. Recall that for graphs G and subsets $Y \subseteq V(G)$, $\partial_G Y = \{v \in Y : \exists u \in V(G-Y), uv \in E(G)\}$. The following was proven in Section 5.2 as a corollary of the proof of the Path-width Theorem (Theorem 5.2.1):

Corollary 7.5.5 (Diestel [33])**.** *For every integer $t \geqslant 1$, every t-vertex forest F, and every graph G with $\mathrm{pw}(G) \geqslant t-1$, there exists $Y \subseteq V(G)$ such that $G[Y]$ has a path-decomposition $(B_1, \dots, B_q)$ of width at most $t-1$ with $\partial_G Y \subseteq B_q$, and $G[Y]$ contains an F minor.*

The proof of the following lemma differs from the proof of Theorem 7.5.2 in [42] only by the addition of the s parameter:

Lemma 7.5.6. *Let $t \geqslant 1$ be an integer and $\mathcal{F}$ be a collection of t-vertex trees. For every graph G and all integers $k \geqslant s \geqslant 1$,*

(1) G contains k pairwise vertex-disjoint subgraphs $M_1, \dots, M_k$ each containing a minor in $\mathcal{F}$, or

(2) there exists $X \subseteq V(G)$ with $|X| \leqslant t(k-s)$ and $G - X$ does not contain s pairwise vertex-disjoint subgraphs each containing a minor in $\mathcal{F}$.

Proof. Proceed by induction on $k \geqslant 1$. In the base case of $k = 1$, it follows that $s = 1$. If (1) does not hold, then G has no minor in $\mathcal{F}$. Hence (2) holds with $X := \emptyset$

since $|X| = 0 = t(k-s)$. Now suppose $k \geqslant 2$ and the result holds for smaller values of k. It may be assumed that G has a minor in $\mathcal{F}$, otherwise (2) holds with $X := \emptyset$ since $|X| = 0 \leqslant t(k-s)$ for all $s \in [k]$.

Claim 7.5.7. *There exists $T \in \mathcal{F}$, $H \subseteq G$, and $B \subseteq V(H)$ such that the following hold:*

$$|B| \leqslant t, \qquad \partial_G V(H) \subseteq B, \qquad T \preccurlyeq H, \qquad T' \not\preccurlyeq H - B \text{ for all } T' \in \mathcal{F}.$$

Subproof. To begin, it is shown that there exists $Y \subseteq V(G)$ such that $G[Y]$ has a path-decomposition $(B_1, \dots, B_q)$ of width at most $t-1$, $\partial_G Y \subseteq B_q$, and $G[Y]$ has a minor in $\mathcal{F}$. If $\operatorname{pw}(G) \leqslant t-1$, then $Y := V(G)$ suffices since G has a minor in $\mathcal{F}$ and $\partial_G Y = \emptyset$. Otherwise $\operatorname{pw}(G) \geqslant t-1$, then choose $F \in \mathcal{F}$ such that $F \preccurlyeq G$, and let Y be given by Corollary 7.5.5.

Let $(B_1, \dots, B_q)$ be a path-decomposition of G of width at most $t-1$ such that $\partial_G Y \subseteq B_q$, and $G[Y]$ has a minor in $\mathcal{F}$. Let $\ell \in [q]$ be minimum such that $H := G[B_1 \cup \cdots \cup B_\ell]$ has a minor in $\mathcal{F}$, and choose $T \in \mathcal{F}$ such that $T \preccurlyeq H$. Put $B := B_\ell$, then $|B| \leqslant t$ and $H - B$ has no minor in $\mathcal{F}$. Furthermore, by properties of path-decompositions $\partial_G V(H) \subseteq B$. ■

Put $G' := G - V(H)$ and let $s \in [k]$. It may be assumed that $s \in [k-1]$ since the case of $s = k$ holds trivially. If G' contains $k-1$ pairwise vertex-disjoint subgraphs $M_1, \dots, M_{k-1}$ each containing a minor in $\mathcal{F}$, then taking $M_k := H$ shows that (1) holds for G. Otherwise, by induction there exists $X' \subseteq V(G')$ with $|X'| \leqslant t(k-1-s)$ and $G' - X'$ does not contain s pairwise vertex-disjoint subgraphs each containing a minor in $\mathcal{F}$. Put $X := X' \cup B$, then $|X| \leqslant t(k-1-s) + t = t(k-s)$. The following shows that (2) holds for G with X, completing the proof: Assume for a contradiction that G contains s pairwise vertex-disjoint subgraphs $M_1, \dots, M_s$ each containing a minor in $\mathcal{F}$. It may be assumed that each $M_1, \dots, M_s$ is subgraph-minimal with a minor in $\mathcal{F}$. Consequently, since the members of $\mathcal{F}$ are connected, $M_1, \dots, M_s$ are all connected. Note that $G - X = (G' - X') \cup (H - B)$. As established earlier, there exists $i \in [s]$ such that M_i is not a subgraph of $G' - X'$. Since $H - B$ has no minor in $\mathcal{F}$, M_i is not a subgraph of $H - B$. Hence M_i meets both $G' - X'$ and $H - B$. As M_i is connected and $\partial_G V(H) \subseteq B$, M_i meets B, a contradiction since $M_i \subseteq G - X \subseteq G - B$. The result follows by induction. ■

Proof of Theorem 7.5.4. Suppose G has no kF minor. Then G has no $(ck)T$ minor. Then by Lemma 7.5.6 with $\mathcal{F} := \{T\}$, for all $s \in [ck]$ there exists $X \subseteq V(G)$ with $|X| \leqslant |T|(ck-s)$ and $G - X$ has no sT minor. In particular, putting $s := c$ shows that $|X| \leqslant |T|(ck-c) = |F|(k-1)$ and $G - X$ has no F minor. It remains to show tightness. Let G be the disjoint union of $ck-1$ copies of $K^{2|T|-1}$, and $H_1, \dots, H_{ck-1}$ denote the components of G. Observe that each H_i has a T minor, but no H_i has two disjoint T minors. Since G has $ck-1$ components, G has no $(ck)T$ minor, hence no kF minor. Now let $X \subseteq V(G)$ with $|X| < |F|(k-1)$. If $|X \cap V(H_i)| \geqslant |T|$ for at least $c(k-1)$ components of G, then $|X| \geqslant |T|c(k-1) = |F|(k-1) > |X|$. Hence, it may be assumed that $|X \cap V(H_i)| \geqslant |T|$ for at most $c(k-1)-1$ components of G. Equivalently, $|X \cap V(H_i)| \leqslant |T|-1$ for at least $ck-1-(c(k-1)-1) = c$ components of G. Therefore $G - X$ has at least c complete graph components of order at least $|T|$, hence an F minor. ■

The bounds on $|X|$ in Lemma 7.5.6 motivate the following conjecture which implies the bound on $|X|$ in Conjecture 7.5.3:

Conjecture 7.5.8. *Let $c \geqslant 1$ be an integer and $T_1, \ldots, T_c$ be trees. For every graph G and all integers $k_1, \ldots, k_c, s_1, \ldots, s_c$ such that $k_i \geqslant s_i \geqslant 1$ for all $i \in [c]$, G contains a $\bigsqcup_{i\in[c]} k_i T_i$ minor, or there exists $X \subseteq V(G)$ with $|X| \leqslant \sum_{i\in[c]} |T_i|(k_i - s_i)$ and $G - X$ has no $\bigsqcup_{i\in[c]} s_i T_i$ minor.*

Lemma 7.5.6 implies the $c = 1$ case of Conjecture 7.5.8 by taking $\mathcal{F} := \{T_1\}$. The next theorem proves another special case:

Theorem 7.5.9. *Let T be a tree with at most one vertex of degree at least 3. For every graph G and all integers x_1, x_2, s_1, s_2 such that $x_i \geqslant s_i \geqslant 1$ for all $i \in [2]$, G contains an $x_1 K^1 \sqcup x_2 T$ minor, or there exists $X \subseteq V(G)$ with $|X| \leqslant (x_1 - s_1) + |T|(x_2 - s_2)$ and $G - X$ has no $s_1 K^1 \sqcup s_2 T$ minor.*

Proof. Suppose G has no $x_1 K^1 \sqcup x_2 T$ minor. If G has no $x_2 T$ minor, then Lemma 7.5.6 with $\mathcal{F} := \{T\}$ shows that there exists $X \subseteq V(G)$ with $|X| \leqslant |T|(x_2 - s_2)$ and $G - X$ has no $s_2 T$ minor, hence no $s_1 K^1 \sqcup s_2 T$ minor. Hence, it may be assumed that G contains a $x_2 T$ minor. Since T has at most one vertex of degree at least 3, G contains a subgraph H isomorphic to $x_2 T$. Let $H_1, \ldots, H_{x_2}$ be the components of H. Since G has no $x_1 K^1 \sqcup x_2 T$ minor, there are at most $x_1 - 1$ vertices in $V(G) \setminus V(H)$. Let $X' \subseteq V(G) \setminus V(H)$ be a set of at most $x_1 - s_1$ vertices so that there are at most $s_1 - 1$ vertices in $V(G) \setminus (V(H) \cup X')$. Put $X := X' \cup V(\bigcup_{i=1}^{x_2 - s_2} H_i)$. Then $|G - X| \leqslant (s_1 - 1) + \sum_{i=x_2-s_2+1}^{x_2} |T| = s_1|K^1| + s_2|T| - 1$, hence $G - X$ has no $s_1 K^1 \sqcup s_2 T$ minor. Finally, verify $|X| \leqslant (x_1 - s_1) + \sum_{i=1}^{x_2-s_2} |T| = (x_1 - s_1) + |T|(x_2 - s_2)$. ■

CHAPTER 8

Coarse Erdős–Pósa Conjecture

Coarse graph theory combines graph-theoretic notions and coarse geometry [55] following Gromov's [58] study of geometric group theory. Gromov's asymptotic dimension is a notion of "large-scale" dimension of a metric space that is invariant under some controlled "distortion of distances". Coarse graph theory attempts to unify much of the recent work [8, 17, 52, 105] blending asymptotic dimension and various graph-theoretic notions to study the "large-scale geometry" of graphs. In this chapter, Section 8.1 briefly introduces the necessary coarse graph theory background and states Georgakopoulos and Papasoglu's Coarse Erdős–Pósa Conjecture, and Sections 8.2 to 8.3 detail some recent steps towards the conjecture. Finally, Section 8.4 discusses the need for coarse analogues of tools from structural graph theory, and proves a recent step in that direction related to graphs with bounded tree-width.

8.1. Coarse Graph Theory

Initiated by Georgakopoulos and Papasoglu [55], *coarse graph theory* is an emerging field concerned with "large-scale geometry" of graphs, how graphs look when viewed "from far away", and the properties that are preserved under some controlled "distortion of distances". The precise notion of "distortion" used in coarse graph theory is described by the following definition:

For reals $M \geqslant 1$ and $A \geqslant 0$, an *(M, A)-quasi-isometry* of a metric-space (X, d) into a metric-space (Y, ρ) is a function $\varphi : X \to Y$ such that the following hold:

- *(Q1) Distortion.* $d(u, v)\frac{1}{M} - A \leqslant \rho(\varphi(u), \varphi(v)) \leqslant d(u, v)M + A$ for all $u, v \in X$.
- *(Q2) Almost onto.* For every $u \in Y$ there exists $v \in X$ such that $\rho(u, \varphi(v)) \leqslant A$.

If such a function φ exists, (X, d) is said to be *(M, A)-quasi-isometric* to (Y, ρ).

Loosely speaking, quasi-isometries send points that are far apart to points that are far apart, and send points that are close together to points that are close together. How far (or close) is controlled by the multiplicative and additive constants M and A. Between graphs, a $(1, 0)$-quasi-isometry is a graph isomorphism, hence quasi-isometries generalise graph isomorphisms.

(X, d) is said to be *quasi-isometric* to (Y, ρ) if there exists reals $M \geqslant 1$ and $A \geqslant 0$ such that (X, d) is (M, A)-quasi-isometric to (Y, ρ). Quasi-isometry is an equivalence relation on any set of metric-spaces.

A motivation for coarse graph theory from the perspective of applicability is the following [54]: It is natural to study a "continuous space" via a discretisation of the space. It turns out that graphs are universal in this respect. That is, any geodesic metric-space (e.g. any connected Riemannian manifold) is quasi-isometric to a connected (possibly infinite) graph [7]. Thus, one could learn about the original space by studying properties of such a discretisation, provided said properties are quasi-isometry invariant. Infinite graphs are discussed later in Section 8.4. Thankfully, coarse graph theory comes in another flavour, one that is closer to the content of this thesis seen thus far: Graph classes. A graph class $\mathcal{G}$ is *quasi-isometric* to a graph class $\mathcal{H}$ if there exists reals $M \geqslant 1$ and $A \geqslant 0$ such that every member of $\mathcal{G}$ is (M, A)-quasi-isometric to some member of $\mathcal{H}$.

Traditional graph-theoretic notions, such as chromatic number or containing a fixed graph H as a minor, may not be quasi-isometry invariants. In such cases, a "coarse version" of that notion is established; usually by replacing disjointness with being far apart. Gromov's [58] asymptotic dimension is a notion of dimension for metric spaces that only sees the "large-scale geometry" of the space. For example, consider the following space described in Chapter 1, that was obtained from $\mathbb{R} \times \mathbb{R}$ by placing at each point of $\mathbb{Z} \times \mathbb{Z}$, a ball from $\mathbb{R}^{100}$. This space, as depicted in Figure 8.1, looks to be 100-dimensional at "small-scales", however, when "zoomed out" the space looks like a plane.

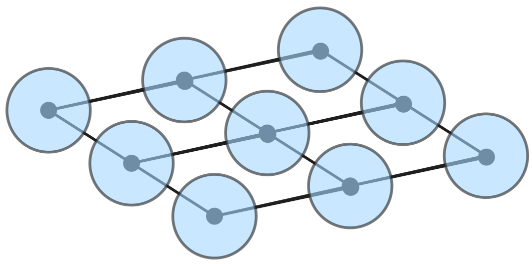

FIGURE 8.1. High dimension at small-scale and low dimension at large-scale.

See [7] for a survey on asymptotic dimension of metric spaces and its application to group theory. For coarse graph theory, asymptotic dimension, and the stricter Assouad–Nagata dimension, measure the "dimension" of a graph class. In some sense, these dimension parameters are coarse versions of the chromatic number. They are defined below for graph classes, and can be defined more generally for metric-spaces.

For integers $r \geqslant 1$ and graphs G, G^r is the graph with vertex set $V(G)$ and edge set $\{uv \in \binom{V(G)}{2} : \text{dist}_G(u, v) \leqslant r\}$. For subsets $X \subseteq V(G)$, the *weak-diameter* of X in G is the number $\max\{\text{dist}_G(u, v) : (u, v) \in X \times X\}$. With respect to a colouring c of G, a subgraph-maximal monochromatic connected subgraph of G is a *monochromatic component* of G.

The *asymptotic dimension* of a graph class $\mathcal{G}$, denoted by $\text{asdim}(\mathcal{G})$, is the minimum integer $n \geqslant 0$ for which there exists a function $D : (0, +\infty) \to (0, +\infty)$, such that for every graph $G \in \mathcal{G}$ and every integer $r \geqslant 1$, there exists a colouring $c : V(G) \to [n+1]$, such that every monochromatic component of G^r has weak-diameter at most $D(r)$. If no such integer exists, then $\text{asdim}(\mathcal{G}) = +\infty$.

The *Assouad–Nagata dimension* of a graph class $\mathcal{G}$, denoted by $\text{ANdim}(\mathcal{G})$, is defined the same as asymptotic dimension except that the function D must satisfy $D(r) \leqslant ar$ for all integers $r \geqslant 1$ and some constant $a \in [0, +\infty)$. Thus $\text{asdim}(\mathcal{G}) \leqslant \text{ANdim}(\mathcal{G})$.

Fujiwara and Papasoglu proved a coarse four[1] colour theorem:

[1] the analogy works as follows: ANdim + 1 is the number of colours. See c in the definition of ANdim.

Theorem 8.1.1 (Fujiwara, Papasoglu [52]). *The class of planar graphs has Assouad–Nagata dimension at most* 3*.*

They conjectured a coarse three colour theorem. Bonamy, Bousquet, Esperet, Groenland, Liu, Pirot, and Scott showed something stronger:

Theorem 8.1.2 (Bonamy, Bousquet, Esperet, Groenland, Liu, Pirot, Scott [17]). *For every integer $g \geqslant 0$, the class of all graphs embeddable in a surface of Euler genus g has Assouad–Nagata dimension at most* 2*.*

The case of $g = 0$ in Theorem 8.1.2 was independently shown by Jørgensen, Lang [72]. Bonamy et al. also proved the following:

Theorem 8.1.3 (Bonamy, Bousquet, Esperet, Groenland, Liu, Pirot, Scott [17]). *Let $\mathcal{F}$ be a minor-closed graph class. Then the following hold:*

- *if $\mathcal{F}$ is a proper minor-closed class, then* $\operatorname{asdim}(\mathcal{F}) \leqslant 2$*, and*
- *if $\mathcal{F}$ has bounded tree-width, then* $\operatorname{asdim}(\mathcal{F}) \leqslant 1$*.*

Results obtained by Distel [38] and by Liu [85] show that Theorem 8.1.3 remains true if "asdim" is replaced by "ANdim".

The following definition is of the coarse version of minors: Let $d \geqslant 1$ be an integer, and G and H be graphs. Suppose $(X_v : v \in V(H)) \cup (P_e : e \in E(H))$ is a collection of (path-)connected subgraphs of G such that the following hold:

- if v is an end of e, then $V(X_v) \cap V(P_e) \neq \emptyset$, and
- for all distinct $A, B \in (X_v : v \in V(H)) \cup (P_e : e \in E(H))$ which are not covered by the first condition, $\operatorname{dist}_G(A, B) \geqslant d$.

Then G is said to contain H as a *d-fat minor*. A graph class $\mathcal{G}$ is said to contain a graph H as an *asymptotic minor*, denoted by $H \prec^{\infty} \mathcal{G}$, if for all integers $d \geqslant 1$, there exists a graph $G \in \mathcal{G}$ such that G contains H as a d-fat minor.

Informally, a minor in a graph is similar to a subgraph except that it is *allowed* to be "spread out". A fat minor on the other hand is *required* to be "spread out".

One of the most famous conjectures in graph theory is Hadwiger's Conjecture [62], which states that $K^{\chi(G)} \preccurlyeq G$, for every graph G. Simply put, Hadwiger's Conjecture tries to generalise the Four Colour Theorem. The depth and history of this conjecture is outside the scope of this thesis, as such, we refer the reader to Seymour's survey [125] on Hadwiger's Conjecture. Georgakopoulos and Papasoglu [55] conjectured a coarse version of Hadwiger's Conjecture, a coarse version of the Kuratowski–Wagner Theorem (Theorem 2.2.2), and much more. We state their conjectures in terms of graphs and graph classes, and remark that they were originally stated more generally for length-spaces.

Conjecture 8.1.4 (Coarse Hadwiger's Conjecture. Georgakopoulos, Papasoglu [55]). *For every graph class $\mathcal{G}$,*

$$K^{\operatorname{ANdim}(\mathcal{G})+1} \prec^{\infty} \mathcal{G}.$$

Conjecture 8.1.5 (Coarse Kuratowski–Wagner Conjecture. Georgakopoulos, Papasoglu [55]). *A graph class $\mathcal{G}$ is quasi-isometric to a planar graph if and only if it has no asymptotic K^5 minor and no asymptotic $K_{3,3}$ minor.*

This concludes the preamble on basic motivations and definitions of coarse graph theory. Now the reader is well equipped to appreciate the following conjecture of Georgakopoulos, Papasoglu [55], which relates coarse graph theory with the topic of this thesis:

Conjecture 8.1.6 (Coarse Erdős–Pósa Conjecture. Georgakopoulos, Papasoglu [55])**.** *There exists a function $f : \mathbb{N} \to \mathbb{N}$ and a constant $c > 0$, such that for all integers $n \geqslant 1$ and $d \geqslant 1$, every graph G contains nK^3 as a d-fat minor, or there exists a subset $X \subseteq V(G)$ with $|X| \leqslant f(n)$ and $G - B_G(X, cd)$ contains no K^3 as a d-fat minor.*

More on coarse graph theory is found in Section 8.4, which discusses the sorts of tools that may be needed to tackle the the various conjectures in coarse graph theory, including conjectures stated in this section. The immediate Section 8.2 proves a recent step in the direction of the Coarse Erdős–Pósa Conjecture.

8.2. Erdős–Pósa Property of Cycles that are Far Apart

This section discusses a step towards the Coarse Erdős–Pósa Conjecture (Conjecture 8.1.6). In a recent paper by Dujmović, Joret, Micek and Morin [41], they proved the following theorem for $f(k) = O(k^{18} \text{polylog}\, k)$ and $g(d) = 19d$:

Theorem 8.2.1 (Dujmović, Joret, Micek, Morin [41])**.** *There exists functions $f, g : \mathbb{N} \to \mathbb{N}$ such that for every graph G and all integers $k, d \geqslant 1$, G contains k cycles at pairwise distance greater than d, or there exists $X \subseteq V(G)$ with $|X| \leqslant f(k)$ and $G - B_G(X, g(d))$ is a forest.*

Ahn, Gollin, Huynh and Kwon [2] solved Theorem 8.2.1 in the case of $k = 2$ and arbitrary d, plus the case of $d = 1$ and arbitrary k. Both sets of authors conjecture that Theorem 8.2.1 remains true with $f(k) = O(k \log k)$. Dujmović et al. [41] asked what is the smallest possible function g for which Theorem 8.2.1 is true? In Section 8.3 we show that $g(d) \geqslant d$ is necessary.

The following "Helly type" theorem for forests was key to the proof of Theorem 8.2.1:

Theorem 8.2.2 (Dujmović, Joret, Micek, Morin [41])**.** *There exists a function $\ell^\star : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, $\ell^\star(k, c) = O(k^{c!\,c})$ such that the following is true: For all integers $c \geqslant 1$, for every forest F and collection $\mathcal{A}$ of non-null subgraphs of F each with at most c components, $(\{F\}, \mathcal{A})$ has the Erdős–Pósa property with bounding function $\ell^\star(\cdot, c)$.*

Theorem 8.2.2 was originally sketched by Gyárfás and Lehel [61]. A contribution of this thesis is pointing out that in the proof of Theorem 8.2.1, one may use the following result of Alon instead of Theorem 8.2.2 to show that Theorem 8.2.1 holds with $f(k) = O(k^3 \log k)$ and $g(d) = 19d$:

Theorem 8.2.3 (Alon [5])**.** *For all integers $k, c \geqslant 1$, for every forest F and collection $\mathcal{A}$ of non-null subgraphs of F each with at most c components, at least one of the following holds:*

- *(a) $\mathcal{A}$ has k pairwise vertex-disjoint members, or*
- *(b) there exists a subset $X \subseteq V(F)$ with $|X| \leqslant 2c^2(k-1)$ and X meets every member of $\mathcal{A}$.*

We remark that Theorem 8.2.3 as originally stated by Alon [5] assumes F is a tree. The following argument shows how one obtains Theorem 8.2.3 from Alon's result on trees: Suppose F is a forest and $\mathcal{A}$ is a collection of non-null subgraphs of F each with at most c components. If $\mathcal{A}$ has k pairwise vertex-disjoint members, then (a) holds, so it may be assumed that this is not the case. Choose a tree F' that contains F as a spanning subgraph. Then $\mathcal{A}$ is a collection of non-null subgraphs of F' each with at most c components. Since $\mathcal{A}$ does not have k pairwise vertex-disjoint members, there exists a subset $X \subseteq V(F')$ with $|X| \leqslant 2c^2(k-1)$ and X meets every member of $\mathcal{A}$. In this case, (b) holds.

As mentioned earlier, using Theorem 8.2.3 instead of Theorem 8.2.2, one may prove Theorem 8.2.1 with $f(k) = O(k^3 \log k)$ and $g(d) = 19d$. To demonstrate this, the remainder of this section states the main definitions and technical lemmas from [41] and combines them to prove Theorem 8.2.1 with $f(k) = O(k^3 \log k)$ and $g(d) = 19d$. Additionally, the remainder of this section also demonstrates a connection between Simonovits' calculation [127] (as in Lemma 3.1.2) and the proof of Theorem 8.2.1.

We remark that we spotted several typos in [41], and found that Claim 8.2.9 (in the proof of Theorem 8.2.1) had some missing details. As such, we notified the authors who corrected the typos and added the details.

Let $d \geqslant 0$ be an integer. A set of cycles $\mathcal{C}$ in a graph G is called a *d-packing* if the members of $\mathcal{C}$ are at pairwise distance greater than d, that is $\operatorname{dist}_G(V(C_1), V(C_2)) > d$ for all distinct $C_1, C_2 \in \mathcal{C}$. A graph G is *unicyclic* if G contains at most one cycle. Let $r \geqslant 0$ be an integer. A cycle C in a graph G is said to be *r-unicyclic* in G if $G[B_G(V(C), r)]$ is unicyclic. Let C be a cycle in a connected graph G. A *C-rooted spanning BFS-unicycle* in G is a spanning unicyclic subgraph U of G such that $C \subseteq U$ and $\operatorname{dist}_G(V(C), v) = \operatorname{dist}_U(V(C), v)$ for all $v \in V(G)$. The first tool gives a way to process a given cycle to either produce an r-unicyclic cycle close to C, or a short cycle close to C.

Lemma 8.2.4 (Dujmović, Joret, Micek, Morin [41]). *Let $r \geqslant 0$ be an integer and G a graph. For every cycle C in G,*

(a) G has an r-unicyclic cycle C' with $V(C') \subseteq B_G(V(C), 2r)$, or
(b) G has a cycle C' of length at most $6r + 2$ with $V(C') \subseteq B_G(V(C), 3r)$.

The second tool is the main lemma of the paper [41]. Recall the following definition from Chapter 3:

$$s(k) := \begin{cases} 4k(\log k + \log\log k + 4) & \text{if } k \geqslant 2, \\ 2 & \text{if } k = 1. \end{cases}$$

Lemma 8.2.5 (Dujmović, Joret, Micek, Morin [41]). *Let d, r, k be integers such that $r \geqslant d \geqslant 1$ and $k \geqslant 1$. Let G be graph and $\mathcal{C}$ a $2d$-packing of d-unicyclic cycles in G. Then*

(a) G has a d-packing of k cycles, or
(b) there exists $X, Y \subseteq V(G)$ such that the following hold:
(i) for all $C \in \mathcal{C}$ there exists a C-rooted spanning BFS-unicycle U of $G[B_G(V(C), r)]$ such that Y contains an endpoint of each edge in $G[B_G(V(C), r)] - E(U)$,

(ii) $B_G(V(C),r) \cap B_G(V(C'),r) \subseteq Y$ *for all distinct* $C, C' \in \mathcal{C}$, *and*
(iii) $|X| \leqslant 2k^2 + (\binom{k}{2}+k)s(k)$ *and* $Y \subseteq B_G(X, 2r+d)$.

The following combines Theorem 8.2.3 and Lemmas 8.2.4 to 8.2.5 to prove that Theorem 8.2.1 holds with $f(k) = O(k^3 \log k)$ and $g(d) = 19d$:

Proof of Theorem 8.2.1. Let f and g be the following functions:

$$f(x) = -18 + 20x + 2x^2 + (\tbinom{x}{2} + x + 9)s(x), \qquad g(x) = 19x.$$

The following shows that for every graph G and all integers $k, d \geqslant 1$, G contains a d-packing of k cycles, or there exists $X \subseteq V(G)$ with $|X| \leqslant f(k)$ and $G - B_G(X, g(d))$ is a forest.

We follow Dujmović, Joret, Micek and Morin's method [41]. It may be assumed that G has no d-packing of k cycles. Let $\mathcal{C} := \{C_1, \dots, C_p\}$ be a maximal $2d$-packing of d-unicyclic cycles in G. Consider the following iterative process which produces a (possibly empty) d-packing of cycles $\mathcal{D} := \{D_1, D_2, \dots\}$ each of length at most $6d+2$: Let D_1 be any cycle in G of length at most $6d+2$ or stop the process if no such cycle exists. Inductively, assume $D_1, \dots, D_j$ have already been defined for some integer $j \geqslant 1$. Either every cycle of G meets $B_G(V(\bigcup \mathcal{C} \cup \bigcup_{i\in[j]} D_i), 4d)$, or there exists a cycle D which avoids it. Stop the process in the former case. In the latter case, apply Lemma 8.2.4 to find (a) a d-unicyclic cycle D' with $V(D') \subseteq B_G(V(D), 2d)$, or (b) a cycle D' of length at most $6d+2$ with $V(D') \subseteq B_G(V(D), 3d)$. In case (a), D' avoids $B_G(V(\bigcup \mathcal{C}), 2d)$, thus $\mathcal{C} \cup \{D'\}$ contradicts the maximality of $\mathcal{C}$. In case (b), D' avoids $B_G(V(\bigcup_{i\in[j]} D_i), d)$, so put $D_{j+1} := D'$. This completes the description of the process.

Put $Y_0 := B_G(V(\bigcup \mathcal{D}), 4d)$. For each $i \in [q]$ arbitrarily choose a vertex $x_i \in V(D_i)$, then $V(D_i) \subseteq B_G(x_i, 3d+1)$. Therefore, putting $X_0 := \{x_i : i \in [q]\}$, shows that:

$$Y_0 \subseteq B_G(X_0, 4d+1). \tag{12}$$

Put $r := 6d$. Applying Lemma 8.2.5 to $\mathcal{C}$, produces subsets $X_1, Y_1 \subseteq V(G)$ such that the following hold: For each $i \in [p]$ there exists a C_i-rooted spanning BFS-unicycle U_i of $G[B_G(V(C_i), r)]$ such that Y_1 contains an endpoint of each edge in $G[B_G(V(C_i), r)] - E(U_i)$; for all distinct $i, j \in [p]$:

$$B_G(V(C_i), r) \cap B_G(V(C_j), r) \subseteq Y_1; \tag{13}$$

$$|X_1| \leqslant 2k^2 + (\tbinom{k}{2} + k)s(k); \tag{14}$$

$$Y_1 \subseteq B_G(X_1, 2r+d). \tag{15}$$

For each $i \in [p]$ arbitrarily choose a vertex $y_i \in V(C_i)$, and put $X_2 := Y_2 := \{y_1, \dots, y_p\}$. Define $\widehat{Y} := Y_0 \cup Y_1 \cup Y_2$ and $\widehat{X} := X_0 \cup X_1 \cup X_2$. Then (12) and (15) imply:

$$\widehat{Y} \subseteq B_G(X_0, 7d+1) \cup B_G(X_1, 2r+d) \cup X_2 \subseteq B_G(\widehat{X}, 13d). \tag{16}$$

Since G has no d-packing of k cycles, p and $q := |\mathcal{D}|$ are less than k. Then (14) implies:

$$|\widehat{X}| \leqslant |X_0| + |X_1| + |X_2| < 2k + 2k^2 + (\tbinom{k}{2} + k)s(k). \tag{17}$$

Define $F_0 := G - \widehat{Y} \cup B_G(V(\bigcup \mathcal{C}), r - 2d)$ and $\widetilde{F_0} := G - \widehat{Y} \cup B_G(V(\bigcup \mathcal{C}), r - d)$. Note that by the stopping condition of the process producing $\mathcal{D}$, every cycle of $G - Y_0$ meets

$B_G(V(\bigcup \mathcal{C}), r-2d)$. This along with $Y_0 \subseteq \widehat{Y}$ shows that F_0 is an induced forest in G. Moreover, $\widetilde{F}_0$ is an induced forest in F_0.

The remainder of the proof relies on the following definitions. For $i \in [p]$, an edge $e \in E(G)$ between $B_G(V(C_i), r-d)$ and $V(\widetilde{F}_0)$ is an *i-exit edge*. For $i, j \in [p]$ and $e, \epsilon \in E(G)$, the tuple (e, i, ϵ, j) is *good* if the following hold:

- $e \neq \epsilon$,
- e is an i-exit edge,
- ϵ is a j-exit edge, and
- e and ϵ are incident to the same component of $\widetilde{F}_0$.

For a good tuple $t := (e, i, \epsilon, j)$, there is a unique walk $W_t := P_1 e P_0 \epsilon P_2$ in G where

- P_1 is the unique shortest path in U_i from $V(C_i)$ to the end of e in $B_G(V(C_i), r-d)$,
- P_0 is the unique path in $\widetilde{F}_0$ from the end of e in $V(\widetilde{F}_0)$ to the end of ϵ in $V(\widetilde{F}_0)$, and
- P_2 is the unique shortest path in U_j from the end of ϵ in $B_G(V(C_j), r-d)$ to $V(C_j)$.

The paths P_0, P_1, P_2 are the *forest leg*, *first leg*, and *second leg* of t respectively. Then both the first and second legs have length $r-d$, and the forest leg may be arbitrarily long. The walk $eP_0\epsilon$ is the *extended forest leg* of t. A good tuple t satisfying $B_G(V(W_t), d) \cap \widehat{Y} = \emptyset$ is *admissible*.

Before proceeding, we briefly sketch the remainder of the proof. Claim 8.2.8 shows that particular forest subgraphs $F_0, F_1, \ldots, F_p$ of G cumulatively contain all d-radius balls centred at legs of admissible tuples. Moreover, if Ψ is the set of such balls, then each member of Ψ is contained entirely in some $V(F_i)$. Define a forest $F = \bigsqcup_{i=0}^{p} F_i$, then there are inclusion maps $\pi_i : V(F_i) \to V(F)$. Define $\mathcal{A}$ to be the images of the members of Ψ under the π_i's. Then applying Theorem 8.2.3 to F and $\mathcal{A}$, one obtains $k + \frac{1}{2}s(k)$ pairwise vertex-disjoint members of $\mathcal{A}$, or a small set $X^\star \subseteq V(F)$ meeting every member of $\mathcal{A}$. In the former case, with the help of Lemma 3.1.2, one finds a d-packing of k cycles in G. In the latter case, inverting the inclusion maps on $X^\star$ produces a subset $X_3 \subseteq V(G)$, which by virtue of Claim 8.2.10 is close enough to the extended forest legs of all admissible tuples. Then any cycle in G that contains the extended forest leg of an admissible tuple is close to X_3, thus such a cycle is hit by $B_G(X_3, g(d))$. On the other hand, Claim 8.2.6 implies that the cycles in G that do not contain the extended forest leg of an admissible tuple are close to $\widehat{X}$. Overall, one finds that $G - B_G(\widehat{X} \cup X_3, g(d))$ is a forest.

Claim 8.2.6. *Every cycle C in G that avoids $B_G(\widehat{Y}, r)$ contains the extended forest leg of some admissible tuple.*

Subproof. Since $\widetilde{F}_0$ is an induced forest in G, $V(C)$ is not a subset of $V(\widetilde{F}_0)$. Thus $V(C) \cap B_G(V(C_i), r-d) \neq \emptyset$ for some $i \in [p]$. Since $Y_1 \cup \{y_i\} \subseteq \widehat{Y}$, $G[B_G(V(C_i), r-d)] - \widehat{Y}$ is a forest. Then $V(C) \cap \widehat{Y} = \emptyset$ implies $V(C)$ is not a subset of $B_G(V(C_i), r-d)$. Hence C has an edge v_0v_1 where $v_0 \in B_G(V(C_i), r-d)$ and $v_1 \notin B_G(V(C_i), r-d)$. Note that $v_1 \in B_G(V(C_i), r)$ since $d \geqslant 1$. One finds $v_1 \notin B_G(V(C_j), r)$ for all $j \in [p] \setminus \{i\}$, otherwise (13) implies $v_1 \in Y_1 \subseteq \widehat{Y}$, contradicting $V(C) \cap \widehat{Y} = \emptyset$. It follows that $v_1 \in \widetilde{F}_0$, so v_0v_1 is an i-exit edge.

Consider a maximal path P in $C \cap \widetilde{F}_0$ starting at v_1, say $P = (v_1, \ldots, v_n)$. Let v_{n+1} be the neighbour of v_n in C different from v_{n-1}. Note that v_{n+1} is not a vertex of $\widetilde{F}_0$, otherwise $v_nv_{n+1} \in E(\widetilde{F}_0)$ ($\widetilde{F}_0$ is an induced subgraph of G), then $v_{n+1} \in \{v_1, \ldots, v_{n-1}\}$ by maximality of P, and so $\widetilde{F}_0$ has a cycle, a contradiction. Furthermore, $v_{n+1} \in V(C)$ implies $v_n \notin \widehat{Y}$. It follows that $v_{n+1} \in B_G(V(C_j), r-d)$ for some $j \in [p]$. Hence v_nv_{n+1} is a j-exit edge, and $t := (v_0v_1, i, v_nv_{n+1}, j)$ is a good tuple whose extended forest leg is contained in C.

It suffices to show that $B_G(V(W_t), d) \cap \widehat{Y} = \emptyset$. Assume not. Let $u \in \widehat{Y}$ such that $\mathrm{dist}_G(V(W_t), u) \leqslant d$. Let P_1 and P_2 be the first and second legs of t respectively, and $P_0 := P$ be the forest leg of t. If $u \in B_G(V(P_1), d)$, then there exists $w \in V(P_q)$ such that $\mathrm{dist}_G(u, w) \leqslant d$. Then $\mathrm{dist}_G(u, v_0) \leqslant \mathrm{dist}_G(u, w) + \mathrm{dist}_G(w, v_0) \leqslant d + (r-d) = r$, which implies $v_0 \in V(C) \cap B_G(\widehat{Y}, r)$, a contradiction. Hence $u \notin B_G(V(P_1), d)$, and similarly one finds $u \notin B_G(V(P_2), d)$. It follows that $u \in B_G(V(P_0), d)$. However, since $P_0 \subseteq C$, $B_G(V(P_0), r) \cap \widehat{Y} = \emptyset$, which implies $u \in B_G(V(P_0), d) \cap \widehat{Y} = \emptyset$, a contradiction. ■

Claim 8.2.7. *Let t be an admissible tuple. Then only the endpoints of W_t meet $\bigcup \mathcal{C}$.*

Subproof. Write $t = (e, i, \epsilon, j)$. Let x be the unique vertex in $V(W_t) \cap V(C_i)$, and let y be the unique vertex in $V(W_t) \cap V(C_j)$. Assume for a contradiction that $W_t - x - y$ meets $\bigcup \mathcal{C}$. Let $v \in V(W_t) \setminus \{x, y\}$ and $C \in \mathcal{C}$ such that $v \in V(C)$. By definition, $V(\widetilde{F}_0)$ is disjoint from $V(\bigcup \mathcal{C})$. Hence v is not in the forest leg of t, which implies v lies in the first or second leg of t. Consequently, $\mathrm{dist}_G(V(C), V(\widetilde{F}_0)) < r - d$, contradicting the definition of $\widetilde{F}_0$. ■

Let $t = (e, i, \epsilon, j)$ be an admissible tuple with the usual notation for W_t. For each $\ell \in \{0, \ldots, p\}$, define the following:

$$\Psi_\ell(t) := \begin{cases} B_G(V(P_0), d) & \text{if } \ell = 0, \\ B_G(V(P_1), d) & \text{if } \ell = i \text{ and } \ell \neq j, \\ B_G(V(P_2), d) & \text{if } \ell \neq i \text{ and } \ell = j, \\ B_G(V(P_1 \cup P_2), d) & \text{if } \ell = i \text{ and } \ell = j, \\ \emptyset & \text{if } \ell \notin \{0, i, j\}. \end{cases}$$

For each $i \in [p]$ put $F_i := G[B_G(V(C_i), r)] - \widehat{Y}$ and $\widetilde{F}_i := G[B_G(V(C_i), r-d)] - \widehat{Y}$. Since $Y_1 \cup Y_2 \subseteq \widehat{Y}$, F_i is a forest, and so is $\widetilde{F}_i$.

Claim 8.2.8. *Let t be an admissible tuple. Then*

(a) $G[\Psi_0(t)]$ is a connected subgraph of F_0, and

(b) for each $\ell \in [p]$, $G[\Psi_\ell(t)]$ is a subgraph of F_ℓ and has $c_\ell \in \{0, 1, 2\}$ components. Moreover, $\sum_{\ell \in [p]} c_\ell \leqslant 2$.

Subproof. Assume the usual notation for t. To see (a), note that $\Psi_0(t) = B_G(V(P_0), d)$ is connected in G, thus $G[\Psi_0(t)]$ is connected. Since t is admissible, $G[\Psi_0(t)]$ avoids $\widehat{Y}$. Since $P_0 \subseteq \widetilde{F}_0$, P_0 avoids $B_G(V(\bigcup \mathcal{C}), r-d)$, thus $G[\Psi_0(t)]$ avoids

$B_G(V(\bigcup\mathcal{C}), r-2d)$. All in all, $G[\Psi_0(t)]$ avoids $\widehat{Y} \cup B_G(V(\bigcup\mathcal{C}), r-2d)$, which implies $G[\Psi_0(t)] \subseteq F_0$.

It remains to show (b). Let $\ell \in [p]$. If $\ell \notin \{i,j\}$ then $c_\ell = 0$ since $\Psi_\ell(t) = \emptyset$. It suffices to show that $B_G(V(P_1), d)$ is connected in F_i and $B_G(V(P_2), d)$ is connected in F_j, then the definition of $\Psi_\ell(t)$ immediately yields $\sum_{\ell\in[p]} c_\ell \leqslant 2$, proving (b). By symmetry, it suffices to show that $B_G(V(P_1), d)$ is connected in F_i. Since t is admissible, $B_G(V(P_1), d)$ is disjoint from $\widehat{Y}$. Since $V(P_1) \subseteq B_G(V(C_i), r-d)$, $B_G(V(P_1), d) \subseteq B_G(V(C_i), r)$. It follows that $B_G(V(P_1), d)$ is connected in F_i, as required. ∎

Claim 8.2.9. *Let t and τ be admissible tuples. If $\Psi_\ell(t) \cap \Psi_\ell(\tau) = \emptyset$ for all $\ell \in \{0, \dots, p\}$, then* $\mathrm{dist}_G(V(W_t), V(W_\tau)) > d$.

Subproof. Write $t = (e, i, \epsilon, j)$ and $\tau = (e', i', \epsilon', j')$. Let P_0, P_1, P_2 be the forest leg, first leg, and second leg of t respectively. Let P_0', P_1', P_2' be the forest leg, first leg, and second leg of τ respectively. Proceed by contraposition and suppose there exists vertices $u \in V(W_t)$ and $v \in V(W_\tau)$ such that $\mathrm{dist}_G(u, v) \leqslant d$. Then either (a) $u \in V(P_0)$ or $v \in V(P_0')$, or (b) $u \notin V(P_0)$ and $v \notin V(P_0')$.

In case (a), without loss of generality it may be assumed that $v \in V(P_0')$. It is immediate that $u \in B_G(v, d) \subseteq \Psi_0(\tau)$, thus it suffices to show that $u \in \Psi_0(t)$. If $u \in V(P_0)$, then $u \in \Psi_0(t)$. Otherwise, u is in the first or second leg of W_t. Without loss of generality $u \in V(P_1)$. Write $e = xy$ where $x \in V(P_1)$ and $y \in V(P_0)$. Let Q be the u-x-subpath of P_1, then the definition of P_1 and $||P_1|| = r - d$ implies $B_G(u, ||Q||) \subseteq B_G(V(C_i), r-d)$. Consequently, $B_G(u, ||Q||)$ is disjoint from $V(\widetilde{F}_0)$. However, $v \in B_G(u, d) \cap V(\widetilde{F}_0)$, and so $||Q|| \leqslant d - 1$. It follows that $u \in B_G(y, d) \subseteq \Psi_0(t)$, as required.

In case (b), without loss of generality it may be assumed that $u \in V(P_1)$ and $v \in V(P_1')$. It is immediate that $v \in B_G(u, d) \subseteq \Psi_i(t)$, thus it suffices to show that $v \in \Psi_i(\tau)$. By Claim 8.2.8, $\Psi_i(t) \subseteq V(F_i)$, thus $v \in B_G(V(C_i), r)$. Recall that $v \in V(P_1') \subseteq B_G(V(C_{i'}), r-d)$, thus $v \in B_G(V(C_i), r) \cap B_G(V(C_{i'}), r)$. Since τ is admissible, $v \notin \widehat{Y}$. Then (13) with $Y_1 \subseteq \widehat{Y}$ implies $i = i'$. Hence $\tau = (e', i, \epsilon', j')$, and so $v \in B_G(P_1', d) \subseteq \Psi_i(\tau)$. ∎

Claim 8.2.10. *Let $S \subseteq V(G)$ and t be an admissible tuple with the usual notation. If $S \cap \Psi_\ell(t) \neq \emptyset$ for some $\ell \in \{0, \dots, p\}$, then $B_G(S, r) \cap V(eP_0\epsilon) \neq \emptyset$.*

Subproof. It suffices to prove the case of $|S| = 1$. Say $S = \{s\}$. If $\ell = 0$, then by hypothesis $s \in B_G(V(P_0), d)$. Hence $B_G(s, r) \cap V(eP_0\epsilon) \neq \emptyset$ since $r > d$. Now consider the case $\ell \in \{i, j\}$. By hypothesis $s \in B_G(V(P_1), d)$ or $s \in B_G(V(P_2), d)$. Without loss of generality $s \in B_G(V(P_1), d)$. Let $u \in V(P_1)$ such that $\mathrm{dist}_G(s, u) \leqslant d$, and let v be the end of e in P_1. Then $\mathrm{dist}_G(s, v) \leqslant \mathrm{dist}_G(s, u) + \mathrm{dist}_G(u, v) \leqslant d + (r - d) = r$, which implies $B_G(s, r) \cap V(eP_0\epsilon) \neq \emptyset$. ∎

The following is in preparation for the application of Theorem 8.2.3: Define the forest $F := \bigsqcup_{i=0}^{p} F_p$. For each $i \in \{0, \dots, p\}$ let π_i be the inclusion map $V(F_i) \to V(F)$. Define a collection of non-null subgraphs of F by

$$\mathcal{A} := (F[\textstyle\bigcup_{i=0}^{p} \pi_i(\Psi_i(t))] : t \text{ is admissible}).$$

By Claim 8.2.8, each member of $\mathcal{A}$ has at most 3 components. Put $k^\star := k + \frac{1}{2}s(k)$. Then by Theorem 8.2.3, at least one of the following holds:

(a) there exists $k^\star$ pairwise vertex-disjoint members of $\mathcal{A}$, or
(b) there exists $X^\star \subseteq V(F)$ with $|X^\star| \leqslant 18(k^\star - 1)$ and $X^\star$ meets every member of $\mathcal{A}$.

In case (b), put $X_3 := \bigcup_{i=0}^{p} \pi_i^{-1}(X^\star)$ and $X := \widehat{X} \cup X_3$. Then (17) implies the following:

$$|X| \leqslant |\widehat{X}| + |X_3| \leqslant 2k + 2k^2 + (\tbinom{k}{2} + k)s(k) + 18(k + \tfrac{1}{2}s(k) - 1) = f(k).$$

Let C be a cycle in G and suppose C avoids $B_G(\widehat{X}, 19d)$. By (16), C avoids $B_G(\widehat{Y}, 6d) \subseteq B_G(\widehat{X}, 19d)$. Then by Claim 8.2.6, C contains the extended forest leg of some admissible tuple t. Since $X^\star$ meets every member of $\mathcal{A}$, $X_3 \cap \Psi_i(t) \neq \emptyset$ for some $i \in \{0, \ldots, p\}$. Then by Claim 8.2.10, $B_G(X_3, 6d)$ meets the extended forest leg of t, thus meets C. It follows that every cycle in G meets $B_G(X, 19d)$, hence $G - B_G(X, g(d))$ is a forest.

In case (a), there exists a collection T of $k^\star$ admissible tuples such that the members of $(F[\bigcup_{i=0}^{p} \pi_i(\Psi_i(t))] : t \in T)$ are pairwise vertex-disjoint. Hence for all distinct $t, \tau \in T$, $\Psi_i(t) \cap \Psi_i(\tau) = \emptyset$ for all $i \in \{0, \ldots, p\}$, which by Claim 8.2.9 implies

$$\operatorname{dist}_G(V(W_t), V(W_\tau)) > d. \tag{18}$$

Since W_t is a walk in G, W_t (as a graph) is a path or contains a cycle. Let $\mathcal{J}$ be the set of all $t \in T$ such that W_t contains a cycle. Since G has no d-packing of k cycles, (18) implies $|\mathcal{J}| < k$. Let $\mathcal{C}'$ be the set of all $C \in \mathcal{C}$ that contain an end of W_t for some $t \in T \setminus \mathcal{J}$. Define $H := \bigcup \mathcal{C}' \cup \bigcup_{t \in T \setminus \mathcal{J}} W_t$. From Claim 8.2.7 and the fact that H does not contain W_t for all $t \in \mathcal{J}$, it follows that $\deg_H(v) \in \{2, 3\}$ for all $v \in V(H)$, and $\deg_H(v) = 3$ if and only if v is an endpoint of W_t for some $t \in T \setminus \mathcal{J}$. Let H' be the 3-regular pseudograph obtained from H by suppressing all vertices of degree 2. Then $|H'| \geqslant 2|T \setminus \mathcal{J}| = 2(k^\star - k) = s(k)$. By Lemma 3.1.2, H' contains k pairwise vertex-disjoint cycles. By appropriately subdividing edges of those cycles, one obtains a collection $\mathcal{P}$ of k pairwise vertex-disjoint cycles in H.

The following shows that $\mathcal{P}$ is a d-packing of cycles in G: By the aforementioned properties of H, the degree 3 vertices of H partition the edges of each member of $\mathcal{P}$ into either a subpath of some $C \in \mathcal{C}'$, or a subpath of W_t for some $t \in T \setminus \mathcal{J}$. Consider any two distinct $D, D' \in \mathcal{P}$, and let P be a shortest $V(D)$-$V(D')$-path in G. It may be assumed that $||P|| \leqslant d$. Let $u \in V(P \cap D)$ and $v \in V(P \cap D')$, then P is a path from u to v. If $u \in V(W_t) \subseteq V(D)$ and $v \in V(W_\tau) \subseteq V(D')$ for some $t, \tau \in T$, then by disjointness of $V(D)$ and $V(D')$, it follows that $t \neq \tau$. Thus $\operatorname{dist}_G(V(D), V(D')) > d$ by (18).

Otherwise, one deduces that at least one of u or v is not an internal vertex of any path in $(W_t : t \in T \setminus \mathcal{J})$. It suffices to derive a contradiction. Without loss of generality, $u \in V(C_i)$ for some $C_i \in \mathcal{C}'$. Since $V(P) \subseteq B_G(V(C_i), d)$ and C_i is d-unicyclic, $v \notin V(C_i)$. Furthermore, since $\mathcal{C}'$ is a $2d$-packing, $v \notin V(\bigcup \mathcal{C}')$. Thus v is an internal vertex of W_τ for some $\tau \in T \setminus \mathcal{J}$. By properties of H, it follows that $W_\tau \subseteq D'$. Since $v \in V(P) \subseteq B_G(V(C_i), r - d)$, v lies in the first or second leg of τ. Without loss of generality v lies in the first leg P_1 of τ. Since τ is admissible, $V(P_1) \cap \widehat{Y} = \emptyset$. Then (13) along with $V(P_1) \cap B_G(V(C_i), r) \supseteq \{v\} \neq \emptyset$ implies $V(P_1) \cap B_G(C', r - d) = \emptyset$

for all $C' \in \mathcal{C} \setminus \{C_i\}$. Consequently, P_1 is a shortest path in U_i from $V(C_i)$ to the end of some i-exit edge in $B_G(V(C_i), r-d)$. Since $\mathrm{dist}_G(V(C_i), v) \leqslant ||P|| \leqslant d$, the $V(C_i)$-v-subpath Q of P_1 has length at most d. If the first edge of P lies on C_i, then along with the two edges of D incident to u, one finds that $\deg_H(u) = 3$. Then u is the endpoint of W_t for some $t \in T \setminus \mathcal{J}$ and $W_t \subseteq D$, contradicting the choice of u. Thus, the first edge of P does not lie on C_i. Since $V(C_i \cup P \cup Q) \subseteq B_G(V(C_i), d)$ and C_i is d-unicyclic, $P = Q$. Therefore $P \subseteq W_t \subseteq D'$, which implies $V(D) \cap V(D') \supseteq \{u\} \neq \emptyset$, a contradiction as announced.

Consequently, for all distinct $D, D' \in \mathcal{P}$, $\mathrm{dist}_G(V(D), V(D')) > d$. This along with $|\mathcal{P}| = k$ shows that $\mathcal{P}$ is a d-packing of k cycles in G, a contradiction. This completes the proof of Theorem 8.2.1. ■

8.3. A Necessary Condition

Dujmović, Joret, Micek and Morin [41] asked what is the smallest possible g satisfying Theorem 8.2.1 (with no restriction on f)? The following theorem is an original contribution of this thesis:

Theorem 8.3.1. *$g(d) \geqslant d$ for every pair of functions $f, g : \mathbb{N} \to \mathbb{N}$ satisfying Theorem 8.2.1.*

Proof. Let $\ell, d \geqslant 1$ be integers. The following constructs a graph G that does not contain two cycles at distance greater than d, and $G - B_G(X, d-1)$ has a cycle for every $X \subseteq V(G)$ with $|X| \leqslant \ell$: Write $v_1, \dots, v_{2\ell+1}$ for the vertices of $K^{2\ell+1}$ and consider the $(d-1)$-subdivision of $K^{2\ell+1}$. For distinct $i, j \in [2\ell+1]$, let $P_{ij} = P_{ji}$ be the subdivided v_i-v_j-edge. For each $i \in [2\ell+1]$, add two new vertices x, y and three new edges so that $T_i := (v_i, x, y)$ is a triangle. We call such triangles *special*. Denote the resulting graph by G. Clearly $G - \{v_1, \dots, v_{2\ell+1}\}$ is a forest, thus every cycle of G contains some v_i. Since $\mathrm{dist}_G(v_i, v_j) = d$ for all $i \neq j$, G does not contain two cycles at distance greater than d. It remains to show that $G - B_G(X, d-1)$ has a cycle for every $X \subseteq V(G)$ with $|X| \leqslant \ell$. To see this, consider an arbitrary $x \in X$. If x lies in some special triangle T_i, then since the paths P_{ij} have length d, T_i is the only triangle that meets $B_G(x, d-1)$. On the other hand, if x does not lie in any special triangle, then x is an internal vertex of some P_{ij}. Then T_i and T_j are the only special triangles that meet $B_G(x, d-1)$. It follows that $B_G(X, d-1) = \bigcup_{x \in X} B_G(x, d-1)$ meets at most $2|X| \leqslant 2\ell$ special triangles, hence missing some special triangle. ■

8.4. Coarse Analogues of Tools from Structural Graph Theory

Coarse graph theory poses many questions that can be thought of as coarse versions of theorems from structural graph theory. As such, one should attempt to make use of the wealth of standard tools from structural graph theory to answer those questions. Tools such as tree-width and the Graph Minor Structure Theorem set the stage for a lot of structural graph theory. For this reason, the present section surveys recent results by Nyugen, Scott, Seymour and by Hickingbotham for producing coarse analogues of tools from structural graph theory. Included, is a characterisation for the graphs which are quasi-isometric to graphs with bounded tree-width.

8.4.1. Excluding a Fat Tree

Nguyen, Scott and Seymour [100] showed a coarse version of the Path-width Theorem (Theorem 5.2.1). To state it, one must define infinite graphs and line-width.

Infinite graphs are defined the same as finite graphs (see Section 2.1), except that their vertex set may be infinite. Subgraphs of infinite graphs are defined the same as for finite graphs. For infinite graphs G and vertices $u, v \in V(G)$, the distance between u and v in G, denoted by $\text{dist}_G(u, v)$, is the minimum number of edges in a connected (finite) subgraph of G which contains both u and v. If there is no such subgraph of G, then $\text{dist}_G(u, v) = +\infty$. Just as it were for finite graphs, $(V(G), \text{dist}_G)$ is a metric-space when G is an infinite graph.

A set $U \subseteq V(G)$ of vertices in an infinite graph G is connected if $\text{dist}_G(x, y)$ is finite for all $x, y \in U$. Minor *models* for infinite graphs are defined the same as for finite graphs, except that there may be infinitely many branch sets, each with possibly infinitely many vertices. For infinite graphs G and H, H is a *minor* of G if G has an H model.

An *interval* S in a total order $(I, \leqslant)$ is a subset of I such that for all $a, c \in S$, if $b \in I$ with $a \leqslant b \leqslant c$, then $b \in S$. A *line-decomposition* of an infinite graph G is a family $(B_i : i \in I)$ of subsets of $V(G)$ indexed by a non-empty total ordering $(I, \leqslant)$, such that following hold:

- *Vertex-property.* For every $v \in V(G)$, $\{i \in I : v \in B_i\}$ is a non-empty interval in $(I, \leqslant)$.
- *Edge-property.* For every $uv \in E(G)$ there exists $i \in I$ such that $\{u, v\} \subseteq B_i$.

The *width* of $(B_i : i \in I)$ is the number $\max_{i \in I} |B_i| - 1$, or $+\infty$ if there is no finite maximum. The *line-width* of G is the minimum width of a line-decomposition of G. Notice that if G is a finite graph, then its line-width equals its path-width.

The main theorems of Chapter 5 can be weakened to the following:

Theorem 8.4.1. *The following statements hold:*

- *For every tree T, there exists an integer $k \geqslant 0$ such that every graph G that is T-minor-free has path-width at most k.*
- *For every integer $k \geqslant 0$, there exists a tree T such that every graph G that contains T as a minor has path-width more than k.*

For infinite graphs G, Chudnovsky, Nguyen, Scott and Seymour [24] showed that G has line-width at most k if and only if every finite subgraph of G has path-width at most k. Therefore, Theorem 8.4.1 still holds when “graph” is replaced by “infinite graph” and “path-width” is replaced by “line-width”. Then the following theorem of Nguyen, Scott and Seymour can be considered as a Coarse Path-width Theorem:

Theorem 8.4.2 (Nguyen, Scott, Seymour [100])**.** *The following statements hold:*

- *For every tree T and integer $d \geqslant 1$, there exists reals $k \geqslant 1$, $M \geqslant 1$, and $A \geqslant 0$ such that every infinite graph that does not contain T as a d-fat minor admits an (M, A)-quasi-isometry into an infinite graph with line-width at most k.*
- *For all reals $k \geqslant 1$, $M \geqslant 1$ and $A \geqslant 0$, there exists a tree T and an integer $d \geqslant 1$ such that every infinite graph that contains T as a d-fat minor does not*

admit an (M, A)-quasi-isometry into an infinite graph with line-width at most k.

8.4.2. Coarse Menger

Menger's Theorem (Theorem 2.1.3) is arguably the most foundational tool in structural graph theory. By looking at the proofs in this thesis, one can see the utility of Menger's Theorem in action. Generally, results about linkages lay the ground work for finding minors in a graph. As such, a coarse Menger's Theorem, if it exists, would be helpful for finding fat minors. This would be a big step in coarse graph theory. Albrechtsen, Huynh, Jacobs, Knappe and Wollan [4], and independently Georgakopoulos and Papasoglu [55] conjectured the following coarse version of Menger's Theorem and proved it for $k = 2$:

False Conjecture 8.4.3 (Coarse Menger Conjecture. Jacobs, Knappe, Wollan [4]; Georgakopoulos, Papasoglu [55]). *For all integers $k, r \geqslant 1$, there exists an integer $d \geqslant 1$ such that for every graph G and all subsets $A, B \subseteq V(G)$, G has k A-B-paths at pairwise distance at least r, or there exists a subset $X \subseteq V(G)$ with $|X| \leqslant k - 1$ and $B_G(X, d)$ is an A-B-separator.*

Nguyen, Scott and Seymour [99] showed that False Conjecture 8.4.3 is false for $k \geqslant 3$. Their counter-examples have tree-width 6, thus the False Conjecture 8.4.3 is even false for graphs with bounded tree-width. Subsequently, the same authors showed that the following weakening of False Conjecture 8.4.3 is also false:

False Conjecture 8.4.4 (Weak Coarse Menger Conjecture. Nguyen, Scott and Seymour [99]). *For all integers $k, r \geqslant 1$, there exists integers $\ell, d \geqslant 1$ such that for every graph G and all subsets $A, B \subseteq V(G)$, G has k A-B-paths at pairwise distance at least r, or there exists a subset $X \subseteq V(G)$ with $|X| \leqslant \ell$ and $B_G(X, d)$ is an A-B-separator.*

In spite of the negative results, there are some in the positive direction. False Conjecture 8.4.3 is false for graphs with bounded tree-width, but is true for graphs with bounded path-width, as shown by the following theorem:

Theorem 8.4.5 (Nguyen, Scott, Seymour [101]). *For all integers $k, r \geqslant 1$ and $\ell \geqslant 0$, there exists an integer $d \geqslant 1$ such that for every graph G with path-width at most ℓ, and all subsets $A, B \subseteq V(G)$, G has k A-B-paths at pairwise distance at least r, or there exists a subset $X \subseteq V(G)$ with $|X| \leqslant k - 1$ and $B_G(X, d)$ is an A-B-separator.*

Another positive result is the following is a version of False Conjecture 8.4.3 for planar graphs when A and B lie on the same face of a plane drawing:

Theorem 8.4.6 (Nguyen, Scott, Seymour [102]). *Let $k, r \geqslant 1$ be integers and G be a graph drawn in the plane. Suppose $A, B \subseteq V(G)$ be sets of vertices on the boundary of the outer face of G. Then there exists k A-B-paths at pairwise distance greater than r, or there exists a collection $\mathcal{H}$ of at most $k - 1$ connected subgraphs of G such that $V(\bigcup \mathcal{H})$ is an A-B-separator and $\sum_{H \in \mathcal{H}} \operatorname{diam}(H) \leqslant 200(k-1)^3 r$.*

8.4.3. Quasi-isometry into Graphs with Bounded Tree-width

When are "complex graph classes" quasi-isometric to "simple graph classes"? Having a bound on the tree-width of graph class reveals some of the "tree-like" structure of graphs in the class. This structure was combined with chordal graphs and the generalised Helly property (Lemma 4.1.2) to prove the Erdős–Pósa Property of Planar Graph Minors (Theorem 7.1.3). This sort of structural use of tree-width makes it a major tool in structural graph theory. A good definition of "simple graph class" is bounded tree-width. Indeed, this was made precise in Section 6.1, which discussed algorithmic motivations for studying tree-width, including the tractability of many difficult graph problems when restricted to graphs with bounded tree-width.

The remainder of this section proves a connection between quasi-isometry and tree-width. The following theorem was shown by Nguyen, Scott and Seymour [103] and independently by Hickingbotham [67]. Both sets of authors obtained different constants so we state it in the following rough way:

Theorem 8.4.7 (Nguyen, Scott, Seymour [103]; Hickingbotham [67])**.** *G admits a tree-decomposition in which each bag is the union of a bounded number of bounded radius balls if and only if G is quasi-isometric to a graph with bounded tree-width.*

We follow Hickingbotham's [67] proof.

For integers $q \geqslant 1$, a *q-quasi-isometry* of a graph G into a graph H is a (q, q)-quasi-isometry from G into H. Let G be a graph, and let $k \geqslant 1$ and $d \geqslant 0$ be integers. A set $S \subseteq V(G)$ is *(k, d)-centred* in G if S has a partition $\mathcal{P}$ into at most k parts, each with weak-diameter at most d in G. A tree-decomposition of G is *(k, d)-centred* if each bag is (k, d)-centred in G. Given a tree T, a *T-decomposition* of a graph G is a tree-decomposition of G whose underlying tree is T. The following theorem implies Theorem 8.4.7:

Theorem 8.4.8 (Hickingbotham [67])**.** *There exists a function $q : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that for all integers $k, d, c \geqslant 1$ and any tree T, the following hold:*

- *If G is a graph that has a (k, d)-centred T-decomposition, then G is $q(k, d)$-quasi-isometric to a graph H that has a T-decomposition with width at most $2k - 1$.*
- *If G is a graph that is c-quasi-isometric to a graph H, where H has a T-decomposition with width at most k, then G has a $(k + 1, 3c^2)$-centred T-decomposition.*

Notice that the underlying tree of the tree-decomposition is maintained across the quasi-isometry in Theorem 8.4.8. This feature is not present in Nguyen, Scott and Seymour's [103] proof. Instead, Nguyen, Scott and Seymour obtain a quasi-isometry of G into a graph with a tree-decomposition whose underlying tree is a subdivision of the tree that indexed the tree-decomposition of G.

As a remark, Hickingbotham's original proof [67] had some typos and an incorrect version of Lemma 8.4.16. As such, we notified the author who has since made the corrections.

Now to the proof. The following lemma proves the second point in Theorem 8.4.8:

Lemma 8.4.9. *Let $k, c \geqslant 1$ be integers and T a tree. If G is a graph that is c-quasi-isometric to a graph H, where H has a T-decomposition with width at most k, then G has a $(k + 1, 3c^2)$-centred T-decomposition.*

Proof. Let φ be a c-quasi-isometry from G to H, and (T, β') be a T-decomposition of H with width at most k. For each $x \in V(H)$ put $A_x := \{v \in V(G) : \mathrm{dist}_H(x, \varphi(v)) \leqslant c\}$. For all $x \in V(H)$ and all $u, v \in A_x$, (Q1) (in the definition of quasi-isometry, see Section 8.1) implies

$$\mathrm{dist}_G(u, v) \leqslant \mathrm{dist}_H(\varphi(u), \varphi(v))c + c^2 \leqslant (\mathrm{dist}_H(x, \varphi(u)) + \mathrm{dist}_H(x, \varphi(v)))c + c^2 \leqslant 3c^2.$$

For each $t \in V(T)$ put $\beta(t) := \bigcup_{x \in \beta'(t)} A_x$, then $\beta(t)$ is the union of at most $k+1$ sets each with weak-diameter at most $3c^2$. It remains to show that (T, β) is a tree-decomposition of G.

Claim 8.4.10. *For all $t \in V(T)$ and all $v \in V(G)$, $v \in \beta(t)$ if and only if $B_H(\varphi(v), c) \cap \beta'(t) \neq \emptyset$.*

Subproof. Suppose $v \in \beta(t)$, then there exists $x \in \beta'(t)$ such that $v \in A_x$. Then $\mathrm{dist}_H(x, \varphi(v)) \leqslant c$, implying that $x \in B_H(\varphi(v), c)$. Hence $x \in B_H(\varphi(v), c) \cap \beta'(t)$. On the contrary, suppose $B_H(\varphi(v), c) \cap \beta'(t) \neq \emptyset$. Let $x \in B_H(\varphi(v), c) \cap \beta'(t)$, then $\mathrm{dist}_H(x, \varphi(v)) \leqslant c$, implying that $v \in A_x$. Since $x \in \beta'(t)$, $v \in A_x \subseteq \beta(t)$. ■

Note that $\varphi(v) \in B_H(\varphi(v), c)$, so $B_H(\varphi(v), c)$ is a non-empty connected set in H. Then Proposition 6.0.4 implies $(T, \beta')[B_H(\varphi(v), c)]$ is a non-empty connected set of vertices in T. Claim 8.4.10 implies $\{t \in V(T) : v \in \beta(t)\} = (T, \beta')[B_H(\varphi(v), c)]$, proving the vertex-property for (T, β). Now consider any edge $uv \in E(G)$. Then by (Q1), $\mathrm{dist}_H(\varphi(u), \varphi(v)) \leqslant \mathrm{dist}_G(u, v)c + c = 2c$. Let $x \in V(H)$ be a centre for a shortest $\varphi(u)$-$\varphi(v)$-path in H, and let $t \in V(T)$ with $x \in \beta'(t)$. Then $\{u, v\} \subseteq A_x \subseteq \beta(t)$. Hence (T, β) is a $(k+1, 3c^2)$-centred tree-decomposition of G. ■

The proof of the first point in Theorem 8.4.8 relies on the following definitions: Let G be a graph. A set of vertices D in G is *dominating* if every vertex of G is in D or is adjacent to a vertex in D. Denote by $\gamma(G)$, the minimum size of a dominating set of vertices in G. Let (T, β) be a tree-decomposition of G. The *independence number* of (T, β) is the number $\max\{\alpha(G[\beta(t)]) : t \in V(T)\}$. The *domination number* of (T, β) is the number $\max\{\gamma(G[\beta(t)]) : t \in V(T)\}$. The *tree-independence number* of G is the minimum independence number of a tree-decomposition of G. A *partition* $\mathcal{P}$ of G is a partition of $V(G)$ into non-empty connected sets. Let $G/\mathcal{P}$ be the graph obtained from G by contracting all edges of A for all $A \in \mathcal{P}$. For each $A \in \mathcal{P}$, the vertex of $G/\mathcal{P}$ formed by contracting the edges of A is referred to as A.

Lemma 8.4.11. *Let $k, d \geqslant 1$ be integers and T a tree. If G is a graph that has a (k, d)-centred T-decomposition, then G is d-quasi-isometric to a graph H that has a T-decomposition with independence number at most k.*

Proof. Let (T, β) be a (k, d)-centred tree-decomposition of G. Let H be obtained from G by adding edges between vertices at distance at most d which lie in a common bag. It is clear that (T, β) is also a tree-decomposition of H. Since (T, β) is (k, d)-centred for G, each $\beta(t)$ is the union of at most k cliques in H, thus (T, β) has independence number at most k with respect to H. It suffices to show that the identity map $V(G) \to V(H)$ is a d-quasi-isometry from G to H. Observe that (Q2) holds trivially. The following shows that (Q1) holds: Since $G \subseteq H$, $\mathrm{dist}_H \leqslant \mathrm{dist}_G$, hence

$\mathrm{dist}_H(u,v) \leqslant \mathrm{dist}_G(u,v)d + d$ for all $u, v \in V(G)$. On the other hand, let $u, v \in V(G)$. Notice that $\mathrm{dist}_H(u,v) = +\infty$ trivially implies $\mathrm{dist}_G(u,v)\frac{1}{d} - d \leqslant \mathrm{dist}_H(u,v)$. Now suppose $\mathrm{dist}_H(u,v) < +\infty$ and let P be a shortest u-v-path in H. Then for every edge $xy \in E(P)$, G contains an x-y-path of length at most d. Consequently, G contains a u-v-walk of length at most $\mathrm{dist}_H(u,v)d$. It follows that $\mathrm{dist}_G(u,v)\frac{1}{d} - d \leqslant \mathrm{dist}_H(u,v)$ for all $u, v \in V(G)$. ∎

The following lemma appeared without proof in [67]:

Lemma 8.4.12. *For every integer $d \geqslant 1$, every graph G, and every partition $\mathcal{P}$ of G into parts of weak-diameter less than d, G is d-quasi-isometric to $G/\mathcal{P}$.*

PROOF. Put $H := G/\mathcal{P}$, and let $\varphi : V(G) \to V(H)$ be the map sending each vertex to its part. The following shows that φ is d-quasi-isometry from G to H: Since H is obtained from G by contracting edges, $\mathrm{dist}_H(\varphi(u), \varphi(v)) \leqslant \mathrm{dist}_G(u,v)$ for all $u, v \in V(G)$. Hence $\mathrm{dist}_H(\varphi(u), \varphi(v)) \leqslant \mathrm{dist}_G(u,v)d + d$ for all $u, v \in V(G)$.

On the other hand, let $u, v \in V(G)$. Notice that $\mathrm{dist}_H(\varphi(u), \varphi(v)) = +\infty$ trivially implies $\mathrm{dist}_G(u,v)\frac{1}{d} - d \leqslant \mathrm{dist}_H(\varphi(u), \varphi(v))$. Now suppose $n := \mathrm{dist}_H(\varphi(u), \varphi(v)) < +\infty$ and let $P = (\varphi(u) = X_1, X_2, \ldots, X_{n+1} = \varphi(v))$ be a shortest $\varphi(u)$-$\varphi(v)$-path in H. Since each $X_1, \ldots, X_{n+1}$ has weak diameter at most $d-1$ in G and there exists an X_i-X_{i+1}-edge in G for all $i \in [n]$, there exists a u-v-walk in G of length at most $(d-1)(n+1) + n = nd + d - 1 \leqslant nd + d^2$. It follows that $\mathrm{dist}_G(u,v)\frac{1}{d} - d \leqslant \mathrm{dist}_H(\varphi(u), \varphi(v))$ for all $u, v \in V(G)$. Hence (Q1) holds for φ. It remains to show (Q2) for φ. Observe that for every $u \in V(H)$, choose any $v \in u$, then $\varphi(v) = u$, so $\mathrm{dist}_H(u, \varphi(v)) \leqslant d$. ∎

The proof of Theorem 8.4.8 relies on the following results from the literature:

Lemma 8.4.13 (Dallard, Milanič, Štorgel [29])**.** *Let $k \geqslant 1$ be an integer and T be a tree. If G is a graph that has a T-decomposition with independence number at most k. Then for every partition $\mathcal{P}$ of G, $G/\mathcal{P}$ has a T-decomposition with independence number at most k.*

Lemma 8.4.14 (Dvořák, Norin [43])**.** *There exist a function $f : \mathbb{N} \to \mathbb{N}$ such that for every integer $k \geqslant 1$, every graph G that has a tree-decomposition with domination number at most k has a partition $\mathcal{P}$ into parts of weak-diameter less than $f(k)$ and $G/\mathcal{P}$ is bipartite.*

Combining the above lemmas shows the following:

Lemma 8.4.15. *There exist a function $f : \mathbb{N} \to \mathbb{N}$ such that for every integer $k \geqslant 1$ and every tree T, every graph G that has a T-decomposition with independence number at most k is $f(k)$-quasi-isometric to a graph H that has a T-decomposition with width at most $2k-1$.*

PROOF. Let f be the function promised by Lemma 8.4.14. Since domination number is at most independence number, G has a partition $\mathcal{P}$ into parts of weak-diameter less than $f(k)$ and $H := G/\mathcal{P}$ is bipartite. By Lemma 8.4.12 G is $f(k)$-quasi-isometric to H. By Lemma 8.4.13 H has a T-decomposition (T, β') with independence number at most k. Since H is bipartite, $\frac{1}{2}|\beta'(t)| \leqslant \alpha(H[\beta'(t)]) \leqslant k$ for all $t \in V(T)$, thus (T, β') has width at most $2k - 1$. ■

An incorrect version of the following lemma originally appeared in [67], here is a correct version:

Lemma 8.4.16. *For all integers $c, q \geqslant 1$, if a graph G_0 is c-quasi-isometric to a graph G_1, and G_1 is q-quasi-isometric to a graph G_2, then G_0 is $(c+2)q$-quasi-isometric to G_2.*

Proof. Let φ be a c-quasi-isometry from G_0 into G_1, and ψ be a q-quasi-isometry from G_1 into G_2. The following shows that $h := \psi \circ \varphi$ is a $(c+2)q$-quasi-isometry from G_0 into G_2: Starting with (Q2) for h. Let $x \in V(G_2)$ be given. By (Q2) for ψ, there exists $u \in V(G_1)$ such that $\mathrm{dist}_{G_2}(x, \psi(u)) \leqslant q$. By (Q2) for φ, there exists $v \in V(G_0)$ such that $\mathrm{dist}_{G_1}(u, \varphi(v)) \leqslant c$. Then by the triangle inequality and (Q1) for ψ, one finds:

$$\begin{aligned}\mathrm{dist}_{G_2}(x, h(v)) &\leqslant \mathrm{dist}_{G_2}(x, \psi(u)) + \mathrm{dist}_{G_2}(\psi(u), h(v))\\ &\leqslant \mathrm{dist}_{G_2}(x, \psi(u)) + \mathrm{dist}_{G_1}(u, \varphi(v))q + q\\ &\leqslant q + cq + q = (c+2)q.\end{aligned}$$

It remains to show (Q1) for h. For all $u, v \in V(G_0)$, one finds:

$$\begin{aligned}\mathrm{dist}_{G_0}(u,v)\frac{1}{(c+2)q} - (c+2)q &\leqslant \mathrm{dist}_{G_0}(u,v)\frac{1}{cq} - (c+2)q\\ &= \left(\mathrm{dist}_{G_0}(u,v)\frac{1}{c} - c + c\right)\frac{1}{q} - (c+2)q.\end{aligned}$$

By (Q1) for φ:

$$\begin{aligned}\mathrm{dist}_{G_0}(u,v)\frac{1}{(c+2)q} - (c+2)q &\leqslant \left(\mathrm{dist}_{G_1}(\varphi(u), \varphi(v)) + c\right)\frac{1}{q} - (c+2)q\\ &= \left(\mathrm{dist}_{G_1}(\varphi(u), \varphi(v))\frac{1}{q} - q\right) + q + \frac{c}{q} - (c+2)q.\end{aligned}$$

By (Q1) for ψ:

$$\mathrm{dist}_{G_0}(u,v)\frac{1}{(c+2)q} - (c+2)q \leqslant \mathrm{dist}_{G_2}(h(u), h(v)) + q + \frac{c}{q} - (c+2)q.$$

Since $q + \frac{c}{q} \leqslant (c+2)q$:

$$\mathrm{dist}_{G_0}(u,v)\frac{1}{(c+2)q} - (c+2)q \leqslant \mathrm{dist}_{G_2}(h(u), h(v)).$$

On the other hand, by (Q1) for ψ and φ:

$$\begin{aligned}\mathrm{dist}_{G_2}(h(u), h(v)) &\leqslant (\mathrm{dist}_{G_0}(u,v)c + c)q + q\\ &= \mathrm{dist}_{G_0}(u,v)cq + (c+1)q\\ &\leqslant \mathrm{dist}_{G_0}(u,v)(c+2)q + (c+2)q.\end{aligned}$$ ■

Now turning to the proof of the first point in Theorem 8.4.8.

Lemma 8.4.17. *There exists a function $q : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that for all integers $k, d \geqslant 1$ and every tree T, if G is a graph that has a (k, d)-centred T-decomposition, then G is $q(k, d)$-quasi-isometric to a graph H that has a T-decomposition with width at most $2k - 1$.*

Proof. Let f be the function promised by Lemma 8.4.15 and put $q(k, d) := (d + 2)f(k)$. By Lemma 8.4.11, G is d-quasi-isometric to a graph G' that has a T-decomposition with independence number at most k. By Lemma 8.4.15 G' is $f(k)$-quasi-isometric to a graph H that has a T-decomposition with width at most $2k - 1$. Then by Lemma 8.4.16 G is $q(k, d)$-quasi-isometric to H. ■

CHAPTER 9

Topics for Future Research

Packing is a topic in mathematics in which one aims to cram as many objects into a container as possible. This thesis investigated Erdős–Pósa type dualities in graph theory, which can be thought of as approximate packing-covering dualities. The main Erdős–Pósa dualities that explored in this thesis were the classic Erdős–Pósa Theorem in Chapter 3, Robertson and Seymour's extension of the Erdős–Pósa Theorem to planar graph minors in Chapter 7, and the Coarse Erdős–Pósa in Conjecture Chapter 8. This chapter discusses some topics for future research.

Erdős–Pósa is well studied area of research with many results established. For this reason, many of the open problems are quantitative, for example:

- As discussed in Section 7.3, for a fixed planar graph H, there exists a constant c_H such that $f(k) = c_H k \log k$ is a bounding function for $\mathcal{F}(H)$. The constant c_H obtained by Cames van Batenburg, Huynh, Joret and Raymond is not even known to be computable. Hence a research direction would be to lower c_H and eliminate or simplify complicated parts of the proof.
- As per Section 7.5, Dujmović, Joret, Micek and Morin showed that for a fixed t-vertex forest F where t' is the maximum number of vertices in F, $f(k) = kt - t'$ a bounding function for $\mathcal{F}(F)$. Moreover, they showed that $f(k) = t(k-1)$ is the absolutely tight bounding function for $\mathcal{F}(T)$ where T is a fixed t-vertex tree. Our Conjecture 7.5.3 says that the same function is the absolutely tight bounding function for $\mathcal{F}(F)$ where F is a fixed t-vertex forest. Additionally, our other conjecture (Conjecture 7.5.8 reproduced below) implies the bounding function in Conjecture 7.5.3:

Conjecture (Conjecture 7.5.8)**.** *Let $c \geqslant 1$ be an integer and $T_1, \dots, T_c$ be trees. For every graph G and all integers $k_1, \dots, k_c, s_1, \dots, s_c$ such that $k_i \geqslant s_i \geqslant 1$ for all $i \in [c]$, G contains a $\bigsqcup_{i\in[c]} k_i T_i$ minor, or there exists $X \subseteq V(G)$ with $|X| \leqslant \sum_{i\in[c]} |T_i|(k_i - s_i)$ and $G - X$ has no $\bigsqcup_{i\in[c]} s_i T_i$ minor.*

Section 8.2 showed that in the proof of the Erdős–Pósa property for cycles that are far apart (Theorem 8.2.1), replacing Theorem 8.2.2 by a result of Alon (Theorem 8.2.3) shows that Theorem 8.2.1 holds with $f(k) = O(k^3 \log k)$ and $g(d) = 19d$. However, Ahn, Gollin, Huynh and Kwon [2], and Dujmović, Joret, Micek and Morin's [41] conjecture that Theorem 8.2.1 holds with $f(k) = O(k \log k)$, remains open.

Open Problem 9.0.1. *Is $g(d) \geqslant (1+\varepsilon)d$ (for some $\varepsilon > 0$) necessary in Theorem 8.2.1?*

Our proof of the necessity of $g(d) \geqslant d$ (Theorem 8.2.1) relied on subdivisions, thus a starting point for answering Open Problem 9.0.1 in the affirmative would be to study d-packing of cycles in graphs with minimum degree at least 3.

The next direction for Erdős–Pósa is the current hot topic in graph theory, coarse graph theory, in which the main conjecture relating to Erdős–Pósa is the Coarse Erdős–Pósa Conjecture (Conjecture 8.1.6). Coarse graph theory as a whole asks many questions that can be thought of as coarse versions of theorems from structural graph theory (see Section 8.1). As such, coarse graph theory would benefit from coarse analogues of tools from structural graph theory (as discussed in Section 8.4). The following lists some related conjectures and open problems:

- A *length space* is a metric space (X, d) such that for every $x, y \in X$ and every $\varepsilon > 0$, there exists an x-y-path (continuous function $[0,1] \to X$) with length at most $d(x,y) + \varepsilon$. A d-fat minor of a length space is defined the same as for graphs (see Section 8.1) except that $(X_v : v \in V(H)) \cup (P_e : e \in E(H))$ is a collection of path-connected sets in X. Georgakopoulos and Papasoglu [55] conjectured that a length space with no d-fat H minor is quasi-isometric to a graph with no H minor. This was shown to be false by Davies, Hickingbotham, Illingworth and McCarty [30], who conjectured the following weakening:

  False Conjecture 9.0.2. *For any graph H, there exists a graph H' and a function $f_H : \mathbb{N} \to \mathbb{N}$ such that if G is a length space with no k-fat H minor, then G is $f_H(k)$-quasi-isometric to a graph with no H' minor.*

  This weakening is also false as shown by Albrechtsen and Davies, who conjectured the following bounded degree version that remains open:

  Conjecture 9.0.3 (Albrechtsen, Davies [3])**.** *For all integers $k, d \geqslant 1$ and any graph H, there exists $M \geqslant 1$, $A \geqslant 0$ and a graph H' such that every graph G with maximum degree at most d and with no k-fat H minor is (M, A)-quasi-isometric to a graph with no H' minor.*

- As discussed in Section 8.1, it is known that proper minor-classes have Assouad–Nagata dimension at most 2. The following is a major open problem related to fat minors posed by Bonamy, Bousquet, Esperet, Groenland, Liu, Pirot and Scott [17]:

  Open Problem 9.0.4. *Is it true that there is a constant d such that for any integer q and graph H, the class of graphs $\mathcal{G}$ with no q-fat H minor has asymptotic dimension at most d?*

  This problem asks whether there is a fat minor version of Theorem 8.1.3. Progress on this problem has been made, for example combining Nguyen, Scott and Seymour's Coarse Path-width Theorem (Theorem 8.4.2) with Theorem 8.1.3 shows that for fixed trees T, the graphs with no fat T minor have asymptotic dimension at most 1. Recently, Hickingbotham [68] made substantial progress on Open Problem 9.0.4 as shown by the theorem:

  Theorem 9.0.5 (Hickingbotham [68])**.** *Let $\Delta, r \geqslant 1$ be integers and $\mathcal{G}$ be a graph class that is closed under taking induced subgraphs and whose members have maximum degree at most Δ. For graphs H, the following hold:*
  – *If $\mathcal{G}$ excludes H as an r-fat minor, then* $\operatorname{asdim}(\mathcal{G}) \leqslant 2$, *and*
  – *if H is planar and $\mathcal{G}$ excludes H as an r-fat minor, then* $\operatorname{asdim}(\mathcal{G}) \leqslant 1$.

- Related to the discussion of coarse Menger in Section 8.4, Nguyen, Scott and Seymour [102] conjectured a coarse version of Menger's Theorem for surfaces:

  Conjecture 9.0.6 (Nguyen, Scott, Seymour [102])**.** *Let $\mathcal{S}$ be a fixed surface. For all integers $k \geqslant 1$ and $r \geqslant 0$, there exists an integer $d \geqslant 1$ such that for every graph G that is embeddable on $\mathcal{S}$, and all subsets $A, B \subseteq V(G)$, G has k A-B-paths at pairwise distance at least r, or there exists a subset $X \subseteq V(G)$ with $|X| \leqslant k-1$ and $B_G(X, d)$ is an A-B-separator.*

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