Examples of Polya Enumeration

G-group   X- set of items   Y- colors

Y^X - set of functions $f: X \to Y$

Weight: weight of a function $f \in Y^X$:

$$W(f) = \prod_{i=1}^{n} x_{f(i)} \qquad n = \# X$$

Inventory of Y^X:

$$F(Y^X) = \sum_{f \in Y^X} W(f)$$

2 functions $f \in Y^X$ are equivalent under the action of G (or in the same G-class) if $\exists\, g \in G$:
$f_2 = g f_1$ ($f_2 \sim_G f_1$)

Configuration: equivalence class of relation $\sim_G$ on Y^X

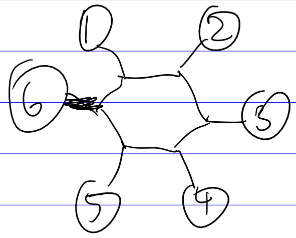
Type of $g \in G$: n-tuple $(b_1 \ldots b_n)$ where b_i is number of cycles of length i

Statement: Inventory of G-classes of functions is:

$$F(C) = \frac{1}{\# G} \sum_{g \in G} \left(\sum_{f \in Y} w(f) \right)^{b_1} \left(\sum_{f \in Y} w(f)^2 \right)^{b_2} \cdots \left(\sum_{f \in Y} w(f)^n \right)^{b_n}$$

Examples:

Enumeration of Isomers

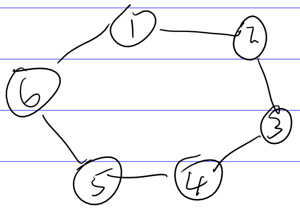


We have 6 carbons & want to attach H or Cl to each carbon.

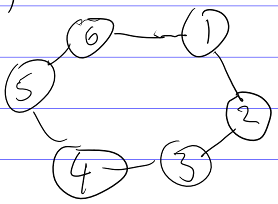
G - cyclic group C_6, only rotations

$$G - \{g^0, g, g^2, g^3, g^4, g^5\}$$

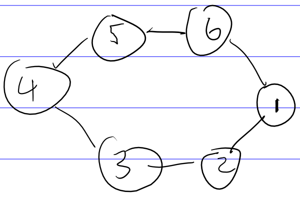
$g^0 = (1)(2)(3)(4)(5)(6)$



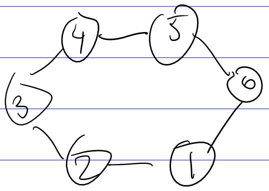
$g = (1,2,3,4,5,6)$



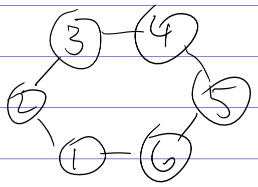
$g^2 = (1,3,5)(2,4,6)$



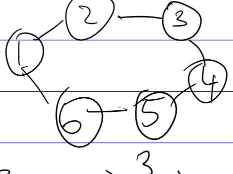
$g^3 = (1,4)(2,5)(3,6)$



$g^4 = (1,5,3)(2,6,4)$



$g^5 = (1,6,5,4,3,2)$



$$F(C) = \frac{1}{6}\left((H+Cl)^6 + (H^2+Cl^2)^3 + 2(H^3+Cl^3)^2 + 2(H^6+Cl^6)\right)$$

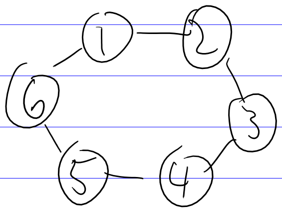
$$= H^6 + H^5Cl + 3H^4Cl^2 + 4H^3Cl^3 + 3H^2Cl^4 + HCl^5 + Cl^6$$

Now suppose we have a 6-bead necklace where we can flip & not just rotate.

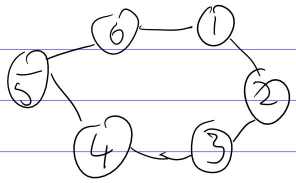
Group D_{2n} "Dihedral necklaces"

Rotations

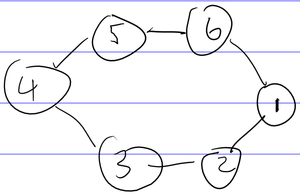
$(1)(2)(3)(4)(5)(6)$



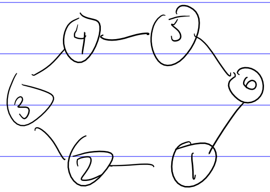
(1, 2, 3, 4, 5, 6)



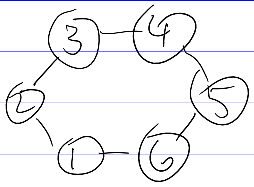
(1, 3, 5) (2, 4, 6)



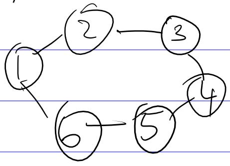
(1, 4) (2, 5) (3, 6)



(1, 5, 3) (2, 6, 4)

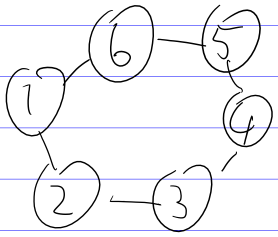


(1, 6, 5, 4, 3, 2)

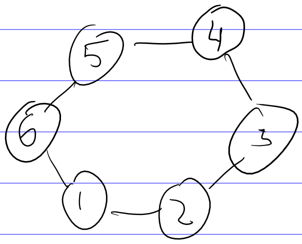


Reflections

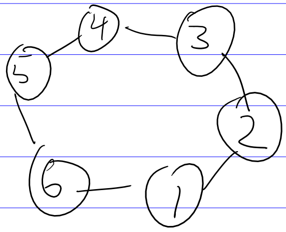
(16) (25) (34)



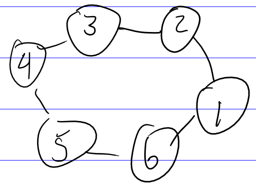
(15) (24)



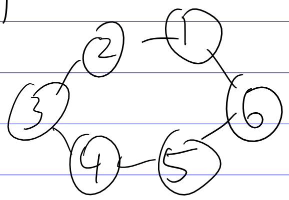
(14) (23) (56)



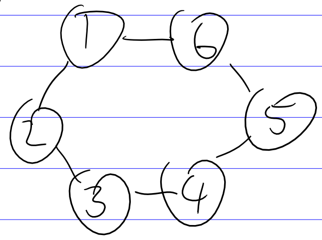
(13)(46)



(12)(36)(45)



(26)(35)



Cycle polynomial:

$$\frac{1}{12}\left(x_1^6 + 3x_1^2 x_2^2 + 4x_2^3 + 2x_3^2 + 2x_6\right)$$

Suppose we have 3 beads r, b, y

$$\frac{1}{12}\Big((r+b+y)^6 + 3(r+b+y)^2(r^2+b^2+y^2)^2 + 4(r^2+b^2+y^2)^3$$
$$+ 2(r^3+b^3+y^3)^2 + 2(r^6+b^6+y^6)\Big)$$

$$= b^6 + b^5r + 3b^4r^2 + 3b^3r^3 + 3b^2r^4 + br^5 + r^6$$
$$+ b^5y + 3b^4ry + 6b^3r^2y + 6b^2r^3y + 3br^4y$$
$$+ r^5y + 3b^4y^2 + 6b^3ry^2 + 11b^2r^2y^2 + 6br^3y^2$$
$$+ 3r^4y^2 + 3b^3y^3 + 6b^2ry^3 + 6br^2y^3 + 3r^3y^3$$
$$+ 3b^2y^4 + 3bry^4 + 3r^2y^4 + by^5 + ry^5 + y^6$$

92 total combinations (coefficients add up)
ex. $11b^2r^2y^2$ means 11 necklaces with 2r, 2b, 2y. Also,

$$92 = \frac{1}{12}\left(3^6 + 3\cdot 3^2 3^2 + 4\cdot 3^3 + 2\cdot 3^2 + 2\cdot 3\right)$$

Cyclic permutation example with permutations of 3 items

123 – (1)(2)(3) (1 2 3) – 312

213 – (1 2)(3) (1 3)(2) – 321 (2 3)(1) – 132

(1 3 2) – 231

⟹ Cycle polynomial

$$\frac{1}{6}\left(X_1^3 + 3X_2X_1 + 2X_3\right)$$