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Graph Product Structure: Theory and Applications

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Abstract. Graph product structure theory describes graphs in complicated graph classes as subgraphs of products of graphs in simpler graph classes. This essay selectively surveys recent results and applications in this field.

One of the aims of the field of Graph Structure Theory is to describe the global structure of graphs in a particular class $\mathcal{G}$ in terms of simpler graphs. Graph Product Structure Theory achieves this goal, by describing graphs in $\mathcal{G}$ as subgraphs of products of graphs in simpler graph classes, typically with bounded treewidth. The *treewidth* of a graph G, denoted by $\text{tw}(G)$, is the standard measure of how similar G is to a tree. Graph classes with bounded treewidth are considered to be simple; for example, numerous NP-complete problems are efficiently solvable on graphs with bounded treewidth. See [5, 39, 52] for background on treewidth, including the definition.

The following Planar Graph Product Structure Theorem by Dujmović, Joret, Micek, Morin, Ueckerdt, and Wood [28] is the classical example of a graph product structure theorem. Here, a graph H is *contained* in a graph G if H is isomorphic to a subgraph of G, written $H \subsetneqq G$. As illustrated in Figure 1, the *strong product* of graphs A and B, denoted by $A \boxtimes B$, is the graph with vertex-set $V(A) \times V(B)$, where distinct vertices $(v, x), (w, y)$ are adjacent if

- $v = w$ and $xy \in E(B)$, or
- $x = y$ and $vw \in E(A)$, or
- $vw \in E(A)$ and $xy \in E(B)$.

Theorem 1 ([28]). *For every planar graph G there is a graph H and a path P such that* $\text{tw}(H) \leqslant 8$ *and* $G \subsetneqq H \boxtimes P$.

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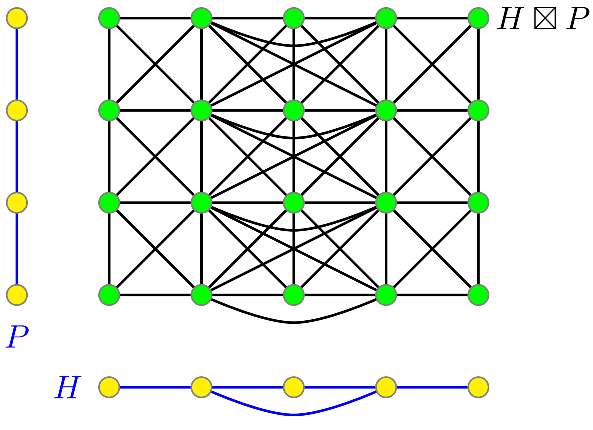

Figure 1. Example of a strong product.

Theorem 1 implies and generalises many results from the literature on the global structure of planar graphs, including the Lipton-Tarjan balanced separator theorem [49], the Baker decomposition and bounded local treewidth [2, 11, 37], and layered treewidth [30].

Theorem 1 has been the key tool for resolving several open problems regarding queue layouts [28], nonrepetitive colourings [24], centred colourings [21], adjacency labelling schemes [6, 23, 38], twin-width [7, 47], infinite graphs [45], clustered colouring [25, 26], and comparable box dimension [35]. Now we give one example.

Motivated by measuring the relative power of stacks and queues as data structures, Heath, Leighton and Rosenberg [40, 41] defined the *stack-number* $\text{sn}(G)$ and *queue-number* $\text{qn}(G)$ of a graph G. They asked whether planar graphs have bounded queue-number, which became the most important open problem in the field, and a key case to consider in the comparison of the relative power of stacks and queues.

Prior to [28] the best known upper bound on the queue-number of n-vertex planar graphs was $O(\log n)$ [13, 22]. With Theorem 1 in hand, Dujmović et al. [28] showed that a $O(1)$ bound is easily proved using already established tools. In particular, Dujmović, Morin, and Wood [29] showed in 2005 that $\text{qn}(G) \leqslant f(\text{tw}(G))$ for some function f. The best known bound is $\text{qn}(G) \leqslant 2^{\text{tw}(G)} - 1$ due to Wiechert [56]. Wood [57] showed in 2005 that $\text{qn}(H \boxtimes P) \leqslant 3\,\text{qn}(H) + 1$ for any graph H and any path P. Now consider any planar graph G. By Theorem 1, $G \subsetneq H \boxtimes P$ where $\text{tw}(H) \leqslant 8$ and P is a path. Since queue-number is subgraph-monotone, the above results together imply that

$$\text{qn}(G) \leqslant \text{qn}(H \boxtimes P) \leqslant 3\,\text{qn}(H) + 1 \leqslant 3(2^{\text{tw}(H)} - 1) + 1 \leqslant 3(2^8 - 1) + 1 = 766.$$

This shows that planar graphs have bounded queue-number, resolving the 28-year old open problem of Heath, Leighton and Rosenberg [40, 41].

The key tool in this proof is the Planar Graph Product Structure Theorem. We emphasise that the authors of [28] were not trying to prove Theorem 1. They first solved the queue-number problem, and then realised that their proof contained a structural description of planar graphs that could be written as a product, as in Theorem 1. It turns out that, back in 2008, Wood [58] had conjectured a version of Theorem 1, where the subgraph relation was replaced by the weaker notion of shallow minor. Theorem 1 implies this conjecture.

Theorem 1 has been extended in various ways. The following result is the state-of-the-art for the product structure of planar graphs.

Theorem 2. *Every planar graph G is contained in:*

(a) $H \boxtimes P$ *where* $\operatorname{tw}(H) \leqslant 6$ *and P is a path* [55],

(b) $H \boxtimes P \boxtimes K_2$ *where* $\operatorname{tw}(H) \leqslant 4$ *and P is a path* [55],

(c) $H \boxtimes P \boxtimes K_3$ *where* $\operatorname{tw}(H) \leqslant 3$ *and P is a path* [28].

Theorem 2(c) is often the most useful version of the Planar Graph Product Structure Theorem since the dependence on $\operatorname{tw}(H)$ in applications is often exponential, as in the queue-number example above. Indeed, using Theorem 2(c), Dujmović et al. [28] showed that planar graphs have queue-number at most 49, which was improved to 42 by Bekos, Gronemann, and Raftopoulou [3]. The treewidth 3 bound in Theorem 2(c) is best possible for any result showing that planar graphs are contained in $H \boxtimes P \boxtimes K_c$ for any constant c (see [28]). Improved bounds are known for special classes of planar graphs, such as squaregraphs [42]. Note that some extra properties hold in Theorems 1 and 2. Notably, H is a minor of the given graph G, and thus H is planar. On the other hand, Dujmović, Joret, Micek, Morin, and Wood [33] proved that in Theorems 1 and 2, H must have unbounded maximum degree, even for planar graphs G with maximum degree 5.

The proofs of Theorems 1 and 2 were inspired by an elegant result of Pilipczuk and Siebertz [51] which uses Sperner's Lemma at its heart. The method is constructive and leads to an efficient algorithm for computing the mapping into the product graph. Indeed, Bose, Morin, and Odak [9] gave a linear-time algorithm for doing so. Note that Illingworth, Scott, and Wood [46] gave a purely combinatorial proof of Theorem 2(c) that avoids planar embeddings and avoids the use of Sperner's Lemma. Roughly speaking, their result describes how any tree-decomposition of a graph excluding a fixed minor can be converted into a product-like structure.

Motivated by Theorem 1, Bose, Dujmović, Javarsineh, Morin, and Wood [8] introduced the following definition. The *row treewidth* of a graph G, denoted by $\operatorname{rtw}(G)$, is the minimum $c \in \mathbb{N}$ such that $G \subsetneq H \boxtimes P$ for some path P and graph H with $\operatorname{tw}(H) \leqslant c$. Theorem 2(a) says that planar graphs have row treewidth at most 6. It is open whether planar graphs have row treewidth at most 3. Bose et al. [8] studied the relationship between row treewidth and layered treewidth $\operatorname{ltw}(G)$. While, it is easily seen that $\operatorname{ltw}(G) \leqslant \operatorname{rtw}(G)$, Bose et al. [8] described a class of graphs G with

$\mathrm{ltw}(G) = 1$ and unbounded row-treewidth. Thus, bounded row-treewidth is a strictly stronger setting than bounded layered treewidth.

Other classes with bounded row treewidth include graphs embeddable on any fixed surface [28]. The best bounds are due to Distel, Hickingbotham, Huynh, and Wood [17].

Theorem 3 ([17]). *Every graph G of Euler genus g is contained in:*

(a) *$H \boxtimes P$ where $\mathrm{tw}(H) \leqslant 2g + 6$ and P is a path, and*

(b) *$H \boxtimes P \boxtimes K_{\max\{2g,3\}}$ where $\mathrm{tw}(H) \leqslant 3$ and P is a path.*

Dujmović et al. [28] established the following characterisation of minor-closed classes with bounded row treewidth. A graph X is *apex* if $X - a$ is planar for some vertex $a \in V(X)$.

Theorem 4 ([28]). *A minor-closed class $\mathcal{G}$ has bounded row treewidth if and only if some apex graph is not in $\mathcal{G}$.*

The product structure of graphs excluding an apex minor has been further studied in [27, 46]. For an apex graph X, let k_X be the minimum integer such that, for some integer c, every X-minor-free graph is contained in $H \boxtimes P \boxtimes K_c$ where H is a graph with $\mathrm{tw}(H) \leqslant k_X$ and P is a path. Illingworth et al. [46] showed that k_X is at most the vertex-cover-number of X. Tightening this result, Dujmović et al. [27] showed that $\mathrm{td}(X) - 2 \leqslant k_X \leqslant 2^{\mathrm{tw}(X)-1}$, where $\mathrm{td}(X)$ is the tree-depth of X. This shows that the tree-depth of X determines the minimum treewidth in such a product structure theorem.

While graphs excluding an apex minor are the largest class with bounded row treewidth, graph products can be used to describe the structure of graphs excluding any fixed minor. Dujmović et al. [25] showed that graphs excluding a fixed minor and with bounded maximum degree have bounded row treewidth. Dujmović et al. [28] showed the following 'Graph Minor Product Structure Theorem', where $+K_a$ means the complete join with a complete graph on a vertices.

Theorem 5 ([28]). *For any graph X there exist integers k, a such that every X-minor-free graph can be obtained from clique-sums of subgraphs of graphs of the form $(H \boxtimes P) + K_a$ where $\mathrm{tw}(H) \leqslant k$.*

This result has enabled many of the above applications, including bounded queue-number, to be extended to the setting of any proper minor-closed class. As another application, Dujmović et al. [28] showed that Theorem 5 implies the following classical theorem of DeVos, Ding, Oporowski, Sanders, Reed, Seymour, and Vertigan [12]: for any graph X and integer $k \geqslant 2$ every X-minor-free has an (improper) vertex-k-colouring such that the union of any $k - 1$ colour classes has bounded treewidth. Another application relates to Hadwiger's Conjecture. Using Theorem 5 as a starting point, Dujmović et al. [26] showed that graphs excluding a fixed K_t-minor are $(t - 1)$-

colourable with monochromatic components of bounded size. This resolves the so-called Clustered Hadwiger Conjecture.

Graph product structure theorems have also been established for various non-minor-closed graph classes [4, 18, 31, 43]. Here Theorems 1 and 2 are used as building blocks for establishing product structure theorems for more general classes, as illustrated by the following example. A graph is *(g,k)-planar* if it has a drawing in a surface of Euler genus g such that each edge is in at most k crossings (and no three edges cross at the same point).

Theorem 6. *Every (g,k)-planar graph G is contained in:*

(a) $H \boxtimes P \boxtimes K_c$ *where* $\mathrm{tw}(H) \leqslant O(k^3)$*, P is a path, and* $c \in O(gk^2)$ [31],

(b) $H \boxtimes P \boxtimes K_c$ *where* $\mathrm{tw}(H) \leqslant 10^{10}$*, P is a path, and* $c = c(g,k)$ [18].

We now briefly summarise some of the recent directions in which graph product structure theory is moving.

- Huynh et al. [45] extended all of the above results to the setting of countably infinite graphs. This is interesting, since in the infinite setting, the graph $H \boxtimes P$ is universal, in the sense that it contains every countable planar graph.
- Several authors have established results which show that graphs of given treewidth are contained in products of simpler graphs [10, 14, 15, 19, 20, 50]. Indeed, this direction predates Theorem 1 through the classical work of Ding and Oporowski [14, 15]. The characterisation of graphs with polynomial growth due to Krauthgamer and Lee [48] is another early example of a graph product structure theorem.
- Hliněný and Jedelský [44] introduced graph product structure theory for dense graph classes and the induced subgraph relation.
- A number of classical theorems about graph minors have recently been revisited in light of graph product structure theory. Dujmović, Hickingbotham, Joret, Micek, Morin, and Wood [32] showed that for every tree T of radius h, there is an integer c such that every T-minor-free graph is contained in $H \boxtimes K_c$ for some graph H with pathwidth at most $2h-1$. This is a qualitative strengthening of the Excluded Tree Minor Theorem of Robertson and Seymour [53]. They also showed that radius is the right parameter to consider in this setting, and $2h-1$ is the best possible bound. Dujmović et al. [27] proved analogous strengthenings of the Grid Minor Theorem of Robertson and Seymour [54]. Here the right parameter of the excluded minor is tree-depth.
- Several recent results show that for various graph classes $\mathcal{G}$, every n-vertex graph in $\mathcal{G}$ is contained in $H \boxtimes K_m$ where H has bounded treewidth and $m \in O(\sqrt{n})$. This direction was initiated by Illingworth et al. [46]. For planar graphs, Distel, Dujmović,

Eppstein, Hickingbotham, Joret, Micek, Morin, Seweryn, and Wood [16] showed this result with $\mathrm{tw}(H) \leqslant 2$ which is best possible. For K_t-minor-free graphs, they showed this result with $\mathrm{tw}(H) \leqslant 4$. These two results are qualitative strengthenings of classical balanced separator results of Lipton and Tarjan [49] and Alon, Seymour, and Thomas [1]. Dvořák and Wood [36] studied analogous questions for any graph class admitting strongly sublinear separators. Recently, Dujmović, Joret, Micek, Morin, and Wood [34] showed that every n-vertex planar graph (and indeed any graph with bounded row treewidth) is contained in $H \boxtimes K_m$ where H is a fan graph (which has pathwidth 2) and $m \in O(\sqrt{n}\log^2 n)$.

We finish with two open problems. The first asks how far can these product structure theorems be generalised. For example, is there a product structure theorem (using other ingredients in addition to products) for any monotone graph class admitting strongly sublinear separators? For example, the Koebe disc packing theorem says every planar graph can be represented as the touching graph of discs in $\mathbb{R}^2$. What can be said about the product structure of touching graphs of spheres in $\mathbb{R}^3$? Our second open problem concerns applications. Does every monotone graph class admitting strongly sublinear separators have bounded queue-number? We expect that a successful answer to this question will unlock the right structure theorem to answer the first question. The interplay between theory and applications is a feature of this field of research.

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