Dynamic Macroeconomics

Instituto Tecnológico Autónomo de México

Maestría en Teoría Económica

Preliminary[1]

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ITAM

Fall 2018

[1]These class notes were written for the Dynamic Macroeconomics course at Instituto Tecnológico Autónomo de México, which makes part of the Masters in Economic Theory. The structure of this course closely follows the class notes written by Carlos Urrutia for this same course. I also include many elements of the class notes written by Dirk Krueger and Jesús Fernández-Villaverde for their macroeconomic theory courses at the University of Pennsylvania. I am extremely grateful with them for allowing me to use their class notes. All comments and corrections are welcome. All errors are mine.

Contents

Chapter 1

Models with no Uncertainty

This section introduces the main tool in quantitative macroeconomic research during the last decades: the neoclassical growth model. Even though the simplest version of this model is not good enough to explain most macroeconomic phenomena, it is a benchmark upon which most of the more complex models are built. Understanding the neoclassical growth model will give the tools to study questions in different fields, such as monetary policy, fiscal policy, unemployment, growth, and business cycles, among others.

The neoclassical growth model was a response to the Solow model, proposed in his 1957 paper, where he explained capital accumulation in the economy. In his model, Solow assumed that the economy was composed of agents that saved an exogenous fraction of their income, which was invested in capital. Even though this model has rich implications on growth and convergence between economies that start from different initial conditions, the assumption that agents save an exogenous and fixed fraction of their income is very strong. The first version of the neoclassical growth model was proposed in the decade of 1960s by David Cass and Tjalling Koopmans, where they assumed that, instead of saving a constant fraction of their income, households solve an intertemporal utility maximization problem, so savings and consumption are endogenous decisions that depend on prices. In particular, the savings decision depends on the interest rate which, in equilibrium, is determined by the aggregate level of capital in the economy, and by the discount factor, among other parameters.

It turns out that most of the models currently used in research are extensions of this model. Real business cycles and the short-term fluctuations in the economy are studied using the neoclassical growth model with uncertainty, where there are shocks to productivity that drive fluctuations. Monetary economics use neokeynesian models, which are real business cycle models with frictions, such as sticky prices and wages. Models that study inequality and income distribution are composed of heterogeneous agents that interact in markets in much the same way as the representative agent does in the neoclassical growth model. Heterogeneity in these models comes in different forms: idiosyncratic risk faced by different agents, life-cycle models in which agents of different cohorts interact, or ex-ante differences such as initial wealth or ability. Similarly, models that study housing and financial markets are extensions of the neoclassical growth model, where individuals have access to multiple assets that differ in returns and risk.

But even thought the neoclassical growth model is such a useful tool, it turns out that by itself it is not good enough to study most of the questions. In order to successfully explain macroeconomic issues, certain extensions and assumptions must be made, depending on the issue of study.

The following sections will present the neoclassical growth model and some of its extensions, such as models with endogenous labor and exogenous technological growth. Every section will first study the Pareto-optimal allocations of the specific model by solving the social planner's problem. Then, every section will define and characterize a descentralized competitive equilibrium, where agents choose consumption, savings and labor by interacting in markets where prices are fixed competitively in equilibrium. In applications where there are no frictions, such as externalities, public goods, taxes, or private information, both the social planner's solution and the competitive equilibrium will coincide, as stated by the welfare theorems. However, in models with such frictions the solution to both problems will usually differ.

1.1 The Neoclassical Growth Model

In the economy there are two types of agents: households and firms. Households live for $T = \infty$ periods. Every period they choose consumption, investment and labor. Firms use labor and capital as inputs to produce the final good in the economy, which is devoted to consumption and investment.

We assume there is a large number of households and firms and all of them are identical.

We call this model a "representative agent" model because all households and firms take the same decision in equilibrium. We denote the population size as L_t. For simplicity, in this section we assume that there is a continuum of households of measure 1 and there is no population growth. This means that $L_t = 1$ for every period t.

The assumption for there being a large number of agents is that in real life there are millions of households and firms, and the effect of the decisions of a single agent have negligible impact on macroeconomic aggregates. For instance, according to Inegi, in Mexico there were 32 million households in 2015, so the actions of one household have negligible effects on the overall economy. We will keep the assumption of identical households for now. In later sections we will assume there are heterogeneous households.

For convention, we will denote aggregate variables with upper-case letters, and per-capita variables with lower-case letters. For example, C_t and c_t will denote aggregate and per-capita consumption, respectively, K_t and k_t are aggregate and per-capita capital, and so on. In this section, given that we assume population size equal to 1 every period, it turns out that $c_t = C_t, k_t = K_t, \ldots$, so we will express the model in per-capita terms. In later sections, where there is population growth, I will make the explicit the difference between aggregate and per-capita variables.

Households maximize a lifetime utility function that depends on consumption in every period of life $\{c_t\}_{t=0}^{\infty}$:

$$U(c_0, c_1, \ldots) = \sum_{t=0}^{\infty} \beta^t u\left(c_t\right)$$

where β is the discount factor and the period utility function u is strictly concave and satisfies the Inada conditions:

$$\begin{array}{llr} 1.\ u'(c) > 0 & 3.\ \lim_{c\to 0} u'(c) = \infty & (1.1) \\ 2.\ u''(c) < 0 & 4.\ \lim_{c\to \infty} u'(c) = 0 & \end{array}$$

Every period, each agent within the household has one unit of time that is devoted inelastically to labor. In addition to time, households start their lives with an amount of capital k_0, which is devoted to production.

Total output in the economy y_t is produced by firms that hire labor l_t and capital k_t in competitive markets as inputs to produce according to a neoclassical production function F, such that $y_t = F(k_t, l_t)$. We assume that F satisfies the following properties: 1) F is homogeneous of degree 1 -has constant returns to scale-, such that $F(\lambda k_t, \lambda l_t) = \lambda F(k_t, l_t)$ for every λ; 2) F is concave, such that $F_k, F_l > 0, F_{kk}, F_{ll} < 0$, and $F_{kl} > 0$; and 3) F satisfies the Inada conditions:

$$\begin{aligned} &1.\ \lim_{k\to 0} F_k(k,l) = \infty \qquad && 3.\ \lim_{k\to\infty} F_k(k,l) = 0 \\ &2.\ \lim_{l\to 0} F_l(k,l) = \infty \qquad && 4.\ \lim_{l\to\infty} F_l(k,l) = 0 \end{aligned} \tag{1.2}$$

Final good production y_t in the economy is devoted to consumption c_t and investment i_t, so the aggregate resource constraint is given by $y_t = c_t + i_t$.

Given an investment choice i_t, capital in the economy evolves according to:

$$k_{t+1} = (1-\delta)k_t + i_t$$

where δ is the depreciation rate of capital. This equation states that the amount of capital at $t+1$ is equal to the undepreciated capital at t plus the total investments made.

1.1.1 Social Planner's Problem

The social planner's problem, whose solution yields the set of Pareto-optimal allocations in this economy, is given by:

$$\max_{c_t, i_t, k_{t+1}} \quad \sum_{t=0}^{\infty} \beta^t u(c_t) \quad s.t.$$

$$\forall t \in \{0,1,\ldots\}: \quad c_t + i_t = F(k_t, 1) \tag{1.3}$$

$$k_{t+1} = (1-\delta)k_t + i_t \tag{1.4}$$

$$c_t, k_{t+1} \geq 0, \quad k_0 \text{ given} \tag{1.5}$$

The social planner maximizes the utility of the representative agent in the economy subject to the resource constraint in the economy (1.3), the capital evolution equation (1.4), non-negativity constraints, and the initial level of capital in the economy k_0 given (1.5). Combining the first two constraints, yields a simplified version of the social planner's problem:

$$\max_{c_t, i_t, k_{t+1}} \quad \sum_{t=0}^{\infty} \beta^t u(c_t) \quad s.t. \tag{1.6}$$

$$\forall t \in \{0, 1, \ldots\}: \quad c_t + k_{t+1} = (1-\delta)k_t + F(k_t, 1)$$

$$c_t, k_{t+1} \geq 0, \quad k_0 \text{ given}$$

The constraint that the social planner faces is such that the total amount of the final good produced $F(k_t, 1)$ plus the undepreciated capital $(1-\delta)k_t$ should be either consumed c_t or allocated to capital for production next period k_{t+1}.

The social planner's problem (1.6) is a maximization problem with restrictions, so we can solve it by setting up the corresponding lagrangean:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t u(c_t) + \sum_{t=0}^{\infty} \lambda_t \left((1-\delta)k_t + F(k_t, 1) - c_t - k_{t+1}\right)$$

Note that there is one resource constraint for every period t, so we assign a lagrange multiplier λ_t for every such restriction. The multiplier λ_t is called the **shadow price of capital**, and represents the marginal value of capital or, equivalently, the additional utility that loosening the resource constraint by one unit would generate to the household. In this lagreangean, we are ignoring the non-negativity constraints $c_t, k_{t+1} \geq 0$ because it will never be optimal for the social planner to assign consumption or capital equal to zero in equilibrium, given the Inada conditions on the utility and production functions, given by conditions (1.1) and (1.2). In equilibrium, if $c_t = 0$, given that the marginal utility of consumption is infinite, increasing consumption by a small amount will generate a large increase in utility so it is optimal for the social planner to assign $c_t > 0$. Similarly, in equilibrium $k_{t+1} > 0$, given that if $k_{t+1} = 0$ the marginal utility of increasing capital is so large that the planner will find it optimal to set $k_{t+1} > 0$.

To characterize the equilibrium, the first order conditions are:

$$[c_t]: \quad \beta^t u'(c_t) - \lambda_t = 0 \tag{1.7}$$

$$[k_{t+1}]: \quad -\lambda_t + \lambda_{t+1}\left(1 - \delta + F_k(k_{t+1}, 1)\right) = 0 \tag{1.8}$$

$$[\lambda_t]: \quad (1-\delta)k_t + F(k_t, 1) - c_t - k_{t+1} = 0 \tag{1.9}$$

By rearranging equations (1.7) and (1.8), we obtain the following optimality condition for the social planner's problem:

$$u'(c_t) = (1 - \delta + F_k(k_{t+1}, 1))\beta u'(c_{t+1}) \tag{1.10}$$

This condition is commonly known as the **Euler equation** or **intertemporal optimality condition**, as it illustrates the inter-temporal trade-off faced by households: allocating one more unit of consumption today c_t yields additional utility $u'(c_t)$, while allocating an additional unit of capital k_{t+1} yields additional utility in present value given by $(1 - \delta + F_k(k_{t+1}, 1))\beta u'(c_{t+1})$. The individual will choose every period t the amount of consumption c_t and capital k_{t+1} that yield exactly the same marginal utility. This equation can be rearranged in the following way:

$$\underbrace{\frac{u'(c_t)}{\beta u'(c_{t+1})}}_{\text{MRS}} = \underbrace{(1 - \delta + F_k(k_{t+1}, 1))}_{\text{MRT}} \tag{1.11}$$

The left-hand side of equation (1.11) represents the marginal rate of subsitution of households, given the utility function u, and the right-hand side represents the marginal rate of transformation, given the production function F. The optimal choice equates the marginal rate of substitution and the marginal rate of transformation.

Given that the utility function is strictly concave and the constraint set is strictly convex, the first order conditions of the planner's problem are necessary conditions for an optimal solution. However, for these to be sufficient conditions we need to add the following **transversality condition** on k_{t+1}:

$$\lim_{t\to\infty} \lambda_t k_{t+1} = 0 \tag{1.12}$$

Stokey et al. (1989) show that the following are necessary and sufficient conditions for the allocations $\{c_t, k_{t+1}\}_{t=0}^{\infty}$ to be optimal and solve the social planner's problem:

$$\textbf{Euler equation:} \qquad \frac{u'(c_t)}{\beta u'(c_{t+1})} = (1 - \delta + F_k(k_{t+1}, 1)), \quad \forall t \geq 0 \tag{1.13}$$

$$\textbf{Resource constraint:} \qquad c_t + k_{t+1} = (1 - \delta)k_t + F(k_t, 1), \quad \forall t \geq 0 \tag{1.14}$$

$$\textbf{Transversality condition:} \qquad \lim_{t\to\infty} \beta^t u'(c_t)k_{t+1} = 0 \tag{1.15}$$

$$\textbf{Initial condition:} \qquad k_0 \text{ given} \tag{1.16}$$

Conditions (1.13) and (1.14) come directly from the first order conditions of the social planner's problem. Note that these equations for all t constitute a set of difference equations in k_t, k_{t+1} and k_{t+2}. For this system of difference equations to have a solution, we need two "boundary conditions". The transversality and initial conditions (1.15) and (1.16) play the role of these boundary conditions. A detailed derivation of the transversality condition and the proof for them to be necessary and sufficient conditions for an optimum are explained in Stokey et al. (1989).

Conditions (1.13)-(1.16) fully characterize the set of Pareto-optimal allocations in the economy. This means that if the allocations $\{c_t, k_{t+1}\}_{t=0}^{\infty}$ are given to households in the economy, there is no way to improve a single household without affecting at least another one.

1.1.2 Competitive Equilibrium

This section characterizes a competitive equilibrium in this environment, where households and firms are left to interact in competitive markets and prices are fixed such that markets clear. The environment is the same as in last section, although here households have access to risk-free financial assets to save in addition to investments in capital.

Definition of Equilibrium

Every period, each agent within the household has one unit of time that is devoted inelastically to labor in exchange for a wage w_t per unit of time. In addition to labor income, households own capital in the economy k_t, which they rent to firms in exchange for a rental rate R_t per unit of capital. Every period the household can save in risk-free bonds a_t, with net returns given by the real interest rate r_t. Finally, households own firms in the economy, so every period they receive the profits generated by the firm π_t. This means that every period the household faces the following budget constraint:

$$c_t + i_t + a_{t+1} = w_t + R_t k_t + (1 + r_t)a_t + \pi_t$$

where i_t denotes investment, w_t is the wage per unit of labor, R_t is the rental price of capital, r_t is the real interest rate on risk-free bonds π_t are the profits of the firm. Note that in this budget constraint the price of the final good is normalized to one or, equivalently, the budget constraint is expressed in terms of the final good. In particular, wages w_t and capital rents r_t are in terms of the final good.

As in last section, investment is denoted by i_t, such that capital evolves according to:

$$k_{t+1} = (1 - \delta)k_t + i_t$$

where δ is the depreciation rate of capital. The amount of capital at $t + 1$ is equal to the undepreciated capital at t plus the total investments made by the household.

The household's problem is given by:

$$\max_{\substack{c_t, k_{t+1}, i_t \\ t \geq 0}} \quad \sum_{t=0}^{\infty} \beta^t u(c_t) \quad s.t. \tag{1.17}$$

$$c_t + i_t + a_{t+1} = w_t + R_t k_t + (1 + r_t)a_t + \pi_t \tag{1.18}$$

$$k_{t+1} = (1 - \delta)k_t + i_t \tag{1.19}$$

$$\lim_{T \to \infty} \left(\prod_{t=0}^{T} \frac{1}{1 + r_t} \right) a_{T+1} \geq 0 \tag{1.20}$$

$$c_t, k_{t+1} \geq 0, \quad k_0 \text{ given}, \quad a_0 = 0 \text{ given} \tag{1.21}$$

Note that there is one budget constraint (1.18) and capital evolution constraint (1.19) for every period t. The constraint given by equation (1.20) is called a **No-Ponzi condition**, and ensures that individuals do not get accumulate increasingly large levels of debt in the long run. Without this condition, individuals could roll-over their debt period after period by borrowing a larger amount every time. This condition rules out those cases.

Given the production function F, the firm chooses the amount of capital and labor to hire every period to maximize profits:

$$\begin{aligned} \pi_t &= \max_{k_t, l_t} \quad y_t - w_t l_t - R_t k_t \qquad (1.22) \\ &= \max_{k_t, l_t} \quad F(k_t, l_t) - w_t l_t - R_t k_t \end{aligned}$$

Definition 1 (Competitive Equilibrium). *A competitive equilibrium are allocations for the household* $\{c_t, k^s_{t+1}, i_t, a_{t+1}\}_{t=0}^{\infty}$*, allocations for the firm* $\{k^d_t, l_t\}_{t=0}^{\infty}$*, and prices* $\{w_t, R_t, r_t\}_{t=0}^{\infty}$ *such that:*

1. *Given* k_0 *and prices* $\{w_t, R_t, r_t\}_{t=0}^{\infty}$*, the allocations* $\{c_t, k^s_{t+1}, i_t, a_{t+1}\}_{t=0}^{\infty}$ *solve the optimization problem of the household, described by (1.17).*
2. *At every* $t \in \{0, 1, \dots\}$*, given prices* w_t, R_t, r_t*, the allocations* k^d_t, l_t *solve the optimization problem of the firm, described by (1.22).*
3. *Markets clear for all* $t \geq 0$*:*
   (a) *Goods:* $c_t + i_t = F(k^d_t, l_t)$
   (b) *Labor:* $l_t = 1$
   (c) *Capital:* $k^d_t = k^s_t$
   (d) *Risk-free bonds:* $a_t = 0$

The market clearing condition for labor states that, in equilibrium, total demand for labor by the firms should be equal to one. This is because we assumed that total population size is equal to one and households supply labor inelastically. Later sections will assume that there is endogenous labor supply, which will change this market clearing condition. The last market clearing condition states that, in equilibrium, there is **zero net supply** for risk-free bonds. This is because we have assumed that in the economy there is a representative agent. Given that in equilibrium the total amount saved must equal total amount borrowed, and all agents are equal to each other, this can only happen if the representative household has no savings. If there were more than one agent in the economy (as will be the case in Section **??**), it could be the case that, in equilibrium, $a^i_{t+1} > 0$ and $a^j_{t+1} < 0$ for agents $i \neq j$, so that agent i is a saver, while agent j is a borrower.

To solve for an equilibrium, it suffices to solve problems (1.17) and (1.22) and set prices $\{w_t, R_t, r_t\}_{t=0}^{\infty}$ that clear the markets in the economy. Next subsections solve the household's and firm's problem.

Household's Problem

Let's start by solving the problem of the household. First of all, notice that, instead of solving problem (1.17), we can combine the first two restrictions such that the budget constraint becomes:

$$c_t + k_{t+1} + a_{t+1} = (1-\delta)k_t + w_t + R_t k_t + (1+r_t)a_t + \pi_t$$

This means that the household's problem can be expressed as:

$$\begin{aligned}
&\max_{c_t, k_{t+1}} && \sum_{t=0}^{\infty} \beta^t u(c_t) \quad s.t. && (1.23) \\
&\forall t \in \{0,1,\dots\}: && c_t + k_{t+1} + a_{t+1} = (1-\delta)k_t + w_t + R_t k_t + (1+r_t)a_t + \pi_t \\
& && c_t, k_{t+1} \geq 0, \quad k_0 \text{ given}, \quad a_0 = 0 \text{ given} \\
& && \lim_{T\to\infty} \left(\prod_{t=0}^{T} \frac{1}{1+r_t} \right) a_{T+1} \geq 0
\end{aligned}$$

In this problem, we got rid of investment i_t by combining the restrictions, so the households only chooses consumption c_t, capital k_{t+1} and risk-free bonds a_{t+1}. Once we solve for the equilibrium, the sequence for investment can be recovered using the capital evolution equation:

$$i_t = k_{t+1} - (1-\delta)k_t$$

As with the social planner's problem, given the Inada conditions for the utility and production functions, in equilibrium the individual will never choose to optimally consume $c_t = 0$ or set capital $k_{t+1} = 0$. Given that the marginal utility of consumption is infinite when $c_t = 0$, it is always optimal to consume a positive amount. On the other hand, if the household sets $k_{t+1} = 0$, the Inada condition on F implies that the real interest rate $r_{t+1} = \infty$. Therefore, it is optimal for the household to choose a positive amount of capital k_{t+1}. This means that we can safely ignore the non-negativity constraints when solving the problem. Similarly, we can ignore the No-Ponzi condition for now.

The associated lagrangean is:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t u(c_t) + \sum_{t=0}^{\infty} \lambda_t \left((1-\delta)k_t + w_t + R_t k_t + (1+r_t)a_t + \pi_t - c_t - k_{t+1} - a_{t+1}\right)$$

There is a lagrange multiplier λ_t for the budget constraint at each period, which represents the shadow price of one unit of wealth, and is interpreted as the additional utility that would bring to the individual having an additional unit of income in equilibrium. The first order conditions for this problem are:

$$
\begin{aligned}
[c_t]: &\quad \beta^t u'(c_t) - \lambda_t = 0 && (1.24)\\
[k_{t+1}]: &\quad -\lambda_t + \lambda_{t+1}(1-\delta+R_{t+1}) = 0 && (1.25)\\
[a_{t+1}]: &\quad -\lambda_t + \lambda_{t+1}(1+r_{t+1}) = 0 && (1.26)\\
[\lambda_t]: &\quad (1-\delta)k_t + w_t + R_t k_t + (1+r_t)a_t + \pi_t - c_t - k_{t+1} - a_{t+1} = 0 && (1.27)
\end{aligned}
$$

To obtain equations (1.25) and (1.26), note that k_{t+1} and a_{t+1} appear in two consecutive budget constraints: the one associated with period t and the one for period $t+1$. Equation (1.27) is simply the period t budget constraint. Combining the first three conditions yields:

$$\beta^t u'(c_t) = (1-\delta+R_{t+1})\beta^{t+1}u'(c_{t+1}) \tag{1.28}$$

$$\beta^t u'(c_t) = (1+r_{t+1})\beta^{t+1}u'(c_{t+1}) \tag{1.29}$$

Equations (1.28) and (1.29) imply that, in equilibrium, $R_t - \delta = r_t$, which is a **non-arbitrage condition**: the returns of capital must equal returns of risk-free bonds in equilibrium. In equilibrium, the household is indifferent between investing in capital or in risk-free bonds. This happens because if, for example, the returns to bonds were larger than returns to capital, households would choose to invest only in bonds and nothing in capital. The Inada condition for capital implies that $R_t = \infty$, so households have incentives to increase capital until both returns to capital and bonds are equal to each other. Rearranging (1.28):

$$\underbrace{\frac{u'(c_t)}{\beta u'(c_{t+1})}}_{\text{MRS}} = \underbrace{(1-\delta+R_{t+1})}_{\text{MRT}} \tag{1.30}$$

This equation is commonly known as the **Euler equation** or the **intertemporal optimality condition** of the household, and has the same interpretation as the Euler condition for the social planner.

Given that the utility function is strictly concave and the constraint set is strictly convex, the first order conditions for the household's problem are necessary conditions for an optimal solution. However, for these to be sufficient conditions we need to add the following two transversality conditions on k_t and a_t:

$$\lim_{t\to\infty} \lambda_t k_{t+1} = 0 \tag{1.31}$$

$$\lim_{t\to\infty} \lambda_t a_{t+1} = 0 \tag{1.32}$$

As for the social planner's problem, it can be shown that equations (1.27), (1.28), (1.29), (1.31) and (1.32) are necessary and sufficient conditions for an optimum, so any allocation that satisfies them constitutes a solution to the household's problem.

This means that, given a sequence of prices $\{w_t, R_t, r_t\}_{t=0}^{\infty}$, the allocations $\{c_t, k_{t+1}, a_{t+1}\}_{t=0}^{\infty}$ that satisfy the following equations characterize the optimal solution for the household:

Euler eq.:

$$\beta^t u'(c_t) = (1 - \delta + R_{t+1})\beta^{t+1} u'(c_{t+1}), \quad \forall t \geq 0 \tag{1.33}$$

$$\beta^t u'(c_t) = (1 + r_{t+1})\beta^{t+1} u'(c_{t+1}), \quad \forall t \geq 0 \tag{1.34}$$

Bud. const.:

$$c_t + k_{t+1} + a_{t+1} = (1 - \delta)k_t + w_t + R_t k_t + (1 + r_t)a_t + \pi_t, \quad \forall t \geq 0 \tag{1.35}$$

Transv. cond.:

$$\lim_{t\to\infty} \beta^t u'(c_t) k_{t+1} = 0 \tag{1.36}$$

$$\lim_{t\to\infty} \beta^t u'(c_t) a_{t+1} = 0 \tag{1.37}$$

Init. cond.:

$$k_0, a_0 \text{ given} \tag{1.38}$$

Firm's Problem

Now, let's solve the firm's problem. Every period t, the problem of the firm is:

$$\max_{k_t, l_t} F(k_t, l_t) - w_t l_t - R_t k_t$$

Note that the firm's problem is completely static. This is because we are assuming that the firm does not own the capital, so it does not have to make any capital accumulation decisions. The dynamic, or inter-temporal, decision is completely made by the household. Extensions of this model that focus on studying corporate finance issues assume that firms have inter-temporal decisions, such as long term debt, equity issuance, and investment decisions.

Given the problem of the firm in the current version of the model, the first order conditions yield:

$$w_t = F_l(k_t, l_t) \tag{1.39}$$

$$R_t = F_k(k_t, l_t) \tag{1.40}$$

Given these conditions, in equilibrium firms make zero profits, so $\pi_t = 0$. This is a consequence of the constant returns to scale assumption, which states that, for every λ:

$$F(\lambda k_t, \lambda l_t) = \lambda F(k_t, l_t)$$

Differentiate the above equation with respect to λ:

$$k_t F_k(\lambda k_t, \lambda l_t) + l_t F_l(\lambda k_t, \lambda l_t) = F(k_t, l_t)$$

This holds, in particular, for $\lambda = 1$, which means that:

$$F(k_t, l_t) = k_t \underbrace{F_k(k_t, l_t)}_{R_t} + l_t \underbrace{F_l(k_t, l_t)}_{w_t} = w_t l_t + R_t k_t \tag{1.41}$$

So $\pi_t = F(k_t, l_t) - w_t l_t - R_t k_t = 0$.

In this way, the equilibrium is fully characterized by prices $\{w_t, R_t, r_t\}_{t=0}^{\infty}$, and allocations $\{c_t, k_{t+1}^s, i_t, a_{t+1}\}_{t=0}^{\infty}$ for the household and $\{k_t^d, l_t\}_{t=0}^{\infty}$ for the firm, that satisfy equations (1.33)-(1.38) of the household, (1.39), (1.40), and (1.41) of the firm, and the market clearing conditions in Definition 1.

Welfare Theorems

Note the similarities between the optimality conditions of the social planner's problem, and the ones associated to the household's problem. It is easy to prove that the allocations of both problems coincide, by noting that, in the competitive equilibrium, the solution to the firm's problem yields $R_t = F_k(k_t, 1)$, given by equation (1.40), and the fact that $l_t = 1$, according to the labor market clearing condition. Similarly, using the solution to the firm's problem and the fact that $\pi_t = 0$, as shown by equation (1.41), the equivalence between the household's budget constraint (1.35) and the resource constraint (1.14) can be shown.

It is not surprising that the solutions to the social planner's problem and the competitive equilibrium are the same. This is because in this environment there are no frictions, such as externalities, public goods, taxes, or information problems, so the two welfare theorems hold. For completeness, I state the two welfare theorems:

Theorem 1 (First Welfare Theorem). *Let $\{c_t, k_{t+1}, i_t, a_{t+1}, l_t\}_{t=0}^{\infty}$ be a competitive equilibrium allocation. Then $\{c_t, k_{t+1}, i_t, a_{t+1}, l_t\}_{t=0}^{\infty}$ is Pareto efficient.*

This theorem states that, whenever we solve for a competitive equilibrium, the resulting allocations are Pareto efficient, which means that those allocations also solve the social planner's problem. In this way, whenever we want to find Pareto-optimal allocations, we can solve either the social planner's problem or the competitive equilibrium.

Theorem 2 (Second Welfare Theorem). *Let $\{c_t, k_{t+1}, i_t, a_{t+1}, l_t\}_{t=0}^{\infty}$ be a Pareto-optimal allocation. Then, there exist prices $\{w_t, R_t, r_t\}_{t=0}^{\infty}$ such that $\{c_t, k_{t+1}, i_t, a_{t+1}, l_t, w_t, R_t, r_t\}_{t=0}^{\infty}$ constitute a competitive equilibrium.*

This means that, whenever we want to solve for an equilibrium, we can explicitly solve the household's and firm's problems as in Definition 1, or we can solve the social planner's problem and construct a sequence of prices accordingly.

1.1.3 Steady state

A steady state in this economy is a competitive equilibrium (Definition 1) in which all variables per capita remain constant across time. Given that population size is constant and is equal to one, aggregate per capita variables are equal, in equilibrium, to the individual choices of households. In a steady state:

$$\begin{aligned}
c_t &= c_{t+1} = c_{t+2} = \ldots = c^* \\
k_t &= k_{t+1} = k_{t+2} = \ldots = k^* \\
y_t &= y_{t+1} = y_{t+2} = \ldots = y^* \\
&\vdots
\end{aligned}$$

Having solved for the equilibrium, we now can characterize how a steady state would look like. Take the Euler equation in the social planner's problem (1.10), and set all variables to their steady state values:

$$\frac{u'(c^*)}{\beta u'(c^*)} = (1 - \delta + F_k(k^*, 1)) \iff \frac{1}{\beta} = (1 - \delta + F_k(k^*, 1))$$

$$\iff F_k(k^*, 1) = \frac{1}{\beta} - 1 + \delta \tag{1.42}$$

Note that in this equation everything is constant except for capital, so capital in steady state k^* is pinned down by (1.42). Moreover, if the production function F is strictly concave, it can be shown that this steady state is unique. This means that there is a unique k^* which solves equation (1.42). Once the capital in steady state k^* is characterized, it is straightforward to find steady state values of all the other variables in the model. For production and consumption we can use the production function and resource constraint:

$$y^* = F(k^*, 1) \tag{1.43}$$

$$c^* = (1 - \delta)k^* + F(k^*, 1) - k^* = F(k^*, 1) - \delta k^* \tag{1.44}$$

Similarly, prices can be computed using the solution to the firm's problem:

$$w^* = F_l(k^*, 1) \tag{1.45}$$

$$R^* = F_k(k^*, 1) \tag{1.46}$$

$$r^* = R^* - \delta \tag{1.47}$$

Note that in this environment there is no growth on per capita variables in steady state. This means that this model cannot successfully explain long-term growth of countries. Further sections will expand the model such that the economy can experience growth in the long run.

1.1.4 The Log Utility Case

There are few cases in which these models can be solved analytically with a closed-form solution. This section presents one such particular case. In most other cases, it is not possible to derive a closed form solution for equilibria and the only option is to solve the model numerically. Section 1.5 is an introduction to solving these models with a computer.

In this example, let the utility function be given by $u(c) = \log(c)$ and the aggregate production function be $F(k, l) = k^\alpha l^{1-\alpha}$. Assume that there is full depreciation, so $\delta = 1$.

Using equation (1.33), the intertemporal optimality condition is given by:

$$\frac{c_{t+1}}{\beta c_t} = R_{t+1}$$

This implies that consumption over time behaves according to:

$$c_{t+1} = R_{t+1}\beta c_t \tag{1.48}$$

Recursively plugging in equation (1.48) we obtain an expression for period t consumption:

$$c_1 = R_1 \beta c_0$$

$$c_2 = R_2 \beta c_1 = R_1 \cdot R_2 \cdot \beta^2 c_0$$

$$\ldots$$

$$c_t = \left(\beta^t \Pi_{j=1}^t R_j\right) c_0 \tag{1.49}$$

If we take the budget constraints in equation (1.35) for periods $t, t+1, t+2 \ldots$ and recursively combine them:

$$c_t + k_{t+1} = w_t + R_t k_t$$
$$c_{t+1} + k_{t+2} = w_{t+1} + R_{t+1} k_{t+1}$$
$$c_{t+2} + k_{t+3} = w_{t+2} + R_{t+2} k_{t+2}$$
$$\ldots$$

We obtain:

$$k_{t+1} = \frac{1}{R_{t+1}} \left(c_{t+1} + k_{t+2} - w_{t+1}\right)$$

$$k_{t+2} = \frac{1}{R_{t+2}} \left(c_{t+2} + k_{t+3} - w_{t+2}\right)$$
$$\ldots$$

Then:

$$
\begin{aligned}
& c_t + \frac{1}{R_{t+1}}\left(c_{t+1} + k_{t+2} - w_{t+1}\right) = w_t + r_t k_t \\
\iff \quad & c_t + \frac{c_{t+1}}{R_{t+1}} + \frac{k_{t+2}}{R_{t+1}} = w_t + \frac{w_{t+1}}{R_{t+1}} + r_t k_t \\
\iff \quad & c_t + \frac{c_{t+1}}{R_{t+1}} + \frac{(c_{t+2} + k_{t+3} - w_{t+2})}{R_{t+1} \cdot R_{t+2}} = w_t + \frac{w_{t+1}}{R_{t+1}} + r_t k_t \\
\iff \quad & c_t + \frac{c_{t+1}}{R_{t+1}} + \frac{c_{t+2}}{R_{t+1} \cdot R_{t+2}} + \frac{k_{t+3}}{R_{t+1} \cdot R_{t+2}} = w_t + \frac{w_{t+1}}{R_{t+1}} + \frac{w_{t+2}}{R_{t+1} \cdot R_{t+2}} + r_t k_t \\
\iff \quad & \ldots \text{ (Summing up to period } t+s) \\
\iff \quad & \sum_{l=0}^{s} \frac{c_{t+l}}{\Pi_{j=0}^{l} R_{t+j}} + \frac{k_{t+s+1}}{\Pi_{j=t}^{t+s} R_j} = \sum_{l=0}^{s} \frac{w_{t+l}}{\Pi_{j=0}^{l} R_{t+j}} + \quad k_t \\
\iff \quad & \ldots \text{ (Taking limit as } s \to \infty) \\
\iff \quad & \sum_{l=0}^{\infty} \frac{c_{t+l}}{\Pi_{j=0}^{l} R_{t+j}} + \lim_{s \to \infty} \left(\frac{1}{\Pi_{j=t}^{t+s} R_j}\right) k_{t+s+1} = \sum_{l=0}^{\infty} \frac{w_{t+l}}{\Pi_{j=0}^{l} R_{t+j}} + \quad k_t \\
\iff \quad & \ldots \text{ (By transversality condition)} \\
\iff \quad & \underbrace{\sum_{l=0}^{\infty} \frac{c_{t+l}}{\Pi_{j=0}^{l} R_{t+j}}}_{\text{Lifetime consumption}} = \underbrace{\sum_{l=0}^{\infty} \frac{w_{t+l}}{\Pi_{j=0}^{l} R_{t+j}}}_{\text{Lifetime labor income}} + \quad k_t
\end{aligned}
$$

The transversality condition is used because:

$$
\begin{aligned}
\lim_{s \to \infty} \lambda_{t+s} k_{t+s+1} &= \lim_{s \to \infty} \beta^{t+s} u'(c_{t+s}) k_{t+s+1} \\
&= \lim_{s \to \infty} \frac{\beta^{t+s}}{c_{t+s}} k_{t+s+1} \\
&= \lim_{s \to \infty} \frac{\beta^{t+s}}{\beta^s \Pi_{j=t}^{t+s} R_j c_t} k_{t+s+1} \\
&= \left(\frac{\beta^t}{c_t}\right) \cdot \lim_{s \to \infty} \left(\frac{1}{\Pi_{j=t}^{t+s} R_j}\right) k_{t+s+1} = 0
\end{aligned}
$$

Plugging in the expression for consumption at period t given by (1.49):

$$\sum_{l=0}^{\infty}\frac{c_{t+l}}{\Pi_{j=0}^{l}R_{t+j}} = \sum_{l=0}^{\infty}\frac{\beta^l\Pi_{j=1}^{l}R_{t+j}c_t}{\Pi_{j=0}^{l}R_{t+j}}$$

$$= \frac{(1-\beta)}{R_t}c_t = \sum_{l=0}^{\infty}\frac{w_{t+l}}{\Pi_{j=0}^{l}R_{t+j}} + k_t$$

$$\iff c_t = \frac{1}{(1-\beta)}\left(\sum_{l=0}^{\infty}\frac{w_{t+l}}{\Pi_{j=1}^{l}R_{t+j}} + R_t k_t\right)$$

In particular:

$$c_0 = \frac{1}{(1-\beta)}\left(\sum_{t=0}^{\infty}\frac{w_t}{\Pi_{j=1}^{t}R_j} + R_0 k_0\right) \tag{1.50}$$

$$c_t = \frac{1}{(1-\beta)}\left(\sum_{t=0}^{\infty}\frac{w_t}{\Pi_{j=1}^{t}R_j} + R_0 k_0\right) \tag{1.51}$$

This is Milton Friedman's (1957) Permanent Income hypothesis. In this model, agents consume every period a fraction of total lifetime income. Instead of making the consumption decision based on present income, they choose how much to consume based on their lifetime income. This means that, in this model, additional income in any single period of life increases consumption at every other period. Note that equation (1.51) illustrates the path of consumption over life. Recall that the discount factor of households is given by a parameter ρ, where:

$$\beta = \frac{1}{1+\rho}$$

Also, $\beta = 1/R_j$ if and only if $\rho = R_j - 1 = r_j$. This means that the path of consumption of households is constant if and only if the discount rate of households equals the discount rate of the markets. If $\beta > 1/R_j = 1/(1+r_j)$, the household is more patient than the market, so consumption follows an increasing path over life. If, instead, $\beta < 1/R_j = 1/(1+r_j)$, households are more impatient than the market, so consumption will decrease over time.

Using the Euler equation (1.48) and the fact that $R_t = F_k(k_t, 1)$, we can see that for a steady state to exist, the following condition must hold:

$$c^* = F_k(k^*, 1)\beta c^* \iff 1 = F_k(k^*, 1)\cdot\beta$$

With the Cobb-Douglas functional form:

$$F_k(k,l) = \alpha \left(\frac{k}{l}\right)^{\alpha-1}$$

So the steady state level of capital is given by:

$$\alpha {k^*}^{\alpha-1} = \frac{1}{\beta} \quad \Longleftrightarrow \quad k^* = (\alpha\beta)^{\frac{1}{1-\alpha}}$$

The other steady state variables are:

$$\begin{aligned} y_t = F(k_t,1) = k^\alpha \quad \Longrightarrow \quad y^* &= (\alpha\beta)^{\frac{\alpha}{1-\alpha}} \\ R^* &= \alpha\left(\frac{1}{\alpha\beta}\right) \\ w^* &= (1-\alpha)\,(\alpha\beta)^{\frac{\alpha}{1-\alpha}} \\ c^* &= w^* + r^* + k^* - k^* = w^* + R^* \end{aligned}$$

1.2 Population Growth

So far, the model presented does not have long-run growth. When the economy reaches its steady state, all aggregate variables stay constant and do not grow. However, this is not what we observe in reality. Countries that are close to their steady state, such as most developed economies, experience growth every year. This section is a first step towards modelling an economy with long-run growth.

The model setup is exactly the same as before, with the only difference that now every household has L_t members, and family size grows at a rate n, such that:

$$L_{t+1} = (1+n)L_t$$

As in last section, we will denote per-capita variables in lower case, and absolute variables in upper case, such that:

$$c_t = \frac{C_t}{L_t}, \quad i_t = \frac{I_t}{L_t}, \quad k_t = \frac{K_t}{L_t}, \quad y_t = \frac{Y_t}{L_t}, \quad \ldots$$

Now, households maximize a lifetime utility function that depends on per-capita consumption rather than on aggregate consumption:

$$\sum_{t=0}^{\infty} \beta^t u\left(\frac{C_t}{L_t}\right) = \sum_{t=0}^{\infty} \beta^t u\left(c_t\right)$$

1.2.1 Social Planner's Problem

The social planner's problem is given by:

$$\begin{aligned} \max_{C_t, K_{t+1}} \quad & \sum_{t=0}^{\infty} \beta^t u\left(\frac{C_t}{L_t}\right) \quad s.t. \\ \forall t \in \{0,1,\ldots\}: \quad & C_t + K_{t+1} = (1-\delta)K_t + F(K_t, L_t) \\ & C_t, K_{t+1} \geq 0, \quad K_0 \text{ given} \end{aligned}$$

By the constant-returns-to-scale property of the production function F, note that:

$$F(K_t, L_t) = L_t \cdot F\left(\frac{K_t}{L_t}, 1\right) = L_t F(k_t, 1)$$

The resource constraint can be re-expressed in per-capita terms by dividing both sides by L_t:

$$\frac{C_t}{L_t} + \frac{K_{t+1}}{L_t} = (1-\delta)\frac{K_t}{L_t} + F\left(\frac{K_t}{L_t}, 1\right)$$

$$\Leftrightarrow \qquad \frac{C_t}{L_t} + \frac{L_{t+1}}{L_t} \cdot \frac{K_{t+1}}{L_{t+1}} = (1-\delta)\frac{K_t}{L_t} + F\left(\frac{K_t}{L_t}, 1\right)$$

$$\Leftrightarrow \qquad c_t + (1+n)k_{t+1} = (1-\delta)k_t + F\left(k_t, 1\right)$$

The social planner's problem can be rewritten in per-capita terms as:

$$\begin{aligned} \max_{c_t, k_{t+1}} \qquad & \sum_{t=0}^{\infty} \beta^t u\left(c_t\right) \quad s.t. \\ \forall t \in \{0, 1, \ldots\}: \qquad & c_t + (1+n)k_{t+1} = (1-\delta)k_t + F\left(k_t, 1\right) \\ & c_t, k_{t+1} \geq 0, \quad k_0 \text{ given} \end{aligned} \tag{1.52}$$

The constraint that the social planner faces is such that per-capita output produced $F(k_t, 1)$ plus the undepreciated capital $(1-\delta)k_t$ is either consumed c_t or allocated to capital for production next period $(1+n)k_{t+1}$. The term $(1+n)$ takes into account that the population will grow from t to $t+1$ at a rate n, so if the economy wants to have k_{t+1} per-capita units of capital per capita in $t+1$, it has to invest $(1+n)k_{t+1}$ at t.

The lagrangean associated to problem (1.52) is:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t u\left(c_t\right) + \sum_{t=0}^{\infty} \lambda_t \left((1-\delta)k_t + F(k_t, 1) - c_t - (1+n)k_{t+1}\right)$$

Again, we are ignoring the non-negativity constraints because the Inada conditions ensure that the social planner will choose positive consumption and capital per capita in equilibrium. The first order conditions are:

$$\begin{aligned} [c_t]: \qquad & \beta^t u'(c_t) - \lambda_t = 0 \\ [k_{t+1}]: \qquad & -\lambda_t(1+n) + \lambda_{t+1}\left(1 - \delta + F_k(k_t, 1)\right) = 0 \\ [\lambda_t]: \qquad & (1-\delta)k_t + F(k_t, 1) - c_t - (1+n)k_{t+1} = 0 \end{aligned}$$

By rearranging the conditions associated to c_t and k_{t+1}, we obtain the following optimality condition:

$$\frac{u'(c_t)}{\beta u'(c_{t+1})} = \frac{(1 - \delta + F_k(k_{t+1}, 1))}{(1+n)} \tag{1.53}$$

This is the Euler condition for the social planner's problem. The right-hand side differs from the Euler equation of the social planner without population growth because now the marginal product of capital k_{t+1} chosen at t has to be split between a larger population at $t+1$. The solution to the social planner's problem is characterized by allocations $\{c_t, k_{t+1}\}_{t=0}^{\infty}$ that satisfy the following necessary and sufficient conditions for an optimum:

Euler eq.: $$\frac{u'(c_t)}{\beta u'(c_{t+1})} = \frac{(1 - \delta + F_k(k_{t+1}, 1))}{(1+n)}, \quad \forall t \geq 0 \tag{1.54}$$

Resource constraint: $$c_t + (1+n)k_{t+1} = (1-\delta)k_t + F(k_t, 1), \quad \forall t \geq 0 \tag{1.55}$$

Transv. cond.: $$\lim_{t \to \infty} \beta^t u'(c_t) k_{t+1} = 0 \tag{1.56}$$

Initial condition: $$k_0 \text{ given} \tag{1.57}$$

Note the similarity between conditions (1.54)-(1.57) and (1.13)-(1.16). In both cases, variables are expressed in per-capita terms. However, recall that in Section 1.1 there was no population growth and population size was normalized to 1, so the optimality conditions did not depend on n. In this section, there is population growth, so the Euler equation depends on the growth rate n. However, if we set population growth $n = 0$, we are back to the original optimality conditions.

1.2.2 Competitive Equilibrium

As in Section 1.1, we assume there is inelastic labor supply so every period each agent within the household supplies one unit of time to labor. The only difference is that now households have L_t agents, all of which supply labor, so total labor income equals $w_t L_t$. Households can invest in capital K_t, which has returns equal to R_t, and in risk-free bonds A_t that have net returns equal to the interest rate r_t. The households are the owners of the firms so every period they receive profits equal to Π_t. However, as shown in equation (1.41), profits are equal to zero in equilibrium whenever the production function satisfies the constant-returns-to-scale assumption and the firm operates in perfect competition. This will be the case throughout most of these lecture notes, so we will omit henceforth the profits Π_t from the households' problem. Every period, the household faces the following budget constraint:

$$C_t + I_t + A_{t+1} = w_t L_t + R_t K_t + (1 + r_t) A_t$$

Combining the budget constraint with the capital evolution equation, $K_{t+1} = (1 - \delta)K_t + I_t$, and expressing the variables in per-capita terms:

$$\begin{aligned}
& \frac{C_t}{L_t} + \frac{K_{t+1}}{L_t} + \frac{A_{t+1}}{L_t} = (1-\delta)\frac{K_t}{L_t} + w_t + R_t \frac{K_t}{L_t} + (1+r_t)\frac{A_t}{L_t} \\
\iff \quad & \frac{C_t}{L_t} + \frac{L_{t+1}}{L_t} \cdot \frac{K_{t+1}}{L_{t+1}} + \frac{L_{t+1}}{L_t} \cdot \frac{A_{t+1}}{L_{t+1}} = (1-\delta)\frac{K_t}{L_t} + w_t + R_t \frac{K_t}{L_t} + (1+r_t)\frac{A_t}{L_t} \\
\iff \quad & c_t + (1+n)k_{t+1} + (1+n)a_{t+1} = (1-\delta)k_t + w_t + R_t k_t + (1+r_t)a_t
\end{aligned}$$

The problem of the household can be conveniently expressed in per-capita terms:

$$\begin{aligned}
\max_{c_t, k_{t+1}, a_{t+1}} \quad & \sum_{t=0}^{\infty} \beta^t u\left(c_t\right) \quad s.t. \qquad (1.58) \\
\forall t \in \{0, 1, \ldots\}: \quad & c_t + (1+n)k_{t+1} + (1+n)a_{t+1} = (1-\delta)k_t + w_t + R_t k_t + (1+r_t)a_t \\
& c_t, k_{t+1} \geq 0, \quad k_0 \text{ given}, \quad a_0 = 0 \text{ given} \\
& \lim_{T \to \infty} \left(\prod_{t=0}^{T} \frac{1}{1+r_t} \right) a_{T+1} \geq 0
\end{aligned}$$

Given the production function, the firm chooses the amount of capital K_t and labor L_t to hire every period to maximize profits:

$$\Pi_t = \max_{K_t, L_t} F(K_t, L_t) - w_t L_t - R_t k_t \tag{1.59}$$

Note that the household maximizes consumption per-capita, so it is more convenient to express its problem in per-capita terms. In contrast, the firm's problem cannot be stated in per-capita terms, because the firm chooses absolute levels of capital and labor, and not only the ratio of capital per-worker. As will be seen later in this section, to compute the steady state, we will re-write the optimality conditions of the firm in per-capita terms.

The following is the definition of a competitive equilibrium in this environment, where there is population growth.

Definition 2 (Competitive Equilibrium)**.** *A competitive equilibrium are allocations for the household* $\{c_t, k_{t+1}^s, i_t, a_{t+1}\}_{t=0}^{\infty}$*, allocations for the firm* $\{K_t^d, L_t^d\}_{t=0}^{\infty}$*, and prices* $\{w_t, R_t, r_t\}_{t=0}^{\infty}$ *such that:*

1. *Given* k_0*,* a_0*, and prices* $\{w_t, R_t, r_t\}_{t=0}^{\infty}$*, the allocations* $\{c_t, k_{t+1}^s, i_t, a_{t+1}\}_{t=0}^{\infty}$ *solve the optimization problem of the household described by (1.58).*

2. *At every* $t \in \{0, 1, \ldots\}$*, given prices* w_t, R_t, r_t*, the allocations* K_t^d, L_t^d *firms solve the optimization problem of the firm, described by (1.59)*

3. *Markets clear for all* $t \geq 0$*:*

   (a) *Goods:* $c_t + i_t = y_t$

   (b) *Labor:* $L_t^d = L_t$

   (c) *Capital:* $K_t^d = K_t^s$

   (d) *Risk-free bonds:* $a_t = 0$

Solving for the equilibrium

The lagrangean associated to the household's problem (1.58) is:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t u(c_t) + \sum_{t=0}^{\infty} \lambda_t \left((1-\delta)k_t + w_t + R_t k_t + (1+r_t)a_t - c_t - (1+n)k_{t+1} - (1+n)a_{t+1}\right)$$

The first order conditions for this problem are:

$$[c_t]: \quad \beta^t u'(c_t) - \lambda_t = 0 \tag{1.60}$$

$$[k_{t+1}]: \quad -\lambda_t(1+n) + \lambda_{t+1}(1-\delta+R_{t+1}) = 0 \tag{1.61}$$

$$[a_{t+1}]: \quad -\lambda_t(1+n) + \lambda_{t+1}(1+r_{t+1}) = 0 \tag{1.62}$$

$$[\lambda_t]: \quad (1-\delta)k_t + w_t + R_t k_t + (1+r_t)a_t - c_t - (1+n)k_{t+1} - (1+n)a_{t+1} = 0 \tag{1.63}$$

Combining equations (1.61) and (1.62), we obtain a non-arbitrage condition analogous to that of Section 1.1:

$$R_t - \delta = r_t$$

Similarly, the Euler equation is:

$$\frac{u'(c_t)}{\beta u'(c_{t+1})} = \frac{(1-\delta+R_{t+1})}{(1+n)} \tag{1.64}$$

The following are sufficient and necessary conditions for an allocation to solve the household's problem:

Euler eq.:

$$\frac{u'(c_t)}{\beta u'(c_{t+1})} = \frac{(1-\delta+R_{t+1})}{(1+n)}, \quad \forall t \geq 0 \tag{1.65}$$

$$\frac{u'(c_t)}{\beta u'(c_{t+1})} = \frac{(1+r_{t+1})}{(1+n)}, \quad \forall t \geq 0 \tag{1.66}$$

Bud. const.:

$$c_t + (1+n)k_{t+1} + (1+n)a_{t+1} = (1-\delta)k_t + w_t + R_t k_t + (1+r_t)a_t \quad \forall t \geq 0 \tag{1.67}$$

Transv. cond.:

$$\lim_{t\to\infty} \beta^t u'(c_t)k_{t+1} = 0 \tag{1.68}$$

$$\lim_{t\to\infty} \beta^t u'(c_t)a_{t+1} = 0 \tag{1.69}$$

Init. cond.:

$$k_0, a_0 \text{ given} \tag{1.70}$$

For the firm's problem, we must be careful in the following: as opposed to the household, which chooses per-capita variables, the firms choose absolute capital and labor. For this reason, we must solve the firms' problem in absolute terms, and then express everything in per-capita terms.

The first order conditions for the firm yield:

$$w_t = F_l(K_t, L_t)$$
$$R_t = F_k(K_t, L_t)$$

Now, we would like to express the firm's optimality conditions in terms of per-capita variables. In this way, the whole model would be expressed in per-capita terms, so we can easily solve for the equilibrium. For this, note that, given that the production function F satisfies the constant-returns-to-scale condition, we can rewrite $F(K, L)$ as:

$$F(K, L) = F(K \cdot \frac{L}{L}, L) = F(L \cdot \frac{K}{L}, L) = F(L \cdot k, L) = L \cdot F(k, 1)$$

This means that:

$$F_K(K, L) = L \cdot \underbrace{F_k\left(\frac{K}{L}, 1\right) \cdot \frac{1}{L}}_{\text{Chain rule}} = F_k(k, 1) \tag{1.71}$$

$$F_l(K, L) = F(k, 1) + L \cdot \underbrace{F_k\left(\frac{K}{L}, 1\right) \cdot \left(\frac{-K}{L^2}\right)}_{\text{Chain rule}} = F(k, 1) - F_k\left(\frac{K}{L}, 1\right) \cdot \frac{K}{L} = F(k, 1) - F_k(k, 1) \cdot k$$

In equilibrium, the firm will hire labor L_t and rent capital K_t such that $k_t = K_t/L_t$ satisfies:

$$w_t = F(k_t, 1) - F_k(k_t, 1) \cdot k_t \tag{1.72}$$
$$R_t = F_k(k_t, 1) \tag{1.73}$$

In equilibrium, given wages (1.72) and rental prices (1.73), the firm makes zero profits, so per-capita profits are $\pi_t = 0$:

$$
\begin{aligned}
\pi_t &= \frac{F(K_t, L_t)}{L_t} - \frac{w_t L_t}{L_t} - \frac{r_t K_t}{L_t} = F(k_t, 1) - w_t - R_t k_t \\
&= F(k_t, 1) - \underbrace{\left(F(k_t, 1) - F_k\left(k_t, 1\right) \cdot k_t\right)}_{w_t} - \underbrace{F_k(k_t, 1)}_{R_t} k_t \\
&= F(k_t, 1) - F(k_t, 1) + F_k\left(k_t, 1\right) \cdot k_t - F_k\left(k_t, 1\right) \cdot k_t = 0
\end{aligned} \tag{1.74}
$$

In this way, the equilibrium is fully characterized by prices $\{w_t, R_t, r_t\}_{t=0}^{\infty}$, and allocations $\{c_t, k_{t+1}, a_{t+1}\}_{t=0}^{\infty}$, that satisfy equations (1.65)-(1.70) of the household, (1.72) and (1.73) of the firm, and the market clearing conditions.

Using the solution to the firm's problem, it can be shown that the allocations of the social planner are equivalent to those of the competitive equilibrium. This happens because in this context the welfare theorems continue to hold.

1.2.3 Steady State

A steady state is a competitive equilibrium in which all variables per capita remain constant over time. In Section 1.1.3 this was trivial, since per-capita variables were equal to aggregate variables, so there was no growth at all. In this section, in steady state per-capita variables remain constant, but aggregate variables grow.

It is straightforward to verify that capital in steady state is uniquely pinned down by:

$$
\frac{1}{\beta} = \frac{(1 - \delta + F_k(k^*, 1))}{(1 + n)}
$$

Once capital per capita k^* is determined, all other variables can be pinned down using the optimality conditions:

$$
\begin{aligned}
c^* &= F(k^*, 1) - (\delta + n)k^* \\
i^* &= \delta k^* \\
w^* &= F(k^*, 1) - F_k(k^*, 1)k^* \\
R^* &= F_k(k^*, 1) \\
r^* &= F_k(k^*, 1) - \delta
\end{aligned}
$$

Now, even though in steady state per-capita variables are constant, aggregate variables are not. Recall that $k_t = K_t/L_t$, $c_t = C_t/L_t, \ldots$, which means that in steady state:

$$\begin{aligned}
K_t &= k^* L_t = (1+n)^t k^* \\
C_t &= c^* L_t = (1+n)^t c^* \\
&\vdots
\end{aligned}$$

In steady state aggregate variables grow at a rate n, the same as population. This model is the first step towards models with positive growth.

1.3 Endogenous labor supply

In Sections 1.1 and 1.2 we assumed that the utility function of households is a function only of consumption and did not depend on leisure. This implies that households allocate all of their time to work and nothing to leisure. This may not be a realistic assumption, given that households decide every day how much time to devote to work and how much to leisure. In particular, the model without labor decisions is not useful to study topics such as labor markets, taxation, human capital, social security systems, among others. In this section we extend the model presented to include endogenous labor decisions.

Assume again that the economy is composed of representative households and firms with the same characteristics as before. In particular, let's assume that households have size L_t and each individual has 1 unit of time available to devote to labor or leisure, so the total amount of time available within the household equals exactly L_t. Household size grows at a constant rate n, such that $L_{t+1} = (1+n)L_t$.

However, assume that now households obtain utility from the fraction of time devoted to leisure by all of its members. This means that households do not necessarily supply inelastically all of their time to labor, but rather choose how much time to work out of the total available time. If we denote by L_t^s the total amount of time devoted to work by all members of a household of size L_t, the total time devoted to leisure is $L_t - L_t^s$. Therefore, the utility function is now:

$$\sum_{t=0}^{\infty} \beta^t u\left(\frac{C_t}{L_t}, \frac{L_t - L_t^s}{L_t}\right)$$

Here, we assume that the utility function is concave in both arguments and the Inada conditions on consumption hold:

1. $u_c, u_l > 0$
2. $u_{cc}, u_{ll} < 0$
3. $u_{cl} > 0$
4. $\lim_{c\to 0} u_c(c,l) = \infty$
5. $\lim_{c\to \infty} u_c(c,l) = 0$

The budget constraint that the household faces every period is:

$$C_t + K_{t+1} + A_{t+1} = (1-\delta)K_t + w_t L_t^s + R_t K_t + (1+r_t)A_t$$

where now $L_t^s \leq L_t$. Again, I omitted including the profits that the household receives for owning the firm, as in equilibrium they turn out to be equal to zero. The problem of the household can be written in per-capita terms as:

$$\max_{c_t, k_{t+1}, l_t^s} \quad \sum_{t=0}^{\infty} \beta^t u\left(c_t, 1 - l_t^s\right) \quad s.t. \tag{1.75}$$

$$\begin{aligned} \forall t \in \{0, 1, \ldots\}: \quad & c_t + (1+n)k_{t+1} + (1+n)a_{t+1} = (1-\delta)k_t + w_t l_t^s + R_t k_t + (1+r_t)a_t \\ & c_t, k_{t+1} \geq 0, \quad 0 \leq l_t^s \leq 1, \quad k_0 \text{ given}, \quad a_0 = 0 \text{ given} \\ & \lim_{T\to\infty} \left(\prod_{t=0}^{T} \frac{1}{1+r_t}\right) a_{T+1} = 0 \end{aligned}$$

Now, the average fraction of time devoted to labor within the household l_t^s is a choice. Moreover, there is an additional restriction that bounds the total fraction of time devoted to labor: $0 \leq l_t^s \leq 1$. This means that households cannot work more than the total amount of time available, equal to 1.

In this section, I omit the definition of a competitive equilibrium, as it is analogous to Definition 2. The only difference is that households now choose labor supply l_t^s, so the labor market clearing condition becomes $l_t^s = l_t^d$.

The lagragean associated to the household's problem is:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t u\left(c_t, 1 - l_t^s\right) + \sum_{t=0}^{\infty} \lambda_t \left((1-\delta)k_t + w_t l_t^s + R_t k_t + (1+r_t)a_t - c_t - (1+n)k_{t+1} - (1+n)a_{t+1}\right)$$

The first order conditions are:

$$
\begin{aligned}
[c_t]: &\quad \beta^t u_c(c_t, 1-l_t^s) - \lambda_t = 0 \\
[l_t]: &\quad -\beta^t u_l(c_t, 1-l_t^s) + \lambda_t w_t = 0 \\
[k_{t+1}]: &\quad -\lambda_t(1+n) + \lambda_{t+1}\left(1-\delta+R_{t+1}\right) = 0 \\
[a_{t+1}]: &\quad -\lambda_t(1+n) + \lambda_{t+1}\left(1+r_{t+1}\right) = 0 \\
[\lambda_t]: &\quad (1-\delta)k_t + w_t l_t^s + R_t k_t + (1+r_t)a_t - c_t - (1+n)k_{t+1} - (1+n)a_{t+1} = 0
\end{aligned}
$$

With endogenous labor choice, we have an additional first order condition, associated with the optimal choice of labor supply. Combining the first two conditions:

$$w_t \cdot u_c(c_t, 1-l_t^s) = u_l(c_t, 1-l_t)$$

This equation is often called the **intra-temportal** optimality condition. In the optimum, the individual is indifferent between having an additional unit of leisure, or working an additional unit of time and consuming the income. The left hand side of this equation represents the marginal utility of working an additional unit of time: an additional unit of time will generate income w_t to the household that can be spent in consumption, whose marginal utility is $u_c(c_t, 1-l_t)$. The right hand side is the marginal utility of leisure. In equilibrium, the marginal gains of working or enjoying leisure are the same.

As in past sections, the other optimality conditions are the Euler equations associated to capital and risk-free assets, the budget constraint, and the transversality conditions. The optimal allocation $\{c_t, l_t^s, k_{t+1}, a_{t+1}\}_{t=0}^{\infty}$ for the consumer, given prices $\{w_t, R_t, r_t\}_{t=0}^{\infty}$, is fully characterized by:

$$\textbf{Euler eq.:} \quad \frac{u_c(c_t, 1-l_t^s)}{\beta u_c(c_{t+1}, 1-l_{t+1}^s)} = \frac{(1-\delta+R_{t+1})}{(1+n)}, \quad \forall t \geq 0 \tag{1.76}$$

$$\frac{u_c(c_t, 1-l_t^s)}{\beta u_c(c_{t+1}, 1-l_{t+1}^s)} = \frac{(1+r_{t+1})}{(1+n)}, \quad \forall t \geq 0 \tag{1.77}$$

$$\textbf{Intratemporal:} \quad u_c(c_t, 1-l_t^s) = \frac{u_l(c_t, 1-l_t)}{w_t}, \quad \forall t \geq 0 \tag{1.78}$$

$$\textbf{Bud. const.:} \quad c_t + (1+n)k_{t+1} + (1+n)a_{t+1} = (1-\delta)k_t + w_t l_t^s + R_t k_t + (1+r_t)a_t \quad \forall t \geq 0 \tag{1.79}$$

$$\textbf{Transv. cond.:} \quad \lim_{t\to\infty} \beta^t u_c(c_t, 1-l_t^s)k_{t+1} = 0 \tag{1.80}$$

$$\lim_{t\to\infty} \beta^t u_c(c_t, 1-l_t^s)a_{t+1} = 0 \tag{1.81}$$

$$\textbf{Init. cond.:} \quad k_0, a_0 \text{ given} \tag{1.82}$$

The firm's problem is analogous to that of past sections, so I omit it in this section. In particular, equilibrium factor prices w_t and R_t are pinned down by equations (1.72) and (1.73).

1.3.1 Steady State

It can be verified that in the stationary steady state the values of c^*, y^*, k^* and l^* are uniquely pinned down by the following equations:

$$\frac{1}{\beta} = \frac{(1-\delta+F_K(k^*, l^*))}{(1+n)}$$

$$\cdot u_c(c^*, 1-l^*) = \frac{u_l(c^*, 1-l^*)}{F_L(k^*, l^*)}$$

$$c^* = F(k^*, l^*) - (n+\delta)k^*$$

$$\vdots$$

1.4 Exogenous Growth and Technological Change

Up to now, the model does not allow for long term growth; in steady state absolute variables grow at exactly the same rate as population, so per capita variables remain constant. This is not what we observe in reality. Most economies experience growth beyond the increase in population size. This section describes a model in which there is exogenous growth generated in steady state. Section 1.7 will introduce models in which long-term growth is endogenously generated.

In this section, for simplicity, we will again assume that leisure does not generate utility to households and labor is inelastically supplied. The model can be extended to have endogenous labor supply as in Section 1.3.

Let's assume now that the production function has a technological parameter A_t that augments labor productivity $F(K_t, A_t L_t)$, and grows at an exogenous rate g, such that:

$$A_{t+1} = (1+g)A_t$$

If we assume that productivity in period 0 is $A_0 = 1$, this means that $A_t = (1+g)^t$.

Note that, naturally, our economy will never reach a steady state, as productivity is growing in the long run, so the economy will be capable of increasing production, consumption and capital per worker period after period. Instead, we will study the economy when it reaches a state in which per-capita variables *grow at a constant rate*, which is commonly known as a **balanced growth path**. Note that a steady state, which is when all per-capita variables remain constant, is a particular case of a balanced growth path, in which variables grow at a rate exactly equal to zero.

Before solving the model, let's understand what a balanced growth path implies, and analyze the conditions under which a balanced growth path can exist. First, on a balanced growth path all variables grow at a constant rate. In principle, growth rates need to be the same for different variables. However, as the next theorem shows, on a balanced growth path all variables grow at the same rate.

Theorem 3. *Along a balanced growth path, all per-capita variables grow at the same rate.*

Proof. Assume we solve the model in per-capita terms, as in Section 1.2 with population growth. The resulting resource constraint is one of the necessary and sufficient conditions for an optimum:

$$c_t = F(k_t, A_t) + (1-\delta)k_t - (1+n)k_{t+1} \tag{1.83}$$

Define the growth rate of per-capita consumption along a balanced growth path as γ_c, such that $c_t = (1+\gamma_c)^t c_0$, assuming the economy starts at period 0 on the balanced growth path. Similarly, define γ_k and γ_y as the growth rates of per-capita capital and income on a balanced growth path, such that $k_t = (1+\gamma_k)^t k_0$ and $y_t = (1+\gamma_y)^t y_0$.

Using these definitions, we can express the resource constraint (1.83) as:

$$(1+\gamma_c)^t c_0 = F((1+\gamma_k)^t k_0, (1+g)^t) + (1-\delta)(1+\gamma_k)^t k_0 - (1+n)(1+\gamma_k)^{t+1} k_0$$

Dividing both sides by $(1+\gamma_k)^t$:

$$\left(\frac{1+\gamma_c}{1+\gamma_k}\right)^t c_0 = F\left(k_0, \left(\frac{1+g}{1+\gamma_k}\right)^t\right) + (1-\delta)k_0 - (1+n)(1+\gamma_k)k_0$$

The only way for this equation to hold for all t is if the terms that depend on t do not vary, which implies that:

$$\left(\frac{1+\gamma_c}{1+\gamma_k}\right) = \left(\frac{1+g}{1+\gamma_k}\right) = 1$$

This implies that $\gamma_c = \gamma_k = g$. Also, output per-capita is:

$$y_t = F((1+g)^t k_0, (1+g)^t) = (1+g)^t F(k_0, 1) = (1+\gamma_y)^t y_0$$

This implies that $\gamma_y = \gamma_c = \gamma_k = g$.

□

Re-writing the resource constraint:

$$
\begin{aligned}
& (1+\gamma_y)^t y_0 = (1+\gamma_c)^t c_0 + (1+n)(1+\gamma_k)^{t+1} k_0 - (1-\delta)(1+\gamma_k)^t k_0 \\
\iff \quad & y_0 = c_0 + (1+n)(1+g)k_0 - (1-\delta)k_0 \\
\iff \quad & y_0 = c_0 + \underbrace{(n+g+ng+\delta)k_0}_{\text{Investment}}
\end{aligned}
$$

This means that, as long as population and technological growth are small (say, around $1-2\%$, which means $ng \approx 0.0002 \approx 0$), investment in the balanced growth path replaces depreciated capital (δk_0), and augments the capital stock such that capital per capita and per level of technology remains constant next period ($(n+g)k_0$).

Now, let's analyze the conditions under which a balanced growth path can exist. Take the Euler equation resulting from solving the model in per-capita terms:

$$\frac{u'(c_t)}{\beta u'(c_{t+1})} = \frac{1-\delta+F_k(k_{t+1},A_{t+1})}{(1+n)} \tag{1.84}$$

On the right-hand side, the only non-constant term is $F_k(k_{t+1},A_{t+1})$. Because of the constant-returns-to-scale assumption, $F_k(k_{t+1},A_{t+1}) = F_k(k_{t+1}/A_{t+1},1)$. On a balanced growth path, given that $k_t = (1+g)^t k_0$ and $A_t = (1+g)^t$, the term $k_{t+1}/A_{t+1} = k_0$ remains constant. Therefore, on a balanced growth path, the term $u'(c_t)/\beta u'(c_{t+1})$ must be constant. This implies the following result:

Theorem 4. *There exists a balanced growth path if and only if u is CRRA.*

Proof. ($\Longleftarrow$) :Assume the utility function u is CRRA, which means that $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$. This means that $u'(c) = c^{-\sigma}$, and the Euler equation (1.84) becomes:

$$\frac{1}{\beta}\cdot\left(\frac{c_t}{c_{t+1}}\right)^{-\sigma} = \frac{1-\delta+F_k(k_0,1)}{(1+n)}$$

On a balanced growth path, $c_{t+1} = (1+g)c_t$, so there exists a balanced growth path as long as the economy starts at a capital level k_0, such that:

$$\frac{1}{\beta}\cdot(1+g)^{\sigma} = \frac{1-\delta+F_k(k_0,1)}{(1+n)}$$

($\Longrightarrow$) : Now, assume there exists a balanced growth path, such that $c_t = (1+g)c_{t+1}$. Given the Euler equation (1.84), this means that:

$$\frac{u'(c_t)}{\beta u'((1+g)c_t)} = M$$

For some constant M. Rearranging and differentiating with respect to c_t:

$$
\begin{aligned}
& u'(c_t) = M \cdot \beta u'((1+g)c_t) \\
\iff \quad & u''(c_t) = M \cdot \beta(1+g)u''((1+g)c_t) \\
\iff \quad & u''(c_t) = (1+g)\frac{u'(c_t)}{\beta u'((1+g)c_t)} \cdot \beta u''((1+g)c_t) \\
\iff \quad & \frac{u''(c_t)c_t}{u'(c_t)} = \frac{u''((1+g)c_t)(1+g)c_t}{u'((1+g)c_t)} = \frac{u''(c_{t+1})c_{t+1}}{u'(c_{t+1})}
\end{aligned}
$$

This means that $\frac{u''(c_t)c_t}{u'(c_t)}$ is constant, which is the definition of CRRA. □

Therefore, the only models in which a balanced growth path can exist are those in which the utility function takes the CRRA form. Moreover, on a balanced growth path all variables grow at the same rate as the technological change.

1.4.1 Solving for a Competitive Equilibrium

This section solves for the competitive equilibrium when there is technological change. As before, I omit the definition of a competitive equilibrium, given that it is analogous to the definitions in past sections.

Note that we can re-state the definition by saying that the economy reaches a balanced growth path, whenever all variables *per effective unit of labor* are constant. Henceforth, we will denote variables per effective unit of labor with a tilde:

$$
\begin{aligned}
\tilde{c}_t &= \frac{C_t}{A_t L_t} = \frac{c_t}{A_t} \\
\tilde{y}_t &= \frac{Y_t}{A_t L_t} = \frac{y_t}{A_t} \\
\tilde{k}_t &= \frac{K_t}{A_t L_t} = \frac{k_t}{A_t} \\
&\vdots
\end{aligned}
$$

We now write the household's problem in terms of per-effective-unit-of-labor variables.

The household's budget constraint becomes:

$$\tilde{c}_t + \tilde{i}_t + (1+g)(1+n)\tilde{a}_{t+1} = \tilde{w}_t + r_t\tilde{k}_t + \tilde{\pi}_t + \tilde{a}_t$$

where $\tilde{w}_t = w_t/A_t$. Now, to express the capital evolution equation in terms of variables per efficiency unit, note that:

$$\frac{K_{t+1}}{A_tL_t} = \frac{K_{t+1}}{A_tL_t}\left(\frac{A_{t+1}L_{t+1}}{A_{t+1}L_{t+1}}\right) = \frac{K_{t+1}}{A_{t+1}L_{t+1}}\left(\frac{A_{t+1}}{A_t}\right)\left(\frac{L_{t+1}}{L_t}\right) = (1+g)(1+n)\tilde{k}_{t+1}$$

This means that the capital evolution equation can be written as:

$$(1+g)(1+n)\tilde{k}_{t+1} = (1-\delta)\tilde{k}_t + \tilde{i}_t$$

The total output in terms of per-effective-units-of-labor is:

$$\tilde{y}_t = \frac{F(K_t, A_tL_t)}{A_tL_t} = F\left(\frac{K_t}{A_tL_t}, \frac{A_tL_t}{A_tL_t}\right) = F(\tilde{k}_t, 1)$$

Now, we explicitly use the CRRA utility function, as stated in the previous theorem:

$$u(c) = \frac{c^{1-\sigma}}{1-\sigma}$$

If we multiply and divide by A_t, we get:

$$\begin{aligned}
\sum_{t=0}^{\infty}\beta^t u(c_t) &= \sum_{t=0}^{\infty}\beta^t \frac{\left(\frac{A_t}{A_t}c_t\right)^{1-\sigma}}{1-\sigma} \\
&= \sum_{t=0}^{\infty}\beta^t \frac{(A_t\tilde{c}_t)^{1-\sigma}}{1-\sigma} \\
&= \sum_{t=0}^{\infty}\beta^t A_t^{1-\sigma}\cdot\frac{\tilde{c}_t^{1-\sigma}}{1-\sigma} \\
&= \sum_{t=0}^{\infty}\hat{\beta}^t \frac{\tilde{c}_t^{1-\sigma}}{1-\sigma}
\end{aligned}$$

where $\hat{\beta} = \beta \cdot (1+g)^{1-\sigma}$. The household's problem is given by:

$$\max_{\tilde{c}_t, \tilde{k}_{t+1}} \quad \sum_{t=0}^{\infty} \hat{\beta}^t u(\tilde{c}_t) \quad s.t. \tag{1.85}$$

$$\begin{aligned}
\forall t \in \{0,1,\ldots\}: \quad & \tilde{c}_t + (1+g)(1+n)\tilde{k}_{t+1} + (1+g)(1+n)\tilde{a}_{t+1} = \\
& \qquad (1-\delta)\tilde{k}_t + \tilde{w}_t + R_t\tilde{k}_t + (1+r)\tilde{a}_t \\
& \tilde{c}_t, \tilde{k}_{t+1} \geq 0, \quad \tilde{k}_0 \text{ given} \\
& \lim_{T\to\infty} \left(\prod_{t=0}^{T} \frac{1}{1+r_t} \right) \tilde{a}_{T+1} \geq 0
\end{aligned}$$

The solution to the household's problem yields equations that are analogous to past sections'. In particular, the solution yields the following Euler equation:

$$\frac{u'(\tilde{c}_t)}{\hat{\beta}u'(\tilde{c}_{t+1})} = \frac{(1-\delta+R_{t+1})}{(1+n)(1+g)}$$

The firm's problem is:

$$\max_{L_t, K_t} \quad F(K_t, A_tL_t) - w_tL_t - r_tK_t$$

So prices are set by:

$$\begin{aligned}
w_t &= A_t \cdot F_l(K_t, A_tL_t) \\
r_t &= F_k(K_t, A_tL_t)
\end{aligned}$$

To express prices in per-efficiency units of labor:

$$F(K_t, A_tL_t) = A_tL_tF\left(\frac{K_t}{A_tL_t}, 1\right) = A_tL_tF(\tilde{k}_t, 1)$$

So:

$$\begin{aligned}
A_tF_l(K_t, A_tL_t) &= A_t \cdot \left[F\left(\frac{K_t}{A_tL_t}, 1\right) + L_tF_k\left(\frac{K_t}{A_tL_t}, 1\right)\left(\frac{-K_t}{A_tL_t^2}\right) \right] \\
&= A_t \left[F\left(\frac{K_t}{A_tL_t}, 1\right) + F_k\left(\frac{K_t}{A_tL_t}, 1\right)\left(\frac{-K_t}{A_tL_t}\right) \right] \\
&= A_tF(\tilde{k}_t, 1) + A_tF_k(\tilde{k}_t, 1)\tilde{k}_t
\end{aligned}$$

Therefore:

$$\begin{aligned} \tilde{w}_t &= F(\tilde{k}_t, 1) + F_k(\tilde{k}_t, 1)\tilde{k}_t \\ r_t &= F_k(\tilde{k}_t, 1) \end{aligned}$$

1.4.2 Balanced Growth Path

On a balanced growth path, all variables per effective unit of labor remain constant:

$$\begin{aligned} \tilde{c}_t &= \tilde{c_{t+1}} = \ldots = \tilde{c}^* \\ \tilde{k}_t &= \tilde{k_{t+1}} = \ldots = \tilde{k}^* \\ &\vdots \end{aligned}$$

This implies that, on a balanced growth path:

$$F_K(\tilde{k}^*, 1) = \frac{(1+n)(1+g)}{\hat{\beta}} - 1 + \delta \tag{1.86}$$

$$\tilde{c}^* = F(\tilde{k}^*, 1) - (\delta + n + g + ng)\tilde{k}^* \tag{1.87}$$

Once we have solved for the balanced growth path variables, we can obtain the per-capita variables by multiplying by $A_t = (1+g)^t$:

$$k_t = (1+g)^t k^*, \quad c_t = (1+g)^t c^*, \quad \ldots$$

If we plot $\log(y_t), \log(k_t), \ldots$ against time t, we obtain a linear slope, as is the case for many developed countries in the last century. Figure 1.1 illustrates per-capita GDP over the last century, where the vertical axis represents constant U.S. dollars in logarithmic scale. According to this Figure, the neoclassical growth model seems a sufficiently good tool to analyze growth in economies that seem to have reached a "balanced growth path", such as the U.S. or some European countries.

The main advantage of this section, as opposed to past sections, is that with exogenous growth we have a model in which per-capita variables grow in the long run, as is observed for most nations. However, so far we have assumed that growth is driven by growth in technology, which follows a completely exogenous process. Later sections will study models in which technological progress is an endogenous variable in the model.

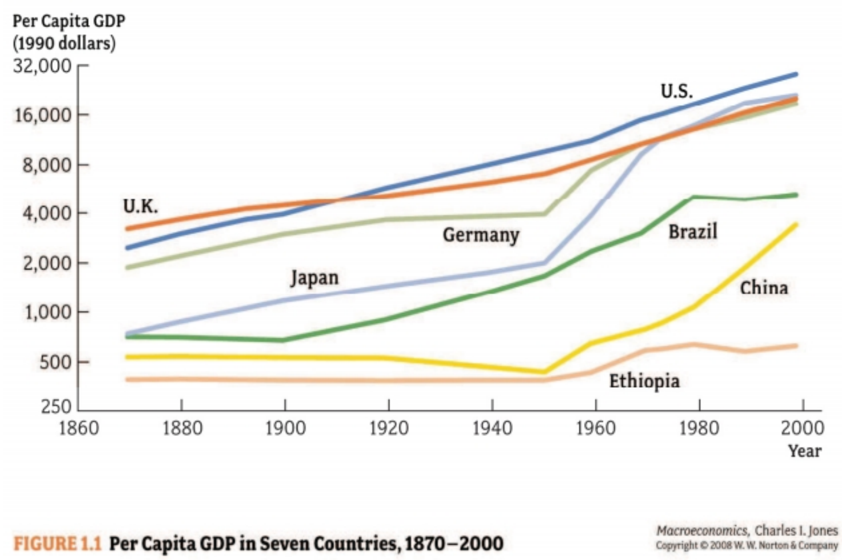

Figure 1.1: Per-capita GDP.

1.5 Numerical Methods I

Up to now, we have studied economies in which the equilibrium and/or steady state is fully characterized by a set of equations but, in most cases, there is no closed-form solution for each of the variables. This section allows us to compute the actual values of the variables using a method to solve non-linear equations called the Newton-Raphson method. The importance of this method is illustrated in the following two examples:

1.5.1 Example I - Computation of Steady State/Balanced Growth Path

Recall that in past sections we computed the steady state/balanced growth path values of capital k^*, consumption c^*, production y^*, and prices w^*, R^* and r^* in the economy (see equations (1.42)-(1.47)). To obtain consumption and production, we first had to solve for capital in steady state/balanced growth path, implicitly defined by equation (1.42). The question that arises is, given parameters for the models described (β, n, g, $\ldots$), how can we numerically obtain the value of k^*?

For example, note that for the model with technological change (equation (1.86)), the level of capital per-efficiency unit of labor in steady state is given by the value $\tilde{k}^*$ that solves:

$$F_k(\tilde{k}^*,1) = \frac{(1+n)(1+g)}{\hat{\beta}} - 1 + \delta$$

This equation can be rearranged, so that $\tilde{k}^*$ solves:

$$F_k(\tilde{k}^*,1) - \frac{(1+n)(1+g)}{\hat{\beta}} + 1 - \delta = 0$$

So the capital per-efficiency unit of labor is determined by the zero, or root, of the non-linear equation:

$$g(x) = F_k(x,1) - \frac{(1+n)(1+g)}{\hat{\beta}} + 1 - \delta$$

That is, $\tilde{k}^*$ is such that $g(\tilde{k}^*) = 0$. The Newton-Raphson method studied in this section is a first approach to finding the zero of this equation, which will allow us to compute the values of the variables in steady state.

1.5.2 Example II - Computation of Equilibrium

The second example to illustrate the importance of solving systems of non-linear equations is a simplified two-period version of the neoclassical growth model. Assume that the individual only lives for two periods, and chooses consumption every period and the amount of capital to accumulate from the first to the second period. Further, assume there is no population growth. The problem of the household is:

$$\max_{c_0,c_1,k_1} \quad u(c_0) + \beta u(c_1) \quad s.t.$$

$$c_0 + k_1 = (1-\delta)k_0 + w_0 + r_0 k_0$$

$$c_1 = (1-\delta)k_1 + w_1 + r_1 k_1$$

$$c_0, c_1, k_1 \geq 0, \quad k_0 \text{ given}$$

The lagrangean is:

$$\mathcal{L} = u(c_0) + \beta u(c_1) + \lambda_0 \left((1-\delta)k_0 + w_0 + r_0 k_0 - c_0 - k_1\right) + \lambda_1 \left((1-\delta)k_1 + w_1 + r_1 k_1 - c_1\right)$$

The first order conditions are:

$$
\begin{aligned}
[c_0]: \quad & u_c(c_0) - \lambda_0 = 0 \\
[c_1]: \quad & \beta u_c(c_1) - \lambda_1 = 0 \\
[k_1]: \quad & -\lambda_0 + \lambda_1 (1 - \delta + r_1) = 0 \\
[\lambda_0]: \quad & (1-\delta)k_0 + w_0 + r_0 k_0 - c_0 - k_1 = 0 \\
[\lambda_1]: \quad & (1-\delta)k_1 + w_1 + r_1 k_1 - c_1 = 0
\end{aligned}
$$

Which, getting rid of the λ's, yields the following optimality conditions:

$$u_c(c_0) = (1 - \delta + r_1)\, \beta u_c(c_1) \tag{1.88}$$

$$(1-\delta)k_0 + w_0 + r_0 k_0 - c_0 - k_1 = 0 \tag{1.89}$$

$$(1-\delta)k_1 + w_1 + r_1 k_1 - c_1 = 0 \tag{1.90}$$

The first order conditions for the firm are:

$$w_0 = F_l(k_0, 1) \tag{1.91}$$

$$w_1 = F_l(k_1, 1) \tag{1.92}$$

$$r_0 = F_k(k_0, 1) \tag{1.93}$$

$$r_1 = F_k(k_1, 1) \tag{1.94}$$

In order to compute the equilibrium of the economy described we must find $(c_0, c_1, k_1, w_0, w_1, r_0, r_1)$ that solve the system of equations (1.88)-(1.94). We can rearrange each equation, to define the function f in the following way:

$$
f(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \\ f_3(x) \\ f_4(x) \\ f_5(x) \\ f_6(x) \\ f_7(x) \end{pmatrix} = \begin{pmatrix} u_c(c_0) - (1 - \delta + r_1)\, \beta u_c(c_1) \\ (1-\delta)k_0 + w_0 + r_0 k_0 - c_0 - k_1 \\ (1-\delta)k_1 + w_1 + r_1 k_1 - c_1 \\ w_0 - F_l(k_0, 1) \\ w_1 - F_l(k_1, 1) \\ r_0 - F_k(k_0, 1) \\ r_1 - F_k(k_1, 1) \end{pmatrix}
$$

Where $x = (c_0, c_1, k_1, w_0, w_1, r_0, r_1)$. Finding an equilibrium to such economy is equivalent to finding a root, or zero, of the function f. That is, an equilibrium is x^* such that

$f(x^*) = 0$. We can also get rid of the last 4 equations, by plugging in the prices directly in the optimality conditions of the household:

$$f(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \\ f_3(x) \end{pmatrix} = \begin{pmatrix} u_c(c_0) - (1 - \delta + F_k(k_1, 1))\, \beta u_c(c_1) \\ (1-\delta)k_0 + F_l(k_0, 1) + F_k(k_0, 1)k_0 - c_0 - k_1 \\ (1-\delta)k_1 + F_l(k_1, 1) + F_k(k_1, 1)k_1 - c_1 \end{pmatrix}$$

As in last example, the Newton-Raphson will be useful to solve for the equilibrium of the economy.

1.5.3 Newton-Raphson

The reasoning behind the method is as follows. Assume we have a vector $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, and a function $f : \mathbb{R}^n \to \mathbb{R}^n$, which can be written as:

$$f(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_n(x) \end{pmatrix}$$

Suppose we are interested in finding a root to that function. That is, suppose we are interested in finding $\bar{x} \in \mathbb{R}^n$, such that $f(\bar{x}) = 0$, and assume we start from a point $\hat{x}$ which is sufficiently close to $\bar{x}$. Consider the following Taylor expansion of the function f around the point $\bar{x}$:

$$f(\bar{x}) \approx f(\hat{x}) + J(\hat{x}) \cdot (\bar{x} - \hat{x})$$

Where J is the Jacobian, or matrix of first derivatives of f. Given that $f(\bar{x}) = 0$, we can rewrite it as:

$$0 \approx f(\hat{x}) + J(\hat{x}) \cdot (\bar{x} - \hat{x}) \quad \iff \quad \bar{x} \approx \hat{x} - J^{-1}(\hat{x}) \cdot f(\hat{x}) \tag{1.95}$$

This means that, if we start from a guess $\hat{x}$ of the root of f that is sufficiently close to $\bar{x}$, we can approximate $\bar{x}$ using equation (1.95).

In matrix form, this can be written as:

$$\begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \vdots \\ \bar{x}_n \end{pmatrix} \approx \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \vdots \\ \hat{x}_n \end{pmatrix} - \begin{pmatrix} \frac{\partial f_1(\hat{x})}{\partial x_1} & \dots & \frac{\partial f_1(\hat{x})}{\partial x_n} \\ \frac{\partial f_2(\hat{x})}{\partial x_1} & \dots & \frac{\partial f_2(\hat{x})}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n(\hat{x})}{\partial x_1} & \dots & \frac{\partial f_n(\hat{x})}{\partial x_n} \end{pmatrix}^{-1} \cdot \begin{pmatrix} f_1(\hat{x}) \\ f_2(\hat{x}) \\ \vdots \\ f_n(\hat{x}) \end{pmatrix}$$

Therefore, the Newton-Raphson algorithm is as follows:

Algorithm 1 (Newton-Raphson)**.**

1. *Set $s = 0$, and start with a guess x^0 for the root of f.*

2. *Compute the function $f(x^s)$ and the Jacobian $J(x^s)$.*

3. *Set x^{s+1} as follows:*

$$x^{s+1} = x^s - J(x^s)^{-1} f(x^s)$$

4. *If $\left\|x^{s+1} - x^s\right\| < \epsilon$, stop. Else, set $s = s + 1$ and go to 2.*

The algorithm is illustrated in Figure 1.2[1]. In this algorithm, the norm $\|\cdot\|$ can be the euclidean norm, or any other norm in $\mathbb{R}^n$. If the starting point x^0 is sufficiently close to the root $\bar{x}$, it can be shown that the method converges and the result obtained is close to the root. However, if we start the method far from the root, the algorithm can converge to another root, or diverge.

With this algorithm, we can approximate numerically the roots to non-linear systems of equations. How can we apply the method to solve for the steady state values in Section 1.5.1, and for the equilibrium in Section 1.5.2?

Example I - Computation of Steady State

Recall that to find the level of capital in steady state, it suffices to find the value x such that $g(x) = 0$, where:

$$g(x) = F_k(x, 1) - \frac{(1+n)(1+g)}{\hat{\beta}} + 1 - \delta$$

[1] Source: https://www.geeksforgeeks.org/program-for-newton-raphson-method/

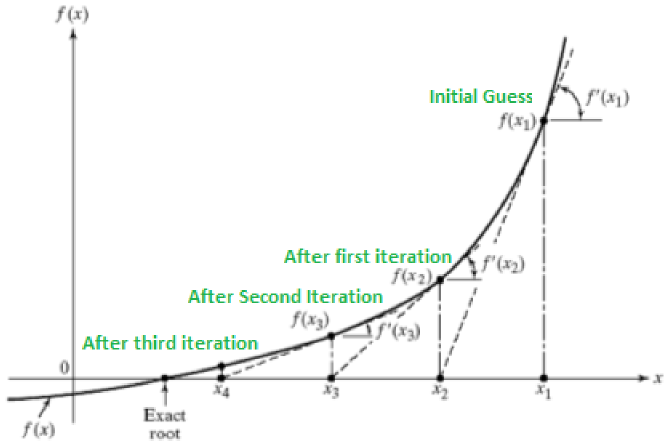

Figure 1.2: Illustration of Newton-Raphson method.

The Jacobian of g is simply its derivative with respect to x, equal to:

$$J(x) = F_{kk}(x, 1)$$

Therefore, in every iteration the algorithm updates the value of x as:

$$x^{s+1} = x^s - F_{kk}(x^s, 1) \cdot \left(F_k(x^s, 1) - \frac{(1+n)(1+g)}{\hat{\beta}} + 1 - \delta \right)$$

If we start at a point x^0 which is sufficiently close to the value $\tilde{k}^*$, the Newton-Raphson algorithm will converge to an approximation of $\tilde{k}^*$.

This algorithm can be implemented in the way described above, by computing the derivative and iterating until convergence. However, most programming languages already have built-in functions that solve non-linear systems of equations. In `Matlab`, a function to solve non-linear equations is `fsolve`, which receives as input a function g and an initial guess of a root x_0. For example, if we assume that the production function takes the form of a Cobb-Douglas function, such that $F(K, L) = K^{\alpha} L^{1-\alpha}$ and $F_K(k, 1) = \alpha k^{\alpha-1}$, the code in Box 1.1 computes the steady state value of capital.

Example II - Computation of Equilibrium

Recall that finding an equilibrium in the two-period environment is equivalent to finding a root for f, where:

```
% Parameter values
aalpha = 0.3;
bbeta  = 0.95;
n      = 0.02;
g      = 0.03;

% Defining the function g
g = @(x) aalpha*(x^(aalpha-1))-((1+n)*(1+g))/bbeta+1-ddelta
x0 = 1;

% Non-linear equation solver
fsolve(g, x0);
```

Box 1.1: `MATLAB` code to compute steady state level of capital.

$$f(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \\ f_3(x) \\ f_4(x) \\ f_5(x) \\ f_6(x) \\ f_7(x) \end{pmatrix} = \begin{pmatrix} u_c(c_0) - (1-\delta+r_1)\,\beta u_c(c_1) \\ (1-\delta)k_0 + w_0 + r_0 k_0 - c_0 - k_1 \\ (1-\delta)k_1 + w_1 + r_1 k_1 - c_1 \\ w_0 - F_l(k_0, 1) \\ w_1 - F_l(k_1, 1) \\ r_0 - F_k(k_0, 1) \\ r_1 - F_k(k_1, 1) \end{pmatrix}$$

In order to apply the algorithm directly, we can compute the Jacobian matrix by finding all the cross derivatives of f. For example:

$$\begin{aligned} f_{11}(x) &= u_{cc}(c_0) \\ f_{12}(x) &= -(1-\delta+r_1)\beta u_{cc}(c_1) \\ f_{21}(x) &= -1 \\ f_{22}(x) &= 0 \\ &\vdots \end{aligned}$$

As in the past example, we can use the `fsolve` function. Box 1.2 illustrates the definition of function f.

```
function F = sys_equations(c0, c1, k1, w0, w1, r0, r1)
 F(1) = c0^(-ssigma) - (1-ddelta+r1)*bbeta*c1^(-ssigma);
 F(2) = (1-ddelta)*k0+w0+r0*k0-c0-k1;
 F(3) = (1-ddelta)*k1+w1+r1*k1-c1;
 F(4) = w0 - (1-aalpha)*k0^aalpha;
 F(5) = w1 - (1-aalpha)*k1^aalpha;
 F(6) = r0 - aalpha*k0^(aalpha-1);
 F(7) = r1 - aalpha*k1^(aalpha-1);
end
```

Box 1.2: `MATLAB` code to compute the equilibrium.

1.5.4 Transitional Dynamics

The last example was a simplified, two-period version of the economy described in Section 1.1, where households live for an infinite number of periods. Our ultimate goal is to compute the equilibrium in such an economy. It turns out that solving for that equilibrium is not very different from what we have done so far.

Recall that the optimality conditions in Section 1.1 at every period t are given by:

$$\frac{u'(c_t)}{\beta u'(c_{t+1})} = (1 - \delta + r_{t+1})$$

$$c_t + k_{t+1} = (1 - \delta)k_t + F(k_t, 1)$$

$$w_t = F_l(k_t, l_t)$$

$$r_t = F_k(k_t, l_t)$$

By plugging the prices in the household's optimality conditions, the equilibrium allocations are fully characterized by the following two equations for every t:

$$\Psi_1(c_t, c_{t+1}, k_t, k_{t+1}) = \frac{u'(c_t)}{\beta u'(c_{t+1})} - (1 - \delta + F_k(k_{t+1}, 1)) = 0$$

$$\Psi_2(c_t, c_{t+1}, k_t, k_{t+1}) = (1 - \delta)k_t + F_l(k_t, 1) + F_k(k_t, 1)k_t - c_t - k_{t+1} = 0$$

The problem is that now, as opposed to the second example above, we have an infinite number of periods, which means that to compute the equilibrium we must solve a system of infinitely many equations. The way around this difficulty is the following. Recall that, no matter what is the starting point k_0, we assumed this economy converges to a steady state in the long run, in which all variables are constant. Technically speaking, if the economy starts *out* of steady state, it will never *exactly* reach steady state, but will get arbitrarily close to it. This means that, for a sufficiently large T, we can assume that the economy effectively reaches steady state, which means that $k_t = k^*$ for all $t \geq T$. Therefore, since variables remain constant after T, the system of equations we need to solve will only include the equations up to period T (those corresponding to $t > T$ are redundant). This assumption reduces the number of equations in our system significantly, to a finite number of equations, namely:

$$\begin{aligned}
&\Psi_1(c_0, c_1, k_0, k_1) = 0 \\
&\Psi_2(c_0, c_1, k_0, k_1) = 0 \\
&\Psi_1(c_1, c_2, k_1, k_2) = 0 \\
&\Psi_2(c_1, c_2, k_1, k_2) = 0 \\
&\ldots \\
&\Psi_1(c_{T-1}, c_T, k_{T-1}, k_T) = 0 \\
&\Psi_2(c_{T-1}, c_T, k_{T-1}, k_T) = 0
\end{aligned}$$

This means that, to solve for the equilibrium variables, we should solve for $\{c_t, k_{t+1}\}_{t=0}^{T}$. In the above system, there are $2 \cdot (T + 1)$ unknowns and $2T$ equations. Recall that k_0 is given and $k_T = k^*$, so can effectively solve for an equilibrium by finding the root of the system of equations.

Figure 1.3 illustrates the behavior of an example economy that starts at 10% the capital of the steady state, assuming the utility function is $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$ and the production function is $f(k, 1) = \alpha k^{1-\alpha}$. After 10 periods, the economy has already converged to steady state.

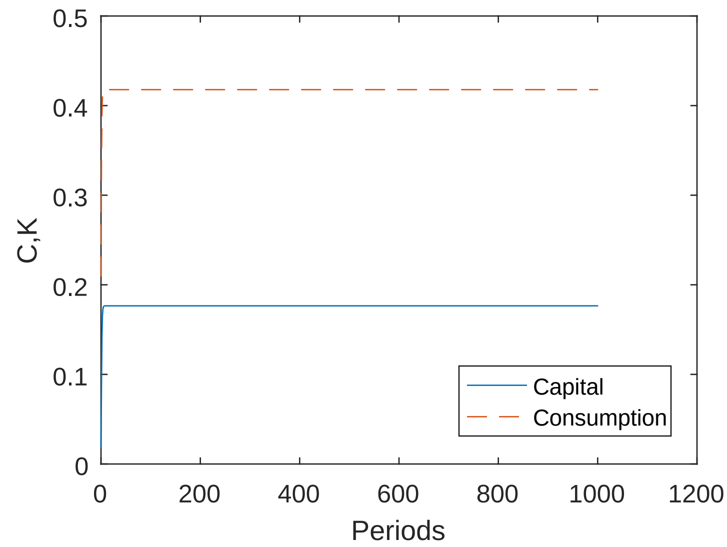

Figure 1.3: Transitional dynamics to steady state.

1.5.5 Approximating the Jacobian

If we want to apply the Newton-Raphson method we must know the Jacobian matrix J of f, which means that we must be able to obtain the derivatives of f with respect to every dimension. In many cases, as the two examples described above, we can obtain the derivatives of f and apply directly Algorithm 1. However, there are cases in which it is not possible to obtain the Jacobian matrix.

If we cannot explicitly compute the derivatives of f, we can obtain a numerical approximation to J. At every iteration, instead of computing $J(x)$, we will have to approximate it.

The Jacobian can be written as:

$$J(x) = [J_1(x), J_2(x), \dots, J_n(x)]$$

Where $J_i(x)$ is the derivative of f with respect to x_i:

$$J_i(x) = \begin{pmatrix} \frac{\partial f_1(x)}{\partial x_i} \\ \frac{\partial f_2(x)}{\partial x_i} \\ \frac{\partial f_3(x)}{\partial x_i} \\ \vdots \\ \frac{\partial f_n(x)}{\partial x_i} \end{pmatrix}$$

Making a Taylor approximation of f, for a sufficiently small $h \geq 0$:

$$\begin{aligned}
f(x_1 + h_1, x_2, x_3, \ldots, x_n) &\approx f(x_1, x_2, x_3, \ldots, x_n) + J_1(x)h_1 \\
f(x_1, x_2 + h_2, x_3, \ldots, x_n) &\approx f(x_1, x_2, x_3, \ldots, x_n) + J_2(x)h_2 \\
f(x_1, x_2, x_3 + h_3, \ldots, x_n) &\approx f(x_1, x_2, x_3, \ldots, x_n) + J_3(x)h_3 \\
&\vdots \\
f(x_1, x_2, x_3, \ldots, x_n + h_n) &\approx f(x_1, x_2, x_3, \ldots, x_n) + J_n(x)h_n
\end{aligned}$$

Which means that we can approximate $J_1, \ldots, J_n$ as:

$$\begin{aligned}
J_1(x) &\approx \frac{1}{h_1}\left(f(x_1 + h_1, x_2, x_3, \ldots, x_n) - f(x_1, x_2, x_3, \ldots, x_n)\right) \\
J_2(x) &\approx \frac{1}{h_2}\left(f(x_1, x_2 + h_2, x_3, \ldots, x_n) - f(x_1, x_2, x_3, \ldots, x_n)\right) \\
&\vdots \\
J_n(x) &\approx \frac{1}{h_n}\left(f(x_1, x_2, x_3, \ldots, x_n + h_n) - f(x_1, x_2, x_3, \ldots, x_n)\right)
\end{aligned}$$

If $h \geq 0$, we are approximating J from the right. Analogously, we can approximate J

from the left:

$$
\begin{aligned}
J_1(x) &\approx \frac{1}{h_1}\left(f(x_1, x_2, x_3, \ldots, x_n) - f(x_1 - h_1, x_2, x_3, \ldots, x_n)\right) \\
J_2(x) &\approx \frac{1}{h_2}\left(f(x_1, x_2, x_3, \ldots, x_n) - f(x_1, x_2 - h_2, x_3, \ldots, x_n)\right) \\
&\vdots \\
J_n(x) &\approx \frac{1}{h_n}\left(f(x_1, x_2, x_3, \ldots, x_n) - f(x_1, x_2, x_3, \ldots, x_n - h_n)\right)
\end{aligned}
$$

Finally, we can approximate J as an average of the right and left approximations:

$$
\begin{aligned}
J_1(x) &\approx \frac{1}{2h_1}\left(f(x_1 + h_1, x_2, x_3, \ldots, x_n) - f(x_1 - h_1, x_2, x_3, \ldots, x_n)\right) \\
J_2(x) &\approx \frac{1}{2h_2}\left(f(x_1, x_2 + h_2, x_3, \ldots, x_n) - f(x_1, x_2 - h_2, x_3, \ldots, x_n)\right) \\
&\vdots \\
J_n(x) &\approx \frac{1}{2h_n}\left(f(x_1, x_2, x_3, \ldots, x_n + h_n) - f(x_1, x_2, x_3, \ldots, x_n - h_n)\right)
\end{aligned}
$$

In the `MATLAB` examples above, we did not explicitly state whether the algorithm should take a numerical approximation of the Jacobian at every iteration, or use an explicit expression to compute it. Having to approximate J at every iteration might slow down the algorithm, given the additional step needed for this. To avoid this extra cost, we can explicitly tell `MATLAB` that we are going to supply the Jacobian matrix. The code in Box 1.3 activates the option in which we supply the `fsolve` function with the Jacobian matrix.

```
options = optimoptions(options,'SpecifyObjectiveGradient',
   true);
fsolve(@(x) equilibrium(K0, Kss, x), X, options);
```

Box 1.3: `MATLAB` code to supply `fsolve` with Jacobian function.

In order for the `SpecifyObjectiveGradient` to work, the objective function `equilibrium` must generate two outputs: the output of the function and the Jacobian matrix. Table 1.1 illustrates the difference in computation times with and without gradient in the above example for certain parameter values. In most cases, the computation time is faster when the user provides the Jacobian matrix to the `fsolve` function.

T	Time (s) with gradient	Time (s) without gradient
50	0.047	0.061
100	0.088	0.159
200	0.34	0.51
500	1.93	2.79
1000	14.4	12.3

Table 1.1: Average computation times over 20 repetitions with and without gradient.

1.5.6 Gauss-Seidel Algorithm

The method described above to solve for an equilibrium can be computationally costly to implement. Assume, for example, that we have an economy with endogenous labor and different types of assets such that, for every period t, we have three or more optimality conditions. Assume also that the transition from the initial state k_0 to the steady state takes 30 or more periods. This means that in such an economy we would have to find the root of a system of 90+ equations. This can take quite some time, so the method described may not be very efficient.

In the Gauss-Seidel algorithm, instead of solving for all equilibrium conditions at the same time, we solve successively the system corresponding to every period t. In this way, instead of solving the system of all equations at the same time, we solve many systems of equations, one after the other.

To implement the algorithm, we can simplify the system of equations Ψ_1 and Ψ_2 by solving for c_t in the budget constraint and plugging it in the Euler equation in the following way:

$$\Psi(k_t, k_{t+1}, k_{t+2}) = \frac{u'((1-\delta)k_t + F_l(k_t, 1) + F_k(k_t, 1)k_t - k_{t+1})}{\beta u'((1-\delta)k_{t+1} + F_l(k_{t+1}, 1) + F_k(k_{t+1}, 1)k_{t+1} - k_{t+2})} - (1-\delta+F_k(k_{t+1}, 1)) = 0$$

Now, to find an equilibrium we must find the capital sequence that satisfies the system of equations $\Psi(k_t, k_{t+1}, k_{t+2})$ for every t. The following algorithm does that:

Algorithm 2 (Gauss-Seidel)**.**

1. *Set* $s = 0$*, and start with a guess* $k_1^0, k_2^0, \ldots, k_{T-1}^0$ *for the sequence of capital along the transition, starting from* k_0 *and ending in* $k_T = k^*$*.*

2. *For every* $j \in \{1, \ldots, T-1\}$*, find the root* x^* *of* $\Psi(k_{j-1}^s, x, k_{j+1}^s)$*, such that* $\Psi(k_{j-1}^s, x^*, k_{j+1}^s) = 0$*. Set* $k_j^{s+1} = x^*$*.*

3. *If* $\left\|(k_1^{s+1}, \ldots, k_{T-1}^{s+1}) - (k_1^s, \ldots, k_{T-1}^s)\right\| < \epsilon$*, stop. Else, set* $s = s + 1$ *and go to 2.*

Once we have found the sequence of capital along the transition from k_0 to k^*, we can construct the sequence of all other variables using the production function, the budget constraint, the capital evolution equation, and so on:

$$
\begin{aligned}
y_t &= f(k_t, 1) \\
c_t &= (1-\delta)k_t + F_l(k_t, 1) + F_k(k_t, 1)k_t - k_{t+1} \\
i_t &= k_{+1} - (1-\delta)k_t \\
&\vdots
\end{aligned}
$$

1.6 Recursive Representation and Dynamic Programming

The main problem with the model described in Sections 1.1-1.4 is that problems such as the household's, described by equation (1.17), require maximizing the utility function over an infinite space. The solution to the consumer's problem are infinite sequences $\{c_t, k_t, i_t\}_{t=0}^{\infty}$ for consumption, capital and investment. When the functional forms are such that we do not have closed-form solutions, as the one obtained for the log utility case in Section 1.1.4, the problem can become infeasible to estimate. The purpose of this section is to transform the consumer's problem in such a way that it can be solved with the use of dynamic programming.

1.6.1 Social Planner's Problem

First, let's take the social planner's problem (1.6) and define the function V in the following way:

$$V(k_0) \quad := \quad \max_{\substack{c_t, k_{t+1} \geq 0 \\ t \geq 0 \\ k_0 \text{ given}}} \quad \sum_{t=0}^{\infty} \beta^t u\left(c_t\right) \quad s.t. \tag{1.96}$$

$$c_t + k_{t+1} = (1-\delta)k_t + F(k_t, 1)$$

The function V is the maximum lifetime discounted utility attainable by an individual that starts life with capital level equal to k_0 and chooses optimally every period. Henceforth, we are going to call the function V the **value function**. Note that the maximum utility attainable V is a function of the stock of capital k_0 owned by the individual at the beginning of her life. In our setup, it can be shown that the value function is increasing in k_0; a consumer that starts life with a larger level of capital will be able to consume more over her life. We refer to k_0 as an **individual state variable**. In this problem, the only state variable is k_0, as this is the only value needed in period 0 to compute the lifetime utility and policy functions.

Now, let's rewrite the social planner's problem in the following way:

$$
\begin{aligned}
V(k_0) &= \max_{\substack{c_t, k_{t+1} \geq 0 \\ t \geq 0 \\ k_0 \text{ given}}} \quad \sum_{t=0}^{\infty} \beta^t u\left(c_t\right) \quad s.t. \qquad (1.97) \\
& \quad c_t + k_{t+1} = (1-\delta) k_t + F(k_t, 1) \\
&= \max_{\substack{c_t, k_{t+1} \geq 0 \\ t \geq 0 \\ k_0 \text{ given}}} \quad u\left(c_0\right) + \beta \cdot \sum_{t=1}^{\infty} \beta^{t-1} u\left(c_t\right) \quad s.t. \\
& \quad c_t + k_{t+1} = (1-\delta) k_t + F(k_t, 1) \\
&= \max_{\substack{c_0, k_1 \geq 0 \\ k_0 \text{ given}}} \quad u\left(c_0\right) + \beta \cdot \left(\max_{\substack{c_t, k_{t+1} \geq 0 \\ t \geq 1 \\ k_1 \text{ given}}} \quad \sum_{t=1}^{\infty} \beta^{t-1} u\left(c_t\right) \right) \quad s.t. \\
& \quad c_t + k_{t+1} = (1-\delta) k_t + F(k_t, 1) \\
&= \max_{c_0, k_1} \quad u\left(c_0\right) + \beta \cdot \left(\max_{\substack{c_{t+1}, k_{t+2} \geq 0 \\ t \geq 0 \\ k_1 \text{ given}}} \quad \sum_{t=0}^{\infty} \beta^{t} u\left(c_{t+1}\right) \right) \quad s.t. \\
& \quad c_t + k_{t+1} = (1-\delta) k_t + F(k_t, 1)
\end{aligned}
$$

Note the similarities between the problem inside brackets in the last line and the problem in the first line of equation (1.97). Given that, in our environment, the utility and production functions remain constant over time, these two problems are equivalent to each other, except for the fact the in the first line the initial level of capital is given by k_0, while in the problem inside brackets the initial level of capital is given by k_1. This means that the problem inside brackets is simply $V(k_1)$, and we can rewrite the original problem as:

$$
\begin{aligned}
V(k_0) = \max_{c_0, k_1 \geq 0} \quad & u\left(c_0\right) + \beta V(k_1) \quad s.t. \\
& c_0 + k_1 = (1-\delta) k_0 + F(k_0, 1)
\end{aligned}
$$

To simplify notation, we will not use subscripts of time and, instead, will denote variables in the present period without apostrophe $(c, k, \ldots)$, and variables that correspond to one period ahead with one apostrophe $(c', k', \ldots)$. In this way, the social planner's problem is:

$$V(k) = \max_{c,k' \geq 0} \qquad u(c) + \beta V(k') \quad s.t. \tag{1.98}$$

$$c + k' = (1-\delta)k + F(k, 1)$$

This form of writing the social planner's problem is called the **recursive formulation** and can be shown to be equivalent to the original problem. The interpretation of this equation is that the maximum attainable lifetime utility is the sum of today's maximum utility $u(c)$ plus the discounted maximum utility attainable in the future $\beta V(k')$, given today's choice for k'. This problem is equivalent to the original problem, because it is equivalent to choose simultaneously the lifetime consumption and capital accumulation streams $\{c_t, k_{t+1}\}_{t=0}^{\infty}$, or to choose optimally today's consumption and capital accumulation, given that you are going to choose optimally in the future. Equation (1.98) is called the **Bellman equation**.

What are the unknowns in the Bellman equation? The Bellman equation is a *functional equation*, which means that, as opposed to the equations we are used to in which unknowns are scalars or vectors, in this case the unknowns are functions. There are three unknowns. The first unknown is a function $V(k)$, which gives the maximum utility for every level of individual capital k. This function appears on both sides of the equation, so solving for it means finding a function V that satisfies (1.98). Second, in order to achieve the utility level $V(k)$, we need to know what are the optimal levels of consumption and capital accumulation for every capital stock k. This means that the solution to the Bellman equation are three functions of k: $V(k)$, $c(k)$ and $k'(k)$. These last two functions are called **policy functions**, as they instruct the social planner what to choose, given a capital stock k.

Note that, given that the recursive representation is equivalent to the original problem (1.6), we can construct the value function, and lifetime consumption and capital accumulation sequences using the policy functions:

$$\begin{aligned}
c_0 &= c(k_0), & k_1 &= k'(k_0) \\
c_1 &= c(k_1), & k_2 &= k'(k_1) = k'(k'(k_0)) \\
c_2 &= c(k_2), & k_3 &= k'(k_2) = k'(k'(k_1)) = k'(k'(k'(k_0))) \\
& \vdots
\end{aligned}$$

Also, we know that the total lifetime utility, given k_0, is given by:

$$V(k_0) = \sum_{t=0}^{\infty} \beta^t u(c_t)$$

The matter of how to solve the Bellman equation is the subject of Section 1.6.3. The techniques used to solve for it are a consequence of mathematical results on functional analysis, that ensure that the methods used yield a correct solution. A review of the mathematical results is the subject of another section.

1.6.2 Recursive Competitive Equilibrium

Having defined the recursive formulation of the social planner, we can define the equivalent concept for the household's problem (1.17). There are two main differences. First, in the competitive equilibrium the household has access to risk-free bonds a_t, which determine the lifetime consumption and utility, so they must be included as an individual state variable.

Second, as opposed to the social planner, who only needs to know the current level of capital k_0 of the household to determine lifetime discounted utility and policy functions, in a competitive equilibrium the household also needs to know what the prices in the economy will be for every t. Recall the prices are determined by aggregate capital in the economy, such that $w_t = F_l(K_t, 1), R_t = F_k(K_t, 1)$, and $r_t = R_t - \delta$. This means that the household needs to know what the aggregate level of capital will be in the economy, to be able to forecast correctly the sequence of prices. Therefore, the relevant state variables in the competitive equilibrium set-up are individual capital k_0, individual risk-free bonds a_0, and aggregate capital in the economy K_t every period.

We have to explicitly separate individual and aggregate capital because, since we assumed there is a continuum of households, individual capital k_t need not be exactly equal to aggregate capital K_t. In equilibrium, in a representative agent economy, they will be equal to each other, but ex-ante they are not. Therefore, when describing the state variables we must distinguish between individual and aggregate state variables. In this case, the state variables are k_0, a_0, and K_t for every t.

We can write the household's problem as:

$$
\begin{aligned}
V(k_0, a_0, \{K_t\}_{t=0}^{\infty}) &= \max_{\substack{c_t, k_{t+1} \geq 0 \\ t \geq 0 \\ k_0, a_0 \text{ given}}} \quad \sum_{t=0}^{\infty} \beta^t u\left(c_t\right) \quad s.t. \qquad (1.99) \\
& \quad c_t + k_{t+1} + a_{t+1} = (1-\delta)k_t + w_t + R_t k_t + (1+r_t)a_t \\
& \quad \lim_{T \to \infty} \left(\prod_{t=0}^{T} \frac{1}{1+r_t} \right) a_{T+1} = 0 \\
&= \max_{\substack{c_t, k_{t+1} \geq 0 \\ t \geq 0 \\ k_0, a_0 \text{ given}}} \quad u\left(c_0\right) + \beta \cdot \sum_{t=1}^{\infty} \beta^{t-1} u\left(c_t\right) \quad s.t. \\
& \quad c_t + k_{t+1} + a_{t+1} = (1-\delta)k_t + w_t + R_t k_t + (1+r_t)a_t \\
& \quad \lim_{T \to \infty} \left(\prod_{t=0}^{T} \frac{1}{1+r_t} \right) a_{T+1} = 0 \\
&= \max_{\substack{c_0, k_1 \geq 0 \\ k_0, a_0 \text{ given}}} \quad u\left(c_0\right) + \beta \cdot \left(\max_{\substack{c_t, k_{t+1} \geq 0 \\ t \geq 1 \\ k_1, a_1 \text{ given}}} \quad \sum_{t=1}^{\infty} \beta^{t-1} u\left(c_t\right) \right) \quad s.t. \\
& \quad c_t + k_{t+1} + a_{t+1} = (1-\delta)k_t + w_t + R_t k_t + (1+r_t)a_t \\
& \quad \lim_{T \to \infty} \left(\prod_{t=0}^{T} \frac{1}{1+r_t} \right) a_{T+1} = 0 \\
&= \max_{c_0, k_1} \quad u\left(c_0\right) + \beta \cdot \left(\max_{\substack{c_{t+1}, k_{t+2} \geq 0 \\ t \geq 0 \\ k_1, a_1 \text{ given}}} \quad \sum_{t=0}^{\infty} \beta^{t} u\left(c_{t+1}\right) \right) \quad s.t. \\
& \quad c_t + k_{t+1} + a_{t+1} = (1-\delta)k_t + w_t + R_t k_t + (1+r_t)a_t \\
& \quad \lim_{T \to \infty} \left(\prod_{t=0}^{T} \frac{1}{1+r_t} \right) a_{T+1} = 0
\end{aligned}
$$

The household's problem in recursive form is:

$$V(a,k,K)=\max_{c,k'\geq 0,a'} \quad u\left(c\right)+\beta V\left(a',k',K'\right) \quad s.t. \tag{1.100}$$

$$c+k'+a'=(1-\delta)k+w(K)+R(K)k+(1+r(K))a$$

$$K'=H(K)$$

Again, we will denote k and a as individual state variables, and K as an **aggregate state variable**. Recall that in this economy there is a continuum of households, where each household realizes that it is small enough, so its decisions do not affect aggregates in the economy. For this reason, prices in the economy $w(K), r(K)$ depend on aggregate capital K, which is independent on the choice of capital of the household k.

The last equation in this recursive formulation is called an **aggregate law of motion**, and states the evolution of aggregate capital given by H. We assume that each household has perfect foresight and can perfectly forecast how aggregate capital will evolve -that is, every household knows H-, and knows that the evolution is independent of its own decisions. Solving for a recursive competitive equilibrium will entail solving for a function $H(K)$, as well as for $V(a,k,K), c(a,k,K), k'(a,k,K)$ and $a'(a,k,K)$.

We can now define a recursive competitive equilibrium in this setting:

Definition 3 (Recursive Competitive Equilibrium)**.** *A recursive competitive equilibrium are a value function* $V(a,k,K)$*, policy functions* $c(a,k,K), k'(a,k,K), a'(a,k,K)$*, pricing functions* $w(K), R(K), r(K)$*, and an aggregate law of motion* $H(K)$ *such that:*

1. *Given* (a,k,K) *and functions* $w(K), R(K), r(K), H(K)$*, the value function* V *solves the problem of the household (1.100), with* $c(a,k,K), k'(a,k,K), a'(a,k,K)$ *being the corresponding policy functions*

2. *Given* K*, prices are such that:*

$$\begin{aligned} w(K) &= F_L(K,L) \\ R(K) &= F_K(K,L) \\ r(K) &= R(K)-\delta \end{aligned}$$

3. *Given K, the prices $w(K), R(K), r(K)$ are such that the markets clear:*

$$c(A, K, K) + k'(A, K, K) = (1 - \delta)K + F(K, L)$$
$$a'(A, K, K) = 0$$

4. *For every K, the aggregate law of motion $H(K)$ is consistent with the optimal choice of households:*

$$H(K) = k'(A, K, K)$$

The first two points in this definition are simply the fact that households and firms maximize. The market clearing conditions are such that, in equilibrium, when individual state variables are equal to aggregate state variables ($k = K$ and $a = A$), markets clear. The last condition states that the law of motion that households use to forecast the evolution of capital coincides with the individual household's decisions, such that the forecasting is correct. Next section describes a method to numerically solve the problems described so far.

1.6.3 Value Function Iteration

In Section 1.1.4, we could solve analytically the model and find closed-form solutions for all the variables over time. However, this is uncommon as often, this type of models are much more convoluted and do not allow for a closed-form solution. In this section we explain the most common method to solve a Bellman equation, such as (1.98) and (1.100). This method always works under certain conditions, that will be explained later. The main idea of the value function iteration approach is to start with a guess of the value function for the household, and iterate until the problem converges. The algorithm is as follows:

Algorithm 3 (Value Function Iteration)**.**

1. *Start with a guess for the value function $V_0(k)$ for the social planner's problem (1.98). For example, set $V_0(k) = 0, \forall k \geq 0$.*

2. *Compute an updated guess $V_1(k)$, by solving the right-hand side of the Bellman equation using $V_0(k)$:*

$$V_1(k) = \max_{c, k' \geq 0} \quad u(c) + \beta V_0(k') \quad s.t.$$
$$c + k' = (1 - \delta)k + F(k, 1)$$

This can be computed directly, as now there is nothing unknown on the right-hand side. For the particular case where $V_0(k) = 0$, note that it suffices to find c and k' that maximize:

$$u((1-\delta)k + F(k,1) - k')$$

3. *In general, for $n \geq 1$, compute $V_{n+1}(k)$ using the guess $V_n(k)$ and the equation:*

$$V_{n+1}(k) = \max_{c,k' \geq 0} \quad u(c) + \beta V_n(k') \quad s.t.$$

$$c + k' = (1-\delta)k + F(k,1)$$

If $\|V_{n+1} - V_n\| < \epsilon$, stop and set the policy functions c and k' by:

$$\{c(k), k'(k)\} = \arg\max_{c,k' \geq 0} \quad u(c) + \beta V_n(k') \quad s.t.$$

$$c + k' = (1-\delta)k + F(k,1)$$

Otherwise, set $n = n+1$ and repeat 3.

Under very general conditions, the algorithm described above works and the value function converges to a unique function V^*. This means that we will be able to solve the model and compute the value and policy functions, even though we don't have a closed form solution for them.

But, even if this algorithm seems simple, how can we perform this computation on a computer? It turns out that even with a computer it is not possible to find an exact solution, so the best we can aim for is to compute an approximation to the value function. In particular, we will only be able to solve the problem for a finite number of points k. The algorithm to approximate the solution is as follows:

1. Set a grid for the state variables $K = \{k_1, k_2, \ldots, k_m\}$ of size m.

2. Set $n = 0$ and start with a guess for the value function $V_0(k), k \in \{k_1, \ldots, k_m\}$.

3. For $n \geq 1$, compute $V_{n+1}(k), k \in \{k_1, \ldots, k_m\}$ using the Bellman equation:

$$V_{n+1}(k) = \max_{c,k' \geq 0} \quad u(c) + \beta V_n(k') \quad s.t.$$

$$c + k' = (1-\delta)k + F(k,1)$$

If $\sum_{i=1}^{m} |V_{n+1}(k_i) - V_n(k_i)| < \epsilon$, stop and set the policy functions c and k' by:

$$\{c(k), k'(k)\} = \arg\max_{c,k' \geq 0} \quad u(c) + \beta V_n(k') \quad s.t.$$

$$c + k' = (1-\delta)k + F(k,1)$$

Otherwise, set $n = n+1$ and repeat 3.

For example, let's take an economy where the utility function is logarithmic $u(c) = \ln(c)$ and the production function takes the Cobb-Douglas form, such that $F(k,1) = k^{\alpha}$. Assume that the parameters are $\alpha = 0.3$, $\beta = 0.95$, and $\delta = 1$. Finally, the grid for capital choices is given by the set $\mathcal{K} = \{0.04, 0.08, 0.12, 0.16, 0.2\}$. The purpose is to compute the value function at each of these capital grid points, such that we end up with an estimate of the values of $V(0.04), V(0.08), V(0.12), V(0.16)$, and $V(0.2)$. The algorithm goes as follows:

1. Start with a guess $V_0(k) = 0$ for all $k \in \mathcal{K}$.

2. Define $V_1(k)$ by solving:

$$\begin{aligned} V_1(k) &= \max_{k' \geq 0} u\left((1-\delta)k + F(k,1) - k'\right) + \beta V_0(k') \\ &= \max_{k' \geq 0} u\left(F(k,1) - k'\right) \end{aligned}$$

Given that $V_0(k) = 0$ for all k, it is optimal to choose the minimum level of capital in the grid, so $k'(k) = 0.04$ for all $k \in \mathcal{K}$. Therefore:

$$\begin{aligned} V_1(0.04) &= u\left(0.04^{0.3} - 0.04\right) = -1.077 \\ V_1(0.08) &= u\left(0.08^{0.3} - 0.04\right) = -0.847 \\ V_1(0.12) &= u\left(0.12^{0.3} - 0.04\right) = -0.715 \\ V_1(0.16) &= u\left(0.16^{0.3} - 0.04\right) = -0.622 \\ V_1(0.20) &= u\left(0.20^{0.3} - 0.04\right) = -0.55 \end{aligned}$$

3. Define $V_2(k)$ by solving:

$$V_2(k) = \max_{k' \geq 0} u\left((1-\delta)k + F(k,1) - k'\right) + \beta V_1(k')$$

Now, given that $V_1(k) \neq 0$, for every $k \in \mathcal{K}$ we have to evaluate the utility for every $k' \in \mathcal{K}$. If the planner chooses $k' = 0.04$:

$$V_2(0.04) = u\left(0.04^{0.3} - 0.04\right) + \beta V_1(0.04)$$

If the planner chooses $k = 0.08$:

$$V_2(0.04) = u\left(0.04^{0.3} - 0.08\right) + \beta V_1(0.08)$$

If the planner chooses $k = 0.12$:

$$V_2(0.04) = u\left(0.04^{0.3} - 0.12\right) + \beta V_1(0.12)$$

If the planner chooses $k = 0.16$:

$$V_2(0.04) = u\left(0.04^{0.3} - 0.16\right) + \beta V_1(0.16)$$

If the planner chooses $k = 0.20$:

$$V_2(0.04) = u\left(0.04^{0.3} - 0.20\right) + \beta V_1(0.20)$$

The optimal choice is given by the highest of these. In particular, for $k = 0.04$ it is optimal to choose $k' = 0.08$ and $v_2(0.08) = -1.710$. This has to be done for each $k \in \mathcal{K}$.

4. Continue until convergence. That is, until the point in which V_{n+1} is sufficiently close to V_n.

A natural question that arises is how to choose the grid for capital and how many points m to use. There is no correct answer to this. A larger number of grid points improves the accuracy obtained in the approximation of the value function V, but at the cost of a slower computation speed.

Note also that there are values for k that are not feasible. Take problem (1.98). The maximization on the right hand side of the equation is made over all k' that satisfy two constraints:

$$\begin{aligned} k' &\geq 0 \\ k' &\leq (1-\delta)k + F(k,1) \end{aligned}$$

The second inequality comes from the fact that $c \geq 0$. This means that choosing levels of capital outside the interval $[0, (1-\delta)k + F(k,1)]$ is not feasible, so we can restrict the points in the grid to being in that interval.

1.7 Endogenous growth

Sections 1.1-1.6 dealt mainly with the neoclassical growth model, where technological progress was assumed exogenous. Namely, we assumed that labor productivity A_t grew exogenously at a constant rate g, such that $A_{t+1} = (1+g)A_t$. Introducing this productivity growth allowed the model to generate growth when the economy reached the balanced growth path. That is, this model was able to explain long-run growth of economies. However, this model is not useful to understand where does productivity growth come from. Why is there productivity growth at all? What is the engine of productivity growth? Why does productivity grow faster in some nations than in others? The purpose of this section is to try to understand the source of productivity growth and what are its determinants. This will allow us to talk about policies that governments can implement to increase productivity growth in the long run.

The following subsections present different models in which there is long-run growth. The first three sections do not explicitly model the creation of new technologies or the expansion of the productivity A_t. Instead, these models remove some of the assumptions made so far, which prevent the economy from experiencing long-run growth. Namely, in the first two sections, we remove the assumption of decreasing returns to scale on capital. The third section studies a model in which there are externalities in capital accumulation. The last two sections present models in which the productivity growth is explicitly modelled.

1.7.1 The AK Model

Up to now we have assumed that the production function F had constant returns to scale, was strictly concave in both of its inputs K and L and satisfied the Inada conditions. The most commonly used production function that satisfies these assumptions is the Cobb-Douglas function $F(K_t, L_t) = K_t^{\alpha}(A_t L_t)^{1-\alpha}$. In this section, we will assume that $\alpha = 1$, such that $F(K_t, L_t) = AK_t$, where A is a constant technological parameter, often normalized to 1. This function has constant returns on capital, as opposed to the diminishing returns to capital that the previous Cobb-Douglas specification satisfied. Also, production now does not depend on labor L_t, so wages are $w_t = 0$. In equilibrium, the capital rental rate will be exactly $R_t = A$ for every t.

Solving the social planner's problem yields the usual first order conditions:

$$\frac{u'(c_t)}{\beta u'(c_{t+1})} = \frac{(1-\delta+A)}{(1+n)}$$

$$c_t + (1+n)k_{t+1} = (1-\delta)k_t + Ak_t$$

First, note that in this model there is generally no steady state. That is, there is no point in which variables per-capita remain constant. For instance, for consumption to be constant over time, we would need the following equality to hold:

$$\frac{1}{\beta} = \frac{(1-\delta+A)}{(1+n)}$$

This equation does not hold in general, unless the model parameters are specifically chosen to make it hold. Having dismissed the existence of a steady state, let's analyse the cases in which the economy reaches a balanced growth path. That is, a point in which all per-capita variables grow at a constant rate. Recall that the existence of a balanced growth path requires that the utility function satisfies the CRRA assumption, in which case the Euler equation becomes:

$$\frac{1}{\beta}\left(\frac{c_t}{c_{t+1}}\right)^{-\sigma} = \frac{(1-\delta+A)}{(1+n)} \tag{1.101}$$

This equation implies that the growth of consumption is always constant, such that $c_{t+1} = (1+\gamma_c)c_t$. In a balanced growth path, γ_c is then pinned down by the following equation:

$$\frac{1}{\beta}(1+\gamma_c)^{\sigma} = \frac{(1-\delta+A)}{(1+n)} \tag{1.102}$$

That is, the growth rate of consumption along the balanced growth path is:

$$\gamma_c = \left(\beta \cdot \frac{(1-\delta+A)}{1+n}\right)^{\frac{1}{\sigma}} - 1 \tag{1.103}$$

This growth rate can be positive, even without technological growth. In particular, it is positive when β is sufficiently large, given that individuals are very patient and choose to save a large part of their income, so capital is rapidly accumulated over time and the constant returns to scale ensure that production continues to grow linearly - the marginal product of capital does not decrease over time. The same happens if depreciation δ is low, as capital accumulates at a fast rate. Finally, if the households have a large elasticity of substitution (if σ is close to 0), they will prefer to postpone consumption for the future by accumulating more capital, leading to larger growth rates.

If we take logarithms to (1.103) and express $\beta = 1/(1+\rho)$:

$$\log(1+\gamma_c) = \frac{1}{\sigma}\left(\log(1-\delta+A) - \log(1+n) - \log(1+\rho)\right)$$

If $A-\delta$, ρ and n are sufficiently small (say, below 10%), the growth rate can be approximated as:

$$\gamma_c \approx \frac{1}{\sigma}\left(A-\delta-n-\rho\right) \tag{1.104}$$

Note that, according to the Euler equation (1.101), the growth rate of consumption is *always* constant, independently of the initial point. This means that in the AK model there is no trasition to a balanced growth path; the economy is always on a balanced growth path. Furthermore, as opposed to the neoclassical growth model, where all variables grow at the rate of technological change g, in the AK model the growth rate of the economy depends on other parameters: A, δ, n, ρ.

1.7.2 Human Capital

Up to now, we have assumed that firms only use unskilled labor L_t and capital K_t as factors of production, where labor is understood as the amount of physical workers. The limitation of this is that the amount of labor is bounded by the population size and households cannot increase their labor supply beyond that. Moreover, this assumption does not allow for improvements in the quality of labor. Certainly, workers have become more productive over the last century, given the increase in schooling and accumulation of human capital. This section introduces a model with human capital, where households can invest in improving the quality of labor they supply.

Assume that the production function depends on physical capital K_t and human capital H_t, such that $Y_t = F(K_t, H_t)$, where F satisfies the same assumptions stated in Section

1.1. Here, we assume there is no technological growth. Households can accumulate human capital in a similar way as they choose to accumulate physical capital. Namely, human capital evolves according to a human-capital evolution equation:

$$H_{t+1} = (1 - \delta_h)H_t + I_t^h$$

Where δ_h is the depreciation rate of human capital and I_t^h are total investments in human capital. As before, physical capital evolves according to the usual capital evolution equation, where we denote I_t as total investments and δ as the depreciation rate of physical capital. The aggregate resource constraint is thus:

$$C_t + I_t + I_t^h = (1 - \delta)K_t + (1 - \delta_h)H_t + F(K_t, H_t)$$

In this resource constraint we are implicitly assuming that human capital can be converted with a linear technology into the final good, so total undepreciated capital $(1-\delta_h)H_t$ can be consumed. The model can be re-expressed in per-capita terms, such that the resource constraint is:

$$c_t + (1 + n)k_{t+1} + (1 + n)h_{t+1} = (1 - \delta)k_t + (1 - \delta_h)h_t + F(k_t, h_t)$$

Where $h_t = H_t/L_t$ is the average human capital per worker in this economy. The social planner's problem is:

$$\begin{aligned}
\max_{c_t, k_{t+1}, h_{t+1}} \quad & \sum_{t=0}^{\infty} \beta^t u\left(c_t\right) \quad s.t. \\
\forall t \in \{0, 1, \ldots\}: \quad & c_t + (1 + n)k_{t+1} + (1 + n)h_{t+1} = (1 - \delta)k_t + (1 - \delta_h)h_t + F(k_t, h_t) \\
& c_t, k_{t+1}, h_{t+1} \geq 0, \quad h_0, k_0 \text{ given}
\end{aligned}$$

Setting the appropriate lagrangean and solving for the optimality conditions yields:

$$\begin{aligned}
[c_t]: \quad & \beta^t u'(c_t) - \lambda_t = 0 \\
[k_{t+1}]: \quad & -(1 + n)\lambda_t + \lambda_{t+1}\left(1 - \delta + F_k(k_{t+1}, h_{t+1})\right) = 0 \\
[h_{t+1}]: \quad & -(1 + n)\lambda_t + \lambda_{t+1}\left(1 - \delta_h + F_h(k_{t+1}, h_{t+1})\right) = 0 \\
[\lambda_t]: \quad & (1 - \delta)k_t + (1 - \delta_h)h_t + F(k_t, h_t) - c_t - (1 + n)(k_{t+1} + h_{t+1}) = 0
\end{aligned}$$

The Euler equations are:

$$\frac{u'(c_t)}{\beta u'(c_{t+1})} = \frac{(1 - \delta + F_k(k_{t+1}, h_{t+1}))}{(1+n)}$$

$$\frac{u'(c_t)}{\beta u'(c_{t+1})} = \frac{(1 - \delta_h + F_h(k_{t+1}, h_{t+1}))}{(1+n)}$$

These optimality conditions imply that, in the optimum, $F_k(k_{t+1}, h_{t+1}) - \delta = F_h(k_{t+1}, h_{t+1}) - \delta_h$, so the social planner is indifferent between investing in human or physical capital. In particular, using the same reasoning as in (1.71), we can re-write this condition as:

$$F_k\left(\frac{k_{t+1}}{h_{t+1}}, 1\right) - \delta = F_h\left(\frac{k_{t+1}}{h_{t+1}}, 1\right) - \delta_h \tag{1.105}$$

In the optimum, the social planner will choose a physical-to-human capital ratio that satisfies (1.105). In particular, the physical-to-human capital ratio should remain constant at every period t, independently on whether the economy has reached a balanced growth path or not. In addition, the fact that the physical-to-human capital ratio is constant, implies that human and physical capital must grow at exactly the same rate. Denote by $\kappa = \frac{k_{t+1}}{h_{t+1}}$, so the Euler equation is:

$$\frac{u'(c_t)}{\beta u'(c_{t+1})} = \frac{(1 - \delta + F_k(\kappa, 1))}{(1+n)}$$

On a balanced growth path, denote by γ_c the growth rate of consumption. Recall that we must assume that the utility function takes the CRRA form, so γ_c is pinned down by:

$$1 + \gamma_c = \left(\frac{(1 - \delta + F_k(\kappa, 1))}{(1+n)}\right)^{\frac{1}{\sigma}}$$

If we assume that F takes the Cobb-Douglas form $F(k, h) = k^\alpha h^{1-\alpha}$, and that physical and human capital depreciate at the same rate, such that $\delta = \delta_h$, the optimal physical-to-human capital ratio is:

$$\kappa = \frac{\alpha}{(1-\alpha)}$$

And the growth rate of the economy is:

$$\gamma_c = \left(\frac{\left(1 - \delta + \alpha^{\alpha}(1-\alpha)^{1-\alpha}\right)}{(1+n)} \right)^{\frac{1}{\sigma}} - 1$$

As in the AK model, depending on the parameters of the model, the economy can reach a balanced growth path with positive growth (compare to equation (1.103)). The reason for this result is that in this model, as in the AK model, there are *constant returns to scale to factors of production that can be accumulated*. In previous sections this was not the case, as labor could not be accumulated beyond population growth, and there were diminishing returns to scale on capital.

1.7.3 Externalities in Production

In this model, firms produce according to the usual production function $F(K_t, A_t L_t)$, where there is a labor-augmenting technology A_t. In this section, growth of the technology parameter A_t is driven endogenously within the model, as opposed to Section 1.4 where technological growth was exogenous. In particular, we assume that the technology A_t equals the average capital used across firms: $A_t = \bar{K}_t$. This means that $Y_t = F(K_t, \bar{K}_t L_t)$. If we assume a Cobb-Douglas production function: $Y_t = K_t^{\alpha}(\bar{K}_t L_t)^{1-\alpha}$.

Recall that in the economy there is a continuum of identical firms of size 1. This means that every firm acts independently of each other, and the decision of each single firm does not affect aggregate outcomes in the economy. In particular, if a firm decides to increase its capital, the average capital used by firms in the economy stays unmodified at the same level. Average capital is affected only when a set of firms of positive measure change their capital. Therefore, every firm takes average capital in the economy as given and beyond its control.

Given these assumptions, the production function has constant returns to scale at the firm level. If a single firm decides to increase all factors of production by a proportion λ, production increases in that same proportion: $F(\lambda K_t, A_t(\lambda L_t)) = \lambda F(K_t, A_t L_t)$. This happens because the decision of a single firm does not affect aggregate capital, so increasing labor and capital by a factor λ leaves $\bar{K}_t$ unchanged.

However, note that from the social planner's perspective, if the factors of production of all firms increase by a factor λ, total production increases more than that, as average capital in the economy will also increase. That is, if the social planner increases capital used by firms by a factor λ, not only does K_t increases, but also $\bar{K}_t$. In the case of a Cobb-Douglas production function:

$$F(\lambda K_t, \lambda \bar{K}_t \lambda L_t) = (\lambda K_t)^{\alpha} (\lambda \bar{K}_t \lambda L_t)^{1-\alpha} = \lambda^{2-\alpha} K_t^{\alpha} L_t^{1-\alpha} > \lambda F(K_t, \bar{K}_t L_t)$$

This means that the social planner faces increasing returns to scale on its factors or production. This is driven by the fact that average capital in the economy increases the productivity of labor, so increasing capital has an additional indirect effect in the economy, beyond the additional production of that capital.

What drives the difference is that there is an externality driven by capital accumulation. When firms accumulate more capital, they do not internalize the fact that increasing average capital also increases the productivity of all other firms in the economy. Therefore, the level of capital chosen privately by each firm is below the socially optimal level. In contrast, the social planner knows that increasing capital has a direct effect, by increasing production, but also an indirect effect, by increasing productivity. Therefore, the social planner chooses the socially optimal level of capital. In this context, the welfare theorems do not hold, so the allocations that solve the social planner's problem and the competitive equilibrium are not equal to each other.

Social Planner's Problem

The social planner knows that there is a continuum of firms of size 1 so that, if all firms choose capital K_t, the average capital is $\bar{K}_t = K_t$. Therefore, the production function becomes:

$$Y_t = F(K_t, \bar{K}_t L_t) = K_t^{\alpha} (\bar{K}_t L_t)^{1-\alpha} = K_t L_t^{1-\alpha}$$

Assume there is no population growth, such that $L_t = 1$ for all t. We can write total per-capita production as $y_t = k_t$. This is exactly the AK model, with $A = 1$ and population growth $n = 0$. This means that along a balanced growth path all variables grow at the rate:

$$\gamma_c = (\beta(2-\delta))^{\frac{1}{\sigma}} - 1$$

Competitive Equilibrium

In a competitive equilibrium, households solve the same problem as in Section 1.23. Recall that if we want to study balanced growth paths, we have to assume that $u = \frac{c^{1-\sigma}}{1-\sigma}$, so the household's optimality conditions are:

$$\frac{1}{\beta}\left(\frac{c_{t+1}}{c_t}\right)^{\sigma} = 1 - \delta + R_{t+1}$$

$$c_t + k_{t+1} + a_{t+1} = (1-\delta)k_t + w_t + R_t k_t + (1+r_t)a_t$$

The firm's problem is:

$$\max_{K_t, L_t} \quad K_t^{\alpha}(\bar{K}_t L_t)^{1-\alpha} - w_t L_t - R_t K_t$$

So prices of factors of production are:

$$\begin{aligned} R_t &= \alpha K_t^{\alpha-1}(\bar{K}_t L_t)^{1-\alpha} \\ w_t &= (1-\alpha)K_t^{\alpha}\bar{K}_t^{1-\alpha}L_t^{-\alpha} \end{aligned}$$

In equilibrium, $\bar{K}_t = K_t$ and $L_t = 1$, so equilibrium prices become:

$$\begin{aligned} R_t &= \alpha \\ w_t &= (1-\alpha)K_t \end{aligned}$$

Plugging these equilibrium prices in the household's Euler equation yields:

$$\frac{1}{\beta}\left(\frac{c_{t+1}}{c_t}\right)^{\sigma} = 1 - \delta + \alpha$$

So the growth rate is:

$$\gamma_c = (\beta(1-\delta+\alpha))^{\frac{1}{\sigma}} - 1$$

The growth rate on a competitive equilibrium is smaller than under the social planner's solution, given that $1 - \delta + \alpha < 2 - \delta$, as $\alpha < 1$. The reason is that, given that household's do not internalize that their own capital accumulation increases overall productivity in the economy, they under-accumulate capital. In contrast, the social planner internalizes this and choose the socially optimal level of capital.

In this model, there is long-run growth because, even though households face decreasing returns to capital accumulation, there are constant returns to capital accumulation in the economy. Therefore, capital can be accumulated at a constant rate, generating production growth at that same rate in the long run.

1.8 R&D Models - Romer (1990)

This section describes the first model in which productivity growth is explicitly modelled. In this setting, productivity A_t is a function of the number of different varieties of intermediate goods used in production.

Every period t, the economy is composed of $N_t + 1$ production sectors: a firm that produces the final good Y_t, and N_t firms that produce intermediate goods $X_1, \ldots, X_{N_t}$. The final good production firm uses as inputs each of the N_t intermediate goods and labor, to produce Y_t according to the production function: $Y_t = \left(\sum_{i=1}^{N_t} X_{it}^{\alpha}\right) L_t^{1-\alpha}$. The final good is produced in a perfectly competitive market and is used for consumption and investment. The problem of the final good firm is:

$$\max_{X_1,\ldots,X_{N_t},L_t} \quad \left(\sum_{i=1}^{N_t} X_{it}^{\alpha}\right) L_t^{1-\alpha} - \sum_{i=1}^{N_t} P_{it} X_{it} - w_t L_t \tag{1.106}$$

Where P_{it} is the price of the intermediate good X_i in period t. The optimality conditions for the final good firm are:

$$w_t = (1-\alpha)\left(\sum_{i=1}^{N_t} X_{it}^{\alpha}\right) L_t^{-\alpha} \tag{1.107}$$

$$P_{it} = \alpha X_{it}^{\alpha-1} L_t^{1-\alpha}, \quad i \in \{1, \ldots, N_t\} \tag{1.108}$$

Intermediate good firms operate on a monopolistic market. To produce good X_{it}, the intermediate producer uses the final good as an input for production and faces a constant marginal cost of production equal to 1. Every period the producer of X_{it} chooses the price of its good P_{it}, to solve:

$$\pi_{it} = \max_{P_{it}} \quad (P_{it} - 1) X_{it}(P_{it}) \tag{1.109}$$

Where $X_{it}(P_{it})$ is the demand function for the intermediate good X_{it}, taken as given by the monopolist. Namely, the demand function can be obtained from (1.108):

$$X_{it}(P_{it}) = \left(\frac{\alpha}{P_{it}}\right)^{\frac{1}{1-\alpha}} \tag{1.110}$$

So the intermediate good producer solves:

$$\pi_{it} = \max_{P_{it}} \quad (P_{it} - 1)\left(\frac{\alpha}{P_{it}}\right)^{\frac{1}{1-\alpha}} \tag{1.111}$$

Which yields the optimality condition for the intermediate good producer i:

$$P_{it} = \frac{1}{\alpha} \quad \Rightarrow \quad \pi_{it} = (1-\alpha)\alpha^{\frac{1+\alpha}{1-\alpha}} \tag{1.112}$$

Households can engage in the creation of a new intermediate good sector j, beyond the existing ones $1, \ldots, N_t$. The creation of a new sector has a fixed cost equal to $\eta > 0$, which has to be paid at period t, and will allow the household to produce the new intermediate good X_j starting from period $t+1$. The owner of sector j has lifetime monopolistic power over that sector, which means that the returns of creating a new sector j are equal to $\sum_{s=t+1}^{\infty}\left(\prod_{l=t}^{s}\frac{1}{1+r_l}\right)\pi_{j,s}$. Given that the production of intermediate goods generates positive profits, households will be attracted to create new sectors as long as total profits of owning a new sector are larger than the cost of creating the sector. The problem of the household is:

$$\max_{c_t, a_{t+}, N_{t+1}} \quad \sum_{t=0}^{\infty}\beta^t u(c_t), \quad s.t. \tag{1.113}$$

$$\begin{aligned} \forall t \in \{0,1,\ldots\}: \qquad & c_t + a_{t+1} = w_t + (1+r_t)a_t + \sum_{i=1}^{N_t}\pi_{it} - \eta(N_{t+1} - N_t) \\ & c_t, N_{t+1} \geq 0, a_0 = 0 \end{aligned}$$

Given that profits of each intermediate good firm are $\pi_{it} = (1-\alpha)\alpha^{\frac{1+\alpha}{1-\alpha}}$, the household's problem can be re-written as:

$$\max_{c_t, a_{t+}, N_{t+1}} \quad \sum_{t=0}^{\infty}\beta^t u(c_t), \quad s.t. \tag{1.114}$$

$$\begin{aligned} \forall t \in \{0,1,\ldots\}: \qquad & c_t + a_{t+1} = w_t + (1+r_t)a_t + (1-\alpha)\alpha^{\frac{1+\alpha}{1-\alpha}}N_t - \eta(N_{t+1} - N_t) \\ & c_t, N_{t+1} \geq 0, a_0 = 0 \end{aligned}$$

Note that in this environment households can move wealth across time through two different channels. They can invest in risk-free bonds a_{t+1}, or they can invest in the creation of new intermediate goods X_{jt}, which generates profits every period in the future. Solving problem (1.114) yields the following two Euler equations for the household:

$$\frac{u'(c_t)}{\beta u'(c_{t+1})} = 1 + r_{t+1} \tag{1.115}$$

$$\frac{u'(c_t)}{\beta u'(c_{t+1})} = 1 + \frac{(1-\alpha)\alpha^{\frac{1+\alpha}{1-\alpha}}}{\eta} \tag{1.116}$$

These two equations imply a non-arbitrage condition between risk-free assets and the creation of new intermediate sectors, such that:

$$r_t = \frac{(1-\alpha)\alpha^{\frac{1+\alpha}{1-\alpha}}}{\eta} \tag{1.117}$$

That is, the returns to an additional risk-free bond a_{t+1} are equal, in equilibrium, to the returns of creating a new sector, net of the cost of creating that sector η. On a balanced growth path, where the interest rate is constant $r_t = r^*$, this non-arbitrage condition is equivalent to a zero-profit condition on the creation of a new intermediate good sector, given by the free entry condition:

$$\underbrace{\eta}_{\text{Cost of creating new sector}} = \underbrace{\sum_{s=t+1}^{\infty} \left(\prod_{l=t}^{s} \frac{1}{1+r^*} \right) \pi_{jt}}_{\text{Benefit of creating new sector}} \tag{1.118}$$

That is, in equilibrium the household is indifferent between accumulating an additional risk-free bond, or creating a new intermediate good sector. Along a balanced growth path, the growth rate of consumption is given by:

$$\gamma_c = \left(\beta \left(1 + \frac{(1-\alpha)\alpha^{\frac{1+\alpha}{1-\alpha}}}{\eta} \right) \right)^{1/\sigma} - 1 \tag{1.119}$$

Note that in this economy the welfare theorems do not hold, given that intermediate good firms have monopolistic power. This means that intermediate goods are produced below the socially-optimal level. The social planner produces intermediate goods up to the socially optimal level of production, achieving larger growth rates in the long run.

Bibliography

Stokey, N., Lucas, R., and Prescott, E. (1989). Recursive methods in economic dynamics. *Harvard University Press*.

Tauchen, G. (1986). Finite state markov-chain approximations to univariate and vector autoregresions. *Economics Letters*.