Pólya's Enumeration Theorem

Roadmap

1) Establish a Galois Connection between the lattice of subgroups of a permutation group & the lattice of partitions of the set on which it acts

2) Computations carried out in the smaller lattice of closed partitions & transferred to permutation group. Central computation is a double Möbius Inversion.

3) Introduce generating functions for sets of functions (colorings) & certain counting functions, including a generalization of Euler φ.

4) Combine these relations to derive a generating function for equivalence classes of functions under a group action & then derive PET.

Section 1 – Galois theory for permutations & partitions

G – group of permutations of a set X

$L(G)$ – lattice of subgroups ordered by inclusion

ex. $X = \{1,2,3\}$ $\quad G - S_3 \quad L(G)$:

$\{(), (1,2), (1,3), (2,3), (1,2,3), (1,3,2)\} \in S_3$

Alternating Group (even permutations) $\quad \{(), (2,3)\} \quad \{(), (1,3)\} \quad \{(), (1,2)\}$

$\{(), (1,2,3)(1,3,2)\}$

$\{()\}$

Def

A partition of a finite n element set X is a collection $\{\pi_1, \pi_2, \dots\}$ of non-empty mutually disjoint subsets of X, $\pi_i \cap \pi_j = \emptyset$ if $i \neq j$, $\bigcup_i \pi_i = X$. The sets π_i are called blocks of the partion.

Let P – set of all partitions of X and

$$A = \{a_1, \dots\} \in P \quad B = \{b_1, \dots\} \in P$$

A is a refinement of B if every block $a_i \in A$ is contained in some $b_i \in B$.

A is a refinement of B if every block b_i can be formed by merging blocks a_i.

The Poset of partitions
$\leq$: $A \leq B$ iff A is a refinement of B, ordered by refinement

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Let $\Pi(X)$ be lattice of partitions of X, ordered by refinement

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ex: $X = \{1, 2, 3\}$

$$\begin{tikzcd}
 & \{\{1,2,3\}\} \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
\{\{1\},\{2,3\}\} \arrow[dr, dash] & \{\{2\}\{1,3\}\} \arrow[d, dash] & \{\{3\}\{1,2\}\} \arrow[dl, dash] \\
 & \{\{1\}\{2\}\{3\}\} &
\end{tikzcd}$$

We can identify a partition of X with an equivilence relation on X if needed

Galois Connection
- A relation between posets, weaker than order-isomorphism

Def
Let $(P, \leq_P)$ and $(Q, \leq_Q)$ be posets. Suppose $F: P \to Q$ and $G: Q \to P$ are a pair of functions s.t. $\forall\, p \in P,\ q \in Q$:

$$F(p) \leq_Q q \quad \text{iff} \quad p \leq_P G(q)$$

Then $\langle F, G \rangle$ form a Galois connection between P and Q.

Define: $\varphi : L(G) \to \Pi(X)$: If H is a subgroup of G $(H \leq G)$ then $F(H)$ is the partition whose blocks are the H-orbits in X, that is, $a \equiv b \bmod F(H)$ iff $b = g(a)$ for some $g \in H$. We call this partition the period of H.

Define: $\Theta : \Pi(X) \to L(G)$: if π is a partition of X, $\Theta(\pi)$ is the set of group elements g which leave the blocks of π invariant (or, equivalently, for which the cycles of g are contained within blocks of π)

$\Theta(\pi)$ forms a subgroup of G.

F and G are increasing mappings:

$$H \subseteq \Theta\tau(H) \quad \forall H \in L(G)$$

$$\pi \geq \tau\Theta(\pi) \quad \forall \pi \in \Pi(X)$$

Def

Residuated mapping - If A, B are Posets, a function $f: A \to B$ is residuated iff the preimage under f of every principal down set ($\downarrow\{b\} = \{b' \in B ; b' \leq b\}$) of B is a principal down-set of A

Def

Dual of a poset P P^* is a poset "flipped" i.e. if $x \leq y \in P \Rightarrow y \leq x \in P^*$

τ is a residual mapping with residual Θ.

If we consider $\Pi(X)^*$ dual to $\Pi(X)$ then (τ, Θ) is a Galois connection between $L(G)$ and $\Pi(X)^*$ (There is a result about this).

$\Theta\tau$ and $\tau\Theta$ are closure operators on $L(G)$ and $\Pi(X)^*$ i.e. idempotent (no effect if called more than once), increasing mappings $x \mapsto \overline{x}$ s.t. $\overline{x} \geq x$ $\forall x$.

Closed subgroups of G are called periodic

Closed partitions of X are called periods.

These periods are the partitions which are periods of subgroups of G.

Def

The period of an element $g \in G$ is the smallest possible integer m s.t. $a^m = e$ (identity)

Denote $\overline{\pi} = \tau\Theta(\pi)$. $\overline{\pi}$ refines π by above inequality since $\pi \mapsto \overline{\pi}$ is a closure operator on $\Pi(X)$.

Let $P(G, X)$ be the lattice of periods of G in X, i.e.

$$P(G, X) = \{\pi \in \Pi(X) \mid \overline{\pi} = \pi\}$$

with induced refinement ordering (Not necessarily a sublattice of $\Pi(X)$).

F & G are inverse order isomorphisms (with original refinement ordering) between lattice of periodic subgroups of G and $P(G, X)$

Section 2 - Colorings & Generating functions

- Polya theory - enumeration of equivilence classes of colorings under the operation of groups of symmetry of the objects being colored.

X - Set of objects (finite) $\quad$ Y - at most countable set of colors
G - group of symmetries

The colorings of X are functions $f: X \to Y$.

Def

Kernel (set theory)

Let X, Y be sets & $f: X \to Y$. Elements x_1 and x_2 are equivilent if $f(x_1)$ and $f(x_2)$ map to same element in Y. The Kernal of f is this equivilence relation "equivilent as far as the function f can tell"

So each function f is an equivilence relation on X, or a partition.

Let the closure of f be called the "G-period of f in X" denoted per (f).

Then $\Theta(\ker f) = \{g \in G \mid fg = f\}$ and per(f) is the partition whose blocks are the orbits of $\Theta(\ker f)$.

Associate with each $f \in Y^X$ a monomial:

$$M(f) = \prod_{i \in X} x^i_{f(i)}$$

x^i_j $\quad$ i - ranges over X $\quad$ j - ranges over Y.

This is a formal way to list ordered pairs of the function.

Let S be a set of functions. Associate a generating function:

$$M(S) = \sum_{f \in S} M(f)$$

A set of functions F is called a proper class (with respect to G) if $fg \in F$ whenever $f \in F$ and $g \in G$. Examples include Y^X, onto functions, on-to-one function, set of functions f s.t. $hf \in fG$ where h is some fixed permutation of Y.

For fixed proper class

π - partition of X, Define generating functions:

$$A(\pi) = M(\{f \in F \mid \ker f = \pi\})$$
$$A_G(\pi) = M(\{f \in F \mid per(f) = \pi\})$$
$$B(\pi) = M(\{f \in F \mid \ker f \geq \pi\})$$

These are functions on $\Pi(X)$ with values in ring of formal power series with variables x_j^i ($R\{[x]\}$)

Let $\mathcal{M}$ - additive group of functions $\Pi(X) \to R\{[X]\}$, and $I(\Pi(X), R\{[X]\})$, or $I(\Pi(X))$ be the incidence algebra of $\Pi(X)$ over $R\{[X]\}$. Then $\mathcal{M}$ is both a right & left $I(\Pi(X))$-module:

$$K \in I(\Pi(X)) \qquad F \in \mathcal{M}$$

$$KF(\pi) = \sum_{6 \geq \pi} K(\pi, 6) F(6)$$

$$FK(\pi) = \sum_{6 \leq \pi} F(6) K(6, \pi)$$

We will use δ, ζ, μ function from incidence algebra

lattice of periods of G in X

Incidence algebra $I(G,X)$ of $P(G,X)$ is inbedded (contained) in $I(\Pi(X))$. (In a paper "Incidence functions as generalized arithmetic functions").

& we let δ', ζ', μ' be the functions for $I(G,X)$.

We also define

$$\bar{\delta} = \delta(\pi, \bar{6}), \qquad \pi, 6 \in \Pi(X)$$

& then $\delta = \mu' \zeta$ (Theorem in another paper).

We also have $B = \zeta A$

Theorem

For any proper class F, $A_G = \mu' B = \bar{\delta} A$, or for any $\pi \in \Pi(X)$,

$$A_G(\pi) = \sum_{6 \geq \pi} \mu'(\pi, 6) B(6) = \sum_{\tau = \pi} A(\tau)$$

2 Counting Functions

1) For each period π let

$\nu(\pi) = |\Theta(\pi)|$, the number of group elements g whose cycles are contained in blocks of π

2) $\varphi(\pi) = |\{g \in G \mid \text{cycles of } g \text{ are blocks of } \pi\}|$, a generalization of euler φ.

We can extend these functions to all of $\Pi(X)$ by defining them to be 0 if π is not a period. Then we can treat them as elements of the module $\mathcal{M}$.

We also have

$$\nu = \varphi \zeta' \quad \text{and} \quad \varphi = \nu \mu'$$

Example

Let G be the cyclic group generated by a single cyclic permutation $g = (1, 2, 3, \dots n)$.

Then $L(G) \cong D_n$ (Lattice of divisors of n).

A typical subgroup has the form (g^m) where $m|n$.

Closure of (g^m) is the subgroup of powers of g leaving invariant the cycles of g. Every subgroup is closed (periodic). So τ is an isomorphism between $L(G)$ and $P(G, X)$, with the image of (g^m) being the partition whose blocks are cycles of g^m.

If π is that partition, then π has m blocks, each with n/m elements. The group elements having the same period are the generators of (g^m) so $\varphi(\pi) = \varphi(\frac{n}{m})$ (classic euler). Then the Möbius function μ', thought of as defined on D_n is the classic Möbius function and $\nu(\pi) = \frac{n}{m}$, the number of distint powers of g^m.

So we can think of $\varphi = \nu\mu'$ in this context as

$$\varphi(n) = \sum_{d|n} d\, \mu\left(\frac{n}{d}\right)$$

Section 3 - Pólya Theory

F - fixed proper class of functions $f: Y \to X$, thought of as "admissible colorings".

The group G acts on F by composition: $f \mapsto fg$. Orbit under this action is denoted fG.

Lemma: If $f_1 G = f_2 G \Rightarrow \operatorname{per} f_1 = \operatorname{per} f_2$

Pf

$H_i = \Theta(\ker f_i) = \{g \in G \mid f_i g = f_i\}$ $i = 1, 2$.

$f_2 = f_1 g_0$ for some g_0, so $H_1 = g_0 H_2 g_0^{-1}$. Conjugate subgroups have same orbits, so

$$\operatorname{per} f_1 = \eta(H_1) = \eta(H_2) = \operatorname{per} f_2.$$

Lemma: If $\pi = \operatorname{per} f$ the number of distinct functions in fG is $[G : \Theta(\pi)]$ and only depends on π.

Right cosets?

Pf

$fg_1 = fg_2$ iff $g_1 g_2^{-1} \in \Theta(\pi)$ so fG is injective with the right cosets of $\Theta(\pi)$

Functions with period π are represented by generating function $A_G(\pi)$. By Lemma 1 & 2, the formal power series $\dfrac{A_G(\pi)}{[G : \Theta(\pi)]}$ represents the G-equivilence classes of functions with period π. Summing over all periods π, we get a generating function for G-classes of functions in F. Following result is how to evaluate that generating function, & is a generalization of Polyas theorem

Theorem

$$\sum_{\pi \in P(G,X)} \frac{A_G(\pi)}{[G : \Theta(\pi)]} = \frac{1}{|G|} \sum_{\sigma \in P(G,X)} \varphi(\sigma)\, B(\sigma)$$

Pf

$$\sum_{\pi} \frac{A_G(\pi)}{[G : \Theta(\pi)]} = \frac{1}{|G|} \sum_{\pi} v(\pi)\, A_G(\pi) \quad \text{by } v(\pi) = |\Theta(\pi)|$$

$$= \frac{1}{|G|} \sum_{\pi} v(\pi)\, \mu^{i}\, B(\pi) \quad \text{by } A_G = \mu^{i} B$$

$$= \frac{1}{|G|} \sum_{\pi} \sum_{\sigma \geq \pi} v(\pi)\, \mu'(\pi, \sigma)\, B(\sigma)$$

$$= \frac{1}{|G|} \sum_{\sigma} \sum_{\pi \leq \sigma} v(\pi)\, \mu'(\pi, \sigma)\, B(\sigma)$$

$$= \frac{1}{|G|} \sum_{\sigma} v\mu'(\sigma) B(\sigma)$$

$$= \frac{1}{|G|} \sum_{\sigma} \varphi(\sigma) B(\sigma) \qquad \text{by } \varphi = v\mu'$$

The RHS is easy to calculate as soon as the group G is known: $\varphi(\sigma) = 0$ unless the blocks of σ are the cycles of an element of G. No need to know what other partitions are periods.

Proof features double mobius inversion on $P(G, X)$, but μ' not known explicitly.

We have an algebraic homomorphism T

$T(x_j^{(i)}) = x_i$ from monomials above

Weight of a function f:

$$W(f) = \prod_{i=1}^{|X|} x_{f(i)} = T(M(f))$$

Functions in the same G-class have the same weight, so $W(fG) = W(f)$. If S is a set of functions (subset of F), the inventory of S is the formal power series

$$W(S) = \sum_{f \in S} W(f) = T(M(S))$$

The inventory of a set of G-classes is the sum of the weights of classes.

Polya theory seeks to determine the Inventory of the set of all classes.

Theorem

Let $g \in G$ and $\pi = \tau((g))$.

Let $B'(g) = T(B(\pi))$. So $B'(g)$ is the inventory of the set of $f \in F$ which are constant on the cycles of g, or equivalently, $fg = f$.

The inventory of the set of all G-classes:

$$\frac{1}{|G|} \sum_{g \in G} B'(g)$$

Proof in paper.

If Y is finite, we can set all $x_i = 1$ and get: Burnside's lemma

Corollary 1: The number of G-classes in F is

$$\frac{1}{|G|} \sum_{g \in G} (\text{number of fixed points of } g)$$

If F is the proper class Y^X of all functions, the inventory of classes can be given more explicitly. If $\pi = \tau((g))$, write $\pi = \{\pi_1, \pi_2, \dots, \pi_k\}$ where $k = \#$ blocks of π (cycles of g).

If $fg = f$ then

$$T(m(f)) = \prod_{i=1}^{k} x_{f(\pi_i)}^{|\pi_i|}$$

All possible k-tuples in Y occur as subscripts for such functions f, so

$$B'(g) = \sum_{(j_1 \dots j_k)} \prod_{i=1}^{k} x_{j_i}^{|\pi_i|}$$

$$= \prod_{i=1}^{k} \sum_{j \in Y} x_j^{|\pi_i|}$$

Last expression depends only on the lengths of the cycles in g, so we introduce the type of g, the n-tuple $(b_1, b_2, \dots b_n)$ where b_i is the number of cycles of length i.

Pólya's Theorem:

If $F = f^{x_j}$, the inventory of G-classes of functions is

$$\frac{1}{|G|} \sum_{g \in G} \left(\sum x_j\right)^{b_1} \left(\sum x_j^2\right)^{b_2} \cdots \left(\sum x_j^n\right)^{b_n}$$

where for each g, $(b_1, b_2 \ldots b_n)$ is the type of g.