Model

In this document, we explain the data requirements and notation for the Quantitative Spatial Model from Ahlfeldt et al. (2015). This model helps to simulate the effect of different policies related to transportation and land use in an urban setting. The minimum data requirements are:

- L_{Ri}: population by spatial unit i.
- L_{Mj}: workers by spatial unit j.
- K: size of the spatial unit.
- Q: vector of floor space prices by spatial unit i.
- t_{ij}: travel times between location i and j in minutes.

The model parameters are:

- α: exp. share in the consumption good (the price is normalized to 1).
- $1 - \alpha$: exp. share in housing.
- θ: commuting and migration elasticity.
- β: output elasticity w.r.t labor.
- $1 - \beta$: output elasticity w.r.t land.
- φ_i: density development.
- μ: prod. function of floorspace, output elasticity w.r.t. capital.
- $1 - \mu$: prod. function of floorspace, output elasticity w.r.t. land.
- δ: decay parameter agglomeration forces.
- λ: agglomeration externality.
- ρ: decay parameter amenities.
- η: congestion force.
- ν: convergence parameter of the contraction mapping.

The endogenous variable of the model are:

- w_j: wages.
- Q: floor space prices.

- $\tilde{\theta}$: share of floor space used commercially.
- L_{Ri}: residents living in each location.
- L_{Mj}: workers living in each location.
- F_{Mi}: commercial floorspace.
- F_{Ri}: residential floorspace.

1 Setup

1.1 Utility

All workers are ex-ante the same, but they differ in random utility draws that are drawn from an extreme value type distribution. The indirect utility of worker ω that lives in i and works in j is:

$$U_{ij\omega} = \frac{B_i w_j \epsilon_{ij\omega}}{d_{ij} Q_i^{1-\alpha}}, \tag{1}$$

- B_i: amenity component in location i
- Q_i: housing price in location i
- w_j: wage per efficiency unit in location j
- α: exp. share in the consumption good (the price is normalized to 1).
- $1 - \alpha$: exp. share in housing.
- d_{ij}: Iceberg commuting costs
- $\epsilon_{ij\omega}$: idiosyncratic shock with dispersion parameter θ.

1.2 Travel times to commuting costs

To transform travel times to commuting costs, we use the following equation:

$$d_{ij} = \exp(\epsilon t_{ij}) \tag{2}$$

- ϵ: parameter that transforms travel times to commuting costs.
- t_{ij}: travel times in minutes.

1.3 Commuting shares

By the properties of extreme value type shocks, the commuting share from location i to j is:

$$\lambda_{ij|i} = \frac{w_j^{\theta} d_{ij}^{-\theta}}{\sum_r w_r^{\theta} d_{ir}^{-\theta}} \tag{3}$$

1.4 Average income

The average income in location i is:

$$\bar{y}_i = \sum_j \lambda_{ij|i} w_j \tag{4}$$

1.5 Population shares

The share of workers that decide to live in i is:

$$\lambda_i = \frac{B_i^{\theta} Q_i^{-\theta(1-\beta)} \sum_j w_j^{\theta} d_{ij}^{-\theta}}{\sum_l B_l^{\theta} Q_l^{-\theta(1-\beta)} \sum_r w_r^{\theta} d_{lr}^{-\theta}} \tag{5}$$

1.6 Production of the homogeneous good

All firms produce an homogeneous good (price normalized to 1). The production function in each location is:

$$Y_j = A_j L_{Mj}^{\beta} F_{Mj}^{1-\beta} \tag{6}$$

1.7 Land Market

The total floor space supplied by location i is:

$$F_i = \varphi_i K_i^{1-\mu} \tag{7}$$

The floor space used by commercial purposes is:

$$F_{Mi} = \frac{(1-\beta) Y_i}{Q_i} \text{ if } A_i > 0,$$

and the total floor space used for residential purposes is:

$$F_{Ri} = \frac{(1-\alpha)\bar{y}_i L_{Ri}}{Q_i} \text{ if } B_i > 0$$

The land market clearing condition implies

$$F_{Mi} + F_{Ri} = F_i$$

1.8 Labor Market Clearing

The labor demand is:

$$L_{Mj} = \frac{\beta Y_j}{w_j} \tag{8}$$

The labor market clears when:

$$\frac{\beta Y_j}{w_j} = \sum_i L\lambda_i \lambda_{ij|i}$$

1.9 Agglomeration forces

Ahlfeldt et al. (2015) include agglomeration forces that affects total factor productivity in each location. In particular:

$$A_i = a_i \Upsilon_i^{\lambda}, \quad \Upsilon_i = \sum_l \exp(-\delta t_{il}) \left(\frac{L_{Ml}}{K_l}\right) \tag{9}$$

1.10 Congestion externalities

Similarly, there is a congestion force that affects the amenity value in each location:

$$B_i = b_i \Omega_i^{\eta}, \quad \Omega_i = \sum_l \exp(-\rho t_{il}) \left(\frac{L_{Rl}}{K_l}\right) \tag{10}$$

1.11 Average welfare

Average welfare is normalized to a constant term:

$$\bar{U} = \gamma \left(\sum_i \sum_j B_i^\theta Q_i^{-\theta(1-\beta)} w_j^\theta d_{ij}^{-\theta} \right)^{\frac{1}{\theta}}, \tag{11}$$

where γ is a constant term.

2 Model Inversion

The first part of the code inverts the model such that the observed data corresponds to the initial equilibrium. In this part, the model recovers:

- A vector w_j that equalizes the labor supply to the workers we observe in the data.
- The productivity vector a_i by matching the number of workers implied by the model and data.
- The amenity vector b_i by matching the number of residents implied by the model and data.
- The vector of land development density in each location by matching the floor space implied by the model and the data.

2.1 Wage distribution

In the first part, we can recover the wage distribution by matching the number of workers implied by the model and the data. In particular given: the number of residents L_{Ri}, workers L_{Mj}, and travel times t_{ij}, and a set of exogenous parameters ϵ and θ. We then solve:

$$L_{Mj}^{\text{model}} = \sum_i L_{Ri} \frac{w_j^\theta d_{ij}^{-\theta}}{\sum_l w_l^\theta d_{il}^{-\theta}} \tag{12}$$

Algorithm

- Start with an initial vector of wages $\mathbf{w}_0$.
- Construct the shares
$$\lambda_{ij|i}^1 = \sum_i L_{Ri} \frac{w_{j,0}^\theta d_{ij}^{-\theta}}{\sum_l w_{l,0}^\theta d_{il}^{-\theta}}$$
- Construct the total employment implied by the model:
$$L_{Mj}^{\text{model},1} = \sum_i L_{Ri} \lambda_{ij|i}^1$$

- New vector of wages:

$$w_{j,1} = \left(\frac{L_{Mj}}{L_{Mj}^{\text{model},1} / w_{j,0}} \right)^{\frac{1}{\theta}}$$

- Update the new vector of wages using a contraction mapping:

$$w_{j,2} = \nu w_{j,1} + (1 - \nu) w_{j,0}$$

- Normalize the vector of wages such that the geometric mean is equal to 1.

$$\exp\left(\frac{1}{N} \sum_{i=1}^{N} \ln w_j \right) = 1$$

- Repeat the algorithm until the difference $|\mathbf{w}_{r+1} - \mathbf{w}_r| < tol$, where tol is some tolerance factor.

2.2 Density development

In the second step, the inversion recovers the development density φ. Given data on: floorspace prices $\mathbf{Q}$, wages $\mathbf{w}$, number of workers L_{Mj}, number of residents L_{Ri}, and a set of exogenous parameters β, α, μ we recover the density development φ by solving the following equations:

- Normalize the housing prices:

$$\bar{Q} = \exp\left(\frac{1}{N} \sum_{i} \log Q_i \right)$$

$$\tilde{Q}_i = \frac{Q_i}{\bar{Q}}$$

- Compute the residential and commercial floorspace:

$$F_{Mi} = \left(\frac{1 - \beta}{\beta} \right) \left(\frac{w_i L_{Mi}}{\tilde{Q}_i} \right)$$

$$F_{Ri} = (1 - \alpha) \frac{\bar{y}_i L_i}{\tilde{Q}_i}$$

- The total demand for floorspace and the share used commercially are:

$$F_i = F_{Mi} + F_{Ri}$$

$$\tilde{\theta} = \frac{F_{Mi}}{F_i}$$

- Recover the development density:

$$\varphi_i = \frac{K_i^{1-\mu}}{F_i}$$

- Then the total amount of floorspace supplied is:

$$F_i^{\text{supply}} = \varphi_i K_i^{1-\mu}$$

2.3 Productivity vector

The third step consists to recover the productivity vector a_i. Given data on housing prices $\mathbf{Q}$, wages $\mathbf{w}$, workers $\mathbf{L}_M$, land $\mathbf{K}$, and travel times t_{ij} and the set of parameters λ, δ we can recover the productivity vector by solving the following equations:

- Recover the productivity vector A_i based on the FOC of the cost minimization problem of firms:

$$A_i = \beta^{-\beta}(1-\beta)^{-(1-\beta)}\tilde{Q}_i^{1-\beta} w_i^{\beta}$$

- Construct the level of externalities:

$$\Upsilon_i = \sum_l \exp(-\delta t_{il}) \left(\frac{L_{Ml}}{K_l}\right)$$

- Recover the productivity term a_i

$$a_i = \frac{A_i}{\Upsilon_i}$$

- Transform the productivity term to zero if there are no workers in location i

$$a_i = \frac{A_i}{\Upsilon_i} \times \mathbf{1}\{L_{Mi} > 0\}$$

2.4 Amenity Parameters

The fourth step of the model inversion consists of recovering the amenity distribution. Given data on the number of residents L_{Ri}, wages w_i, housing prices $\tilde{Q}_i$, and travel times t_{ij}, and a set of exogenous parameters θ, α, ρ, and η, we can invert the model and recover the vector of parameters b_i.

- Normalize the total number of residents

$$\bar{L} = \exp\left(\frac{1}{N}\sum_i \log L_i\right)$$

$$\tilde{L}_{Ri} = \frac{L_i}{\bar{L}}$$

- Normalize the market access measure

$$W_i = \left(\sum_j w_j^{\theta} d_{ij}^{-\theta}\right)^{\frac{1}{\theta}}$$

$$\bar{W} = \exp\left(\frac{1}{N}\sum_i \log W_i\right)$$

$$\tilde{W}_i = \frac{W_i}{\bar{W}}$$

- Recover the amenity vector $\mathbf{B}$:

$$B_i = \left(\frac{\tilde{L}_{Ri}\tilde{Q}_{Ri}^{1-\alpha}}{\tilde{W}_i}\right)^{\frac{1}{\theta}}$$

- Compute the congestion externalities:

$$\Omega = \sum_l \exp(-\rho t_{il})\left(\frac{L_{Rl}}{K_l}\right)$$

- Recover the term b_i

$$b_i = \frac{B_i}{\Omega^{-\eta}}$$

- Transform the amenity term to zero if there are no residents:

$$b_i = \frac{B_i}{\Omega^{-\eta}} \times \mathbf{1}\{L_{Ri} > 0\}$$

3 Counterfactuals

After the algorithm recovers the fundamentals of the economy then we are able to run the different counterfactuals. There can be different counterfactuals to run such as reduction in travel times t_{ij} due to new infrastructure investments, changes in total floorspace supplied φ, changes in TFP a_i, or changes in amenities b_i. The algorithm is the following:

Given a set of exogenous fundamentals:

$$\{a_i, b_i, t_{ij}, K_i\},$$

and a set of exogenous parameters:

$$\{\alpha, \beta, \theta, \delta, \rho, \lambda, \eta, \varphi, \epsilon, \mu\},$$

we can solve for the set of endogenous variables: $\{\tilde{\theta}_i, w_i, Q_i, L_{Ri}, L_{Mi}\}$.

Algorithm

- Start with an initial vector of endogenous variables $\{\tilde{\theta}_{i,0}, w_{i,0}, Q_{i,0}, L_{Ri,0}\}$
- Compute iceberg commuting costs:

$$d_{ij} = \exp(\epsilon t_{ij})$$

- Compute the endogenous amenities with the residential externalities:

$$\Omega_{i,1} = \sum_l \exp(-\rho t_{il}) \left(\frac{L_{Rl,0}}{K_l}\right)$$

$$B_i = b_i \Omega^{-\eta}$$

- Compute the residential and employment shares:

$$\lambda_{ij,1} = \frac{B_i^{\theta} w_{j,0}^{\theta} Q_{i,0}^{-\theta(1-\alpha)} d_{ij}^{-\theta}}{\sum_l \sum_r B_l^{\theta} w_{r,0}^{\theta} Q_{l,0}^{-\theta(1-\alpha)} d_{lr}^{-\theta}}$$

- Compute the conditional commuting shares:

$$\lambda_{ij|i,1} = \frac{w_{j,0}^{\theta} d_{ij}^{-\theta}}{\sum_r w_{r,0}^{\theta} d_{ir}^{-\theta}}$$

- Compute the total number of workers in location j:

$$L_{Mj,1} = \sum_i \lambda_{ij|i,1} L_i = \sum_i \lambda_{ij,1} \bar{L}$$

- Compute TFP A_i and agglomeration externalities:

$$\Upsilon_{i,1} = \sum_l \exp(-\delta t_{il}) \left(\frac{L_{Ml,0}}{K_l}\right)$$

$$A_{i,1} = a_i \Upsilon_{i,1}^{\lambda}$$

- Compute total output in each location:

$$Y_{i,1} = A_{i,1}(L_{Mi,1})^{\beta}(\tilde{\theta}_{i,0}\varphi_i K_i^{1-\mu})^{1-\alpha}$$

- Compute average income and commuter market access measure:

$$\bar{y}_{i,1} = \sum_j \lambda_{ij|i,1} w_{j,0}$$

$$W_{i,1} = \left(\sum_j w_{j,0}^{\theta} d_{ij}^{-\theta}\right)^{\frac{1}{\theta}}$$

- update new vector of residents:

$$L_{Ri,1} = \sum_j \lambda_{ij} \bar{L}$$

- Update new vector of wages:

$$w_{i,1} = \frac{\beta Y_{i,1}}{L_{Mi,1}}$$

- Update new vector of housing prices:

$$Q_{i,1} = \frac{(1-\beta)Y_{i,1}}{\tilde{\theta}_{i,0}\varphi_i K_i^{1-\mu}} \quad \text{if} \quad a_i > 0, b_i = 0 \text{ or } a_i > 0, b_i > 0$$

$$Q_{i,1} = \frac{(1-\alpha)\bar{y}_{i,1}L_{i,1}}{(1-\tilde{\theta}_{i,0})\varphi K_i^{1-\mu}} \quad \text{if} \quad a_i = 0, b_i > 0 \text{ or } a_i > 0, b_i > 0$$

- Update new vector of the share of floorspace used commercially:

$$\tilde{\theta,1} = 1 \quad \text{if} \quad a_i > 0, b_i = 0$$

$$\tilde{\theta,1} = 0 \quad \text{if} \quad a_i = 0, b_i > 0$$

$$\tilde{\theta,1} = \frac{(1-\beta)Y_{i,1}}{Q_{i,1}\varphi_i K_i^{1-\mu}} \quad \text{if} \quad a_i > 0, b_i > 0$$

- Compute the new vector of endogenous variables to repeat the algorithm:

$$L_{Ri,2} = \nu L_{Ri,0} + (1-\nu)L_{Ri,1}$$

$$w_{i,2} = \nu w_{i,0} + (1-\nu) w_{i,1}$$

$$Q_{i,2} = \nu Q_{i,0} + (1-\nu) Q_{i,1}$$

$$\tilde{\theta}_{i,2} = \nu \tilde{\theta}_{i,0} + (1-\nu) \tilde{\theta}_{i,1}$$

- Compute the difference for the endogenous variables between the simulations:

$$z_{i,x} = |x_{i,1} - x_{i,0}|,$$

  where x corresponds to $L_{Ri}, w_i, Q_i, \tilde{\theta}_i$.

- Repeat the algorithm until:

$$\max(\mathbf{z}_L, \mathbf{z}_w, , \mathbf{z}_Q, , \mathbf{z}_{\tilde{\theta}}) < tol$$

- For the case of the closed city, compute aggregate welfare:

$$\bar{U} = \gamma \left(\sum_i \sum_j B_i^{\theta} Q_i^{-\theta(1-\beta)} w_j^{\theta} d_{ij}^{-\theta} \right)^{\frac{1}{\theta}}$$

- For the case of the open city update $\bar{L}$, until welfare is equalized to the initial $\bar{U}$.

References

Ahlfeldt, G. M., Redding, S. J., Sturm, D. M., and Wolf, N. (2015). The Economics of Density: Evidence From the Berlin Wall. *Econometrica*, 83:2127–2189.