Exploring Sparse and Hereditary Graph Classes via Products and Tree-Decompositions

Robert Hickingbotham

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School of Mathematics
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Abstract

This thesis explores the global structure of sparse and hereditary graph classes via products and tree-decompositions. We focus on two areas.

First, we advance graph product structure theory by systematically studying the structural properties of graph products while also establishing new product structure theorems. Graph product structure theory describes complex graphs as subgraphs of products of simpler graphs. In 2020, Dujmović, Joret, Micek, Morin, Ueckerdt, and Wood [J. ACM 2020] established that every planar graph is contained in the strong product (denoted $\boxtimes$) of a graph with bounded treewidth and a path. This seminal result, known as the *Planar Graph Product Structure Theorem*, has been the key tool to resolve several major open problems regarding queue layouts, nonrepetitive colourings, centred colourings, clustered colourings, adjacency labelling schemes, vertex rankings, and infinite graphs.

Inspired by this result, we explore the product structure of various sparse graph classes. First, we prove that for every tree T of radius h, there is an integer c such that every T-minor-free graph is contained in $H \boxtimes K_c$ for some graph H with pathwidth at most $2h-1$. This is a qualitative strengthening of the Excluded Tree Minor Theorem of Robertson and Seymour (GM I). Second, we show that every graph with Euler genus g is contained in $H \boxtimes P \boxtimes K_{\max\{2g,3\}}$ for some graph H with treewidth at most 3 and for some path P. This improves upon an earlier result of Dujmović et al. [J. ACM 2020]. Finally, we use shallow minors to prove product structure theorems for various beyond-planar graph classes. In particular, we show that powers of bounded degree planar graphs, k-planar graphs, fan-planar graphs, and k-fan-bundle planar graphs have a product structure of the form $H \boxtimes P$ for some graph H with bounded treewidth and for some path P.

Graph product structure theory comes under the wider field of graph sparsity theory. One of the most fundamental tools in this area is colouring numbers. Using this tool, we present improved bounds on the following graph parameters: cop-width, flip-width, odd chromatic number, and proper conflict-free chromatic number.

The second goal of this thesis is to explore the pathwidth and treewidth of hereditary graph classes. Recently, there has been substantial interest in understanding the unavoidable induced subgraphs for graphs with large treewidth. In this direction, we make two contributions. First, we initiate the study of induced subgraphs and path-decompositions. We show that for several natural graph classes, the unavoidable induced subgraphs for graphs of large pathwidth are subdivisions of complete binary trees and line graphs of subdivisions of complete binary trees. Second, we describe the unavoidable induced subgraphs for circle graphs with large treewidth. A circle graph is the intersection graph of a set of chords on a circle. Our result is the first to describe the unavoidable induced subgraphs for a natural hereditary graph class when they are not the so-called obvious candidates.

Declaration

This thesis contains no material which has been accepted for the award of any other degree or diploma at any university or equivalent institution and that, to the best of my knowledge and belief, this thesis contains no material previously published or written by another person, except where due reference is made in the text of the thesis.

Robert Hickingbotham
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Publications during enrolment

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6. R. Hickingbotham and D. R. Wood: Structural properties of graph products. *Journal of Graph Theory*, accepted in 2023 [197]
7. R. Campbell, M. Distel, J. P. Gollin, D. Harvey, K. Hendrey, R. Hickingbotham, B. Mohar, and D. R. Wood: Graphs of linear growth have bounded treewidth. *Electronic Journal of Combinatorics* 30:P3.1 (2023) [64]
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This thesis contains results from the papers [62, 105, 120, 189–193, 196, 197]. I have only included results in which I contributed key creative ideas. I am very grateful for all my collaborators.

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Contents

Chapter 1

Introduction

1.1 Overview

This thesis explores the structure of sparse and hereditary graph classes via products and tree-decompositions. We focus on two research themes. See Section 1.2 for undefined terms.

Theme #1: Graph Product Structure Theory

Graph product structure theory describes complex graphs as subgraphs of products of simpler graphs, typically with bounded treewidth. In 2020, Dujmović, Joret, Micek, Morin, Ueckerdt and Wood [121] established that every planar graph is contained in the strong product (denoted $\boxtimes$) of a graph with bounded treewidth and a path. This seminal result has been the key tool to resolve several major open problems regarding queue layouts [121], nonrepetitive colourings [116], centred colourings [95], clustered colourings [117, 118], adjacency labelling schemes [46, 115, 148], vertex rankings [51], twin-width [49], infinite graphs [203], and comparable box dimension [129].

This thesis seeks to advance graph product structure theory by systematically studying the structural properties of graph products (Chapter 2) while also establishing new product structure theorem for various graph classes (Chapters 3–5). In Chapter 2, we study various structural properties of graph products such as degeneracy, pathwidth, and treewidth. In Chapter 3, we prove a qualitative strengthening of Robertson and Seymour's Excluded Tree Minor Theorem via the lens of product structure theory. We show that for every tree T of radius h, there exists $c \in \mathbb{N}$ such that every T-minor-free graph G is contained in $H \boxtimes K_c$ for some graph H with pathwidth at most $2h-1$. In Chapter 4, we establish product structure theorems for graphs on surfaces. We show that every square-graph is contained in $H \boxtimes P$ for some outerplanar graph H and some path P. We also show that every graph with Euler genus g is contained in $H \boxtimes P \boxtimes K_{\max\{2g,3\}}$ for some planar graph H with treewidth 3 and for some path P. This strengthens an earlier result of Dujmović et al. [121]. In Chapter 5, we use shallow minors to establish product structure theorems for various beyond-planar graph classes such as powers of bounded degree planar graphs, k-planar graphs, fan-planar graphs, and k-fan-bundle planar graphs.

Graph product structure theory comes under the wider field of *graph sparsity theory*. One of the most fundamental tools in this area is colouring numbers which characterise bounded expansion and nowhere dense graph classes. In Chapter 6, we further demonstrate the power of colouring numbers by presenting several original applications of this tool to newly introduced graph parameters.

Theme #2: Pathwidth and Treewidth of Hereditary Graph Classes

The second goal of this thesis is to explore the pathwidth and treewidth of hereditary graph classes. Pathwidth and treewidth are fundamental parameters in structural and algorithmic graph theory [40, 185, 282]. What substructures force a graph to have large pathwidth or large treewidth? Under the graph minor and subgraph relations, these substructures are well understood. For pathwidth, it follows from the Excluded Tree Minor Theorem (Theorem 1.4) [285] that a minor-closed class has bounded pathwidth if and only if it excludes a tree. This implies that graphs with sufficiently large pathwidth contain a subdivision of a large complete binary tree as a subgraph. Similarly, for treewidth, it follows from the Excluded Grid Minor Theorem (Theorem 1.2) [288] that a minor-closed class has bounded treewidth if and only if it excludes a planar graph. This implies that graphs with sufficiently large treewidth contain a subdivision of a large wall as a subgraph. So the picture is complete for the unavoidable minors and subgraphs for graphs with large pathwidth and graphs with large treewidth.

In recent years, there has been substantial interest in answering analogous questions with respect to the induced subgraph relation [1–9, 45, 225, 243, 275, 303]. For example, Korhonen [225] showed that graphs with bounded maximum degree and sufficiently large treewidth contain a subdivision of a large wall or the line graph of a subdivision of a large wall as an induced subgraph.

In this thesis, we make two contributions in this area. First, we initiate the study of induced subgraphs and path-decompositions. In Chapter 7, we show that for graphs with bounded maximum degree and graphs that exclude a given graph as a minor, the unavoidable induced subgraphs for such graphs with large pathwidth are subdivisions of large complete binary trees and line graphs of subdivisions of large complete binary trees. In Chapter 8, we describe the unavoidable induced subgraphs for circle graphs with large treewidth. A *circle graph* is the intersection graph of a set of chords of a circle. Circle graphs are a widely studied graph class [89, 92, 94, 128, 161, 220, 226]. To the best of our knowledge, our result is the first to describe the unavoidable induced subgraphs for a natural hereditary graph class when they are not the so-called "obvious candidates": complete graphs, complete bipartite graphs, subdivisions of walls, and line graphs of subdivisions of walls.

1.2 Background

1.2.1 Graph Basics

Unless specified otherwise, this thesis studies undirected simple finite graphs G with vertex-set $V(G)$ and edge-set $E(G)$. For undefined terms and notations, see the textbook by Diestel [101]. For $m, n \in \mathbb{Z}$ with $m \leqslant n$, let $[m, n] := \{m, m+1, \dots, n\}$ and $[n] := [1, n]$. Let $\mathbb{N} := \{1, 2, \dots\}$ and $\mathbb{N}_0 := \{0, 1, \dots\}$.

Let G be a graph. For a vertex $v \in V(G)$, let $N_G(v)$ denote the set of vertices in $V(G)$ adjacent to v, and let $N_G[v]$ denote $N_G(v) \cup \{v\}$. When the graph G is clear, we drop the subscript G and use the notation $N(v)$ and $N[v]$. For a set $S \subseteq V(G)$, let $N_G(S) := (\bigcup N_G(v) : v \in S) \setminus S$ and let $N_G[S] := N_G(S) \cup S$. The *line graph* $L(G)$ of G has $V(L(G)) := E(G)$ where two vertices in $L(G)$ are adjacent if their corresponding edges are incident to a common vertex in G. A *vertex c-colouring* of G is any function $\phi : V(G) \to C$ where $|C| \leqslant c$. If $\phi(u) \neq \phi(v)$ whenever $uv \in E(G)$, then ϕ is *proper*. The *chromatic number* $\chi(G)$ of G is the minimum $c \in \mathbb{N}_0$ such that G has a proper c-colouring.

The *radius* $\operatorname{rad}(G)$ of a connected graph G is the minimum integer $r \geqslant 0$ such that, for some vertex $v \in V(G)$, for every vertex $w \in V(G)$, we have $\operatorname{dist}_G(v, w) \leqslant r$.

A graph G is *d-degenerate* if every subgraph of G has minimum degree at most d. The *degeneracy* $\operatorname{degen}(G)$ of G is the minimum integer d such that G is d-degenerate. Degeneracy is an important graph parameter as it is a primary measure of the sparsity of a graph. Moreover, d-degenerate graphs are $(d+1)$-colourable and in fact $(d+1)$-choosable.

A key theme of this thesis is exploring the structure of graphs that do not contain a given substructure. We are particularly interested in the following containment relations: induced subgraph, subgraph, and minor. A graph H is an *induced subgraph* of G if H can be obtained from G by deleting vertices. If we also allow the deletion of edges, then H is a *subgraph* of G. Throughout this thesis, we say H is *contained* in G if H is isomorphic to a subgraph of G, written $H \subsetneq G$. A graph H is a *minor* of a graph G if a graph isomorphic to H can be obtained from G by vertex deletion, edge deletion, and edge contraction. A graph G is *H-minor-free* if H is not a minor of G. The *Hadwiger number* $h(G)$ of a graph G is the maximum integer t such that K_t is a minor of G.

A *graph class* is a collection of graphs closed under isomorphism. A graph class is *minor-closed* if it is closed under taking minors. A graph class is *hereditary (monotone)* if it is closed under taking induced subgraphs (subgraphs). Clearly every minor-closed class is monotone and every monotone class is hereditary. A graph class is *proper* if it is not the class of all graphs. In a proper minor-closed graph class $\mathcal{G}$, there is a graph X such that every graph in $\mathcal{G}$ is X-minor-free; we say that $\mathcal{G}$ *excludes* X as a minor.

Let $\mathcal{G}$ be a graph class. A *graph parameter* is a function β such that $\beta(G) \in \mathbb{R}$ for every graph G and $\beta(G_1) = \beta(G_2)$ for all isomorphic graphs G_1 and G_2. We say that β is *unbounded* on $\mathcal{G}$ if $\sup\{\beta(G) : G \in \mathcal{G}\} = \infty$, otherwise we say that it is *bounded*. We say that β is a *minor-closed parameter* if $\beta(H) \leqslant \beta(G)$ whenever H is a minor of

G. Two graph parameters α and β are *tied on $\mathcal{G}$* if there exists a function f such that $\alpha(G) \leqslant f(\beta(G))$ and $\beta(G) \leqslant f(\alpha(G))$ for every graph $G \in \mathcal{G}$. If α and β are tied on the class of all graphs, then we say that α and β are *tied*. Moreover, α and β are *linearly/quadratically/polynomially tied on $\mathcal{G}$* if f may be taken to be linear/quadratic/polynomial.

A forest is *rooted* if each component has a root vertex (which defines the ancestor relation). The *vertex-height* of a rooted forest F is the maximum number of vertices in a root-leaf path in F. The *closure* of a rooted forest F is the graph with $V(G) := V(F)$ with $vw \in E(G)$ if and only if v is an ancestor of w (or vice versa). The *tree-depth* $\operatorname{td}(G)$ of a graph G is the minimum vertex-height of a rooted forest F such that G is contained in the closure of F.

Let Σ be a surface; that is, a 2-dimensional manifold. A *drawing* of a graph G in Σ is a function ϕ that maps each vertex $v \in V(G)$ to a point $\phi(v) \in \Sigma$ and maps each edge $e = vw \in E(G)$ to a non-self-intersecting curve $\phi(e)$ in Σ with endpoints $\phi(v)$ and $\phi(w)$, such that:

- $\phi(v) \neq \phi(w)$ for all distinct vertices v and w;
- $\phi(x) \notin \phi(e)$ for each edge $e = vw$ and each vertex $x \in V(G) \setminus \{v, w\}$;
- each pair of edges intersect at a finite number of points: $\phi(e) \cap \phi(f)$ is finite for all distinct edge e, f; and
- no three edges internally intersect at a common point: for distinct edges e, f, g the only possible element of $\phi(e) \cap \phi(f) \cap \phi(g)$ is $\phi(v)$ where v is a vertex incident to all of e, f, g.

A *crossing* of distinct edges $e = uv$ and $f = xy$ is a point in $(\phi(e) \cap \phi(f)) \setminus \{\phi(u), \phi(v), \phi(x), \phi(y)\}$; that is, an internal intersection point. A drawing is *simple* if any two edges share at most one point in common, including endpoints. An *embedding* of G on Σ is a drawing of G on Σ with no crossings. The *Euler genus* of a surface with h handles and c cross-caps is $2h + c$. The *Euler genus* of a graph G is the minimum integer $g \geqslant 0$ such that there is an embedding of G in a surface of Euler genus g; see [253] for more about graph embeddings in surfaces.

A *plane graph* is a graph G equipped with a drawing of G in the plane $\mathbb{R}^2$ with no crossings. The *outer face* of G is the face with unbounded area. An *internal* face is a face with bounded area. If every vertex of G lies on the outer face, then G is an *outerplane* graph. A graph is *planar* if it is isomorphic to a plane graph. A graph is *outerplanar* if it is isomorphic to an outerplane graph.

1.2.2 Planar Graphs and Minors

Planar graphs are one of the most renowned graph classes. Historically, they were central to the development of graph theory due to the 4-Colour Theorem [16, 284] which states that every planar graph has a proper 4-colouring. Outside of graph theory, planar graphs have applications in other fields of mathematics such as complex analysis [186, 223, 309], classical geometry [308], knot theory [78, 233], hyperbolic geometry [68, 305], topology [317, 318], quantum physics [20, 176, 246], and computational geometry [166].

While planar graphs are often easy to visualise, they are structurally complex. For example, many NP-complete problems such as MAXIMUM INDEPENDENT SET and 3-COLOURING, remain NP-complete for planar graphs. Nevertheless, planar graphs have many rich structural properties. For example, Wagner's theorem characterises planar graphs in terms of forbidden minors.

Theorem 1.1 ([330]). *A graph is planar if and only if it excludes K_5 and $K_{3,3}$ as a minor.*

Planar graphs are a quintessential example of a minor-closed graph class. Theorem 1.1 therefore characterises a proper minor-closed graph class in terms of a finite set of forbidden minors. This theorem is particularly useful in showing that a graph is not planar.

In a far-reaching conjecture, Wagner conjectured that every proper minor-closed graph class can be characterised by a finite set of forbidden minors [331]. Robertson and Seymour proved this conjecture in their seminal *Graph Minors* series where they developed a rich theory for the structure of graphs that forbid a minor. This monumental series of twenty-three papers laid the foundation for modern structural graph theory, earning Robertson and Seymour the 2006 Fulkerson Prize.

When working with graph minors, it is often easier to use the equivalent notion of models. Let G and H be graphs. A *model* of H in G is a function μ with domain $V(H)$ such that:

1. $\mu(v)$ (called a *branch set*) is a connected subgraph of G;
2. $\mu(v) \cap \mu(w) = \varnothing$ for all distinct $v, w \in V(G)$; and
3. for every edge $vw \in E(H)$, there is an edge $xy \in E(G)$ such that $x \in V(\mu(v))$ and $y \in V(\mu(w))$.

It is folklore that H is a minor of G if and only if G contains a model of H. For $r \in \mathbb{N}$, if there exists a model μ of H in G such that $\mu(v)$ has radius at most r for all $v \in V(H)$, then H is an *r-shallow minor* of G.

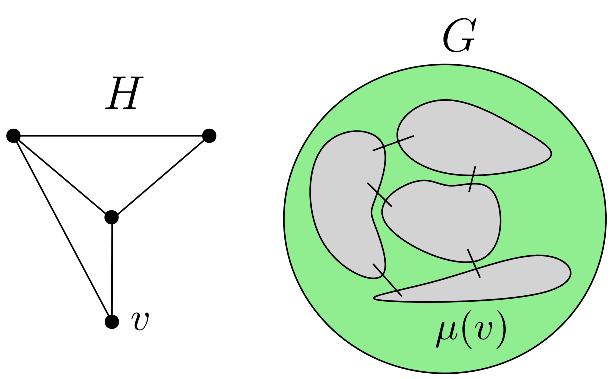


Figure 1.1. A model of a graph H in a graph G.

Related to minors and models are the notions of topological minors and subdivisions. Let $r \geqslant 0$ be an integer and $s \geqslant 0$ be a half-integer (that is, $2s$ is an integer). A graph $\tilde{G}$ is a *subdivision* of a graph G if $\tilde{G}$ can be obtained from G by replacing each edge vw by a path P_{vw} with endpoints v and w (internally disjoint from the rest of $\tilde{G}$). If each of these paths has the same length, then $\tilde{G}$ is said to be *uniform*. If each of the paths have length at most $r+1$, then $\tilde{G}$ is an *$(\leqslant r)$-subdivision* of G. A graph H is a *topological minor* of G if a subgraph of G is isomorphic to a subdivision of H. A graph G is *H-topological minor-free* if H is not a topological minor of G. We say that H is an *s-shallow topological minor* of G if a subgraph of G is isomorphic to a $(\leqslant 2s)$-subdivision of H. Since every topological minor of a graph is also a minor, it follows that if a graph H is an s-shallow topological minor of a graph G, then H is also an r-shallow minor of G whenever $s \leqslant r$. The *Hajós number $h'(G)$* of G is the maximum integer t such that K_t is a topological minor of G. Clearly $h'(G) \leqslant h(G)$ for every graph G. Conversely, there are graph classes with unbounded Hadwiger number but bounded Hajós number. For example, the class of graphs with maximum degree 3 excludes K_5 as a topological minor yet has unbounded Hadwiger number.

1.2.3 Treewidth and Pathwidth

In their proof of Wagner's conjecture, Robertson and Seymour first considered the special case when a planar graph is forbidden as a minor. To describe the structure of such graphs, we need the following definitions.

For a graph H, an *H-decomposition* of a graph G is a collection $\mathcal{W} = (W_x : x \in V(H))$ of subsets of $V(G)$ (called *bags*) indexed by the nodes of H such that:

1. for every edge $vw \in E(G)$, there exists a node $x \in V(H)$ with $v, w \in W_x$; and
2. for every vertex $v \in V(G)$, the set $\{x \in V(H) : v \in W_x\}$ induces a connected subgraph of H.

The *width* of $\mathcal{W}$ is $\max\{|W_x| : x \in V(H)\} - 1$. The *torso* of a bag B_x (with respect to $\mathcal{W}$), denoted by $G\langle B_x\rangle$, is the graph obtained from the induced subgraph $G[B_x]$ by adding edges so that $B_x \cap B_y$ is a clique for each edge $xy \in E(H)$. A *tree-decomposition* is a T-decomposition for any tree T. The *treewidth* $\mathrm{tw}(G)$ of a graph G is the minimum width of a tree-decomposition of G.

Treewidth is a minor-closed parameter which measures how similar a graph is to a tree. In fact, a connected graph has treewidth at most 1 if and only if it is a tree. Tree-decompositions were introduced by Robertson and Seymour [287] where they proved the celebrated Excluded Grid Minor Theorem. Let $n \in \mathbb{N}$. The *$(n \times n)$-grid* is the graph with vertex-set $\{(i,j) : i, j \in [n]\}$ and edge-set

$$\{(i,j)(i+1,j) : i \in [n-1], j \in [n]\} \cup$$
$$\{(i,j)(i,j+1) : i \in [n], j \in [n-1]\}.$$

The *$(n \times n)$-wall* is the graph with vertex-set $\{(i,j) \colon i,j \in [n]\}$ and edge-set

$$\{(i,j)(i+1,j) \colon i \in [n-1], j \in [n]\} \cup$$
$$\{(i,j)(i,j+1) \colon i \in [n], j \in [n-1],\ i+j \text{ even}\}.$$

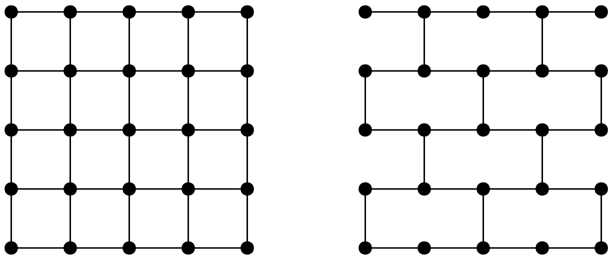


Figure 1.2. The (5×5)-grid and the (5×5)-wall.

Grids and walls are canonical examples of graphs with large treewidth. Indeed, for every $n \in \mathbb{N}$, the $(n \times n)$-grid has treewidth equal to n [185]. Conversely, the Excluded Grid Minor Theorem of Robertson and Seymour [287] states that every graph with sufficiently large treewidth contains a large grid as a minor.

Theorem 1.2 (Excluded Grid Minor Theorem [288]). *There is a function f such that for every $n \in \mathbb{N}$, every graph with treewidth at least $f(n)$ contains the $(n \times n)$-grid as a minor.*

Since every planar graph is a minor of a sufficiently large grid, Theorem 1.2 implies that a minor-closed graph class has bounded treewidth if and only if it excludes a planar graph. This was a key ingredient in Robertson and Seymour's proof of Wagner's conjecture. The asymptotics of f have been substantially improved since the original work. Most significantly, Chekuri and Chuzhoy [75] showed that f can be chosen to be polynomial in n. The current best bound is $f(n) \in \mathcal{O}(n^9 \operatorname{polylog} n)$, which follows from a result of Chuzhoy and Tan [84].

The Excluded Grid Minor Theorem also describes the unavoidable subgraphs for graphs with large treewidth. By Theorem 1.2, graphs with sufficiently large treewidth contain a large wall as a minor. But since walls have maximum degree 3, a folklore result implies that containing a wall as a minor is equivalent to containing a wall as a topological minor. So Theorem 1.2 implies the following.

Corollary 1.3 ([288]). *There is a function f such that, for every $n \in \mathbb{N}$, every graph with treewidth at least $f(n)$ contains a subdivision of the $(n \times n)$-wall as a subgraph.*

So by the Excluded Grid Minor Theorem, we have a clear picture of the unavoidable minors and subgraphs for graphs with large treewidth. In contrast, the picture for the unavoidable induced subgraphs for graphs with large treewidth is more opaque; see Section 1.4 for a discussion on this.

Treewidth and tree-decomposition are central tools in algorithmic and structural graph theory. Compared to planar graphs, graphs with bounded treewidth are a simpler class of graphs. Many problems that are NP-complete in general become tractable when parameterised by treewidth. For example, MAXIMUM INDEPENDENT SET and k-COLOURING are linear-time solvable for graphs with bounded treewidth. More generally, Courcelle's powerful metatheorem [86] states that every graph property definable in monadic second-order logic can be decided in linear-time on graphs of bounded treewidth.

Akin to treewidth, we have the notion of pathwidth. A *path-decomposition* of a graph is a P-decomposition for any path P. The *pathwidth* $\mathrm{pw}(G)$ of a graph G is the minimum width of a path-decomposition of G. Pathwidth is graph parameter introduced by Robertson and Seymour [285] which measures how similar a graph is to a path. Since every path is a tree, $\mathrm{tw}(G) \leqslant \mathrm{pw}(G)$ for every graph G.

Complete binary trees are the canonical example of graphs with large pathwidth: for every $h \in \mathbb{N}$, the complete binary tree with height h has pathwidth $\lceil h/2 \rceil$ [295]. So the class of all forests has unbounded pathwidth. Conversely, Robertson and Seymour [285] proved that for every tree T there is an integer c such that every T-minor-free graph has pathwidth at most c. Bienstock, Robertson, Seymour, and Thomas [34] showed the same result with $c = |V(T)| - 2$, which is best possible, since the complete graph on $|V(T)| - 1$ vertices is T-minor-free and has pathwidth $|V(T)| - 2$.

Theorem 1.4 (Excluded Tree Minor Theorem [34])**.** *For every tree T, every graph with pathwidth at least $|V(T)| - 1$ contains T as a minor.*

So Theorem 1.4 implies that a minor-closed graph class has bounded pathwidth if and only if it excludes a forest. Moreover, the Excluded Tree Minor Theorem describes the unavoidable subgraphs for graphs with large pathwidth. By Theorem 1.4, graphs with large pathwidth contain a large complete binary tree as a minor. Since binary trees have maximum degree 3, it follows that minor-containment for binary trees is equivalent to topological-minor containment. Therefore, Theorem 1.4 implies the following. Let T_h denote the complete binary tree with height h.

Corollary 1.5. *For every $h \in \mathbb{N}$, every graph with pathwidth at least $|V(T_h)| - 1$ contains a subdivision of T_h as a subgraph.*

See Chapter 7 for a discussion on the unavoidable induced subgraphs for graphs with large pathwidth.

Graphs with bounded pathwidth and treewidth have many useful structural properties [39, 185, 280]. For example, the following folklore lemma implies that for any graph with bounded treewidth and for any collection of connected subgraphs, either there is a small hitting set or there are many vertex-disjoint copies of graphs in the collection; see [205] for a proof.

Lemma 1.6. *For every graph G, for every tree-decomposition $\mathcal{D}$ of G, for every collection $\mathcal{F}$ of connected subgraphs of G, and for every $\ell \in \mathbb{N}$, either:*

(a) *there are ℓ vertex-disjoint subgraphs in $\mathcal{F}$, or*
(b) *there is a set $S \subseteq V(G)$ consisting of at most $\ell-1$ bags of $\mathcal{D}$ such that $S \cap V(F) \neq \varnothing$ for all $F \in \mathcal{F}$.*

Lemma 1.6 is used in Chapter 3 to prove a product structure strengthening of the Excluded Tree Minor Theorem.

1.2.4 Graph Minor Structure Theorem

By the Excluded Grid Minor Theorem (Theorem 1.2), we have a rich understanding of the structure of graphs that exclude a planar graph as minor. What about graphs that exclude a non-planar graph as a minor? The Graph Minor Structure Theorem of Robertson and Seymour [289] describes the structure of such graphs by a tree-decomposition where each torso can be constructed using three ingredients: graphs on surfaces, vortices, and apex vertices. To describe this formally, we need the following definitions.

Let G_0 be a graph embedded in a surface Σ. Let F be a facial cycle of G_0. An F-vortex (with respect to G_0) is an F-decomposition $(B_x \subseteq V(H) \colon x \in V(F))$ of a graph H such that $V(G_0 \cap H) = V(F)$ and $x \in B_x$ for each $x \in V(F)$. For $g, p, a \geqslant 0$ and $k \geqslant 1$, a graph G is *(g, p, k, a)-almost embeddable* if for some set $A \subseteq V(G)$ with $|A| \leqslant a$, there are graphs $G_0, G_1, \dots, G_p$ such that:

- $G - A = G_0 \cup G_1 \cup \dots \cup G_p$;
- $G_1, \dots, G_p$ are pairwise vertex-disjoint;
- G_0 is embedded in a surface Σ of Euler genus at most g;
- there are p pairwise disjoint G_0-clean closed discs $D_1, \dots, D_p$ in Σ; and
- for $i \in [p]$, there is an F_i-vortex $(B_x \subseteq V(G_i) \colon x \in V(F_i))$ of G_i of width at most k.

The vertices in A are called *apex* vertices—they can be adjacent to any vertex in G. For $\ell \in \mathbb{N}$, a graph is *ℓ-almost-embeddable* if it is (g, p, k, a)-almost-embeddable for some $g, p, k, a \leqslant \ell$.

Theorem 1.7 (Graph Minor Structure Theorem [289])**.** *For every proper minor-closed class $\mathcal{G}$, there is a constant $\ell \geqslant 1$ such that every graph $G \in \mathcal{G}$ has a tree-decomposition $(B_x \colon x \in V(T))$ such that for every node $x \in V(T)$, the torso $G\langle B_x \rangle$ is ℓ-almost-embeddable.*

This powerful theorem reduces problems on proper minor-closed graph classes (complex graphs) to graphs on surfaces (simpler graphs). As such, it has found a plethora of theoretical and algorithmic applications. However, this theorem tells us nothing about the structure of planar graphs since they are part of the building blocks. Yet for many problems, planar graphs are the first hard case. This motivates the need for a structure theorem for planar graphs; one that describes planar graphs in terms of more basic building blocks. With this in mind, we now turn to graph product structure theory.

1.3 Graph Product Structure Theory

Graph product structure theory describes complex graphs as subgraphs of products of simpler graphs. As this is a rapidly expanding field, we provide an up-to-date survey on this topic, along with a discussion of the original contributions that this thesis makes to this area. Note that there are several results mentioned in this survey which I have co-authored but have not been included in this thesis.

For two graphs G and H, the *strong product* $G \boxtimes H$ is the graph with vertex-set $V(G) \times V(H)$ and an edge between two vertices (v, w) and (v', w') if and only if $v = v'$ and $ww' \in E(H)$, or $w = w'$ and $vv' \in E(G)$, or $vv' \in E(G)$ and $ww' \in E(H)$ (see Figure 1.3 for example).

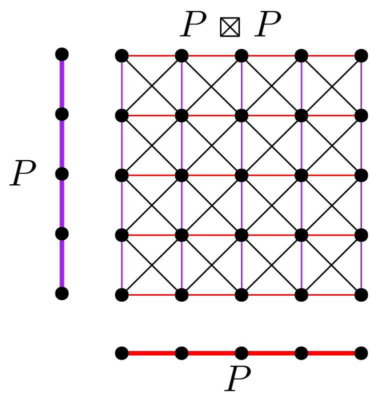


Figure 1.3. The strong product of two paths.

Typically within graph product structure theory, we are interested in showing that for a particular graph class $\mathcal{G}$, there are integers $t, \ell \geqslant 0$ such that every graph in $\mathcal{G}$ is contained in $H \boxtimes P \boxtimes K_\ell$ for some graph H with treewidth at most t and for some path P. Here the primary goal is to minimise t, where minimising ℓ is a secondary goal. Indeed, for some applications of graph product structure theory, the main dependency is on $\text{tw}(H)$ rather than the K_ℓ term; see the upcoming discussion on centred colouring in Section 1.3.8 as an example.

1.3.1 Planar Graphs

The seminal result in this field is the *Planar Graph Product Structure Theorem.*[1] In 2020, Dujmović, Joret, Micek, Morin, Ueckerdt and Wood [121] showed that every planar graph is contained in the strong product of a graph with bounded treewidth and a path.

Theorem 1.8 ([121]). *Every planar graph is contained in $H \boxtimes P$ for some planar graph H with $\text{tw}(H) \leqslant 8$ and for some path P.*

Theorem 1.8 has been the key tool to resolve several major open problems regarding queue layouts [121], nonrepetitive colourings [116], centred colourings [95], clustered

[1]Note that there are several variants of the Planar Graph Product Structure Theorem (Theorems 1.8–1.11). Where it is important, we specify which variant we are referring to.

colourings [117, 118], adjacency labelling schemes [46, 115, 148], vertex rankings [51], twin-width [49], infinite graphs [203] and comparable box dimension [129]. See Section 1.3.8 for a discussion on the applications of graph product structure theory.

In addition to Theorem 1.8, Dujmović et al. [121] also proved the following product structure theorem for planar graphs.

Theorem 1.9 ([121]). *Every planar graph is contained in $H \boxtimes P \boxtimes K_3$ for some planar graph H of treewidth at most 3 and for some path P.*

If a graph H has treewidth at most 3, then $\text{tw}(H \boxtimes K_3) \leqslant 3(\text{tw}(H) + 1) - 1 = 11$ (see Lemma 2.18). So neither Theorem 1.9 nor Theorem 1.8 are strictly stronger than the other. The proofs for the above two theorems build heavily on the earlier work of Pilipczuk and Siebertz [276].

Using the same proof method, Dujmović [53] proved the following variant of the Planar Graph Product Structure Theorem.

Theorem 1.10 ([53]). *Every planar graph is contained in $H \boxtimes P \boxtimes K_2$ for some planar graph H of treewidth at most 4 and for some path P.*

Bose et al. [52] defined the *row treewidth* $\text{rtw}(G)$ of a graph G to be the minimum treewidth of a graph H such that $G \subsetneq H \boxtimes P$ for some path P. Similarly, for a graph class $\mathcal{G}$, the *row treewidth* of $\mathcal{G}$ is the minimum $k \in \mathbb{N}_0$ such that every graph in $\mathcal{G}$ has row treewidth at most k. Theorem 1.8 says that planar graphs have row treewidth at most 8. Refining the proof of Theorem 1.8, Ueckerdt, Wood and Yi [324] proved that planar graphs have row treewidth at most 6.

Theorem 1.11 ([324]). *Every planar graph is contained in $H \boxtimes P$ for some planar graph H of treewidth at most 6 and for some path P.*

The proofs for Theorems 1.8–1.11 are constructive and give a $\mathcal{O}(n^2)$ time algorithm for finding H and the mapping of $V(G)$ onto $V(H \boxtimes P)$. Morin [254] refined the decomposition algorithm of Theorem 1.8 to give a $\mathcal{O}(n \log n)$ time algorithm. Bose et al. [53] further refined the algorithm to give an optimal $\mathcal{O}(n)$ time algorithm for Theorems 1.8–1.11.

An important open problem is to determine the minimum $k \in \mathbb{N}$ such that every planar graph has row treewidth at most k. Dujmović et al. [121] showed that there are planar graphs with row treewidth at least 3; see Section 1.3.4. So $k \in \{3, 4, 5, 6\}$. We now discuss a special family of planar graphs where we achieve tight bounds for their row treewidth.

Squaregraphs

A *squaregraph* is a plane graph in which each internal face is a 4-cycle and each internal vertex has degree at least 4. Squaregraphs were introduced in 1973 by Soltan et al. [306] and have many interesting structural and metric properties. For example, Bandelt et al. [24]

showed that squaregraphs are median graphs and are thus partial cubes, and that every squaregraph can be isometrically embedded[2] into the cartesian product of five trees. See the survey by Bandelt and Chepoi [23] for background on metric graph theory.

In Chapter 4, we prove the following product structure theorem for squaregraphs, as illustrated in Figure 1.4. For graphs G and H, the *semi-strong product* $G \bowtie H$ is the graph with vertex-set $V(G) \times V(H)$ with an edge between two vertices (v, w) and (v', w') if $v = v'$ and $ww' \in E(H)$, or $vv' \in E(G)$ and $ww' \in E(H)$; see [159, 202] for example. Note that $G \bowtie H \subseteq G \boxtimes H$ for all G and H.

Theorem 1.12. *Every squaregraph is contained in $H \bowtie P$ for some outerplanar graph H and some path P.*

Since outerplanar graphs have treewidth at most 2, Theorem 1.12 is stronger than Theorem 1.11 in the case of squaregraphs. Theorem 1.12 is also stronger than Theorem 1.11 in the sense that Theorem 1.12 uses $\bowtie$ whereas Theorem 1.11 uses $\boxtimes$. That said, it is well-known that in the case of bipartite planar graphs G, the proof of Theorem 1.11 can be adapted to show that $G \subsetneq H \bowtie P$ where H has treewidth at most 6 and P is a path.

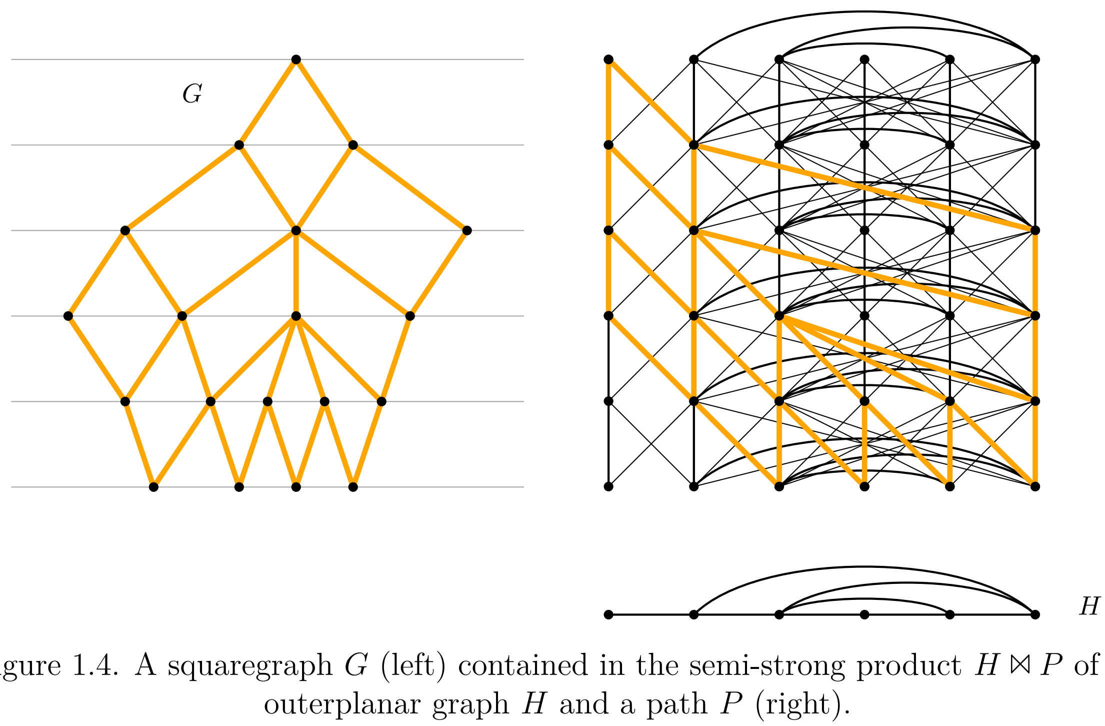


Figure 1.4. A squaregraph G (left) contained in the semi-strong product $H \bowtie P$ of an outerplanar graph H and a path P (right).

We in fact prove a more general sufficient condition for a planar graph to have such a product structure which implies Theorem 1.12; see Theorem 4.3.

We also show that Theorem 1.12 is best possible in the sense that "outerplanar graph" cannot be replaced by "forest". Moreover, this lower bound holds for strong products. In fact, we prove that for every integer $\ell \in \mathbb{N}$, there is a squaregraph G such that for any

[2] A graph H can be *isometrically embedded* into a graph G if there exists an isomorphism ϕ from $V(H)$ to the vertex-set of a subgraph of G such that $\text{dist}_H(u, v) = \text{dist}_G(\phi(u), \phi(v))$ for all $u, v \in V(H)$.

graph H and path P, if $G \subsetneq H \boxtimes P \boxtimes K_\ell$ then H contains a cycle (and is therefore not a forest); see Theorem 4.8 in Section 4.3.2. Also note that Theorem 1.12 cannot be strengthened by replacing "outerplanar graph" by "graph with bounded pathwidth". Indeed, Bose et al. [52] showed that for every $k \in \mathbb{N}$, there is a tree T (which is a squaregraph) such that for any graph H and path P, if $T \subsetneq H \boxtimes P$, then $\text{pw}(H) \geqslant k$.

1.3.2 Minor-Closed Graph Classes

We now consider extensions of the Planar Graph Product Structure Theorem (Theorem 1.8) to other minor-closed graph classes.

Graphs on Surfaces

Dujmović et al. [121] generalised Theorem 1.9 for graphs embeddable in any fixed surface as follows.

Theorem 1.13 ([121])**.** *Every graph with Euler genus g is contained in $H \boxtimes P \boxtimes K_{\max\{2g,3\}}$ for some graph H with treewidth at most 4 and for some path P.*

In Chapter 4, we improve the bound on the treewidth of H from 4 to 3 and with H planar.

Theorem 1.14. *Every graph with Euler genus g is contained in $H \boxtimes P \boxtimes K_{\max\{2g,3\}}$ for some planar graph H with treewidth at most 3 and for some path P.*

The bound on the treewidth of H in Theorem 1.14 is optimal since Dujmović et al. [121] showed that for every integer $\ell \geqslant 0$, there is a planar graph G such that if G is contained in $H \boxtimes P \boxtimes K_\ell$, then H has treewidth at least 3. See [234] for another product structure theorem for graphs on surfaces.

Apex-Minor-Free Graphs

Graphs with Euler genus at most g form a minor-closed class of graphs. Moreover, it follows from Euler's formula that graphs with Euler genus at most g exclude $K_{3,2g+3}$ as a minor. Now $K_{3,2g+2}$ is an example of an apex graph. A graph X is an *apex graph* if it contains a vertex $v \in V(X)$ such that $X - v$ is planar. Generalising the Planar Graph Product Structure Theorem even further, Dujmović et al. [121] proved the following for apex-minor-free graphs.

Theorem 1.15 ([121])**.** *For every apex graph X, there exists $k \in \mathbb{N}$ such that every X-minor-free graph G is contained in $H \boxtimes P$ for some graph H with treewidth at most k and for some path P.*

Theorem 1.15 is best possible in the sense that if X is not an apex graph, then there are X-minor-free graphs with unbounded row treewidth. In particular, the family of apex-grids (grids plus a dominant vertex) has unbounded row treewidth. Note that the bound for k in Theorem 1.15 is large as it depends on constants from the Graph Minor Structure Theorem (Theorem 1.7).

The *vertex-cover number* $\tau(G)$ of a graph G is the size of a smallest set $S \subseteq V(G)$ such that every edge of G has at least one end-vertex in S. Illingworth et al. [205] showed that for every apex graph X, there exists $\ell \in \mathbb{N}$ such that every X-minor-free graph G is contained in $H \boxtimes P \boxtimes K_\ell$ for some graph H with $\text{tw}(H) \leqslant \tau(X)$ and for some path P. This result was improved upon by Dujmović, Hickingbotham, Hodor, Joret, La, Micek, Morin, Rambaud and Wood [119] who showed that $\text{tw}(H)$ can be bounded by a function of the tree-depth of X.

Theorem 1.16 ([119])**.** *For every apex graph X, there exists $\ell \in \mathbb{N}$ such that every X-minor-free graph G is contained in $H \boxtimes P \boxtimes K_\ell$ for some graph H with $\text{tw}(H) \leqslant 2^{\text{td}(X)+1} - 1$ and for some path P.*

Note that $\text{td}(G) \leqslant \tau(G) + 1$ for every graph G. Conversely, there are graph classes with bounded tree-depth but unbounded vertex-cover number (e.g. the class of all trees with vertex-height 3). So Theorem 1.16 is more general than the aforementioned result of Illingworth et al. [205].

Proper Minor-Closed Graph Classes

While the class of H-minor-free graphs does not have bounded row treewidth when H is a non-apex graph, Dujmović et al. [121] showed the following product structure theorem for such graphs.

Theorem 1.17 (Graph Minor Product Structure Theorem [121])**.** *For every proper minor-closed class $\mathcal{G}$ there exists $k, a \in \mathbb{N}$ such that every graph $G \in \mathcal{G}$ can be obtained by clique-sums of graphs $G_1, \dots, G_n$ such that for each $i \in [n]$,*

$$G_i \subseteq (H_i \boxtimes P_i) + K_a,$$

for some graph H_i with treewidth at most k and some path P_i.

Here $A + B$ is the complete join of the graphs A and B. If we also assume that our graph G has bounded maximum degree, then Dujmović, Esperet, Morin, Walczak and Wood [117] showed that we in fact have bounded row treewidth.

Theorem 1.18 ([117])**.** *For every proper minor-closed class $\mathcal{G}$, there exists $c, k \in \mathbb{N}$ such that every graph in $\mathcal{G}$ with maximum degree Δ is a subgraph of $H \boxtimes P \boxtimes K_{c\Delta}$ for some graph H of treewidth at most k and for some path P.*

1.3.3 Non-Minor-Closed Graph Classes

We now discuss product structure theorems for non-minor-closed classes.

(g,k)-Planar Graphs

A graph is *k-planar* if it has a drawing in the plane such that each edge is involved in at most k crossings. Such graphs have been extensively studied; see [99, 222] for surveys. This definition has a natural extension for other surfaces Σ. A graph is *(Σ,k)-planar* if it has a drawing in Σ where each edge is involved in at most k crossings. A graph is *(g,k)-planar* if it is (Σ,k)-planar for some surface Σ with Euler genus at most g.

Refining a result of Dujmović, Morin and Wood [125], in Section 5.4.2, we prove the following product structure theorem for (g,k)-planar graphs.

Theorem 1.19. *Every (g,k)-planar graph G is contained in $H \boxtimes P \boxtimes K_{2\max\{2g,3\}(k+1)^2}$ for some graph H with treewidth at most $\binom{k+4}{3}-1$ and for some path P, and thus G has row treewidth at most $2\max\{2g,3\}(k+1)^2\binom{k+4}{3}-1$.*

In the case when $g=0$ and $k=1$, Dujmović et al. [125] and Bekos, Da Lozzo, Hliněný and Kaufmann [29] independently proved the following, stronger product structure theorem using *framed graphs*; see Section 1.3.4 for a discussion on this tool.

Theorem 1.20 ([29, 125])**.** *Every 1-planar graph is contained in $H \boxtimes P \boxtimes K_7$ for some planar graph H with treewidth at most 3 and for some path P.*

Theorem 1.20 is significantly stronger for $k=1$ since H has treewidth at most 3 which is best possible. In Section 4.4, we extend this result to graphs on surfaces.

Theorem 1.21. *Every $(g,1)$-planar graph is contained in $H \boxtimes P \boxtimes K_{\max\{4g,7\}}$ for some planar graph H with treewidth at most 3 and for some path P.*

As mentioned previously, for some applications of graph product structure theory, the main dependency is on $\operatorname{tw}(H)$ with the K_ℓ term being negligible. Inspired by this, Dujmović et al. [125] asked whether there exist an absolute constant C and a function f such that every k-planar graph is contained in $H \boxtimes P \boxtimes K_{f(k)}$ for graph H with $\operatorname{tw}(H) \leqslant C$ and for some path P. Using *blocking partitions*, Distel, Hickingbotham, Seweryn and Wood [106] answered this question; see Section 1.3.4 for a discussion on this tool.

Theorem 1.22 ([106])**.** *There is a function f such that every (g,k)-planar graph G is contained in $H \boxtimes P \boxtimes K_{f(g,k)}$ for some graph H with $\operatorname{tw}(H) \leqslant 963\,922\,179$ and for some path P.*

The point of Theorem 1.22 is that $\operatorname{tw}(H)$ is bounded by an absolute constant, whereas in Theorem 1.19, $\operatorname{tw}(H) \in \mathcal{O}(k^3)$.

Powers of Planar Graphs

For $k \in \mathbb{N}$, the *k-th power G^k* of a graph G is the graph with vertex-set $V(G)$, where $vw \in E(G^k)$ if and only if $\operatorname{dist}_G(v,w) \in \{1,\dots,k\}$. Refining a result of Dujmović et al. [125], we prove the following product structure theorem in Section 5.4.1.

Theorem 1.23. *Let G be a planar graph. Let $k \in \mathbb{N}$ and $d := \Delta(G^{\lfloor k/2 \rfloor})$. Then G^k is contained in $H \boxtimes P \boxtimes K_{3(2\lfloor k/2 \rfloor+1)^2(d+1)}$ for some graph H with treewidth at most $\binom{2\lfloor k/2 \rfloor+4}{3} - 1$ and for some path P.*

Note that dependence on Δ is unavoidable since, for example, if G is the complete $(\Delta-1)$-ary tree of height k, then G^{2k} is a complete graph on roughly $(\Delta-1)^k$ vertices.

Ossona de Mendez [261] asked whether there exists an absolute constant C and a function f such that, for every planar graph with maximum degree Δ, the graph G^k is contained in $H \boxtimes P \boxtimes K_{f(k,\Delta)}$ for some graph H with $\text{tw}(H) \leqslant C$ and for some path P. Using blocking partitions, Distel, Hickingbotham, Seweryn and Wood [106] answered this question with the following product structure theorem.

Theorem 1.24 ([106])**.** *There is a function f such that for any integers $k, \Delta \geqslant 1$, for every planar graph G with maximum degree Δ, the graph G^k is contained in $H \boxtimes P \boxtimes K_{f(k,\Delta)}$ for some graph H with $\text{tw}(H) \leqslant 963\,922\,179$ and for some path P.*

String Graphs

A *string graph* is the intersection graph of a set of curves in the plane with no three curves meeting at a single point. Such graphs are widely studied; see [154, 155, 247, 269, 294] for example. For an integer $\delta \geqslant 2$, if each curve is involved in at most δ intersections with other curves, then the corresponding string graph is called a *δ-string graph*. A *(g, δ)-string graph* is defined analogously for curves on a surface with Euler genus at most g.

Refining a result of Dujmović et al. [125], we prove the following product structure theorem in Section 5.4.3.

Theorem 1.25. *Every (g, δ)-string graph G is contained in $H \boxtimes P \boxtimes K_{2\max\{2g,3\}(\delta+1)^2}$ for some graph H with treewidth at most $\binom{2\lfloor \delta/2 \rfloor+4}{3} - 1$, and thus G has row treewidth at most $2\max\{2g, 3\}(\delta+1)^2\binom{2\lfloor \delta/2 \rfloor+4}{3} - 1$.*

Fan-Planar Graphs

A graph is *fan-planar* if it has a drawing in the plane such that for each edge $e \in E(G)$, the edges that cross e have a common end-vertex and they cross e from the same side (when directed away from their common end-vertex). Fan-planar graphs were introduced by Kaufmann and Ueckerdt [212]. In Section 5.4.5, we prove the following product structure theorem.

Theorem 1.26. *Every fan-planar graph G is contained in $H \boxtimes P \boxtimes K_{81}$ for some graph H with treewidth at most 19, and thus G has row treewidth at most 1619.*

Fan-Bundle Planar Graphs

A *fan* in a graph is a set of edges incident to a common end-vertex. In a k-fan-bundle drawing of a graph in the plane the edges of a fan may be bundled together at their end-vertices and crossings between bundles are allowed as long as each bundle is crossed by at most k other bundles. More formally, in a *k-fan-bundle planar drawing* of a graph G, each edge has three parts; the first and the last parts are *fan-bundles*, which may be shared by several edges in a fan, while the middle part is unbundled. Each fan-bundle can cross at most k other fan-bundles, while the unbundled parts are crossing-free. A graph is *k-fan-bundle planar* if it admits a k-fan-bundle planar drawing. Fan-bundle planar graphs were introduced by Angelini et al. [15] where they studied their density and algorithmic properties. The following product structure theorem, proved in Section 5.4.6, is the first to consider this graph class from a structural perspective.

Theorem 1.27. *Every k-fan-bundle planar graph G is contained in $H \boxtimes P \boxtimes K_{6(2k+3)^2}$ for some graph H with treewidth at most $\binom{2k+6}{3} - 1$ and some path P, and thus G has row treewidth at most $\binom{2k+6}{3}6(2k+3)^2 - 1$.*

Building upon Theorem 1.27, Distel, Hickingbotham, Seweryn and Wood [106] used blocking partitions to show that the treewidth of H can be bounded by an absolute constant.

Theorem 1.28 ([106])**.** *There is a function f such that every k-fan-bundle planar graph G is contained in $H \boxtimes P \boxtimes K_{f(k)}$ for some graph H with $\operatorname{tw}(H) \leqslant 963\,922\,179$ and for some path P.*

Map Graphs

Start with a graph G embedded in a surface Σ without crossings, with each face labelled a 'nation' or a 'lake', where each vertex of G is incident with at most d nations. Let M be the graph whose vertices are the nations of G, where two vertices are adjacent in G if the corresponding faces in G share a vertex. Then M is called a *(Σ, d)-map graph*. If Σ has Euler genus at most g, then M is called a *(g, d)-map graph*. Graphs embeddable in Σ are precisely the $(\Sigma, 3)$-map graphs [114]. So map graphs are a natural generalisation of graphs embeddable in surfaces.

Using framed graphs, Dujmović et al. [125] and Bekos et al. [29] independently proved the following product structure theorem (where the result of [29] gives slightly stronger bounds for the K_ℓ term).

Theorem 1.29 ([29, 125])**.** *For every integer $d \geqslant 3$, every $(0, d)$-map graph is contained in $H \boxtimes P \boxtimes K_{3\lfloor d/2 \rfloor + \lfloor d/3 \rfloor - 1}$ for some planar graph H with treewidth at most 3 and for some path P.*

In Section 4.4, we lift this result to map graphs on arbitrary surfaces.

Theorem 1.30 ([105])**.** *Every (g, d)-map graph is contained in $H \boxtimes P \boxtimes K_\ell$ for some planar graph H with treewidth at most 3 and for some path P, where $\ell = \max\{2g\lfloor \frac{d}{2} \rfloor, d+3\lfloor \frac{d}{2} \rfloor - 3\}$.*

The attraction of Theorem 1.30 is that it generalises the Planar Graph Product Structure Theorem ($g = d = 0$) and $\mathrm{tw}(H)$ is independent of d and g, and $\mathrm{tw}(H)$ is in fact best possible.

1.3.4 Tools

We now discuss some of the main tools used to prove product structure theorems.

Graph Partitions

Let G be a graph. A *partition* of G is a collection $\mathcal{P}$ of sets of vertices in G such that each vertex of G is in exactly one element of $\mathcal{P}$. Each element of $\mathcal{P}$ is called a *part*. Empty parts are allowed. The *width* of $\mathcal{P}$ is the maximum number of vertices in a part. For $\ell \in \mathbb{N}$, we say that $\mathcal{P}$ is an *ℓ-partition* if the width of $\mathcal{P}$ is at most ℓ. We say that $\mathcal{P}$ is *connected* if each part induces a connected subgraph of G. The *quotient* of $\mathcal{P}$ (with respect to G) is the graph, denoted by $G/\mathcal{P}$, whose vertices are the non-empty parts in $\mathcal{P}$, where distinct non-empty parts $A, B \in \mathcal{P}$ are adjacent in $G/\mathcal{P}$ if and only if some vertex in A is adjacent in G to some vertex in B. For a graph H, an *H-partition* of G is a partition $\mathcal{P} = (A_x \subseteq V(G) : x \in V(H))$ of G indexed by $V(H)$, such that for each edge $vw \in E(G)$, if $v \in A_x$ and $w \in A_y$ then $x = y$ or $xy \in E(H)$. That is, $G/\mathcal{P}$ is contained in H.

A *layering* of a graph G is a partition $\mathcal{L}$ of G, whose parts are ordered $\mathcal{L} = (L_0, L_1, \dots)$ such that for each edge $vw \in E(G)$, if $v \in L_i$ and $w \in L_j$ then $|i - j| \leqslant 1$. Equivalently, a layering is a P-partition for some path P. Often we are interested in BFS-layerings. For a connected graph G, let $r \in V(G)$ and let $L_i := \{v \in V(G) : \mathrm{dist}_G(v, r) = i\}$ for each $i \geqslant 0$. Then $(L_0, L_1, \dots)$ is a *BFS-layering* of G, said to be *rooted* at r. A path P is *vertical* (with respect to $\mathcal{L}$) if $|V(P) \cap L_i| \leqslant 1$ for all $i \geqslant 0$. A *BFS-spanning tree* T of G is a spanning tree of G, where for each non-root vertex $v \in L_i$ there is exactly one edge vw in T with $w \in L_{i-1}$.

A *layered partition* $(\mathcal{P}, \mathcal{L})$ of a graph G consists of a partition $\mathcal{P}$ and a layering $\mathcal{L}$ of G. If $\mathcal{P} = (A_x : x \in V(H))$ is an H-partition, then $(\mathcal{P}, \mathcal{L})$ is a *layered H-partition* with *width* $\max\{|A_x \cap L| : x \in V(H), L \in \mathcal{L}\}$. Layered partitions were introduced by Dujmović et al. [121] who observed the following connections between partition and products (which follow directly from the definitions).

Observation 1.31 ([121])**.** *For all graphs G and H, G is contained in $H \boxtimes P \boxtimes K_\ell$ for some path P if and only if G has a layered H-partition $(\mathcal{P}, \mathcal{L})$ with width at most ℓ.*

Observation 1.32 ([121])**.** *For all graphs G and H and any $p \in \mathbb{N}$, G is contained in $H \boxtimes K_p$ if and only if G has an H-partition with width at most p.*

Observations 1.31 and 1.32 are fundamental to graph product structure theory. Indeed, essentially every product structure theorem mentioned in this thesis is in fact proved via partitions.

Shallow Minors

Recall that a shallow minor of a graph is obtained by contracting disjoint subgraphs with small radii and then deleting vertices and edges. In Chapter 5, we show that product structure is well-behaved under shallow minors.

Theorem 1.33. *If G is an r-shallow minor of $H \boxtimes P \boxtimes K_\ell$ where H has treewidth at most t and P is a path, then $G \subsetneq J \boxtimes P \boxtimes K_{\ell(2r+1)^2}$ where J has treewidth at most $\binom{2r+1+t}{t} - 1$, and thus $\operatorname{rtw}(G) \leqslant \binom{2r+1+t}{t}\ell(2r+1)^2 - 1$.*

Theorem 1.33 is useful because many beyond-planar graphs can be described as a shallow minor of the strong product of a planar graph with a small complete graph. In particular, we show that the following beyond-planar graph classes have such a shallow minor structure:

- powers of bounded degree planar graphs (Section 5.4.1);
- (g, k)-planar graphs (Section 5.4.2);
- (g, δ)-string graphs (Section 5.4.3);
- (k, p)-cluster planar graphs (Section 5.4.4);
- fan-planar graphs (Section 5.4.5); and
- k-fan-bundle planar graphs (Section 5.4.6).

To prove these results, we first planarise the given graph. For most of the above graph classes, we use the standard planarisation technique of inserting dummy vertices at crossing points. For fan-planar graphs, however, our planarisation procedure is more substantial and is of independent interest (see Lemma 5.22).

Using Theorem 1.33, we conclude that the above beyond-planar graph classes have bounded row treewidth. For fan-planar graphs and k-fan-bundle planar graphs, it was not known whether they had a product structure. For the other graph classes, Dujmović et al. [125] used the concept of shortcut systems to show that they have bounded row treewidth; see Section 5.3.1 for the definition of shortcut systems. However, shortcut systems are limited in that they only apply to graph classes with linear crossing number (see Lemma 5.3). Shallow minors subsume and generalise shortcut systems. In particular, the result for fan-planar graphs uses shallow minors in their full generality since fan-planar graphs have super-linear crossing numbers.

Framed Graphs

Let G be a multigraph embedded in a surface Σ without crossings, where each face is bounded by a cycle. For any integer $d \geqslant 3$, let $G^{(d)}$ be the multigraph embedded in Σ obtained from G as follows: for each face F of G bounded by a cycle C of length at most

d, for all distinct non-adjacent vertices v, w in C, add an edge vw across F to $G^{(d)}$. We say that $G^{(d)}$ is a *(Σ, d)-framed multigraph* with *frame* G. If Σ has Euler genus at most g, then $G^{(d)}$ is a *(g, d)-framed multigraph*.

Framed graphs (for $g = 0$) were introduced by Bekos et al. [31] and are useful because they include several interesting graph classes. In particular:

- every graph with Euler genus g is a subgraph of a $(g, 3)$-framed multigraph;
- every $(\Sigma, 1)$-planar graph is contained in some $(\Sigma, 4)$-framed multigraph (see Lemma 4.15); and
- every (Σ, d)-map graph is a spanning subgraph of some (Σ, d)-framed multigraph (see Lemma 4.14).

Bekos et al. [29] and Dujmović et al. [125] independently proved the following product structure results for $(0, d)$-framed graphs (where the result of [29] gives slightly stronger bounds than that of [125]).

Theorem 1.34 ([29, 125])**.** *For every integer $d \geqslant 3$, every $(0, d)$-framed graph is contained in $H \boxtimes P \boxtimes K_{3\lfloor d/2\rfloor + \lfloor d/3\rfloor - 1}$ for some planar graph H with treewidth at most 3.*

In Section 4.4, we lift Theorem 1.34 to framed graphs on arbitrary surfaces.

Theorem 1.35. *For all integers $g \geqslant 0$ and $d \geqslant 3$, every (g, d)-framed multigraph is contained in $H \boxtimes P \boxtimes K_\ell$ for some planar graph H with treewidth 3 and for some path P, where $\ell = \max\{2g\lfloor \frac{d}{2} \rfloor, d + 3\lfloor \frac{d}{2} \rfloor - 3\}$.*

Theorems 1.34 and 1.35 immediately imply product structure theorems for $(g, 1)$-planar graphs and map graphs (Theorems 1.20, 1.21, 1.29 and 1.30).

Blocking Partitions

Blocking partitions are a tool introduced by Distel, Hickingbotham, Seweryn and Wood [106] to prove product structure theorems where $\text{tw}(H)$ is bounded by an absolute constant rather than depending on a parameter defining $\mathcal{G}$. Let G be a graph and let $\mathcal{P}$ be a connected partition of G. A path P in G is *$\mathcal{P}$-clean* if $|V(P) \cap B| \leqslant 1$ for each part $B \in \mathcal{P}$. We say that $\mathcal{P}$ is *ℓ-blocking* if every $\mathcal{P}$-clean path in G has length at most ℓ. Distel, Hickingbotham, Seweryn and Wood [106] proved that every graph G with bounded Euler genus admits a 894-blocking partition where the width of the partition is bounded by a function of the maximum degree of G.

Theorem 1.36 ([106])**.** *Every graph G with Euler genus g and maximum degree Δ has a 894-blocking partition with width at most*

$$f(\Delta, g) := \max\{10\Delta^{80}(3612\Delta^{452} + 900), 8950g^2 + 1796g\}.$$

Let G and H be graphs and let $r, s \geqslant 0$ be integers. A *rooted model* $((B_x, v_x) \colon x \in V(H))$ of H is a model of H where each B_x has a corresponding root $v_x \in V(B_x)$. If for every $x \in V(H)$ and for every $u \in V(B_x) \setminus \{v_x\}$, we have $\operatorname{dist}_{B_x}(v_x, u) \leqslant r$ and $\deg_{B_x}(u) \leqslant s$, then we say that H is an *(r,s)-shallow minor* of G.

Using Theorems 1.33 and 1.36, Distel, Hickingbotham, Seweryn and Wood [106] proved the following product structure theorem.

Theorem 1.37 ([106]). *There is a function f such that for every graph G of Euler genus g, every (r,s)-shallow minor H of $G \boxtimes K_d$ is contained in $J \boxtimes P \boxtimes K_{f(d,g,r,s)}$ for some graph J with $\operatorname{tw}(J) \leqslant 963\,922\,179$.*

The key point of Theorem 1.37 is that $\operatorname{tw}(J)$ is independent of the parameters g, d, r and s. Building upon previous observations of Hickingbotham and Wood [196], Distel et al. [106] observed that the following graph classes can be described as (r,s)-shallow minors of the strong product of a graph with bounded genus and a small complete graph:

- (g,k)-planar graphs;
- powers of graphs with bounded genus and bounded maximum degree; and
- k-fan-bundle planar graphs.

So Theorem 1.37 implies Theorems 1.22, 1.24 and 1.28.

Lower Bounds

We now introduce an important graph family that is useful for proving lower bounds. For a graph G and integer $\ell \geqslant 1$, let $\widehat{\ell G}$ be the graph obtained from ℓ disjoint copies of G by adding one dominant vertex. For $c, \ell \in \mathbb{N}$, we define $G_{c,\ell}$ recursively as follows. First, $G_{1,\ell} := P_{\ell+1}$ is the path on $\ell + 1$ vertices. Further, for $c \geqslant 2$, let $G_{c,\ell} := \widehat{\ell\, G_{c-1,\ell}}$. The graph family $(G_{c,\ell} \colon c, \ell \in \mathbb{N})$ is common in the literature, and is particularly important for clustered and defective colouring [147, 217, 260, 326].

The next lemma collects together some useful and well-known properties of $G_{c,\ell}$.

Lemma 1.38. *For all $c, \ell \in \mathbb{N}$,*

(i) $\operatorname{tw}(G_{c,\ell}) = c$ *and* $\operatorname{rad}(G_{c,\ell}) = 1$;
(ii) *for any ℓ-partition of $G_{c,\ell}$, there is a $(c+1)$-clique in $G_{c,\ell}$ whose vertices are in distinct parts;*
(iii) *If $G_{c,\ell}$ is contained in $H \boxtimes K_\ell$ for some graph H, then* $\operatorname{tw}(H) \geqslant c$;
(iv) *$G_{3,\ell}$ is planar; and*
(v) *$G_{c,\ell}$ is K_{c+2}-minor-free.*

Proof. Since $\operatorname{tw}(\widehat{\ell G}) = \operatorname{tw}(G) + 1$ and $\operatorname{rad}(\widehat{\ell G}) = 1$ for any graph G and $\ell \in \mathbb{N}$, part (i) follows by induction.

We establish (ii) by induction on c. In the case $c = 1$, every ℓ-partition of $P_{\ell+1}$ contains an edge whose endpoints are in different parts, and we are done. Now assume the claim for $c-1$ ($c \geqslant 2$). Consider an ℓ-partition of $\widehat{\ell\, G_{c-1,\ell}}$. At most $\ell - 1$ copies of $G_{c-1,\ell}$ contain a vertex in the same part as the dominant vertex v. Thus, some copy G_0 of $G_{c-1,\ell}$ contains no vertices in the same part as v. By induction, G_0 contains a c-clique K whose vertices are in distinct parts. Since v is dominant, $K \cup \{v\}$ satisfies the induction hypothesis.

Consider an H-partition of $G_{c,\ell}$ of width at most ℓ. By (ii), $G_{c,\ell}$ contains a $(c+1)$-clique whose vertices are in distinct parts. So $\omega(H) \geqslant c+1$, implying $\text{tw}(H) \geqslant c$. This establishes (iii).

Observe that $G_{2,\ell}$ is outerplanar. The disjoint union of outerplanar graphs is outerplanar and the graph obtained from any outerplanar graph by adding a dominant vertex is planar; thus $G_{3,\ell}$ is planar. This proves (iv).

We next show that $G_{c,\ell}$ is K_{c+2}-minor-free. $G_{1,\ell}$ is a path and so has no K_3-minor. $G_{2,\ell}$ is outerplanar and so has no K_4-minor. Let $c \geqslant 3$ and assume the result holds for smaller c. Suppose that $G_{c,\ell}$ contains a K_{c+2}-minor. Since K_{c+2} is 2-connected, some copy of $G_{c-1,\ell}$ in $G_{c,\ell}$ contains a K_{c+1}-minor. This contradiction establishes (v). □

Since $G_{c,\ell}$ has radius 1, every layering of $G_{c,\ell}$ has at most three non-empty layers. So in particular, if $G_{c,\ell}$ has a layered H-partition with width at most w, then $G_{c,\ell}$ has an H-partition with width at most $3w$. So Items (iii) and (iv) of Lemma 1.38 imply that there is a planar graph with row treewidth 3. More generally, they imply that for every $\ell \in \mathbb{N}$, there is a planar graph G such that if $G \lesssim H \boxtimes P \boxtimes K_\ell$ for some graph H and for some path P, then $\text{tw}(H) \geqslant 3$.

1.3.5 Graphs with Bounded Treewidth

We now discuss product structure theorems of the form $H \boxtimes K_\ell$ for some graph H with bounded treewidth. The study of such results actually predates the Planar Graph Product Structure Theorem, though they were not described with such language. As Observation 1.32 shows, partitions and products are equivalent. So earlier results concerning partitions can in fact be understood as product structure theorems.

Tree-Partitions

One rich area of research in this direction is tree-partitions. A *tree-partition* is a T-partition for some tree T. The *tree-partition-width* $\text{tpw}(G)$ of a graph G is the minimum width of a tree-partition of G. By Observation 1.32, $\text{tpw}(G)$ equals the minimum $\ell \in \mathbb{N}_0$ such that G is contained in $T \boxtimes K_\ell$ for some tree T. Tree-partitions were independently introduced by Seese [298] and Halin [178], and have since been widely investigated [41, 42, 102, 103, 138, 335, 336]. Tree-partition-width has also been called *strong treewidth* [42, 298]. Applications of tree-partitions include graph drawing [65, 97, 123, 126, 341], nonrepetitive graph colouring [26], clustered graph colouring [11], monadic second-order logic [236], network emulations [37, 38, 44, 152], statistical learning theory [346], size-Ramsey numbers [110], and the edge Erdős-Pósa property [73, 162, 279].

As noted by Seese [298], bounded tree-partition-width implies bounded treewidth (this also follows from Lemma 2.18). But in general, tree-partition-width can be much larger than treewidth. For example, a fan graph (a path plus a dominant vertex) on n vertices has treewidth 2 and tree-partition-width $\Omega(\sqrt{n})$. On the other hand, the referee of [102] showed that if the maximum degree and treewidth are both bounded, then so is the tree-partition-width, which is one of the most useful results about tree-partitions. This result was further refined, first by Wood [336] and then by Distel and Wood [107], culminating in the following product structure theorem.

Theorem 1.39 ([102, 107, 336])**.** *For $k, \Delta \in \mathbb{N}$, every graph G of treewidth less than k and maximum degree at most Δ is contained in $T \boxtimes K_m$ where T is a tree with maximum degree 6Δ and $m := 18k\Delta$.*

The bound on m is best possible up to the multiplicative constant [336]. Note that bounded maximum degree is not necessary for bounded tree-partition-width. For example, trees trivially have bounded tree-partition-width, yet they have unbounded maximum degree. Ding and Oporowski [103] characterised graph classes with bounded tree-partition-width in terms of excluded topological minors. Extending the proof method of Theorem 1.39, Campbell, Clinch, Distel, Gollin, Hendrey, Hickingbotham, Huynh, Illingworth, Tamitegama, Tan and Wood [63] gave an alternative characterisation of bounded tree-partition-width using disjointed partitions.

Disjointed Partitions

A partition $\mathcal{P}$ of a graph G is *d-disjointed* if, for every part $B \in \mathcal{P}$ and every component X of $G - B$, we have $N_G(B) \cap V(X) \leqslant d$. Campbell et al. [63] proved that disjointed partitions characterise bounded tree-partition-width.

Theorem 1.40 ([63])**.** *Let $d, k, \ell \in \mathbb{N}$. For any graph G, if $\operatorname{tw}(G) < k$ and G has a d-disjointed partition with width at most ℓ, then G has tree-partition-width at most $24dk\ell$.*

If our graph G has maximum degree Δ, then the singleton partition of G is a Δ-disjointed partition. So Theorem 1.40 generalises Theorem 1.39.

Campbell et al. [63] in fact proved a much more general result which characterises c-tree-partition-width. For $c \in \mathbb{N}$, the *c-tree-partition-width* $\operatorname{tpw}_c(G)$ of a graph G is the minimum width of an H-partition for some graph H with $\operatorname{tw}(H) \leqslant c$. By Observation 1.32, this is equivalent to the minimum $\ell \in \mathbb{N}_0$ such that G is contained in $H \boxtimes K_\ell$ for some graph H with $\operatorname{tw}(H) \leqslant c$.

A partition $\mathcal{P}$ of a graph G is *(c, d)-disjointed* if, for every c-tuple $(B_1, \dots, B_c) \in \mathcal{P}^c$ of distinct parts in $\mathcal{P}$, for every component X of $G - (B_1 \cup \dots \cup B_c)$, there exists $Q \subseteq V(X)$ with $|Q| \leqslant d$ such that, for each component Y of $X - Q$, we have $V(Y) \cap N_G(B_i) = \varnothing$

for some $i \in [c]$. Note that a $(1, d)$-disjointed partition is equivalent to a d-disjointed partition.

Campbell et al. [63] characterised c-tree-partition-width as follows.

Theorem 1.41 ([63]). *Let $c, d, k, \ell \in \mathbb{N}$. For any graph G, if $\operatorname{tw}(G) < k$ and G has a (c, d)-disjointed partition of width at most ℓ, then $\operatorname{tpw}_c(G) \leqslant 2cd\ell(12k)^c$.*

Underlying Treewidth

The above results motivate the following definitions due to Campbell et al. [63]. The *underlying treewidth* $\operatorname{utw}(\mathcal{G})$ of a graph class $\mathcal{G}$ is the minimum $c \in \mathbb{N}_0$ such that, for some function f, for every graph $G \in \mathcal{G}$ there is a graph H with $\operatorname{tw}(H) \leqslant c$ such that G is contained in $H \boxtimes K_{f(\operatorname{tw}(G))}$. If there is no such c, then $\mathcal{G}$ has *unbounded* underlying treewidth. We call f the *treewidth-binding function*. For example, Theorem 1.39 says that any graph class with bounded maximum degree has underlying treewidth at most 1 with treewidth-binding function $\mathcal{O}(\operatorname{tw}(G))$.

Using Theorem 1.41, Campbell et al. [63] proved the following results concerning underlying treewidth:

- The class of planar graphs has underlying treewidth 3.
- The class of graphs embeddable on any fixed surface has underlying treewidth 3.
- The class of K_t-minor-free graphs has underlying treewidth $t - 2$.
- For $t \geqslant \max\{s, 3\}$, the class of $K_{s,t}$-minor-free graphs has underlying treewidth s.

In all these results, the treewidth-binding function is $\mathcal{O}(\operatorname{tw}(G)^2 \log(\operatorname{tw}(G)))$ for fixed s and t. Using a different method, Illingworth, Scott and Wood [205] reproved the above results with a linear treewidth binding function. Their results are in fact corollaries of a more general tool which describes how to convert a tree-decomposition of a graph in a minor-closed class into an H-partition where $\operatorname{tw}(H)$ is bounded. This tool also gives an alternative proof of the Planar Graph Product Structure Theorem (Theorem 1.9). See Section 1.3.6 for a discussion on another application of their method.

For a graph X, let $\mathcal{G}_X$ be the class of graphs that exclude X as a minor. Campbell et al. [63] asked for the underlying treewidth of $\mathcal{G}_X$. In particular, what structural properties of X determine the underlying treewidth of $\mathcal{G}_X$? Since the class of K_t-minor-free graphs has underlying treewidth $t - 2$, it follows that $\operatorname{utw}(\mathcal{G}_X) \leqslant |V(X)| - 2$. By Lemma 1.38, it follows that $\operatorname{td}(X) - 2 \leqslant \operatorname{utw}(\mathcal{G}_X)$. Dujmović, Hickingbotham, Hodor, Joret, La, Micek, Morin, Rambaud and Wood [119] showed that the tree-depth of X is tied to the underlying treewidth of $\mathcal{G}_X$.

Theorem 1.42 ([119]). *For every graph X and integer $k \geqslant 1$, every graph $G \in \mathcal{G}_X$ with $\operatorname{tw}(G) < k$ is contained in $H \boxtimes K_{\mathcal{O}(k)}$ for some graph H with $\operatorname{tw}(H) \leqslant 2^{\operatorname{td}(X)+1} - 4$.*

If X is planar, then by the Excluded Grid Minor Theorem (Theorem 1.2), there exists an absolute constant C such that every graph in $\mathcal{G}_X$ has treewidth at most C. So Theorem 1.42 implies the following.

Theorem 1.43 ([119])**.** *For every planar graph X, there exists an integer $c \geqslant 0$ such that every graph $G \in \mathcal{G}_X$ is contained in $H \boxtimes K_c$ for some graph H with $\operatorname{tw}(H) \leqslant 2^{\operatorname{td}(X)+1} - 4$.*

The point of Theorem 1.43 is that the treewidth of H only depends on the tree-depth of X, not on $|V(X)|$. Note that for the Excluded Grid Minor Theorem, dependence on $|V(X)|$ is unavoidable, since the complete graph on $|V(X)| - 1$ vertices is X-minor-free, but has treewidth $|V(X)| - 2$. Theorem 1.43 is a qualitative strengthening of the Excluded Grid Minor Theorem since $\operatorname{tw}(G) \leqslant \operatorname{tw}(H \boxtimes K_c) \leqslant c(\operatorname{tw}(H) + 1) - 1 \leqslant c(2^{\operatorname{td}(X)+1} - 3) - 1$ (see Lemma 2.18).

Graphs with Bounded Pathwidth

In Chapter 3, we prove the following analogous result to Theorem 1.43 for graphs with bounded pathwidth.

Theorem 1.44. *For every tree T of radius h, there exists $c \in \mathbb{N}$ such that every T-minor-free graph G is contained in $H \boxtimes K_c$ for some graph H with pathwidth at most $2h - 1$.*

Theorem 1.44 is a qualitative strengthening of the Excluded Tree Minor Theorem (Theorem 1.4) since $\operatorname{pw}(G) \leqslant \operatorname{pw}(H \boxtimes K_c) \leqslant c(\operatorname{pw}(H)+1)-1 \leqslant 2ch-1$ (see Lemma 2.18). Note that the proof of Theorem 1.44 depends on the Excluded Tree Minor Theorem. The point of Theorem 1.44 is that $\operatorname{pw}(H)$ only depends on the radius of T, not on $|V(T)|$ which may be much greater than the radius. Moreover, we also show that radius is the right parameter of T to consider here; see Proposition 3.1.

1.3.6 $\mathcal{O}(\sqrt{n})$-Blow Ups

As mentioned previously, Illingworth et al. [205] proved that the class of K_t-minor-free graphs has underlying treewidth $t - 2$ with a linear treewidth binding function. In one of the cornerstone results of Graph Minor Theory, Alon, Seymour, and Thomas [13] proved that every n-vertex K_t-minor-free graph G has $\operatorname{tw}(G) < t^{3/2} n^{1/2}$. For fixed $t \geqslant 5$, this bound is asymptotically tight since the $n^{1/2} \times n^{1/2}$ grid is K_5-minor-free and has treewidth $n^{1/2}$. Using a similar proof strategy for their results for underlying treewidth, Illingworth et al. [205] showed the following product structure strengthening of the Alon–Seymour–Thomas Theorem.

Theorem 1.45 ([205])**.** *Every n-vertex K_t-minor-free graph G is contained in $H \boxtimes K_{\lfloor m \rfloor}$ for some graph H with $\operatorname{tw}(H) \leqslant t - 2$, where $m = 2\sqrt{(t-3)n}$.*

Theorem 1.45 implies the result of Alon, Seymour, and Thomas [13] since

$$\operatorname{tw}(G) \leqslant \operatorname{tw}(H \boxtimes K_{\lfloor m \rfloor}) \leqslant (\operatorname{tw}(H)+1)m - 1 < t\sqrt{(t-3)n}.$$

The following definition naturally arises. For a proper minor-closed graph class $\mathcal{G}$, let $\beta(\mathcal{G})$ be the minimum integer such that for some $c \geqslant 0$, every n-vertex graph $G \in \mathcal{G}$ is contained in $H \boxtimes K_m$, for some graph H with treewidth at most $\beta(\mathcal{G})$, where $m \leqslant c\sqrt{n}$. Theorem 1.45 implies that if $\mathcal{G}_t$ is the class of K_t-minor-free graphs, then $\beta(\mathcal{G}_t) \leqslant t-2$. Illingworth et al. [205] asked whether $\beta(\mathcal{G})$ is upper bounded by an absolute constant. Distel, Dujmović, Eppstein, Hickingbotham, Joret, Micek, Morin, Seweryn and Wood [104] answered this question in the affirmative.

Theorem 1.46 ([104])**.** *Every n-vertex K_t-minor-free graph G is contained in $H \boxtimes K_m$ for some graph H of treewidth at most 4, where $m \in \mathcal{O}_t(\sqrt{n})$.*

Theorem 1.46 implies that $\beta(\mathcal{G}) \leqslant 4$ for every proper minor-closed class $\mathcal{G}$. The proof of Theorem 1.46 actually shows that $\operatorname{tw}(H-v) \leqslant 3$ for some vertex $v \in V(H)$. Distel et al. [104] also showed improved bounds on $\beta(\mathcal{G})$ for particular minor-closed graph classes. First consider the class $\mathcal{L}$ of planar graphs. Since planar graphs are K_5-minor-free, the above result of Illingworth et al. [205] shows that $\beta(\mathcal{L}) \leqslant 3$. Distel et al. [104] showed that $\beta(\mathcal{L}) \leqslant 2$, resolving an open problem of Illingworth et al. [205].

Theorem 1.47 ([104])**.** *Every n-vertex planar graph is contained in $H \boxtimes K_m$, where H is a graph with treewidth 2 and $m \in \mathcal{O}(\sqrt{n})$.*

The Lipton–Tarjan Separator Theorem [240] is one of the most important structural results about planar graphs, with numerous algorithmic applications [241]. It is equivalent to saying that every n-vertex planar graph has treewidth $\mathcal{O}(\sqrt{n})$ (see [134]). Theorem 1.47 is a product structure strengthening of the Lipton–Tarjan Separator Theorem.

Distel et al. [104] in fact proved a more general result than Theorem 1.47 for graphs that exclude a $K_{3,t}$ minor.

Theorem 1.48 ([104])**.** *Every $K_{3,t}$-minor-free n-vertex graph is contained in $H \boxtimes K_m$, where H is a graph with treewidth 2 and $m \in \mathcal{O}(t\sqrt{n})$.*

Since $K_{3,3}$ is not planar, Theorem 1.48 with $t = 3$ implies Theorem 1.48. More generally, Theorem 1.48 also implies results for graphs embeddable in any fixed surface. As previously mentioned, graphs with Euler genus g excludes $K_{3,2g+3}$ as a minor. Thus Theorem 1.48 implies the following.

Theorem 1.49 ([104])**.** *Every n-vertex graph with Euler genus g is contained in $H \boxtimes K_m$, where H is a graph with treewidth 2 and $m \in \mathcal{O}((g+1)\sqrt{n})$.*

Note that Gilbert et al. [163] and Djidjev [109] proved that n-vertex graphs with Euler genus g admit balanced separators of order $\mathcal{O}(\sqrt{(g+1)n})$ and thus have treewidth

$\mathcal{O}(\sqrt{(g+1)n})$. Theorem 1.49 is a qualitative strengthening of these results, with slightly worse dependence on g.

1.3.7 Polynomial and Linear Growth

We now discuss product structure theorems for graphs with polynomial growth.

The *growth* of a graph G is the function $f_G\colon \mathbb{N} \to \mathbb{N} \cup \{\infty\}$ where $f_G(r)$ is the supremum of $|V(H)|$ taken over all subgraphs H of G with radius at most r. Growth in graphs is an important topic in group theory [167, 168, 174, 175, 251, 291, 333], where growth of a finitely generated group is defined through the growth of the corresponding Cayley graphs. Growth of graphs also appears in metric geometry [232], algebraic graph theory [164, 165, 206–208, 321], and in models of random infinite planar graphs [14, 137]. A graph class $\mathcal{G}$ has *linear*/*polynomial growth* if $\sup\{f_G(r)\colon G \in \mathcal{G}\}$ is bounded from above and below by a linear/polynomial function of r.

Krauthgamer and Lee [232] proved the following product structure theorem for graphs with polynomial growth.

Theorem 1.50 ([232])**.** *For every $d > 0$, there exists $k \in \mathcal{O}(d\log(d))$ such that, for every graph G, if $f_G(r) \leqslant r^d$ for every integer $r \geqslant 2$, then G is contained in $P^{(1)} \boxtimes P^{(2)} \boxtimes \cdots \boxtimes P^{(k)}$ for some paths $P^{(1)}, P^{(2)}, \dots, P^{(k)}$.*

Theorem 1.50 is a rough characterisation for graphs with polynomial growth since $P^{(1)} \boxtimes P^{(2)} \boxtimes \cdots \boxtimes P^{(k)}$ has growth at most $r^{\mathcal{O}(d\log(d))}$.

Campbell, Distel, Gollin, Harvey, Hendrey, Hickingbotham, Mohar and Wood [64] proved the following product structure theorem for graphs with linear growth.

Theorem 1.51 ([64])**.** *For any $c \geqslant 1$, every graph G with growth $f_G(r) \leqslant cr$ for every integer $r \geqslant 2$, is contained in $T \boxtimes K_{\lfloor 882c^3 \rfloor}$ for some tree T.*

The growth of $T \boxtimes K_{\lfloor 882c^3 \rfloor}$ is at least the growth of T which can be super polynomial, for example if T is a complete binary tree. Campbell et al. [64] conjecture the following rough characterisation of graphs of linear growth.

Conjecture 1.52 ([64])**.** There exist functions $g\colon \mathbb{R} \to \mathbb{N}$ and $h\colon \mathbb{R} \to \mathbb{R}$ such that for any $c \geqslant 1$, every graph G with growth $f_G(r) \leqslant cr$ is contained in $T \boxtimes K_{g(c)}$ for some tree T with growth $f_T(r) \leqslant h(c)r$.

This conjecture (if true) would characterise graphs of linear growth in the sense that every subgraph H of $T \boxtimes K_{g(c)}$ has growth $f_H(r) \leqslant g(c)h(c)r \in \mathcal{O}(r)$.

More generally, for graphs of polynomial growth, Campbell et al. [64] conjectured the following product structure characterisation.

Conjecture 1.53 ([64])**.** There exist functions $g\colon \mathbb{R} \times \mathbb{N} \to \mathbb{N}$ and $h\colon \mathbb{R} \times \mathbb{N} \to \mathbb{R}$ such that for any $c \geqslant 1$ and $d \in \mathbb{N}$, every graph G with growth $f_G(r) \leqslant cr^d$ is contained in $T_1 \boxtimes \cdots \boxtimes T_d \boxtimes K_{g(c,d)}$, where each T_i is a tree of growth $f_{T_i}(r) \leqslant h(c,d)r$.

1.3.8 Applications

We now move on to discuss several key applications of graph product structure theory. See Chapter 2 for further discussion on the properties of graph products.

Queue Layouts

Heath, Leighton and Rosenberg [187, 188] introduced queue layouts as a way to measure the power of queues to represent graphs. Let G be a graph and $\preceq$ be a total order on $V(G)$. Two distinct edges $vw, xy \in E(G)$ with $v \prec w$ and $x \prec y$ *nest* with respect to $\preceq$ if $v \prec x \prec y \prec w$ or $x \prec v \prec w \prec y$; and *overlap* with respect to $\preceq$ if $v = x$ or $w = y$. Consider a function $\varphi : E(G) \to [k]$ for some integer $k \geqslant 1$. Then $(\preceq, \varphi)$ is a *k-queue layout* of G if vw and xy do not nest for all edges $vw, xy \in E(G)$ with $\varphi(vw) = \varphi(xy)$. If vw and xy neither nest nor overlap for all edges $vw, xy \in E(G)$ with $\varphi(vw) = \varphi(xy)$, then $(\preceq, \varphi)$ is a *strict k-queue layout* of G. The minimum integer $q \geqslant 0$ for which G has a q-queue layout is the *queue-number* $\operatorname{qn}(G)$ of G. The minimum integer $q \geqslant 0$ for which G has a strict q-queue layout is the *strict-queue-number* $\operatorname{sqn}(G)$ of G.

Given a k-queue layout $(\preceq, \varphi)$ of a graph G, for each $i \in [k]$, the set $\varphi^{-1}(i)$ behaves like a queue, in the sense that each edge $vw \in \varphi^{-1}(i)$ with $v \prec w$ corresponds to an element in a sequence of queue operations, such that if we traverse the vertices in the order of $\preceq$, then vw is enqueue at v and then dequeue at w. Since no two edges in $\varphi^{-1}(i)$ nest, this ensures that the operation behaves in a first-in-first-out manner. In this way, the queue-number measures the power of queues to represent graphs.

Wood [334] showed that $\operatorname{qn}(G \boxtimes H) \leqslant 2\operatorname{sqn}(H) \cdot \operatorname{qn}(G) + \operatorname{sqn}(H) + \operatorname{qn}(G)$ for all graphs G and H. Observe that $1 \preceq 2 \preceq \cdots \preceq \ell$ together with $\phi(ij) = |i - j|$ defines a strict $(\ell - 1)$-queue layout of K_ℓ (assuming $V(K_\ell) = [\ell]$); thus $\operatorname{sqn}(K_\ell) \leqslant \ell - 1$. Hence, for every graph G,

$$\operatorname{qn}(G \boxtimes K_\ell) \leqslant (2\ell - 1)\operatorname{qn}(G) + \ell - 1. \tag{$\star$}$$

Dujmović et al. [123] proved that graphs of bounded treewidth have bounded queue-number. The best known bound is $\operatorname{qn}(G) \leqslant 2^{\operatorname{tw}(G)} - 1$, due to Wiechert [332]. Using similar techniques to the result of Wood [334], Dujmović et al. [121] proved the following.

Lemma 1.54 ([121]). *For every graph H and every path P, if a graph G is contained in $H \boxtimes P \boxtimes K_\ell$, then*

$$\operatorname{qn}(G) \leqslant 3\ell\, 2^{\operatorname{tw}(H)} + \lfloor \tfrac{3}{2}\ell \rfloor.$$

For planar graphs, Theorem 1.9 and Lemma 1.54 imply the following.

Theorem 1.55 ([121]). *Every planar graph has queue-number at most* 49.

Theorem 1.55 resolved a long-standing open problem of Heath et al. [187] asking whether planar graphs have bounded queue-number. Refining the proof in [121], Bekos et al. [27] strengthened the upper bound in Theorem 1.55 to 42. Using the Graph Minor

Product Structure Theorem (Theorem 1.17), Theorem 1.55 was quickly extended to show that every proper minor-closed graph class has bounded queue-number.

Theorem 1.56 ([121]). *For every proper minor-closed class $\mathcal{G}$, there exists $k \in \mathbb{N}$ such that every graph in $\mathcal{G}$ has queue-number at most k.*

Theorems 1.55 and 1.56 demonstrate the power of product structure theory by showing that planar graphs are the main bottleneck in preventing larger scale generalisation.

Lemma 1.54 is applicable to any graph class with bounded row-treewidth. So together with Theorems 1.23, 1.26 and 1.27, it follows that for every integer $k \geqslant 1$ and graph G:

- if G is planar, then $\text{qn}(G^k) \in \mathcal{O}_k(\Delta(G^{\lfloor k/2 \rfloor}))$;
- if G is fan-planar, then $\text{qn}(G) \leqslant 127402104$; and
- if G is k-fan-bundle planar, then $\text{qn}(G) \in 2^{\mathcal{O}(k^3)}$.

Observe that for all integers $k, \Delta \geqslant 2$, the complete $(\Delta - 1)$-ary tree T of height $\lfloor \frac{k}{2} \rfloor$ has diameter at most k and maximum degree Δ. Since T^k is a complete graph, $\text{qn}(T^k) \geqslant \lfloor \frac{|V(T)|}{2} \rfloor = \lfloor \frac{\Delta(T^{\lfloor k/2 \rfloor})+1}{2} \rfloor$ [188]. Therefore, for fixed $k \in \mathbb{N}$, the above upper bound $\text{qn}(G^k) \in \mathcal{O}_k(\Delta(G^{\lfloor k/2 \rfloor}))$ on the queue-number of k-powers of planar graphs G is asymptotically best possible.

Nonrepetitive Colourings

The next application of graph product structure theory is to nonrepetitive colourings. Thue [319] constructed arbitrarily long words $w_1 w_2 \dots$ on an alphabet of three symbols with no repeated consecutive blocks; that is, there are no integers $i, k \in \mathbb{N}$ such that $w_i w_{i+1} \dots w_{i+k-1} = w_{i+k} w_{i+k+1} \dots w_{i+2k-1}$. Such a word is called *square-free*. This result is fundamental in the combinatorics of words. Nonrepetitive colourings was introduced by Alon et al. [12] as a generalisation of square-free words to the graph-theoretic setting. A vertex c-colouring ϕ of a graph G is *nonrepetitive* if, for every path $v_1, \dots, v_{2h}$ in G, there exists $i \in [h]$ such that $\phi(v_i) \neq \phi(v_{i+h})$. The *nonrepetitive chromatic number* $\pi(G)$ is the minimum integer $c \geqslant 0$ such that G has a nonrepetitive c-colouring. Thue's theorem says that the nonrepetitive chromatic number of any path is at most 3. Nonrepetitive colourings have been widely studied; see the survey [340].

Alon et al. [12] first asked whether planar graphs have bounded nonrepetitive chromatic number. For many years, this was considered the most important open problem in the study of nonrepetitive colouring. Product structure theory resolves this question. First, Kündgen and Pelsmajer [235] showed that $\pi(G) \leqslant 4^{\text{tw}(G)}$ for every graph G. Building upon this result, Dujmović et al. [116] proved the following.

Lemma 1.57 ([116]). *For every graph H and path P, if a graph G is contained in $H \boxtimes P \boxtimes K_\ell$, then*

$$\pi(G) \leqslant \ell\, 4^{\text{tw}(H)+1}.$$

So Lemma 1.57 implies that graph classes with bounded row treewidth have bounded nonrepetitive chromatic number. For planar graphs, Theorem 1.9 and Lemma 1.57 gives the following.

Theorem 1.58. *Every planar graph G has nonrepetitive chromatic number at most* 768.

In the special case when when G is a squaregraph, Theorem 1.12 and Lemma 1.57 imply that $\pi(G) \leqslant 4^3 = 64$.

Using a structure theorem of Grohe and Marx [173], Dujmović et al. [116] further generalised Theorem 1.58 to graphs excluding a fixed topological minor.

Theorem 1.59 ([116])**.** *Every graph excluding a fixed topological-minor has bounded nonrepetitive chromatic number.*

Applying Lemma 1.57 with Theorems 1.23, 1.26 and 1.27, it follows that for every integer $k \geqslant 1$ and graph G:

- if G is planar, then $\pi(G^k) \in \mathcal{O}_k(\Delta(G^{\lfloor k/2 \rfloor}))$;
- if G is fan-planar, then $\pi(G) \leqslant 81 \times 4^{20}$; and
- if G is k-fan-bundle planar, $\pi(G) \in 2^{\mathcal{O}(k^3)}$.

Centred Colourings

Nešetřil and Ossona de Mendez [258] introduced the following definitions. A vertex c-colouring ϕ of a graph G is *p-centred* if, for every connected subgraph $X \subseteq G$, we have $|\{\phi(v) : v \in V(X)\}| > p$ or there exists some $v \in V(X)$ such that $\phi(v) \neq \phi(w)$ for every $w \in V(X) \setminus \{v\}$. That is, every connected subgraph in G receives more than p colours or has a vertex with a unique colour. For an integer $p \geqslant 1$, the *p-centred chromatic number* $\chi_p(G)$ is the minimum integer $c \geqslant 0$ such that G has a p-centred c-colouring. Centred colourings are important within graph sparsity theory as they characterise graph classes with bounded expansion [258]. They are used in the design of parameterised algorithms in classes of bounded expansion; see [132, 157, 257, 258].

Debski et al. [95] established that $\chi_p(G \boxtimes H) \leqslant \chi_p(G)\chi(H^p)$ for all graphs G and H. Pilipczuk and Siebertz [276, Lemma 15] showed that $\chi_p(G) \leqslant \binom{p+t}{t} \leqslant (p+1)^t$ for every graph G with treewidth at most t. As centred colourings are closed under taking subgraphs, the next result follows.

Lemma 1.60 ([95])**.** *For every graph H and path P, if a graph G is contained in $H \boxtimes P \boxtimes K_\ell$ then*

$$\chi_p(G) \leqslant \ell(p+1)\chi_p(H) \leqslant \ell(p+1)\binom{p+\operatorname{tw}(H)}{\operatorname{tw}(H)} \in \mathcal{O}_{\ell,\operatorname{tw}(H)}(p^{\operatorname{tw}(H)+1}).$$

Observe that the exponent of p in Lemma 1.60 only depends on $\operatorname{tw}(H)$. This highlights why minimising $\operatorname{tw}(H)$ is often the primary goal in graph product structure theory.

Using Lemma 1.60 and the Planar Graph Product Structure Theorem ([?]), Debski et al. [95] showed the following.

Theorem 1.61 ([95])**.** *For every planar graph G and for every $p \in \mathbb{N}$, $\chi_p(G) \in \mathcal{O}(p^3 \log(p))$.*

This improved upon a previous bound of $\mathcal{O}(p^{19})$ by Pilipczuk and Siebertz [276].

Applying Lemma 1.60 with Theorems 1.23, 1.26 and 1.27, it follows that for every integer $k \geqslant 1$ and every graph G:

- $\chi_p(G^k) \in \mathcal{O}_k(p^{k^3}\Delta(G^{\lfloor k/2 \rfloor}))$ if G is planar;
- $\chi_p(G) \leqslant 81(p+1)\binom{p+19}{19} \in \mathcal{O}(p^{20})$ if G is fan-planar; and
- $\chi_p(G) \leqslant 6(2k+3)^2(p+1)\binom{p+\binom{2k+6}{3}-1}{\binom{2k+6}{3}-1} \in p^{\mathcal{O}(k^3)}$ if G is k-fan-bundle planar.

For k-planar graphs and k-fan-bundle planar graphs, Distel, Hickingbotham, Seweryn and Wood [106] showed that Theorems 1.22 and 1.28 and Lemma 1.60 give the following.

Theorem 1.62 ([106])**.** *For every $k \in \mathbb{N}$, for every k-planar graph G and for every $p \in \mathbb{N}$, $\chi_p(G) \in \mathcal{O}_k(p^{15288900})$.*

Theorem 1.63 ([106])**.** *For every $k \in \mathbb{N}$, for every k-fan-bundle planar graph G and for every $p \in \mathbb{N}$, $\chi_p(G) \in \mathcal{O}_k(p^{15288900})$.*

The key point of these two theorems is that the exponent of p is independent of k.

Universal Graphs and Adjacency Labelling Scheme

Product structure is also useful for constructing sparse universal graphs. Let $\mathcal{F}$ be a family of graphs. A graph G is *universal* for $\mathcal{F}$ if every graph in $\mathcal{F}$ is isomorphic to a subgraph of G. Similarly, a graph U is *induced universal* for $\mathcal{F}$ if every graph in $\mathcal{F}$ is isomorphic to an induced subgraph of U. What is the minimum number of edges in a universal graph for the family of n-vertex planar graphs? What is the minimum number of vertices in an induced universal graph for the family of n-vertex planar graphs? Here, product structure theory provides the state-of-the-art bounds. Esperet, Joret, and Morin [148] used product structure to construct universal graphs with a near-linear number of edges.

Theorem 1.64 ([148])**.** *For every $k \in \mathbb{N}$, for every class of graphs $\mathcal{G}$ with row treewidth at most k, the family of n-vertex graphs in $\mathcal{G}$ has a universal graph with $(1+o(1))n$ vertices and at most $k^2 \cdot n \cdot 2^{\mathcal{O}(\sqrt{\log(n)\cdot\log(\log(n))})}$ edges.*

So when $\mathcal{G}$ is the class of planar graphs, we have the following.

Theorem 1.65 ([148])**.** *The family of n-vertex planar graphs has a universal graph with $(1+o(1))n$ vertices and at most $n \cdot 2^{\mathcal{O}(\sqrt{\log(n)\cdot\log(\log(n))})}$ edges.*

The previous best known bound on the number of edges for the family of n-vertex planar graphs was $\mathcal{O}(n^{3/2})$ due to Babai et al. [21] in 1982. So product structure provides a significant improvement.

Dujmović, Esperet, Gavoille, Joret, Micek, and Morin [115] constructed an induced universal graph with near-linear number of vertices for any graph class with bounded row treewidth.

Theorem 1.66 ([115])**.** *For every $k \in \mathbb{N}$, for every class of graphs $\mathcal{G}$ with row treewidth at most k, the family of n-vertex graphs in $\mathcal{G}$ has an induced universal graph with $n^{1+o(1)}$ vertices.*

So for planar graphs, we have the following.

Theorem 1.67 ([115])**.** *The family of n-vertex planar graphs has an induced universal graph with $n^{1+o(1)}$ vertices.*

The family of n-vertex planar graphs has been shown to have an induced universal graph with $n^{c+o(1)}$ vertices for successive values of $c = 6$ [255], 4 [211], 2 [160], $\frac{4}{3}$ [46]. Note that the proof for $c = \frac{4}{3}$ also exploits product structure theory [46]. Theorem 1.67 is the first asymptotically optimal result for induced universal graph for planar graphs.

Induced universal graphs have an equivalent interpretation in terms of adjacency labelling schemes. Let $\{0,1\}^*$ denote the space of all finite binary strings. A graph class $\mathcal{G}$ has an *$f(n)$-bit adjacency labelling scheme* if there exists a function $A\colon (\{0,1\}^*)^2 \to \{0,1\}$ such that, for every n-vertex graph $G \in \mathcal{G}$, there exists $\ell\colon V(G) \to \{0,1\}^*$ such that $|\ell(v)| \leqslant f(n)$ for each vertex v of G and such that, for every two vertices v, w of G

$$A(\ell(v), \ell(w)) = \begin{cases} 0, & \text{if } vw \notin E(G); \\ 1, & \text{if } vw \in E(G). \end{cases}$$

Kannan et al. [211] observed that induced universal graph and adjacency labelling schemes are equivalent; see [307, Section 2.1] for details. So in particular, Theorems 1.66 and 1.67 are equivalent to the following.

Theorem 1.68 ([115])**.** *Every class of graphs with bounded row treewidth has a $(1 + o(1))\log_2(n)$-bit adjacency labelling scheme.*

Theorem 1.69 ([115])**.** *The class of planar graphs has a $(1 + o(1))\log_2(n)$-bit adjacency labelling scheme.*

As Theorems 1.64, 1.66 and 1.68 are applicable to any graph class with bounded row-treewidth, it immediately follows that the following graph classes have universal graphs with near-linear edges, induced universal graphs with near-linear number of edges, and $(1 + o(1))\log_2(n)$-bit adjacency labelling scheme:

- graphs with Euler genus at most g (via Theorem 1.14);

- apex-minor-free graphs (via Theorem 1.15);
- (g, k)-planar graphs (via Theorem 1.19);
- (g, δ)-string graphs (via Theorem 1.25);
- fan-planar graphs (via Theorem 1.26);
- k-fan-bundle planar graphs (via Theorem 1.27); and
- (g, d)-map graphs (via Theorem 1.30).

1.3.9 Graph Sparsity Theory

All the graph classes that we have considered in this section are examples of sparse graph classes. But what is the right notion of sparsity? Bounded degeneracy is a first necessary condition, however, this is not the full answer. For instance, the 1-subdivision of complete graphs are 2-degenerate, yet they are not considered to be sparse. This is because, in many respects, they preserve the dense structure of complete graphs as they contain a dense substructure at 'low-depth'.

Nešetřil and Ossona de Mendez [257] introduced the following measure of graph sparsity. Let G be a graph and $r \geqslant 0$ be an integer. Let $G \,\triangledown\, r$ be the set of all r-shallow minors of G, and let $\nabla_r(G) := \max\{|E(H)|/|V(H)|\colon H \in G \,\triangledown\, r \text{ and } V(H) \neq \varnothing\}$. A hereditary graph class $\mathcal{G}$ has *bounded expansion* with *expansion function* $f_{\mathcal{G}} : \mathbb{N}_0 \to \mathbb{R}$ if $\nabla_r(G) \leqslant f_{\mathcal{G}}(r)$ for every $r \geqslant 0$ and graph $G \in \mathcal{G}$. Bounded expansion is a robust measure of sparsity with many characterisations [257, 258, 347]. Many natural sparse graph classes have bounded expansion such as proper minor-closed classes [258], classes with bounded maximum degree [258], bounded stack number [259], bounded queue-number [259], or bounded nonrepetitive chromatic number [259]. See the book by Nešetřil and Ossona de Mendez [257] for further background on bounded expansion and sparsity theory.

One of the most useful tools in graph sparsity theory is colouring numbers. Kierstead and Yang [215] introduced the following definitions. For a graph G, a total order $\preceq$ of $V(G)$, a vertex $v \in V(G)$, and an integer $r \geqslant 1$, let $R(G, \preceq, v, r)$ be the set of vertices $w \in V(G)$ for which there is a path $v = w_0, w_1, \dots, w_{r'} = w$ of length $r' \in [0, r]$ such that $w \preceq v$ and $v \prec w_i$ for all $i \in [r' - 1]$, and let $Q(G, \preceq, v, r)$ be the set of vertices $w \in V(G)$ for which there is a path $v = w_0, w_1, \dots, w_{r'} = w$ of length $r' \in [0, r]$ such that $w \preceq v$ and $w \prec w_i$ for all $i \in [r' - 1]$. For a graph G and integer $r \geqslant 1$, the *r-strong colouring number* $\operatorname{scol}_r(G)$ of G, is the minimum $k \in \mathbb{N}_0$ such that there is a total order $\preceq$ of $V(G)$ with $|R(G, \preceq, v, r)| \leqslant k$ for every vertex v of G. Likewise, the *r-weak colouring number* $\operatorname{wcol}_r(G)$ of G is the minimum $k \in \mathbb{N}_0$ such that there is a total order $\preceq$ of $V(G)$ with $|Q(G, \preceq, v, r)| \leqslant k$ for every vertex v of G.

Colouring numbers are fundamental to graph sparsity theory because they characterise bounded expansion [347] and nowhere dense classes [171], and have several algorithmic applications [131, 172]. Moreover, they provide upper bounds on several graph parameters of interest. First note that $\operatorname{scol}_1(G) = \operatorname{wcol}_1(G)$ which equals the degeneracy of G plus

1, implying $\chi(G) \leqslant \mathrm{scol}_1(G)$. A proper graph colouring is *acyclic* if the union of any two colour classes induces a forest; that is, every cycle is assigned at least three colours. For a graph G, the *acyclic chromatic number* $\chi_a(G)$ of G is the minimum integer k such that G has an acyclic k-colouring. Kierstead and Yang [215] proved that $\chi_{\mathrm{a}}(G) \leqslant \mathrm{scol}_2(G)$ for every graph G. Other parameters that can be bounded by strong and weak colouring numbers include game chromatic number [214, 215], Ramsey numbers [76], oriented chromatic number [229], arrangeability [76], and boxicity [150].

Another attractive aspect of strong colouring numbers is that they interpolate between degeneracy and treewidth. As previously noted, $\mathrm{scol}_1(G)$ equals the degeneracy of G plus 1. At the other extreme, Grohe et al. [171] showed that $\mathrm{scol}_r(G) \leqslant \mathrm{tw}(G) + 1$ for every $r \in \mathbb{N}$, and indeed $\mathrm{scol}_r(G) \to \mathrm{tw}(G) + 1$ as $r \to \infty$.

In Chapter 6, we continue this line of research by presenting several new applications of colouring numbers. First, we bound the cop-width and flip-width of a graph by its strong colouring numbers. Cop-width and flip-width are new families of graph parameters introduced by Torunczyk [320] that generalise treewidth, degeneracy, generalised colouring numbers, clique-width and twin-width; see Section 6.3 for their definitions. Our main contribution is the following.

Theorem 1.70. *For every $r \in \mathbb{N}$, every graph G has* $\mathrm{copwidth}_r(G) \leqslant \mathrm{scol}_{4r}(G)$.

Theorem 1.70 is used to show that graph classes with linear strong colouring numbers have linear cop-width and linear flip-width. This implies that proper minor-closed classes have linear cop-width and linear flip-width; see Section 6.3 for the details.

Next, we bound the odd chromatic number and the conflict-free chromatic number of a graph by its strong colouring numbers. The odd chromatic number χ_o and the conflict-free chromatic number χ_{pcf} are new graph parameters introduced by Petruševski and Škrekovski [274] and Fabrici et al. [151] respectively; see Section 6.4 for their definitions. Since their introduction, they have received widespread attention from the graph colouring community [66, 67, 82, 87, 122, 151, 151, 242, 273, 274].

We prove the following.

Theorem 1.71. *For every graph G, $\chi_o(G) \leqslant \chi_{pcf}(G) \leqslant 2\,\mathrm{scol}_2(G) - 1$.*

Theorem 1.71 implies that every graph class with bounded expansion has bounded conflict-free chromatic number and bounded odd chromatic number. This result substantially generalises previous works in this direction [87, 122]; see Section 6.4 for the details.

1.4 Hereditary Graph Classes

We now move on to discuss the second theme of this thesis which is exploring hereditary graph classes via tree-decompositions. While hereditary graph classes can be dense in that they may contain arbitrarily large complete graphs, they may still possess a reasonable global structure.

Historically, the study of hereditary graph classes was motivated by Berge's perfect graph conjectures. A graph G is *perfect* if, for every induced subgraph H of G, $\chi(H) = \omega(H)$ where $\omega(H)$ is the order of the largest clique in H. It is easy to see that complete graphs and bipartite graphs are perfect, but odd cycles of length at least 5 are not. See [322] for a survey on perfect graphs.

A *hole* in a graph is an induced cycle of length at least 4. An *antihole* in a graph is an induced subgraph isomorphic to the compliment of a cycle of length at least 4. A hole (antihole) is *odd* if it has an odd number of vertices. Berge [32] conjectured that odd holes and odd antiholes are the only obstructions for a graph being perfect. This conjecture, known as the *Strong Perfect Graph Conjecture*, stimulated a great body of research into the structure of hereditary graph classes. In 2006, Chudnovsky, Robertson, Seymour, and Thomas [83] resolved this conjecture.

Theorem 1.72 (Strong Perfect Graph Theorem [83])**.** *A graph is perfect if and only if it contains neither an odd hole nor an odd antihole.*

The proof for this landmark result spans over 150 pages, earning the authors the 2009 Fulkerson Prize.

In recent years, there has been substantial interest in studying hereditary graph classes via tree-decompositions [1–9, 45, 225, 243, 275, 303]. The following question naturally arises: what are the unavoidable induced subgraphs for graphs with large treewidth? First, since complete graphs have large treewidth ($\mathrm{tw}(K_n) = n-1$), and every induced subgraph of a complete graphs is also a complete graph, it follows that the family of complete graphs is a candidate. By an analogous argument, the family of complete bipartite graphs is also a candidate. Furthermore, by Corollary 1.3, subdivisions of walls are a candidate since they exclude K_3 and $K_{2,2}$ as induced subgraphs. Finally, line graphs of subdivision of walls are also a candidate as they have large treewidth and exclude $K_4, K_{2,2}$ and $K_{1,3}$ as induced subgraphs; see Figure 1.5. These four families of graphs — complete graphs, complete bipartite graphs, subdivisions of walls and line graphs of subdivisions of walls — constitute the so-called *obvious candidates* for the unavoidable induced subgraphs for graphs with large treewidth. However, this list of candidates is known to not be exhaustive [45, 88, 277, 303]. Nevertheless, for several natural graph classes, it has been shown that the unavoidable induced subgraphs for graphs with large treewidth in the class are indeed the obvious candidates. For example, Korhonen [225] showed that graphs with bounded maximum degree and sufficiently large treewidth contain a subdivision of a large wall or the line graph of a subdivision of a large wall as an induced subgraph.

In this thesis, we make two contributions in this research direction. First, we initiate the exploration of induced subgraphs and path-decompositions. Second, we describe the unavoidable induced subgraphs for circle graphs with large treewidth.

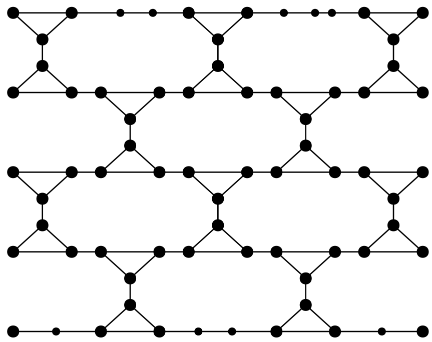


Figure 1.5. A line graph of a subdivision of a wall.

1.4.1 Induced Subgraphs and Path-Decompositions

Consider the following problem for pathwidth: what are the unavoidable induced subgraphs for graphs with large pathwidth? Due to the Excluded Forest Minor Theorem (Theorem 1.4) [285], obvious candidates for the unavoidable induced subgraphs for graphs with large pathwidth are subdivisions of large complete binary trees and line graphs of subdivisions of large complete binary trees; see Figure 1.6. While determining the full list of candidates in the general setting looks challenging, we show that these graphs suffice in the bounded maximum degree setting (Section 7.4) as well as for K_n-minor-free graphs (Section 7.5). Let T_k denote the complete binary tree of height k.

Theorem 1.73. *There is a function f such that every graph G with maximum degree Δ and pathwidth at least $f(k, \Delta)$ contains a subdivision of T_k or the line graph of a subdivision of T_k as an induced subgraph.*

Theorem 1.74. *For every fixed $n \in \mathbb{N}$, there is a function f such that every K_n-minor-free graph G with pathwidth at least $f(k)$ contains a subdivision of T_k or the line graph of a subdivision of T_k as an induced subgraph.*

In addition, we characterise when a hereditary graph class defined by a finite set of forbidden induced subgraphs has bounded pathwidth. For a finite set of graphs $\mathcal{S}$, let $\mathcal{I}_{\mathcal{S}}$ be the class of graphs that contain no graph in $\mathcal{S}$ as an induced subgraph. We call $K_{1,3}$

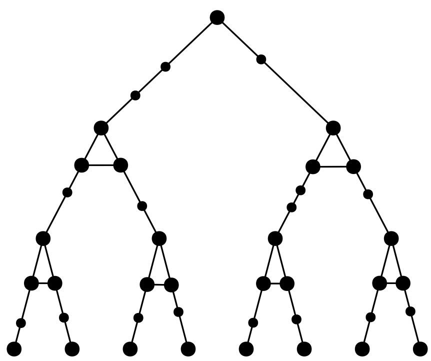


Figure 1.6. A line graph of a subdivision of a complete binary tree.

a *claw*. A *fork* is either a path or a subdivision of a claw, and a *semi-fork* is the line graph of a fork. A *tripod* is a forest where each component is a fork, and a *semi-tripod* is a graph where each component is a semi-fork; see Figure 1.7. We prove the following characterisation for when $\mathcal{I}_{\mathcal{S}}$ has bounded pathwidth.

Theorem 1.75. *For a finite set of graphs $\mathcal{S}$, $\mathcal{I}_{\mathcal{S}}$ has bounded pathwidth if and only if $\mathcal{S}$ includes a complete graph, a complete bipartite graph, a tripod and a semi-tripod.*

See Chapter 7 for the proofs of Theorems 1.73–1.75.

1.4.2 Circle Graphs

In Chapter 8, we study the treewidth of graphs that are defined by circular drawings.

A *circle graph* is the intersection graph of a set of chords of a circle. Circle graphs form a widely studied graph class [89, 92, 94, 128, 161, 220, 226], and there have been several recent breakthroughs concerning them. In the study of graph colourings, Davies and McCarty [92] showed that circle graphs are quadratically χ-bounded, improving upon a previous longstanding exponential upper bound. Davies [89] further improved this bound to $\chi(G) \in \mathcal{O}(\omega(G) \log \omega(G))$, which is best possible. Circle graphs are also fundamental to the study of vertex-minors and are conjectured to lie at the heart of a global structure theorem for vertex-minor-closed graph classes (see [248]). To this end, Geelen, Kwon, McCarty, and Wollan [161] recently proved an analogous result to the Excluded Grid Minor Theorem for vertex-minors using circle graphs. In particular, they showed that a vertex-minor-closed graph class has bounded rankwidth if and only if it excludes a circle graph as a vertex-minor. For further motivation and background on circle graphs, see [90, 248].

Our main contribution in Chapter 8 essentially determines when a circle graph has large treewidth.

Theorem 1.76. *Let $t \in \mathbb{N}$ and let G be a circle graph with treewidth at least $12t+2$. Then G contains an induced subgraph H that consists of t vertex-disjoint cycles $(C_1, \dots, C_t)$ such that, for all $i < j$, every vertex of C_i has at least two neighbours in C_j. Moreover, every vertex of G has at most four neighbours in any C_i $(1 \leqslant i \leqslant t)$.*

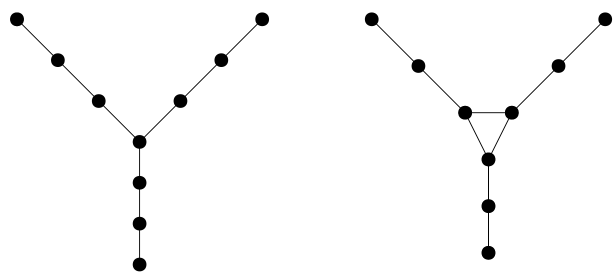


Figure 1.7. A fork and a semi-fork.

Observe that, in Theorem 1.76, the subgraph H has a K_t-minor obtained by contracting each of the cycles C_i to a single vertex, implying that H has treewidth at least $t-1$. Moreover, since circle graphs are closed under taking induced subgraphs, H is also a circle graph. We now highlight several consequences of Theorem 1.76.

First, Theorem 1.76 describes the unavoidable induced subgraphs of circle graphs with large treewidth. To date, most of the results concerning treewidth and induced subgraphs have focused on graph classes where the unavoidable induced subgraphs are the obvious candidates. Circle graphs contain neither subdivisions of large walls nor line graphs of subdivisions of large walls, and there are circle graphs of large treewidth that contain neither large complete graphs nor large complete bipartite graphs (see Theorem 8.17). To the best of our knowledge, this is the first result to describe the unavoidable induced subgraphs of the large treewidth graphs in a natural hereditary class when they are not the obvious candidates. Later we show that the unavoidable induced subgraphs of graphs with large treewidth in a vertex-minor-closed class $\mathcal{G}$ are the obvious candidates if and only if $\mathcal{G}$ has bounded rankwidth (see Theorem 8.19).

Second, the subgraph H in Theorem 1.76 is an explicit witness to the large treewidth of G (with only a multiplicative loss). Circle graphs being χ-bounded says that circle graphs with large chromatic number must contain a large clique witnessing this. Theorem 1.76 can therefore be considered to be a treewidth analogue to the χ-boundedness of circle graphs. We also prove an analogous result for circle graphs with large pathwidth (see Theorem 8.18).

Third, since the subgraph H has a K_t-minor, it follows that every circle graph contains a complete minor whose order is at least one twelfth of its treewidth. This is in stark contrast to the general setting where there are K_5-minor-free graphs with arbitrarily large treewidth (for example, grids). Theorem 1.76 also implies the following relationship between the treewidth, Hadwiger number and Hajós number of circle graphs (see Section 8.5).

Theorem 1.77. *For the class of circle graphs, the treewidth and Hadwiger number are linearly tied. Moreover, the Hajós number is quadratically tied to both of them. Both 'linear' and 'quadratic' are best possible.*

1.4.3 Circular Graph Drawings

The second theme of Chapter 8 aims to understand the relationship between circular drawings of graphs and their crossing graphs. A *circular drawing* (also called *convex drawing*) of a graph places the vertices on a circle with edges drawn as straight line segments. Circular drawings are a well-studied topic; see [158, 221, 304] for example. The *crossing graph* of a drawing D of a graph G has vertex-set $E(G)$ where two vertices are adjacent if the corresponding edges cross. Circle graphs are precisely the crossing graphs of circular drawings. If a graph has a circular drawing with a well-behaved crossing graph, must the graph itself also have a well-behaved structure? Graphs that have a circular drawing with no crossings are exactly the outerplanar graphs, which have treewidth at most 2. Put another way, outerplanar graphs are those that have a circular drawing whose crossing graph is K_2-minor-free. The next result extends this fact, relaxing 'K_2-minor-free' to 'K_t-minor-free'.

Theorem 1.78. *For every integer $t \geqslant 3$, if a graph G has a circular drawing where the crossing graph has no K_t-minor, then G has treewidth at most $12t - 23$.*

Theorem 1.78 says that G having large treewidth is sufficient to force a complicated crossing graph in every circular drawing of G. A topological $K_{2,4t}$-minor also suffices.

Theorem 1.79. *If a graph G has a circular drawing where the crossing graph has no K_t-minor, then G contains no $K_{2,4t}$ as a topological minor.*

Outerplanar graphs are exactly those graphs that have treewidth at most 2 and exclude a topological $K_{2,3}$-minor. As such, Theorems 1.78 and 1.79 extend these structural properties of outerplanar graphs to graphs with circular drawings whose crossing graphs are K_t-minor-free. We also prove a product structure theorem for such graphs, showing that every graph that has a circular drawing whose crossing graph has no K_t-minor is contained in $H \boxtimes K_{\mathcal{O}(t^3)}$ for some graph H with treewidth at most 2 (see Corollary 8.7).

Part I

Sparse Graph Classes

Chapter 2

Graph Products

2.1 Overview

In this chapter, we systematically study the structural properties of graph products. We explore the following properties of cartesian, direct and strong products: complete multipartite subgraphs, degeneracy, pathwidth and treewidth. Our key contributions are the following:

Complete multipartite subgraphs: We characterise the presence of complete multipartite subgraphs in cartesian, direct and strong products. These results are presented in Section 2.3.

Degeneracy: We present tight upper and lower bounds on the degeneracy of direct and strong products. See Section 2.4 for these results.

Pathwidth and treewidth of cartesian and strong products: We establish two new lower bounds for the treewidth of cartesian and strong products. The first bound states that if one graph does not admit small ε-separations and the other graph is connected with many vertices, then the cartesian product has large treewidth. The second bound states that if one graph has large treewidth and the other graph has large Hadwiger number, then the strong product has large treewidth. In addition, we characterise when the cartesian and strong product of two monotone graph classes has bounded treewidth and when it has bounded pathwidth. These results are presented in Section 2.5.1.

Pathwidth and treewidth of direct products: We characterise when the direct product of two monotone graph classes has bounded treewidth and when it has bounded pathwidth. For treewidth, the characterisation states that the direct product of two graph classes has bounded treewidth if and only if the connected graphs in one of the classes have a bounded number of vertices while the graphs in the other class have bounded treewidth; or if the connected graphs in one of the classes have bounded vertex cover number while the graphs in the other class have bounded treewidth and bounded maximum degree. For pathwidth, our characterisation is directly analogous to that for treewidth with the stronger condition that the second class has bounded pathwidth. We also demonstrate that the treewidth of a graph is polynomially tied to the treewidth of the direct product of the graph with K_2. To our knowledge, it was previously open whether these two parameters were tied. These results are presented in Sections 2.5.2 and 2.5.3.

This line of research has previously been explored for the following properties of graph products: connectivity [60, 61, 329, 342]; queue-number [334]; stack-number [113, 144, 199, 238, 278]; thinness [50]; boxicity and cubicity [69]; polynomial growth [130]; bounded expansion and colouring numbers [130]; chromatic number [112, 216, 290, 302, 312, 327, 328, 348]; and Hadwiger number [17, 71, 209, 230, 250, 272, 337, 345]. See the handbook by Hammack et al. [180] for an in-depth treatment of graph products.

This chapter is based on joint work with Wood [197].

2.2 Preliminaries

Let G_1 and G_2 be graphs. A *graph product* $G_1 \bullet G_2$ is defined with vertex-set:

$$V(G_1 \bullet G_2) := \{(a, v) : a \in V(G_1), v \in V(G_2)\}.$$

The *cartesian product* $G_1 \,\square\, G_2$ consists of edges of the form $(a, v)(b, u)$ where either $ab \in E(G_1)$ and $v = u$, or $uv \in E(G_2)$ and $a = b$. The *direct product* $G_1 \times G_2$ consists of edges of the form $(a, v)(b, u)$ where $ab \in E(G_1)$ and $uv \in E(G_2)$. This product is also known as the *tensor product*, the *Kronecker product* and the *cross product*. The *lexicographic product* $G_1 \circ G_2$ consists of edges of the form $(a, v)(b, u)$ where either $ab \in E(G_1)$, or $a = b$ and $uv \in E(G_2)$. The *strong product* $G_1 \boxtimes G_2$ is defined as $(G_1 \,\square\, G_2) \cup (G_1 \times G_2)$. For a graph product $\bullet \in \{\square, \times, \circ, \boxtimes\}$, graph classes $\mathcal{G}_1$ and $\mathcal{G}_2$, and graph H, let $\mathcal{G}_1 \bullet \mathcal{G}_2$ be the graph class $\{G_1 \bullet G_2 : G_1 \in \mathcal{G}_1, G_2 \in \mathcal{G}_2\}$ and let $\mathcal{G}_1 \bullet H$ be the graph class $\{G_1 \bullet H : G_1 \in \mathcal{G}_1\}$.

We frequently make use of the well-known fact that $\operatorname{tw}(G \boxtimes K_n) \leqslant (\operatorname{tw}(G) + 1)n - 1$ for every graph G and integer $n \geqslant 1$ (see [43] for an implicit proof).

Let $\mathsf{v}(G) := |V(G)|$ be the *order of G*. Let $\tilde{\mathsf{v}}(G)$ be the maximum order of a connected component of G.

The following well-known properties of graph products are a straightforward consequence of their definition:

- for all $\bullet \in \{\square, \times, \boxtimes\}$ and graphs G_1 and G_2 with subgraphs $H_1 \subseteq G_1$ and $H_2 \subseteq G_2$, we have $H_1 \bullet H_2 \subseteq G_1 \bullet G_2$;
- for all hereditary graph classes $\mathcal{G}_1$ and $\mathcal{G}_2$, we have $\mathcal{G}_1 \subseteq \mathcal{G}_1 \,\square\, \mathcal{G}_2$ and $\mathcal{G}_2 \subseteq \mathcal{G}_1 \,\square\, \mathcal{G}_2$;
- for all graphs G_1 and G_2 and vertices $v_i \in V(G_i)$ where $\deg_{G_i}(v_i) = d_i$ for $i \in \{1, 2\}$:
  - $\deg_{G_1 \square G_2}((v_1, v_2)) = d_1 + d_2$;
  - $\deg_{G_1 \times G_2}((v_1, v_2)) = d_1 d_2$; and
  - $\deg_{G_1 \boxtimes G_2}((v_1, v_2)) = d_1 + d_2 + d_1 d_2$.

For a subgraph $Z \subseteq G_1 \bullet G_2$ of a graph product where $\bullet \in \{\square, \times, \boxtimes\}$, the *projection of Z onto G_1* is the subgraph of G_1 induced by the set of vertices $u_1 \in V(G_1)$ such that

$(u_1, u_2) \in V(Z)$ for some $u_2 \in V(G_2)$, and the *projection of Z onto G_2* is the subgraph of G_2 induced by the set of vertices $v_2 \in V(G_2)$ such that $(v_1, v_2) \in V(Z)$ for some $v_1 \in V(G_1)$.

2.3 Complete Multipartite Subgraphs

For integers $d \geqslant 2$ and $n_1, \ldots, n_d \geqslant 1$, a *complete d-partite graph* $K_{n_1,\ldots,n_d}$ has a partition of its vertex-set, $(A_1, \ldots, A_d)$, such that for all distinct $i, j \in [d]$ and $a_i \in A_i$ and $a_j \in A_j$, we have $a_i a_j \in E(K_{n_1,\ldots,n_d})$ and $|A_i| = n_i$. When $d = 2$, it is a *complete bipartite graph*. Observe that the graph $K_{1,\ldots,1}$ is a complete graph. In this section we characterise when a cartesian, direct and a strong product contains a given complete multipartite subgraph.

2.3.1 Cartesian Product

We begin with cartesian products. Let G_1 and G_2 be graphs. We say that $(u_1, v_1), \ldots, (u_d, v_d) \in V(G_1 \square G_2)$ are *aligned* if $u_1 = u_2 = \ldots = u_d$ or $v_1 = v_2 = \ldots = v_d$. Observe that for a subgraph $H \subseteq G_1 \square G_2$, if the vertex-set $V(H)$ is aligned, then H is contained in G_1 or G_2. For a set $X \subseteq V(G_1 \square G_2)$, if every pair of vertices in X is aligned, then X is aligned. Furthermore, if x and y are adjacent vertices in $G_1 \square G_2$, then x, y are aligned. Hence, we have the following.

Lemma 2.1. *For all graphs G_1 and G_2, every clique in $G_1 \square G_2$ is aligned.*

The next two lemmas are used in our characterisation of complete multipartite subgraphs in cartesian products.

Lemma 2.2. *For all graphs G_1 and G_2, if $X \subseteq V(G_1 \square G_2)$ and $|\bigcap(N(x) \colon x \in X)| > 2$, then X is aligned.*

Proof. Let $X := \{(u_1, v_1), \ldots, (u_k, v_k)\}$ for some integer $k \geqslant 1$. If $k = 1$ then the claim holds trivially. So assume that $k \geqslant 2$. For the sake of contradiction, suppose there exists distinct $i, j \in [k]$ such that $u_i \neq u_j$ and $v_i \neq v_j$. Then

$$\begin{aligned} N((u_i, v_i)) \cap N((u_j, v_j)) =& (\{(u_i, v) : vv_i \in E(G_2)\} \cup \{(u, v_i) : uu_i \in E(G_1)\}) \\ & \cap (\{(u_j, v) : vv_j \in E(G_2)\} \cup \{(u, v_j) : uu_j \in E(G_1)\}) \\ \subseteq& \{(u_i, v_j), (u_j, v_i)\}. \end{aligned}$$

But this contradicts the assumption that the intersection of the neighbourhoods of (u_i, v_i) and (u_j, v_j) has size greater than 2. Thus, every pair of vertices in X is aligned and hence, X is aligned. $\square$

Lemma 2.3. *If $(u_1, v_1), (u_2, v_2)$ are distinct vertices that are aligned in $V(G_1 \square G_2)$, then $N[(u_1, v_1)] \cap N[(u_2, v_2)]$ is aligned.*

Proof. Without loss of generality, $v_1 = v_2 = v^\star$. Then

$$\begin{aligned}N((u_1, v^\star)) \cap N((u_2, v^\star)) =&(\{(u_1, v) : vv^\star \in E(G_2)\} \cup \{(u, v^\star) : uu_1 \in E(G_1)\})\\ &\cap (\{(u_2, v) : vv^\star \in E(G_2)\} \cup \{(u, v^\star) : uu_2 \in E(G_1)\})\\ \subseteq&\{(x, v^\star) : x \in V(G_1)\}.\end{aligned}$$

Hence $N[(u_1, v^\star)] \cap N[(u_2, v^\star)]$ is aligned. □

We now characterise when a cartesian product contains a given complete multipartite subgraph.

Theorem 2.4. *For all integers $d \geqslant 2$ and $n_1, \dots, n_d \geqslant 1$, and for all graphs G_1 and G_2 with non-empty vertex-sets and maximum degree Δ_1 and Δ_2 respectively, $K_{n_1,\dots,n_d} \subseteq G_1 \square G_2$ if and only if at least one of the following conditions hold:*

- *$K_{n_1,\dots,n_d}$ is a subgraph of G_1 or G_2;*
- *$d = 2$ and $(n_1, n_2) = (2, 2)$ and $K_2 \subseteq G_1$ and $K_2 \subseteq G_2$; or*
- *$d = 2$ and $(n_1, n_2) = (1, s)$ for some integer $s \geqslant 1$ and $\Delta_1 + \Delta_2 \geqslant s$.*

Proof. Clearly if $K_{n_1,\dots,n_d}$ is a subgraph of either G_1 or G_2, then it is a subgraph of $G_1 \square G_2$.

Observe that $K_{1,s}$ is a subgraph of a graph G if and only if the maximum degree of G is at least s. As such, $K_{1,s}$ is a subgraph of $G_1 \square G_2$ if and only if $\Delta_1 + \Delta_2 \geqslant s$. Now consider the bipartite graph $K_{2,2}$. Suppose $xy \in E(G_1)$ and $uv \in E(G_2)$. Let $A = \{(x, u), (y, v)\}$ and $B = \{(y, u), (x, v)\}$. Then (A, B) defines a $K_{2,2}$ subgraph in $G_1 \square G_2$. Now suppose that $K_{2,2} \subseteq G_1 \square G_2$ and K_2 is not a subgraph of G_1. Then $E(G_1) = \varnothing$ and hence $V(G_1)$ is an independent set. As such, the connected components of $G_1 \square G_2$ are isomorphic to the connected components of G_2. Since $K_{2,2}$ is connected, it is a subgraph of G_2, as required.

It remains to consider the cases when $(n_1, \dots, n_d)$ is not equal to $(1, s)$ or $(2, 2)$. First, suppose $d \geqslant 3$ and that $K_{n_1,\dots,n_d} \subseteq G_1 \square G_2$ with partition $(A_1, \dots, A_d)$. For every $i \in [d]$, let $(x_i, y_i) \in A_i$. Then $X := \{(x_1, y_1), \dots, (x_d, y_d)\}$ induces a K_d subgraph in $G_1 \square G_2$. By Lemma 2.1, X is aligned. Furthermore, for every $(u, v) \in V(K_{n_1,\dots,n_d})$, there exists distinct $i, j \in [d]$ such that $(u, v) \in N[(x_i, y_i)] \cap N[(x_j, y_j)]$. By Lemma 2.3, $V(K_{n_1,\dots,n_d})$ is aligned and hence $K_{n_1,\dots,n_d}$ is a subgraph of G_1 or G_2, as required.

Now suppose $d = 2$. Let (A_1, A_2) be the bipartition of K_{n_1,n_2} where $n_1 = |A_1| \geqslant 2$ and $n_2 = |A_2| \geqslant 3$. By Lemma 2.2, A_1 is aligned. By Lemma 2.3, $V(K_{n_1,n_2})$ is aligned and hence K_{n_1,n_2} is a subgraph of G_1 or G_2. □

2.3.2 Direct Product

The next theorem characterises when a direct product contains a given complete multipartite subgraph.

Theorem 2.5. *For all integers $d \geqslant 2$ and $n_1, \ldots, n_d \geqslant 1$, and for all graphs G_1 and G_2 with non-empty edge-sets, $K_{n_1,\ldots,n_d} \subseteq G_1 \times G_2$ if and only if $K_{a_1,\ldots,a_d} \subseteq G_1$ and $K_{b_1,\ldots,b_d} \subseteq G_2$ where a_i, b_i are positive integers and $n_i \leqslant a_i b_i$ for all $i \in [d]$.*

Proof. Suppose $K_{a_1,\ldots,a_d} \subseteq G_1$ and $K_{b_1,\ldots,b_d} \subseteq G_2$. Let $(A_1, \ldots A_d)$ be the partition of $K_{a_1,\ldots,a_d}$ and $(B_1, \ldots, B_d)$ be the partition of $K_{b_1,\ldots,b_d}$. Consider the graph $K_{a_1,\ldots,a_d} \times K_{b_1,\ldots,b_d}$. For all distinct $i, j \in [d]$, every vertex $v_i \in (A_i, B_i)$ is adjacent to every vertex $v_j \in (A_j, B_j)$. Hence $((A_1, B_1), \ldots, (A_d, B_d))$ defines a $K_{a_1b_1,\ldots,a_db_d}$ subgraph in $K_{a_1,\ldots,a_d} \times K_{b_1,\ldots,b_d}$. Since $n_i \leqslant a_i b_i$ for all $i \in [d]$, we have $K_{n_1,\ldots,n_d} \subseteq G_1 \times G_2$.

For the second direction, suppose $K_{n_1,\ldots,n_d} \subseteq G_1 \times G_2$. Let $(N_1, \ldots, N_d)$ correspond to the partition of $K_{n_1,\ldots,n_d}$. For $i \in \{1, 2\}$ and $j \in [d]$, let $N_j^{(i)}$ be the vertices in N_j projected onto G_i. By the definition of the direct product, $\{(x, v_2) : x \in V(G_1)\}$ and $\{(v_1, y) : y \in V(G_2)\}$ are independent sets for all $v_1 \in V(G_1)$ and $v_2 \in V(G_2)$. As such, $N_j^{(i)}$ and $N_k^{(i)}$ are disjoint for distinct $j, k \in [d]$ and $i \in \{1, 2\}$. Moreover, as every vertex in N_j is adjacent to every vertex in N_k, it follows that $K_{a_1\ldots,a_d} \subseteq G_1[N_1^{(1)} \cup \ldots N_d^{(1)}]$ and $K_{b_1,\ldots,b_d} \subseteq G_2[N_1^{(2)} \cup \ldots N_d^{(2)}]$ where $a_j = |N_j^{(1)}|$, and $b_j = |N_j^{(2)}|$ for all $j \in [d]$. Since there are $a_j b_j$ vertices in $(N_j^{(1)}, N_j^{(2)})$, we have $n_j \leqslant a_j b_j$ for all $j \in [d]$, as required. □

2.3.3 Strong Product

Let $K_{n_1,\ldots,n_d,\overline{x}}$ be the graph obtained from the complete multipartite graph $K_{n_1,\ldots,n_d,x}$ by adding an edge between each pair of vertices in the part of size $x \geqslant 0$. More formally, $V(K_{a_1,\ldots,a_d,\overline{x}}) = A_1 \cup \cdots \cup A_d \cup X$, such that $A_1, \ldots, A_d, X$ are pairwise disjoint sets where for distinct $j, k \in [d]$, we have $|A_j| = a_j$, $|X| = x$, and $uv, vw_1, w_1w_2 \in E(K_{a_1,\ldots,a_d,\overline{x}})$ where $u \in A_j$, $v \in A_k$, and $w_1, w_2 \in X$. Observe that $K_{n_1,\ldots,n_d,\overline{x}} = K_{n_1,\ldots,n_d,1,\ldots,1}$ and $K_{n+c} = K_{1,\ldots,1,\overline{c}}$.

Lemma 2.6. *If $K_{a_1,\ldots,a_d,\overline{x}} \subseteq G_1$ and $K_{b_1,\ldots,b_d,\overline{y}} \subseteq G_2$, then*

$$K_{a_1b_1+a_1y+b_1x,\ldots,a_db_d+a_dy+b_dx,\overline{xy}} \subseteq G_1 \boxtimes G_2.$$

Proof. Let $A_1, \ldots, A_d, X$ be the subsets of $V(G_1)$ defining a $K_{a_1,\ldots,a_d,\overline{x}}$ subgraph of G_1. Let $B_1, \ldots, B_d, Y$ be the subsets of $V(G_2)$ defining a $K_{b_1,\ldots,b_d,\overline{y}}$ subgraph of G_2. Then $(A_1 \times B_1) \cup (A_1 \times Y) \cup (B_1 \times X), \ldots, (A_d \times B_d) \cup (A_d \times Y) \cup (B_d \times X), (X \times Y)$ define a $K_{a_1b_1+a_1y+b_1x,\ldots,a_db_d+a_dy+b_dx,\overline{xy}}$ subgraph in $G_1 \boxtimes G_2$. □

The next theorem characterises when a strong product contains a given complete multipartite subgraph.

Theorem 2.7. *For all integers $d \geqslant 2$ and $n_1, \ldots, n_d \geqslant 1$, and for all graphs G_1 and G_2, we have $K_{n_1,\ldots,n_d} \subseteq G_1 \boxtimes G_2$ if and only if there exists non-negative integers $a_1, \ldots, a_d, b_1, \ldots, b_d$, $z_1, \ldots, z_d, x, y \geqslant 0$ such that:*

- *$K_{a_1,\ldots,a_d,\overline{x}} \subseteq G_1$;*

- $K_{b_1,\dots,b_d,\overline{y}} \subseteq G_2$;
- $n_j \leqslant a_j b_j + a_j y + b_j x + z_j$ *for all* $j \in [d]$*; and*
- $z_1 + \dots + z_d \leqslant xy$.

Proof. First, suppose that $K_{n_1,\dots,n_d}$ is a subgraph in $G_1 \boxtimes G_2$. Let $(N_1,\dots,N_d)$ be the partition of the $K_{n_1,\dots,n_d}$ subgraph. For all $i \in \{1,2\}$ and $j \in [d]$, let $N_j^{(i)}$ be the vertex-set for the projection of $G_1 \boxtimes G_2[N_j]$ onto G_i, let $\tilde{N}_j^{(i)} := N_j^{(i)} \setminus (\bigcup_{h\in[d],h\neq j} N_h^{(i)})$ and let $Z^{(i)} := \bigcup_{j\in[d]}(N_j^{(i)} \setminus \tilde{N}_j^{(i)})$. Then $(\tilde{N}_1^{(i)},\dots,\tilde{N}_d^{(i)}, Z^{(i)})$ are the colour classes of a complete multipartite subgraph of G_i. For all $j \in [d]$, let $a_j := |\tilde{N}_j^{(1)}|$, $b_j := |\tilde{N}_j^{(2)}|$, $x := |Z^{(1)}|$ and $y := |Z^{(2)}|$. Thus $K_{a_1,\dots,a_d,x} \subseteq G_1$ and $K_{b_1,\dots,b_d,y} \subseteq G_2$. Consider distinct vertices $u_1, v_1 \in Z^{(1)}$. Then there exists distinct $j,k \in [d]$ such that $(u_1,u_2) \in N_j$ and $(v_1,v_2) \in N_k$ for some $u_2, v_2 \in V(G_2)$ which implies that $u_1v_1 \in E(G_1)$. Similarly, the vertices in $Z^{(2)}$ are also pairwise adjacent in G_2. Hence, $K_{a_1,\dots,a_d,\overline{x}} \subseteq G_1$ and $K_{b_1,\dots,b_d,\overline{y}} \subseteq G_2$. Let $Z := Z^{(1)} \times Z^{(2)}$ and $z_j := |Z \cap N_j|$ for all $j \in [d]$. Then $z_1 + \dots z_d \leqslant |Z| = xy$. Since $N_j \subseteq (\tilde{N}_j^{(1)} \times N_j^{(2)}) \cup (\tilde{N}_j^{(1)} \times Z_2) \cup (Z_1 \cap N_j^{(2)})$ we have $n_j \leqslant a_j b_j + a_j y + b_j x + z_j$ for all $j \in [d]$, as required.

Now suppose that $K_{a_1,\dots,a_d,\overline{x}} \subseteq G_1$ and $K_{b_1,\dots,b_d,\overline{y}}$ where for all $j \in [d]$, a_j, b_j, x, y are non-negative integers. By Lemma 2.6, we have $K_{n_1,\dots,n_d,\overline{xy}} \subseteq G_1 \boxtimes G_2$. Splitting the colour class of size xy into sets of size $z_1,\dots,z_d$, and combining z_j with the j^{th} colour classes for all j, we obtain a $K_{a_1b_1+a_1y+b_1x+z_1,\dots,a_db_d+a_dy+b_dx+z_d}$ subgraph in $G_1 \boxtimes G_2$. Since $n_j \leqslant a_j b_j + a_j y + b_j x + z_j$ for all $j \in [d]$, we have $K_{n_1,\dots,n_d} \subseteq G_1 \boxtimes G_2$. □

Recall that for a graph G, $\omega(G)$ is the maximum size of a clique in G. To illustrate the usefulness of Theorem 2.7, we show how it implies the following well-known result.

Corollary 2.8. *For all graphs G and H, $\omega(G \boxtimes H) = \omega(G)\omega(H)$.*

Proof. Let $c := \omega(G)$ and $d := \omega(H)$. Since $K_{0,\dots,0,\overline{c}} \subseteq G$ and $K_{0,\dots,0,\overline{d}} \subseteq H$, by Theorem 2.7 we have $K_{1,\dots,1} = K_{cd} \subseteq G \boxtimes H$ by setting $z_i := 1$ for all $i \in [cd]$. Hence, $\omega(G \boxtimes H) \geqslant \omega(G)\omega(H)$. Now suppose that $K_n = K_{1,\dots,1} \subseteq G \boxtimes H$. By Theorem 2.7, there exists non-negative integers $a_1,\dots,a_n,b_1,\dots,b_n,z_1,\dots,z_n,x,y$ such that $K_{a_1,\dots,a_n,\overline{x}} \subseteq G_1$; and $K_{b_1,\dots,b_n,\overline{y}} \subseteq G_2$; and:

$$n_i \leqslant a_i b_i + a_i y + b_i x + z_i \text{ for all } i \in [n]; \text{ and} \tag{2.1}$$

$$z_1 + \dots + z_n \leqslant xy. \tag{2.2}$$

Let $A := \{i \in [n] : a_i \geqslant 1\}$, $B := \{i \in [n] : b_i \geqslant 1\}$ and $Z := \{i \in [n] : z_i \geqslant 1\}$. Then $|A| + |B| + |Z| \geqslant n$ since by Equation (2.1), a_i, b_i or z_i is at least 1 for all $i \in [n]$. By Equation (2.2), $|Z| \leqslant xy$. Now $K_{1,\dots,1,\overline{x}} = K_c \subseteq G$, and $K_{1,\dots,1,\overline{y}} = K_d \subseteq H$ where $c := |A| + x$ and $d := |B| + y$. Therefore, $cd = (|A| + x)(|B| + y) \geqslant |A| + |B| + xy \geqslant n$. Hence $\omega(G)\omega(H) \geqslant \omega(G \boxtimes H)$, as required. □

The case of complete bipartite subgraphs is of particular interest. The following result generalises a lemma due to Bonnet et al. [48, Lemma 7.2].

Corollary 2.9. *For all integers $t \geqslant s \geqslant 1$ and $\Delta \geqslant 1$, for all graphs G and H where G is $K_{s,t}$-free and H has maximum degree Δ, $G \boxtimes H$ is $K_{(s-1)(\Delta+1)+1,(s+t)(\Delta+1)}$-free. Moreover, for every integer $n \geqslant 1$, there exists a $K_{s,t}$-free graph $\tilde{G}$ and a graph $\tilde{H}$ with maximum degree Δ such that $K_{(s-1)(\Delta+1),n} \subseteq \tilde{G} \boxtimes \tilde{H}$.*

Proof. For the sake of contradiction, suppose that $K_{(s-1)(\Delta+1)+1,(s+t)(\Delta+1)} \subseteq G \boxtimes H$. By Theorem 2.7, there exists non-negative integers $a_1, a_2, b_1, b_2, z_1, z_2, x, y$ such that $K_{a_1,a_2,\overline{x}} \subseteq G_1$; and $K_{b_1,b_2,\overline{y}} \subseteq G_2$; and

$$(s-1)(\Delta+1)+1 \leqslant a_1(b_1+y)+b_1x+z_1; \tag{2.3}$$
$$(s+t)(\Delta+1) \leqslant a_2(b_2+y)+b_2x+z_2; \text{ and} \tag{2.4}$$
$$z_1+z_2 \leqslant xy. \tag{2.5}$$

Observe that if b_1 and y are both equal to 0 then Equations (2.3) and (2.5) cannot both be satisfied. As such, $b_2 + y \leqslant \Delta + 1$ since H has maximum degree Δ. Similarly, b_2 and y cannot both equal to 0 as this will violate Equation (2.4). Thus, $b_1 + y \leqslant \Delta + 1$. By Equation (2.5), $z_1 \leqslant xy$ and $z_2 \leqslant xy$. By Equation (2.3), we have $(\Delta+1)(s-1)+1 \leqslant a_1(b_1+y)+b_1x+xy \leqslant (a_1+x)(\Delta+1)$. As such, $a_1 + x > s-1$. Similarly, by Equation (2.4), $(s+t)(\Delta+1) \leqslant a_2(b_2+y)+b_2x+xy \leqslant (a_2+x)(\Delta+1)$. As such, $a_2+x \geqslant s+t$. This forces $K_{s,t} \subseteq G$, a contradiction.

Finally, let $\tilde{G} := K_{s-2,n,1}$ and $\tilde{H} := K_{\Delta,0,1}$. Then $\tilde{G}$ is $K_{s,t}$-free and H has maximum degree Δ. By Theorem 2.7, $K_{n_1,n_2} \subseteq \tilde{G} \boxtimes \tilde{H}$ where $n_1 = \Delta(s-2)+(s-2)+\Delta+1 = (s-1)(\Delta+1)$ and $n_2 = n$. ☐

2.4 Degeneracy

Bickle [33] determined the degeneracy of cartesian products. In particular, for all graphs G_1 and G_2, $\operatorname{degen}(G_1 \square G_2) = \operatorname{degen}(G_1) + \operatorname{degen}(G_2)$.

2.4.1 Direct Product

We now prove tight bounds for the degeneracy of direct products. We make use of the following observation.

Observation 2.10. *For all integers $t_i \geqslant s_i \geqslant 1$ where $i \in \{1, 2\}$,*

$$\operatorname{degen}(K_{s_1,t_1} \times K_{s_2,t_2}) = \min\{s_1t_2, s_2t_1\}.$$

This follows from the fact that $K_{s_1,t_1} \times K_{s_2,t_2}$ is the disjoint union of $K_{s_1t_2,s_2t_1}$ and $K_{s_1s_2,t_1t_2}$, and that $\operatorname{degen}(K_{s,t}) = \min\{s,t\}$.

Theorem 2.11. *For $i \in \{1,2\}$, let G_i be a graph with maximum degree Δ_i and* $\mathrm{degen}(G_i) = d_i$ *that contains K_{s_i,t_i} as a subgraph where $s_i \leqslant t_i$. Then:*

$$\max\{d_1 d_2, \min\{s_1 t_2, s_2 t_1\}, \min\{\Delta_1, \Delta_2\}\} \leqslant \mathrm{degen}(G_1 \times G_2) \leqslant \min\{d_1 \Delta_2, d_2 \Delta_1\}.$$

Proof. We first prove the lower bound. Let Q_i be a subgraph of G_i with $\delta(Q_i) = d_i$. Then $\delta(Q_1 \times Q_2) = d_1 d_2$ and thus, $\mathrm{degen}(G_1 \times G_2) \geqslant d_1 d_2$. Furthermore, since G_i contains K_{1,Δ_i} and K_{s_i,t_i} as a subgraph, by Observation 2.10, $\mathrm{degen}(G_1 \times G_2) \geqslant \min\{s_1 t_2, s_2 t_1\}$ and $\mathrm{degen}(G_1 \times G_2) \geqslant \min\{\Delta_1, \Delta_2\}$.

Now we prove the upper bound. Without loss of generality, assume $d_1 \Delta_2 \leqslant d_2 \Delta_1$. Let Z be a subgraph of $G_1 \times G_2$. Our goal is to show that $\delta(Z) \leqslant d_1 \Delta_2$. Let X be the projection of Z onto G_1. Since G_1 is d_1-degenerate, there exists a vertex $v_1 \in V(X)$ with $\deg_X(v_1) \leqslant d_1$. By construction of X, we have $(v_1, v_2) \in V(Z)$ for some $v_2 \in V(G_2)$. The neighbourhood of (v_1, v_2) in Z is a subset of $\{(u_1, u_2) : u_1 v_1 \in E(X), u_2 v_2 \in E(G_2)\}$. Now $|\{(u_1, u_2) : u_1 v_1 \in E(X), u_2 v_2 \in E(G_2)\}| \leqslant |\{u_1 \in V(X) : u_1 v_1 \in E(X)\}||\{u_2 \in V(X) : u_2 v_2 \in E(G_2)\}| \leqslant d_1 \Delta_2$. Thus (v_1, v_2) has degree at most $d_1 \Delta_2$ in Z and hence $\delta(Z) \leqslant d_1 \Delta_2$, as required. □

When both graphs are regular, the upper and lower bounds in Theorem 2.11 are equal. Furthermore, by Observation 2.10, the direct product of two complete bipartite graphs realises the upper bound in Theorem 2.11. We now show that there is a family of graphs that realises the lower bound in Theorem 2.11.

Lemma 2.12. *For all integers $d_i, \Delta_i, s_i, t_i \geqslant 1$ where $s_i \leqslant d_i \leqslant \Delta_i$ and $s_i \leqslant t_i \leqslant \Delta_i$ for $i \in \{1,2\}$, there exists graphs G_1 and G_2 where for $i \in \{1,2\}$, G_i has maximum degree Δ_i,* $\mathrm{degen}(G_i) = d_i$*, and G_i contains K_{s_i,t_i} as a subgraph, such that*

$$\mathrm{degen}(G_1 \times G_2) = \max\{d_1 d_2, \min\{s_1 t_2, s_2 t_1\}, \min\{\Delta_1, \Delta_2\}\}.$$

Proof. Let $\tilde{d} := \max\{d_1 d_2, \min\{s_1 t_2, s_2 t_1\}, \min\{\Delta_1, \Delta_2\}\}$. For $i \in \{1,2\}$, let H_i be a d_i-regular graph and let G_i be the disjoint union of H_i, K_{s_i,t_i} and K_{1,Δ_i}. Then G_i has maximum degree Δ_i, $\mathrm{degen}(G_i) = d_i$, and G_i contains K_{s_i,t_i} and K_{1,Δ_i} as subgraphs.

We now show that $\mathrm{degen}(G_1 \times G_2) = \tilde{d}$. Theorem 2.11 provides the lower bound. For the upper bound, our goal is to show that $\mathrm{degen}(J_1 \times J_2) \leqslant \tilde{d}$ whenever $J_i \in \{H_i, K_{s_i,t_i}, K_{1,\Delta_i}\}$ for $i \in \{1,2\}$. This implies that $\mathrm{degen}(G_1 \times G_2) \leqslant \tilde{d}$ since every connected subgraph of $G_1 \times G_2$ is a subgraph of $J_1 \times J_2$ for some choice of $J_i \in \{H_i, K_{s_i,t_i}, K_{1,\Delta_i}\}$ where $i \in \{1,2\}$. We proceed by case analysis.

First, $\operatorname{degen}(H_1 \times H_2) = d_1 d_2$ since H_1 and H_2 are regular. By Observation 2.10,

$$\begin{aligned}
\operatorname{degen}(K_{s_1,t_1} \times K_{s_2,t_2}) &= \min\{s_1 t_2, s_2 t_1\},\\
\operatorname{degen}(K_{s_1,t_1} \times K_{1,\Delta_2}) &= \min\{t_1, \Delta_2\},\\
\operatorname{degen}(K_{1,\Delta_1} \times K_{s_2,t_2}) &= \min\{\Delta_1, t_2\}, \text{ and}\\
\operatorname{degen}(K_{1,\Delta_1} \times K_{1,\Delta_2}) &= \min\{\Delta_1, \Delta_2\}.
\end{aligned}$$

Clearly the degeneracy is at most $\tilde{d}$ for each of the above graphs. Now consider the graph $H_1 \times K_{s_2,t_2}$. Each vertex has degree $d_1 s_2$ or $d_1 t_2$. Furthermore, the set of vertices with degree equal to $d_1 t_2$ are an independent set. Hence, every subgraph of $H_1 \times K_{s_2,t_2}$ contains a vertex with degree at most $d_1 s_2$. As such, $\operatorname{degen}(H_1 \times K_{s_2,t_2}) = d_1 s_2$. By the same reasoning: $\operatorname{degen}(H_1 \times K_{1,\Delta_2}) = d_1$, $\operatorname{degen}(K_{s_1,t_1} \times H_2) = s_1 d_2$, and $\operatorname{degen}(K_{1,\Delta_1} \times H_2) = d_2$. Again, the degeneracy is at most $\tilde{d}$ for each of the above graphs. Hence, $\operatorname{degen}(J_1 \times J_2) \leqslant \tilde{d}$ whenever $J_i \in \{H_i, K_{s_i,t_i}, K_{1,\Delta_i}\}$ for $i \in \{1,2\}$, as required. □

We conclude that the upper and lower bounds in Theorem 2.11 are tight.

2.4.2 Strong Product

We now consider the degeneracy of strong products. As the next lemma illustrates, more cases arise for this graph product compared to the other two.

Lemma 2.13. *For all integers $t_i \geqslant s_i \geqslant 1$, $\operatorname{degen}(K_{s_1,t_1} \boxtimes K_{s_2,t_2}) = f(s_1,t_1,s_2,t_2)$ where*

$$f(s_1,t_1,s_2,t_2) := \max\{s_1+s_2+s_1 s_2, \min\{t_1+t_2, s_1(t_2+1), s_2(t_1+1)\}, \min\{s_1 t_2, s_2 t_1\}\}.$$

Proof. Let $\hat{d} := f(s_1,t_1,s_2,t_2)$. Let (S_1,T_1) and (S_2,T_2) be the bipartition of K_{s_1,t_1} and K_{s_2,t_2} respectively. Let $G := K_{s_1,t_1} \boxtimes K_{s_2,t_2}$. Let $A := S_1 \times S_2$, $B := S_1 \times T_2$, $C := T_1 \times S_2$, and $D := T_1 \times T_2$.

We first prove the lower bound by showing that there exists a subgraph of G with minimum degree $\hat{d}$. Observe that $\delta(G) = s_1 + s_2 + s_1 s_2$. By Observation 2.10, there exists a subgraph J of $K_{s_1,t_1} \times K_{s_2,t_2} \subseteq G$ with $\delta(J) = \min\{s_1 t_2, s_2 t_1\}$. Let H be the subgraph of G induced by $A \cup B \cup C$. For this subgraph, $\deg_H(a) = t_1 + t_2$, $\deg_H(b) = s_2(t_1+1)$, and $\deg_H(c) = s_1(t_2+1)$ for every $a \in A$, $b \in B$ and $c \in C$. As such, $\delta(H) = \min\{t_1+t_2, s_1(t_2+1), s_2(t_1+1)\}$. Therefore, either G, J or H has minimum degree $\hat{d}$.

We now prove the upper bound. Let Z be a subgraph of G. We proceed by case analysis to show that the minimum degree of Z is at most $\hat{d}$.

First, if $V(Z)$ is a subset of either A, B, C or D, then $\delta(Z) = 0$ since these are independent sets. So we may assume that $V(Z)$ intersects at least two of those sets. Now if there exists a vertex $v \in V(Z) \cap D$, then $\deg_Z(v) \leqslant s_1+s_2+s_1 s_2 \leqslant \hat{d}$. So we may assume that $V(Z) \cap D = \varnothing$. Now if $V(Z) \cap A$, $V(Z) \cap B$, and $V(Z) \cap C$ are all non-empty, then

$\delta(Z) \leqslant \min\{t_1 + t_2, s_1(t_2+1), s_2(t_1+1)\}$. Furthermore, if $V(Z) \subseteq B \cup C$, then $\delta(Z) \leqslant \min\{s_1 t_2, s_2 t_1\}$. Finally, if $V(Z) \subseteq A \cup B$ or $V(Z) \subseteq A \cup C$, then $\delta(Z) \leqslant \max\{s_1, s_2\}$. In each case, $\delta(Z) \leqslant \hat{d}$ as required. □

The next theorem provides tight bounds for the degeneracy of strong products.

Theorem 2.14. *For $i \in \{1, 2\}$, let G_i be a graph with maximum degree Δ_i and $\operatorname{degen}(G_i) = d_i$ that contains K_{s_i,t_i} as a subgraph where $s_i \leqslant t_i$. Then*

$$g(d_1, \Delta_1, s_1, t_1, d_2, \Delta_2, s_2, t_2) \leqslant \operatorname{degen}(G_1 \boxtimes G_2) \leqslant h(d_1, \Delta_1, d_2, \Delta_2)$$

where f is specified by Lemma 2.13 and

$$g(d_1, \Delta_1, s_1, t_1, d_2, \Delta_2, s_2, t_2) = \max\{d_1 + d_2 + d_1 d_2, f(s_1, t_1, s_2, t_2), \min\{\Delta_1, \Delta_2\} + 1\},$$
$$h(d_1, \Delta_1, d_2, \Delta_2) = d_1 + d_2 + \min\{d_1 \Delta_2, d_2 \Delta_1\}.$$

Proof. We first prove the lower bound. Let Q_i be a subgraph of G_i with $\delta(Q_i) = d_i$. Then $\delta(Q_1 \boxtimes Q_2) = d_1 + d_2 + d_1 d_2$ and thus, $\operatorname{degen}(G_1 \boxtimes G_2) \geqslant d_1 + d_2 + d_1 d_2$. Furthermore, since G_i contains K_{1,Δ_i} and K_{s_i,t_i} as a subgraph, by Lemma 2.13, $\operatorname{degen}(G_1 \boxtimes G_2) \geqslant f(s_1, t_1, s_2, t_2)$ and $\operatorname{degen}(G_1 \boxtimes G_2) \geqslant \min\{\Delta_1, \Delta_2\} + 1$.

We now prove the upper bound. Let Z be a subgraph of $G_1 \boxtimes G_2$. Our goal is to show that $\delta(Z) \leqslant h(d_1, \Delta_1, d_2, \Delta_2)$. Without loss of generality, $d_1 \Delta_2 \leqslant d_2 \Delta_1$. Let Z be a subgraph of $G_1 \boxtimes G_2$ and let X be the projection of Z onto G_1. Since G_1 is d_1-degenerate, there exists a vertex $v_1 \in V(X)$ with $\deg_X(v_1) \leqslant d_1$. Let Y be the subgraph of G_2 induced by the set of vertices v_2 in G_2 such that $(v_1, y) \in V(Z)$. Since G_2 is d_2-degenerate, there exists a vertex $v_2 \in V(Y)$ with $\deg_Y(v_2) \leqslant d_2$. By construction of Y, $(v_1, v_2) \in V(Z)$. By the definition of the strong product,

$$\deg_Z((v_1, v_2)) \leqslant |N_X(v_1)| + |N_Y(v_2)| + |N_X(v_1)||N_{G_2}(v_2)| \leqslant d_1 + d_2 + d_1 \Delta_2.$$

Hence $\delta(Z) \leqslant h(d_1, \Delta_1, d_2, \Delta_2)$ as required. □

We now construct a family of graphs that realises the lower bound in Theorem 2.14.

Lemma 2.15. *For all integers $d_i, \Delta_i, s_i, t_i \geqslant 1$ where $s_i \leqslant d_i \leqslant \Delta_i$ and $s_i \leqslant t_i \leqslant \Delta_i$ for $i \in \{1, 2\}$, there exists graphs G_1 and G_2 where for $i \in \{1, 2\}$, G_i has maximum degree Δ_i, $\operatorname{degen}(G_i) = d_i$, and G_i contains K_{s_i,t_i} as a subgraph, such that*

$$\operatorname{degen}(G_1 \boxtimes G_2) = g(d_1, \Delta_1, s_1, t_1, d_2, \Delta_2, s_2, t_2)$$

where g is specified in Theorem 2.14.

Proof. Let $\tilde{d} := g(d_1, \Delta_1, s_1, t_1, d_2, \Delta_2, s_2, t_2)$. Our proof parallels the proof of Lemma 2.12. For $i \in \{1, 2\}$, let H_i be a d_i-regular graph and let G_i be the disjoint union of H_i, K_{s_i,t_i}

and K_{1,Δ_i}. Then G_i has maximum degree Δ_i, $\text{degen}(G_i) = d_i$, and G_i contains K_{s_i,t_i} and K_{1,Δ_i} as subgraphs.

We now show that $\text{degen}(G_1 \boxtimes G_2) = \tilde{d}$. Theorem 2.14 provides the lower bound. For the upper bound, our goal is to show that $\text{degen}(J_1 \boxtimes J_2) \leqslant \tilde{d}$ whenever $J_i \in \{H_i, K_{s_i,t_i}, K_{1,\Delta_i}\}$ for $i \in \{1,2\}$. We proceed by case analysis.

First, $\text{degen}(H_1 \boxtimes H_2) = d_1 + d_2 + d_1 d_2$ since H_1 and H_2 are regular. By Lemma 2.13,

$$\begin{aligned}
&\text{degen}(K_{s_1,t_1} \boxtimes K_{s_2,t_2}) = f(s_1,t_1,s_2,t_2) \text{ where } f \text{ is specifed by Lemma 2.13},\\
&\text{degen}(K_{s_1,t_1} \boxtimes K_{1,\Delta_2}) = \max\{2s_1+1, \min\{t_1, s_1\Delta_2\}\},\\
&\text{degen}(K_{1,\Delta_1} \boxtimes K_{s_2,t_2}) = \max\{2s_2+1, \min\{s_2\Delta_1, t_2\}\}, \text{ and}\\
&\text{degen}(K_{1,\Delta_1} \boxtimes K_{1,\Delta_2}) = \min\{\Delta_1, \Delta_2\} + 1.
\end{aligned}$$

For each of the above graphs, the degeneracy is at most $\tilde{d}$. Now consider the graph $H_1 \boxtimes K_{s_2,t_2}$. Each vertex in $H_1 \boxtimes K_{s_2,t_2}$ has degree $d_1 + s_2 + d_1 s_2$ or $d_1 + t_2 + d_1 t_2$. Let $A := \{v \in V(H_1 \boxtimes K_{s_2,t_2}) : \deg(v) = d_1 + s_2 + d_1 s_2\}$ and $B := \{v \in V(H_1 \boxtimes K_{s_2,t_2}) : \deg(v) = d_1 + t_2 + d_1 t_2\}$. Let Z be a subgraph of $H_1 \boxtimes K_{s_2,t_2}$. If $V(Z) \cap A$ is non-empty, then $\delta(Z) \leqslant d_1 + s_2 + d_1 s_2$. Otherwise, $V(Z) \subseteq B$ in which case $\delta(Z) \leqslant d_1$. Thus, $\text{degen}(H_1 \boxtimes K_{s_2,t_2}) = d_1 + s_2 + d_1 s_2$. By the same reasoning: $\text{degen}(H_1 \boxtimes K_{1,\Delta_2}) = 2d_1 + 1$, $\text{degen}(K_{s_1,t_1} \boxtimes H_2) = d_2 + s_1 + d_2 s_1$, and $\text{degen}(K_{1,\Delta_1} \boxtimes H_2) = 2d_2 + 1$. Again, the degeneracy is at most $\tilde{d}$ for each of the above graphs. Having considered all possibilities, it follows that $\text{degen}(J_1 \boxtimes J_2) \leqslant \tilde{d}$ whenever $J_i \in \{H_i, K_{s_i,t_i}, K_{1,\Delta_i}\}$ for $i \in \{1,2\}$, as required. □

The next lemma describes a family of graphs that realises the upper bound in Theorem 2.14.

Lemma 2.16. *For all integers $k_1, k_2, d_1, d_2 \geqslant 1$ there exists graphs G_1 and G_2 where for $i \in \{1,2\}$, G_i has maximum degree $\Delta_i := k_i d_i$, and* $\text{degen}(G_i) = d_i$*, such that*

$$\text{degen}(G_1 \boxtimes G_2) = d_1 + d_2 + \min\{d_1\Delta_2, d_2\Delta_1\}.$$

Proof. For $i \in \{1,2\}$, let G_i be a graph with vertex partition (A_i, B_i) where A_i is a clique on $d_i + 1$ vertices and B_i is an independent set of size $(d_i + 1)(k_i - 1)$ such that each $a_i \in A_i$ has $d_i(k_i - 1)$ neighbours in B_i and each $b_i \in B_i$ has d_i neighbours in A_i. Then G_i has maximum degree Δ_i and $\text{degen}(G_i) = d_i$.

We now show that $\text{degen}(G_1 \boxtimes G_2) = d_1 + d_2 + \min\{d_1\Delta_2, d_2\Delta_1\}$. Theorem 2.14 provides the upper bound. Let $X := A_1 \times A_2$, $Y := B_1 \times A_2$, and $Z := A_1 \times B_2$. Let H

be the subgraph of $G_1 \boxtimes G_2$ induced on $X \cup Y \cup Z$. Let $x \in X$, $y \in Y$ and $z \in Z$. Then

$$\begin{aligned}
\deg_H(x) &= |N_H(x) \cap X| + |N_H(x) \cap Y| + |N_H(x) \cap Z| \\
&= \Big((d_1+1)(d_2+1) - 1\Big) + \Big(d_2(\Delta_1 - d_1)\Big) + \Big(d_1(\Delta_2 - d_2)\Big), \\
\deg_H(y) &= |N_H(y) \cap X| + |N_H(y) \cap Y| + |N_H(y) \cap Z| \\
&= \Big(d_1(d_2+1)\Big) + \Big(d_2\Big) + \Big(d_1(\Delta_2 - d_2)\Big) = d_1 + d_2 + \Delta_2 d_1, \text{ and} \\
\deg_H(z) &= |N_H(z) \cap X| + |N_H(z) \cap Y| + |N_H(z) \cap Z| \\
&= \Big(d_2(d_1+1)\Big) + \Big(d_2(\Delta_1 - d_1)\Big) + \Big(d_1\Big) = d_1 + d_2 + d_2\Delta_1.
\end{aligned}$$

In which case, the minimum degree of H is $d_1 + d_2 + \min\{d_1\Delta_2, d_2\Delta_1\}$ as required. □

We conclude that the upper and lower bounds in Theorem 2.14 are tight.

2.5 Pathwidth and Treewidth

We now consider the treewidth and pathwidth of graph products. Many papers have studied the treewidth and pathwidth for various families of graphs that are defined by a graph product; see [70, 85, 108, 153, 182–184, 224, 231, 239, 252, 263]. In this section, we continue this work by presenting new lower bounds for cartesian and strong product as well as characterising when the cartesian, direct and strong products have bounded treewidth and when they have bounded pathwidth.

Let G be a graph. We say that $X, Y \subseteq V(G)$ *touch* if $X \cap Y \neq \varnothing$ or there is an edge of G between X and Y. A *bramble*, $\mathcal{B}$, is a set of pairwise touching connected subgraphs. A set $S \subseteq V(G)$ is a *hitting set* of $\mathcal{B}$ if S intersects every element of $\mathcal{B}$. The *order* of $\mathcal{B}$ is the minimum size of a hitting set of $\mathcal{B}$. The canonical example of a bramble of order ℓ is the set of crosses (union of a row and column) in the $\ell \times \ell$ grid. The following Treewidth Duality Theorem shows the intimate relationship between treewidth and brambles.

Theorem 2.17 ([299]). *A graph G has treewidth at least ℓ if and only if G contains a bramble of order at least $\ell + 1$.*

2.5.1 Cartesian and Strong Product

We now consider the treewidth of the cartesian and strong products. The following upper bound is well-known (see [43] for an implicit proof).

Lemma 2.18. *For all graphs G_1 and G_2,*

$$\operatorname{tw}(G_1 \,\square\, G_2) \leqslant \operatorname{tw}(G_1 \boxtimes G_2) \leqslant (\operatorname{tw}(G_1) + 1)\mathsf{v}(G_2) - 1, \text{ and}$$
$$\operatorname{pw}(G_1 \,\square\, G_2) \leqslant \operatorname{pw}(G_1 \boxtimes G_2) \leqslant (\operatorname{pw}(G_1) + 1)\mathsf{v}(G_2) - 1.$$

The proof of Lemma 2.18 in [43] shows the following, more general result which we use in Section 2.5.3.

Lemma 2.19. *For all graphs G_1, G_2 and H, if G_1 has an H-decomposition with width at most k, then $G_1 \boxtimes G_2$ has an H-decomposition with width at most $(k+1)\mathsf{v}(G_2) - 1$.*

Proof Sketch. Let $(H, \mathcal{W})$ be an H-decomposition of G_1 with width k. Modify $(H, \mathcal{W})$ to obtain a tree-decomposition $(H, \mathcal{B})$ of $G_1 \boxtimes G_2$ by setting $B_t := \{(v,u) : v \in W_t, u \in V(G_2)\}$ for all $t \in V(H)$. Then $(H, \mathcal{B})$ is an H-decomposition of $G_1 \boxtimes G_2$ with width at most $(k+1)\mathsf{v}(G_2) - 1$. □

A natural question is whether the upper bound in Lemma 2.18 is tight up to a constant factor. The following result shows that this is not the case.

Proposition 2.20. *For all $n \geqslant k+1 \geqslant 0$, there exists a connected graph $G_{k,n}$ such that $\operatorname{tw}(G_{k,n}) = k$ and $\mathsf{v}(G_{k,n}) = n$ and*

$$\operatorname{tw}(G_{k,n} \boxtimes G_{k,n}) = \Theta(n + k^2).$$

Proof. We make no attempt to optimise the constants in this proof. Let $G_{k,n}$ be the graph that consists of a path $P_{\tilde{n}} = (v_0, \dots, v_{\tilde{n}-1})$ on $\tilde{n} = n - k$ vertices and a complete graph K_{k+1} where $V(P_n) \cap V(K_k) = \{v_0\}$. Then $\operatorname{tw}(G_{k,n}) = k$ and $\mathsf{v}(G_{k,n}) = n$.

We now show that $\operatorname{tw}(G_{k,n} \boxtimes G_{k,n}) = \Theta(n + k^2)$. For the lower bound, since $\mathcal{G}_{k,n}$ is connected, by Theorem 2.21, $\operatorname{tw}(G_{k,n} \boxtimes G_{k,n}) \geqslant n - 1$. Moreover, since $K_k \boxtimes K_k \simeq K_{k^2}$, $\operatorname{tw}(G_{k,n} \boxtimes G_{k,n}) \geqslant \operatorname{tw}(K_{k^2}) = k^2 - 1$. Thus $\operatorname{tw}(G_{k,n} \boxtimes G_{k,n}) = \Omega(n + k^2)$.

For the upper bound, we construct a tree-decomposition of $G_{k,n} \boxtimes G_{k,n}$ with width $O(n + k^2)$. We begin by specifying the bags. For $i, j, \ell \in [\tilde{n} - 1]$, let

$$\begin{aligned}
X &:= \{(v_0, u), (u, v_0) : u \in V(G_{k,n})\}, \\
W_y &:= \{(a,b) : a, b \in K_k\} \cup X, \\
C_{u_i} &:= \{(a, v_{j-1}), (a, v_j) : a \in K_k\} \cup X, \\
D_{w_j} &:= \{(v_{i-1}, a), (v_i, a) : a \in K_k\} \cup X, \text{ and} \\
L_{z_\ell} &:= \{(v_{\ell-1}, v_s), (v_\ell, v_s), (v_s, v_{\ell-1}), (v_s, v_\ell) : s \in [\ell, n-1]\} \cup X.
\end{aligned}$$

Observe that $|X| \leqslant 2n$, $|W_y| \leqslant 2n + k^2$, $|C_i| = |D_i| \leqslant 2n + 2k$ and $|L_{w_\ell}| \leqslant 6n$. Moreover, observe that $V(K_k \boxtimes K_k) \subseteq W_y$, $V(K_k \boxtimes P_n) \subseteq \bigcup(C_{u_i} : i \in [n-1])$, $V(P_n \boxtimes K_k) \subseteq \bigcup(D_{z_i} : i \in [\tilde{n}-1])$, and $V(P_n \boxtimes P_n) \subseteq \bigcup(L_{w_\ell} : \ell \in [\tilde{n}-1])$. Thus, every bag has size $O(n+k^2)$ and every vertex is in a bag. For the tree to index the decomposition, let $P^{(C)} := (u_1, \dots, u_{\tilde{n}-1})$, $P^{(D)} := (w_1, \dots, w_{\tilde{n}-1})$, and $P^{(L)} := (z_1, \dots, z_{\tilde{n}-1})$ be paths on $\tilde{n} - 1$ vertices. Let T be the tree obtained by taking the disjoint union of $P^{(C)}$, $P^{(D)}$ and $P^{(L)}$, then adding the vertex y and the edges yu_1, yw_1, yz_1 (see Figure 2.1). Let $\mathcal{W} := \{W_t : t \in V(T)\}$. We claim that $(T, \mathcal{W})$ is a tree-decomposition of $G_{k,n} \boxtimes G_{k,n}$.

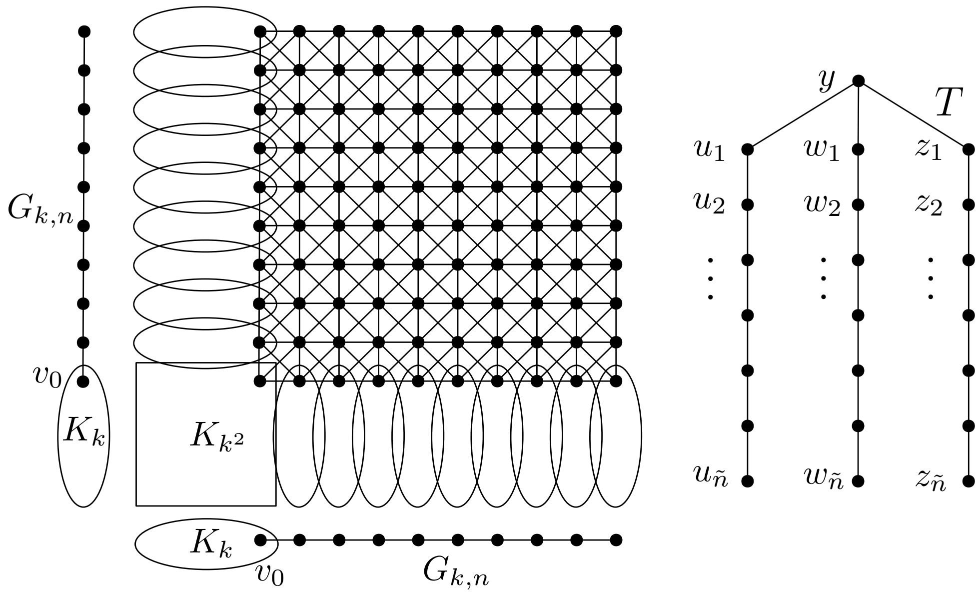


Figure 2.1. $G_{k,n} \boxtimes G_{k,n}$ with the tree T.

As noted earlier, every vertex is in a bag. Let $(a_1, a_2)(b_1, b_2) \in E(G_{k,n} \boxtimes G_{k,n})$. Suppose that a_1 or a_2 is equal to v_0. Then $(a_1, a_2) \in X$. Thus, since $(b_1, b_2) \in W_{t'}$ for some $t' \in V(T)$ and $X \subseteq W_t$ for all $t \in V(T)$, we have $(a_1, a_2)(b_1, b_2) \in W_{t'}$. Now suppose that $a_1, a_2 \in K_{k+1} \setminus \{v_0\}$. Then $N((a_1, a_2)) \subseteq K_{k+1} \boxtimes K_{k+1}$ in which case $(a_1, a_2), (b_1, b_2) \in W_y$. Now if $a_1 \in K_{k+1} \setminus \{v_0\}$ and $a_2 = v_i$ for some $i \in [\tilde{n} - 1]$, then $N((a_1, a_2)) \subseteq C_{u_{i-1}} \cup C_{u_i}$ in which case $(a_1, a_2), (b_1, b_2) \in W_{u_{i'}}$ for $i' \in \{i - 1, i\}$. Similarly, if $a_2 \in K_{k+1} \setminus \{v_0\}$ and $a_1 = v_i$ for some $j \in [\tilde{n} - 1]$, then $(a_1, a_2), (b_1, b_2) \in W_{w_{j'}}$ for $j' \in \{j - 1, j\}$. Finally, if $a_1 = v_i$ and $a_2 = v_j$ for some $i, j \in [\tilde{n} - 1]$, then $N(a_1, a_2) \subseteq L_{w_{\ell-1}} \cup L_{w_\ell}$ and thus $(a_1, a_2), (b_1, b_2) \in W_{u_{\ell'}}$ for some $\ell' \in [\ell - 1, \ell]$ where $\ell := \min\{i, j\}$. As such, every edge is in a bag.

It remains to show that for any $(a_1, a_2) \in V(G_{k,n} \boxtimes G_{k,n})$, the subtree $T^{(a_1,a_2)} := T[t \in V(T) : (a_1, a_2) \in W_t]$ is connected. If $(a_1, a_2) \in X$, then $T^{(a_1,a_2)} = V(T)$. If $a_1, a_2 \in V(K_k) \setminus \{v_0\}$, then $V(T^{(a_1,a_2)}) = \{y\}$. If $a_1 \in V(K_k) \setminus \{v_0\}$ and $a_2 = v_i$ for some $i \in [\tilde{n} - 1]$, then $V(T^{(a_1,a_2)}) = \{u_{i-1}, u_i\}$. If $a_1 = v_j$ for some $j \in [\tilde{n} - 1]$ and $a_2 \in V(K_k) \setminus \{v_0\}$, then $V(T^{(a_1,a_2)}) = \{w_{j-1}, w_j\}$. If $a_1 = v_j$ and $a_2 = v_i$ for some $i, j \in [\tilde{n} - 1]$ then $V(T^{(a_1,a_2)}) = \{z_{\ell-1}, z_\ell\}$ where $\ell := \min\{i, j\}$. In each case, $T^{(a_1,a_2)}$ is connected. Therefore, $(T, \mathcal{W})$ is indeed a tree-decomposition of $G_{k,n} \boxtimes G_{k,n}$ with width $O(n + k^2)$, as required. □

We now consider lower bounds for the treewidth of cartesian and strong products. Wood [338] established the following lower bound for highly-connected graphs.

Theorem 2.21 ([338]). *For all k-connected graphs G_1 and G_2 each with at least n vertices,*

$$\operatorname{tw}(G_1 \boxtimes G_2) \geqslant \operatorname{tw}(G_1 \,\square\, G_2) \geqslant k(n - 2k + 2) - 1.$$

We now present two new lower bounds. For a graph G and $\varepsilon \in [\frac{2}{3}, 1)$, a partition (A, S, B) of $V(G)$ is an *ε-separation* if $1 \leqslant |A|, |B| \leqslant \varepsilon|V(G)|$ and there is no edge between A and B. The *order* of (A, S, B) is $|S|$. Robertson and Seymour [287] showed that graphs with small treewidth have separations with small order.

Lemma 2.22 ([287]). *For every $\varepsilon \in [\frac{2}{3}, 1)$, every graph with treewidth k has an ε-separation of order at most $k + 1$.*

Lemma 2.23. *For all $\varepsilon, \beta \in [\frac{2}{3}, 1)$ where $\beta > \varepsilon$ and integers $k, n, m \geqslant 1$ where $m \geqslant kn$ and $n \geqslant \frac{1}{1-\beta}$ and for all connected graphs G and H with m and n vertices respectively such that every β-separation of G has order at least k, every ε-separation of $G \square H$ has order at least $(1 - \frac{\varepsilon}{\beta})kn$.*

Proof. Let $V(H) := [n]$. Let (A, S, B) be an ε-separation of $G \square H$. Our goal is to show that $|S| \geqslant (1 - \frac{\varepsilon}{\beta})kn$. For $i \in [n]$, let $G^{(i)}$ be the copy of G in $G \square H$ induced by $\{(v, i) : v \in V(G)\}$. We say that $G^{(i)}$ *belongs* to A if $|A \cap V(G^{(i)})| \geqslant \beta m$, and $G^{(i)}$ *belongs* to B if $|B \cap V(G^{(i)})| \geqslant \beta m$.

Suppose that some copy $G^{(i)}$ belongs to A and some copy $G^{(j)}$ belongs to B. Let $X := \{v \in V(G) : (v, i) \in A, (v, j) \in B\}$. Thus $|X| \geqslant (2\beta - 1)m > 0$, which implies $i \neq j$. Since H is connected, for each $x \in X$ there is a path from (x, i) to (x, j) contained within the subgraph of $G \square H$ induced by $\{(x, \ell) : \ell \in V(H)\}$. Since these paths are pairwise disjoint and each path contains a vertex from S, we have $|S| \geqslant |X| \geqslant (2\beta - 1)kn \geqslant (1 - \frac{\varepsilon}{\beta})kn$.

Now assume, without loss of generality, that no copy of G belongs to B. Say t copies of G belong to A. Then $\beta tm \leqslant |A| \leqslant \varepsilon nm$, implying that $t \leqslant \frac{\varepsilon n}{\beta}$. Thus, at least $(1 - \frac{\varepsilon}{\beta})n$ copies of G belong to neither A nor B. Now consider such a copy $G^{(i)}$. If $G^{(i)}$ is contained in $S \cup A$, then $|S \cap V(G^{(i)})| \geqslant (1 - \beta)m \geqslant k$. Similarly, if $G^{(i)}$ is contained in $S \cup B$, then $|S \cap V(G^{(i)})| \geqslant k$. Otherwise, $G^{(i)}$ contains vertices in both A and B, in which case $(A \cap V(G^{(i)}), S \cap V(G^{(i)}), B \cap V(G^{(i)}))$ is a β-separation of $G^{(i)}$ and thus, $|S \cap V(G^{(i)})| \geqslant k$. Therefore,

$$|S| = \sum_{i=1}^{n} |V(G^{(i)}) \cap S| \geqslant (1 - \tfrac{\varepsilon}{\beta})kn$$

as required. □

Lemmas 2.22 and 2.23 imply the following.

Theorem 2.24. *For all $\varepsilon, \beta \in [\frac{2}{3}, 1)$ where $\beta > \varepsilon$ and integers $k, n, m \geqslant 1$ where $m \geqslant kn$ and $n \geqslant \frac{1}{1-\beta}$ and for all connected graphs G and H with m and n vertices respectively such that every β-separation of G has order at least k,*

$$\operatorname{tw}(G \square H) \geqslant (1 - \tfrac{\varepsilon}{\beta})kn - 1.$$

By applying Lemmas 2.23 and 2.18, we determine the treewidth of d-dimensional grid graphs up to a constant factor.

Corollary 2.25. *For fixed $d \geqslant 2$ and all integers $n_1 \geqslant \ldots \geqslant n_d \geqslant 1$,*

$$\operatorname{tw}(P_{n_1} \square \cdots \square P_{n_d}) = \Theta\left(\prod_{j=2}^{d} n_j\right) \quad \textit{and} \quad \operatorname{tw}(P_{n_1} \boxtimes \cdots \boxtimes P_{n_d}) = \Theta\left(\prod_{j=2}^{d} n_j\right).$$

Proof Sketch. We proceed by induction on $d \geqslant 2$. For the upper bound, apply Lemma 2.18 by setting $G_1 := P_{n_1} \boxtimes \cdots \boxtimes P_{n_{d-1}}$ and $G_2 := P_{n_d}$ with the induction hypothesis that $\operatorname{tw}(P_{n_1} \boxtimes \cdots \boxtimes P_{n_d}) \leqslant \prod_{j=2}^{d} n_j$ for all $d \geqslant 2$. For the lower bound, apply Lemma 2.23 by setting $G := P_{n_1} \square \cdots \square P_{n_{d-1}}$ and $H := P_{n_d}$ with the induction hypothesis that for all $d \geqslant 2$ and $\varepsilon \in [\frac{2}{3}, 1)$, there is a constant $c(d, \varepsilon)$ such that for all sufficiently large n_d (as a function of d and ε), every ε-separation in $P_{n_1} \square \cdots \square P_{n_d}$ has order at least $c(d, \varepsilon) \prod_{j=2}^{d} n_j$. The lower bound then follows by Lemma 2.22. □

This result demonstrates that d-dimensional grids are a family of graphs for which Theorem 2.24 is tight up to a constant factor, whereas the lower bound given by Theorem 2.21 is not of the correct order for this family.

We now present another lower bound in terms of the Hadwiger number.

Theorem 2.26. *For all graphs G and H,*

$$\operatorname{tw}(G \boxtimes H) \geqslant h(H)(\operatorname{tw}(G) + 1) - 1.$$

Proof. Let $(B_i : i \in V(K_t))$ be a model of K_t in H where $t := h(H)$. By Theorem 2.17, there is a bramble $\mathcal{B}$ in G of such that every hitting set of $\mathcal{B}$ has order at least $\operatorname{tw}(G) + 1$. For $X \in \mathcal{B}$ and $i \in [t]$, let $(X, i) := \{(v, u) \in V(G \boxtimes H) : v \in X, u \in B_i\}$. Let $G_{X,i}$ denote the subgraph of $G \boxtimes H$ that is induced by (X, i). Since $G[X]$ and $H[B_i]$ are connected, $G_{X,i}$ is connected. For $i \in [t]$, let $\mathcal{B}_i := \{(X, B_i) : X \in \mathcal{B}\}$. Then $\mathcal{B}_i$ is a bramble for $G \boxtimes (H[B_i])$.

Let $\mathcal{C} := \{(X, B_i) : X \in \mathcal{B}, i \in [t]\}$. Consider $(X, B_i), (Y, B_j) \in \mathcal{C}$. Since X and Y touch in G, for some vertices $v \in X$ and $w \in Y$, either $v = w$ or $vw \in E(G)$. Moreover, there exists $u_i \in B_i$ and $u_j \in B_j$ such that $u_i = u_j$ (if $i = j$) or $u_i u_j \in E(H)$. Thus, in $G \boxtimes H$, the vertices (v, u_i) and (w, u_j) are adjacent or equal. Since $(v, u_i) \in (X, B_i)$ and $(v, u_j) \in (Y, B_j)$, the sets (X, B_i) and (Y, B_j) touch. Hence $\mathcal{C}$ is a bramble in $G \boxtimes H$. Let J be a hitting set for $\mathcal{C}$. For each $i \in [t]$, let $J_i := \{(v, u) \in J : u \in B_i\}$. Thus J_i is a hitting set for the bramble $\mathcal{B}_i$ (in $G \boxtimes B_i$). By Theorem 2.17, $|J_i| \geqslant \operatorname{tw}(G) + 1$. Since the J_i's are pairwise disjoint, $|J| \geqslant t(\operatorname{tw}(G) + 1)$. Hence $\operatorname{tw}(G \boxtimes H) \geqslant t(\operatorname{tw}(G) + 1) - 1$ by Theorem 2.17. □

Theorem 2.26 and Lemma 2.18 together imply that for every graph G,

$$\operatorname{tw}(G \boxtimes K_n) = (\operatorname{tw}(G) + 1)n - 1.$$

The next two theorems characterise when the cartesian and strong products have bounded treewidth and when they have bounded pathwidth.

Theorem 2.27. *The following are equivalent for monotone graph classes $\mathcal{G}_1$ and $\mathcal{G}_2$:*

1. *$\mathcal{G}_1 \boxtimes \mathcal{G}_2$ has bounded treewidth;*
2. *$\mathcal{G}_1 \square \mathcal{G}_2$ has bounded treewidth;*
3. *both $\mathcal{G}_1$ and $\mathcal{G}_2$ have bounded treewidth and $\tilde{\mathsf{v}}(\mathcal{G}_1)$ or $\tilde{\mathsf{v}}(\mathcal{G}_2)$ is bounded; and*
4. *$\mathcal{G}_1 \boxtimes \mathcal{G}_2$ has bounded Hadwiger number;*

Proof. Building on the work of Wood [337], Pecaninovic [272] showed that (3) and (4) are equivalent. Since treewidth is closed under subgraphs, (1) implies (2). By Lemma 2.18, (3) implies (1). To show that (2) implies (3), suppose that $\mathcal{G}_1 \square \mathcal{G}_2$ has bounded treewidth. If either $\mathcal{G}_1$ or $\mathcal{G}_2$ have unbounded treewidth, then $\mathcal{G}_1 \square \mathcal{G}_2$ has unbounded treewidth, a contradiction. If the connected graphs in both $\mathcal{G}_1$ and $\mathcal{G}_2$ have unbounded order, then by Theorem 2.21, $\mathcal{G}_1 \square \mathcal{G}_2$ has unbounded treewidth, a contradiction. As such, both $\mathcal{G}_1$ and $\mathcal{G}_2$ have bounded treewidth and $\tilde{\mathsf{v}}(\mathcal{G}_1)$ or $\tilde{\mathsf{v}}(\mathcal{G}_2)$ is bounded. □

We omit the proof for the following theorem as it is identical to Theorem 2.27.

Theorem 2.28. *The following are equivalent for monotone graph classes $\mathcal{G}_1$ and $\mathcal{G}_2$:*

1. *$\mathcal{G}_1 \boxtimes \mathcal{G}_2$ has bounded pathwidth;*
2. *$\mathcal{G}_1 \square \mathcal{G}_2$ has bounded pathwidth; and*
3. *both $\mathcal{G}_1$ and $\mathcal{G}_2$ have bounded pathwidth and $\tilde{\mathsf{v}}(\mathcal{G}_1)$ or $\tilde{\mathsf{v}}(\mathcal{G}_2)$ is bounded.*

We conclude this subsection with some open problems. First and foremost, can we determine the treewidth of $G \boxtimes H$ up to a constant factor? A first step in resolving this question is to determine if Theorem 2.26 can be strengthened by showing that there exists a constant $c > 0$ such that $\operatorname{tw}(G \boxtimes H) \geqslant c \operatorname{tw}(G) \operatorname{tw}(H)$ for all graphs G and H. Similar questions arise for the cartesian product. It is also open whether there exist a constant $c > 0$ such that $\operatorname{tw}(G \boxtimes H) \leqslant c \operatorname{tw}(G \square H)$ for all graphs G and H.

2.5.2 Direct Product with K_2

We now consider when a direct product has bounded treewidth and when it has bounded pathwidth. In comparison with the other two products, characterisation for this product is more involved. Before considering the direct product of two classes of graphs, we first investigate the direct product of a class of graphs with a single graph. Lemma 2.18 immediately provides an upper bound for the treewidth and pathwidth of the direct product of a class of graphs with a single graph. The challenge therefore lies in proving a lower bound. In related results, Bottreau and Métivier [54] demonstrated that for every graph G and non-bipartite graph H, G is a minor of $G \times H$, and hence $\operatorname{tw}(G \times H) \geqslant \operatorname{tw}(G)$ and $\operatorname{pw}(G \times H) \geqslant \operatorname{pw}(G)$. This lower bound for treewidth was independently shown by

Eppstein et al. [142]. It remains to consider the case when H is bipartite. A case of particular interest is when $H = K_2$. Thomassen [316] showed the following.

Theorem 2.29 ([316])**.** *There exists a function f such that every graph G with $\mathrm{tw}(G) \geqslant f(k)$ contains a bipartite subgraph $\hat{G}$ such that $\mathrm{tw}(\hat{G}) \geqslant k$.*

Note that the function f in Theorem 2.29 is exponential with respect to k. Since every bipartite subgraph of G is also a subgraph of $G \times K_2$, we have the following.

Corollary 2.30. *For every graph class $\mathcal{G}$ with unbounded treewidth, $\mathcal{G} \times K_2$ also has unbounded treewidth.*

Together with Lemma 2.18, Corollary 2.30 demonstrates that $\mathrm{tw}(G)$ and $\mathrm{tw}(G \times K_2)$ are tied. Similarly, for pathwidth, we now show how an analogous result to Theorem 2.29 follows from the Excluded Tree Minor Theorem (Theorem 1.4).

Proposition 2.31. *There exists a function f such that every graph G with $\mathrm{pw}(G) \geqslant f(k)$ contains a bipartite subgraph $\hat{G}$ such that $\mathrm{pw}(\hat{G}) \geqslant k$.*

Proof. The complete binary tree T_k with height k has $2^k - 1$ vertices and pathwidth $\lceil k/2 \rceil$ [295]. Let $f(k) := 2^{(2k+1)} - 2 = |V(T_{2k+1})| - 1$ and G be a graph with $\mathrm{pw}(G) \geqslant f(k)$. By Theorem 1.4, G contains T_{2k+1} as a minor. Since T_{2k+1} has maximum degree 3, G contains a subdivision of T_{2k+1} as a subgraph which is our desired bipartite subgraph with pathwidth k. □

Let G be a graph with sufficiently large pathwidth. By Proposition 2.31, G contains a bipartite subgraph $\hat{G}$ with large pathwidth. Since $\hat{G}$ is a subgraph of $G \times K_2$, it follows that $G \times K_2$ has large pathwidth. As such, Proposition 2.31 implies the following.

Corollary 2.32. *For every graph class $\mathcal{G}$ with unbounded pathwidth, $\mathcal{G} \times K_2$ also has unbounded pathwidth.*

Together with Lemma 2.18, it follows that $\mathrm{pw}(G)$ and $\mathrm{pw}(G \times K_2)$ are tied. Note that the function f in Proposition 2.31 is exponential with respect to k. We now strengthen these results by demonstrating that $\mathrm{tw}(G)$ and $\mathrm{tw}(G \times K_2)$ are polynomially tied and then showing that this implies that $\mathrm{pw}(G)$ and $\mathrm{pw}(G \times K_2)$ are polynomially tied.

Theorem 2.33. *There exists a positive constant c such that $\mathrm{tw}(G \times K_2) \geqslant k$ for every graph G with $\mathrm{tw}(G) \geqslant ck^4 \log^{5/2}(k)$.*

To prove Theorem 2.33, we make use of grid-like-minors which were introduced by Reed and Wood [281]. A *grid-like-minor* of order ℓ in a graph G is a set $\mathcal{P}$ of paths in G such that the intersection graph of $\mathcal{P}$ is bipartite and contains a K_ℓ minor.[1] Reed and Wood [281] showed the following.

[1]The intersection graph of a set X, whose elements are sets, has vertex-set X where distinct vertices are adjacent whenever the corresponding sets have a non-empty intersection.

Theorem 2.34 ([281]). *For some positive constant c, every graph with treewidth at least $c\ell^4\sqrt{\log(\ell)}$ contains a grid-like-minor of order ℓ.*

We now explain how to adapt the proof for Theorem 2.34 to show that if a graph G has sufficiently large treewidth, then $G \times K_2$ contains a grid-like-minor of large order. A key lemma that Reed and Wood used to prove Theorem 2.34 is the following.

Lemma 2.35 ([281]). *For all integers $k, \ell \geqslant 1$, every graph G with treewidth at least $k\ell - 1$ contains ℓ disjoint paths $P_1, \dots, P_\ell$, and for distinct $i, j \in [\ell]$, G contains k disjoint paths between P_i and P_j.*

Given a black-white colouring of the vertices of a bipartite graph, to *switch* the colouring means to recolour the black vertices white and the white vertices black. For a graph G with vertex-disjoint subgraphs H_1 and H_2, a path $P = (v_1, \dots, v_n)$ in G *joins* H_1 and H_2 if $V(P) \cap V(H_1) = \{v_1\}$ and $V(P) \cap V(H_2) = \{v_n\}$.

Lemma 2.36. *Let G be a graph and let $H_1, \dots, H_t$ be vertex-disjoint bipartite subgraphs in G. Suppose there exists a set of internally disjoint paths $\mathcal{P} = \{P_1, \dots, P_k\}$ in G such that for all $i \in [k]$, P_i joins H_j and H_ℓ for some distinct $j, \ell \in [t]$ and that P_i and H_b are disjoint for all $b \in [t] \setminus \{j, \ell\}$. Then there exists a subset $\mathcal{S} \subseteq \mathcal{P}$ where $|\mathcal{S}| \geqslant k/2$ such that the graph $\hat{G} := (\bigcup_{i \in [t]} H_i) \cup (\bigcup_{S \in \mathcal{S}} S)$ is a bipartite subgraph of G.*

Proof. For all $j \in [t]$, let $\mathcal{P}^{(j)}$ be the set of paths in $\mathcal{P}$ that join any H_i and $H_{i'}$ where $i, i' \in [j]$. Note that $\varnothing = \mathcal{P}^{(1)} \subseteq \mathcal{P}^{(2)} \subseteq \dots \subseteq \mathcal{P}^{(t)} = \mathcal{P}$. We prove the following by induction on j.

Claim: For all $j \in [t]$, there exists a subset $\mathcal{S}^{(j)} \subseteq \mathcal{P}^{(j)}$ where $|\mathcal{S}^{(j)}| \geqslant |\mathcal{P}^{(j)}|/2$ such that the graph $\hat{G}^{(j)} := (\bigcup_{i \in [j]} H_i) \cup (\bigcup_{S \in \mathcal{S}^{(j)}} S)$ is a bipartite subgraph of G.

For $j = 1$, the claim holds trivially since $\mathcal{P}^{(1)} = \varnothing$. Now assume it holds for $j - 1$. Then there exists a set of paths $\mathcal{S}^{(j-1)}$ with size at least $|\mathcal{P}^{(j-1)}|/2$ such that $\hat{G}^{(j-1)}$ is bipartite. Since $\hat{G}^{(j-1)}$ and H_j are vertex-disjoint bipartite subgraphs, we may take a proper black-white colouring of their vertices. Let $A^{(j)} := \mathcal{P}^{(j)} \setminus \mathcal{P}^{(j-1)}$. Then each path in $A^{(j)}$ joins H_j to some H_i where $i \in [j-1]$. We say that a path $P = (v_1, \dots, v_n) \in A^{(j)}$ is *agreeable* if there exists a proper black-white colouring of the vertices of P such that the colouring of v_1 and v_n corresponds with the colour they were previously assigned. Otherwise we say that P is *disagreeable*. Observe that if we switch the black-white colouring for the vertices in H_j, then all the agreeable paths in $A^{(j)}$ become disagreeable and all the disagreeable paths become agreeable. So if there are more disagreeable paths than agreeable, then switch the black-white colouring of the vertices in H_j. In doing so, the set of agreeable paths $B^{(j)} \subseteq A^{(j)}$ has size at least $|A^{(j)}|/2$. Let $\mathcal{S}^{(j)} := \mathcal{S}^{(j-1)} \cup (\bigcup_{P \in B^{(j)}} P)$. Then $\hat{G}^{(j)}$ is bipartite and $|\mathcal{S}^{(j)}| = |\mathcal{S}^{(j-1)} \cup B^{(j)}| \geqslant |\mathcal{P}^{(j-1)}|/2 + |A^{(j)}|/2 = |\mathcal{P}^{(j)}|/2$ as required.

The result follows when $j = t$. □

Note that Lemma 2.36 generalises Erdős' result [146]: if $H_1, \dots, H_t$ are the vertices of a graph G and $\mathcal{P}$ is the set of the edges in G, then by Lemma 2.36, G contains a bipartite subgraph with at least half the edges.

The next lemma is the main original contribution for this subsection in which we adapt Lemma 2.35 to the setting of a direct product with K_2.

Lemma 2.37. *Let G be a graph with treewidth at least $2k\ell - 1$ for some integers $k, l \geqslant 1$. Then there exists a set $X \subseteq E(K_\ell)$ where $|X| \geqslant |E(K_\ell)|/2$ such that $G \times K_2$ contains ℓ-disjoint paths $P_1, \dots, P_\ell$ and for all $ij \in X$, $G \times K_2$ contains k-disjoint paths between P_i and P_j.*

Proof. By Lemma 2.35, G contains ℓ disjoint paths $\tilde{P}_1, \dots, \tilde{P}_\ell$, and a set $\mathcal{Q}_{i,j}$ of $2k$ disjoint paths between $\tilde{P}_i$ and $\tilde{P}_j$ where $i, j \in [\ell]$ are distinct. Take a proper black-white colouring of $\tilde{P}_1, \dots, \tilde{P}_j$. For a path $Q = (v_1^{(i,j)}, \dots, v_n^{(i,j)}) \in \mathcal{Q}_{i,j}$, we say that it is *agreeable* if there exists a black-white colouring of Q such that the colour of v_1 and v_n corresponds with the colour they were previously assign. Otherwise, we say that Q is *disagreeable*. Observe that if we switch the black-white colouring of $\tilde{P}_j$, then every agreeable path becomes disagreeable and every disagreeable path becomes agreeable. If there are more agreeable paths than disagreeable, then let $\mathcal{M}_{i,j}$ be the set of agreeable paths, otherwise let $\mathcal{M}_{i,j}$ be the set of disagreeable path. By the pigeon-hole principle, $|\mathcal{M}_{i,j}| \geqslant k$. Note that the set $\mathcal{M}_{i,j}$ of paths are *pairwise-agreeable*, in the sense that if one path in $\mathcal{M}_{i,j}$ is agreeable then all the paths in $\mathcal{M}_{i,j}$ are agreeable.

For each distinct $i, j \in [\ell]$, let $R_{i,j} = (v_1^{(i,j)}, \dots, v_{n_{i,j}}^{(i,j)})$ be an arbitrary path in $\mathcal{M}_{i,j}$. We now define an auxiliary graph J as follows. Let J consist of the ℓ-disjoint paths $\tilde{P}_1, \dots, \tilde{P}_\ell$. For each distinct $i, j \in [\ell]$, add a path $\tilde{R}_{i,j}$ from $v_1^{(i,j)} \in V(\tilde{P}_i)$ to $v_{n_{i,j}}^{(i,j)} \in V(\tilde{P}_j)$ of length $n_{i,j}$ that is internally disjoint from all other vertices in J. Let $\mathcal{R} = \{\tilde{R}_{i,j} : i, j \in [\ell], i \neq j\}$. By Lemma 2.36, there exists a set $\mathcal{S} \subseteq \mathcal{R}$ where $|\mathcal{S}| \geqslant |\mathcal{R}|/2$ such that $\tilde{J} = (\bigcup_{i \in [\ell]} \tilde{P}_i) \cup \mathcal{S}$ is bipartite. Let $X = \{ij : R_{i,j} \in \mathcal{S}\}$ and note that $|X| \geqslant |E(K_\ell)|/2$. Now for all $ij \in X$, since the set of paths $\mathcal{M}_{i,j}$ are pairwise agreeable, it follows that for all $Q = (v_1, \dots, v_n) \in \mathcal{M}_{i,j}$, we can add a path $\hat{Q}$ of length $|Q|$ from $v_1 \in V(\tilde{P}_i)$ to $v_n \in V(\tilde{P}_j)$ to the graph $\tilde{J}$ that is internally disjoint to all other paths without compromising $\tilde{J}$ being bipartite. Do this for all paths in $\mathcal{M}_{i,j}$ that has not yet been considered whenever $ij \in X$ and let $\hat{J}$ be the bipartite graph obtained. Let $\phi : V(\hat{J}) \to [2]$ be a proper 2-colouring of $\hat{J}$.

Now consider $G \times K_2$. For each $i \in [\ell]$, let P_i be the path in $G \times K_2$ induced by $\{(v, \phi(v)) : v \in V(\tilde{P}_i)\}$. Since $\tilde{P}_i$ and $\tilde{P}_j$ are disjoint for distinct $i, j \in [\ell]$, it follows that $P_1 \dots, P_\ell$ are ℓ-disjoint paths in $G \times K_2$. It remains to show that for all distinct $ij \in X$, there exists k disjoint paths from P_i to P_j. Let $Q = (v_1^{(i,j)}, \dots, v_n^{(i,j)}) \in \mathcal{M}_{i,j}$. By construction of $\hat{J}$, for each $v \in V(\hat{Q})$ there exists a corresponding $\hat{v} \in V(\tilde{Q})$. Let Y be the path in $G \times K_2$ that is induced by $\{(v, \phi(\hat{v})) : v \in V(Q)\}$ and let $\mathcal{Y}_{i,j}$ be the set of such paths. Since the set of paths $\mathcal{M}_{i,j}$ are internally-disjoint, it follows that the set of paths $\mathcal{Y}_{i,j}$ are also internally-disjoint. Moreover, $|\mathcal{Y}_{i,j}| = |\mathcal{M}_{i,j}| \geqslant k$ as required. $\square$

Let $d(\ell)$ be the minimum integer such that every graph with no K_ℓ minor is $d(\ell)$-degenerate. Kostochka [227, 228] and Thomason [313] independently proved that $d(\ell) \in \Theta(\ell\sqrt{\log(\ell)})$.

Theorem 2.38 ([228, 313, 314])**.** *Every graph that does not contain K_ℓ as a minor is $d(\ell)$-degenerate where $d(\ell) := c\ell\sqrt{\log(\ell)}$ and thus has average degree at most $2c\ell\sqrt{\log(\ell)}$ where c is a positive constant.*

The proof for the following theorem is a simple adaption of the proof of Theorem 2.34 in [281].

Theorem 2.39. *For every graph G with treewidth at least $c\ell^4 \log^{5/2}(\ell)$, $G \times K_2$ contains a grid-like-minor of order ℓ for some constant c.*

Proof. Let $t := \lceil 2(2c\ell\sqrt{\log(\ell)} + 1)\rceil$ and $k := \lceil 4e\binom{t}{2}d(t)\rceil$ where c is specified by Theorem 2.38. Let G be a graph with treewidth at least $c\ell^4 \log^{5/2}(\ell)$ which is at least $2kt - 1$ for an appropriate value of c. By Lemma 2.37, there exists a set $X \subseteq E(K_t)$ where $|X| \geqslant |E(K_t)|/2$ such that $G \times K_2$ contains t-disjoint paths $P_1, \dots, P_t$ and for all $ij \in X$, $G \times K_2$ contains a set $\mathcal{Q}_{i,j}$ of k-disjoint paths between P_i and P_j. Let J be the subgraph of K_t with vertex-set $V(K_t)$ and edge-set X. By Theorem 2.38, J contains K_ℓ as a minor. From here on in, the rest of the proof follows [281, Theorem 1.2].

For distinct $ij, ab \in X$, let $H_{i,j,a,b}$ be the intersection graph of $\mathcal{Q}_{i,j} \cup \mathcal{Q}_{a,b}$. Since $H_{i,j,a,b}$ is bipartite, if K_t is a minor of $H_{i,j,a,b}$ then we are done. So assume that K_t is not a minor of $H_{i,j,a,b}$. By Theorem 2.38, $H_{i,j,a,b}$ is $d(t)$-degenerate. Let H be the intersection graph of $\cup\{\mathcal{Q}_{i,j} : ij \in X\}$. Then H is $|X|$-colourable where each colour class is some $\mathcal{Q}_{i,j}$. Since each colour class has k vertices, and each pair of colour-classes in H induce a $d(t)$-degenerate subgraph, then by [281, Lemma 4.3], H has an independent set with one vertex from each colour class. That is, in each set $\mathcal{Q}_{i,j}$ there is a path $Q_{i,j}$ such that $Q_{i,j} \cap Q_{a,b} = \varnothing$ for distinct $ij, ab \in X$. Consider the set of paths $\mathcal{P} := \{P_i : i \in [t]\} \cup \{Q_{i,j} : ij \in X\}$. The intersection graph of $\mathcal{P}$ is bipartite and contains the 1-subdivision of J. Since J contains K_ℓ as a minor, $\mathcal{P}$ is a grid-like-minor of order ℓ in G. $\square$

Reed and Wood [281] showed that the treewidth of a grid-like-minor of order ℓ is at least $\lfloor \ell/2 \rfloor - 1$. As such, Theorem 2.39 implies Theorem 2.33, and $\text{tw}(G)$ and $\text{tw}(G \times K_2)$ are polynomially tied.

We now show that Theorem 2.33 implies a polynomial lower bound for the pathwidth of $G \times K_2$. Groenland et al. [170] proved the following.

Theorem 2.40 ([170])**.** *For all integers $t, h \geqslant 1$, for every graph G with $\text{tw}(G) \geqslant t - 1$, either $\text{pw}(G) \leqslant th + 1$ or G contains a subdivision of a complete binary tree of height h.*

It is well-known that the pathwidth of a subdivision of a complete binary tree with height $2k$ is at least k (see [295]). Therefore, Theorems 2.40 and 2.33 imply the following.

Theorem 2.41. *There exists some positive constant c such that* $\mathrm{pw}(G \times K_2) \geqslant k$ *for every graph G with* $\mathrm{pw}(G) \geqslant ck^5 \log^{5/2}(k)$.

By Lemma 2.18 and Theorem 2.41, we conclude that $\mathrm{pw}(G)$ and $\mathrm{pw}(G \times K_2)$ are polynomially tied.

2.5.3 Direct Product

In this subsection, we characterise when the direct product of two classes of graphs has bounded treewidth (Theorem 2.50) and when it has bounded pathwidth (Theorem 2.51). We begin with some definitions. For every integer $k \geqslant 0$, the k-daddy-longlegs $W^{(k)}$ is the tree with $V(W^{(k)}) = \{r, u_1, \dots, u_k, v_1, \dots, v_k\}$ and $E(W^{(k)}) = \{ru_i, u_iv_i : i \in [k]\}$; see Figure 2.2. Let G be a graph. The *daddy-longlegs number* $\mathrm{dll}(G)$ of G is the maximum integer $k \geqslant 0$ such that $W^{(k)}$ is a minor of G. The *path number* $\mathrm{path}(G)$ of G is the maximum integer $n \geqslant 0$ such that G contains a path on n vertices. Clearly the path number of a graph is at least its diameter plus 1. A set $A \subseteq V(G)$ is a *vertex cover* of G if $G - A$ is an independent set. The *vertex cover number* $\tau(G)$ of G is the minimum size of a vertex cover of G.

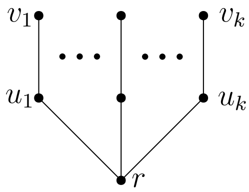


Figure 2.2. The k-daddy-longlegs $W^{(k)}$.

We now work towards showing that graphs with bounded daddy-longlegs number and bounded path-number have bounded vertex cover number. We begin with some basic lemmas.

Lemma 2.42. *For all integers $j, n \geqslant 1$, if a tree T has at most j leaves and* $\mathrm{path}(T) \leqslant n$, *then* $\mathsf{v}(T) \leqslant \lceil \frac{j}{2}\,\mathrm{path}(T) \rceil$.

Proof. We proceed by induction on n. For $n = 1$, the claim holds trivially. Now suppose the claims holds up to $n - 1$. Let T be a tree with at most j leaves and $\mathrm{path}(T) \leqslant n$. Let T_L be the subtree of T obtained by deleting its leaves. Then T_L has at most j leaves. If T contains a path P with n vertices, then the endpoints of P are leaves in T. If T contains a path P with $n - 1$ vertices, then at least one of the endpoints of T is a leaf. As such, $\mathrm{path}(T_L) \leqslant n - 2$. By induction, $\mathsf{v}(T_L) \leqslant \lceil j(n-2)/2 \rceil$. Since at most j vertices were deleted from T to obtain T_L, we have $\mathsf{v}(T) \leqslant \mathsf{v}(T_L) + j \leqslant \lceil jn/2 \rceil$ as required. □

The next lemma is important for our upcoming characterisations.

Lemma 2.43. *For every connected graph G,* $\tau(G) \leqslant \lceil (\mathrm{dll}(G) + 1)\,\mathrm{path}(G)/2 \rceil$.

Proof. Let T be a DFS-spanning tree of G rooted at some $v \in V(G)$. Note that v may be a leaf vertex in T. Let L be the set of leaves of T excluding v and let $T_L := T - L$. Since T is a DFS-tree, L is an independent set, and hence $V(T_L)$ is a vertex cover of G. We now bound $\mathsf{v}(T_L)$. Let $\tilde{L} := \{\tilde{u}_1, \dots, \tilde{u}_j\}$ be the leaves in T_L excluding v. If $j = 0$ then $V(T_L) = \{v\}$ and we are done. Otherwise, for every $i \in [j]$, let $\tilde{v}_i \in L$ where $\tilde{u}_i\tilde{v}_i \in E(T)$. Note that $\tilde{v}_i \neq \tilde{v}_j$ whenever $i \neq j$. Let $T_r := T_L - \tilde{L}$ and let μ be a model of $W^{(j)}$ in G defined by $\mu(r) = T_r$, $\mu(u_i) = \tilde{u}_i$ and $\mu(v_i) = \tilde{v}_i$ for all $i \in [j]$. Then $W^{(j)}$ is a minor of G and hence $j \leqslant \operatorname{dll}(G)$. Thus T_L has at most $j+1$ leaves. Since $\operatorname{path}(T_L) \leqslant \operatorname{path}(G)$, by Lemma 2.42, $\mathsf{v}(T_L) \leqslant \lceil (\operatorname{dll}(G)+1)\operatorname{path}(G)/2 \rceil$ as required. □

We now present several lemmas for direct products in the general framework of H-decompositions. The motivation for doing so is that once results are established within this framework, we can quickly deduce bounds for both treewidth and pathwidth. As such, this framework provides a unified approach for which we can tackle both parameters at once.

The *square* of a graph G, denoted G^2, is the graph with $V(G^2) = V(G)$ where $uv \in E(G^2)$ if $\operatorname{dist}_G(u,v) \in \{1,2\}$. For our purposes, the key property of this graph is that $N_G[v]$ is a clique in G^2 for every vertex $v \in V(G)$. The next basic lemma concerning G^2 is useful.

Lemma 2.44. *For all graphs G and H where G has maximum degree Δ and has an H-decomposition with width at most k, G^2 has an H-decomposition with width at most $(k+1)(\Delta+1)-1$.*

Proof. Let $(H, \mathcal{W})$ be an H-decomposition of G with width at most k. For all $t \in V(H)$, let $B_t := \bigcup (N_G[v] \colon v \in W_t)$ and let $\mathcal{B} := \{B_t : t \in V(H)\}$. We claim that $(H, \mathcal{B})$ is an H-decomposition of G^2 with width at most $(k+1)(\Delta+1)-1$. Note that for all $v \in V(G^2)$ there exists a node $t_1 \in V(H)$ such that $v \in W_{t_1}$. By construction, $W_{t_1} \subseteq B_{t_1}$ and hence $v \in B_{t_1}$. Hence every vertex in G^2 is in a bag. Let $uv \in E(G^2)$. Now if uv is also an edge in $E(G)$ then there exists a node $t_2 \in V(H)$ such that $\{u,v\} \subseteq W_{t_2}$ and hence $\{u,v\} \subseteq B_{t_2}$. Otherwise, $\operatorname{dist}_G(u,v) = 2$ and hence u and v share a common neighbour $y \in V(G)$ in G. Now since $(H, \mathcal{W})$ is an H-decomposition, there exists a node $t_3 \in V(H)$ such that $y \in W_{t_3}$. So by construction, $\{u,v\} \subseteq N_G[y] \subseteq B_{t_3}$, and hence the endpoints of each edge in G^2 is in a bag.

It remains to show that for all $v \in V(G^2)$, the induced subgraph $H[\{t : v \in B_t\}]$ is connected. For all $v \in V(G^2)$, let $H^{(v)} := H[\{t : v \in W_t\}]$. By construction,

$$H[\{t : v \in B_t\}] = H[\{t : N[v] \cap W_t \neq \varnothing\}] = H[\bigcup_{u \in N[v]} \{t : u \in W_t\}] = \bigcup_{u \in N[v]} H^{(u)}.$$

Now for all $u \in N_G(v)$, there exists $x \in V(H)$ such that $\{u,v\} \in W_x$ and as such, $V(H^{(v)}) \cap V(H^{(u)}) \neq \varnothing$. Hence $H[\{t : v \in B_t\}]$ is a connected subgraph of H for all $v \in V(G^2)$. We conclude that $(H, \mathcal{B})$ is indeed an H-decomposition of G^2. Note that the width of $(H, \mathcal{B})$ is at most $(k+1)(\Delta+1)-1$ since for every vertex in a bag of $(H, \mathcal{W})$, we add at most Δ other vertices to that bag in order to obtain $(H, \mathcal{B})$. $\square$

The next lemma is the heart of our characterisation of when a direct product has bounded treewidth or when it has bounded pathwidth. For the sake of simplicity, we shall consider a path to be a subdivision of K_1.

Lemma 2.45. *For all connected graphs G_1, G_2 and H where $\tau(G_1) = t$, and G_2 has maximum degree Δ and has an H-decomposition with width at most k, there exists an H'-decomposition of $G_1 \times G_2$ with width at most $t(k+1)(\Delta+1)$ where H' is a subdivision of H.*

Proof. Let $A \subseteq V(G_1)$ be a vertex cover of G_1 where $|A| = t$. We may assume that A is a clique in G_1. Let $\tilde{G}_1 := G[A]$ and let $L := V(G_1) - A$. By Lemma 2.44, there exists an H-decomposition of G_2^2 with width at most $(k+1)(\Delta+1)-1$. By Lemma 2.19, there exists an H-decomposition $(H, \mathcal{W})$ of $\tilde{G}_1 \boxtimes G_2^2$ that has width at most $t(k+1)(\Delta+1)-1$.

If $L = \varnothing$, then $G_1 \boxtimes G_2 \subseteq \tilde{G}_1 \times G_2^2$ and we are done. Otherwise, let $(\ell, v) \in L \times V(G_2)$. Now $N_{G_1}(\ell)$ is a clique in K_t and $N_{G_2}(v)$ is a clique in G_2^2. Thus, $N_{G_1 \times G_2}((\ell, v))$ is a clique in $\tilde{G}_1 \boxtimes G_2^2$. As such, there exists a node $x \in V(H)$ such that $N_{G_1 \times G_2}((\ell, v)) \subseteq W_x$. We now explain how to modify $(H, \mathcal{W})$ to obtain an H'-decomposition $(H', \mathcal{B})$ of $G_1 \times G_2$ where H' is a subdivision of H.

Let $(\ell_1, v_1), (\ell_2, v_2), \ldots, (\ell_j, v_j)$ be an arbitrary ordering of the vertices in $L \times V(G_2)$ where $j = |L \times V(G_2)|$. For each $i \in [j]$, let $x_i \in V(H)$ be a vertex where $N_{G_1 \times G_2}((\ell, v)) \subseteq W_{x_i}$. Initialise $i := 1$, $H_i := H$, $V_i := V(\tilde{G}_1 \boxtimes G_2^2)$ and $\mathcal{W}_i := \mathcal{W}$. For $i = 1, 2, \ldots, j+1$, apply the following procedure: let y_i be a neighbour of x_i in H_i. Subdivide the edge $x_i y_i \in E(H_i)$ and let H_{i+1} be the graph obtained and let z_i be the new vertex from the subdivided edge (see Figure 2.3). Note that if no such $x_i y_i$ edge exists then $H_i = K_1$. In which case, let H_{i+1} be obtained from H_i by adding the vertex z_i and edge $x_i z_i$. Let $W_{z_i} := W_{x_i} \cup \{(\ell_i, v_i)\}$ and $\mathcal{W}_{i+1} := \mathcal{W}_i \cup \{W_{z_i}\}$ and $V_{i+1} := V_i \cup \{(\ell_i, v_i)\}$.

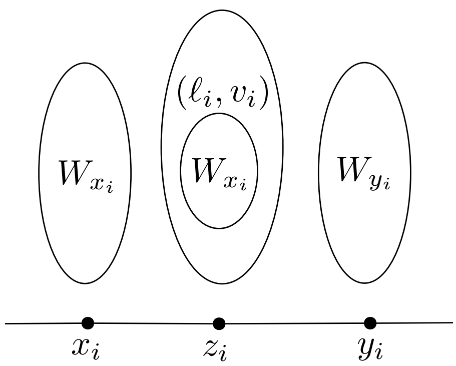


Figure 2.3. Subdividing the $x_i y_i$ edge in H_i to obtain H_{i+1}.

Once the above procedure has completed, let $H' := H_{j+1}$ and for all $x \in V(H')$, let $B_x := W_x$ and $\mathcal{B} := \{B_x : x \in V(H')\}$. Note that H' is a subdivision of H since H_{i+1} is obtained by subdividing an edge of H_i for all $i \in [j]$.

We now demonstrate that $(H', \mathcal{B})$ is an H'-decomposition of $G_1 \times G_2$. Let $(u_1, u_2) \in V(G_1 \times G_2)$. If $u_1 \in V(G_1) - L$, then $(u_1, u_2) \in V(\tilde{G}_1 \boxtimes G_2^2)$ and hence there is a $x \in V(H)$ such that $(u_1, u_2) \in W_x$. By the construction of $(H', \mathcal{B})$, it follows that $(u_1, u_2) \in B_x$. Otherwise, $u_i \in L$ in which case $(u_1, u_2) = (l_i, v_i)$ for some $i \in [j]$ and $(u_1, u_2) \in W_{z_i}$. Hence every vertex is in a bag.

Let $(u_1, u_2)(u_3, v_4) \in E(G_1 \times G_2)$. Since L is an independent set, at most one of u_1 and u_3 is in L. Suppose that $u_1 \in L$. Then there exists $i \in [j]$ such that $(u_1, u_2) = (\ell_i, v_i)$. Since $(u_3, u_4) \in N_{G_1 \times G_2}[(\ell_i, v_i)]$, by the construction of B_{z_i} it follows that $(u_1, u_2), (u_3, v_4) \in B_{z_i}$. Now if neither u_1 nor u_3 is in L, then there is a node $x \in V(H)$ such that $(u_1, u_2), (u_3, u_4) \in W_x$ and hence $(u_1, u_2), (u_3, u_4) \in B_x$. Hence, the endpoints of each edge is in a bag.

It remains to show that for all $(u_1, u_2) \in V(G_1 \times G_2)$, the subgraph $H'[\{x \in V(H') : (u_1, u_2) \in B_x\}]$ is connected. We prove the following claim by induction on $i \geqslant 1$:

Claim: For every vertex $(u_1, u_2) \in V_i$, the induced subgraph $H_i[\{x \in V(H') : (u_1, u_2) \in W_x\}]$ is connected.

For $i = 1$, this claim holds since $(H_1, \mathcal{W}_1)$ is an H-decomposition of $\tilde{G}_1 \boxtimes G_2^2$. Now assume that it holds up to $i - 1$ where $i \geqslant 2$. The graph H_i is obtained from H_{i-1} by replacing the edge $x_i y_i$ by the path x_i, z_i, y_i. Furthermore, $V_i \setminus V_{i-1} = \{(\ell_i, v_i)\}$. Now the vertex (ℓ_i, v_i) is only in the bag W_{z_i} and hence $H_i[x : (\ell_i, v_i) \in W_x]$ is connected. Now if $(u_1, u_2) \in V_{i-1}$ then by induction $H_{i-1}[x : (u_1, u_2) \in W_x]$ is connected. For the sake of contradiction, suppose that $H_i[x : (u_1, u_2) \in W_x]$ is disconnected. The only way for this to occur is if $(u_1, u_2) \in W_{x_i}$ and $(u_1, u_2) \notin W_{z_i}$. However, by the construction of W_{z_i}, we have $W_{x_i} \subseteq W_{z_i}$, a contradiction. Hence $H_i[\{t : (u_1, u_2) \in W_t\}]$ is connected for all $(u_1, u_2) \in V_i$.

This completes our proof that $(H', \mathcal{B})$ is an H'-decomposition of $G_1 \times G_2$ where H' is a subdivision of H. We conclude by noting that the width of this H'-decomposition is at most 1 more than the width of $(H, \mathcal{W})$, as required. □

When the graph H in Lemma 2.45 realises the treewidth or pathwidth of G_2, we have the following.

Corollary 2.46. *For all connected graphs G_1 and G_2,*

$$\operatorname{tw}(G_1 \times G_2) \leqslant \tau(G_1)(\operatorname{tw}(G_2) + 1)(\Delta(G_2) + 1), \text{ and}$$
$$\operatorname{pw}(G_1 \times G_2) \leqslant \tau(G_1)(\operatorname{pw}(G_2) + 1)(\Delta(G_2) + 1).$$

The next two lemmas are key cases for when a direct product has unbounded treewidth.

Lemma 2.47. *For every integer $k \geqslant 1$ and graph G with $\operatorname{dll}(G) \geqslant k$, $K_{k,k}$ is a minor of $G \times P_{2k}$ and thus $\operatorname{tw}(G \times P_{2k}) \geqslant k$.*

Proof. Since $\operatorname{dll}(G) \geqslant k$, it follows that G contains a tree T such the subtree T_L obtained by deleting the leaves set of T has at least k leaves. We claim that $T \times P_{2k}$ contains $K_{k,k}$

as a minor. Since $T \times P_{2k}$ is a subgraph of $G \times P_{2k}$, this implies that $K_{k,k}$ is also a minor of $G \times P_{2k}$.

Let $P_{2k} = (p_1, \dots, p_{2k})$. Let $M = \{u_1v_1, u_2v_2, \dots, u_kv_k\}$ be a matching in T such that for each $i \in [k]$, u_i is a leaf of T_L and v_i is a leaf of T. Let T' be the subtree of T_L obtained by deleting $u_1, \dots, u_k$. Take a proper black-white colouring of $V(T)$. Let B be the set of black vertices and let W be the set of white vertices. For each $j \in [k]$, let $x_j \in \{u_j, v_j\} \cap B$ and $y_j \in \{u_j, v_j\} \cap W$. Let X_j be the subgraph of $T \times P_{2k}$ induced by

$$\{(x_j, p_i), (y_j, p_{i-1}) \colon i \in \{2, 4, \dots, 2k\}\}.$$

Since $u_jv_j \in E(T)$ and P_{2j} is bipartite, it follows that X_j is a copy of P_{2j}. Moreover, X_i and X_j are vertex-disjoint whenever $i \neq j$ since M is a matching.

For each $i \in [k]$, let Y_i be the subgraph of $T \times P_{2k}$ induced by

$$\{(x, p_{2i}) \colon x \in V(T') \cap B\} \cup \{(y, p_{2i-1}) \colon y \in V(T') \cap W\}.$$

Since $p_{2i-1}p_{2i} \in E(P_{2k})$ and T' is bipartite, it follows that Y_i is a copy of T'. Moreover, since $\{p_{2i}, p_{2i-1}\} \cap \{p_{2j}, p_{2j-1}\} = \varnothing$ whenever $i \neq j$, it follows that Y_i and Y_j are vertex-disjoint.

We claim that $(\{X_j \colon j \in [k]\}, \{Y_i \colon i \in [k]\})$ is a $K_{k,k}$ model in $T \times P_{2k}$. Since $V(T') \cap \{u_1, v_1, \dots, u_k, v_k\} = \varnothing$, it follows the X_j and Y_i's are pairwise vertex-disjoint. It therefore remains to show that X_j and Y_i are adjacent. For each $j \in [k]$, let $z_j \in V(T')$ be the neighbour of u_j. Then for all $i, j \in [k]$, either the edge $(u_j, p_{2i-1})(z_j, p_{2i})$ or the edge $(u_j, p_{2i})(z_j, p_{2i-1})$ is between X_i and Y_j (depending on whether u_i is white or black). In either case, X_i and Y_j are adjacent which completes the proof fo the claim that $K_{k,k}$ is a minor of $T \times P_{2k}$. $\square$

The next lemma is folklore and uses the well-known fact that $\operatorname{tw}(P_n \,\square\, P_n) = n$ [40].

Lemma 2.48. *For integers $b, n \geqslant 1$, let S_b be the star with b leaves and P_n be the path on n vertices. Then $\operatorname{tw}(S_b \times S_b) \geqslant b$ and $\operatorname{tw}(P_{2n-1} \times P_{2n-1}) \geqslant n$.*

Proof. For the direct product of stars, $K_{b,b} \subseteq S_b \times S_b$ by Theorem 2.5 and hence $\operatorname{tw}(S_b \times S_b) \geqslant b$.

Now consider the direct product of two paths. Since $\operatorname{tw}(P_n \,\square\, P_n) \geqslant n$, it suffice to show that $P_n \,\square\, P_n \subseteq P_{2n-1} \times P_{2n-1}$. Let $V(P_{2n-1}) = [2n-1]$. For each $i \in [n]$, let $H^{(i)}$ be the subgraph of $P_{2n-1} \times P_{2n-1}$ induced by $\{(i-k, n+i-k) : k \in [n]\}$. For each $j \in [n]$, let $P^{(j)}$ be the subgraph of $P_{2n-1} \times P_{2n-1}$ induced by $\{(j+k, n-j+k) : k \in [n]\}$. Then $(\bigcup(H^{(i)} \colon i \in [n])) \cup (\bigcup(P^{(j)} \colon j \in [n]))$ defines a $P_n \,\square\, P_n$ subgraph in $P_{2n-1} \times P_{2n-1}$ where for all $i, j \in [n]$, $H^{(i)}$ is the i^{th}-horizontal path in $P_n \,\square\, P_n$ and $P^{(j)}$ is the j^{th}-vertical path in $P_n \,\square\, P_n$, as required (see Figure 2.4). $\square$

The next lemma is the Moore bound; see [249] for a survey.

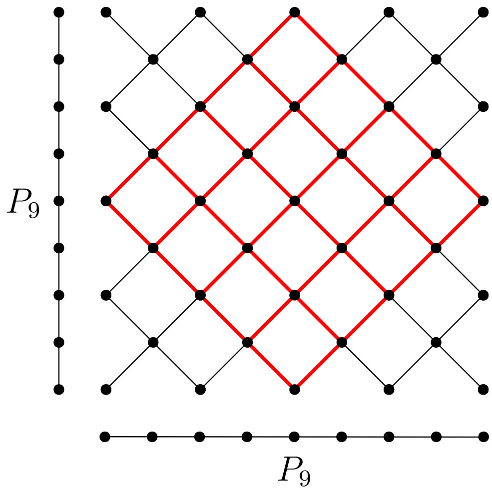


Figure 2.4. The (5×5)-grid in $P_9 \times P_9$.

Lemma 2.49. *For every connected graph G with maximum degree $\Delta > 1$ and diameter d,*

$$\mathsf{v}(G) \leqslant \begin{cases} 1 + \Delta \frac{(\Delta-1)^d - 1}{\Delta - 2}, & \text{if } \Delta > 2, \\ 2d+1, & \text{if } \Delta = 2. \end{cases}$$

For a graph G, let $\tilde{\tau}(G)$ be the maximum vertex cover number of a component of G. We now prove our characterisation for when a direct product has bounded treewidth.

Theorem 2.50. *Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be monotone graph classes that contains K_2. Then $\mathrm{tw}(\mathcal{G}_1 \times \mathcal{G}_2)$ is bounded if and only if $\mathrm{tw}(\mathcal{G}_1)$ and $\mathrm{tw}(\mathcal{G}_2)$ are bounded and at least one of the following conditions hold:*

- *$\tilde{\mathsf{v}}(\mathcal{G}_1)$ or $\tilde{\mathsf{v}}(\mathcal{G}_2)$ is bounded;*
- *$\tilde{\tau}(\mathcal{G}_1)$ and $\Delta(\mathcal{G}_2)$ are bounded; or*
- *$\tilde{\tau}(\mathcal{G}_2)$ and $\Delta(\mathcal{G}_1)$ are bounded.*

Proof. Assume that $\mathrm{tw}(\mathcal{G}_1 \times \mathcal{G}_2)$ is bounded. By Corollary 2.30, if $\mathcal{G}_1$ or $\mathcal{G}_2$ has unbounded treewidth, then $\mathcal{G}_1 \times \mathcal{G}_2$ has unbounded treewidth. Hence, we may assume that there exists an integer $k \geqslant 1$ such that $\mathrm{tw}(\mathcal{G}_1) \leqslant k$ and $\mathrm{tw}(\mathcal{G}_2) \leqslant k$. Now suppose there exists an integer $c_1 \geqslant 1$ such that $\tilde{\mathsf{v}}(\mathcal{G}_1) \leqslant c_1$ or $\tilde{\mathsf{v}}(\mathcal{G}_2) \leqslant c_1$. By Lemma 2.18, $\mathrm{tw}(\mathcal{G}_1 \times \mathcal{G}_2) \leqslant (k+1)c_1 - 1$ and we are done.

It remains to consider the case when neither $\tilde{\mathsf{v}}(\mathcal{G}_1)$ nor $\tilde{\mathsf{v}}(\mathcal{G}_2)$ is bounded. Suppose that $\mathcal{G}_1$ and $\mathcal{G}_2$ both have unbounded maximum degree. Then for every integer $b \geqslant 1$, the star S_b is a member of both $\mathcal{G}_1$ and $\mathcal{G}_2$ and $(S_b \times S_b)_{b \in \mathbb{N}} \subseteq \mathcal{G}_1 \times \mathcal{G}_2$. However, by Lemma 2.48, this is a contradiction since $(S_b \times S_b)_{b \in \mathbb{N}}$ has unbounded treewidth. Therefore either $\mathcal{G}_1$ or $\mathcal{G}_2$ has bounded maximum degree.

Without loss of generality, there exists an integer $c_2 \geqslant 1$ such that $\Delta(\mathcal{G}_2) \leqslant c_2$. Since $\tilde{\mathsf{v}}(\mathcal{G}_2)$ is unbounded, by Lemma 2.49 it follows that $\mathrm{path}(\mathcal{G}_2)$ is unbounded. Now if $\mathrm{path}(\mathcal{G}_1)$ is also unbounded, then $(P_n \times P_n)_{n \in \mathbb{N}}$ is a family of graphs in $\mathcal{G}_1 \times \mathcal{G}_2$. This contradicts $\mathrm{tw}(\mathcal{G}_1 \times \mathcal{G}_2)$ being bounded since by Lemma 2.48, $(P_n \times P_n)_{n \in \mathbb{N}}$ has unbounded treewidth. Similarly, if $\mathrm{dll}(\mathcal{G}_1)$ is unbounded, then by Lemma 2.47, $\mathrm{tw}(\mathcal{G}_1 \times \mathcal{G}_2)$ is unbounded. So assume there exists integers $j, n \geqslant 1$ such that for every connected graph $G \in \mathcal{G}_1$, we have $\mathrm{dll}(G) \leqslant j$ and $\mathrm{path}(G) \leqslant n$. By Lemma 2.43, $\hat{\tau}(\mathcal{G}_1) \leqslant \lceil (j+1)n/2 \rceil$. By Corollary 2.46, $\mathrm{tw}(\mathcal{G}_1 \times \mathcal{G}_2) \leqslant \lceil (j+1)n/2 \rceil (k+1)(\Delta + 1)$ and thus, is bounded. As we have considered all possibilities, this completes our proof. $\square$

The next theorem characterises when a direct product has bounded pathwidth. We omit the proof as it is identical to Theorem 2.50 except we use Corollary 2.32 instead of Corollary 2.30.

Theorem 2.51. *Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be monotone graph classes that contains K_2. Then $\mathcal{G}_1 \times \mathcal{G}_2$ has bounded pathwidth if and only if $\mathcal{G}_1$ and $\mathcal{G}_2$ both have bounded pathwidth and at least one of the following holds:*

- *$\mathsf{v}(\hat{\mathcal{G}}_1)$ or $\mathsf{v}(\hat{\mathcal{G}}_2)$ is bounded;*
- *$\tau(\hat{\mathcal{G}}_1)$ and $\Delta(\mathcal{G}_2)$ are bounded; or*
- *$\tau(\hat{\mathcal{G}}_2)$ and $\Delta(\mathcal{G}_1)$ are bounded.*

Chapter 3

Product Structure of Graphs with Bounded Pathwidth

3.1 Overview

Robertson and Seymour's Excluded Tree Minor Theorem (Theorem 1.4) states that for every tree T there is an integer c such that every T-minor-free graph has pathwidth at most c. Bienstock, Robertson, Seymour, and Thomas [34] and Diestel [100] showed the same result with $c = |V(T)| - 2$, which is best possible, since the complete graph on $|V(T)| - 1$ vertices is T-minor-free and has pathwidth $|V(T)| - 2$. Inspired by graph product structure theory, we prove the following result in this chapter.

Theorem 1.44. *For every tree T of radius h, there exists $c \in \mathbb{N}$ such that every T-minor-free graph G is contained in $H \boxtimes K_c$ for some graph H with pathwidth at most $2h - 1$.*

Theorem 1.44 is a qualitative strengthening of the Excluded Tree Minor Theorem of Robertson and Seymour [285] since $\text{pw}(G) \leqslant \text{pw}(H \boxtimes K_c) \leqslant c(\text{pw}(H) + 1) - 1 \leqslant 2ch - 1$. Note that the proof of Theorem 1.44 depends on Theorem 1.4. The point of Theorem 1.44 is that $\text{pw}(H)$ only depends on the radius of T, not on $|V(T)|$ which may be much greater than the radius. Moreover, radius is the right parameter of T to consider here, as we now explain.

For a tree T, let $g(T)$ be the minimum $k \in \mathbb{N}$ such that, for some $c \in \mathbb{N}$, every T-minor-free graph G is contained in $H \boxtimes K_c$ where $\text{pw}(H) \leqslant k$. Theorem 1.44 shows that if T has radius h, then $g(T) \leqslant 2h - 1$. Now we show a lower bound.

Proposition 3.1. *For all $h \in \mathbb{N}_0$ and $c \in \mathbb{N}$, there exists a tree T with radius at most h such that if T is contained in $H \boxtimes K_c$ then $\text{pw}(H) \geqslant h$.*

Let T be any tree with radius h. Thus T contains a path on $2h$ vertices, and so every tree with radius at most $h - 1$ is T-minor-free. So Proposition 3.1 implies that for every

$c \in \mathbb{N}$, there is a T-minor-free graph X such that if X is contained in $H \boxtimes K_c$, then $\operatorname{pw}(H) \geqslant h - 1$. Hence

$$h - 1 \leqslant g(T) \leqslant 2h - 1. \tag{3.1}$$

This says that the radius of T is the right parameter to consider in Theorem 1.44. Moreover, both the lower and upper bounds in (3.1) can be achieved. Let $T_{h,d}$ denote the complete d-ary tree of height h.

Proposition 3.2. *For all $h, c \in \mathbb{N}$, there is a $T_{h,3}$-minor-free graph G, such that for every graph H, if G is contained in $H \boxtimes K_c$, then H has a clique of size $2h$, implying $\operatorname{pw}(H) \geqslant \operatorname{tw}(H) \geqslant 2h - 1$.*

The next result improves Theorem 1.44 for an excluded path. It shows that the lower bound in (3.1) is achieved when T is a path, since P_{2h+1} has radius h, and a graph has no path on $2h + 1$ vertices if and only if it is P_{2h+1}-minor-free.

Proposition 3.3. *For any $h \in \mathbb{N}$, every graph G with no path on $2h + 1$ vertices is contained in $H \boxtimes K_{4h}$ for some graph H with $\operatorname{pw}(H) \leqslant h - 1$.*

This chapter is based on joint work with Dujmović, Joret, Micek, Morin and Wood [120].

3.2 Proofs

We prove the following quantitative version of Theorem 1.44.

Theorem 3.4. *Let T be a tree with t vertices, radius h, and maximum degree d. Then every T-minor-free graph G is contained in $H \boxtimes K_{(d+h-2)(t-1)}$ for some graph H with pathwidth at most $2h - 1$.*

The following lemma is folklore (see [205] for a proof).

Lemma 3.5. *For every graph G, for every tree-decomposition $\mathcal{D}$ of G, for every collection $\mathcal{F}$ of connected subgraphs of G, and for every $\ell \in \mathbb{N}$, either:*

- *(a) there are ℓ vertex-disjoint subgraphs in $\mathcal{F}$, or*
- *(b) there is a set $S \subseteq V(G)$ consisting of at most $\ell - 1$ bags of $\mathcal{D}$ such that $S \cap V(F) \neq \varnothing$ for all $F \in \mathcal{F}$.*

Observation 1.32 and the next lemma imply Theorem 3.4, since the tree T in Theorem 3.4 is a subtree of $T_{h,d}$, and every T-minor-free graph G satisfies $\operatorname{tw}(G) \leqslant \operatorname{pw}(G) \leqslant t - 2$ by the result of Bienstock, Robertson, Seymour, and Thomas [34].

Lemma 3.6. *For any $h, d \in \mathbb{N}$ with $d + h \geqslant 3$, for every $T_{h,d}$-minor-free graph G, for every tree-decomposition $\mathcal{D}$ of G, and for every vertex r of G, the graph G has a partition $\mathcal{P}$ such that:*

- *each part of $\mathcal{P}$ is a subset of the union of at most $d + h - 2$ bags of $\mathcal{D}$,*
- *$\{r\} \in \mathcal{P}$, and*
- *$G/\mathcal{P}$ has a path-decomposition of width at most $2h-1$ in which the first bag contains $\{r\}$.*

Proof. We proceed by induction on pairs $(h, |V(G)|)$ in a lexicographic order. Fix h, d, G, $\mathcal{D}$, and r as in the statement. We may assume that G is connected. The statement is trivial if $|V(G)| \leqslant 1$. Now assume that $|V(G)| \geqslant 2$.

For the base case, suppose that $h = 1$. For $i \geqslant 0$, let $V_i := \{v \in V(G) : \mathrm{dist}_G(v, r) = i\}$. So $V_0 = \{r\}$. If $|V_i| \geqslant d$ for some $i \geqslant 1$, then contracting $G[V_0 \cup \cdots \cup V_{i-1}]$ into a single vertex gives a $T_{1,d}$-minor. So $|V_i| \leqslant d-1 = d+h-2$ for each $i \geqslant 0$. Thus $\mathcal{P} := (V_i : i \geqslant 0)$ is a partition of G, and each part of $\mathcal{P}$ is a subset of the union of at most $d + h - 2$ bags of $\mathcal{D}$. Moreover, the quotient $G/\mathcal{P}$ is a path, which has a path-decomposition of width 1, in which the first bag contains $\{r\}$.

Now assume that $h \geqslant 2$ and the result holds for $h - 1$. Let R be the neighbourhood of r in G. Let $\mathcal{F}$ be the set of all connected subgraphs of $G - r$ that contain a vertex from R and contain a $T_{h-1,d+1}$-minor. If there are d pairwise vertex-disjoint subgraphs $S_1, \dots, S_d$ in $\mathcal{F}$, then we claim that G contains a $T_{h,d}$-minor. Indeed, for each $i \in [d]$ consider a $T_{h-1,d+1}$-model $(W^i_x : x \in V(T_{h-1,d+1}))$ in S_i. Since S_i is connected, we may assume that all vertices of S_i are in the model. For each $i \in [d]$, let y_i be a node of $T_{h-1,d+1}$ such that $W^i_{y_i}$ contains a vertex from R, and let Y^i be the union of W^i_x for all ancestors x of y_i in $T_{h-1,d+1}$. Observe that there is a $T_{h-1,d}$-model in S_i such that the root of $T_{h-1,d}$ is mapped to the set Y^i. Therefore $G - r$ contains d pairwise disjoint models of $T_{h-1,d}$ such that each root branch set contains a vertex from R. So G contains a model of $T_{h,d}$, as claimed.

So $\mathcal{F}$ contains no d pairwise vertex-disjoint elements. By Lemma 1.6, there is a minimal set $X \subseteq V(G - r)$, such that X is a subset of the union of $d - 1 \leqslant d + h - 2$ bags of $\mathcal{D}$, and $G - r - X$ contains no element of $\mathcal{F}$.

Let $G_1, \dots, G_p$ be the components of $G - r - X$ that contain a vertex from R. By construction of X, the graph G_i contains no $T_{h-1,d+1}$-minor. By induction, G_i has a partition $\mathcal{P}_i$ such that:

- each part of $\mathcal{P}_i$ is a subset of the union of at most $(d + 1) + (h - 1) - 2 = d + h - 2$ bags of $\mathcal{D}$, and
- $G_i/\mathcal{P}_i$ has a path-decomposition $\mathcal{B}_i$ of width at most $2h - 3$.

Let $Z := V(G - r - X) \setminus V(G_1 \cup \cdots \cup G_p)$; that is, Z is the set of vertices of all components of $G - r - X$ that have no vertex in R.

Consider a vertex $v \in X$. By the minimality of X, the graph $G - r - (X \setminus \{v\})$ contains a connected subgraph Y_v that contains v and a vertex $r_v \in R$ (and contains a $T_{h-1,d+1}$-minor). Let P_v be a path from v to r_v in Y_v plus the edge $r_v r$. So $P_v - \{v, r\}$ is contained in some G_i, and thus P_v avoids Z. So $\cup\{P_v : v \in X\}$ is a connected subgraph in $G - Z$. Let G' be obtained from G by contracting $\cup\{P_v : v \in X\}$ into a vertex r', and deleting any remaining vertices not in Z. So $V(G') = \{r'\} \cup Z$. Since G' is a minor of G, the graph G' is $T_{h,d}$-minor-free. Let $\mathcal{D}'$ be the tree-decomposition of G' obtained from $\mathcal{D}$ by replacing each instance of each vertex in $\cup\{P_v : v \in X\}$ by r' then removing the other vertices in $V(G) \setminus V(G')$. Observe that for every bag B in $\mathcal{D}'$, we have $B - \{r'\}$ contained in some bag of $\mathcal{D}$. By induction, G' has a partition $\mathcal{P}'$ such that:

- each part of $\mathcal{P}'$ is a subset of the union of at most $d + h - 2$ bags of $\mathcal{D}'$,
- $\{r'\} \in \mathcal{P}'$, and
- $G'/\mathcal{P}'$ has a path-decomposition $\mathcal{B}'$ of width at most $2h - 1$ in which the first bag contains $\{r'\}$.

Let $\mathcal{P} := \{\{r\}\} \cup \{X\} \cup \mathcal{P}_1 \cup \cdots \cup \mathcal{P}_p \cup (\mathcal{P}' \setminus \{\{r'\}\})$. Then $\mathcal{P}$ is a partition of G such that each part is a subset of the union of at most $d + h - 2$ bags of $\mathcal{D}$. Let $\mathcal{B}$ be a sequence of subsets of vertices of $G/\mathcal{P}$ obtained from the concatenation of $\mathcal{B}_1, \ldots, \mathcal{B}_p$, and $\mathcal{B}'$ by adding $\{r\}$ and X to every bag that comes from $\mathcal{B}_1, \ldots, \mathcal{B}_p$ and replacing $\{r'\}$ by X. Now we argue that $\mathcal{B}$ is a path-decomposition of $G/\mathcal{P}$. Indeed, each part of $\mathcal{P}$ is contained in consecutive bags of $\mathcal{B}$, specifically $\{r\}$ and X are added to all bags across $\mathcal{B}_1, \ldots, \mathcal{B}_p$, and X is in the first bag of $\mathcal{B}'$. Since $G_1, \ldots, G_p$ are components of $G - r - X$, the neighbourhood in $G/\mathcal{P}$ of a part in $\mathcal{P}_i$ is contained in $\mathcal{P}_i \cup \{\{r\}, X\}$. Note also that the neigbourhood of $\{r\}$ in $G/\mathcal{P}$ is contained in $\mathcal{P}_1 \cup \cdots \cup \mathcal{P}_p \cup \{X\}$. It follows that $\mathcal{B}$ is a path-decomposition of $G/\mathcal{P}$. By construction, the width of $\mathcal{B}$ is at most $2h - 1$ and the first bag contains $\{r\}$, as required. $\square$

We now prove Proposition 3.1 which shows that radius is the right parameter to consider in this setting.

Proof of Proposition 3.1. Let $(X_h, \tilde{x})$ be the complete ternary tree with height h rooted at $\tilde{x}$. We prove the following induction hypothesis: for all $h \in \mathbb{N}_0$ and $c \in \mathbb{N}$, there exists a rooted tree (T_h, r) where $\operatorname{dist}_{T_h}(v, r) \leqslant h$ for all $v \in V(T_h)$ such that for every H-partition $(V_x : x \in V(H))$ of T_h with width at most c, H contains $(X_h, \tilde{x})$ as a subgraph with $r \in V_{\tilde{x}}$. Since $\operatorname{pw}(X_h) = h$, this implies the claim.

We proceed by induction on h. For $h = 0$, the claim holds trivially by setting (T_0, r) to be a single vertex r.

Now suppose $h > 0$. Let (T_{h-1}, r') be given by the induction hypothesis and let $n = |V(T_{h-1})|$. Let $m = (2n + 1)c$ and let (T_h, r) be obtained by taking m copies $((T_{h-1,1}, r_1), \ldots, (T_{h-1,m}, r_m))$ of (T_{h-1}, r') plus a vertex r along with the edges rr_i for all $i \in \{1, \ldots, m\}$. Since $\operatorname{dist}_{T_{h-1,i}}(v_i, r_i) \leqslant h - 1$ for all $v_i \in V(T_{h-1,i})$, we have $\operatorname{dist}_{T_h}(v, r) \leqslant h$ for all $v \in V(T_h)$.

Let $(V_x : x \in V(H))$ be an H-partition of T_h with width at most c. Let $\tilde{x} \in V(H)$ be such that $r \in V_{\tilde{x}}$. Let $X := \{i \in [m] : V(T_{h-1,i}) \cap V_{\tilde{x}} = \varnothing\}$. Since $|V_{\tilde{x}}| \leqslant c$ it follows that $|X| \geqslant m - (c-1) \geqslant 2nc+1$. Choose any $j \in X$ and let $A := \{x \in V(H) : V_x \cap V(T_{h-1,j}) \neq \varnothing\}$. Let $Y := \{i \in X : V(T_{h-1,i}) \cap (\bigcup_{x \in A} V_x) = \varnothing\}$. Since $|A| \leqslant n$, it follows that

$|Y| \geqslant |X| - nc \geqslant nc+1$. Choose any $k \in Y$ and let $B := \{x \in V(H) \colon V_x \cap V(T_{h-1,k}) \neq \varnothing\}$. Let $Z := \{i \in Y \colon V(T_{h-1,i}) \cap (\bigcup_{x \in B} V_x) = \varnothing\}$. As before, $|Z| \geqslant |Y| - nc \geqslant 1$. Choose any $\ell \in Z$ and let $C := \{x \in V(H) \colon V_x \cap V(T_{h-1,\ell}) \neq \varnothing\}$.

By construction, $\{r\}, A, B$ and C are pairwise disjoint. Let $(i, I) \in \{(j, A), (k, B), (\ell, C)\}$. Since $(V_x \cap V(T_{h-1,i}) \colon x \in I)$ is a partition of $T_{h-1,i}$, it follows by induction that $H[I]$ contains $(H_{h-1,i}, x_i)$ as a subgraph where $r_i \in V_{x_i}$. Thus $H[\{\tilde{x}\} \cup A \cup B \cup C]$ contains the desired complete ternary tree. □

We turn to the proof of Proposition 3.2. It is a strengthening of a similar result by Norin, Scott, Seymour, and Wood [260, Lemma 13].

Proof of Proposition 3.2. We proceed by induction on $h \geqslant 1$. First consider the base case $h = 1$. Let G be a path on $n = c + 1$ vertices. Thus G is $T_{1,3}$-minor-free. Suppose that G is contained in $H \boxtimes K_c$. Since $n > c$ and G is connected, $|E(H)| \geqslant 1$ and H has a clique of size 2, as desired.

Now assume $h \geqslant 2$ and the result holds for $h - 1$. Let $t_0 := |V(T_{h-1,3})|$. By induction, there is a $T_{h-1,3}$-minor-free graph G_0, such that for every graph H, if G_0 is contained in $H \boxtimes K_c$, then H has a clique of size $2h - 2$. Let G be obtained from a path P of length $c + 1$ as follows: for each edge vw of P, add $2c$ copies of G_0 complete to $\{v, w\}$.

Suppose for the sake of contradiction that G contains a $T_{h,3}$-model. Let X be the branch set corresponding to the root of $T_{h,3}$. So $G - X$ contains three pairwise disjoint subgraphs Y_1, Y_2, Y_3, each containing a $T_{h-1,3}$-minor. Each Y_i intersects P, otherwise Y_i is contained in some component of $G - P$ which is a copy of G_0. By the construction of G, each Y_i intersects P in a subpath P_i. Without loss of generality, P_1, P_2, P_3 appear in this order in P. Since each component of $G - P$ is only adjacent to an edge of P, no component of $G - P_2$ is adjacent to both Y_1 and Y_3. In particular, X is not adjacent to both Y_1 and Y_3, which is a contradiction. Thus G is $T_{h,3}$-minor-free.

Now suppose that G is contained in $H \boxtimes K_c$. Let $\mathcal{P}$ be the corresponding H-partition of G. Since $|V(P)| > c$ there is an edge v_1v_2 of P with $v_i \in Q_i$ for some distinct parts $Q_1, Q_2 \in \mathcal{P}$. At most $c - 1$ of the copies of G_0 attached to v_1v_2 intersect Q_1, and at most $c - 1$ of the copies of G_0 attached to v_1v_2 intersect Q_2. Thus some copy of G_0 attached to v_1v_2 avoids $Q_1 \cup Q_2$. Let H_0 be the subgraph of H induced by those parts that intersect this copy of G_0. So neither Q_1 nor Q_2 is in H_0. By induction, H_0 has a clique C_0 of size $2(h - 1)$. Since G_0 is complete to v_1v_2, we have that $C_0 \cup \{Q_1, Q_2\}$ is a clique of size $2h$ in H, as desired. □

Finally, we prove Proposition 3.3. We in fact prove a stronger result in terms of tree-depth. It is well-known and easily seen that $\mathrm{pw}(G) \leqslant \mathrm{td}(G) - 1$ for every graph G. Thus, the following lemma implies Proposition 3.3 since every P_{2h+1}-minor-free graph G has $\mathrm{tw}(G) \leqslant \mathrm{pw}(G) \leqslant 2h - 1$ by the Excluded Tree Minor Theorem (Theorem 1.4).

Lemma 3.7. *For any $h, k \in \mathbb{N}$, for every graph G with no path on $2h+1$ vertices, for every tree-decomposition $\mathcal{D}$ of G, the graph G has a partition $\mathcal{P}$ such that $\operatorname{td}(G/\mathcal{P}) \leqslant h$ and each part of $\mathcal{P}$ is a subset of at most two bags of $\mathcal{D}$.*

Proof. We proceed by induction on h. For $h = 1$, G is the disjoint union of copies of K_1 and K_2. Let $\mathcal{P}$ be the partition of G where the vertex-set of each component of G is a part of $\mathcal{P}$. Thus $E(G/\mathcal{P}) = \varnothing$ and $\operatorname{td}(G/\mathcal{P}) = 1$. Each part is a subset of one bag of $\mathcal{D}$.

Now assume $h \geqslant 2$ and the claim holds for $h-1$. We may assume that G is connected. Suppose G contains three vertex-disjoint paths, $P^{(1)}$, $P^{(2)}$ and $P^{(3)}$, each with $2h-1$ vertices. Let G' be the graph obtained by contracting each path $P^{(i)}$ into a vertex v_i. Since G' is connected, there is a (v_i, v_j)-path of length at least 2 in G' for some distinct $i, j \in \{1, 2, 3\}$. Without loss of generality, $i = 1$ and $j = 2$. So there exist vertices $u \in V(P^{(1)})$ and $v \in V(P^{(2)})$ together with a (u, v)-path Q of length at least 2 in G that internally avoids $P^{(1)} \cup P^{(2)}$. Let x be the endpoint of $P^{(1)}$ that is furthest from u (on $P^{(1)}$) and let y be the endpoint of $P^{(2)}$ that is furthest from v (on $P^{(2)}$). Then $(xP^{(1)}uQvP^{(2)}y)$ is a path with at least $2h+1$ vertices, a contradiction.

Now assume that G contains no three vertex-disjoint paths with $2h-1$ vertices. By Lemma 1.6, there is a set $S \subseteq V(G)$ consisting of at most two bags of $\mathcal{D}$ such that $G - S$ is P_{2h-1}-free. By induction, $G - S$ has a partition $\mathcal{P}'$ such that $\operatorname{td}((G-S)/\mathcal{P}') \leqslant h-1$ and each part of $\mathcal{P}'$ is a subset of at most two bags of $\mathcal{D}$. Let $\mathcal{P} := \mathcal{P}' \cup \{S\}$. Then $\mathcal{P}$ is the desired partition of G since $\operatorname{td}(G/\mathcal{P}) \leqslant \operatorname{td}((G-S)/\mathcal{P}') + 1 \leqslant h$. $\square$

Chapter 4

Product Structure of Graphs on Surfaces

4.1 Overview

In this chapter, we prove product structure theorems for graphs on surfaces.

First, we consider squaregraphs. Recall that a *squaregraph* is a plane graph in which each internal face is a 4-cycle and each internal vertex has degree at least 4. We prove the following product structure theorem for such graphs where $\bowtie$ is the semi-strong product (defined in Section 1.3.1).

Theorem 1.12. *Every squaregraph is contained in $H \bowtie P$ for some outerplanar graph H and some path P.*

We show that this is best possible in the sense that "outerplanar graph" cannot be replaced by "forest".

Next, we consider graphs with bounded Euler genus. Dujmović et al. [121] proved that for every graph G with Euler genus g there is a graph H with treewidth at most 4 and a path P such that $G \subsetneq H \boxtimes P \boxtimes K_{\max\{2g,3\}}$. We improve this result by replacing "4" by "3" and with H planar.

Theorem 1.14. *Every graph with Euler genus g is contained in $H \boxtimes P \boxtimes K_{\max\{2g,3\}}$ for some planar graph H with treewidth at most 3 and for some path P.*

Treewidth at most 3 is best possible, even for planar graphs [121].

We in fact prove a stronger result in terms of framed graphs; see Section 1.3.4 for their definition.

Theorem 1.35. *For all integers $g \geqslant 0$ and $d \geqslant 3$, every (g,d)-framed multigraph is contained in $H \boxtimes P \boxtimes K_\ell$ for some planar graph H with treewidth 3 and for some path P, where $\ell = \max\{2g\lfloor \frac{d}{2} \rfloor, d + 3\lfloor \frac{d}{2} \rfloor - 3\}$.*

Framed graphs (for $g = 0$) were introduced by Bekos et al. [31] and are useful because they include several interesting graph classes, as shown by the following three examples.

First, every graph with Euler genus g is a subgraph of a $(g, 3)$-framed multigraph. Thus Theorem 1.35 with $d = 3$ implies Theorem 1.14.

We also show that every (Σ, d)-map graph is a spanning subgraph of $G^{(d)}$ for some multigraph G embedded in Σ without crossings; see Lemma 4.14. Thus Theorem 1.35 implies that (g, d)-map graphs have the following product structure.

Theorem 4.1. *Every (g, d)-map graph is contained in $H \boxtimes P \boxtimes K_\ell$ for some planar graph H with treewidth at most 3 and for some path P, where $\ell = \max\{2g\lfloor \frac{d}{2} \rfloor, d + 3\lfloor \frac{d}{2} \rfloor - 3\}$.*

Furthermore, we show that every $(\Sigma, 1)$-planar graph is contained in $G^{(4)}$ for some multigraph G embedded in Σ without crossings; see Lemma 4.15. Thus Theorem 1.35 implies the following product structure theorem.

Theorem 1.21. *Every $(g, 1)$-planar graph is contained in $H \boxtimes P \boxtimes K_{\max\{4g,7\}}$ for some planar graph H with treewidth at most 3 and for some path P.*

Dujmović et al. [125] proved that every (g, k)-planar graph is contained in $H \boxtimes P \boxtimes K_\ell$, for some graph H with treewidth at most $\binom{k+4}{3} - 1$ where $\ell = \max\{2g, 3\}(6k^2 + 16k + 10)$. In the $k = 1$ case, Theorem 1.21 is significantly stronger since H has treewidth at most 3 instead of at most 9. Note that Dujmović et al. [125] previously proved Theorem 1.21 in the planar case ($g = 0$), and a similar result was independently obtained by Bekos et al. [29].

The results for squaregraphs are based on joint work with Jungeblut, Merker and Wood [193]. The results for graphs on surfaces are based on joint work with Distel, Huynh and Wood [105].

4.2 Preliminaires

For a graph G with $A, B \subseteq V(G)$, let $G[A, B]$ be the subgraph of G with $V(G[A, B]) := A \cup B$ and $E(G[A, B]) := \{uv \in E(G) : u \in A, v \in B\}$.

A *matching* M in a graph G is a set of edges in G such that no two edges in M have a common end-vertex. A matching M *saturates* a set $S \subseteq V(G)$ if every vertex in S is incident to some edge in M.

In a plane graph G, a vertex is *outer* if it is on the outer-face of G and is *inner* otherwise. Let I_G denote the set of inner vertices in G.

Recall that a *layered partition* $(\mathcal{P}, \mathcal{L})$ of a graph G consists of a partition $\mathcal{P}$ and a layering $\mathcal{L}$ of G. If $\mathcal{P}$ is an H-partition, then $(\mathcal{P}, \mathcal{L})$ is a *layered H-partition*. If $\mathcal{P} = (A_x : x \in V(H))$, then the *width* of $(\mathcal{P}, \mathcal{L})$ is $\max\{|A_x \cap L| : x \in V(H), L \in \mathcal{L}\}$. We say that a layered partition of width at most 1 is *thin*.

Analogous to Observation 1.32, we have the following observation which connects layered partitions to $\bowtie$.

Observation 4.2. *For all graphs G and H, $G \lesssim (H \boxtimes K_\ell) \bowtie P$ for some path P if and only if G has a layered H-partition $(\mathcal{P}, \mathcal{L})$ with width at most ℓ, such that each $L \in \mathcal{L}$ is an independent set in G.*

In Observation 4.2 we may use $G \lesssim (H \boxtimes K_\ell) \bowtie P$ instead of $G \lesssim H \boxtimes K_\ell \boxtimes P$ when each $L \in \mathcal{L}$ is an independent set, since no edges in G correspond to edges in $H \boxtimes K_\ell \boxtimes P$ of the form $(v, x, w)(v', y, w)$ where $vv' \in E(H)$, $x, y \in V(K_\ell)$ and $w \in V(P)$.

As mentioned in Section 1.3.1, it is well-known that in the case of bipartite planar graphs G, the proof of Theorem 1.11 can be adapted to show that $G \lesssim H \bowtie P$ for some graph H of treewidth at most 6 and for some path P. To see this, we may assume that G is edge-maximal bipartite planar. Thus G is connected, and each face is a 4-cycle. Let $\mathcal{L} = (L_0, L_1, \dots)$ be a BFS-layering of G. So each L_i is an independent set. Each face can be written as (a, b, c, d) where $a \in L_i$ and $b, d \in L_{i+1}$ and $c \in L_i \cup L_{i+2}$, for some $i \geqslant 0$. Let G' be the planar triangulation obtained from G by adding the edge bd across each such face. Thus $(L_0, L_1, \dots)$ is a layering of G'. The proof of Theorem 1.11 shows that G' has a partition $\mathcal{P}$ such that $\operatorname{tw}(G/\mathcal{P}) \leqslant 6$ and $(\mathcal{P}, \mathcal{L})$ is a thin layered partition. By construction, $(\mathcal{P}, \mathcal{L})$ is a layered partition of G. By Observation 4.2, $G \lesssim H \bowtie P$.

4.3 Squaregraphs

4.3.1 Sufficient Conditions

We now work towards proving Theorem 1.12. We first prove the following, more general sufficient condition for a plane graph to be contained in the strong or semi-strong product of an outerplanar graph and a path. Afterwards, we show that this more general result implies Theorem 1.12.

Theorem 4.3. *Let G be a plane graph with inner vertices I_G. If G has a layering $\mathcal{L} = (L_0, L_1, \dots)$ such that $G[L_{i-1}, L_i]$ has a matching saturating $L_{i-1} \cap I_G$ for each $i \geqslant 1$, then $G \lesssim H \boxtimes P$ for some outerplanar graph H and path P. Moreover, if $V(L_i)$ is an independent set for all $L_i \in \mathcal{L}$, then $G \lesssim H \bowtie P$.*

Proof. By Observations 1.32 and 4.2, it suffices to show that G has a thin layered H-partition $\mathcal{P}$ (with respect to $\mathcal{L}$) for some outerplanar graph H. For each $i \in [n]$, let E_i be a matching in $G[L_{i-1}, L_i]$ that saturates $L_{i-1} \cap I_G$. For vertices $u \in L_{i-1}$ and $v \in L_i$ and an edge $uv \in E_i$, we say that u is the *parent* of v and v is the *child* of u. Observe that each vertex $u \in L_{i-1} \cap I_G$ has exactly one child and each vertex $v \in L_i$ has at most one parent. Let J be the subgraph of G where $V(J) = V(G)$ and $E(J) = \bigcup_{i \in [n]} E_i$.

Let X be a connected component of J. Choose the maximum $j \in [0, n]$ such that there exists some vertex $v \in V(X) \cap L_j$. Vertex v must be outer because each vertex in $L_j \cap I_G$ is adjacent in J to some vertex in L_{j+1}. As illustrated in Figure 4.1, since each vertex in X has at most one child and at most one parent, X is a vertical path with respect to $\mathcal{L}$.

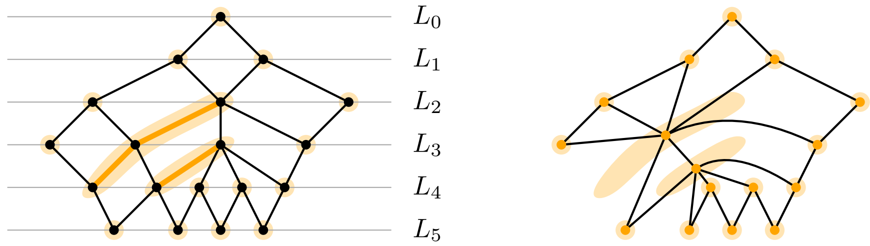


Figure 4.1. Left: A squaregraph with a BFS-layering and a partition $\mathcal{P}$ into vertical paths (thick orange). The vertical paths are constructed from matchings between consecutive layers, where the leftmost vertex in L_i is chosen for each inner vertex in L_{i-1}. Right: The lower endpoint of each path is on the outer-face, so when each path is contracted we obtain an outerplanar graph.

Let $\mathcal{P}$ be the partition of G determined by the connected components of J. Let $H = G/\mathcal{P}$ be the quotient of $\mathcal{P}$. Since each part in $\mathcal{P}$ is a vertical path with respect to $\mathcal{L}$, it follows that $(\mathcal{P}, \mathcal{L})$ is a thin layered H-partition. It remains to show that H is outerplanar. Since each part in $\mathcal{P}$ is connected, H is a minor of G and is therefore planar. Since each part of $\mathcal{P}$ contains a vertex on the outer-face, contracting each part of $\mathcal{P}$ into a single vertex gives a plane embedding of H with each vertex on the outer-face; see Figure 4.1. Therefore H is outerplanar. □

We now work towards showing that squaregraphs satisfy the conditions for Theorem 4.3.

A plane graph G is *leveled* if the edges are straight line-segments and vertices are placed on a sequence of horizontal lines, $(L_0, L_1, \dots)$, called *levels*, such that each edge joins two vertices in consecutive levels. If, in addition, we allow straight-line edges between consecutive vertices on the same level, then G is *weakly leveled*. Observe that the levels in a weakly leveled plane graph G define a layering of G. Leveled plane graphs were first introduced by Sugiyama et al. [310], and have since been well studied [25].

For a weakly leveled plane graph G with levels $(L_0, L_1, \dots)$ and a vertex $v \in L_i$, the *up-degree* of v is $|N_G(v) \cap L_{i-1}|$ and the *down-degree* of v is $|N_G(v) \cap L_{i+1}|$. We now give a more natural condition that forces our desired matching between two consecutive levels.

Lemma 4.4. *Let G be a weakly leveled plane graph with inner vertices I_G. If each vertex in I_G has down-degree at least 2, then $G \subsetneq H \boxtimes P$ for some outerplanar graph H and path P. Moreover, if G is a leveled plane graph, then $G \subsetneq H \bowtie P$.*

Proof. Let $(L_0, L_1, \dots)$ be the levels of G. Observe that if G is a leveled plane graph, then $V(L_i)$ is an independent set for all $i \geqslant 0$. For each $i \in [n]$, let E_i be the set of edges in $G[L_{i-1}, L_i]$ between each vertex $v \in L_{i-1} \cap I_G$ and its leftmost neighbour in L_i; see Figure 4.1. For the sake of contradiction, suppose there exists a vertex $u \in L_{i-1} \cup L_i$ that is incident to two edges in E_i. By construction, each vertex in $L_{i-1} \cap I_G$ is incident to at most one edge in E_i so $u \in L_i$. Let x and y be the neighbours of u in L_{i-1}, where x is to the left of y. Since x has down-degree at least 2, x is adjacent to a vertex v that is to the right of u. However, this contradicts G being weakly leveled plane since uy and vx cross; see Figure 4.2. Therefore, E_i is a matching that saturates $L_{i-1} \cap I_G$. The claim therefore follows by Theorem 4.3. □

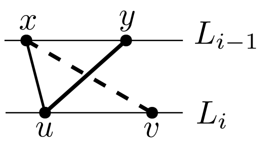


Figure 4.2. Contradiction in the proof of Lemma 4.4.

We are ready to prove Theorem 1.12 which we restate here for convenience.

Theorem 1.12. *Every squaregraph is contained in $H \bowtie P$ for some outerplanar graph H and some path P.*

Proof. We may assume that G is connected (since if each component of G has the desired product structure, then so does G). Bannister et al. [25] showed that G is isomorphic to a leveled plane graph with levels given by a BFS-layering of G rooted at any vertex r on the outer-face. Without loss of generality, assume G is leveled plane with corresponding levels $(L_0, L_1, \dots)$. Below we show that every inner vertex in G has up-degree at most 2. Since each inner vertex has degree at least 4, each inner vertex has down-degree at least 2. The result thus follows from Lemma 4.4.

For the sake of contradiction, suppose there exists an inner vertex with up-degree at least 3. Let $i \in [n]$ be minimum such that there is a vertex $v \in L_i \cap I_G$ with up-degree at least 3. Let u_1, u_2, u_3 be neighbours of v in L_{i-1} ordered left to right. Since the levels are defined by a BFS-layering, there is a (u_1, r)-path and a (u_3, r)-path that does not contain u_2; see Figure 4.3. Hence, u_2 is an inner vertex of G and thus has degree at least 4. However, by planarity, v is the only neighbour of u_2 in L_i. Since u_2 has no neighbours in L_{i-1} (as G is leveled plane), u_2 has three neighbours in L_{i-2}, which contradicts the minimality of i, as required. □

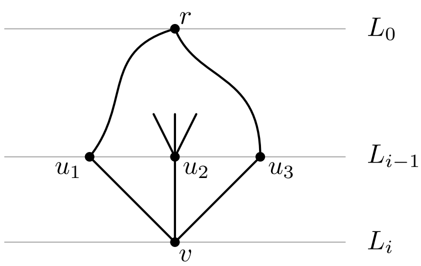


Figure 4.3. Vertex $v \in L_i$ with three neighbours u_1, u_2, u_3 in the preceding layer L_{i-1}. Since u_2 is an inner vertex, it has degree at least 4.

4.3.2 Tightness

In this subsection, we show that Theorem 1.12 is tight by proving a lower bound for the product structure of bipartite graphs.

Recall that the *row treewidth* of a graph G is the minimum integer k such that $G \subsetneq H \boxtimes P$ for some graph H with treewidth k and path P [52]. Theorem 1.11 says that every planar graph has row treewidth at most 6. Dujmović et al. [121] showed that the maximum row treewidth of planar graphs is at least 3. They in fact proved the following stronger result.

Theorem 4.5 ([121]). *For all $k, \ell \in \mathbb{N}$ with $k \geqslant 2$ there is a graph G with pathwidth k such that for any graph H and path P, if $G \subsetneq H \boxtimes P \boxtimes K_\ell$ then $K_{k+1} \subsetneq H$ and thus H has treewidth at least k. Moreover, if $k = 2$ then G is outerplanar, and if $k = 3$ then G is planar.*

Theorem 1.12 says that squaregraphs have row treewidth at most 2. We show that this bound is tight by proving Theorem 4.8 which is an analogous result to Theorem 4.5 for bipartite graphs. As an introduction to the key ideas in the proof of Theorem 4.8, we first establish Proposition 4.7 which is a slight generalisation of Theorem 4.5. We need the following lemma for finding long paths in quotient graphs.

Lemma 4.6. *For every $a, n \in \mathbb{N}$, there exists a sufficiently large $n' \in \mathbb{N}$ such that for every graph G that contains an n'-vertex path and for every H-partition $(A_x : x \in V(H))$ of G where $|A_x| \leqslant a$ for all $x \in V(H)$, for each $w \in V(H)$ the graph $H - w$ contains a path on n vertices.*

Proof. Let m be sufficiently large compared to n and let $n' := (a+1)am + a$. Suppose G has a path on n' vertices. Let $G' = G - A_w$. Since $|V(P) \cap A_w| \leqslant a$, P is split into at most $a + 1$ disjoint subpaths in G'. Thus, there is a path $P_{\max}$ in G' with at least am vertices. Let $\tilde{H}$ be the sub-quotient of H with respect to $P_{\max}$. Observe that $\tilde{H}$ is connected and that $|V(\tilde{H})| \geqslant am/a = m$. Moreover, $\tilde{H} \subseteq H - w$ since $A_w \cap V(P_{\max}) = \varnothing$. Now $\tilde{H}$ has maximum degree at most $2a$ since every vertex in $P_{\max}$ has degree at most 2. Thus, since m is sufficiently large, $\tilde{H}$ contains a path on at least n vertices, as required. □

The following result generalises Theorem 4.5 (which is the $n = 2$ case).

Proposition 4.7. *For all $k, \ell, n \in \mathbb{N}$ there exists a graph G with pathwidth at most $k+1$ such that for any graph H and path P, if $G \subsetneq H \boxtimes P \boxtimes K_\ell$ then $P_n + K_k \subsetneq H$.*

Proof. We proceed by induction on $k \geqslant 1$. Let n' be sufficiently large compared to n. Let $G^{(1)}$ be the graph obtained from a path on n' vertices plus a dominant vertex v. Observe that $G^{(1)}$ has radius 1 and pathwidth at most 2. Suppose $G^{(1)} \subsetneq H \boxtimes P \boxtimes K_\ell$ for some graph H and path P. By Observation 1.32, there is a layered H-partition $(A_x : x \in V(H))$ of G of width at most ℓ. Let $w \in V(H)$ be such that $v \in A_w$. Since

$G^{(1)}$ has radius 1, every layering of $G^{(1)}$ consists of at most three layers so $|A_x| \leqslant 3\ell$ for all $x \in V(H)$. By Lemma 4.6 and since n' is sufficiently large, $H - w$ contains a path on n vertices. As v is dominant in $G^{(1)}$, w is also dominant in H. Thus $P_n + K_1 \subsetneq H$.

Now suppose $k > 1$ and let $G^{(k-1)}$ be a graph that satisfies the induction hypothesis for $k-1$. Let $G^{(k)}$ be obtained by taking 3ℓ disjoint copies of $G^{(k-1)}$ plus a dominant vertex v. Then $G^{(k)}$ has pathwidth at most $k+1$. As in the base case, let $(A_x : x \in V(H))$ be a layered H-partition of $G^{(k)}$ of width ℓ. Let $w \in V(H)$ be such that $v \in A_w$. Since $G^{(k)}$ has radius 1, it follows that $|A_w - \{v\}| \leqslant 3\ell - 1$. Thus, there is a copy of $G^{(k-1)}$ that contains no vertices from A_w. Now consider the sub-quotient $\tilde{H}$ of H with respect to this copy of $G^{(k-1)}$. By induction, $P_n + K_{k-1} \subsetneq \tilde{H}$. Since v is dominant in $G^{(k)}$, w is dominant in H and thus $P_n + K_k \subsetneq H$, as required. □

Note that in Proposition 4.7, the graph $G^{(1)}$ is outerplanar and the graph $G^{(2)}$ is planar for every $n \in \mathbb{N}$.

We now prove our main lower bound which is a bipartite version of Proposition 4.7. A *red-blue colouring* of a bipartite graph G is a proper vertex 2-colouring of G with colours 'red' and 'blue'. For $r \in \mathbb{N}$, a graph H is a *r-small minor* of a graph G, if there is a model μ of H in G such that $|V(\mu(v))| \leqslant r$ for all $v \in V(H)$.

Theorem 4.8. *For all $i, j, k, \ell, n \in \mathbb{N}$ where $i + j = k$, there exists a bipartite graph $G^{(i,j)}$ with pathwidth at most $k+1$ such that for any graph H and path P, if $G^{(i,j)} \subsetneq H \boxtimes P \boxtimes K_\ell$ then $P_n + K_{i,j}$ is a 2-small minor of H. Moreover, $G^{(1,0)}$ is an outerplanar squaregraph and $G^{(1,1)}$ is a bipartite planar graph.*

Proof. Let $P_n = (a_1, \dots, a_n)$ be a path on n vertices. Let $B = \{b_1, \dots, b_i\}$ and $C = \{c_1, \dots, c_j\}$ be the bipartition of $V(K_{i,j})$. We proceed by induction on k with the following hypothesis: for every $i, j, k, \ell, n \in \mathbb{N}$ where $i + j = k$, there exists a red-blue coloured connected bipartite graph G, such that for any graph H, if $(A_x : x \in V(H))$ is a layered H-partition of G of width at most ℓ, then H contains a model μ of $P_n + K_{i,j}$ such that for each $u \in V(P_n + K_{i,j})$ we have $|V(\mu(u))| \leqslant 2$ and $\bigcup(A_a : a \in V(\mu(u)))$ contains:

1. a red vertex when $u \in B$;
2. a blue vertex when $u \in C$; and
3. a red and a blue vertex when $u \in V(P_n)$.

The claimed theorem follows by Observation 1.32.

For $k = 1$ we may assume that $i = 1$ and $j = 0$. Let n' be sufficiently large and let $G^{(1,0)}$ be the bipartite graph obtained from a red-blue coloured path $P_G = (u_1, \dots, u_{n'})$ on n' vertices plus a red vertex v adjacent to all the blue vertices in $V(P_G)$. Observe that $G^{(1,0)}$ has radius 2 and pathwidth at most 2. Let $(A_x : x \in V(H))$ be a layered H-partition of $G^{(1,0)}$ of width ℓ. Let $w \in V(H)$ be such that $v \in A_w$. Then A_w contains a red vertex. Since $G^{(1,0)}$ has radius 2, every layering of $G^{(1,0)}$ has at most five layers, so $|A_x| \leqslant 5\ell$ for all $x \in V(H)$. By Lemma 4.6 and since n' is sufficiently large, $H - w$ contains a path $P_H = (a'_1, \dots, a'_{2n})$ on $2n$ vertices. Now for every edge $a'_i a'_{i+1} \in E(P_H)$, there exists $j \in [n'-1]$ such that $u_j, u_{j+1} \in A_{a'_i} \cup A_{a'_{i+1}}$. As such, $A_{a'_i} \cup A_{a'_{i+1}}$ contains a red and a blue vertex. For all $i \in [n]$, let $\mu(a_i) = H[\{a'_{2i-1}, a'_{2i}\}]$ and $\mu(b_1) = \{w\}$. Then μ is a model of $P_n + K_{1,0}$ in H which satisfies the induction hypothesis.

Now suppose $k > 1$ and that there is a red-blue coloured connected bipartite graph $G^{(i-1,j)}$ such that for any graph H, if $(A_x : x \in V(H))$ is a layered H-partition of G of width at most ℓ, then H contains a model $\tilde{\mu}$ of $P_n + K_{i-1,j}$ where $|V(\tilde{\mu}(u))| \leqslant 2$ for all $u \in V(P_n + K_{i-1,j})$ and $\bigcup(A_a \colon a \in V(\mu(u)))$ contains a red vertex when $u \in B$; a blue vertex when $u \in C$; and a red and a blue vertex when $u \in V(P_n)$. Let $G^{(i,j)}$ be obtained by taking 5ℓ copies of $G^{(i-1,j)}$ plus a red vertex v that is adjacent to all the blue vertices. Then $G^{(i,j)}$ has radius 2 and pathwidth at most $k+1$. As in the base case, let $(A_x : x \in V(H))$ be a layered H-partition of $G^{(i,j)}$ of width ℓ. Let $w \in V(H)$ be such that $v \in A_w$. Then A_w contains a red vertex. Since $G^{(i,j)}$ has radius 2, $|A_w - \{v\}| \leqslant 5\ell - 1$. Thus, there is a copy of $G^{(i-1,j)}$ that contains no vertices from A_w. Now consider the sub-quotient $\tilde{H}$ of H with respect to this copy of $G^{(i-1,j)}$. By induction, $\tilde{H}$ contains a model $\tilde{\mu}$ which satisfies the induction hypothesis. Let $\mu(b_i) = \{w\}$ and $\mu(v) = \tilde{\mu}(v)$ for all $v \in V(P_n + K_{i-1,j})$. Since v is adjacent to all the blue vertices in G, w is adjacent to a vertex in $\bigcup(A_a \colon a \in V(\mu(u)))$ whenever $u \in V(P_n) \cup C$. Thus μ is a model of $P_n + K_{i,j}$ in H which satisfies the induction hypothesis, as required.

As illustrated in Figure 4.4, $G^{(1,0)}$ is an outerplanar squaregraph and $G^{(1,1)}$ is a bipartite planar graph. $\square$

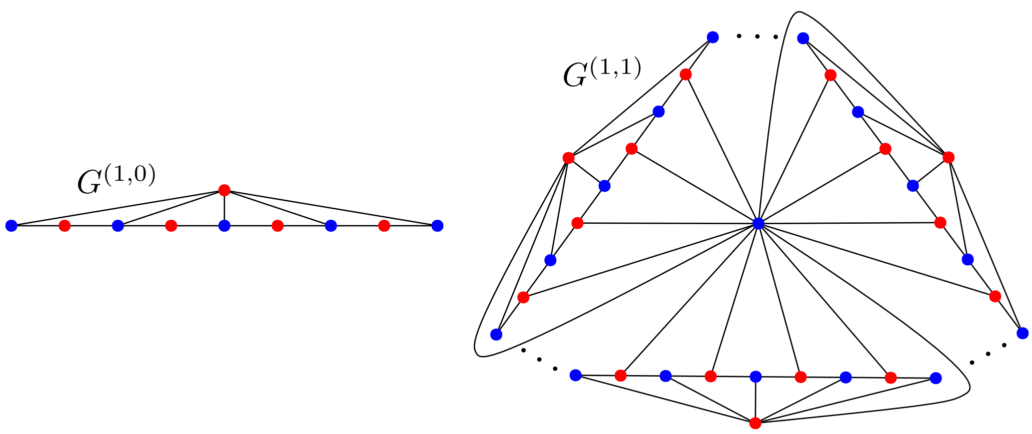


Figure 4.4. The graphs $G^{(1,0)}$ and $G^{(1,1)}$ from Theorem 4.8.

We now highlight several consequences of Theorem 4.8. First, since the graph $G^{(1,0)}$ is an outerplanar squaregraph and $P_2 + K_{1,0}$ is a 3-cycle, we have the following:

Corollary 4.9. *For every $\ell \in \mathbb{N}$, there exists a squaregraph G such that for any graph H and path P, if $G \subsetneq H \boxtimes P \boxtimes K_\ell$ then H contains a cycle of length at most 6.*

Thus Theorem 1.12 is best possible in the sense that "outerplanar graph" cannot be replaced by "forest".

Second, since the graph $G^{(1,1)}$ is a bipartite planar graph and $P_2 + K_{1,1} \cong K_4$ which has treewidth 3, we have the following:

Corollary 4.10. *For every $\ell \in \mathbb{N}$, there exists a bipartite planar graph G such that for any graph H and path P, if $G \subsetneq H \boxtimes P \boxtimes K_\ell$ then H contains a 2-small minor of K_4 and thus $\operatorname{tw}(H) \geqslant 3$.*

Therefore, the maximum row treewidth of bipartite planar graphs is at least 3. We conclude this section with the following open problem: what is the maximum row treewidth of bipartite planar graphs? As in the case of (non-bipartite) planar graphs, the answer is in $\{3, 4, 5, 6\}$.

4.4 Graphs on Surfaces

In this section, we prove Theorem 1.35.

Theorem 1.35. *For all integers $g \geqslant 0$ and $d \geqslant 3$, every (g, d)-framed multigraph is contained in $H \boxtimes P \boxtimes K_\ell$ for some planar graph H with treewidth 3 and for some path P, where $\ell = \max\{2g\lfloor \frac{d}{2} \rfloor, d + 3\lfloor \frac{d}{2} \rfloor - 3\}$.*

We need the following lemma of Dujmović et al. [125], which is a special case of their Lemma 24 (which is an extension of Lemma 17 from [121]).

Lemma 4.11 ([125])**.** *Let G^+ be a plane multigraph in which each face of G^+ is bounded by a cycle with length in $\{3, \dots, d\}$. Let T be a spanning tree of G^+ rooted at some vertex r on the boundary of the outer-face of G^+. Assume there is a vertical path P in T with end-vertices p_1 and p_2 such that the cycle C obtained from P by adding the edge p_1p_2 is a subgraph of $G^+ - r$. Let G be the plane graph consisting of all the vertices and edges of G^+ contained in C and the interior of C. Then $G^{(d)}$ has an H-partition $\mathcal{P}$ such that $P \in \mathcal{P}$ and each part $S_i \in \mathcal{P} \setminus \{P\}$ has a partition $\{X_i, Y_i\}$ where $|X_i| \leqslant d - 3$ and Y_i is the union of at most three vertical paths in T, and H is planar with treewidth at most 3.*

The next lemma is the heart of our proof.

Lemma 4.12. *Let G be a connected multigraph embedded in a surface of Euler genus g without crossings, where each face of G is bounded by a cycle. Then for every spanning tree T of G and every integer $d \geqslant 3$, $G^{(d)}$ has an H-partition $\mathcal{P}$ such that one part $Z \in \mathcal{P}$ is the union of at most $2g$ vertical paths in T and each part $S_i \in \mathcal{P} \setminus \{Z\}$ has a partition $\{X_i, Y_i\}$ where $|X_i| \leqslant d - 3$ and Y_i is the union of at most three vertical paths in T, and H is planar with treewidth at most 3.*

Proof. We start by following the proof of [121, Lemma 21], which is the heart of the proof of Theorem 1.13. Near the end of our proof we follow a different strategy to obtain the stronger result.

If $g = 0$, then the claim follows from Lemma 4.11 by considering an appropriate supergraph G^+ of G. Now assume that $g \geqslant 1$. Say G has n vertices, m edges, and f faces. By Euler's formula, $n - m + f = 2 - g$. Let D be the multigraph with vertex-set the set of faces in G, where for each edge e of $E(G) \setminus E(T)$, if f_1 and f_2 are the faces of G with e on their boundary, then there is an edge joining f_1 and f_2 in D. (Think of D as the spanning subgraph of the dual graph consisting of those edges that do not cross edges in T.) Note that $|V(D)| = f = 2-g-n+m$ and $|E(D)| = m-(n-1) = |V(D)|-1+g$. Since T is a tree, D is connected; see [124, Lemma 11] for a proof. Let T^* be a spanning tree of D. Thus $|E(D) \setminus E(T^*)| = g$. Let $Q = \{a_1b_1, a_2b_2, \dots, a_gb_g\}$ be the set of edges in G dual to the edges in $E(D) \setminus E(T^*)$. Let r be the root of T, and for $i \in \{1, 2, \dots, g\}$, let Z_i be the union of the a_ir-path and the b_ir-path in T, plus the edge a_ib_i. Let $Z := Z_1 \cup Z_2 \cup \cdots \cup Z_g$. By construction, Z is a connected subgraph of G; see Figure 4.5 for an example. In fact, since r is contained in each of the $2g$ vertical paths, $T[V(Z)]$ is connected. Say Z has p vertices and q edges. Since Z consists of a subtree of T plus the g edges in Q, we have $q = p - 1 + g$.

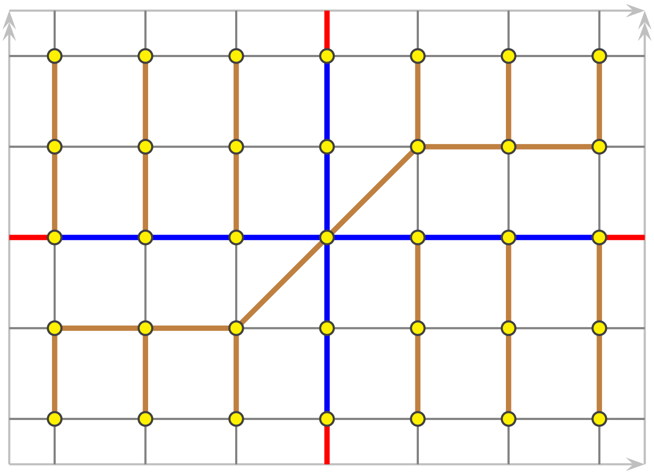


Figure 4.5. Example of the construction in the proof of Lemma 4.12, where brown edges are in T, red edges are in Q, and blue edges are in T and in $Z - E(Q)$.

We now describe how to 'cut' along the edges of Z to obtain a new embedded graph $\tilde{G}$; see Figure 4.6. First, each edge e of Z is replaced by two edges e' and e'' in $\tilde{G}$. Each vertex of G that is not contained in $V(Z)$ is untouched. Consider a vertex $v \in V(Z)$ incident with edges $e_1, e_2, \dots, e_d$ in Z in clockwise order. In $\tilde{G}$ replace v by new vertices $v_1, v_2, \dots, v_d$, where v_i is incident with e'_i, e''_{i+1} and all the edges incident with v clockwise from e_i to e_{i+1} (exclusive). Here e_{d+1} means e_1 and e''_{d+1} means e''_1. This operation defines a cyclic ordering of the edges in $\tilde{G}$ incident with each vertex (where e''_{i+1} is followed by e'_i in the cyclic order at v_i). This in turn defines an embedding of $\tilde{G}$ in some orientable surface[1]. Let Z' be the set of vertices introduced in $\tilde{G}$ by cutting through vertices in Z.

We now show that $\tilde{G}$ is connected. Consider vertices x_1 and x_2 of $\tilde{G}$. Select faces f_1

[1]If G is embedded in a non-orientable surface, then the edge signatures for G are ignored in the embedding of $\tilde{G}$.

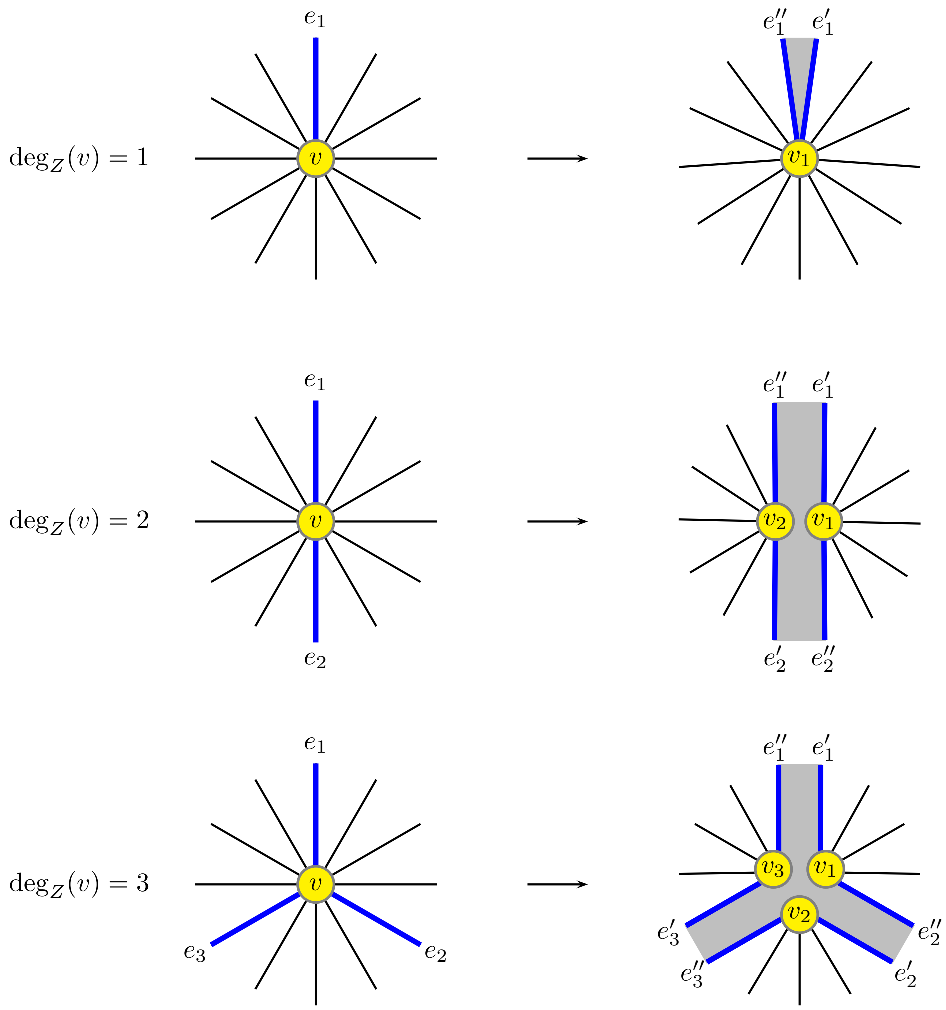


Figure 4.6. Cutting the blue edges in Z at each vertex.

and f_2 of $\tilde{G}$ respectively incident to x_1 and x_2 that are also faces of G. Let P be a path joining f_1 and f_2 in the dual tree T^*. Then the edges of G dual to the edges in P were not split in the construction of $\tilde{G}$. Therefore an x_1x_2-walk in $\tilde{G}$ can be obtained by following the boundaries of the faces corresponding to vertices in P. Hence $\tilde{G}$ is connected.

Say $\tilde{G}$ has n' vertices and m' edges, and the embedding of $\tilde{G}$ has f' faces and Euler genus g'. Each vertex with degree d in Z is replaced by d vertices in $\tilde{G}$. Each edge in Z is replaced by two edges in $\tilde{G}$, while each edge of $E(G) - E(Z)$ is maintained in $\tilde{G}$. Thus

$$n' = n - p + \sum_{v \in V(Z)} \deg_Z(v) = n + 2q - p = n + 2(p - 1 + g) - p = n + p - 2 + 2g$$

and $m' = m + q = m + p - 1 + g$. Each face of G is preserved in $\tilde{G}$. Say s new faces are created by the cutting. Thus $f' = f + s$. Since $\tilde{G}$ is connected, $n' - m' + f' = 2 - g'$ by Euler's formula. Thus $(n + p - 2 + 2g) - (m + p - 1 + g) + (f + s) = 2 - g'$, implying $(n - m + f) - 1 + g + s = 2 - g'$. Hence $(2 - g) - 1 + g + s = 2 - g'$, implying $g' = 1 - s$. Since $g' \geqslant 0$, we have $s \leqslant 1$. Since $g \geqslant 1$, by construction, $s \geqslant 1$. Thus $s = 1$ and $g' = 0$.

Hence $\tilde{G}$ is plane and all the vertices in Z' are on the boundary of a single face, F, of $\tilde{G}$. Moreover, the boundary of F is a cycle C_F and $V(C_F) = Z'$. Consider F to be the outer-face of $\tilde{G}$.

Now construct a supergraph G^+ of $\tilde{G}$ by adding a vertex r^+ in F and edges from r^+ to each vertex in Z'. Then G^+ is a plane multigraph where each face of G^+ is bounded by a cycle.

We now depart from the proof of Dujmović et al. [121, Lemma 21]. Let P^+ be an arbitrary path such that $V(P^+) = V(C_F)$ and let $v^+ \in V(P^+)$ be an end-vertex of P^+. Let T^+ be the following spanning tree of G^+ rooted at r^+. Initialise T^+ to be the path P^+ plus the edge r^+v^+. Let $E' := \{vw \in E(T) : v \in Z, w \in V(G) \setminus V(Z)\}$ and $h := |E'|$. Observe that $T - V(Z)$ is a forest with h components. For each edge $vw \in E'$, w is adjacent to exactly one vertex $v_i \in V(Z')$ introduced when cutting v. Add the edge v_iw to T^+. Finally, add the induced forest $T - V(Z)$ to T^+; see Figure 4.7. Then T^+ is connected since each component of $T - V(Z)$ is adjacent in T^+ to some vertex in $V(P^+)$. Furthermore, since $|V(T^+)| = |V(P^+)| + |V(G) \setminus V(Z)|$ and $|E(T^+)| = |E(P^+)| + h + (|V(G) \setminus V(Z)| - h) = |V(P^+)| + |V(G) \setminus V(Z)| - 1$, it follows that T^+ is indeed a spanning tree of G^+. Consider each component of $T - V(Z)$ to be a subtree of T^+.

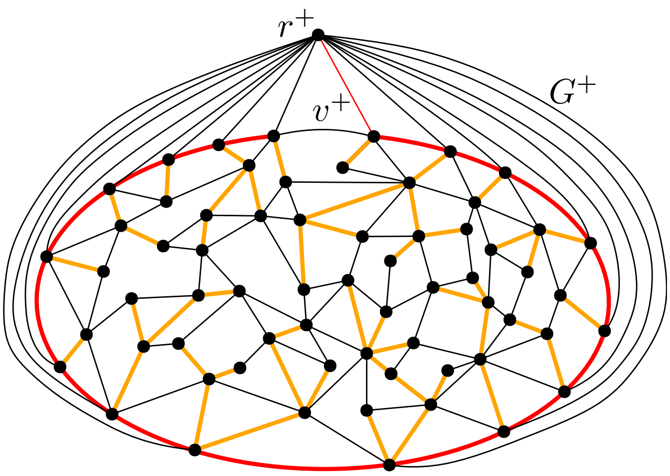


Figure 4.7. Example of the spanning tree T^+ in the graph G^+, where the edges in $E(P^+) \cup \{r^+v^+\}$ are red and the edges that are either in $E(T - V(Z))$ or of the form v_iw are orange.

Now every vertical path in T^+ contained in $V(G) \setminus V(Z)$ corresponds to a vertical path in T. Every maximal vertical path in T^+ consists of the edge r^+v^+, a subpath of P^+, some edge v_iw (where $w \in V(G) \setminus V(Z)$), followed by a path in $T - V(Z)$ from w to a leaf in T. Since every vertical path P in T^+ is contained in some maximal vertical path in T^+, it follows that $P \cap (V(G) \setminus V(Z))$ is a vertical path in T. Thus every vertical path in T^+ that is contained in $V(G) \setminus V(Z)$ is a vertical path in T.

Triangulate every face in G^+ whose facial cycle has length greater than d. Since r^+ is on the boundary of the outer-face of G^+, $V(P^+) = V(C_F)$, every facial cycle has length in $\{3,\dots,d\}$ and P^+ is a vertical path of T^+, Lemma 4.11 is applicable. Let $\mathcal{P}'$ be the H-partition of $\tilde{G}^{(d)}$ given by Lemma 4.11. Therefore, H is planar with treewidth at most 3, where $P^+ \in \mathcal{P}'$ and each part in $S_i \in \mathcal{P}' \setminus \{P^+\}$ has a partition $\{X_i, Y_i\}$ where $|X_i| \leqslant d-3$ and Y_i is the union of at most three vertical paths in T'. Let $\mathcal{P}$ be the partition of $G^{(d)}$ obtained by replacing P^+ by Z. Since $V(P^+) = V(Z')$ and all the split vertices of G are in Z, we have $G^{(d)}/\mathcal{P} \cong \tilde{G}^{(d)}/\mathcal{P}' \cong H$. Hence $\mathcal{P}$ is also an H-partition where H is planar with treewidth at most 3. In addition, since each vertical path in T^+ that is disjoint from $V(Z') \cup \{r^+\}$ is a vertical path in T, each part $S_i \in \mathcal{P} \setminus \{Z\}$ has a partition $\{X_i, Y_i\}$ where $|X_i| \leqslant d-3$ and Y_i is the union of at most three vertical paths in T, as required. □

Theorem 1.35 is an immediate consequence of Observation 1.31 and the next lemma.

Lemma 4.13. *Let G be a multigraph embedded in a surface of Euler genus g without crossings, where each face is bounded by a cycle. Then $G^{(d)}$ has a layered H-partition $(\mathcal{P}, \mathcal{L})$ with width at most $\max\{2g\lfloor \frac{d}{2} \rfloor, d+3\lfloor \frac{d}{2} \rfloor - 3\}$, such that H is planar with treewidth at most 3.*

Proof. Since each face of G is bounded by a cycle, G is connected. Let T be a BFS-spanning tree of G with corresponding BFS-layering $(V_0, V_1, \dots)$. By Lemma 4.12, $G^{(d)}$ has an H-partition $\mathcal{P}$ such that one part $Z \in \mathcal{P}$ is the union of at most $2g$ vertical paths in T and each part $S_i \in \mathcal{P} \setminus \{Z\}$ has a partition $\{X_i, Y_i\}$ where $|X_i| \leqslant d-3$ and Y_i is the union of at most three vertical paths in T, and H is planar with treewidth at most 3. It remains to adjust the layering of G to obtain a layering of $G^{(d)}$. If $uv \in E(G^{(d)})$ then $\operatorname{dist}_G(u,v) \leqslant \lfloor \frac{d}{2} \rfloor$. Thus if $u \in V_i$ and $v \in V_j$ then $|i-j| \leqslant \lfloor \frac{d}{2} \rfloor$. For each $j \in \mathbb{N}$, let $L_j = V_{j\lfloor \frac{d}{2} \rfloor} \cup \dots \cup V_{(j+1)\lfloor \frac{d}{2} \rfloor - 1}$. Then $(\mathcal{P}, \mathcal{L} = (L_0, L_1, \dots))$ is a layered H-partition of $G^{(d)}$ with width at most $\max\{2g\lfloor \frac{d}{2} \rfloor, d+3\lfloor \frac{d}{2} \rfloor - 3\}$, as required. □

We conclude by showing that (Σ, d)-map graphs and $(\Sigma, 1)$-planar graphs are contained in framed graphs. Dujmović et al. [125] proved the following result in the case of plane map graphs (and similar results were previously known in the literature [57, 58, 77]). An analogous proof works for arbitrary surfaces, which we include for completeness. Together with Theorem 1.35, this implies Theorem 4.1.

Lemma 4.14. *For every surface Σ and integer $d \geqslant 3$, every (Σ, d)-map graph is a subgraph of $G^{(d)}$ for some multigraph G embedded in Σ without crossings, where each face of G is bounded by a cycle.*

Proof. Let G_0 be a graph embedded in Σ, with each face labelled a nation or a lake, and where each vertex of G_0 is incident with at most d nations. Let M be the corresponding map graph.

If G_0 has a face F of length 2, then add a new vertex inside F adjacent to both vertices on the boundary of F, which creates two new triangular faces F_1 and F_2. If F is a lake, then make F_1 and F_2 lakes. If F is a nation, then make F_1 a nation and make F_2 a lake. The resulting map graph is still M. So we may assume that G_0 is an edge-maximal multigraph embedded in Σ with no face of length 2 (and with each face labelled a nation or a lake), such that M is the corresponding map graph. This is well-defined since the assumption of having no face of length 2 implies that $|E(G_0)| \leqslant 3(|V(G)|+g-2)$, where g is the Euler genus of Σ.

Suppose that some face f of G_0 has a disconnected boundary. Let v and w be vertices in distinct components of the boundary of f. Add the edge vw to G_0 across f. The corresponding map graph is unchanged, which contradicts the edge-maximality of G_0. Thus each face of G_0 has a connected boundary. Suppose that some face f of G_0 has a repeated vertex v in the boundary walk of f. Let u, v, w be consecutive vertices on the boundary of f. So u, v, w are distinct. Add the edge uw inside f so that uvw bounds a disk. Label the resulting face uvw as a lake. Since v appears elsewhere in the boundary of f, the corresponding map graph is unchanged, which contradicts the edge-maximality of G_0. Thus no facial walk of G_0 has a repeated vertex. Since each facial walk is connected, every face of G_0 is bounded by a cycle.

Let G_0^* be the dual multigraph of G_0. So the vertices of G_0^* correspond to faces of G_0, and each vertex of G_0^* is a nation vertex or a lake vertex. Since every face of G_0 is bounded by a cycle, every face of G_0^* is bounded by a cycle.

Let x be a vertex of G_0, let F_x be the corresponding face of G_0^*, and let $(v_1, \dots, v_s)$ be the facial cycle of F_x. Let $C_x := (w_1, \dots, w_r)$ be the circular subsequence of $(v_1, \dots, v_s)$ consisting of only the nation vertices. Since x is incident to at most d nations, $r \leqslant d$.

Let G be the supergraph of G_0^* obtained by adding an edge between each pair of consecutive vertices in $C_x = (w_1, \dots, w_r)$ for each vertex x of G_0. The graph consisting of C_x plus these added edges is called the *nation cycle* (of x). Note that if $r = 1$ then the nation cycle has no edges, and if $r = 2$ then the nation cycle has one edge. Since every face of G_0^* is bounded by a cycle, every face of G is bounded by a cycle. Moreover, each nation cycle of length at least 3 is now a facial cycle of G with length at most d. By construction, G embeds in Σ with no crossings. Let $G^{(d)}$ be the d-framed graph whose frame is G.

By definition, $V(M) \subseteq V(G^{(d)})$. To prove the claim, it suffices to show that $E(M) \subseteq E(G^{(d)})$. Indeed, if $vw \in E(M)$ then the nation faces corresponding to v and w have a common vertex x on their boundary. The vertex x corresponds to a face F_x in G_0^* and the facial cycle of F_x contains v and w. Therefore, the nation cycle C_x of F_x contains v and w. If C_x has length 2 then $vw \in E(G) \subseteq E(G^{(d)})$. If C_x has length at least 3 then it has length at most d and it bounds a face in G. So $vw \in E(G^{(d)})$. $\square$

Dujmović et al. [125] proved the following result in the case of 1-planar graphs (and similar results were previously known in the literature [31, 57, 58, 77]). An analogous proof works for arbitrary surfaces, which we include for completeness. Together with Theorem 1.35, this implies Theorem 1.21.

Lemma 4.15. *Every $(\Sigma, 1)$-planar graph G with at least three vertices is contained in $G_0^{(4)}$ for some multigraph G_0 embedded in Σ with no crossings where each face of G_0 is bounded by a cycle.*

Proof. We may assume that G is embedded in Σ with at most one crossing on each edge, such that no two edges of G incident to a common vertex cross, since such a crossing can be removed by a local modification to obtain an embedding of G in which the two edges do not cross.

Initialise $G' := G$. Add edges to G' to obtain an edge-maximal multigraph embedded in Σ such that each edge is in at most one crossing, no two edges incident to a common vertex cross, and no face is bounded by two parallel edges. The final condition ensures that G' is well-defined, since it follows from Euler's formula that if G has k crossings, then $|E(G')| \leqslant 3(|V(G)| + k + g - 2) - 2k$.

Consider crossing edges $e_1 = vw$ and $e_2 = xy$ in G'. So v, w, x, y are distinct. Since e_1 is the only edge that crosses e_2 and e_2 is the only edge that crosses e_1, by the edge-maximality of G', there is a cycle $C = (v, x, w, y)$ in G' that bounds a disc whose interior intersects no edge of G' except e_1 and e_2.

Let G_0 be the embedded multigraph obtained from G' by deleting each pair of crossing edges. Thus the above-defined cycle C bounds a face of G_0. By the edge-maximality of G', every other face of G_0 (that is, not arising from a pair of deleted crossing edges) is a triangular face of G'. Thus, G_0 is a multigraph embedded in Σ with no crossings, such that each face of G_0 is bounded by a 3-cycle or a 4-cycle, and G is contained in $G_0^{(4)}$. □

Chapter 5

Product Structure of Beyond-Planar Graphs

5.1 Overview

In this chapter, we use shallow minors to establish product structure theorems for several beyond-planar graphs. The key observation that drives our work is that many beyond-planar graphs can be described as a shallow minor of the strong product of a planar graph with a small complete graph. In particular, we show that powers of bounded degree planar graphs, k-planar, δ-string graphs, (k,p)-cluster planar, fan-planar, and k-fan-bundle planar graphs have such a shallow-minor structure. Our main technical result shows that product structure is well-behaved under shallow minors.

Theorem 5.1. *For all graphs H and L, if a graph G is an r-shallow minor of $H \boxtimes L \boxtimes K_\ell$ where H has treewidth at most t and $\Delta(L^r) \leqslant k$, then $G \subsetneq J \boxtimes L^{2r+1} \boxtimes K_{\ell(k+1)}$ for some graph J with treewidth at most $\binom{2r+1+t}{t} - 1$.*

We also show that k-gap planar graphs do not have bounded local treewidth and, as a consequence, cannot be described as a subgraph of the strong product of a graph with bounded treewidth and a path.

This chapter is based on joint work with Wood [196] except for Theorem 5.26 which is based on joint work with Illingworth, Mohar and Wood [192].

5.2 Preliminaries

Beyond-planar graphs is a vibrant research topic that studies graph classes defined by drawings that forbid certain crossing configurations. See the recent survey by Didimo et al. [99] as well as the monograph by Hong and Tokuyama [201]. A key objective of this chapter is to understand the global structure of beyond-planar graphs.

We now introduce the beyond-planar graphs that are relevant to this chapter. A drawing of a graph G in the plane is:

- *k-planar* if each edge of G is involved in at most k crossings [267];
- *k-quasi planar* if every set of k edges do not mutually cross;
- *k-gap planar* if every crossing can be charged to one of the two edges involved so that at most k crossings are charged to each edge [22];
- *fan-crossing free* if for each edge $e \in E(G)$, the edges that cross e form a matching [80];
- *fan-planar* if for each edge $e \in E(G)$ the edges that cross e have a common end-vertex and they cross e from the same side (when directed away from their common end-vertex) [212]; or
- *right angle crossing (RAC)* if each edge is drawn as a straight line segment and edges cross at right angles [98].

A graph is *k-planar*, *k-quasi planar*, *k-gap planar*, *fan-crossing free*, *fan-planar*, or *RAC* if it respectively has a drawing that is k-planar, k-quasi planar, k-gap planar, fan-crossing free, fan-crossing, or RAC.

The *crossing number* $\mathrm{cr}(G)$ of a graph G is the minimum number of crossings in a drawing of G. A graph class $\mathcal{G}$ has *linear crossing number* if there exists a constant $c > 0$ such that $\mathrm{cr}(G) \leqslant c|V(G)|$ for every $G \in \mathcal{G}$. By the Crossing Lemma [10, 237], this is equivalent to there being a constant $c' > 0$ such that $\mathrm{cr}(G) \leqslant c'|E(G)|$ for every $G \in \mathcal{G}$.

Before proceeding, note that the operation of taking a shallow minor of the product of a graph with a small complete graph has previously been studied within graph sparsity theory. In particular, Har-Peled and Quanrud [181] showed that $\nabla_r(G \circ K_\ell) \leqslant 5\ell^2(r+1)^2\nabla_r(G)$ for every graph G (improving on an earlier result by Nešetřil and Ossona de Mendez [258]).

For $n \in \mathbb{N}$, let $\overline{K_n}$ denote the edge-less graph on n vertices.

5.3 Shallow Minors and Graph Products

5.3.1 Shortcut Systems

We begin by examining the relationship between shallow minors and shortcut systems. Dujmović et al. [125] introduced shortcut systems as a way to prove product structure theorems for various non-minor-closed graph classes. A set $\mathcal{P}$ of paths in a graph G is a *(k, d)-shortcut system* if every path $P \in \mathcal{P}$ has length at most k and every vertex $v \in V(G)$ is an internal vertex for at most d paths in $\mathcal{P}$. Let $G^{\mathcal{P}}$ denote the supergraph of G obtained by adding the edge uv if $\mathcal{P}$ contains a (u, v)-path. Dujmović et al. [125] observed that k-planar graphs, d-map graphs, and several other classes can be described by applying shortcut systems to planar graphs. Using the following theorem, they deduced that these classes have a product structure.

Theorem 5.2 ([125]). *If $G \subsetneq H \boxtimes P \boxtimes K_\ell$, for some graph H of treewidth at most t and $\mathcal{P}$ is a (k,d)-shortcut system for G, then $G^{\mathcal{P}} \subsetneq J \boxtimes P \boxtimes K_{d\ell(k^3+3k)}$ for some graph J of treewidth at most $\binom{k+t}{t} - 1$ and some path P.*

In Section 5.3.2, we adapt the proof of Theorem 5.2 to establish an analogous result in the more general setting of shallow minors.

Huynh and Wood [204] introduced the following variant of shortcut systems. A set $\mathcal{P}$ of paths in a graph G is a *$(k,d)^*$-shortcut system* if every path in $\mathcal{P}$ has length at most k and for every $v \in V(G)$, if M_v is the set of vertices $u \in V(G)$ such that there exists a (u,w)-path in $\mathcal{P}$ in which v is an internal vertex, then $|M_v| \leqslant d$. Note that (k,d)-shortcut systems and $(k,d)^*$-shortcut systems differ in how they count the number of paths that a vertex contributes to. Every $(k,d)^*$-shortcut system is a $(k, \binom{d}{2})$-shortcut system, and every (k,d)-shortcut system is a $(k,2d)^*$-shortcut system. Thus, $(k,d)^*$-shortcut systems can give better bounds compared to (k,d)-shortcut systems. Shallow minors inherit this strength of $(k,d)^*$-shortcut systems.

We show that graphs obtained by applying a shortcut system to a planar graph are k-gap planar and thus have linear crossing number. Later, we show that the class of fan-planar graphs, which has a shallow minor structure, has super-linear crossing number and thus cannot be described by shortcut systems (see Section 5.4.5).

Lemma 5.3. *If $\mathcal{P}$ is a (k,d)-shortcut system of a planar graph G, then $G^{\mathcal{P}}$ is $\big((d-1)(k-1) + 2d\big)$-gap planar.*

Proof. Assume that G is a plane graph. For some $\varepsilon > \delta > 0$, for every vertex $x \in V(G)$ and edge $uv \in E(G)$, let

$$B_x := \{p \in \mathbb{R}^2 : \mathrm{dist}_{\mathbb{R}^2}(p,x) \leqslant \varepsilon\} \quad \text{and} \quad C_{uv} := \{p \in \mathbb{R}^2 : \mathrm{dist}_{\mathbb{R}^2}(p,uv) \leqslant \delta\} \setminus (B_u \cup B_v).$$

Choosing ε and δ to be sufficiently small, we may assume that B_x, B_y, C_{ab} and C_{uv} are non-empty and pairwise-disjoint for all $x, y \in V(G)$ and $ab, uv \in E(G)$. For each (x,y)-path $P = (x = w_0, w_1, \dots, w_{\ell-1}, w_\ell = y) \in \mathcal{P}$, draw the edge xy in the region

$$B_{w_0} \cup C_{w_0 w_1} \cup B_{w_1} \cup \cdots \cup B_{w_{\ell-1}} \cup C_{w_{\ell-1} w_\ell} \cup B_{w_\ell}$$

so that for all $uv, xy \in E(G^{\mathcal{P}})$ with corresponding shortcuts $P_1, P_2 \in \mathcal{P}$, each crossing between uv and xy occurs in some B_w where w is an internal vertex of P_1 or P_2, and uv and xy cross at most once in B_w (see Figure 5.1).

Let E_1 be the edges of G and $E_2 := E(G^{\mathcal{P}}) \setminus E_1$ be the new edges in $G^{\mathcal{P}}$. Since E_1 is a set of non-crossing edges, every crossing involves an edge from E_2. If $uv \in E_1$ and $xy \in E_2$ cross, then charge the crossing to uv. Now suppose $x_1y_1, x_2y_2 \in E_2$ cross in B_w for some $w \in V(G)$. Let $P_1, P_2 \in \mathcal{P}$ be the shortcuts that respectively correspond to x_1y_1 and x_2y_2. If w is an internal vertex of P_1, then charge the crossing to x_2y_2. Otherwise w is an internal vertex of P_2, in which case, charge the crossing to x_1y_1.

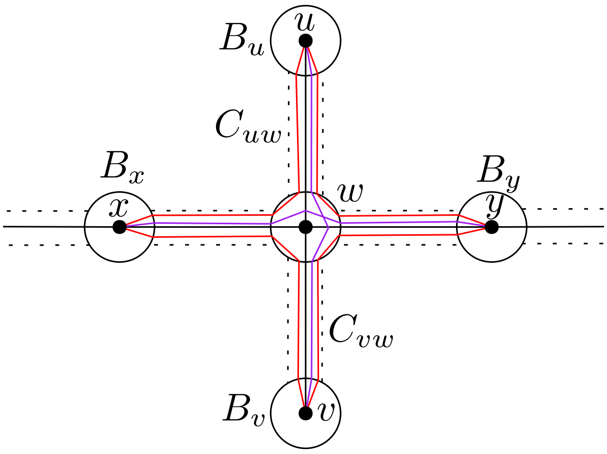


Figure 5.1. Drawing $G^{\mathcal{P}}$ into the plane.

We now upper bound the number of crossings charged to an edge. For an edge $uv \in E_1$, if an edge $xy \in E_2$ crosses uv, then the (x, y)-path in $\mathcal{P}$ contains u or v as an internal vertex. As such, at most $2d$ crossings are charged to uv. Now consider an edge $x_1y_1 \in E_2$ with corresponding path $P_1 \in \mathcal{P}$. If an edge $x_2y_2 \in E_2$ with corresponding path $P_2 \in \mathcal{P}$ crosses x_1y_1 in B_w and the crossing is charged to x_1y_1 for some $w \in V(G)$, then w is an internal vertex of P_2 that is contained in $V(P_1)$. Since each vertex is an internal vertex for at most d paths and $V(P_1)$ has at most $(k-1)$ internal vertices, at most $(d-1)(k-1)+2d$ crossings are charged to x_1y_1. Therefore, $G^{\mathcal{P}}$ is $\big((d-1)(k-1)+2d\big)$-gap planar. $\square$

We now show that shallow minors subsumes shortcut systems.

Lemma 5.4. *For every (k, d)-shortcut system $\mathcal{P}$ of a graph G, $G^{\mathcal{P}}$ is a $(\frac{k-1}{2})$-shallow topological minor of $G \circ \overline{K_{d+1}}$.*

Proof. Assume that $V(K_{d+1}) = [d + 1]$ and that for each $uv \in E(G)$ there is a corresponding (u, v)-path in $\mathcal{P}$ with length 1. We will construct a topological minor of $G^{\mathcal{P}}$ in $G \circ \overline{K_{d+1}}$ where each vertex $v \in V(G^{\mathcal{P}})$ is mapped to $(v, 1) \in V(G \circ \overline{K_{d+1}})$. For each $w \in V(G)$, let M_w be the set of paths in $\mathcal{P}$ that contains w as an internal vertex. Let $\phi_w : M_w \to [2, d + 1]$ be an injective function. For each $v \in V(G)$, extend the domain and range of ϕ_v by letting $\phi_v(P) := 1$ for all paths $P \in \mathcal{P}$ with end-vertex v. For each path $P_{uv} = (u = w_0, w_1, \dots, w_{\ell-1}, w_\ell = v) \in \mathcal{P}$ where $\ell \leqslant k$, let $\tilde{P}_{uv}$ be the path in $G \circ \overline{K_{d+1}}$ defined by $V(\tilde{P}_{uv}) := \{(w_i, \phi_{w_i}(P_{uv})) : i \in [0, \ell]\}$ and $E(\tilde{P}_{uv}) := \{(w_i, \phi_{w_i}(P_{uv}))(w_{i+1}, \phi_{w_{i+1}}(P_{uv})) : i \in [0, \ell-1]\}$. Let $\tilde{\mathcal{P}}$ be the set of such $\tilde{P}_{uv}$ paths.

We claim that $\tilde{\mathcal{P}}$ defines a $(\frac{k-1}{2})$-shallow topological minor of $G^{\mathcal{P}}$ in $G \circ \overline{K_{d+1}}$ where each vertex $v \in V(G^{\mathcal{P}})$ is mapped to $(v, 1) \in V(G \circ \overline{K_{d+1}})$ and each edge $uv \in E(G^{\mathcal{P}})$ is mapped to $\tilde{P}_{uv} \in \tilde{\mathcal{P}}$. Let $uv \in E(G^{\mathcal{P}})$. Then there exists a path $P_{uv} \in \mathcal{P}$ with length at most k and end-vertices u and v. By construction, $\tilde{P}_{uv}$ is a path in $G \circ \overline{K_{d+1}}$ from $(u, 1)$ to $(v, 1)$ with length at most k. Thus, it suffice to show that the paths in $\tilde{\mathcal{P}}$ are internally disjoint. The internal vertices of each path $\tilde{P}_{uv} \in \tilde{\mathcal{P}}$ are of the form $(w, \phi_w(P_{uv}))$ where $\phi_w(P_{uv}) \in [2, d+1]$. Suppose there is another path $\tilde{Q}_{xy} \in \tilde{\mathcal{P}}$ for which $(w, \phi_w(P_{uv}))$ is an internal vertex of $\tilde{Q}_{xy}$. Then $\tilde{P}_{uv} = \tilde{Q}_{xy}$ since ϕ_w is injective. As such, the paths in $\tilde{\mathcal{P}}$ are internally disjoint, as required. $\square$

5.3.2 Key Tool

Having shown that shallow minors subsume shortcut systems, we now show that shallow minors inherit product structure.

Theorem 5.1. *For all graphs H and L, if a graph G is an r-shallow minor of $H \boxtimes L \boxtimes K_\ell$ where H has treewidth at most t and $\Delta(L^r) \leqslant k$, then $G \subsetneq J \boxtimes L^{2r+1} \boxtimes K_{\ell(k+1)}$ for some graph J with treewidth at most $\binom{2r+1+t}{t} - 1$.*

To prove Theorem 5.1, we use the language of H-partitions from Dujmović et al. [121]. Recall the following definitions. Let G be a graph. A *partition* of G is a collection $\mathcal{P}$ of sets of vertices in G such that each vertex of G is in exactly one part of $\mathcal{P}$. The *quotient* of $\mathcal{P}$ (with respect to G) is the graph, denoted by $G/\mathcal{P}$, with vertex-set $\mathcal{P}$ where distinct parts $A, B \in \mathcal{P}$ are adjacent in $G/\mathcal{P}$ if and only if some vertex in A is adjacent in G to some vertex in B. An *H-partition* of G is a partition $\mathcal{P} = (Y_x : x \in V(H))$ indexed by the nodes of H such that $G/\mathcal{P} \subsetneq H$. Note that $\mathcal{P}$ is allowed to contain some empty parts. For our purposes, we require the following extension of H-partitions. An *(H, L)-partition* $(\mathcal{Y}, \mathcal{Z})$ consists of an H-partition $\mathcal{Y}$ and an L-partition $\mathcal{Z}$ of G for some graphs H and L. The *width* of $(\mathcal{Y}, \mathcal{Z})$ is $\max\{|Y_y \cap Z_z| : y \in V(H), z \in V(L)\}$.

The following generalises Observation 1.32.

Observation 5.5. *For all graphs H and L, a graph G is contained in $H \boxtimes L \boxtimes K_\ell$ if and only if G has an (H, L)-partition with width at most ℓ.*

For a rooted tree (T, x_0), we say that a node $a \in V(T)$ is a *T-ancestor* of $x \in V(T)$ (and x is a *T-descendent* of a) if a is contained in the path in T from x_0 to x. If in addition $a \neq x$, then a is a *strict T-ancestor* of x. Note that x is a T-ancestor and a T-descendent of itself.

The proof of Theorem 5.1 is an adaptation of the proof of Theorem 5.2 [125, Theorem 9]. We make use of the following well-known normalisation lemma (see [125, Lemma 2] for a proof).

Lemma 5.6. *For every graph H of treewidth t, there is a rooted tree (T, x_0) with $V(T) = V(H)$ and a width-t tree-decomposition $(T, \{W_x : x \in V(T)\})$ of H that has the following additional properties:*

(T1) for each node $x \in V(H)$, the subtree $T[x] := T[\{y \in V(T) : x \in W_y\}]$ is rooted at x; and consequently

(T2) for each edge $xy \in E(H)$, one of x or y is a T-ancestor of the other.

We now prove our main technical lemma which, together with Observation 5.5, implies Theorem 5.1.

Lemma 5.7. *Let G be a graph having an (H, L)-partition with width ℓ in which H has treewidth at most t and $\Delta(L^r) \leqslant k$. Then every r-shallow minor G' of G has a (J, L^{2r+1})-partition with width at most $\ell(k+1)$ where the graph J has treewidth at most $\binom{2r+1+t}{t} - 1$.*

Proof. Let μ be an r-shallow model of G' in G. Assume that $V(G') \subseteq V(G)$ and that u is a centre of $\mu(u)$ for each $u \in V(G')$. Let $\mathcal{Y} := (Y_x : x \in V(H))$ and $\mathcal{Z} := (Z_z : z \in V(L))$ respectively be an H-partition and an L-partition of G, where $(\mathcal{Y}, \mathcal{Z})$ has width at most ℓ. Let $\mathcal{B} = (B_x : x \in V(T))$ be a tree-decomposition of H that satisfies the conditions of Lemma 5.6. Note that $V(T) = V(H)$. Let x_0 denote the root of T. For a vertex $u \in V(G')$, let $X_u := \{x \in V(H) : V(\mu(u)) \cap Y_x \neq \varnothing\}$; that is, X_u is the set of nodes in H that indexes a part in a $\mathcal{Y}$ which contains a vertex from $\mu(u)$.

Claim 1: For every $u \in V(G')$, there exists a node $a(u) \in X_u$ such that $a(u)$ is a T-ancestor of every node in X_u.

Proof. Since $\mathcal{Y}$ is a partition of G and $\mu(u)$ is connected, $H[X_u]$ is connected. By the transitivity of the T-ancestor relationship, $(T2)$, there exists a node $a(u) \in X_u$ such that $a(u)$ is a T-ancestor of every node in X_u. □

Note that $a(u)$ is the vertex in X_u that is closest (in T) to x_0. We now use $a(u)$ to define a partition of G'. For each $x \in V(T)$, define $S_x := \{u \in V(G') : a(u) = x\}$. Observe that $\mathcal{S} := (S_x : x \in V(T))$ is a partition of $V(G')$. Let $J := G'/\mathcal{S}$ denote the resulting quotient graph, and let $V(J) \subseteq V(T)$ where each $x \in V(J)$ is obtained by identifying S_x in G'. For each $z \in V(L)$, let $Z'_z := Z_z \cap V(G')$ and $\mathcal{Z}' := (Z'_z : z \in V(L))$.

From here on in, we need to show: (i) $\mathcal{S}$ has small width with respect to $\mathcal{Z}'$; (ii), that J has small treewidth; and (iii), that $G'/\mathcal{Z}'$ is contained in L^{2r+1}. The next claim demonstrates (i).

Claim 2: $|S_x \cap Z'_z| \leqslant \ell(k+1)$ for all $x \in V(J)$ and $z \in V(L)$.

Proof. Let $u \in S_x \cap Z'_z$ and $W := \{j \in V(L) : \text{dist}_L(z, j) \leqslant r\}$. By definition, $|W| \leqslant k+1$. Since $a(u) = x$, there is a vertex $w \in V(\mu(u))$ that is contained in the part Y_x (recall that Y_x is the part in the H-partition of G that is index by x). Since $\text{dist}_G(u, w) \leqslant r$ and L is a partition of G, $w \in \bigcup_{j \in W} Z_j$. As such, $w \in Y_x \cap (\bigcup_{j \in W} Z_j)$. Therefore $|S_x \cap Z'_z| \leqslant \ell(k+1)$ since each vertex in $Y_x \cap (\bigcup_{j \in W} Z_j)$ can contribute at most one vertex to $S_x \cap Z_z$ and $|Y_x \cap (\bigcup_{j \in W} Z'_j)| \leqslant \ell(k+1)$. □

The following claim will be useful in bounding the treewidth of J.

Claim 3: For each edge $xy \in E(J)$, one of x or y is a T-ancestor of the other.

Proof. Our goal is to show that there exists a node $w \in V(T)$ such that x and y are both T-ancestors of w. This implies that one of x or y is a T-ancestor of the other.

Since $xy \in E(J)$, G' contains an edge uv with $u \in S_x$ and $v \in S_y$. By the definition of $\mathcal{S}$, it follows that $a(u) = x$ and $a(v) = y$. By Claim 1, x is a T-ancestor of every node in X_u and y is a T-ancestor of every node in X_v. Since $uv \in E(G')$, there is an edge

$\tilde{u}\tilde{v} \in E(G)$ where $\tilde{u} \in V(\mu(u))$ and $\tilde{v} \in V(\mu(v))$. Let $\tilde{x} \in V(T)$ and $\tilde{y} \in V(T)$ respectively be the nodes in T such that $\tilde{u} \in Y_{\tilde{x}}$ and $\tilde{v} \in Y_{\tilde{y}}$. Since $\tilde{u} \in V(\mu(u))$ and $\tilde{v} \in V(\mu(v))$, it follows that $\tilde{x} \in X_u$ and $\tilde{y} \in X_y$. So x is a T-ancestor of $\tilde{x}$, and y is a T-ancestor of $\tilde{y}$. As such, if $\tilde{x} = \tilde{y}$, then we are done by setting $w := \tilde{x}$. Otherwise, $\tilde{x}\tilde{y} \in E(H)$ since $\tilde{u}\tilde{v} \in E(G)$. By $(T2)$, either $\tilde{x}$ or $\tilde{y}$ is a T-ancestor of the other. Without loss of generality, assume that $\tilde{x}$ is a T-ancestor of $\tilde{y}$. Then by setting $w := \tilde{y}$, it follows that x and y are both T-ancestors of w, as required. □

We now show that J has small treewidth.

Claim 4: J has a tree-decomposition in which every bag has size at most $\binom{2r+1+t}{t}$.

Proof. We will define a tree-decomposition of J with small width that is indexed by the tree T. For each node x of T, let C_x be a bag that contains x as well as every T-ancestor a of x such that J contains an edge ax' where x' is a T-descendent of x. Clearly, every vertex is in a bag. Claim 3 ensures that every edge is in a bag. The connectedness of $T[a] := T[\{x \in V(T) : a \in C_x\}]$ follows from the fact that for every edge $x'a \in E(J)$ where x' is a T-descendent of a, every node x on the (x', a)-path in T is a node in $T[a]$. As such, $(C_x : x \in V(T))$ defines a tree-decomposition of J. It remains to bound the size of each bag C_x.

Let H^+ denote the super graph of H with vertex-set $V(H)$ in which $xy \in E(H^+)$ if and only if $x, y \in B$ for some bag $B \in \mathcal{B}$. Let $\overrightarrow{H}^+$ be the directed graph obtained by directing each edge $xy \in E(H^+)$ from its T-descendant x towards its T-ancestor y. Our goal is to show that for every $x \in V(T)$, if $a \in C_x$ then there is a directed path of length at most $2r + 1$ in $\overrightarrow{H}^+$ from x to a. Once we have shown that, we may then appeal to [276, Lemma 13], which states that the number of nodes in $\overrightarrow{H}^+$ that can be reached from any node x by a directed path of length at most $2r+1$ is at most $\binom{2r+1+t}{t}$, thus implying our claim.

Consider an arbitrary node $x \in V(T)$ where $x_0, \dots, x_h$ is the path from the root x_0 of T to $x_h := x$. To simplify our notation, let Y_i and S_i be shorthand for Y_{x_i} and S_{x_i} respectively. For each $i \in [0, h]$, let $V_i := \bigcup(Y_y \colon y$ is a T-descendant of $x_i)$; that is, V_i is the set of vertices in G that are contained in a part that is indexed by a T-descendant of x_i. Observe that $V_i \subseteq V_j$ whenever $j \geqslant i$.

Let $x_\delta \in C_x$. We now work towards showing that there is a directed path of length at most $2r + 1$ in $\overrightarrow{H}^+$ from x to x_δ. Since $x_\delta \in C_x$, it is because x_δ is adjacent in J to some T-descendent of x or $x_\delta = x$. In the case when $x_\delta = x$, there is trivially a directed path in $\overrightarrow{H}^+$ from x to x_δ of length at most $2r + 1$. So we may assume there is a T-descendant x' of x such that $x_\delta x' \in E(J)$. So G' contains an edge uv with $u \in S_{x'}$ and $v \in S_\delta$. Since $a(v) = x_\delta$, there exists $\tilde{v} \in V(\mu(v)) \cap Y_\delta$. Let $\tilde{u} \in V(\mu(u))$ be such that there exists a path $P = (\tilde{u} = w_0, w_1, \dots, w_p = \tilde{v})$ in $G[V(\mu(v)) \cup \{\tilde{u}\}]$ where $p \leqslant 2r + 1$ (see Figure 5.2).

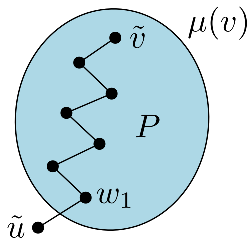


Figure 5.2. Path P from $\tilde{u}$ to $\tilde{v}$.

For each $i \in [0,p]$, let $s_i := \max\{\ell \in [0,h] : w_0, \dots, w_i \subseteq V_\ell\}$, and let $a_i := x_{s_i}$. That is, a_i is the furthest vertex from the root on the (x_0, x_h)-path such that $w_0, \dots, w_i \subseteq V_{a_i}$. Then $s_0, \dots, s_p$ is a non-increasing sequence and $a_0, \dots, a_p$ is a sequence of nodes of T whose distance from the root x_0 is non-increasing.

We claim that $a_0 = x_h$. Since $a(u) = x'$ which is a T-descendant of x_h, it follows by Claim 1 that $\tilde{u} \in V_h$. Since h is trivially the maximum of $\{0, 1, \dots, h\}$, $a_0 = x_h$ as required.

We claim that $a_p = x_\delta$. Since $a(u)$ and $a(v)$ are T-descendent of x_δ, by Claim 1, $w_0, w_1, \dots, w_p \subseteq V_\delta$ and so a_p is a T-descendant of x_δ. Since $\tilde{v} \in Y_\delta$, it follows that a_p is a T-ancestor of x_δ. Hence $a_p = x_\delta$ as required.

We claim that $a_0, \dots, a_p$ is a lazy walk[1] in H^+. Suppose that $a_i \neq a_{i+1}$ for some $i \in [0, p-1]$. Then $w_i \in V_{a_i}$ and $w_{i+1} \notin V_{a_i}$. Let b_i and c_i be the unique nodes where $w_i \in Y_{b_i}$ and $w_{i+1} \in Y_{c_i}$. By definition of V_{a_i}, b_i is a T-descendant of a_i. Now $b_i c_i \in E(H)$ since $w_i w_{i+1} \in E(G)$, and thus by $(T2)$, c_i is a strict T-ancestor of a_i. Since $w_1, \dots, w_i, w_{i+1} \in V_{c_i}$ and $w_{i+1} \in Y_{c_i}$, it follows that $a_{i+1} = c_i$. By $(T1)$, $a_{i+1} \in B_{a_{i+1}}$ and $a_{i+1} \in B_{b_i}$. Since a_i is on the path from b_i to a_{i+1} in T, this implies $a_{i+1} \in B_{a_i}$. Therefore, $a_i a_{i+1} \in E(H^+)$ as required. So by removing repeated vertices from this lazy walk, we obtain a path of length at most $2r+1$ in the directed graph $\overrightarrow{H}^+$ from x to x_δ, as required. $\square$

To finish the proof, note that $\mathcal{Z}'$ may not be an L-partition of G'. In particular, for every edge $vw \in E(G')$ with $v \in Z'_i$ and $w \in Z'_j$, we have $\operatorname{dist}_L(i,j) \leqslant 2r+1$. Thus, by indexing $\mathcal{Z}'$ by L^{2r+1} instead of L, we obtain a valid partition of G'. Therefore, G' has a (J, L^{2r+1})-partition with width at most $\ell(k+1)$ where $\operatorname{tw}(J) \leqslant \binom{2r+t}{t} - 1$. $\square$

Recall that the *row treewidth* $\operatorname{rtw}(G)$ of a graph G is the minimum treewidth of a graph H such that $G \subsetneq H \boxtimes P$ for some path P. Since $\Delta(P^r) \leqslant 2r$ and $P^{2r+1} \subsetneq P \boxtimes K_{2r+1}$, we have the following consequence of Theorem 5.1:

Theorem 1.33. *If G is an r-shallow minor of $H \boxtimes P \boxtimes K_\ell$ where H has treewidth at most t and P is a path, then $G \subsetneq J \boxtimes P \boxtimes K_{\ell(2r+1)^2}$ where J has treewidth at most $\binom{2r+1+t}{t} - 1$, and thus $\operatorname{rtw}(G) \leqslant \binom{2r+1+t}{t}\ell(2r+1)^2 - 1$.*

[1] A *lazy walk* in a graph G is a sequence $(v_1, \dots, v_n)$ of vertices such that v_i and v_{i+1} are adjacent or equal for each $i \in [n-1]$.

5.3.3 Layered Treewidth

Layered tree-decompositions were introduced by Dujmović et al. [124] as a precursor to graph product structure theory. Recall that a *layering* of a graph G is an ordered partition $\mathcal{L} := (L_0, L_1, \dots)$ of $V(G)$ such that for every edge $vw \in E(G)$, if $v \in L_i$ and $w \in L_j$, then $|i-j| \leqslant 1$. A *layered tree-decomposition* $(\mathcal{L}, \mathcal{T})$ consists of a layering $\mathcal{L}$ and a tree-decomposition $\mathcal{T} = (T, \mathcal{B})$ of G. The *layered width* of $(\mathcal{L}, \mathcal{T})$ is $\max\{|L \cap B| : L \in \mathcal{L}, B \in \mathcal{B}\}$. The *layered treewidth* $\operatorname{ltw}(G)$ of G is the minimum layered width of any layered tree-decomposition of G. It is known that $\operatorname{ltw}(G) \leqslant \operatorname{rtw}(G) + 1$ for every graph G [121] but for every integer $k \geqslant 1$, there exists a graph G with $\operatorname{ltw}(G) = 1$ and $\operatorname{rtw}(G) \geqslant k$ [52]. Dujmović et al. [124] showed that every graph G with Euler genus g has $\operatorname{ltw}(G) \leqslant 2g+3$. In addition, Dujmović et al. [124, Lemma 9] proved the following:

Lemma 5.8 ([124])**.** *For every graph G, and for every r-shallow minor H of $G \boxtimes K_\ell$,*

$$\operatorname{ltw}(H) \leqslant \ell(4r+1)\operatorname{ltw}(G).$$

We mention several applications of layered treewidth. For strong colouring-numbers, van den Heuvel and Wood [326] proved that $\operatorname{scol}_s(G) \leqslant \operatorname{ltw}(G)(2s+1)$ for every graph G. The *boxicity* $\operatorname{box}(G)$ of a graph G is the minimum integer $d \geqslant 1$, such that G is the intersection graph of axis-aligned boxes in $\mathbb{R}^d$. Thomassen [315] established that planar graphs have boxicity at most 3. Scott and Wood [297] showed that $\operatorname{box}(G) \leqslant 6\operatorname{ltw}(G)+4$ for every graph G. A closer inspection of their proof gives the following bound.

Lemma 5.9 ([297])**.** $\operatorname{box}(G) \leqslant 6\operatorname{ltw}(G) + 3$ *for every graph G.*

At this stage, it is worth comparing the structural properties of layered treewidth to that of row treewidth. For many applications, row treewidth has superseded layered treewidth by giving qualitatively stronger bounds. For example, Lemmas 1.60 and 6.6 imply that graphs with bounded row treewidth have polynomial weak colouring numbers and polynomial p-centred chromatic numbers. In contrast, we now show that there is a graph family with bounded layered treewidth that has super-polynomial weak colouring numbers and super-polynomial p-centred chromatic numbers.

Theorem 5.10. *For all integers $k \geqslant 2$, the class of graphs with layered treewidth at most k has super-polynomial weak colouring numbers and super-polynomial p-centred chromatic numbers.*

Proof. For a graph G and an integer $t \geqslant 0$, let $G^{(t)}$ be the graph obtained from G by subdividing every edge t times. Let $\mathcal{G} = \{G^{(6\operatorname{tw}(G))} : G \text{ is a graph}\}$. Bose et al. [52] showed that every graph in $\mathcal{G}^{(6\operatorname{tw})}$ has layered treewidth at most 2. Grohe et al. [171] and Dubois et al. [111] respectively showed that $\mathcal{G}^{(6\operatorname{tw})}$ has super-polynomial weak colouring numbers and has super-polynomial p-centred chromatic numbers, as required.[2] □

[2] A more careful analysis of [52, 111, 171] in fact shows that the class of graphs with layered treewidth

While row treewidth is qualitatively stronger than layered treewidth, there are nevertheless several applications (in particular, boxicity and strong colouring numbers) for which row treewidth has not been shown to give better bounds than layered treewidth. Moreover, for beyond-planar graphs, our current tools often give much better bounds for layered treewidth than they do for row treewidth (see Section 5.4). For example, using Lemma 5.8 and Theorem 1.33, we show that fan-planar graphs have row treewidth at most 1619 and layered treewidth at most 45 (see Section 5.4.5). As such, we obtain significantly stronger bounds for the boxicity and strong colouring numbers of fan-planar graphs using layered treewidth than via row treewidth. This highlights the value of layered treewidth, especially for beyond-planar graphs.

5.4 Shallow Minors and Beyond-Planar Graphs

This section shows that several beyond-planar graph classes can be described as a shallow minor of the strong product of a planar graph with a small complete graph. In fact, we show the slightly stronger result that they are shallow minors of the lexicographic product of a planar graph with a small edge-less graph. Recall that the *lexicographic product $G_1 \circ G_2$* of two graphs G_1 and G_2 has vertex-set $V(G_1) \times V(G_2)$ and edges of the form $(a, v)(b, u)$ where either $ab \in E(G_1)$, or $a = b$ and $uv \in E(G_2)$. Using Theorem 5.1, we then deduce product structure for these graphs.

5.4.1 Powers of Planar Graphs

Recall that the *k-th power G^k* of a graph G is the graph with vertex-set $V(G^k) := V(G)$ and $uv \in E(G^k)$ if $\text{dist}_G(u, v) \leqslant k$ and $u \neq v$. Dujmović et al. [125] showed that if a graph G has maximum degree Δ, then $G^k = G^{\mathcal{P}}$ for some $(k, 2k\Delta^k)$-shortcut system $\mathcal{P}$.

Huynh and Wood [204] introduced the following generalisation of low-degree squares of graphs: for a graph G and integer $d \geqslant 1$, let $G^{(d)}$ be the graph obtained from G by adding a clique on $N_G(v)$ for each vertex $v \in V(G)$ with $\deg_G(v) \leqslant d$. Huynh and Wood [204] observed that $G^{(d)} = G^{\mathcal{P}}$ where $\mathcal{P}$ is some $(2, d)^\star$-shortcut system.

We consider a further generalisation of squares of graphs. Let $d \in \mathbb{N}$ and G be a graph. For each vertex $v \in V(G)$, let $M_v \subseteq N(v)$ where $|M_v| \leqslant d$ and let $\mathcal{M} := \{M_v : v \in V(G)\}$. Let $G^{\mathcal{M}}$ denote the graph obtained from G by adding the edge uw to G whenever there exists $v \in V(G)$ such that $u \in M_v$ and $w \in N(v)$. We call $\mathcal{M}$ a *d-lift* of G. Clearly $G^{(d)} \subsetneq G^{\mathcal{M}}$ for some d-lift $\mathcal{M}$. So d-lifts generalise low-degree squares of graphs.

Lemma 5.11. *For every graph G and every d-lift $\mathcal{M}$, the graph $G^{\mathcal{M}}$ is a 1-shallow minor of $G \circ \overline{K_{d+1}}$.*

1 has super-polynomial weak colouring numbers and super-polynomial p-centred chromatic numbers. We omit this for simplicity.

Proof. Say $\mathcal{M} = \{M_w : w \in V(G)\}$. For $u \in V(G)$, let $S_u := \{w \in V(G) : u \in M_w\} \subseteq N_G(u)$ and let $\phi_u : M_u \to \{2, \dots, d+1\}$ be an injective function. For each $u \in V(G)$, let $\mu(u)$ be the subgraph of $G \circ \overline{K_{d+1}}$ induced by $\{(u,1)\} \cup \{(w, \phi_w(u)) : w \in S_u\}$. We claim that μ is a 1-shallow model of $G^{\mathcal{M}}$ in $G \circ \overline{K_{d+1}}$.

Let $u, w \in V(G)$ be distinct. First $\mu(u)$ is connected and has radius at most 1 since $S_u \subseteq N_G(v)$. Second, if $S_u \cap S_w = \varnothing$, then $\mu(u)$ and $\mu(w)$ are disjoint. Otherwise, there exists $v \in M_u \cap M_w$. Say $(v, i) \in \mu(u)$ and $(v, j) \in \mu(w)$. Then $i \neq j$ since ϕ_v is injective and so $\mu(u)$ and $\mu(w)$ are vertex-disjoint.

It remains to show that if $uw \in E(G^{\mathcal{M}})$ then $\mu(u)$ and $\mu(w)$ are adjacent. If $uw \in E(G)$, then $\mu(u)$ and $\mu(w)$ are adjacent since $(u,1)(v,1) \in E(G \circ \overline{K_{d+1}})$. Otherwise, there exists $v \in V(G)$ such that $u, w \in N_G(v)$ and $u \in M_v$ or $w \in M_v$. Assume $u \in M_v$. Then $(v, \phi_v(u)) \in V(\mu(u))$ and hence $\mu(u)$ and $\mu(w)$ are adjacent since $(v, \phi_v(u))(w,1) \in E(G \circ \overline{K_{d+1}})$, as required. $\square$

We now apply Lemma 5.11 to obtain the following product structure theorem for d-lifts of planar graphs.

Theorem 5.12. *For every planar graph G and every d-lift $\mathcal{M}$ of G, the graph $G^{\mathcal{M}}$ is contained in $H \boxtimes P \boxtimes K_{27(d+1)}$ for some graph H with treewidth at most 19 and for some path P, and thus* $\operatorname{rtw}(G^{\mathcal{M}}) \leqslant 540(d+1) - 1$.

Proof. By Theorem 1.9, $G \subsetsim J \boxtimes P \boxtimes K_3$ for some graph J with treewidth at most 3 and some path P. By Lemma 5.11, $G^{\mathcal{M}}$ is a 1-shallow minor of $G \circ \overline{K_{d+1}}$. The claim then immediately follows by Theorem 1.33. $\square$

Since clique-lifts generalise squares of graphs, we have the following corollary.

Corollary 5.13. *For every planar graph G with maximum degree Δ, the graph $G^2 \subsetsim J \boxtimes P \boxtimes K_{27(\Delta+1)}$ for some graph J with treewidth at most 19, and thus* $\operatorname{rtw}(G^2) \leqslant 540(\Delta + 1) - 1$.

More generally, we now show that powers of graphs can be described using shallow minors.

Lemma 5.14. *Let G be a graph and $k \in \mathbb{N}$ and $d := \Delta(G^{\lfloor k/2 \rfloor})$. Then G^k is a $\lfloor \frac{k}{2} \rfloor$-shallow minor of $G \circ \overline{K_{d+1}}$.*

Proof. For a vertex $v \in V(G)$, let N_v be the set of vertices in G at distance at most $\lfloor \frac{k}{2} \rfloor$ from v. Then $|N_v| \leqslant d+1$, and $w \in N_v$ if and only if $v \in N_w$. Let $\phi_v : N_v \to [d+1]$ be an injective function. For each vertex $v \in V(G)$, let $\mu(v)$ be the subgraph of $G \circ \overline{K_{d+1}}$ induced by $\{(w, \phi_w(v)) : w \in N_v\}$. It follows that $\mu(v)$ is a connected subgraph of $G \circ \overline{K_{d+1}}$ with radius at most $\lfloor \frac{k}{2} \rfloor$, and $\mu(v) \cap \mu(w) = \varnothing$ for distinct $v, w \in V(G)$.

Consider an edge $uv \in E(G^k)$. Let $P_{uv} = (u = w_0, \dots, w_p = v)$ be a shortest (u,v)-path in G (and thus $p \leqslant k$). Then $\mathrm{dist}_G(u, w_{\lfloor p/2 \rfloor}) \leqslant \lfloor \frac{k}{2} \rfloor$ and $\mathrm{dist}_G(w_{\lfloor p/2 \rfloor+1}, v) \leqslant \lfloor \frac{k}{2} \rfloor$. Thus $(w_{\lfloor p/2 \rfloor}, \phi_{w_{\lfloor p/2 \rfloor}}(u)) \in \mu(u)$ and $(w_{\lfloor p/2 \rfloor+1}, \phi_{w_{\lfloor p/2 \rfloor}+1}(v)) \in \mu(v)$ so

$$(w_{\lfloor p/2 \rfloor}, \phi_{w_{\lfloor p/2 \rfloor}}(u))(w_{\lfloor p/2 \rfloor+1}, \phi_{w_{\lfloor p/2 \rfloor}+1}(v)) \in E(G \circ \overline{K_{d+1}}).$$

Therefore μ is a $\lfloor \frac{k}{2} \rfloor$-shallow model of G^k in $G \circ \overline{K_{d+1}}$. □

Our next theorem implies that for any fixed $k \in \mathbb{N}$, if a graph G has bounded row-treewidth and bounded maximum degree, then G^k also has bounded row-treewidth. Recall that $P^a \subsetneq P \boxtimes K_a$ where P is a path. Applying Lemma 5.14 with Theorem 5.1 by setting $r = \lfloor \frac{k}{2} \rfloor$ and L to be path, we obtain the following.

Theorem 5.15. *Let G be a graph contained in $H \boxtimes P \boxtimes K_\ell$, where H is a graph with $\mathrm{tw}(H) \leqslant t$, P is a path, and $\ell \in \mathbb{N}$. Let $k \in \mathbb{N}$ and $d := \Delta(G^{\lfloor k/2 \rfloor})$. Then $G^k \subsetneq J \boxtimes P \boxtimes K_{\ell(2\lfloor k/2 \rfloor+1)^2(d+1)}$ for some graph J with treewidth at most $\binom{2\lfloor k/2 \rfloor+1+t}{t} - 1$.*

In the case when G is planar, Theorem 5.15 with Theorem 1.33 (and $t = \ell = 3$) implies the following product structure for powers of G.

Theorem 1.23. *Let G be a planar graph. Let $k \in \mathbb{N}$ and $d := \Delta(G^{\lfloor k/2 \rfloor})$. Then G^k is contained in $H \boxtimes P \boxtimes K_{3(2\lfloor k/2 \rfloor+1)^2(d+1)}$ for some graph H with treewidth at most $\binom{2\lfloor k/2 \rfloor+4}{3} - 1$ and for some path P.*

For layered treewidth and boxicity, by applying Lemma 5.14 with Lemmas 5.8 and 5.9, it follows that $\mathrm{ltw}(G^k) \leqslant 3(\Delta(G^{\lfloor k/2 \rfloor})+1)(2k+1)$ and $\mathrm{box}(G^k) \leqslant 18(\Delta(G^{\lfloor k/2 \rfloor})+1)(2k+1)+3$.

5.4.2 (g,k)-Planar Graphs

Recall that a graph G is *(g,k)-planar* if, for some surface Σ with Euler genus at most g, G has a drawing on Σ such that each edge is involved in at most k crossings.

Dujmović et al. [125] observed that every (g,k)-planar graph is a subgraph of $G^{\mathcal{P}}$ for some graph G with Euler genus at most g and some $(k+1,2)$-shortcut system $\mathcal{P}$. Thus, by Lemma 5.4, every (g,k)-planar graph is a $\frac{k}{2}$-shallow topological minor of $G \circ \overline{K_3}$ where G has Euler genus at most g. We obtain a slightly stronger bound using the standard planarisation method.

Lemma 5.16. *Every (g,k)-planar graph G is a $\frac{k}{2}$-shallow topological minor of $H \circ \overline{K_2}$ where H has Euler genus at most g.*

Proof. Draw G into a surface Σ with Euler genus at most g such that every edge of G is involved in at most k crossings. For each crossing $(p, \{uv, xy\})$ where $uv, xy \in E(G)$, insert a dummy vertex w at p to obtain a graph H with Euler genus at most g. Let

$W := V(H) \setminus V(G)$ be the set of dummy vertices. Each $uv \in E(G)$ corresponds to a path P_{uv} with length at most k in H where each internal vertex of P is a dummy vertex. Let $\mathcal{P}$ be the set of such paths. For each $w \in W$, let M_w be the set of paths in $\mathcal{P}$ that contain w as an internal vertex. Then $|M_w| \leqslant 2$. Let $\phi_w : M_w \to \{1,2\}$ be an injective function. For each $v \in V(G)$, let $\phi_v(P) := 1$ for all paths P_{uv} with end-vertex v. For each path $P_{uv} = (u = w_0, w_1, \dots, w_{j-1}, w_j = v) \in \mathcal{P}$ where $j \leqslant k+1$, let $\tilde{P}_{uv}$ be the path in $H \circ \overline{K_2}$ defined by $V(\tilde{P}_{uv}) := \{(w_i, \phi_{w_i}(P_{uv})) : i \in [0,j]\}$ and $E(\tilde{P}_{uv}) := \{(w_i, \phi_{w_i}(P_{uv}))(w_{i+1}, \phi_{w_{i+1}}(P_{uv})) : i \in [0, j-1]\}$. Let $\tilde{\mathcal{P}}$ be the set of such $\tilde{P}_{uv}$ paths.

We claim that $\tilde{\mathcal{P}}$ defines a $\frac{k}{2}$-shallow topological minor of G in $H \circ \overline{K_2}$ where each vertex $v \in V(G)$ is mapped to $(v,1) \in V(H \circ \overline{K_2})$ and each edge $uv \in E(G)$ is mapped to $\tilde{P}_{uv} \in \tilde{\mathcal{P}}$.

Let $uv \in E(G)$. Then there is a path $P_{uv} \in \mathcal{P}$ with length at most $k+1$ and end-vertices u and v. By construction, $\tilde{P}_{uv}$ is a path in $H \circ \overline{K_2}$ from $(u,1)$ to $(v,1)$ with length at most k. Thus, it suffice to show that the paths in $\tilde{\mathcal{P}}$ are internally disjoint. Now for a path $\tilde{P}_{uv} \in \tilde{\mathcal{P}}$, its internal vertices are of the form $(w, \phi_w(P_{uv}))$ where $w \in W$. Now if there is another path $\tilde{Q}_{xy} \in \tilde{\mathcal{P}}$ for which $(w, \phi_w(P_{uv}))$ is an internal vertex of $\tilde{Q}_{xy}$, then $\tilde{P}_{uv} = \tilde{Q}_{xy}$ since ϕ_w is injective. As such, the paths in $\tilde{\mathcal{P}}$ are internally disjoint, as required. □

Applying Lemma 5.16 with Theorems 1.33 and 1.14, we obtain the following product structure theorems for (g,k)-planar graphs.

Theorem 5.17. *Every (g,k)-planar graph G is contained in $H \boxtimes P \boxtimes K_{2\max\{2g,3\}(k+1)^2}$ for some graph H with treewidth at most $\binom{k+4}{3} - 1$, and thus G has row treewidth at most $2\max\{2g,3\}(k+1)^2\binom{k+4}{3} - 1$.*

Note that Dujmović et al. [125] proved that every (g,k)-planar graph is a subgraph of $H \boxtimes P \boxtimes K_{\max\{2g,3\}(6k^2+16k+10)}$, for some graph H of treewidth at most $\binom{k+4}{3} - 1$. Thus, our results only improve those of Dujmović et al. [125] by a constant factor.

5.4.3 String Graphs

Recall the following definitions. A *string graph* is the intersection graph of a set of curves in the plane with no three curves meeting at a single point. For an integer $\delta \geqslant 2$, if each curve is involved in at most δ intersections with other curves, then the corresponding string graph is called a *δ-string graph*. A *(g,δ)-string graph* is defined analogously for curves on a surface with Euler genus at most g.

Dujmović et al. [125] showed that every (g,δ)-string graph G is a subgraph of $H^{\mathcal{P}}$ for some graph H with Euler genus at most g and some $(\delta+1, \delta+1)$-shortcut system $\mathcal{P}$. By Lemma 5.4, G is a $\frac{\delta}{2}$-shallow topological minor of $H \circ \overline{K}_{\delta+2}$. We obtain a stronger bound by the standard planarisation method.

Lemma 5.18. *Every (g, δ)-string graph G is a $\lfloor \frac{\delta}{2} \rfloor$-shallow minor of $H \circ \overline{K_2}$ for some graph H with Euler genus at most g.*

Proof. Let $\mathcal{C} := \{C_v : v \in V(G)\}$ be a set of curves in a surface of Euler genus at most g such that each curve is involved in at most δ intersections with other curves and the intersection graph is isomorphic to G. For each pair of curves in $\mathcal{C}$ that intersects, add a vertex at their intersection point. Add an edge between each pair of consecutive vertices along a curve in $\mathcal{C}$. If a curve C_v is involved in exactly one crossing, add another vertex c_v to the curve adjacent to the intersection point that C_v contains. If a curve C_v is not involved in any crossing, add a vertex c_v to the curve. Let H be the planar graph obtained by this process. For each vertex $v \in V(G)$, let M_v be the set of vertices in H that are on the curve C_v. For each $w \in V(H)$, let $N_w := \{v \in V(G) \colon w \in M_v\}$. Then $|N_w| \leqslant 2$ for all $w \in V(H)$.

Observe that $H[M_v]$ has radius at most $\lfloor \frac{\delta}{2} \rfloor$ for each vertex $v \in V(G)$. For each $w \in V(H)$, let $\phi_w : N_w \to \{1, 2\}$ be an injective function. For each vertex $v \in V(G)$, let $\mu(v)$ be the subgraph of $H \circ \overline{K_2}$ induced by $\{(w, \phi_w(v)) : w \in M_v\}$.

We claim that μ defines a $\lfloor \frac{\delta}{2} \rfloor$-shallow model of G in $H \circ \overline{K_2}$.

Let $u, v \in V(G)$ be distinct. First, since $H[M_v]$ has radius at most $\lfloor \frac{\delta}{2} \rfloor$ and is connected, so is $\mu(u)$. Second, if u and v do not intersect, then $\mu(u)$ and $\mu(v)$ are disjoint. Otherwise, there exists $w \in C_u \cap C_v$. In which case, $\phi_w(u) \neq \phi_w(v)$ since ϕ_w is injective and hence $\mu(u)$ and $\mu(v)$ are vertex-disjoint. Finally, if $uv \in E(G)$ then there is a vertex $w_1 \in M_u \cap M_v$. Let $w_2 \in M_v$ be a neighbour of w_1. Then $(w_1, \phi_{w_1}(u))(w_2, \phi_{w_2}(v)) \in E(H \circ \overline{K_2})$ so $\mu(u)$ and $\mu(v)$ are adjacent. Therefore μ defines a $\lfloor \frac{\delta}{2} \rfloor$-shallow model of G in $H \circ \overline{K_2}$. □

Applying Lemma 5.18 in conjunction with Theorems 1.33 and 1.14, we deduce the following product structure theorem for string graphs.

Theorem 5.19. *Every (g, δ)-string graph G is contained in $H \boxtimes P \boxtimes K_{2\max\{2g,3\}(\delta+1)^2}$ for some graph H with treewidth at most $\binom{2\lfloor \delta/2 \rfloor + 4}{3} - 1$ and for some path P, and thus G has row treewidth at most $2\max\{2g, 3\}(\delta + 1)^2 \binom{2\lfloor \delta/2 \rfloor + 4}{3} - 1$.*

Note that Dujmović et al. [125] previously showed that every (g, δ)-string graph is contained in $H \boxtimes P \boxtimes K_{\max\{2g,3\}(\delta^4+4\delta^3+9\delta^2+10\delta+4)}$ for some graph H with treewidth at most $\binom{\delta+4}{3} - 1$. Thus, Theorem 5.19 improves this result by a factor of δ^2 in the K_ℓ term.

5.4.4 (k, p)-Cluster Planar Graphs

Di Giacomo et al. [96] introduced the following class of beyond-planar graphs. A graph G is *(k, p)-cluster planar*[3] if $V(G)$ can be partitioned into clusters $C_1, \ldots, C_n$ where $|C_i| \leqslant k$

[3] Note that Di Giacomo et al. [96] called G a *(k, p)-planar graph*. We use the language of (k, p)-cluster planar graphs to avoid confusion with (g, k)-planar graphs.

for all $i \in [n]$ such that G admits a drawing, called a *(k,p)-cluster planar representation*, where:

1. each cluster C_i is associated with a closed, bounded planar region R_i, called a *cluster region*;
2. cluster regions are pairwise disjoint;
3. each vertex $v \in V(G)$ is identified with at most p distinct points, called *ports*, on the boundary of its cluster region;
4. each inter-cluster edge $uv \in E(G)$ is a curve that joins a port of u to a port of v; and
5. inter-cluster edges do not cross or intersect cluster regions except at their endpoints.

We now show that (k,p)-cluster planar graphs have a simple product structure with no dependency on p.

Lemma 5.20. *Every (k,p)-cluster planar graph G is contained in $H \boxtimes K_k$ for some planar graph H.*

Proof. Begin with a (k,p)-cluster planar representation of G. Replace each cluster region R_i by a vertex x_i such that x_i and x_j are adjacent if there is an edge joining a port on the boundary of R_i to a port on the boundary of R_j. In doing so, we obtain a plane graph H (see Figure 5.3). For each cluster C_i, let $\phi_i : C_i \to [k]$ be an injective function. For each cluster C_i and vertex $v \in C_i$ let $\phi(v) = (x_i, \phi_i(v)) \in V(H \boxtimes K_k)$. Observe that ϕ is an injective function whose domain is $V(G)$.

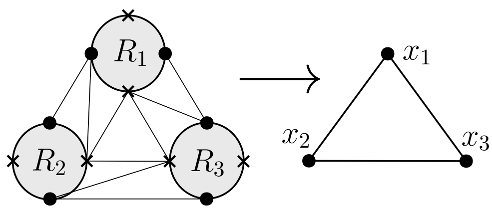


Figure 5.3. Planarising a (k,p)-cluster planar graph.

We claim that $\phi(u)\phi(v) \in E(H \boxtimes K_k)$ whenever $uv \in E(G)$. First, if $u, v \in C_i$ then $\phi(u)\phi(v) = (x_i, \phi_i(u))(x_i, \phi_i(v)) \in E(H \boxtimes K_k)$. Otherwise, if $u \in C_i$ and $v \in C_j$ where $i \neq j$, then $x_i x_j \in E(H)$ since there is an inter-cluster edge from C_i to C_j. As such, $\phi(u)\phi(v) = (x_i, \phi_i(u))(x_j, \phi_j(v)) \in E(H \boxtimes K_k)$. Thus $G \subsetneq H \boxtimes K_k$, as required. $\square$

Since (k,p)-cluster planar graphs have a product structure, the results from Section 1.3.8 apply to this class of graphs. We omit the details.

We highlight the following consequence of Lemma 5.20. A graph is *IC-planar* if it has a drawing in the planar such that every edge is involved in at most one crossing and the set of all edges that cross form a matching. Di Giacomo et al. [96] observed that the class of IC-planar graphs correspond to $(4,1)$-cluster planar graphs. So by Lemma 5.20, every IC-planar graph G is contained in $H \boxtimes K_4$ for some planar graph H.

5.4.5 Fan-Planar Graphs

Recall that a graph G is *fan-planar* if it has a drawing where for each edge $e \in E(G)$, the edges that cross e have a common end-vertex and they cross e from the same side (when directed away from their common end-vertex). Equivalently, G is fan-planar if it has a drawing that forbids the two crossing configurations in Figure 5.4.[4] The class of fan-planar graphs extends 1-planar graphs and is a proper subset of 3-quasi planar graphs.

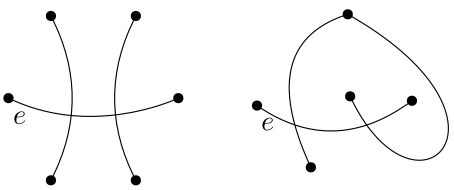


Figure 5.4. Forbidden crossing configurations for fan-planar graphs.

We now work towards showing that fan-planar graphs have a shallow minor structure. Note that compared to the other graph classes that we consider in this chapter, the proof for this result is much more involved and is highly non-trivial.

For a drawing of a fan-planar graph G and a crossed edge $uv \in E(G)$, a common end-vertex of the edges that cross uv is called a *friend* of uv. If uv is crossed at least twice, then it has one friend; otherwise, uv is crossed once and has two friends. A *friend assignment* of G assigns a friend to each crossed edge. For a given friend assignment, a crossed edge uv is *well-behaved* if there exists a non-crossing point $p \in uv$ such that u is the assigned friend of each edge that crosses uv between p and u, and v is the assigned friend of each edge that crosses uv between p and v. If every crossed edge in G is well-behaved, then the friend assignment is *well-behaved*.

Lemma 5.21. *Every simple fan-planar graph G has a well-behaved friend assignment.*

Proof. Let $uv \in E(G)$ be a crossed edge and let $e_1, \dots, e_m$ be the edges that cross uv ordered by their crossing point from u to v. Let w be a common end-vertex of $e_1, \dots, e_m$. If none of $e_1, \dots, e_m$ cross another edge incident to v, then let u be the assigned friend of $e_1, \dots, e_m$ and choose p to be an arbitrary point (along uv) between $e_m \cap uv$ and v. Otherwise, let $i \in [m]$ be minimum such that e_i crosses another edge vz incident to v. Choose p to be an arbitrary point on uv between $uv \cap e_{i-1}$ and $uv \cap e_i$. Since none of the edges $e_1, \dots, e_{i-1}$ cross another edge incident to v, we may let u be their assigned friend.

It remains to show that for each $j \in [i, m]$, v is a common end-vertex of the edges that cross e_j and thus we may let v be the assigned friend of e_j. If e_j is only crossed by uv, then we are done. If e_j crosses vz then v is e_j's only friend. Otherwise, the end-vertex of e_j opposite to w is contained in the region bounded by the edges uv, vz, e_i (since G is

[4]Note that there is another notion of *fan-planarity* in the literature which includes an additional excluded crossing configuration (see [79, 81]). Our results hold for both understandings of fan-planarity.

simple fan-planar). Now suppose that another edge $\tilde{e}$ incident to u crosses e_j. Since $\tilde{e}$ does not cross e_i or vz (as Configuration I is forbidden), $\tilde{e}$ must cross e_j from the opposite side that uv cross e_j, contradicting G being fan-planar (see Figure 5.5). Thus uv is the only edge incident to u that crosses e_j and so v is a common end-vertex of the edges that cross e_j, as required. □

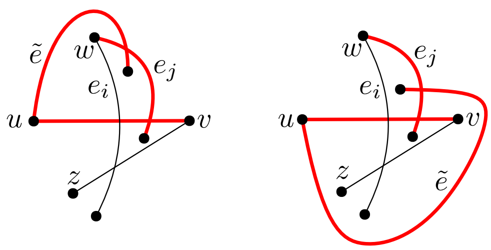


Figure 5.5. Proof of Lemma 5.21.

We now prove our main technical lemma of this subsection.

Lemma 5.22. *Every fan-planar graph G is a 1-shallow minor of $H \circ \overline{K_3}$ for some planar graph H.*

Proof. By [219, Theorem 1] and Lemma 5.21, we may assume that G has a simple fan-planar drawing with a well-behaved friend assignment. Initialise $G^{(0)} := G$. Arbitrarily choose an edge $u_1v_1 \in E(G^{(0)})$ that is involved in at least two crossings and let $w_1 \in V(G^{(0)})$ be the assigned friend of u_1v_1. If no such edge exists, then G is 1-planar and we are done by Lemma 5.16. Let E_1 be a maximal subset of $E(G^{(0)})$ such that:

- $u_1v_1 \in E_1$;
- each edge in E_1 has u_1, v_1 or w_1 as its assigned friend; and
- $E_1 \setminus V(G^{(0)})$ is a connected subset of the plane.

Observe that every edge in E_1 crosses another edge in E_1 and thus every edge in E_1 is incident to u_1, v_1 or w_1.

Let V_1 be the set of vertices incident to edges in E_1. Consider the subgraph of $G^{(0)}$ with vertex-set V_1 and edge-set E_1. At each crossing point, add a dummy vertex to planarise the subgraph. Let $\tilde{G}_1$ be the plane graph obtained and let $\tilde{D}_1$ be the set of dummy vertices added. Let $G'_1 := \tilde{G}_1 \cup (G^{(0)} - E_1)$; see Figure 5.6.

Claim 1: If an edge $e \in E(\tilde{G}_1)$ is crossed in G'_1, then e is incident to some $z \in V_1 \setminus \{u_1, v_1, w_1\}$.

Proof. Since $\tilde{G}_1$ is plane, e is crossed by some edge $e' \in E(G^{(0)}) \setminus E_1$. Now e is a segment of some edge $xz \in E_1$ where $x \in \{u_1, v_1, w_1\}$ and $z \in V_1$. For the sake of contradiction, suppose that e is not incident to any vertex in $V_1 \setminus \{u_1, v_1, w_1\}$. Then there exists an edge $\tilde{e} \in E_1$ and a dummy vertex $d = (xz \cap \tilde{e}) \in \tilde{D}_1$ such that e is between x and d (along xz). Now if the assigned friend of $\tilde{e}$ is x, then the assigned friend of e' is also x since xz is well-behaved. Otherwise, if the assigned friend of $\tilde{e}$ is z then $z \in \{u_1, v_1, w_1\} \setminus \{x\}$. Thus, the

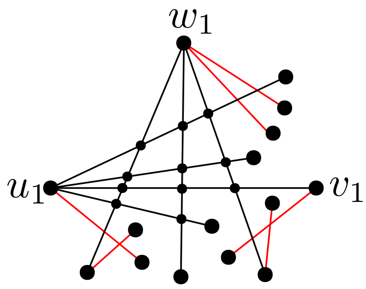


Figure 5.6. The graph $G_1' := \tilde{G}_1 \cup (G^{(0)} - E_1)$ where black edges are from $E(\tilde{G}_1)$ and red edges are from $E(G^{(0)}) - E_1$.

assigned friend of e' is in $\{x, z\} \subseteq \{u_1, v_1, w_1\}$. However, this contradicts the maximality of E_1 since e' crosses an edge in E_1 and the assigned friend of e' is in $\{u_1, v_1, w_1\}$, as required. $\square$

For $\varepsilon > \delta > 0$, for every vertex $x \in V(G_1')$ and edge $uv \in E(G_1')$, let

$$B_x := \{p \in \mathbb{R}^2 : \mathrm{dist}_{\mathbb{R}^2}(p, x) \leqslant \varepsilon\} \quad \text{and} \quad C_{uv} := \{p \in \mathbb{R}^2 : \mathrm{dist}_{\mathbb{R}^2}(p, uv) \leqslant \delta\} \setminus (B_u \cup B_v).$$

Choosing ε and δ to be sufficiently small, we may assume that $B_x \cap B_y = \varnothing$, $B_x \cap C_{uv} = \varnothing$, and $C_{uv} \cap C_{ab} = \varnothing$ for all $x, y \in V(G_1')$ and pairwise non-crossing edges $ab, uv \in E(G_1')$. Let $T^{(1)}$ be a spanning tree of $\tilde{G}_1$ rooted at some $d_1 \in \tilde{D}_1$. Using a standard blow-up trick (e.g. see [204, Lemma 5.5]), we may replace $T^{(1)}$ by a subdivided star $S^{(1)}$ drawn in the region $(\bigcup_{x \in V(\tilde{G}_1)} B_x) \cup (\bigcup_{uv \in E(\tilde{G}_1)} C_{uv})$ where each root to leaf path in $S^{(1)}$ corresponds to a root to leaf path in $T^{(1)}$; see Figure 5.7.

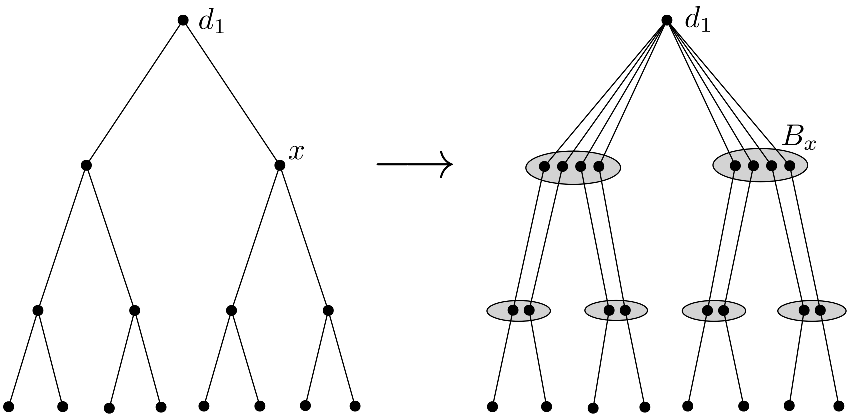


Figure 5.7. Replacing a tree by a subdivided star.

Let $G^{(1)}$ be the graph obtained from G_1' by replacing $\tilde{G}_1$ by $S^{(1)}$ then removing the subdivision vertices in $S^{(1)}$; see Figure 5.8. Charge u_1, v_1 and w_1 to d_1. By Claim 1, for every crossed edge of the form $d_1 z \in E(G^{(1)})$ where $z \in V_1$, there exists an edge $xz \in E_1$ where $x \in \{u_1, v_1, w_1\}$ such that every edge that cross $d_1 z$ (in $G^{(1)}$) also crosses xz (in $G^{(0)}$) in the same direction.

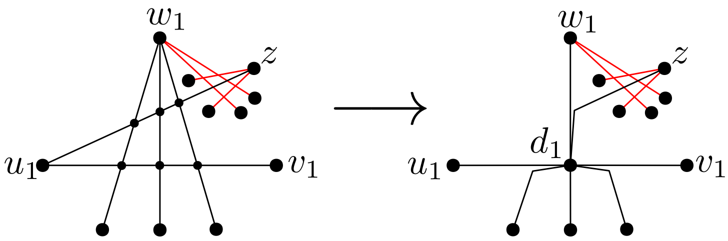


Figure 5.8. Obtaining $G^{(1)}$ from G_1'.

Claim 2: $G^{(1)}$ is simple fan-planar and there is a well-behaved friend assignment of $G^{(1)}$ where d_1 is not an assigned friend of any edge.

Proof. We first show that $G^{(1)}$ is simple fan-planar. Let $e \in E(G^{(1)})$ be a crossed edge and let A_e be the set of edges in $E(G^{(1)})$ that cross e. If $\{e\} \cup A_e \subseteq E(G^{(0)})$, then Configuration I and Configuration II are forbidden and e does not cross any edge at least twice since $G^{(0)}$ is simple fan-planar. So assume that $(\{e\} \cup A_e) \cap (E(G^{(1)}) \setminus E(G^{(0)})) \neq \varnothing$. There are two cases to consider.

First, suppose $e = d_1 z$ for some $z \in V_1$. Since the edges in $E(G^{(1)}) \setminus E(G^{(0)})$ do not pairwise cross, it follows that $A_e \subseteq E(G^{(0)})$ and e does not cross any edge incident to d_1. By Claim 1, there is an edge $xz \in E_1$ where $x \in \{u_1, v_1, w_1\}$ such that every edge that crosses e (in $G^{(1)}$) also crosses xz (in $G^{(0)}$) in the same direction. Thus the edges that cross e have a common end-vertex and they all cross e in the same direction and e does not cross any edge more than once or any edge incident to z.

Second, suppose $e \in E(G^{(1)}) \cap E(G^{(0)})$ and $A_e \cap (E(G^{(1)}) \setminus E(G^{(0)})) \neq \varnothing$. Let y be the assigned friend of e in $G^{(0)}$. Since e crosses an edge in $E(G^{(1)}) \setminus E(G^{(0)})$ and E_1 is maximal, it follows that $y \in V_1 \setminus \{u_1, v_1, w_1\}$. Thus $A_e \cap (E(G^{(1)}) \setminus E(G^{(0)})) = \{d_1 y\}$ since $y \notin \{u_1, v_1, w_1\}$. Furthermore, every edge in $A_e \cap E(G^{(0)})$ is incident to y since y is the friend of e in $G^{(0)}$. As such, every edge that crosses e (in $G^{(1)}$) is incident to y. By Claim 1, there is an edge $xy \in E_1$ where $x \in \{u_1, v_1, w_1\}$ such that every edge that crosses $d_1 y$ (in $G^{(1)}$) crosses xy (in $G^{(0)}$) in the same direction. Thus Configuration I and II are forbidden and e does not cross any edge more than once since $G^{(0)}$ is simple fan-planar, as required.

We now specify the friend assignment. By Lemma 5.21, G has a well-behaved friend assignment since it is simple fan-planar. Now suppose d_1 is the assigned friend of some edge e. Since the edges $u_1 d_1$, $v_1 d_1$ and $w_1 d_1$ are uncrossed in $G^{(1)}$, e crosses an edge of the form $d_1 y$ where $y \in V_1 \setminus \{u_1, v_1, w_1\}$. By an analogous argument to that in the previous paragraph, every edge that crosses e in $G^{(1)}$ is also incident to y, and so we may modify the friend assignment such that y is the assigned friend of e. By doing so for every such edge, we obtain the desired friend assignment. $\square$

Observe the following properties of $G^{(1)}$:

- edges that are uncrossed in $G^{(0)}$ remain uncrossed in $G^{(1)}$;

- the edges u_1d_1, v_1d_1 and w_1d_1 are uncrossed in $G^{(1)}$; and
- $G^{(1)}$ has less crossings than $G^{(0)}$.

Let $(u_1, v_1, w_1, E_1, d_1, G^{(1)}), (u_2, v_2, w_2, E_2, d_2, G^{(2)}), \dots, (u_m, v_m, w_m, E_m, d_m, G^{(m)})$ be the sequence obtained by iterating the above procedure where $u_iv_i \in E(G^{(i-1)})$ is an arbitrary edge crossed at least twice in $G^{(i-1)}$, d_i is not an assigned friend of any edge in $G^{(j)}$ for all $i \leqslant j$ and each edge in $G^{(m)}$ is crossed at most once. Then $G^{(m)}$ is 1-planar. Note that this sequence is well-defined since $G^{(i)}$ has less crossings than $G^{(i-1)}$.

Add a dummy vertex at each crossing point in $G^{(m)}$ and let H be the plane graph obtained. For every edge of the form $d_iz \in E(G^{(m)})$ that is crossed by another edge e, charge z as well as the end-vertices of e to the dummy vertex at $d_iz \cap e$. For crossed edges of the form $e', \tilde{e} \in E(G) \cap E(G^{(m)})$, charge an end-vertex of e' and an end-vertex of $\tilde{e}$ to the dummy vertex at $e' \cap \tilde{e}$.

Let $D := V(H) \setminus V(G)$ be the set of dummy vertices in H. Observe that every dummy vertex $d_i \in D$ has at most three vertices charged to it, u_i, v_i, w_i, and that these vertices are neighbours of d_i in H. In addition, the original edges in G have the following key properties:

Claim 3: If $xy \in E(G)$ then either:

Case 1. $xy \in E(H)$;
Case 2. there is a path $P_{xy} = (x, d, y)$ in H where x or y is charged to d and $d \in D$; or
Case 3. there is a path $P_{xy} = (x, d_1, d_2, y)$ in H where x is charged to d_1 and y is charged to d_2 and $d_1, d_2 \in D$.

Proof. First, consider an edge $xy \in E(G^{(m)}) \cap E(G)$. If xy is uncrossed in $G^{(m)}$, then $xy \in E(H)$ and we are in Case 1. Otherwise xy is involve in a single crossing in $G^{(m)}$ and so we are in Case 2. So assume $xy \in E_i$ for some $i \in [m]$. Since every edge in E_i is incident to u_i, v_i or w_i, we may assume that $x \in \{u_i, v_i, w_i\}$. Thus x is charged to d_i and $d_ix \in E(H)$. If $d_iy \in E(G^{(m)})$ and it is uncrossed, then we are in Case 2. If $d_iy \in E(G^{(m)})$ and it is crossed, then we are in Case 3. So it remains to consider the case when $d_iy \in E_j$ for some $j \in [i+1, m]$. Since d_i is not the assigned friend of any edge in $G^{(j)}$, it follows that $y \in \{u_j, v_j, w_j\}$. Thus every edge e that crosses d_iy is contained in E_j as y is the assigned friend of e. As such, after planarising E_j (by replacing it with a star rooted at d_j), y will be charged to d_j and the edges d_id_j and d_jy will be uncrossed in $G^{(j)}$, and hence we are in Case 3. $\square$

To finish the proof, we now specify the model μ for the 1-shallow minor of G in $H \circ \overline{K_3}$. For each dummy vertex $d \in D$, let $\phi_d : N_d \to \{1, 2, 3\}$ be an injective function where N_d is the set of vertices that are charged to d. For a vertex $u \in V(G)$, let M_u be the set of dummy vertices that u is charged to. For each $u \in V(G)$, let $\mu(u)$ be the subgraph of $H \circ \overline{K_3}$ induced by $\{(u, 1)\} \cup \{(d, \phi_d(u)) : d \in M_u\}$.

Let $u, v \in V(G)$ be distinct. First $\mu(u)$ is connected and has radius at most 1 since $ud \in E(H)$ for all $d \in M_u$. Second, if $M_u \cap M_v = \varnothing$, then clearly $\mu(u)$ and $\mu(v)$ are disjoint. Otherwise, if there exists some $d \in M_u \cap M_v$, then $\phi_d(u) \neq \phi_d(v)$ since ϕ_d is injective. Thus, $\mu(u)$ and $\mu(v)$ are vertex-disjoint. Finally, Claim 3 implies that if $uv \in E(G)$ then $\mu(u)$ and $\mu(v)$ are adjacent. Therefore μ defines a 1-shallow model of G in $H \circ \overline{K_3}$. □

Applying Lemma 5.22 with Theorems 1.33 and 1.9, we obtain the following product structure theorem for fan-planar graphs.

Theorem 1.26. *Every fan-planar graph G is contained in $H \boxtimes P \boxtimes K_{81}$ for some graph H with treewidth at most 19, and thus G has row treewidth at most 1619.*

For layered treewidth and boxicity, by applying Lemma 5.22 with Lemmas 5.8 and 5.9, it follows that $\operatorname{ltw}(G) \leqslant 45$ and $\operatorname{box}(G) \leqslant 276$ for every fan-planar graph G.

We now show that fan-planar graphs cannot be described by applying a shortcut system to a planar graph.

Binucci et al. [35, Theorem 3] proved that for every integer $k \geqslant 1$, there is a fan-planar graph that is not k-planar. Essentially the same proof gives the following, stronger result.

Proposition 5.23. *For every integer $k \geqslant 1$, there is a fan-planar graph that is not k-gap planar.*

Proof. For each $h \in \mathbb{N}$, the complete tripartite graph $K_{1,3,h}$ is fan-planar and $|E((K_{1,3,h}))| = 4h+3$; see Figure 5.9. Moreover, it is known [18, Theorem 1] that $\operatorname{cr}(K_{1,3,h}) = \Omega(h^2) = \Omega(|E((K_{1,3,h}))|^2)$. Thus, this graph family has super-linear crossing number and so it is not k-gap planar. □

Lemma 5.3 and Proposition 5.23 imply that fan-planar graphs cannot be described by shortcut systems applied to a planar graph. This highlights the value and power of shallow minors.

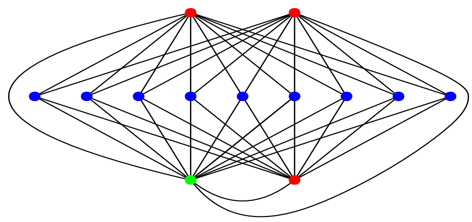


Figure 5.9. A fan-planar drawing of $K_{1,3,h}$.

As a converse to Proposition 5.23, Binucci et al. [35, Theorem 4] showed that there exists a 2-planar graph that is not fan-planar. Since every 2-planar graph is 1-gap planar graph [22, Theorem 7], we have the following.

Proposition 5.24. *For every integer $k \geqslant 1$, there is a k-gap planar graph that is not fan-planar.*

Proposition 5.23 and Proposition 5.24 demonstrate that for any integer $k \geqslant 1$, the class of fan-planar graphs and k-gap planar graphs are incomparable.[5] This resolves an open problem of Didimo et al. [99, Problem 12] who asked for the relationship between k-gap planar graphs and fan-planar graphs.

[5]Graph classes $\mathcal{G}_1$ and $\mathcal{G}_2$ are *incomparable* if $\mathcal{G}_1 \setminus \mathcal{G}_2 \neq \varnothing$ and $\mathcal{G}_2 \setminus \mathcal{G}_1 \neq \varnothing$.

5.4.6 k-Fan-Bundle Planar Graphs

A *fan* is a set of edges incident to a common end-vertex. In a k-fan-bundle drawing of a graph in the plane the edges of a fan may be bundled together at their end vertices and crossings between bundles are allowed as long as each bundle is crossed by at most k other bundles. More formally, in a k-fan-bundle planar drawing of a graph G, each edge has three parts; the first and the last parts are *fan-bundles*, which may be shared by several edges in a fan, while the middle part is unbundled. Each fan-bundle can cross at most k other fan-bundles, while the unbundled parts are crossing-free. A vertex $u \in V(G)$ can be incident to more than one bundle. Let B_u be one such bundle. We say that B_u is *anchored* at u which is the *origin* of B_u. The endpoint of B_u that is different from u is the *terminal* of B_u. A graph is *k-fan-bundle planar* if it admits a *k-fan-bundle planar drawing*. Fan bundle planar graphs were introduced by Angelini et al. [15] where they studied their density and algorithmic properties. Our results are the first to consider these graphs from a structural perspective.

Lemma 5.25. *Every k-fan-bundle planar graph G is a $(k+1)$-shallow minor of $H \circ \overline{K_2}$ for some planar graph H.*

Proof. Begin with a k-fan-bundle-planar drawing of G. For each bundle $B_u^{(i)}$, add a dummy vertex at the terminal of $B_u^{(i)}$ to obtain a k-planar graph H'. Let W be the set of dummy vertices added. Replace each crossing in H' by a dummy vertex to obtain a plane graph H (see Figure 5.10) and let D be the new dummy vertices that are added at this step. In doing so, each bundle $B_u^{(i)}$ is replaced by a (u, w)-path $P_u^{(i)}$ on at most $k+2$ vertices for some dummy vertex $w \in W$. For each $d \in D$, let $\phi_d : \{u, v\} \to \{1, 2\}$ be an injective function where u and v are the origins of the bundles that cross at d. For each $u \in V(G)$, let $W_u := (\bigcup(V(P_u^{(i)}) \colon i \in [c_u]))) \cap W$ and $D_u := (\bigcup(V(P_u^{(i)}) \colon i \in [c_u]) \cap D$ where c_u is the number of bundles anchored at u. For each $u \in V(G)$, let $\mu(u)$ be the subgraph of $H \circ \overline{K_2}$ induced by $\{(u,1)\} \cup \{(w,1) : w \in W_u\} \cup \{(d, \phi_w(d)) : d \in D_u\}$.

We claim that μ is a $(k+1)$-shallow model of G in $H \circ \overline{K_2}$. Let $u, v \in V(G)$ be distinct. First, $\mu(u)$ is connected with radius at most $k+1$ as it is the union of paths on at most $k+2$ vertices that share a common end-vertex. Now if no bundle anchored at u crosses a bundle anchored at v, then $\mu(u)$ and $\mu(v)$ are clearly disjoint. Otherwise, if there is a bundle $B_u^{(i)}$ that crosses a bundle $B_v^{(i')}$, then there is some $d \in D_u \cap D_v$. In which case, $\phi_d(u) \neq \phi_d(v)$ since ϕ_d is injective and so $\mu(u)$ and $\mu(v)$ are vertex-disjoint. Now if $uv \in E(G)$, then there is a bundle $B_u^{(i)}$ anchored at u and a bundle $B_v^{(i')}$ anchored at v that are adjacent. Let $w_1, w_2 \in W$ be the dummy vertices that are respectively added to the terminals of $B_u^{(i)}$ and $B_v^{(i')}$. Then $(w_1, 1) \in \mu(u)$ and $(w_2, 1) \in \mu(v)$ so $(w_1, 1)(w_2, 1) \in E(H \circ \overline{K_2})$ and thus

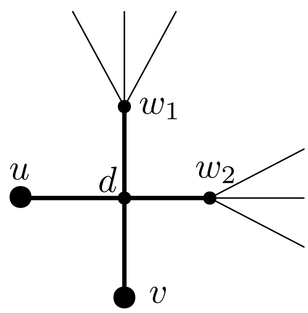


Figure 5.10. Planarising a 1-fan-bundle planar graph.

$\mu(u)$ and $\mu(v)$ are adjacent, as required. □

Applying Lemma 5.25 with Theorems 1.33 and 1.9, we obtain the following product structure theorem for k-fan-bundle planar graphs.

Theorem 1.27. *Every k-fan-bundle planar graph G is contained in $H \boxtimes P \boxtimes K_{6(2k+3)^2}$ for some graph H with treewidth at most $\binom{2k+6}{3} - 1$ and some path P, and thus G has row treewidth at most $\binom{2k+6}{3} 6(2k+3)^2 - 1$.*

For layered treewidth and boxicity, it follows from Lemmas 5.25, 5.8 and 5.9, that $\operatorname{ltw}(G) \leqslant 24k + 25$ and $\operatorname{box}(G) \leqslant 144k + 154$ for every k-fan-bundle planar graph G.

We raise the following open problems concerning k-fan-bundle graphs. Angelini et al. [15] showed that 1-fan-bundle graphs are incomparable with 2-planar graphs. What is the relationship between 1-fan-bundle graphs and k-planar graphs? Does there exist an integer $k \geqslant 1$ such that every 1-fan bundle graph is k-planar? Or more weaker, does there exists an integer $k \geqslant 1$ such that every 1-fan bundle graph is k-gap planar?

5.5 Lower Bounds

Having described several beyond-planar graph classes as shallow minors of the strong product of a planar graph with a small complete graph, we now give examples of classes that cannot be described in this manner.

Recall that a hereditary graph class $\mathcal{G}$ has *bounded expansion* with *expansion function* $f_{\mathcal{G}} : \mathbb{N} \cup \{0\} \to \mathbb{R}$ if $\nabla_r(G) \leqslant f_{\mathcal{G}}(r)$ for every $r \geqslant 0$ and graph $G \in \mathcal{G}$. If $f_{\mathcal{G}}$ is polynomial, then $\mathcal{G}$ has *polynomial expansion*. A hereditary graph class $\mathcal{G}$ is *somewhere dense* if there exists an integer $r \geqslant 0$ such that every graph H is an r-shallow minor of some graph $G \in \mathcal{G}$. If $\mathcal{G}$ is not somewhere dense, then it is *nowhere dense*.

5.5.1 k-Gap Planar Graphs

Recall that a graph G is *k-gap planar* if it has a drawing where each crossing is charged to one of the two edges involved and each edge has at most k crossings charged to it. This class of graphs has been implicitly studied for some time [143, 262]. The language of k-gap planar graphs was introduced by Bae et al. [22]. We show that k-gap planar graphs have unbounded row treewidth and thus cannot be described as a shallow minor of the strong product of a planar graph with a small complete graph, even with $k = 1$. This result is of particular interest since k-gap planar graphs have polynomial expansion [143, Theorem 6.9]. We in fact show a stronger result in terms of local treewidth.

Eppstein [140] defined a graph class $\mathcal{G}$ to have the *treewidth-diameter property*, more recently called *bounded local treewidth*, if there is a function f such that, for every graph $G \in \mathcal{G}$, for every vertex $v \in V(G)$ and for every integer $r \geqslant 0$, the subgraph of G induced by the vertices at distance at most r from v has treewidth at most $f(r)$. If f is linear, then $\mathcal{G}$ has *linear local treewidth*.

Graph classes with bounded row treewidth have linear local treewidth [121]. We show that k-gap planar graphs do not have bounded local treewidth in a stronger sense. Dawar et al. [93] defined a graph class $\mathcal{G}$ to *locally exclude a minor* if for each $r \in \mathbb{N}$ there is a graph H_r such that for every graph $G \in \mathcal{G}$ every subgraph of G with radius at most r contains no H_r-minor. Observe that if $\mathcal{G}$ has bounded local treewidth, then $\mathcal{G}$ locally excludes a minor.

The next theorem shows that 1-gap planar graphs do not locally exclude a minor.

Theorem 5.26. *For every $t \in \mathbb{N}$, there exists a 1-gap planar graph G with radius 1 which contains K_{t+1} as a minor.*

Proof. Let $\phi\colon E(K_t) \to \{1, \dots, \binom{t}{2}\}$ be a bijection. As illustrated in Figure 5.11, for each $ij \in E(K_t)$, embed vertices at $(\phi(ij), i), (\phi(ij), j) \in \mathbb{R}^2$ together with a straight vertical edge between them (red edges in Figure 5.11).

For each $i \in [t]$, draw a straight horizontal edge between each pair of consecutive vertices along the $y = i$ line. Let G_0 be the graph obtained. Let P_i be the subgraph of G_0 induced by the vertices on the $y = i$ line. Then P_i is a path on $t - 1$ vertices (green edges in Figure 5.11).

For each vertex v in $P_1 \cup \dots \cup P_t$ add a 'vertical' edge from v to a new vertex v' drawn with y-coordinate $t + 1$ (brown edges in Figure 5.11).

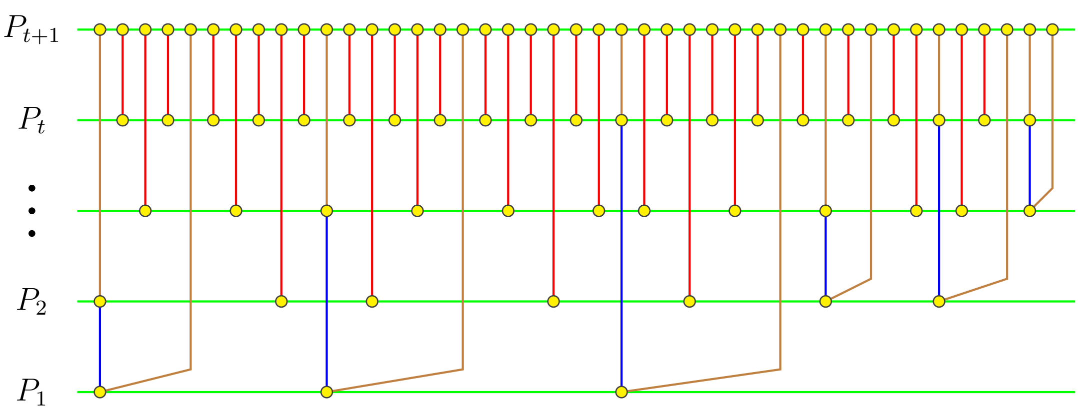


Figure 5.11. The graph G_1 in the proof of Theorem 5.26.

For $i = 1, 2, \dots, t$ complete the following step. If two vertical edges e and f cross an edge g in P_i at points x and y respectively, and no other vertical edge crosses g between x and y, then subdivide g between x and y, introducing a new vertex v, and add a new vertical edge from v to a new vertex v' with y-coordinate $t + 1$ (red edges in Figure 5.11).

Finally, add a path P_{t+1} through all the vertices with y-coordinate $t + 1$. We obtain a drawing of a graph G_1. Each crossing is between a vertical and a horizontal edge, and each horizontal edge is crossed by at most one edge. Thus, G_1 is 1-gap planar.

By construction, no edge in P_{t+1} is crossed in the drawing, and every vertex has a neighbour in P_{t+1}. Thus, contracting P_{t+1} to a single vertex gives a 1-gap planar graph G with radius 1. Finally, G contains a K_{t+1}-minor, obtained by contracting each horizontal path P_i into a single vertex. $\square$

5.5.2 RAC, Fan-Crossing Free, and k-Quasi Planar Graphs

Recall that a graph G is:

- *k-quasi planar* if it has a drawing in the plane where every set of k edges do not mutually cross;
- *fan-crossing free* if it has a drawing in the plane where for each edge $e \in E(G)$, the edges that cross e form a matching; or
- *right angle crossing (RAC)* if it has a drawing in the plane where each edge is drawn as a straight line segment and all crossings are at right angles.

Using known results, Brandenburg [59] proved that for every graph G, the 3-subdivision of G is an RAC graph; the 2-subdivision of G is fan-crossing free; and the 1-subdivision of G is 3-quasi planar. This implies that each of these classes are somewhere dense. Note that Eppstein [141] previously observed that the class of RAC graphs is somewhere dense. Since graph classes with bounded row treewidth have polynomial expansion [130], these beyond-planar classes have unbounded row treewidth. Thus, they cannot be described as a shallow minor of the strong product of a planar graph with a small complete graph. Additionally, Brandenburg [59] asked what is the queue-number and stack-number of these classes. Since graph classes with bounded stack-number or bounded queue-number have bounded expansion [259], we conclude that RAC, fan-crossing free, and k-quasi planar graphs (where $k \geqslant 3$) have unbounded stack-number and unbounded queue-number.

Chapter 6

Colouring Numbers

6.1 Overview

As introduced in Chapter 1, colouring numbers are important families of graph parameters which characterise bounded expansion [347] and nowhere dense classes [171], and have several algorithmic applications [131, 172]. To further illustrate their usefulness, we present several original applications of colouring numbers in this chapter.

First, we explore the colouring numbers of shallow minors. In this direction, we prove the following.

Theorem 6.1. *For every graph G, for every r-shallow minor H of $G \boxtimes K_\ell$, and for every integer $s \geqslant 1$,*

$$\operatorname{scol}_s(H) \leqslant \ell \operatorname{scol}_{2rs+2r+s}(G) \quad \textit{and} \quad \operatorname{wcol}_s(H) \leqslant \ell \operatorname{wcol}_{2rs+2r+s}(G).$$

Using the shallow minor structure theorems from the previous chapter, we show that Theorem 6.1 implies that several beyond-planar graph classes have linear strong colouring numbers and cubic weak colouring numbers.

Second, we bound the cop-width and flip-width of a graph by its strong colouring numbers. Cop-width and flip-width are recently introduced graph parameters introduced by Torunczyk [320] that generalise treewidth, degeneracy, generalised colouring numbers, clique-width and twin-width. Our main contribution is the following.

Theorem 6.2. *For every $r \in \mathbb{N}$, every graph G has $\operatorname{copwidth}_r(G) \leqslant \operatorname{scol}_{4r}(G)$.*

Next, we bound the odd chromatic number and the conflict-free chromatic number of a graph by its strong colouring numbers. The odd chromatic number χ_o and the conflict-free chromatic number χ_{pcf} are new graph parameters introduced by Petruševski and Škrekovski [274] and Fabrici et al. [151] respectively. We prove the following.

Theorem 6.3. *For every graph G, $\chi_{pcf}(G) \leqslant 2 \operatorname{scol}_2(G) - 1$.*

Finally, we use colouring numbers to bound the defective choice number of $K_{s,t}$-subgraph-free graphs (Theorem 6.18 below).

Sections 6.3 and 6.4 are based on my single authored papers [189, 191] while Section 6.2 is based on joint work with Wood [196] and Section 6.5 is based on joint work with Briański and Wood [62].

6.2 Shallow Minors

In this section, we prove Theorem 6.1.

Theorem 6.1. *For every graph G, for every r-shallow minor H of $G \boxtimes K_\ell$, and for every integer $s \geqslant 1$,*

$$\operatorname{scol}_s(H) \leqslant \ell \operatorname{scol}_{2rs+2r+s}(G) \quad \textit{and} \quad \operatorname{wcol}_s(H) \leqslant \ell \operatorname{wcol}_{2rs+2r+s}(G).$$

Sergey Norin (see [149]) observed that every r-shallow minor of a graph G has average degree at most $2\operatorname{scol}_{4r+1}(G)$. The following lemma generalises this observation (the case $s = 1$ implies the above observation):

Lemma 6.4. *For every graph G, for every r-shallow minor H of G, and for every integer $s \geqslant 1$,*

$$\operatorname{scol}_s(H) \leqslant \operatorname{scol}_{2rs+2r+s}(G).$$

Proof. Let $\preceq$ be a total order of $V(G)$ such that $|R(G, u, \preceq, 2rs+2r+s)| \leqslant \operatorname{scol}_{2rs+2r+s}(G)$ for every vertex u of G. Let μ be an r-shallow model of H in G. Let ℓ_u be the leftmost vertex in $V(\mu(u))$ with respect to $\preceq$. Since the $\mu(u)$'s are pairwise disjoint, the ℓ_u are pairwise distinct. Let $\preceq'$ be the total order of $V(H)$ where $u \prec v$ if and only if $\ell_u \prec \ell_v$.

Let $v \in V(H)$. We will show that for each $u \in R(H, v, \preceq', s)$, there exists $\tilde{u} \in V(\mu(u))$ such that $\tilde{u} \in R(G, \ell_v, \preceq, 2rs + 2r + s)$. Since the $\mu(u)$'s are pairwise disjoint, this will imply that the $\tilde{u}$'s are distinct, and that $|R(H, v, \preceq', s)| \leqslant |R(G, \ell_v, \preceq, 2rs + 2r + s)| \leqslant \operatorname{scol}_{2rs+2r+s}(G)$ for all $v \in V(H)$.

Consider a vertex $u \in R(H, v, \preceq', s)$. So $u \preceq' v$ and there is a path $(v = w_0, w_1, w_2, \dots, w_t = u)$ of length $t \leqslant s$ in H, such that $v \prec' w_i$ for every $i \in [t-1]$. Thus $\ell_u \preceq \ell_v \prec \ell_{w_i}$ for every $i \in [t-1]$. For $i \in [0, t-1]$ there is an edge $x_i y_{i+1}$ of G where $x_i \in V(\mu(w_i))$ and $y_{i+1} \in V(\mu(w_{i+1}))$. For $i \in [1, t-1]$ there is an (y_i, x_i)-path P_i in $\mu(w_i)$ of length at most $2r$ (since $\mu(w_i)$ has radius at most r). Similarly, there is an (ℓ_v, x_0)-path P_0 in $\mu(v)$ of length at most $2r$, and there is an (y_t, ℓ_u)-path P_t in $\mu(u)$ of length at most $2r$. Thus $P = \ell_v P_0 x_0, y_1 P_1 x_1, y_2 P_2 x_2, \dots, y_{t-1} P_{t-1} x_{t-1}, y_t P_t \ell_u$ is an (ℓ_v, ℓ_u)-path in G of length at most $(2r+1)(t+1) - 1 \leqslant (2r+1)(s+1) - 1 = 2rs + 2r + s$.

Let $\tilde{u}$ be the first vertex in $V(\mu(u))$ on P where $\tilde{u} \preceq \ell_v$. This is well-defined since ℓ_u is a candidate. Let P' be the subpath of P from ℓ_v to $\tilde{u}$. Consider an internal vertex x of P'. So x is in $\mu(w_i)$ for some $i \in [t]$. If $i = t$ then $\ell_v < x$ by the definition of $\tilde{u}$. If

$i \in [t-1]$ then $\ell_v \prec \ell_{w_i} \preceq x$ by the definition of ℓ_{w_i}. Thus, every internal vertex of P' is to the right of ℓ_v in $\preceq$ and so $\tilde{u} \in R(G, \ell_v, \preceq, 2rs+2r+s)$, as required. □

Note that a result of Zhu [347, Corollary 3.5] implies that there is a function f that bounds the s-strong colouring numbers of an r-shallow minor H of a graph G by a function of its $f(s)$-strong colouring number. The bound obtained in Lemma 6.4 is significantly better.

For weak colouring numbers, Grohe et al. [171] showed that if H is an r-shallow topological minor of G then $\operatorname{wcol}_s(H) \leqslant 2\operatorname{wcol}_{2rs+s}(G)$ for every integer $s \geqslant 1$. The following lemma extends this result to shallow minors. We omit the proof since it is directly analogous to Lemma 6.4.

Lemma 6.5. *For every graph G, for every r-shallow minor H of G, and for every integer $s \geqslant 1$,*

$$\operatorname{wcol}_s(H) \leqslant \operatorname{wcol}_{2rs+2r+s}(G).$$

For graph products, Dvořák et al. [130] showed that $\operatorname{scol}_s(G \boxtimes H) \leqslant \operatorname{scol}_s(G)(\Delta(H)+2)^s$ for all graphs G and H. Their proof technique also applies to weak colouring numbers. A closer inspection of their proof reveals the following stronger upper bound.

Lemma 6.6 ([130])**.** *For all graphs G and H and every integer $s \geqslant 1$,*

$$\operatorname{scol}_s(G \boxtimes H) \leqslant \operatorname{scol}_s(G)(\Delta(H^s)+1) \quad \textit{and}$$
$$\operatorname{wcol}_s(G \boxtimes H) \leqslant \operatorname{wcol}_s(G)(\Delta(H^s)+1).$$

Theorem 6.1 immediately follows from Lemmas 6.4–6.6.

6.2.1 Applications of Theorem 6.1

We now discuss applications of Theorem 6.1 to various beyond-planar graphs.

We say a graph class $\mathcal{G}$ has *linear strong colouring numbers* if there is a constant $c > 0$ such that $\operatorname{scol}_s(G) \leqslant cs$ for every graph $G \in \mathcal{G}$ and for every integer $s \geqslant 1$. Similarly, we say a graph class $\mathcal{G}$ has *cubic weak colouring numbers* if there is a constant $c > 0$ such that $\operatorname{wcol}_s(G) \leqslant cs^3$ for every graph $G \in \mathcal{G}$ and for every integer $s \geqslant 1$. Now suppose we are given a graph class $\mathcal{G}$ with linear strong colouring numbers. For $r, \ell \in \mathbb{N}$, Let $\mathcal{G}_r^\ell$ be the class of r-shallow minors of graphs in $\mathcal{G} \boxtimes K_\ell$. Then for all graphs $H \in \mathcal{G}_r^\ell$ and integer $s \geqslant 1$, we have

$$\operatorname{scol}_s(H) \leqslant \ell \operatorname{scol}_{2rs+2r+s}(G) \leqslant c\ell(2rs+2r+s) \leqslant c\ell(4r+1)s.$$

So $\mathcal{G}_r^\ell$ has linear strong colouring numbers (with corresponding constant $c\ell(4r+1)$). An analogous argument shows that if $\mathcal{G}$ has cubic weak colouring numbers, then $\mathcal{G}_r^\ell$ also has cubic strong colouring numbers. Now van den Heuvel et al. [325] proved that for every graph G with Euler genus g and for every integer $s \geqslant 1$, we have $\operatorname{scol}_s(G) \leqslant (4g+5)s+2g+1$ and $\operatorname{wcol}_s(G) \leqslant \left(2g+\binom{s+2}{2}\right)(2s+1)$. Thus Theorem 6.1 implies the following.

Theorem 6.7. *For every graph G with Euler genus g, if H is an r-shallow minor of $G \boxtimes K_\ell$ then:*

$$\operatorname{scol}_s(H) \leqslant \ell\Big((4g+5)(2rs+2r+s)+2g+1\Big) \quad \textit{and}$$
$$\operatorname{wcol}_s(H) \leqslant \ell\Big(2g+\binom{(2r+1)s+2r+2}{2}\Big)((4r+2)s+4r+1).$$

Therefore, any class of graphs where each graph in the class can be described as a shallow minor of a strong product of a graph with bounded Euler genus with a small complete graphs has linear strong colouring numbers and cubic weak colouring numbers. As shown in Chapter 5, the following graph classes have such a shallow minor structure:

- (g,k)-planar graphs (Lemma 5.16);
- (g,δ)-string graphs (Lemma 5.18);
- fan-planar graphs (Lemma 5.22); and
- k-fan-bundle planar graphs (Lemma 5.25).

So by Theorem 6.7, each of these graph classes have linear strong colouring numbers and cubic weak colouring numbers. In the case when G is a fan-planar graph, Lemma 5.22 and Theorem 6.7 give the following bounds:

$$\operatorname{scol}_s(G) \leqslant 45s+33 \quad \text{and} \quad \operatorname{wcol}_s(G) \leqslant \binom{3s+4}{2}(18s+15).$$

Previously, it was known that the above graph classes have linear strong colouring numbers since graphs with bounded layered treewidth have linear strong colouring numbers [326]; see Section 5.3.3. More importantly, these are the first known polynomial bound on the weak colouring numbers for these classes. As Theorem 5.10 shows, layered treewidth can not be used to deduce polynomial bounds for weak-colouring numbers. Superpolynomial bounds were known for these classes since they have bounded strong colouring numbers [347].

6.3 Cop-Width and Flip-Width

In this section, we bound the cop-width and flip-width of a graph by its strong colouring numbers. Cop-width and flip-width are new families of graph parameters introduced by Torunczyk [320] that extend colouring numbers to the dense setting. Their definitions are inspired by a game of cops and robber by Seymour and Thomas [299]:

"The robber stands on a vertex of the graph, and can at any time run at great speed to any other vertex along a path of the graph. He is not permitted to run through a cop,

however. There are k cops, each of whom at any time either stands on a vertex or is in a helicopter (that is, is temporarily removed from the game). The objective of the player controlling the movement of the cops is to land a cop via helicopters on the vertex occupied by the robber, and the robber's objective is to elude capture. (The point of the helicopters is that cops are not constrained to move along paths of the graph – they move from vertex to vertex arbitrarily.) The robber can see the helicopter approaching its landing spot and may run to a new vertex before the helicopter actually lands" [299].

Seymour and Thomas [299] showed that the least number of cops needed to win this game on a graph G is in fact equal to $\mathrm{tw}(G) + 1$, thus giving a min-max theorem for treewidth. Torunczyk [320] introduced the following parameterised version of this game: for some fixed $r \in \mathbb{N}$, the robber runs at speed r. So in each round, after the cops have taken off in their helicopters to their new positions (they may also choose to stay put), which are known to the robber, and before the helicopters have landed, the robber may traverse a path of length at most r that does not run through a cop that remains on the ground. This variant is called the *cop-width game with radius r and width k*, if there are k cops, and the robber can run at speed r. For a graph G, the *radius-r cop-width* $\mathrm{copwidth}_r(G)$ of G is the least number $k \in \mathbb{N}$ such that the cops have a winning strategy for the cop-width game played on G with radius r and width k.

Say a class of graphs $\mathcal{G}$ has *bounded cop-width* if there is a function f such that for every $r \in \mathbb{N}$ and graph $G \in \mathcal{G}$, $\mathrm{copwidth}_r(G) \leqslant f(r)$. Torunczyk [320] showed that bounded cop-width coincides with bounded expansion.

Theorem 6.8 ([320])**.** *A class of graphs has bounded expansion if and only if it has bounded cop-width.*

As such, only sparse graph classes have bounded cop-width. Flip-width is then defined as a dense analog of cop-width. Here, the cops have enhanced power where they are allowed to perform flips on subsets of the vertex-set of the graph with the goal of isolating the robber. For a fixed graph G, applying a flip between a pair of sets of vertices $A, B \subseteq V(G)$ results in the graph obtained from G by inverting the adjacency between any pair of vertices a, b with $a \in A$ and $b \in B$. If G is a graph and $\mathcal{P}$ is a partition of $V(G)$, then call a graph G' a *$\mathcal{P}$-flip* of G if G' can be obtained from G by performing a sequence of flips between pairs of parts $A, B \in \mathcal{P}$ (possibly with $A = B$). Finally, call G' a *k-flip* of G, if G' is a $\mathcal{P}$-flip of G, for some partition $\mathcal{P}$ of $V(G)$ with $|\mathcal{P}| \leqslant k$.

The *flip-width game with radius $r \in \mathbb{N}$ and width $k \in \mathbb{N}$* is played on a graph G. Initially, $G_0 = G$ and v_0 is a vertex of G chosen by the robber. In each round $i \geqslant 1$ of the game, the cops announce a new k-flip G_i of G. The robber, knowing G_i, moves to a new vertex v_i by running along a path of length at most r from v_{i-1} to v_i in the previous graph G_{i-1}. The game terminates when v_i is an isolated vertex in G_i. For a fixed $r \in \mathbb{N}$, the *radius-r flip-width* $\mathrm{flipwidth}_r(G)$ of a graph G is the least number $k \in \mathbb{N}$ such that the cops have a winning strategy in the flip-width game of radius r and width k on G.

In contrast to cop-width, flip-width is well-behaved on dense graphs. For example, one can easily observe that for all $r \in \mathbb{N}$, the radius-r flip-width of a complete graph is equal to 1. Moreover, to demonstrate the robustness of flip-width, Torunczyk [320] proved the following results.

Theorem 6.9 ([320]).

- *Every class of graphs with bounded expansion has bounded flip-width.*
- *Every class of graphs with bounded twin-width has bounded flip-width.*
- *If a class of graphs $\mathcal{G}$ has bounded flip-width, then any first-order interpretations of $\mathcal{G}$ also has bounded flip-width.*
- *There is a slicewise polynomial algorithm that approximates the flip-width of a given graph graphs G.*

These results provide evidence to the view that flip-width is the right analog of generalised colouring numbers for dense graphs. See [72, 145, 320] for further results and conjectures concerning flip-width.

6.3.1 Our Results

Our main result in this section states that the cop-width of a graph is bounded by its strong colouring numbers.

Theorem 6.2. *For every $r \in \mathbb{N}$, every graph G has* $\operatorname{copwidth}_r(G) \leqslant \operatorname{scol}_{4r}(G)$.

Previously, the best known bounds for the cop-width of a sparse graph was through its weak colouring numbers. Torunczyk [320] showed that for every $r \in \mathbb{N}$, every graph G has

$$\operatorname{copwidth}_r(G) \leqslant \operatorname{wcol}_{2r}(G) + 1.$$

Moreover, if G excludes $K_{t,t}$ as a subgraph, then $\operatorname{flipwidth}_r(G) \leqslant (\operatorname{copwidth}_r(G))^t$. While graph classes with bounded strong colouring numbers have bounded weak colouring numbers, strong colouring numbers often give much better bounds than weak colouring numbers. In fact, Grohe et al. [171] and Dvořák et al. [135] have both shown that there are graph classes with polynomial strong colouring numbers and exponential weak colouring numbers.

We now present a couple of applications of Theorem 6.2. First, we use Theorem 6.2 to show that graph classes with linear strong colouring numbers have linear cop-width and linear flip-width.

Theorem 6.10. *Every class of graphs with linear strong colouring numbers has linear cop-width and linear flip-width.*

Second, Theorem 6.2 gives improved bounds for the cop-width of many well-studied sparse graphs. For example, Van den Heuvel, Ossona de Mendez, Quiroz, Rabinovich and

Siebertz [325] showed that for every $r \in \mathbb{N}$, every K_t-minor-free graph G has $\operatorname{scol}_r(G) \leqslant \binom{t-1}{2}(2r+1)$. So Theorem 6.2 implies the following.

Theorem 6.11. *For all $r, t \in \mathbb{N}$, every K_t-minor-free graph G has*

$$\operatorname{copwidth}_r(G) \leqslant \binom{t-1}{2}(8r+1).$$

By Theorem 6.10, K_t-minor-free graphs also have linear flip-width; see Corollary 6.14 for an explicit bound. In regard to the previous best known bound for this class of graphs, van den Heuvel et al. [325] showed that for every $r \in \mathbb{N}$, every K_t-minor-free graph G has $\operatorname{wcol}_r(G) \in O_t(r^t)$. By the aforementioned result of Torunczyk [320], the previous best known bound for the cop-width and flip-width of K_t-minor-free graph G was:

$$\operatorname{copwidth}_r(G) \in O_t(r^{t-1}) \text{ and } \operatorname{flipwidth}_r(G) \in O_t(r^{(t-1)^2}).$$

Theorem 6.2 is also applicable to non-minor-closed graph classes. Van den Heuvel and Wood [326] showed that, for every $r \in \mathbb{N}$, every (g, k)-planar graph G has $\operatorname{scol}_r(G) \leqslant (4g+6)(k+1)(2r+1)$. So Theorem 6.2 implies the following.

Theorem 6.12. *For all $g, k, r \in \mathbb{N}$, every (g, k)-planar graph G has*

$$\operatorname{copwidth}_r(G) \leqslant (4g+6)(k+1)(8r+1).$$

See [133, 196, 325, 326] for other graph classes that Theorem 6.2 applies to.

6.3.2 Proofs

Theorem 6.2. *For every $r \in \mathbb{N}$, every graph G has* $\operatorname{copwidth}_r(G) \leqslant \operatorname{scol}_{4r}(G)$.

Proof. Let $n := |V(G)| - 1$ and let $(v_0, v_1, \dots, v_n)$ be a total order $\preceq$ of $V(G)$ where $|R(G, \preceq, v, 4r)| \leqslant \operatorname{scol}_{4r}(G)$ for every $v \in V(G)$. For every $s \in \mathbb{N}$ and $v_i, v_j \in V(G)$ where $i \leqslant j$, let $M(v_i, v_j, s)$ be the set of vertices $w \in V(G)$ for which there is a path $v_j = w_0, w_1, \dots, w_{s'} = w$ of length $s' \in [0, s]$ such that $w \preceq v_i$ and $v_i \prec w_\ell$ for all $\ell \in [s'-1]$.

Claim. *For all $v_i, v_j \in V(G)$ where $i \leqslant j$, $|M(v_i, v_j, 2r)| \leqslant \operatorname{scol}_{4r}(G)$.*

Proof. Let $k \in [i, j]$ be minimal such that $v_k \in Q(G, \preceq, v_j, 2r)$. So G contains a path $P = (v_j = w_0, \dots, w_{r'} = v_k)$ of length $r' \in [0, 2r]$ such that $v_k \prec w_\ell$ for all $\ell \in [r'-1]$. We claim that $M(v_i, v_j, 2r) \subseteq R(G, \preceq, v_k, 4r)$. Let $u \in M(v_i, v_j, 2r)$. Then there is a path $P' = (v_j = u_0, \dots, u_{s'} = u)$ of length $s' \in [0, 2r]$ such that $u \preceq v_i$ and $v_i \prec u_\ell$ for all $\ell \in [s'-1]$. Suppose there is an $\ell \in [s'-1]$ such that $u_\ell \prec v_k$. Choose ℓ to be minimum. Then $u_\ell \in Q(G, \preceq, v_j, 2r)$ since $u_\ell \prec u_a$ for each $a \in [0, \ell-1]$, which contradicts the choice of k. So $v_k \preceq u_\ell$ for all $\ell \in [s'-1]$. By taking the union of P and P', it follows that G

contains a (v_k, u)-walk W of length at most $4r$ such that $v_k \prec w$ for all $w \in V(W)\setminus\{v_k, u\}$. Therefore $w \in R(G, \preceq, v_k, 4r)$ and so $|M(v_i, v_j, 2r)| \leqslant |R(G, \preceq, v_k, 4r)| \leqslant \mathrm{scol}_{4r}(G)$. □

For each round $i \geqslant 0$ until the robber is caught, we define a tuple $(v_i, x_i, C_i, D_i, V_i, P_i)$ where:

(1) $x_i \in V(G)$ is the location of the robber at the end of round i;
(2) $C_0 := \{v_0\}$ and $C_i := M(v_i, x_{i-1}, 2r)$ is the set of vertices that the cops are on at the end of round i for each $i \geqslant 1$;
(3) $D_0 := \varnothing$ and $D_i := C_{i-1} \cap C_i$ is the set of vertices where cops remain put throughout round i for each $i \geqslant 1$;
(4) $V_i := \{v_0, \dots, v_i\}$; and
(5) $P_0 := \varnothing$ and P_i is the (x_{i-1}, x_i)-path of length at most r that the robber traverse during round i for each $i \geqslant 1$.

We will also maintain the following invariants for each round $i \geqslant 0$:

(6) $v_i \preceq x_i$;
(7) for $i \geqslant 1$, every path in G from x_{i-1} to a vertex in V_{i-1} of length at most r contains a vertex from D_i;
(8) $M(v_i, x_i, r) \subseteq C_i$;
(9) $V(P_i) \cap V_{i-1} = \varnothing$ for each $i \geqslant 1$; and
(10) if $v_i = x_i$, then the robber is caught.

Together with the previous claim, (2), (6) and (10) imply that the robber is caught within n rounds using at most $\mathrm{scol}_{4r}(G)$ cops.

We construct our sequence of tuples using induction on $i \geqslant 0$. Initialise the game of cops and robber with the robber on some vertex in $V(G)$, one cop on v_0 and the remaining cops all in the helicopters. Define the tuple $(v_0, x_0, C_0, D_0, V_0, P_0)$ according to (1)–(5). Clearly such a tuple is well-defined. Moreover, it is easy to see that the tuple satisfies (6)–(10).

Now suppose we are at round $i \geqslant 1$ and the robber has not yet been caught. By induction, we may assume that there is a tuple $(v_{i-1}, x_{i-1}, C_{i-1}, D_{i-1}, V_{i-1}, P_{i-1})$ for round $i-1$ which satisfies (1)–(10). Since the robber has not yet been caught, (6) and (10) imply that $v_{i-1} \prec x_{i-1}$, so $v_i \preceq x_{i-1}$. Therefore, there is a well-defined tuple $(v_i, x_i, C_i, D_i, V_i, P_i)$ which satisfies (1)–(5). We now show that $(v_i, x_i, C_i, D_i, V_i, P_i)$ satisfies the additional invariants.

We first verify (7). Let $F_i := M(v_{i-1}, x_{i-1}, r)$. Let $u \in V_{i-1}$ and suppose there is a $(x_{i-1} = w_0, w_1, \dots, w_{r'} = u)$-path $P^\star$ in G where $r' \in [0, r]$. Consider the minimal $j \in [r']$ such that $w_j \in V_{i-1}$. Since $\{w_1, \dots, w_{j-1}\} \cap V_{i-1} = \varnothing$, it follows that $w_j \in F_i$. So for every $u \in V_{i-1}$, every (x_{i-1}, u)-path in G of length at most r contains a vertex from F_i. By (2) and (8) (from the $i-1$ case), it follows that $F_i \subseteq C_{i-1} \cap C_i$. So (7) follows from (3). Now since the robber is not allowed to run through a cop that stays put, (9) follows by (3), (5) and (7). Property (6) then immediately follows from (9) since $x_i \in V(P_i)$. Now consider a vertex $y \in M(v_i, x_i, r)$. Then G contains a $(x_i = w_0, w_1, \ldots, w_{r'} = y)$-path P' of length $r' \in [0, r]$ such that $v_i \preceq w_j$ for all $j \in [r' - 1]$. By taking the union of P' and P, it follows that G contains an (x_{i-1}, y)-walk W of length at most $2r$. Moreover, by (7), $v_i \preceq z$ for all $z \in V(W) \setminus \{x_{i-1}, y\}$. So $u \in M(v_i, x_{i-1}, 2r)$ and thus (2) implies (8). Finally, if $v_i = x_i$ then (8) implies $x_i \in C_i$, so (10) follows by (1) and (2), as required. □

Theorem 6.2 implies that graph classes with linear strong colouring numbers have linear cop-width. To complete the proof of Theorem 6.10, we leverage known results concerning neighbourhood diversity. Neighbourhood diversity is a well-studied concept with various applications [47, 49, 139, 156, 210, 271, 283]. Let G be a graph. For a set $S \subseteq V(G)$, let $\pi_G(S) := |\{N_G(v) \cap S \colon v \in V(G) \setminus S\}|$. For $k \in \mathbb{N}$, let $\pi_G(k) := \max\{\pi_G(S) \colon S \subseteq V(G), |S| \leqslant k\}$.

Lemma 6.13. *For all $k, r \in \mathbb{N}$, every graph G with $\operatorname{copwidth}_r(G) \leqslant k$ has*

$$\operatorname{flipwidth}_r(G) \leqslant \pi_G(k) + k.$$

Proof. We claim that for every set $S \subseteq V(G)$ where $|S| \leqslant k$, there is a $(\pi_G(k)+k)$-flip that isolates S while leaving $G - S$ untouched. Let $\mathcal{P}$ be a partition of $V(G)$ that partition S into singleton and vertices in $v \in V(G) \setminus S$ according to $N_G(v) \cap S$. Then $|\mathcal{P}| \leqslant \pi_G(k) + k$. Moreover, every vertex $s \in S$ can be isolated by flipping $\{s\}$ with every part of $\mathcal{P}$ that is complete to $\{s\}$. Thus, a winning strategy for the cops in the cop-width game with radius r and width k can be transformed into a winning strategy for the flip-width graph with radius r and width $c + k$, as required. □

Reidl et al. [283] showed that for every graph class $\mathcal{G}$ with bounded expansion, there exists $c > 0$ such that $\pi_G(k) \leqslant ck$ for every $G \in \mathcal{G}$. Since graph classes with linear strong colouring numbers have bounded expansion, Theorem 6.2 and Lemma 6.13 imply Theorem 6.10. As a concrete example, Bonnet et al. [49] showed that for every K_t-minor-free graph G and for every set $A \subseteq V(G)$,

$$\pi_G(A) \leqslant 3^{2t/3+o(t)}|A| + 1.$$

So Theorem 6.11 and Lemma 6.13 imply the following.

Corollary 6.14. *For all $r, t \in \mathbb{N}$, every K_t-minor-free graph G has*

$$\operatorname{flipwidth}_r(G) \leqslant (3^{2t/3+o(t)} + 1)(t-2)(t-3)(8r+1) + 1.$$

6.4 Odd Colouring and Conflict-Free Colouring

In this section, we use strong colouring numbers to bound the odd chromatic number and proper conflict-free chromatic number of a graph. Let G be a graph and $\psi : V(G) \to C$ be a proper colouring of G. If $N(v) := \{w \in V(G) : vw \in E(G)\}$ is the neighbourhood of a vertex v, then ψ is an *odd colouring* if for each $v \in V(G)$ with $|N(v)| > 0$, there exists a colour $\alpha \in C$ such that $|\{w \in N(v) : \psi(w) = \alpha\}|$ is odd. Similarly, ψ is a *conflict-free colouring* of G if for each $v \in V(G)$ with $|N(v)| > 0$, there exists a colour $\alpha \in C$ such that $|\{w \in N(v) : \psi(w) = \alpha\}| = 1$. The *odd chromatic number* $\chi_o(G)$ of G is the minimum integer c such that G has a proper odd c-colouring. Likewise, the *conflict-free chromatic number* $\chi_{pcf}(G)$ of G is the minimum integer c such that G has a proper conflict-free c-colouring. Clearly $\chi_o(G) \leqslant \chi_{pcf}(G)$ since a conflict-free colouring is an odd colouring.

Motivated by connections to hypergraph colouring, the odd chromatic number and the conflict-free chromatic number were recently introduced by Petruševski and Škrekovski [274] and Fabrici et al. [151] respectively. These parameters have gained significant traction with a particular focus on determining a tight upper bound for planar graphs. Petruševski and Škrekovski [274] showed that the odd chromatic number of planar graphs is at most 9 and conjectured that their odd chromatic number is at most 5. Petr and Portier [273] improved this upper bound to 8. For conflict-free colourings, Fabrici et al. [151] proved a matching upper bound of 8 for planar graphs. For proper minor-closed classes, a result of Cranston et al. [87] implies that the odd chromatic number of K_t-minor free graphs is $O(t\sqrt{\log t})$. For non-minor closed classes, Cranston et al. [87] showed that the odd chromatic number of 1-planar graphs is at most 23. Dujmović et al. [122] proved a more general upper bound of $O(k^5)$ for the odd chromatic number of k-planar graphs. See [66, 67, 82] for other results concerning these new graph parameters.

In this section, we bound the conflict-free chromatic number of a graph by its 2-strong colouring number. Our key contribution is the following.

Theorem 6.3. *For every graph G, $\chi_{pcf}(G) \leqslant 2\,\mathrm{scol}_2(G) - 1$.*

Note that Theorem 6.3 is best possible in the sense that the conflict-free chromatic number is not bounded by the 1-strong colouring number [67]. Before proving Theorem 6.3, we highlight several noteworthy consequences.

First, Theorem 6.3 implies that graph classes with bounded expansion have bounded conflict-free chromatic number and bounded odd chromatic number. A result of Zhu [347] implies that $\mathrm{scol}_2(G) \leqslant 8(\nabla_1(G))^3 + 1$ for every graph G. Thus, we have the following consequence of Theorem 6.3.

Corollary 6.15. *For every graph G, $\chi_{pcf}(G) \leqslant 16(\nabla_1(G))^3 + 1$.*

Second, Theorem 6.3 implies a stronger bound for the odd chromatic number and the conflict-free chromatic number of k-planar graphs. Van den Heuvel and Wood [326] showed that $\mathrm{scol}_2(G) \leqslant 30(k+1)$ for every k-planar graph G. Thus we have the following consequence of Theorem 6.3:

Theorem 6.16. *For every k-planar graph G, $\chi_{pcf}(G) \leqslant 60k + 59$.*

Theorem 6.16 is the first known upper bound for the conflict-free chromatic number of k-planar graphs. For the odd chromatic number, the previous best known upper bound for k-planar graphs was $\chi_o(G) \in O(k^5)$ due to Dujmović et al. [122].

Finally, Theorem 6.3 gives the first known upper bound for the conflict-free chromatic number of K_t-minor free graphs. Van den Heuvel, Ossona de Mendez, Quiroz, Rabinovich and Siebertz [325] showed that $\mathrm{scol}_2(G) \leqslant 5\binom{t-1}{2}$ for every K_t-minor free graph G. Thus Theorem 6.3 implies the following:

Theorem 6.17. *For every K_t-minor free graph G, $\chi_{pcf}(G) \leqslant 5(t-1)(t-2) - 1$.*

See [133, 196, 325, 326] for other graph classes that Theorem 6.3 applies to.

Proof of Theorem 6.3. We may assume that G has no isolated vertices. Let $\preceq$ be the ordering $(v_1, \dots, v_n)$ of $V(G)$ where $|R(G, v_i, \preceq, 2)| \leqslant \mathrm{scol}_2(G)$ for every vertex v_i of G. For each vertex $v_i \in V(G)$, let $N^-(v_i) := R(G, v_i, \preceq, 1) \setminus \{v_i\}$ be the *left neighbours* of v_i, and let $v_j \in N(v_i)$ where $j = \min\{\ell \in [n] : v_\ell \in N(v_i)\}$ be the *leftmost neighbour* of v_i. Let $\pi(v_i)$ denote the leftmost neighbour of v_i.

We now specify the conflict-free colouring $\psi : V(G) \to [2\,\mathrm{scol}_2(G)+1]$ by colouring the vertices left to right. For $i = 1$, let $\psi(v_1) = 1$. Now suppose $i > 1$ and that $v_1, \dots, v_{i-1}$ are coloured. Let $X_i := \{\psi(v_j) : v_j \in R(G, u, \preceq, 2) \setminus \{v_i\}\}$ and $Y_i := \{\psi(\pi(v_j)) : v_j \in N^-(v_i)\}$. Observe that $|X_i| \leqslant |R(G, u, \preceq, 2) \setminus \{v_i\}| \leqslant \mathrm{scol}_2(G) - 1$ and $|Y_i| \leqslant |R(G, v_i, \preceq, 1) \setminus \{v_i\}| \leqslant \mathrm{scol}_2(G) - 1$ and so $|X_i \cup Y_i| \leqslant 2\,\mathrm{scol}_2(G) - 2$. As such, there exists some colour $\alpha \in [2\,\mathrm{scol}_2(G) - 1] \setminus (X_i \cup Y_i)$. Let $\psi(v_i) := \alpha$.

Now ψ is proper as each vertex receives a different colour to its left neighbours. We now show that it is conflict-free. Let $v_i \in V(G)$ and let $v_j = \pi(v_i)$. We claim that $\psi(v_j) \neq \psi(v_\ell)$ for every $v_\ell \in N(v_i) \setminus \{v_j\}$. Since v_j is the leftmost neighbour of v_i, $j < \ell$. If $\ell < i$, then $v_j \in R(G, \preceq, v_\ell, 2)$ (by the path v_ℓ, v_i, v_j) and so $\psi(v_j) \in X_\ell$. Otherwise $i < \ell$ so $v_i \in N^-(v_\ell)$ and thus $\psi(v_j) \in Y_\ell$. As such, $\psi(v_j) \in X_\ell \cup Y_\ell$ and hence $\psi(v_j) \neq \psi(v_\ell)$, as required. $\square$

6.5 Defective Colouring

We conclude this chapter by applying colouring numbers to defective colourings.

For a colouring of a graph G, a *monochromatic component* of G is a connected component of the subgraph of G induced by all the vertices assigned a single colour. A colouring has *defect* d if every monochromatic component has maximum degree at most d. Note that a colouring with defect 0 is precisely a proper colouring. The *defective chromatic number* of a graph class $\mathcal{G}$ is the minimum integer k for which there exists an integer d such that every graph in $\mathcal{G}$ is k-colourable with defect d. Defective colouring is a well-studied topic, see [339] for a survey.

Eaton and Hull [136] introduced a list colouring variant of defective colouring. A *list assignment* for a graph G is a function L that assigns a set $L(v)$ of colours to each vertex $v \in V(G)$. For a list-assignment L of a graph G and integer $d > 0$, we say that G is *L-colourable with defect d* if there is a colouring of G with defect d such that each vertex $v \in V(G)$ is assigned a colour in $L(v)$. A list assignment L is a *k-list assignment* if $|L(v)| \geqslant k$ for each vertex $v \in V(G)$. Define G to be *k-choosable with defect d* if G is L-colourable with defect d for every k-list assignment L of G. The *defective choice number* of a graph class $\mathcal{G}$ is the minimum integer k for which there exists an integer $d \geqslant 0$, such that every graph $G \in \mathcal{G}$ is k-choosable with defect d.

We now show that colouring numbers can be used to bound the defective choice number of graphs with no $K_{s,t}$ subgraph.

Theorem 6.18. *Every graph G with no $K_{s,t}$ subgraph is s-choosable with defect at most $\operatorname{scol}_2(G) + (t-1)\binom{\operatorname{scol}_2(G)}{s-1}$.*

Proof. Let L be an s-list-assignment for G. Let $\preceq$ be a total ordering of $V(G)$ witnessing $\operatorname{scol}_2(G)$. Consider each $v \in V(G)$ in the order given by $\preceq$, and choose $\operatorname{scol}(v) \in L(v)$ distinct from the colour assigned to the $s-1$ leftmost neighbours of v. Suppose that v has monochromatic degree at least $\operatorname{scol}_2(G)+(t-1)\binom{\operatorname{scol}_2(G)}{s-1}+1$. At most $\operatorname{scol}_2(G)$ neighbours of v are to the left of v. Thus there is a set W of $t\binom{\operatorname{scol}_2(G)}{s-1} + 1$ neighbours w of v with $\operatorname{scol}(w) = \operatorname{scol}(v)$ and $v \prec w$ for each $w \in W$. For each $w \in W$, by the choice of $\operatorname{scol}(w)$, there is a set A_w of $s-1$ neighbours of w to the left of v. Thus $A := \bigcup(A_w \colon w \in W)$ is 2-reachable from v. Hence $|A| \leqslant \operatorname{scol}_2(G)$. For each $(s-1)$-subset A' of A there are at most $t-1$ vertices $w \in W$ with $A_w = A'$, as otherwise with v there would be a $K_{s,t}$ subgraph in G. Hence $|W| \leqslant (t-1)\binom{\operatorname{scol}_2(G)}{s-1}$, which is the desired contradiction. Thus each vertex v has monochromatic degree at most $\operatorname{scol}_2(G) + (t-1)\binom{\operatorname{scol}_2(G)}{s-1}$. Hence G is s-choosable with defect $\operatorname{scol}_2(G) + (t-1)\binom{\operatorname{scol}_2(G)}{s-1}$. □

Theorem 6.18 is in fact equivalent to an earlier result by Ossona de Mendez et al. [262]. To express their result, we need the following definitions. Let $s \geqslant 0$ be a half-integer (that is, $2s$ is an integer). Recall that a graph H is an *s-shallow topological minor* of a graph G if a $(\leqslant 2s)$-subdivision of H is a subgraph of G. For a graph G, let $G\widetilde{\triangledown}s$ be the set of all s-shallow topological minors of G. The *topological greatest reduced average density* $\widetilde{\nabla}_s(G)$ with rank s of a graph G is defined as

$$\widetilde{\nabla}_s(G) := \max_{H \in G\widetilde{\triangledown}s, |V(H)| \neq \varnothing} \frac{|E(H)|}{|V(H)|}.$$

Ossona de Mendez et al. [262] proved the following.

Theorem 6.19 ([262])**.** *There is a function f such that, for all integers $s, t \geqslant 1$, every graph G with no $K_{s,t}$ subgraph is s-choosable with defect at most $f(s, t, \widetilde{\nabla}_{1,2}(G))$.*

Theorem 6.19 was the key tool used by Ossona de Mendez et al. [262] in bounding the defective choice number for various sprase graph classes. Now Zhu [347] proved that $\tilde{\nabla}_{1,2}(G)$ and $\mathrm{scol}_2(G)$ are tied. So Theorem 6.18 and Theorem 6.19 are equivalent (up to the defect term). The advantage of Theorem 6.18 over Theorem 6.19 is its elementary proof.

Part II

Hereditary Graph Classes

Chapter 7

Pathwidth and Induced Subgraphs

7.1 Overview

Pathwidth is a fundamental parameter in structural and algorithmic graph theory [40, 185, 282]. This chapter studies the unavoidable induced subgraphs for graphs with large pathwidth. Due to the Excluded Tree Minor Theorem (Theorem 1.4) [285], obvious candidates for the unavoidable induced subgraphs include subdivisions of large complete binary trees and line graphs of subdivisions of large complete binary trees. While determining the other candidates in the general setting looks challenging, we show that these graphs suffice in the bounded degree setting (Section 7.4) as well as for K_n-minor-free graphs (Section 7.5). Let T_k denote the complete binary tree of height k.

Theorem 1.73. *There is a function f such that every graph G with maximum degree Δ and pathwidth at least $f(k, \Delta)$ contains a subdivision of T_k or the line graph of a subdivision of T_k as an induced subgraph.*

Theorem 1.74. *For every fixed $n \in \mathbb{N}$, there is a function f such that every K_n-minor-free graph G with pathwidth at least $f(k)$ contains a subdivision of T_k or the line graph of a subdivision of T_k as an induced subgraph.*

In addition, we characterise when a hereditary graph class defined by a finite set of forbidden induced subgraphs has bounded pathwidth. Recall that for a finite set of graphs $\mathcal{S}$, $\mathcal{I}_{\mathcal{S}}$ denotes the class of graphs that contain no graph in $\mathcal{S}$ as an induced subgraph.

Theorem 1.75. *For a finite set of graphs $\mathcal{S}$, $\mathcal{I}_{\mathcal{S}}$ has bounded pathwidth if and only if $\mathcal{S}$ includes a complete graph, a complete bipartite graph, a tripod and a semi-tripod.*

See [277] for other results concerning induced subgraphs and path-decompositions. This chapter is based on my single-authored paper [190].

7.2 Preliminiaries

Let G and H be graphs and let $r \geqslant 0$ be an integer. H is an *induced minor* of G if a graph isomorphic to H can be obtained from G by vertex deletion and edge contraction. An *induced minor model* $(X_v : v \in V(H))$ of H in G is a collection of non-empty subsets of $V(G)$ such that:

- $G[X_v]$ is a connected subgraph of G;
- $X_u \cap X_v = \varnothing$ for all distinct $u, v \in V(H)$;
- $G[X_u \cup X_v]$ is connected for every edge $uv \in E(H)$; and
- $G[X_u \cup X_v]$ is disconnected whenever $uv \notin E(H)$

It is folklore that H is an induced minor of G if and only if G contains an induced minor model of H. Recall that a graph G' is a *subdivision* of G if G' can be obtained from G by replacing each edge uv of G by a path of length at least one with end-vertices u and v whose internal vertices are new vertices private to that path. We call the vertices $V(G) \subseteq V(G')$ the *original vertices* in G'.

We restate the Excluded Tree Minor Theorem for convenience.

Theorem 1.4 (Excluded Tree Minor Theorem [34])**.** *For every tree T, every graph with pathwidth at least $|V(T)| - 1$ contains T as a minor.*

Note that Theorem 1.4 implies that every graph with pathwidth at least $|V(T_k)| - 1$ contains a subdivision of T_k as a subgraph since T_k has maximum degree 3.

7.3 From Induced Minors to Induced Subgraphs

To prove Theorems 1.73 and 1.74, we first construct an induced minor of a large complete binary tree in our graph. We then use the following lemma to find our desired induced subgraphs.

Lemma 7.1. *Every graph G that contains T_{8k} as an induced minor contains a subdivision of T_k or the line graph of a subdivision of T_k as an induced subgraph.*

The rest of this subsection is dedicated to proving Lemma 7.1. A *net-graph* is a semi-fork obtained from a triangle by appending disjoint paths of length 1 at each vertex of the triangle. For a graph G and a vertex $v \in V(G)$ with degree 3 and neighbours a, b, c, a *net-graph replacement* at v is the graph G', with $V(G') = V(G) \setminus \{v\} \cup \{x, y, z\}$ where x, y, z are new vertices and $E(H) = E(G - v) \cup \{xy, yz, zx, xa, yb, zc\}$. A *wattle* $\tilde{T}_k$ is obtained from a subdivision of T_k by picking a (possibly empty) subset X of the degree 3-vertices and performing net-graph replacements at each vertex in X. The following lemma from Aboulker et al. [1] will be useful in constructing an induced wattle $\tilde{T}_k$ from a large induced minor.

Lemma 7.2 ([1]). *Let G be a connected graph whose vertex-set is partitioned into connected sets $A, B, C, \{a\}, \{b\}, \{c\}$ and S. Suppose that every edge of G has either both ends in one of the sets, or is from $\{a\}$ to A, from $\{b\}$ to B, from $\{c\}$ to C, or from S to $A \cup B \cup C$. Then a, b, c are the degree one vertices of some induced fork or semi-fork in G.*

For a rooted tree (T, r), a vertex $u \in V(T)$ is an *ancestor* of $v \in V(T)$ (and v is a *descendant* of u) if u is a vertex on the (v, r)-path in T. If $p_v \in V(T)$ is the neighbour of v on the (v, r)-path in T, then p_v is the *parent* of v (and v is a *child* of p_v). For a rooted complete binary tree (T_k, r), every vertex $u \in V(T_k)$ with degree at least 2 has a *left child* ℓ_v and a *right child* c_v. A vertex $v \in V(T_k)$ is a *left (right) descendant* of u if it is a descendant of the left (right) child of u.

Lemma 7.3. *If a graph G contains T_{4k} as an induced minor, then G contains a wattle $\tilde{T}_k$ as an induced subgraph.*

Proof. Let $(X_v : v \in V(T_{4k}))$ be an induced minor model of (T_{4k}, r) in G. Let $(V_0, V_1, \dots, V_{4k})$ be a BFS-layering of T_{4k} where $V_0 = \{r\}$. For a vertex $v \in V(T_{4k})$ with degree 3, let $P_v := (w_{v,0}, w_{v,1} \dots, w_{v,m_v})$ be a vertex-minimal path in $G[X_v]$ such that $N_G(w_{v,0}) \cap X_{p_v}$ and $N_G(w_{v,m_v}) \cap X_{\ell_v}$ are non-empty. By minimality, $w_{v,0}$ is the only vertex in $V(P_v)$ adjacent to vertices in X_{p_v} and w_{v,m_v} is the only vertex in $V(P_v)$ adjacent to vertices in X_{ℓ_v}.

We prove the following claim by induction on $k \geqslant 0$: If a graph G contains T_{4k} as an induced minor, then G contains a wattle $\tilde{T}_k$ as an induced subgraph whose leaves are contained in $\{w_{v,0} : v \in V_{4k}\}$ and every vertex in $V(\tilde{T}_k) \cap (\bigcup(X_v : v \in V_{4k}))$ is a leaf.

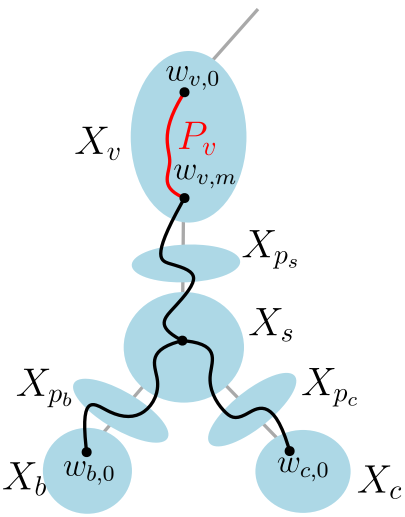


Figure 7.1. Extending the wattle $\tilde{T}_k$.

For $k = 0$, the claim holds trivially by letting $V(\tilde{T}_k)$ be an arbitrary vertex in X_r. For $k = 1$, let $a, b \in V_4$ respectively be a left and right descendant of r in T_{4k}. By taking $\tilde{T}_k$ to be a vertex minimal $(w_{a,0}, w_{b,0})$-path whose internal vertices are contained in $\bigcup(X_v : v \in V_1 \cup V_2 \cup V_3)$, we are done.

Now suppose the claim holds for $k - 1$. By induction, T_{4k-4} contains a wattle $\tilde{T}_{k-1}$ as an induced subgraph whose leaves are contained in $\{w_{v,0} : v \in V_{4k-4}\}$ and every vertex in $V(\tilde{T}_{k-1}) \cap (\bigcup(X_v : v \in V_{4k-4}))$ is a leaf. Now consider a leaf $w_{v,0} \in V(\tilde{T}_{k-1})$. First append the path P_v to $\tilde{T}_{k-1}$. Let $s \in V_{4k-2}$ be a left descendant of v and let $b, c \in V_{4k}$ respectively be left and right descendants of s. Let $A = X_{p_s}$, $B = X_{p_b}$, $C = X_{p_c}$ and $S = X_s$. Since $(X_v : v \in V(T_{4k}))$ is an induced minor, we may apply Lemma 7.2 on $G[\{w_{v,m_v}, w_{b,0}, w_{c,0}\} \cup A \cup B \cup C \cup S]$ to obtain an induced fork or semi-fork with end points $w_{v,m_v}, w_{b,0}, w_{c,0}$. Add this induced fork or semi-fork to $\tilde{T}_{k-1}$ and repeat for all leaves in $\tilde{T}_{k-1}$ to obtain an induced wattle $\tilde{T}_k$ that satisfies the induction hypothesis (see Figure 7.1). □

For a rooted tree (T, r), we say that a rooted subtree (T', r') is *vertical* if r' is an ancestor (with respect to (T, r)) of every vertex in $V(T')$. We use the following lemma to clean up our wattle $\tilde{T}_k$. A *red-blue colouring* of a bipartite graph G is a proper vertex 2-colouring of G with colours 'red' and 'blue'.

Lemma 7.4. *For every red-blue colouring of (T_{2k}, r), there exists a subdivision of a vertical (T_k, r') whose original vertices are monochromatic.*

Proof. We prove the following by induction on k: for every red-blue colouring of (T_k, r), there exists a subdivision of a vertical (T_h, r') whose original vertices are red and a subdivision of a vertical (T_j, r'') whose original vertices are blue such that $h + j \geqslant k$.

For $k = 0$, the claim is trivial. Now suppose the claim holds for $k - 1$. Let (T^1_{k-1}, r^1) and (T^2_{k-2}, r^2) be the components of $T_k - r$. By induction, for each $i \in \{1, 2\}$ there exists a subdivision of a vertical (T_{h_i}, r'_i) in (T^i_{k-1}, r^i) whose original vertices are red and a subdivision of a vertical (T_{j_i}, r''_i) in (T^i_{k-1}, r^i) whose original vertices are blue such that $h_i + j_i \geqslant k - 1$. If $\max\{h_1, h_2\} + \max\{j_1, j_2\} \geqslant k$ then we are done. Otherwise, $h_1 = h_2$, $j_1 = j_2$ and $h_1 + j_1 = k - 1$. If r is coloured red, then together with the (r'_1, r'_2)-path in T_k (which goes through r), we have a vertical red (T_{h_1+1}, r). If r is coloured blue, then together with the (r''_1, r''_2)-path in T_k, we have a vertical blue (T_{j_1+1}, r), as required. □

Lemma 7.5. *Every wattle $\tilde{T}_{2k}$ contains a subdivision of T_k or the line graph of a subdivision of T_k as an induced subgraph.*

Proof. Let T'_{2k} be an auxiliary copy of T_{2k} obtained from the wattle $\tilde{T}_{2k}$ by first contracting each triangle into a red vertex then contracting each subdivided path to an edge and colouring the remaining vertices blue. By Lemma 7.4, T'_{2k} contains a subdivision T'_k of T_k whose original vertices are monochromatic. If the original vertices of T'_k are red, then the wattle $\tilde{T}_{2k}$ contains the line graph of a subdivision of T_k as an induced subgraph (where each triangle in the line graph corresponds to an original red vertex in T'_k). Otherwise, the original vertices of T'_k are blue, and thus the wattle $\tilde{T}_{2k}$ contains a subdivision of T_k as an induced subgraph (where the original vertices correspond to the original blue vertices in T'_k). □

Lemma 7.1 immediately follows from Lemmas 7.3 and 7.5.

7.4 From Bounded Degree to Induced Minors

We now show that graphs with bounded degree and sufficiently large pathwidth contain a large complete binary tree as an induced minor. Theorem 1.73 immediately follows from the next theorem together with Lemma 7.1.

Theorem 7.6. *There is a function f such that every graph with maximum degree Δ and pathwidth at least $f(k,\Delta)$ contains T_k as an induced minor.*

To prove Theorem 7.6, we use sparsifable graphs which is a new technique introduced by Korhonen [225]. For a graph G, a vertex $v \in V(G)$ is *sparsifable* if it satisfies one of the following conditions:

1. v has degree at most 2;
2. v has degree 3 and all of its neighbours have degree at most 2;
3. v has degree 3, one of its neighbours has degree at most 2, and the two other neighbours form a triangle with v.

We say that G is *sparsifable* if all of its vertices are sparsifable. Sparsifable graphs are useful since minors and induced minors are roughly equivalent in this setting. More precisely, Korhonen [225] showed that if a sparsifable graph G contains a graph H with minimum degree at least 3 as a minor, then G contains H as an induced minor. We prove a slightly stronger version of this result that relaxes the minimum degree condition for H.

Lemma 7.7. *Let H and H^+ be graphs such that H is an induced subgraph of H^+ and each vertex in $V(H)$ has degree at least 3 in H^+. If a sparsifable graph G contains H^+ as a minor, then G contains H as an induced minor.*

Proof. Since G is sparsifable, it has maximum degree at most 3. For a model $(X_v : v \in V(H^+))$ of H^+ in G, we say that an edge $ab \in E(G)$ is *H-violating* if there are vertices $u, v \in V(H)$ such that $a \in X_u$, $b \in X_v$ and $uv \notin E(H)$. Choose $(X_v : v \in V(H^+))$ such that the number of H-violating edges is minimised. We claim that there are no H-violating edges, which implies that $(X_v : v \in V(H))$ is an induced minor of H in G.

For the sake of contradiction, suppose $ab \in E(G)$ is an H-violating edge with $a \in X_u$ and $b \in X_v$. Since a has degree at most 3 in G, u has degree at least 3 in H^+ and $uv \notin E(H^+)$, it follows that a has a neighbour in X_u. Likewise, b has a neighbour in X_v. Now if a has degree at most 2 in G, then $G[X_u \setminus \{a\}]$ is connected, so we may replace X_u by $X_u \setminus \{a\}$ to obtain a model of H^+ with strictly less H-violating edges, a contradiction. Thus a must have degree 3 in G and b likewise must also have degree 3 in G.

Now since G is sparsifable and a and b are adjacent with degree 3, they have a common neighbour c in G. If $c \notin \bigcup(X_w : w \in N_{H^+}[u])$, then $G[X_u \setminus \{a\}]$ is connected, and so we may replace X_u by $X_u \setminus \{a\}$ to obtain a model of H^+ with strictly less H-violating edges, a contradiction. By symmetry, a contradiction also occurs if $c \notin \bigcup(X_w : w \in N_{H^+}[v])$. Since $uv \notin E(H^+)$ it remains to consider the case when there exists $w \in V(H^+) \setminus \{u,v\}$ such that $c \in X_w$ and $uw, vw \in E(H^+)$. Since $w \notin \{u,v\}$, it follows that $G[X_u \setminus \{a\}]$ is connected and contains a neighbour of a and $G[X_v \setminus \{b\}]$ is connected and contains a neighbour of b. As such, by replacing X_u by $X_u \setminus \{a\}$, X_v by $X_v \setminus \{b\}$ and X_w by $X_w \cup \{a,b\}$, we obtain a model of H^+ with strictly less H-violating edges, a contradiction. $\square$

A *distance-5 independent set* $\mathcal{I} \subseteq V(G)$ in a graph G is a set of vertices such that the distance between any pair of vertices in $\mathcal{I}$ is at least 5. For a distance-5 independent set $\mathcal{I}$ in a graph G, let $\mathcal{I}(G)$ be the 2-shallow minor of G obtained by contracting each of the balls of radius 2 that are centred at vertices in $\mathcal{I}$ with corresponding model $(X_v : v \in V(\mathcal{I}(G))$. Observe that if G has maximum degree Δ, then $|X_v| \leqslant \Delta^2 + 1$ for all $v \in V(\mathcal{I}(G))$ and so $\mathrm{pw}(\mathcal{I}(G)) \geqslant (\mathrm{pw}(G)+1)/(\Delta^2+1) - 1$ (see Lemma 2.18). We use the following lemma implicitly proved by Korhonen [225].

Lemma 7.8 ([225])**.** *Let G be a graph and $\mathcal{I} \subseteq V(G)$ be a distance-5 independent set. Then for any subgraph H' of $\mathcal{I}(G)$ with maximum degree 3, there exists an induced subgraph $G[S]$ of G such that $G[S]$ contains H' as a minor and every vertex in $\mathcal{I} \cap S$ is sparsifable in $G[S]$.*

Lemma 7.9. *There exists a constant δ such that every graph G with $\mathrm{pw}(G) \geqslant 2k^2 \log^{\delta}(k)$ contains a subgraph H with maximum degree 3 and $\mathrm{pw}(H) \geqslant k$.*

Proof. By a result of Chekuri and Chuzhoy [74], if $\mathrm{tw}(G) \geqslant k \log^{\delta} k$ for some constant δ, then G contains a subgraph H with maximum degree 3 and treewidth at least k. Since pathwidth is bounded from below by treewidth, we are done. So assume that $\mathrm{tw}(G) < k \log^{\delta} k$. In which case, by a result of Groenland et al. [170], G contains a subdivision of the complete binary tree T_{2k} as a subgraph which has the desired pathwidth [295]. □

Let T_k^+ be the tree obtained from T_{k+1} by adding a leaf vertex adjacent to the root. Observe that T_k is an induced subtree of T_k^+ where each vertex in $V(T_k)$ has degree at least 3 in T_k^+. We are now ready to prove Theorem 7.6.

Proof of Theorem 7.6. Let $g(\Delta^4+1) := 2^{k+2} - 1$, $g(i) := \Big(2(g(i+1))^2 \log^{\delta}(g(i+1)) + 1\Big)(\Delta^2+1) - 1$ and $f(k, \Delta) := g(0)$ where δ is from Lemma 7.9. Let G be a graph with maximum degree Δ and $\mathrm{pw}(G) \geqslant f(k, \Delta)$. We first construct a sparsifable induced subgraph of G with large pathwidth. Using a greedy algorithm, partition $V(G)$ into $\Delta^4 + 1$ distance-5 independent sets $\mathcal{I}_0, \dots, \mathcal{I}_{\Delta^4}$. Initialise $i := 0$ and $G_i := G$. We construct G_{i+1} as an induced subgraph of G_i such that every vertex in $\mathcal{I}_i \cap V(G_{i+1})$ is sparsifable in G_{i+1} and $\mathrm{pw}(G_{i+1}) \geqslant g(i+1)$. Since G_i has maximum degree at most Δ, $\mathrm{pw}(\mathcal{I}_i(G_i)) \geqslant (\mathrm{pw}(G_i)+1)/(\Delta^2+1) - 1 \geqslant 2(g(i+1))^2 \log^{\delta}(g(i+1))$. By Lemma 7.9, $\mathcal{I}_i(G_i)$ contains a subgraph H_i with maximum degree 3 where $\mathrm{pw}(H_i) \geqslant g(i+1)$. By Lemma 7.8, there exists an induced subgraph $G_i[S_i]$ of G_i that contains H_i as a minor and every vertex in $\mathcal{I}_i \cap S_i$ is sparsifable in $G_i[S_i]$. As pathwidth is closed under minors, $\mathrm{pw}(G_i[S_i]) \geqslant \mathrm{pw}(H_i) \geqslant g(i+1)$. Set $G_{i+1} := G_i[S_i]$.

Now consider $\tilde{G} := G_{d^4+1}$. By the above procedure, $\tilde{G}$ is a sparsifable induced subgraph of G with pathwidth at least $g(\Delta^4+1) = 2^{k+2} - 1 = |V(T_k^+)| - 1$. By Theorem 1.4, $\tilde{G}$ contains T_k^+ as a minor. Therefore, by Lemma 7.7, $\tilde{G}$ contains T_k as an induced minor and hence G contains T_k as an induced minor. □

7.5 From Minor-Free to Induced Minors

Let $\mathcal{T}_k$ be the rooted tree of height k in which every non-leaf node has k children and every path from the root to a leaf has k edges. We prove the following result for K_n-minor-free graphs.

Theorem 7.10. *There is a function f such that every K_n-minor-free graph G with pathwidth at least $f(k,n)$ contains $\mathcal{T}_k$ as an induced minor.*

Since T_k is an induced subgraph of $\mathcal{T}_k$, Lemma 7.1 and Theorem 7.10 imply Theorem 1.74. Theorem 7.10 quickly follows from the following Ramsey-type result due to Kierstead and Penrice [213] that was recently re-proven by Atminas and Lozin [19].

Lemma 7.11 ([19, 213])**.** *There is a function g such that any graph that contains $\mathcal{T}_{g(n)}$ as a subgraph contains K_n, $K_{n,n}$ or $\mathcal{T}_n$ as an induced subgraph.*

Proof of Theorem 7.10. Let $f(k,n) := |V(\mathcal{T}_{g(\max\{k,n\})})| - 1$ where g is from Lemma 7.11.[1] By Theorem 1.4, G contains a minor model $(X_v : v \in V(\mathcal{T}_{g(\max\{k,n\})}))$ of $\mathcal{T}_{g(\max\{k,n\})}$. Let G' be the induced minor of G obtained from contracting each of the X_v's. Since G' contains $\mathcal{T}_{g(\max\{k,n\})}$ as a subgraph, it follows by Lemma 7.11 that G' contains K_n, $K_{n,n}$ or $\mathcal{T}_k$ as an induced subgraph. Since G excludes K_n as a minor, G' does not contain K_n or $K_{n,n}$ as subgraphs. Hence G' contains $\mathcal{T}_k$ as an induced subgraph and thus G contains $\mathcal{T}_k$ as an induced minor. $\square$

7.6 Finitely Many Forbidden Induced Subgraphs

In this section we characterise when a hereditary graph class defined by a finite set $\mathcal{S}$ of forbidden induced subgraphs has bounded pathwidth. Lozin and Razgon [243] showed that the graph class $\mathcal{I}_\mathcal{S}$ has bounded treewidth if and only if $\mathcal{S}$ includes a complete graph, a complete bipartite graph, a tripod and a semi-tripod. We strengthen this result to show that $\mathcal{I}_\mathcal{S}$ in fact has bounded pathwidth if and only if $\mathcal{S}$ includes a complete graph, a complete bipartite graph, a tripod and a semi-tripod.

Theorem 1.75. *For a finite set of graphs $\mathcal{S}$, $\mathcal{I}_\mathcal{S}$ has bounded pathwidth if and only if $\mathcal{S}$ includes a complete graph, a complete bipartite graph, a tripod and a semi-tripod.*

Proof. If $\mathcal{S}$ does not include a complete graph, a complete bipartite graph, a tripod or a semi-tripod, then by the observations of Lozin and Razgon [243], $\mathcal{I}_\mathcal{S}$ has unbounded treewidth and thus has unbounded pathwidth.

Now assume S contains a complete graph, a complete bipartite graph, a tripod and a semi-tripod. Observe that for every tripod (semi-tripod), there exists $k \in \mathbb{N}$ such that every (line graph of a) subdivision of T_k contains the tripod (semi-tripod) as an induced

[1] Note that $|V(\mathcal{T}_k)| = (k^{k+1} - 1)/(k-1)$ whenever $k \geqslant 2$.

subgraph. For the sake of contradiction, suppose $\mathcal{I}_\mathcal{S}$ has unbounded pathwidth. By the results of Lozin and Razgon [243], there exists $w \in \mathbb{N}$ such that every graph in $\mathcal{I}_\mathcal{S}$ has treewidth at most w. Thus every graph in $\mathcal{I}_\mathcal{S}$ is K_{w+2}-minor-free. Since $\mathcal{I}_\mathcal{S}$ has unbounded pathwidth, Theorem 1.74 implies that for every $k \in \mathbb{N}$, there exists a graph $G_k \in \mathcal{I}_\mathcal{S}$ that contains a subdivision of T_k or the line graph of a subdivision of T_k as an induced subgraph. Therefore, for k sufficiently large, G_k contains the tripod or the semi-tripod in $\mathcal{S}$ as an induced subgraph, a contradiction. $\square$

Chapter 8

Treewidth, Circle Graphs and Circular Drawings

8.1 Overview

This chapter studies the treewidth of graphs that are defined by circular drawings. Recall that a *circle graph* is an intersection graph of a set of chords of a circle. Our first contribution essentially determines when a circle graph has large treewidth.

Theorem 1.76. *Let $t \in \mathbb{N}$ and let G be a circle graph with treewidth at least $12t+2$. Then G contains an induced subgraph H that consists of t vertex-disjoint cycles $(C_1, \dots, C_t)$ such that, for all $i < j$, every vertex of C_i has at least two neighbours in C_j. Moreover, every vertex of G has at most four neighbours in any C_i $(1 \leqslant i \leqslant t)$.*

Since the subgraph H has a K_t-minor, it follows that every circle graph contains a complete minor whose order is at least one twelfth of its treewidth. We show that Theorem 1.76 implies the following relationship between the treewidth, Hadwiger number and Hajós number of circle graphs (see Section 8.5).

Theorem 1.77. *For the class of circle graphs, the treewidth and Hadwiger number are linearly tied. Moreover, the Hajós number is quadratically tied to both of them. Both 'linear' and 'quadratic' are best possible.*

The second aim of this chapter is to understand the relationship between circular drawings of graphs and their crossing graphs. A *circular drawing* of a graph places the vertices on a circle with edges drawn as straight line segments. If a graph has a circular drawing with a well-behaved crossing graph, must the graph itself also have a well-behaved structure? In this direction, we show the following.

Theorem 1.78. *For every integer $t \geqslant 3$, if a graph G has a circular drawing where the crossing graph has no K_t-minor, then G has treewidth at most $12t - 23$.*

Theorem 1.78 says that G having large treewidth is sufficient to force a complicated crossing graph in every circular drawing of G. A topological $K_{2,4t}$-minor also suffices.

Theorem 1.79. *If a graph G has a circular drawing where the crossing graph has no K_t-minor, then G contains no $K_{2,4t}$ as a topological minor.*

We also show that the assumption in Theorem 1.78 that the crossing graph has bounded Hadwiger number cannot be weakened to bounded degeneracy. In particular, we construct graphs with arbitrarily large complete graph minors that have a circular drawing whose crossing graph is 2-degenerate (Theorem 8.16).

Our proofs of Theorems 1.76, 1.78 and 1.77 are all based on the same core lemmas proved in Section 8.3. The results about circle graphs are in Section 8.5, while the results about graph drawings are in Section 8.4.

This chapter is based on joint work with Illingworth, Mohar and Wood [192].

8.2 Preliminaries

The *crossing graph* of a drawing D of a graph G is the graph X_D with vertex-set $E(G)$, where for each crossing between edges e and f in D, there is an edge of X_D between the vertices corresponding to e and f. Note that X_D is actually a multigraph, where the multiplicity of ef equals the number of times e and f cross in D. In most drawings that we consider, each pair of edges cross at most once, in which case X_D has no parallel edges.

Numerous papers have studied graphs that have a drawing whose crossing graph is well-behaved in some way. Here we give some examples. The *crossing number* $\mathrm{cr}(G)$ of a graph G is the minimum number of crossings in a drawing of G; see the surveys [268, 293, 311] or the monograph [292]. Obviously, $\mathrm{cr}(G) \leqslant k$ if and only if G has a drawing D with $|E(X_D)| \leqslant k$. Tutte [323] defined the *thickness* of a graph G to be the minimum number of planar graphs whose union is G; see [200, 256] for surveys. Every planar graph can be drawn with its vertices at prespecified locations [179, 270]. It follows that a graph G has thickness at most k if and only if G has a drawing D such that $\chi(X_D) \leqslant k$. A graph is *k-planar* if G has a drawing D in which every edge is in at most k crossings; that is, X_D has maximum degree at most k; see [114, 124, 169, 267] for example. More generally, Eppstein and Gupta [143] defined a graph G to be *k-degenerate crossing* if G has a drawing D in which X_D is k-degenerate. Bae et al. [22] defined a graph G to be *k-gap planar* if G has a drawing D in which each crossing can be assigned to one of the two involved edges and each edge is assigned at most k of its crossings. This is equivalent to saying that every subgraph of X_D has average degree at most $2k$. It follows that every k-degenerate crossing graph is k-gap-planar, and every k-gap-planar graph is a $2k$-degenerate crossing graph [204].

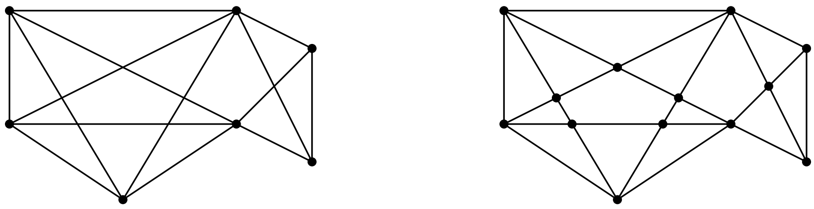
A drawing is *circular* if the vertices are positioned on a circle and the edges are straight line segments. A theme of this chapter is to study circular drawings D in which X_D is well-behaved in some way. Many papers have considered properties of X_D in this setting. The *convex crossing number* of a graph G is the minimum number of crossings in a circular drawing of G; see [293] for a detailed history of this topic. Obviously, G

has convex crossing number at most k if and only if G has a circular drawing D with $|E(X_D)| \leqslant k$. The *book thickness* (also called *page-number* or *stack-number*) of a graph G can be defined as the minimum, taken over all circular drawings D of G, of $\chi(X_D)$. This parameter is widely studied; see [28, 30, 113, 127, 343, 344] for example.

8.3 Tools

In this section, we introduce two auxiliary graphs that are useful tools for proving our main theorems.

For a drawing D of a graph G, the *planarisation*, P_D, of D is the plane graph obtained by replacing each crossing with a dummy vertex of degree 4, as illustrated in Figure 8.1. Note that P_D depends upon the drawing D (and not just upon G).



(a) Drawing D of a graph G. (b) Planarisation P_D.

Figure 8.1. A drawing and its planarisation.

Let G be a graph drawn in the plane without crossings where each face is labelled a 'nation' or a 'lake'. Recall that the *map graph* of G is the graph whose vertices are the nations of G, where two vertices are adjacent in G if the corresponding faces in G share a vertex. Throughout this chapter, we will implicitly assume that every face is a nation. Note that every map graph where each face is a nation is connected. For a drawing D of a graph G, we say that the *map graph*, M_D, of D is the map graph of the planarisation P_D of D. Figure 8.2 shows the map graph M_D for the drawing D in Figure 8.1.

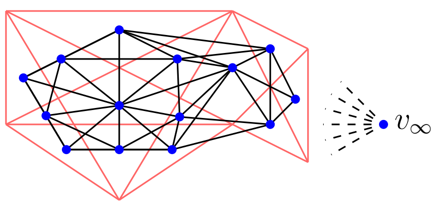


Figure 8.2. Map graph M_D. v_∞ is the vertex corresponding to the outer face: it is adjacent to all vertices except the unique vertex of degree 10.

In Section 8.3.1, we show that the radius of the map graph M_D acts as an upper bound for the treewidths of G and X_D. In Section 8.3.2, we show that if D is a circular drawing and the map graph M_D has large radius, then X_D contains a useful substructure. Thus the radius of M_D provides a useful bridge between the treewidth of G, the treewidth of X_D, and the subgraphs of X_D.

8.3.1 Map Graphs with Small Radii

Here we prove that for any drawing D of a graph G, the radius of M_D acts as an upper bound for both the treewidth of G and the treewidth of X_D.

Theorem 8.1. *For every drawing D of a graph G,*

$$\operatorname{tw}(G) \leqslant 6\operatorname{rad}(M_D) + 7 \quad \text{and} \quad \operatorname{tw}(X_D) \leqslant 6\operatorname{rad}(M_D) + 7.$$

Wood and Telle [341, Prop. 8.5] proved that if a graph G has a circular drawing D such that whenever edges e and f cross, e or f crosses at most d edges, then G has treewidth at most $3d+11$. This assumption implies $\operatorname{rad}(M_D) \leqslant \lfloor d/2 \rfloor + 1$ and so the first inequality of Theorem 8.1 generalises this result.

It is not surprising that treewidth and radius are related for drawings. A classical result of Robertson and Seymour [286, (2.7)] says that $\operatorname{tw}(G) \leqslant 3\operatorname{rad}(G) + 1$ for every connected planar graph G. Several authors improved this bound as follows.

Lemma 8.2 ([36, 124])**.** *For every connected planar graph G,*

$$\operatorname{tw}(G) \leqslant 3 \operatorname{rad}(G).$$

We now prove that if a planar graph G has large treewidth, then the map graph of any plane drawing of G has large radius. A *triangulation* of a plane graph G is a plane supergraph of G on the same vertex-set and where each face is a triangle.

Lemma 8.3. *Let G be a plane graph with map graph M_G. Then there is a plane triangulation H of G with $\operatorname{rad}(H) \leqslant \operatorname{rad}(M_G) + 1$. In particular,*

$$\operatorname{tw}(G) \leqslant 3 \operatorname{rad}(M_G) + 3.$$

Proof. Let F_0 be a face of G such that every vertex in M_G has distance at most $\operatorname{rad}(M_G)$ from F_0. For each face F of G, let $\operatorname{dist}_0(F)$ be the distance of F from F_0 in M_G.

Fix a vertex v_0 of G in the boundary of F_0, and set $\rho(v_0) := -1$. For every other vertex v of G, let

$$\rho(v) = \min\{\operatorname{dist}_0(F) \colon v \text{ is on the boundary of face } F\}.$$

Note that ρ takes values in $\{-1, 0, \dots, \operatorname{rad}(M_G)\}$.

We now construct a triangulation H of G such that every vertex $v \neq v_0$ is adjacent (in H) to a vertex u with $\rho(u) < \rho(v)$. In particular, the distance from v to v_0 in H is at most $\rho(v) + 1 \leqslant \operatorname{rad}(M_G) + 1$, and so H has the required radius. For each face F, let v_F be a vertex of F with smallest ρ-value. Note that $v_{F_0} = v_0$. Triangulate G as follows. First, consider one-by-one each face F. For every vertex v of F that is not already adjacent to v_F, add the edge vv_F. Finally, let H be obtained by triangulating the resulting graph.

Consider any vertex $v \neq v_0$. Let F be a face on whose boundary v lies and with $\rho(v) = \mathrm{dist}_0(F)$. If $F = F_0$, then $\rho(v) = 0$ and $vv_0 \in E(H)$, as required. Otherwise assume $F \neq F_0$. By considering a shortest path from F to F_0 in M_G, there is some face $F' \in V(M_G)$ that shares a vertex with F and has $\mathrm{dist}_0(F') < \mathrm{dist}_0(F)$. Let v' be a vertex on the boundary of both F and F'. Then $\rho(v_F) \leqslant \rho(v') \leqslant \mathrm{dist}_0(F') < \mathrm{dist}_0(F) = \rho(v)$. Furthermore, by construction, v and v_F are adjacent in H, as required.

By Lemma 8.2, $\mathrm{tw}(G) \leqslant \mathrm{tw}(H) \leqslant 3\,\mathrm{rad}(H) \leqslant 3\,\mathrm{rad}(M_G) + 3$. □

Note that a version of Lemma 8.3 with $\mathrm{rad}(M_G)$ replaced by the eccentricity of the outerface in M_G can be proved via outerplanarity[1].

We use the following lemma about planarisations to extend Lemma 8.3 from plane drawings to arbitrary drawings.

Lemma 8.4. *For every drawing D of a graph G, the planarisation P_D of D satisfies*

$$\mathrm{tw}(G) \leqslant 2\,\mathrm{tw}(P_D) + 1 \quad \text{and} \quad \mathrm{tw}(X_D) \leqslant 2\,\mathrm{tw}(P_D) + 1.$$

Proof. Consider a tree-decomposition $(T, \mathcal{W})$ of P_D in which each bag has size at most $\mathrm{tw}(P_D) + 1$. Now, we prove the first inequality. Arbitrarily orient the edges of G. Each dummy vertex x of P_D corresponds to a crossing between two oriented edges ab and cd of G. For each dummy vertex x, replace each instance of x in the tree-decomposition $(T, \mathcal{W})$ by b and d. It is straightforward to verify this gives a tree-decomposition $(T, \mathcal{W}')$ of G with bags of size at most $2\,\mathrm{tw}(P_D) + 2$. Hence $\mathrm{tw}(G) \leqslant 2\,\mathrm{tw}(P_D) + 1$.

Now, we prove the second inequality. Each dummy vertex x of P_D corresponds to a crossing between two edges e and f of G. For each dummy vertex x, replace each instance of x in $(T, \mathcal{W})$ by e and f. Also, for each vertex v of G, delete all instances of v from $(T, \mathcal{W})$. This gives a tree-decomposition $(T, \mathcal{W}'')$ of X_D with bags of size at most $2\,\mathrm{tw}(P_D) + 2$. Hence $\mathrm{tw}(X_D) \leqslant 2\,\mathrm{tw}(P_D) + 1$. □

We are now ready to prove Theorem 8.1.

Proof of Theorem 8.1. Let P_D be the planarisation of D. By definition, $M_D \cong M_{P_D}$. Lemma 8.3 implies

$$2\,\mathrm{tw}(P_D) + 1 \leqslant 2(3\,\mathrm{rad}(M_{P_D}) + 3) + 1 = 6\,\mathrm{rad}(M_D) + 7.$$

Lemma 8.4 now gives the required result. □

[1] Say a plane graph G is *k-outerplane* if removing all the vertices on the boundary of the outerface leaves a $(k-1)$-outerplane subgraph, where a plane graph is *0-outerplane* if it has no vertices. Consider a plane graph G, where v_∞ is the vertex of M_G corresponding to the outerface. Then one can show that if v_∞ has eccentricity k in M_G, then G is $(k+1)$-outerplane, and conversely, if G is k-outerplane, then v_∞ has eccentricity at most k in M_G. Bodlaender [40] showed that every k-outerplanar graph has treewidth at most $3k-1$. The same proof shows that every k-outerplane graph has treewidth at most $3k-1$ (which also follows from [124]).

8.3.2 Map Graphs with Large Radii

The next lemma is a cornerstone of this chapter. It shows that if the map graph of a circular drawing has large radius, then the crossing graph contains a useful substructure. For $a, b \in \mathbb{R}$ where $a < b$, let (a, b) denote the open interval $\{r \in \mathbb{R} \colon a < r < b\}$.

Lemma 8.5. *Let D be a circular drawing of a graph G. If the map graph M_D has radius at least $2t$, then the crossing graph X_D contains t vertex-disjoint induced cycles $C_1, \dots, C_t$ such that, for all $i < j$, every vertex of C_i has at least two neighbours in C_j. Moreover, every vertex of X_D has at most four neighbours in any C_i $(1 \leqslant i \leqslant t)$.*

Proof. Let $F \in V(M_D)$ be a face with distance at least $2t$ from the outer face of G. Let p be a point in the interior of F. Let R_0 be the infinite ray starting at p and pointing vertically upwards. More generally, for $\theta \in \mathbb{R}$, let R_θ be the infinite ray with endpoint p that makes a clockwise angle of θ (radians) with R_0. In particular, R_π is the ray pointing vertically downwards from p and $R_{\theta+2\pi} = R_\theta$ for all θ.

In the statement of the following claim, and throughout this chapter, "cross" means internally intersect.

Claim. *Every R_θ crosses at least $2t - 1$ edges of G.*

Proof. Consider moving along R_θ from p to the outer face. The distance in M_D only changes when crossing an edge or a vertex of G and changes by at most 1 when doing so. Since each R_θ contains at most one vertex of G, it must cross at least $2t - 1$ edges. □

For each edge e of G, define $I_e \coloneqq \{\theta \colon e \text{ crosses } R_\theta\}$. Since each edge is a line segment not passing through p, each I_e is of the form $(a, a') + 2\pi\mathbb{Z}$ where $a < a' < a + \pi$. Also note that edges e and f cross exactly if $I_e \cap I_f \neq \varnothing$, $I_e \not\subseteq I_f$, and $I_f \not\subseteq I_e$.

For a set of edges $E' \subseteq E(G)$, define $I_{E'} = \bigcup\{I_e \colon e \in E'\}$. We say that E' is *dominant* if $I_{E'} = \mathbb{R}$ and is *minimally dominant* if no proper subset of E' is dominant. Note that if $e, f \in E'$ and E' is minimally dominant, then e and f cross exactly if $I_e \cap I_f \neq \varnothing$.

Claim. *If E' is minimally dominant, then*

(i) *every R_θ crosses at most two edges of E',*
(ii) *E' induces a cycle in X_D,*
(iii) *every edge of G crosses at most four edges of E'.*

Proof. We first prove (i). Suppose that there is some R_θ crossing distinct edges $e_1, e_2, e_3 \in E'$. Then $\theta \in I_{e_1} \cap I_{e_2} \cap I_{e_3}$ and $\theta + \pi \notin I_{e_1} \cup I_{e_2} \cup I_{e_3}$. Hence we may write

$$I_{e_i} = (a_i, a_i') + 2\pi\mathbb{Z}, \qquad i = 1, 2, 3$$

where $\theta - \pi < a_i < \theta < a_i' < \theta + \pi$. By relabelling, we may assume that $a_1 < a_2 < a_3 < \theta$. Now, if a_3' is not the largest of a_1', a_2', a_3', then $(a_3, a_3') \subseteq (a_1, a_1') \cup (a_2, a_2')$ and so $I_{e_3} \subseteq I_{e_1} \cup I_{e_2}$ which contradicts the minimality of E'. Hence $a'_3 \geqslant a'_1, a'_2$. But then $(a_2, a'_2) \subseteq (a_1, a'_1) \cup (a_3, a'_3)$ and so $I_{e_2} \subseteq I_{e_1} \cup I_{e_3}$ which again contradicts minimality. This proves (i).

We next show that E' induces a connected subgraph of X_D. If E' does not, then there is a partition $E_1 \cup E_2$ of E' into non-empty sets such that no edge in E_1 crosses any edge in E_2. Since E' is minimally dominant, this means $I_{E_1} \cap I_{E_2} = \varnothing$. Consider $\mathbb{R}$ with the topology induced by the Euclidean metric, which is a connected space. But I_{E_1} and I_{E_2} are non-empty open sets that partition $\mathbb{R}$. Hence, E' induces a connected subgraph.

We now show that E' induces a 2-regular graph in X_D, which together with connectedness establishes (ii). Let $e \in E'$ and write $I_e = (a, a') + 2\pi\mathbb{Z}$ where $a < a' < a + \pi$. Since E' is dominant, there are $f, f' \in E'$ with $a \in I_f$ and $a' \in I_{f'}$. If $f = f'$, then $I_e \subseteq I_f$ which contradicts minimality. Hence f, f' are distinct and so e has degree at least two in X_D. Suppose that e has some neighbour f'' in X_D distinct from f, f'. Since $I_{f''}$ is not a subset of I_e, it must contain at least one of a, a'. By symmetry, we may assume that $I_{f''}$ contains a. But then, for some sufficiently small $\varepsilon > 0$, all of $I_e, I_f, I_{f''}$ contain $a + \varepsilon$ and so $R_{a+\varepsilon}$ crosses three edges of E', which contradicts (i). Hence e has exactly two neighbours in E' which establishes (ii).

Finally consider an arbitrary edge $e = uv$ of G. Let R_u be the infinite ray from p that contains u and R_v be the infinite ray from p that contains v. Observe that every edge of G that crosses e also crosses R_u or R_v. By (i), at most four edges in E' cross e which proves (iii). □

For a set of edges $E' \subseteq E(G)$, say an edge $e \in E'$ is *maximal in E'* if there is no $f \in E' \setminus \{e\}$ with $I_e \subseteq I_f$. Suppose E' is dominant. Let $E'_{\max}$ be the set of maximal edges in E'. Clearly, $E'_{\max}$ is still dominant and so has a minimally dominant subset. In particular, every dominant set of edges E' has a subset E_1 that is minimally dominant and all of whose edges are maximal in E'.

Claim. *Let $E' \subseteq E(G)$ and $E_1, E_2 \subseteq E'$. Suppose that all the edges of E_1 are maximal in E' and that E_2 is dominant. Then every edge in E_1 crosses at least two edges in E_2.*

Proof. Let $e_1 \in E_1$ and write $I_{e_1} = (a, a') + 2\pi\mathbb{Z}$ where $a < a' < a + \pi$. Since E_2 is dominant, there are $e_2, e_3 \in E_2$ with $a \in I_{e_2}$ and $a' \in I_{e_3}$. If $e_2 = e_3$, then $I_{e_1} \subseteq I_{e_2}$, which contradicts the maximality of e_1 in E'.

By symmetry, it suffices to check that e_1 and e_2 cross. Note that for some sufficiently small $\varepsilon > 0$, $a + \varepsilon$ is in both I_{e_1} and I_{e_2} and so $I_{e_1} \cap I_{e_2} \neq \varnothing$. As $a \in I_{e_2} \setminus I_{e_1}$ we have $I_{e_2} \not\subseteq I_{e_1}$. Finally, the maximality of e_1 in E' means $I_{e_1} \not\subseteq I_{e_2}$. Hence e_1 and e_2 do indeed cross. □

We are now ready to complete the proof. Note that a set of edges is dominant exactly if it crosses every R_θ. By the first claim, $E = E(G)$ is dominant. Let $E_1 \subseteq E$ be minimally dominant such that every edge of E_1 is maximal in E. By part (i) of the second claim, every R_θ crosses at most two edges of E_1 and so, by the first claim, crosses at least $2t-3$ edges of $E \setminus E_1$. Hence, $E \setminus E_1$ is dominant. Let $E_2 \subseteq E \setminus E_1$ be minimally dominant such that every edge of E_2 is maximal in $E \setminus E_1$. Continuing in this way, we obtain pairwise disjoint $E_1, E_2, \dots, E_t \subset E$ such that, for all $i \geqslant 1$:

- E_i is minimally dominant;
- every edge of E_i is maximal in $E \setminus (\bigcup_{i'<i} E_{i'})$;
- every R_θ crosses at most two edges of E_i; and
- every R_θ crosses at least $2(t-i)-1$ edges of $E \setminus (\bigcup_{i' \leqslant i} E_{i'})$.

By part (ii) of the second claim, every E_i induces a cycle C_i in X_D. Let $i < j$ and $E' := E \setminus (\bigcup_{i'<i} E_{i'})$. Then $E_i, E_j \subseteq E'$ and every edge of E_i is maximal in E'. Hence, by the third claim, every edge in E_i crosses at least two edges in E_j. In particular, every vertex of C_i has at least two neighbours in C_j.

Finally, by part (iii) of the second claim, every vertex of X_D has at most four neighbours in any C_i. □

8.4 Structural Properties of Circular Drawings

Theorem 8.1 says that for any drawing D of a graph G, the radius of M_D provides an upper bound for $\text{tw}(G)$ and $\text{tw}(X_D)$. For a general drawing it is impossible to relate $\text{tw}(X_D)$ to $\text{tw}(G)$. Firstly, planar graphs can have arbitrarily large treewidth (for example, the $(n \times n)$-grid has treewidth n) and admit drawings with no crossings. In the other direction, $K_{3,n}$ has treewidth 3 and crossing number $\Omega(n^2)$, as shown by Kleitman [218]. In particular, the crossing graph of any drawing of $K_{3,n}$ has average degree linear in n and thus has arbitrarily large complete minors [244, 245] and so arbitrarily large treewidth.

Happily, this is not true for circular drawings. Using the tools in Section 8.3, we show that if a graph G has large treewidth, then the crossing graph of any circular drawing of G has large treewidth. In fact, the crossing graph must contain a large (topological) complete graph minor (see Theorems 1.78 and 8.6). In particular, if X_D is K_t-minor-free, then G has small treewidth. We further show that if X_D is K_t-minor-free, then G does not contain a subdivision of $K_{2,4t}$ (Theorem 1.79). Using these results, we deduce a product structure theorem for G (Corollary 8.7).

In the other direction, we ask what properties of a graph G guarantee that it has a circular drawing D where X_D has no K_t-minor. Certainly G must have small treewidth. Adding the constraint that G does not contain a subdivision of $K_{2,f(t)}$ is not sufficient (see Lemma 8.13) but a bounded maximum degree constraint is: we show that if G has bounded maximum degree and bounded treewidth, then G has a circular drawing where the crossing graph has bounded treewidth (Proposition 8.14).

We also show that there are graphs with arbitrarily large complete graph minors that admit circular drawings whose crossing graphs are 2-degenerate (see Theorem 8.16).

8.4.1 Necessary Conditions for K_t-Minor-Free Crossing Graphs

This subsection studies the structure of graphs that have circular drawings whose crossing graph is (topological) K_t-minor-free. Much of our understanding of the structure of these graphs is summarised by the next four results (Theorems 1.78, 1.79 and 8.6 and Corollary 8.7).

Theorem 1.78. *For every integer $t \geqslant 3$, if a graph G has a circular drawing where the crossing graph has no K_t-minor, then G has treewidth at most $12t - 23$.*

Theorem 1.79. *If a graph G has a circular drawing where the crossing graph has no K_t-minor, then G contains no $K_{2,4t}$ as a topological minor.*

Theorem 8.6. *If a graph G has a circular drawing where the crossing graph has no topological K_t-minor, then G has treewidth at most $6t^2 + 6t + 1$.*

From these, we may deduce a product structure theorem for graphs that have a circular drawing whose crossing graph is K_t-minor-free. Campbell et al. [63, Prop. 55] showed that if a graph G is $K_{2,t}$-topological minor-free and has treewidth at most k, then G is contained in $H \boxtimes K_{\mathcal{O}(t^2k)}$ where $\operatorname{tw}(H) \leqslant 2$. Thus, Theorems 1.78 and 1.79 imply the following product structure result.

Corollary 8.7. *If a graph G has a circular drawing where the crossing graph has no K_t-minor, then G is contained in $H \boxtimes K_{\mathcal{O}(t^3)}$ where $\operatorname{tw}(H) \leqslant 2$.*

En route to proving these results, we use the cycle structure built by Lemma 8.5 to find (topological) complete minors in the crossing graph of circular drawings. We first show that the treewidth and Hadwiger number $h(X_D)$ of X_D as well as the radius of M_D are all linearly tied.

Lemma 8.8. *For every circular drawing D,*

$$\operatorname{tw}(X_D) \leqslant 6 \operatorname{rad}(M_D) + 7 \leqslant 12\, h(X_D) - 11 \leqslant 12 \operatorname{tw}(X_D) + 1.$$

Proof. The first inequality is exactly Theorem 8.1, while the final one is the well-known fact that $h(G) \leqslant \operatorname{tw}(G) + 1$ for every graph G. To prove the middle inequality, we need to show that for any circular drawing D,

$$\operatorname{rad}(M_D) \leqslant 2\, h(X_D) - 3. \tag{8.1}$$

Let $t := h(X_D)$ and suppose, for a contradiction, that $\operatorname{rad}(M_D) \geqslant 2t - 2$. By Lemma 8.5, X_D contains $t - 1$ vertex-disjoint cycles $C_1, \dots, C_{t-1}$ such that, for all $i < j$, every vertex of C_i has a neighbour in C_j. Contracting C_1 to a triangle and each C_i ($i \geqslant 2$) to a vertex gives a K_{t+1}-minor in X_D. This is the required contradiction. ∎

Clearly, the Hajós number of a graph is at most the Hadwiger number. Our next lemma implies that the Hajós number $h'(X_D)$ of X_D is quadratically tied to the radius of M_D and to the treewidth and Hadwiger number of X_D.

Lemma 8.9. *For every circular drawing D,*

$$\mathrm{rad}(M_D) \leqslant h'(X_D)^2 + 3\,h'(X_D) + 1.$$

Proof. Let $t = h'(X_D) + 1$ and suppose, for a contradiction, that $\mathrm{rad}(M_D) \geqslant t^2 + t$. By Lemma 8.5, X_D contains $(t^2 + t)/2$ vertex-disjoint cycles $C_1, \ldots, C_{(t^2+t)/2}$ such that, for all $i < j$, every vertex of C_i has a neighbour in C_j. For each $i \in [t]$, let $v_i \in V(C_i)$. We assume that $V(K_t) = [t]$ and let $\phi \colon E(K_t) \to [t+1, (t^2+t)/2]$ be a bijection. Then, for each $ij \in E(K_t)$, there is a (v_i, v_j)-path P_{ij} in X_D whose internal vertices are contained in $V(C_{\phi(ij)})$. Since ϕ is a bijection, it follows that $(P_{ij} \colon ij \in E(K_t))$ defines a topological K_t-minor in X_D, a contradiction. $\square$

We are now ready to prove Theorems 1.78 and 8.6.

Proof of Theorem 1.78. Let D be a circular drawing of G with $h(X_D) \leqslant t - 1$. By (8.1), $\mathrm{rad}(M_D) \leqslant 2t - 5$. Finally, by Theorem 8.1, $\mathrm{tw}(G) \leqslant 12t - 23$. $\square$

Proof of Theorem 8.6. Let D be a circular drawing of G with $h'(X_D) \leqslant t - 1$. By Lemma 8.9, $\mathrm{rad}(M_D) \leqslant t^2 + t - 1$. Finally, by Theorem 8.1, $\mathrm{tw}(G) \leqslant 6t^2 + 6t + 1$. $\square$

We now show that the bound on $\mathrm{tw}(G)$ in Theorem 1.78 is within a constant factor of being optimal. Let G_n be the $(n \times n)$-grid, which has treewidth n (see [185]). Theorem 1.78 says that in every circular drawing D of G_n, the crossing graph X_D has a K_t-minor, where $t = \Omega(n)$. On the other hand, let D be the circular drawing of G_n obtained by ordering the vertices $R_1, R_2, \ldots, R_n$, where R_i is the set of vertices in the i-th row of G_n (ordered arbitrarily). Let E_i be the set of edges in G_n incident to vertices in R_i; note that $|E_i| \leqslant 3n - 1$. If two edges cross, then they have end-vertices in some E_i. Thus $(E_1, \ldots, E_n)$ is a path-decomposition of X_D of width at most $3n$. In particular, X_D has no K_{3n+2}-minor. Hence, the bound on $\mathrm{tw}(G)$ in Theorem 1.78 is within a constant factor of optimal. See [300, 301] for more on circular drawings of grid graphs.

Now we turn to subdivisions and the proof of Theorem 1.79. As a warm-up, we give a simple proof in the case of no division vertices.

Proposition 8.10. *For every $k \in \mathbb{N}$, for every circular drawing D of $K_{2,4k-1}$, X_D contains $K_{k,k}$ as a subgraph.*

Proof. Let the vertex classes of $K_{2,4k-1}$ be X and Y, where $X = \{x, y\}$ and $|Y| = 4k - 1$. Vertices x and y split the circle into two arcs, one of which must contain at least $2k$ vertices from Y. Label these vertices $x, v_1, \ldots, v_s, y$ where $s \geqslant 2k$ in order around the circle. For every $i \in [k]$, define the edges $e_i = yv_i$ and $f_i = xv_{k+i}$. The e_i and f_i are vertices in X_D, and for all i and j, edges e_i and f_j cross, as required. $\square$

We now work towards the proof of Theorem 1.79.

A *linear drawing* of a graph G places the vertices on the x-axis with edges drawn as semi-circles above the x-axis. In such a drawing, we consider the vertices of G to be elements of $\mathbb{R}$ given by their x-coordinates. Such a drawing can be wrapped to give a circular drawing of G with an isomorphic crossing graph. For an edge $uv \in E(G)$ where $u < v$, define I_{uv} to be the open interval (u, v). For a set of edges $E' \subseteq E(G)$, define $I_{E'} := \bigcup\{I_e \colon e \in E'\}$. Two edges $uv, xy \in E(G)$ where $u < v$ and $x < y$ are *nested* if $u < x < y < v$ or $x < u < v < y$.

Lemma 8.11. *Let $a, b \in \mathbb{R}$ where $a < b$, and let D be a linear drawing of a graph G where G consists of two internally vertex-disjoint paths $P_1 = (v_1, \dots, v_n)$ and $P_2 = (u_1, \dots, u_m)$ such that $u_1, v_1 \leqslant a < b \leqslant u_m, v_n$. Then there exists $E' \subseteq E(G)$ such that $(a, b) \subseteq I_{E'}$ and E' induces a connected graph in X_D. Moreover, for $x \in \{a, b\}$, if $x \notin V(P_1) \cap V(P_2)$, then $x \in I_{E'}$.*

Proof. We first show the existence of E'. Observe that $(a, b) \subseteq I_{E(P_1)} \cup \{v_1, \dots, v_n\}$. If G contains an edge uv where $u \leqslant a < b \leqslant v$, then we are done by setting $E' = \{uv\}$. So assume that G has no edge of that form. Then there is a vertex $v \in V(P_1)$ such that $a < v < b$. Each such vertex v is not in $V(P_2)$, implying $v \in I_{E(P_2)}$. Therefore $(a, b) \subseteq I_{E(G)}$. Let E' be a minimal set of edges of $E(G)$ such that $(a, b) \subseteq I_{E'}$. By minimality, no two edges in E' are nested. We claim that $X_D[E']$ is connected. If not, then there is a partition $E_1 \cup E_2$ of E' into non-empty sets such that no edge in E_1 crosses any edge in E_2. Since E' is minimal, this means $I_{E_1} \cap I_{E_2} = \varnothing$. Consider (a, b) with the topology induced by the Euclidean metric, which is a connected space. But $I_{E_1} \cap (a, b)$ and $I_{E_2} \cap (a, b)$ are non-empty open sets that partition (a, b), a contradiction. Hence, $X_D[E']$ is connected.

Finally, let $x \in \{a, b\}$ and suppose that $x \notin V(P_1) \cap V(P_2)$. Then G has an edge uv such that $u < x < v$. If $x \in I_{E'}$, then we are done. Otherwise, E' contains an edge incident to x. Since $a < u < b$ or $a < v < b$, it follows that uv crosses an edge in E'. So adding uv to E' maintains the connectivity of $X_D[E']$ and now $x \in I_{E'}$. $\square$

Lemma 8.12. *Let G be a subdivision of $K_{2,3}$ and let $x, y \in V(G)$ be the vertices with degree 3. For every circular drawing D of G, there exists a component Y in X_D that contains an edge incident to x and an edge incident to y.*

Proof. Let P_1, P_2, P_3 be the internally disjoint (x, y)-paths in G. Let $\mathcal{U} = (u_1, \dots, u_m)$ be the sequence of vertices on the clockwise arc from x to y (excluding x and y). Let $\mathcal{V} = (v_1, \dots, u_n)$ be the sequence of vertices on the anti-clockwise arc from x to y (excluding x and y). Say an edge $uv \in E(G)$ is *vertical* if $u \in \mathcal{U}$ and $v \in \mathcal{V}$.

Suppose that no edge of G is vertical. By the pigeonhole principle, we may assume that $V(P_1) \cup V(P_2) \subseteq \mathcal{U} \cup \{x, y\}$. The claim then follows by applying Lemma 8.11 along the clockwise arc from x to y.

Now assume that $E(G)$ contains at least one vertical edge. Let $e_1, \ldots, e_k$ be an ordering of the vertical edges of G such that if e_i is incident to $u_{i'}$ and e_{i+1} is incident to $u_{j'}$, then $i' \leqslant j'$. In the case when $u_{i'} = u_{j'}$, then e_i and e_{i+1} are ordered by their endpoints in $\mathcal{V}$.

Claim. *For each $i \in [k-1]$, there exists $E_i \subseteq E(G)$ such that $E_i \cup \{e_i, e_{i+1}\}$ induces a connected subgraph of X_D.*

Proof. Clearly, the claim holds if e_i and e_{i+1} cross or if there is an edge in G that crosses both e_i and e_{i+1}. So assume that e_i and e_{i+1} do not cross and no edge crosses both e_i and e_{i+1}. Assume $e_i = u'v'$ and $e_{i+1} = u''v''$, where $u', u'' \in \mathcal{U}$ and $v', v'' \in \mathcal{V}$. Let $j \in \{1, 2, 3\}$. If P_j does not contain e_i, then P_j contains neither endpoint of e_i. Since e_i separates x from y in the drawing, P_j contains e_i or an edge that crosses e_i. Likewise, P_j contains e_{i+1} or an edge that crosses e_{i+1}. Let $P'_j = (p_1, \ldots, p_m)$ be a vertex-minimal subpath of P_j such that p_1p_2 is e_i or crosses e_i, and $p_{m-1}p_m$ is e_{i+1} or crosses e_{i+1}. By minimality, no edge in $E(P'_j) \setminus \{p_1p_2, p_{m-1}p_m\}$ crosses e_i or e_{i+1}. Therefore, by the ordering of the vertical edges, no edge in $E(P'_j) \setminus \{p_1p_2, p_{m-1}p_m\}$ is vertical. As such, either $\{p_2, \ldots, p_{m-1}\} \subseteq \mathcal{U}$ or $\{p_2, \ldots, p_{m-1}\} \subseteq \mathcal{V}$. By the pigeonhole principle, without loss of generality, $V(P'_1) \cup V(P'_2) \subseteq \mathcal{U}$. Since $V(P'_1)$ and $V(P'_2)$ have distinct endpoints, the claim then follows by applying Lemma 8.11 along the clockwise arc between u' and u''. □

It follows from the claim that all the vertical edges are contained in a single component Y of X_D. Now consider the three edges in G incident to x. By the pigeonhole principle, without loss of generality, two of these edges are of the form xu_i, xu_j where $i < j$. Let u_a be the vertex in $\mathcal{U}$ incident to the vertical edge e_1. If $a < j$, then e_1 crosses xu_j. If $a = j$, then by the ordering of the vertical edges, the path P_i that contains the edge xu_i also contains an edge that crosses both e_1 and xu_j. Otherwise, $j < a$ and applying Lemma 8.11 to the clockwise arc between u_j and u_a, it follows that xu_j is also in Y. By symmetry, there is an edge incident to y that is in Y, as required. □

We are now ready to prove Theorem 1.79.

Proof of Theorem 1.79. Let G be a subdivision of $K_{2,4t}$ and let D be a circular drawing of G. We show that X_D contains a K_t-minor. Let x, y be the degree $4t$ vertices in G. Let $\mathcal{U} = (u_1, \ldots, u_m)$ be the sequence of vertices on the clockwise arc from x to y (excluding x and y). Let $\mathcal{V} = (v_1, \ldots, v_n)$ be the sequence of vertices on the anti-clockwise arc from x to y (excluding x and y). Say an edge $uv \in E(G)$ is *vertical* if $u \in \mathcal{U}$ and $v \in \mathcal{V}$.

Let ℓ be the number of vertical edges in G. Let $k := \min\{\ell, t\}$ and let $d := t - k$. Then G contains $4d$ paths $P_1, \ldots, P_{4d}$ that contain no vertical edge. We say that P_i is a *$\mathcal{U}$-path* (respectively, *$\mathcal{V}$-path*) if it contains an edge incident to a vertex in $\mathcal{U}$ ($\mathcal{V}$). By the pigeonhole principle, without loss of generality, $P_1, \ldots, P_{2d}$ are $\mathcal{U}$-paths. By pairing the paths and then applying Lemma 8.11 to the clockwise arc from x to y, it follows that X_D contains d vertex-disjoint connected subgraphs $Y_1, \ldots, Y_d$ in X_D where each Y_i contains an edge (in G) incident to x and an edge incident to y. Consider distinct $i, j \in \{1, \dots, d\}$. Let $xu_{i'} \in V(Y_i)$ and $xu_{j'} \in V(Y_j)$ and assume that $i' < j'$. Since $xu_{j'}$ separates $u_{i'}$ from y in the drawing, and $P_1, \dots, P_{2d}$ are internally disjoint, it follows that there is an edge in $V(Y_i)$ that crosses $xu_{j'}$. So $Y_1, \dots, Y_d$ are pairwise adjacent, which form a K_d-minor in X_D.

Let $\tilde{E} := \{e_1, \dots, e_k\}$ be any set of k vertical edges in G. Since $t = d + k$, there are $4k$ internally disjoint (x, y)-paths distinct from $P_1, \dots, P_{4d}$, at least $3k$ of which avoid $\tilde{E}$. Grouping these paths into k sets each with three paths, it follows from Lemma 8.12 that there exists k vertex-disjoint connected subgraphs $Z_1, \dots, Z_k$ in X_D where each Z_i contains an edge (in G) incident to x and an edge incident to y. Since each $e \in \tilde{E}$ separates x and y in the drawing, it follows that each $V(Y_i)$ and $V(Z_j)$ contains an edge (in G) that crosses e. Thus, by contracting each Y_i into a vertex and each $Z_j \cup \{e_j\}$ into a vertex and then deleting all other vertices in X_D, we obtain the desired K_t-minor in X_D. $\square$

8.4.2 Sufficient Conditions for K_t-Minor-Free Crossing Graphs

It is natural to consider whether the converse of Theorems 1.78 and 1.79 holds. That is, does there exist a function f such that if a $K_{2,t}$-topological minor-free graph G has treewidth at most k, then there is a circular drawing of G whose crossing graph is $K_{f(t,k)}$-minor-free? Our next result shows that this is false in general. A *t-rainbow* in a circular drawing of a graph is a non-crossing matching consisting of t edges between two disjoint arcs in the circle.

Lemma 8.13. *For every $t \in \mathbb{N}$, there exists a $K_{2,4}$-topological minor-free graph G with* $\text{tw}(G) = 2$ *such that, for every circular drawing D of G, the crossing graph X_D contains a K_t-minor.*

Proof. Let T be any tree with maximum degree 3 and sufficiently large pathwidth (as a function of t). Such a tree exists as the complete binary tree of height $2h$ has pathwidth h. Let G be obtained from T by adding a vertex v complete to $V(T)$, so G has treewidth 2. Since $G - v$ has maximum degree 3, it follows that G is $K_{2,4}$-topological minor-free.

Let D be a circular drawing of G and let D_T be the induced circular drawing of T. Since T has sufficiently large pathwidth, a result of Pupyrev [278, Thm. 2] implies that X_D has large chromatic number or a $4t$-rainbow[2]. Since the class of circle graphs is χ-bounded [177], it follows that if X_D has large chromatic number, then it contains a large clique and we are done. So we may assume that D_T contains a $4t$-rainbow. By the pigeonhole principle, there is a subset $\{a_1b_1, \dots, a_{2t}b_{2t}\}$ of the rainbow edges such that a_ib_i topologically separates v from a_j and b_j whenever $i < j$. As such, a_ib_i crosses the edges va_j and vb_j in D whenever $i < j$. Therefore X_D contains a $K_{t,2t}$ subgraph with bipartition $(\{a_1b_1, \dots, a_tb_t\}, \{va_{t+1}, vb_{t+1}, \dots, va_{2t}, vb_{2t}\})$ and this contains a K_t-minor. $\square$

[2] The result of Pupyrev [278] is in terms of stacks and queues but is equivalent to our statement.

Lemma 8.13 is best possible in the sense that $K_{2,4}$ cannot be replaced by $K_{2,3}$. An easy exercise shows that every biconnected $K_{2,3}$-topological minor-free graph is either outerplanar or K_4. It follows (by considering the block-cut tree) that every $K_{2,3}$-minor-free graph has a circular 1-planar drawing, so the crossing graph consists of isolated edges and vertices.

While $K_{2,k}$-topological minor-free and bounded treewidth is not sufficient to imply that a graph has a circular drawing whose crossing graph is K_t-minor-free, we now show that bounded degree and bounded treewidth is sufficient.

Proposition 8.14. *For $k, \Delta \in \mathbb{N}$, every graph G with treewidth less than k and maximum degree at most Δ has a circular drawing in which the crossing graph X_D has treewidth at most $(6\Delta + 1)(18k\Delta)^2 - 1$.*

Proof. By Theorem 1.39, G is contained in $T \boxtimes K_m$ where T is a tree with maximum degree $\Delta_T := 6\Delta$ and $m := 18k\Delta$. Since the treewidth of the crossing graph does not increase when deleting edges and vertices from the drawing, it suffices to show that $T \boxtimes K_m$ admits a circular drawing in which the crossing graph X_D has treewidth at most $(\Delta_T + 1)m^2 - 1$. Without loss of generality, assume that $V(K_m) = \{1, \dots, m\}$. Take a circular drawing of T such that no two edges cross (this can be done since T is outerplanar). For each vertex $v \in V(T)$, replace v by $((v, 1), \dots, (v, m))$ to obtain a circular drawing D of $T \boxtimes K_m$. Observe that if two edges $(u, i)(v, j)$ and $(x, a)(y, b)$ cross in D, then $\{u, v\} \cap \{x, y\} \neq \varnothing$. For each vertex $v \in V(T)$, let W_v be the set of edges of $T \boxtimes K_m$ that are incident to some (v, i). We claim that $(W_v : v \in V(T))$ is a tree-decomposition of X_D with the desired width. Clearly, each vertex of X_D is in a bag and for each vertex $e \in V(X_D)$, the set $\{x \in V(T) : e \in W_x\}$ induces a graph isomorphic to either K_2 or K_1 in T. Moreover, by the above observation, if $e_1e_2 \in E(X_D)$, then there exists some node $x \in V(T)$ such that $e_1, e_2 \in W_x$. Finally, since there are $\binom{m}{2}$ intra-K_m edges and $\Delta_T \cdot m^2$ cross-K_m edges, it follows that $|W_v| \leqslant (\Delta_T + 1)m^2$ for all $v \in V(T)$, as required. □

We conclude this subsection with the following open problem:

Open Problem 8.15. Does there exist a function f such that every $K_{2,k}$-minor-free graph G has a circular drawing D in which the crossing graph X_D is $K_{f(k)}$-minor-free?

8.4.3 Circular Drawings and Degeneracy

Theorems 1.78 and 8.6 say that if a graph G has a circular drawing D where the crossing graph X_D excludes a fixed (topological) minor, then G has bounded treewidth. Graphs excluding a fixed (topological) minor have bounded average degree and degeneracy [244, 245]. Despite this, we now show that X_D having bounded degeneracy is not sufficient to bound the treewidth of G. In fact, it is not even sufficient to bound the Hadwiger number of G.

Theorem 8.16. *For every $t \in \mathbb{N}$, there is a graph G_t and a circular drawing D of G_t such that:*

- *G_t contains a K_t-minor;*
- *G_t has maximum degree 3; and*
- *X_D is 2-degenerate.*

Proof. We draw G_t with vertices placed on the x-axis (x-coordinate between 1 and t) and edges drawn on or above the x-axis. This can then be wrapped to give a circular drawing of G_t.

For real numbers $a_1 < a_2 < \cdots < a_n$, we say a path P is drawn as a *monotone path* with vertices $a_1, \ldots, a_n$ if it is drawn as follows where each vertex has x-coordinate equal to its label:

In all our monotone paths, $a_1, a_2, \ldots, a_n$ will be an arithmetic progression. We construct our drawing of G_t as follows (see Figure 8.3 for the construction with $t = 4$).

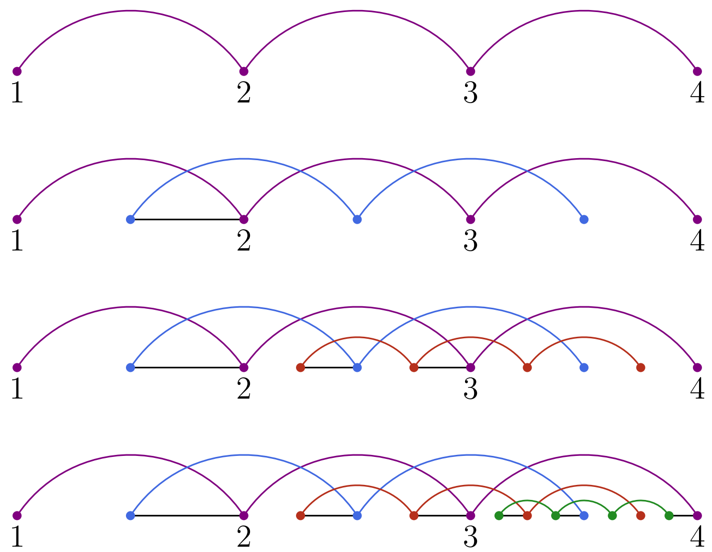


Figure 8.3. G_4 built up path-by-path, where P_0 is purple, P_1 is blue, P_2 is red, P_3 is green, and the $e_{r,s}$ are black.

First let P_0 be the monotone path with vertices $1, 2, \ldots, t$. For $s \in \{1, 2, \ldots, t-1\}$, let P_s be the monotone path with vertices

$$s + 2^{-s}, s + 3 \cdot 2^{-s}, s + 5 \cdot 2^{-s}, \ldots, t - 2^{-s}.$$

Observe that these paths are vertex-disjoint. For $0 \leqslant r < s \leqslant t-1$, let $I_{r,s}$ be the interval

$$[s + 2^{-r} - 2^{-s}, s + 2^{-r}].$$

Note that the lower end-point of $I_{r,s}$ is a vertex in P_s and the upper end-point is a vertex in P_r. Also note that no vertex of any P_i lies in the interior of $I_{r,s}$. Indeed, for $i > s$, the vertices of P_i have value at least $s + 2^{-r}$ and for $i \leqslant s$, the denominator of the vertices of P_i precludes them from being in the interior. Hence for all $r < s$ we may draw a horizontal edge $e_{r,s}$ between the end-points of $I_{r,s}$.

Graph G_t and the drawing D are obtained as a union of the P_s together with all the $e_{r,s}$. The paths P_s are vertex-disjoint and edge $e_{r,s}$ joins P_r to P_s, so G_t contains a K_t-minor. We now show that the $I_{r,s}$ are pairwise disjoint. Note that $I_{r,s} \subset (s, s+1]$ so two I with different s values are disjoint. Next note that $I_{r,s} \subset (s + 2^{-(r+1)}, s + 2^{-r}]$ for $r \leqslant s-2$ while $I_{s-1,s} = [s + 2^{-s}, s + 2^{-(s-1)}]$ and so two I with the same s but different r values are disjoint. In particular, any vertex v is the end-point of at most one $e_{r,s}$ and so has degree at most three. Hence, G_t has maximum degree three.

Each edge $e_{r,s}$ is horizontal and crosses no other edges so has no neighbours in X_D. Next consider an edge aa' of P_s. We have $a' = a + 2 \cdot 2^{-s}$. Exactly one vertex in $V(P_0) \cup V(P_1) \cup \cdots \cup V(P_s)$ lies between a and a': their midpoint, $m = a + 2^{-s}$. Vertex m has at most two non-horizontal edges incident to it and so, in X_D, every $aa' \in E(P_s)$ has at most two neighbours in $E(P_0) \cup E(P_1) \cup \cdots \cup E(P_s)$. Thus, X_D is 2-degenerate, as required. □

8.5 Structural Properties of Circle Graphs

Recall that a *circle graph* is the intersection graph of a set of chords of a circle. More formally, let C be a circle in $\mathbb{R}^2$. A *chord* of C is a closed line segment with distinct endpoints on C. Two chords of C either cross, are disjoint, or have a common endpoint. Let S be a set of chords of a circle C such that no three chords in S cross at a single point. Let G be the crossing graph of S. Then G is called a *circle graph*. Note that a graph G is a circle graph if and only if $G \cong X_D$ for some circular drawing D of a graph H, and in fact one can take H to be a matching.

We are now ready to prove Theorems 1.76 and 1.77. While the treewidth of circle graphs has previously been studied from an algorithmic perspective [220], to the best of our knowledge, these theorems are the first structural results on the treewidth of circle graphs.

Theorem 1.76. *Let $t \in \mathbb{N}$ and let G be a circle graph with treewidth at least $12t+2$. Then G contains an induced subgraph H that consists of t vertex-disjoint cycles $(C_1, \dots, C_t)$ such that, for all $i < j$, every vertex of C_i has at least two neighbours in C_j. Moreover, every vertex of G has at most four neighbours in any C_i $(1 \leqslant i \leqslant t)$.*

Proof. Let D be a circular drawing of a graph such that $G \cong X_D$. Let M_D be the map graph of D. Since $\mathrm{tw}(X_D) = \mathrm{tw}(G) \geqslant 12t + 2$, it follows by Theorem 8.1 that M_D has radius at least $2t$. The claim then follows from Lemma 8.5. □

Theorem 1.77. *For the class of circle graphs, the treewidth and Hadwiger number are linearly tied. Moreover, the Hajós number is quadratically tied to both of them. Both 'linear' and 'quadratic' are best possible.*

Proof. Let G be a circle graph and let D be a circular drawing with $G \cong X_D$. By Lemma 8.8,

$$\operatorname{tw}(G) \leqslant 6\operatorname{rad}(M_D) + 7 \leqslant 12\,h(G) - 11 \leqslant 12\operatorname{tw}(G) + 1.$$

So the Hadwiger number and treewidth are linearly tied for circle graphs. This inequality and Lemma 8.9 imply

$$h'(G) - 1 \leqslant h(G) - 1 \leqslant \operatorname{tw}(G) \leqslant 6\operatorname{rad}(M_D) + 7 \leqslant 6h'(G)^2 + 18\,h'(G) + 13.$$

Hence the Hajós number is quadratically tied to both the treewidth and Hadwiger number for circle graphs. Finally, $K_{t,t}$ is a circle graph which has treewidth t, Hadwiger number $t+1$, and Hajós number $\Theta(\sqrt{t})$. Hence, 'quadratic' is best possible. □

We now discuss several noteworthy consequences of Theorems 1.77 and 1.76. Recently, there has been significant interest in understanding the unavoidable induced subgraphs of graphs with large treewidth [2–9, 45, 243, 275, 303]. Obvious candidates of unavoidable induced subgraphs include complete graphs, complete bipartite graphs, subdivision of large walls, and line graphs of subdivision of large walls. We say that a hereditary class of graphs $\mathcal{G}$ is *induced-*tw*-bounded* if there is a function f such that, for every graph $G \in \mathcal{G}$ with $\operatorname{tw}(G) \geqslant f(t)$, G contains K_t, $K_{t,t}$, a subdivision of the $(t \times t)$-wall, or a line graph of a subdivision of the $(t \times t)$-wall as an induced subgraph[3]. While the class of all graphs is not induced-tw-bounded [45, 88, 277, 303], many natural graph classes are. For example, Aboulker et al. [1] showed that every proper minor-closed class is induced-tw-bounded and Korhonen [225] showed that the class of graphs with bounded maximum degree is induced-tw-bounded. We now show the following.

Theorem 8.17. *The class of circle graphs is not induced-*tw*-bounded.*

Proof. We first show that for all $t \geqslant 50$, no circle graph contains a subdivision of the $(t \times t)$-wall or a line graph of a subdivision of the $(t \times t)$-wall as an induced subgraph. As the class of circle graphs is hereditary, it suffices to show that for all $t \geqslant 50$, these two graphs are not circle graphs. These two graphs are planar (so K_5-minor-free) and have treewidth $t \geqslant 50$. However, Lemma 8.8 implies that every K_5-minor-free circle graph has treewidth at most 49, which is the required contradiction.

Now consider the family of couples of graphs $((G_t, X_t) \colon t \in \mathbb{N})$ given by Theorem 8.16 where X_t is the crossing graph of the drawing of G_t. Then $(X_t \colon t \in \mathbb{N})$ is a family of circle graphs. Since $(G_t \colon t \in \mathbb{N})$ has unbounded treewidth, Theorem 1.78 implies that

[3]This definition is motivated by analogy to χ-boundedness; see [296]. Note that while the language of 'induced tw-bounded' is original to this thesis, Abrishami et al. [7] previously used this definition under the guise of 'special' and Abrishami et al. [3] used it under the guise of 'clean'.

$(X_t : t \in \mathbb{N})$ also has unbounded treewidth. Moreover, since X_t is 2-degenerate for all $t \in \mathbb{N}$, it excludes K_4 and $K_{3,3}$ as (induced) subgraphs, as required. □

While the class of circle graphs is not induced-tw-bounded, Theorem 1.76 describes the unavoidable induced subgraphs of circle graphs with large treewidth. To the best of our knowledge, this is the first theorem to describe the unavoidable induced subgraphs of a natural hereditary graph class that is not induced-tw-bounded. In fact, it does so with a linear lower bound on the treewidth of the unavoidable induced subgraphs.

Theorem 1.76 can also be used to describe the unavoidable induced subgraphs of circle graphs with large pathwidth.

Theorem 8.18. *There exists a function f such that every circle graph G with $\mathrm{pw}(G) \geqslant f(t)$ contains:*

- *a subdivision of a complete binary tree with height t as an induced subgraph, or*
- *the line graph of a subdivision of a complete binary tree with height t as an induced subgraph, or*
- *an induced subgraph H that consists of t vertex-disjoint cycles $(C_1, \dots, C_t)$ such that, for all $i < j$, every vertex of C_i has at least two neighbours in C_j. Moreover, every vertex of G has at most four neighbours in any C_i $(1 \leqslant i \leqslant t)$.*

Proof. If $\mathrm{tw}(G) \geqslant 12t + 2$, then the claim follows from Theorem 1.76. Now assume $\mathrm{tw}(G) < 12t + 2$. So G excludes K_{12+3} as a minor. By Lemma 7.1 and Theorem 7.10, it follows that there is a function $g(k, t)$ such that every graph with treewidth less than k and pathwidth at least $g(k, t)$ contains a subdivision of a complete binary tree with height t as an induced subgraph or the line graph of a subdivision of a complete binary tree with height t as an induced subgraph. The result follows with $f(t) := \max\{g(12t + 2, t), 12t + 2\}$. □

We now discuss applications of Theorem 1.76 to vertex-minor-closed classes. For a vertex v of a graph G, to *locally complement* at v means to replace the induced subgraph on the neighbourhood of v by its complement.[4] A graph H is a *vertex-minor* of a graph G if H can be obtained from G by a sequence of vertex deletions and local complementations. Vertex-minors were first studied by Bouchet [55, 56] under the guise of isotropic systems. The name 'vertex-minor' is due to Oum [264]. Circle graphs are a canoncial example of a vertex-minor-closed class.

We now show that a vertex-minor-closed graph class is induced-tw-bounded if and only if it has bounded rank-width. Rank-width is a graph parameter introduced by Oum and Seymour [266] that describes whether a graph can be decomposed into a tree-like structure by simple cuts. For a formal definition and surveys on this parameter, see [198, 265]. Oum [264] showed that rank-width is closed under vertex-minors.

[4]The *complement* of a graph H, $\overline{H}$, is defined with $V(\overline{H}) = V(H)$ where $uv \in E(\overline{H})$ if and only if $uv \notin E(H)$.

Theorem 8.19. *A vertex-minor-closed class $\mathcal{G}$ is induced-*tw*-bounded if and only if it has bounded rankwidth.*

Proof. Suppose $\mathcal{G}$ has bounded rankwidth. By a result of Abrishami, Chudnovsky, Hajebi, and Spirkl [7], there is a function f such that every graph in $\mathcal{G}$ with treewidth at least $f(t)$ contains K_t or $K_{t,t}$ as an induced subgraph. Thus, $\mathcal{G}$ is induced-tw-bounded. Now suppose $\mathcal{G}$ has unbounded rank-width. By a result of Geelen, Kwon, McCarty, and Wollan [161], $\mathcal{G}$ contains all circle graphs. It therefore follows by Theorem 8.17 that $\mathcal{G}$ is not induced-tw-bounded. □

We conclude with the following question:

Open Problem 8.20. Let $\mathcal{G}$ be a vertex-minor-closed class with unbounded rank-width. What are the unavoidable induced subgraphs of graphs in $\mathcal{G}$ with large treewidth?

The cycle structure (or variants thereof) in Theorem 1.76 must be included in the list of unavoidable induced subgraphs.

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Index