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The Matter Power Spectrum On Small Scales: An Approach To The Asymptotics Of Dark Matter Within Kinetic Field Theory

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ABSTRACT

The Matter Power Spectrum $P(k)$ is an important object in cosmology to study density fluctuations on different scales and times. In the small-scale limit, we know that it decreases as k^{-3}. In this thesis, we try to approach the explicit amplitude of this decay by using a two-particle model and the saddle point approximation. These assumptions allow us to compute the amplitude analytically as we will show for both the interacting and non-interacting case. We show that the amplitude depends on the second derivative of the interaction term in the particles' trajectories.

ZUSAMMENFASSUNG

Das Leistungspektrum $P(k)$ von Materie beschreibt Dichteschwankungen im Universum auf unterschiedlichen Skalen und Zeiten und ist daher ein essentielles Objekt in der Kosmologie. Es ist bekannt, dass im Grenzfall kleiner Skalen eine k^{-3}-Proportionalität vorliegt. In dieser Arbeit versuchten wir, uns mit einem Zwei-Teilchen-Modell und der Sattelpunktapproximation der exakten Amplitude dieses Abfalls zu widmen. Diese Annahmen gestatten es uns, sie analytisch zu berechnen, wie wir sowohl für den interaktionsfreien als auch den interagierenden Fall darlegen. Wir zeigen, dass diese Amplitude von der zweiten Ableitung des Interaktionsterms der Trajektorie der Teilchen abhängt.

ACKNOWLEDGMENTS

There are some people without whom this work would not have been possible and I want to thank them.

The two people who by far had the biggest influence on the thesis and always were of help were Dr. Sara Konrad and my supervisor Prof. Matthias Bartelmann.

Matthias Bartelmann is to be given credit for the idea of this work, for the broad overview of the connections to other theories and for the kinetic field theory itself. He entrusted me with the development of this theory and gave me the freedom to explore it in my own way. When I needed help, he was always there with advice and support, even though I sometimes needed time to understand his vision.

Sara Konrad probably had an even bigger influence on this project. It is her work on the small-scale limit of the power spectrum that I built upon. She was the address for all the questions I had, always had an open ear, usually a solution for my problems and most importantly patience and took her time for me. As I was new to this theory she explained everything to me, multiple times if necessary, and I benefited a lot from both her experience and intuition for this topic and also her encouraging words when I got thrown back and was losing hope.

I want to thank both of them for their help from the bottom of my heart.

The rest of Matthias Bartelmann's group was also always helpful and supportive, whether with technical answers or relaxed chats during the breaks. These people made the work environment a pleasant and motivating place to be. Hannes Heisler and Johanna Velinova are to be mentioned here, as they were there to talk things through with me every now and then, and especially Johanna Velinova was an anchor for me in the madness of such abstract work.

I also benefited from the discussions with my family and friends, who were not at all familiar with the topic. Their questions and the need to explain things in simple words helped me to take a step back, realize what I was actually doing and understand it better.

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1 INTRODUCTION

Since the ancient cultures of Mesopotamia and Greece, humans have been fascinated by the night sky and have used scientific methods to analyse the universe around us. The methods and tools grew more precise over time and new objects and structures have been discovered, while with every discovery unravelling how much more there is to find and determine. From the movements of our own solar system to our location in the Milky Way, all the other galaxies and the discovery of black holes, each of these findings gave us as humanity a new perspective and understanding of our universe. We now obviously know much more than a few hundred or even thousands of years ago but also have many more open questions yet to be answered.

One of those current questions is the nature of dark matter. The visible matter in the universe is not "enough" to explain for example the strength of the gravitational lensing that we see or the formation of galaxies. We have postulated dark matter as a way to explain these by assuming that it doesn't interact with light and is thus "dark" and unseen to us. Although we can not see it directly, we can see its gravitational effect on the universe. This only presents little information to us and there are countless speculations about its constitution and behaviour.

We work with the assumption that dark matter is "cold", i.e. nonrelativistic. In this thesis, we will try to find out more about the structures that dark matter forms or, to be precise, their size. One can express the density distribution of dark matter in the universe, take the correlation of that and calculate the Fourier transform of this function, the so-called "power spectrum". Ginat et al. [5] have derived that in the limit of small distances ($q \to 0, k \to \infty$) the power spectrum decreases with k^{-3}. In the following, we will try to compute the explicit amplitude of this limit. We will follow Konrad et al. [7] and approach the problem through statistical methods from kinetic field theory. A quantity of dark matter is modelled as a cloud of N particles and we average over all initial distributions to find reasonable results.

2 THEORETICAL BACKGROUND

2.1 KINETIC FIELD THEORY

For the purpose of this thesis, the methods we are using do not require much of the actual kinetic field theory (KFT). However, as we start with expressions derived from KFT, we will use this opportunity to present the most general ideas and framework of the theory.

The theory aims to address questions about structure formation in the universe like "How do the same structures and geometrical forms occur over and over again?", "Which length scales dominate cosmic structures and why?", "How do systems of different length scales differ and how are they alike?". While the vast majority of studies on structure formation in cosmology uses numerical simulations, this is not satisfactory as it does not answer the questions of how exactly the universe shapes and why it does so. The natural desire for an analytical theory to explain the evolution of cosmic structures is what KFT tries to satisfy. But how?

2.1.1 *The Generating Functional*

The most important object in KFT is the so-called "generating functional". We assume an N-particle ensemble and define the generating functional as a path-integral over all trajectories $\mathbf{x}$, augmented by a source field $\mathbf{J}$. It takes the form

$$Z[\mathbf{J}] = \int \mathcal{D}\mathbf{x} \int \mathrm{d}\mathbf{x}^{(i)} \delta_D \left[\mathbf{x} - \mathbf{x}^{cl}\right] P\left(\mathbf{x}^{(i)}\right) e^{i(\mathbf{J},\mathbf{x})} \tag{2.1}$$

The delta function δ_D enforces the classical trajectory $\mathbf{x}^{cl}$ and $P\left(\mathbf{x}^{(i)}\right)$ is the probability distribution of the initial conditions $\mathbf{x}^{(i)}$. If we integrate the paths out with the delta function, we get

$$Z[\mathbf{J}] = \int \mathrm{d}\Gamma e^{i(\mathbf{J},\mathbf{x})} \tag{2.2}$$

The scalar product in the exponent is a time integral

$$(\mathbf{J}, \mathbf{x}) = \int_0^t \mathrm{d}t' \langle \mathbf{J}(t'), \mathbf{x}(t') \rangle \tag{2.3}$$

and $d\Gamma$ is the phase-space measure ([1], Chapter 4).

Taking functional derivatives and applying other operators to this functional gives us trajectories, translations and other physical properties in our system. The freedom of choice of **J** makes it possible to realize different conditions in this formalism.

For this work, it is not important to understand where this comes from or how exactly this works out and produces correct results.

2.1.2 *Further Generalization - The Propagator*

In principle, KFT tries to take a step back. The generating functional is a very general object and can be used to describe a variety of systems.

From general relativity and cosmology, we know that even time is relative and has to be treated in different settings. To make up for this, KFT uses a so-called "propagator" g that generalizes time. It can just be the "normal" time $g = t - t'$ or something more complex like the cosmic growth factor $g = D_+(t) - D_+(t')$ [3] depending on our system. [1]

For our purposes, we will just keep in mind that it grows monotonically with time and leave it mostly unspecified.

2.1.3 *Conceptual Framework*

An important difference between KFT and other models for cosmic structure formation is the approach not to model the universe as a fluid but as a collection of particles. By doing so, we take a somewhat microscopic approach to large scale-problems instead of a macroscopic one, e.g. fluid dynamics.

In this work, we also deal with N-particle systems and the explicit trajectories for two of them, using a statistical approach only for their initial distribution. [3, 7] In a homogeneous and isotropic universe, the ansatz that our particles are drawn from a certain initial distribution relative to an arbitrary reference position is justified.

2.2 THE CDM POWER SPECTRUM

What we will deal with now is the power spectrum of cold dark matter (CDM). Dark matter is matter that only or almost exclusively interacts gravitationally. Cold dark matter is a specific form of dark matter that is non-relativistic and thus moves slowly compared to the speed of light.
But what is a matter power spectrum?

2.2.1 *The Correlation Function and the Power Spectrum*

We want to describe the clustering of matter in the universe. The first tool we need is the normalized density contrast

$$\delta(\mathbf{x}) = \frac{\rho(\mathbf{x}) - \bar{\rho}}{\bar{\rho}} \tag{2.4}$$

[3] that is the normalized difference of the local density at a specific location to the mean density.

We furthermore define the autocorrelation function as

$$\xi_2(|\mathbf{x}_1 - \mathbf{x}_2|) = \langle \delta(\mathbf{x}_1)\delta(\mathbf{x}_2) \rangle = \frac{1}{V} \int \mathrm{d}^3\mathbf{x}_1 \ \delta(\mathbf{x}_1)\delta(\mathbf{x}_2) \tag{2.5}$$

for two particles and similarly for N particles.

This function $\xi(r)$ can be interpreted as quantifying the conditional probability of finding matter at distance r to some reference matter (with the right normalization). It measures the clustering of matter structures like galaxies and something like "common distances" between them.

Now we would like to measure how frequently clusters of matter with a certain size occur. We do this by taking the Fourier transform of the autocorrelation function and getting some form of measure for how often structures of a certain size, expressed by wavenumber k, appear.

This is the power spectrum $P(k)$.

2.2.2 *The Model For Cold Dark Matter*

We simulate cold dark matter by taking N particles which have a certain initial distribution that we get from [3]

$$\mathcal{P}(\{q\},\{p\}) = \frac{V^{-N}\mathcal{C}(\{q\},\{p\})}{\sqrt{(2\pi)^{3N}\det C_{pp}^N}} e^{-\{p\}^T (C_{pp}^N)^{-1}\{p\}/2} \tag{2.6}$$

V^N normalizes the integral about the positions of the N particles. C_{pp}^N is the momentum correlation matrix which gives a probability distribution of the momenta and depends on the relative positions of the particles (see 3). $\mathcal{C}(\{q\},\{p\})$ is a function that will be set to unity for simplicity.

For now, it is sufficient to imagine a Gaussian distribution in the momenta and a uniform distribution in the space coordinate $\mathbf{q}$.

For N particles the previously defined power spectrum is given by

$$P(k,t) \propto \prod_{n=1}^{N} \int \mathrm{d}^3q_n \mathrm{d}^3p_n \mathcal{P}(\{q\},\{p\}) e^{i\mathbf{k}[\mathbf{x}_1(t)-\mathbf{x}_2(t)]} \tag{2.7}$$

We have to integrate over all N particles' trajectories. In the explicit form of $\mathcal{P}(\{q\},\{p\})$ we get a dependence on all the relative positions of the particles so the trajectories depend on each other.

In this thesis, we use $\mathbf{x}(t)$ as the position of a particle at time t while $\mathbf{q}$ is reserved exclusively for the initial positions. $\mathbf{q}_{21}$ is the relative initial position of two particles and $\mathbf{p}_{21}$ the relative initial momentum.

2.3 SMALL SCALE ASYMPTOTICS AND THE SADDLE POINT APPROACH

The power spectrum looks very different for different length scales. We work within the limit of small scales, i.e. $\mathbf{k} \to \infty$ and as we will see also $\mathbf{q} \to 0$. This will help us in dealing with the obscure matrix C_{pp}^N that we saw in 2.6.

An important aspect in order to deal with the integrals in the power spectrum is the so-called "saddle point approximation". The logic is simple: Suppose that we have a function $f(x) = e^{\lambda g(x)}$ we want to integrate over. If we have the case $\lambda \to \infty$, we can approximate the integral by the maximum of $g(x)$. λ and the exponential function make the contribution of the maximum of g much bigger than the contribution from rest of the function.
In the complex case, the argument is similar and leads us to saddle points of the exponent. Let us take a closer look.

2.3.1 *Real Exponent*

For $g(x)$ real, we see that the maximum of $g(x)$ is the maximum of $f(x)$. But more significantly, $\lambda \to \infty$ "stretches" the function g vertically. So, as the exponential function grows steeper with growing exponents, the maximum of f becomes much higher relative to the rest of the function. To give an intuitive example, see here a plot of the function $g(x) = (-0.01(x-10)^2 + 1)\cos(x-10)$ from $x = 0$ to $x = 20$:

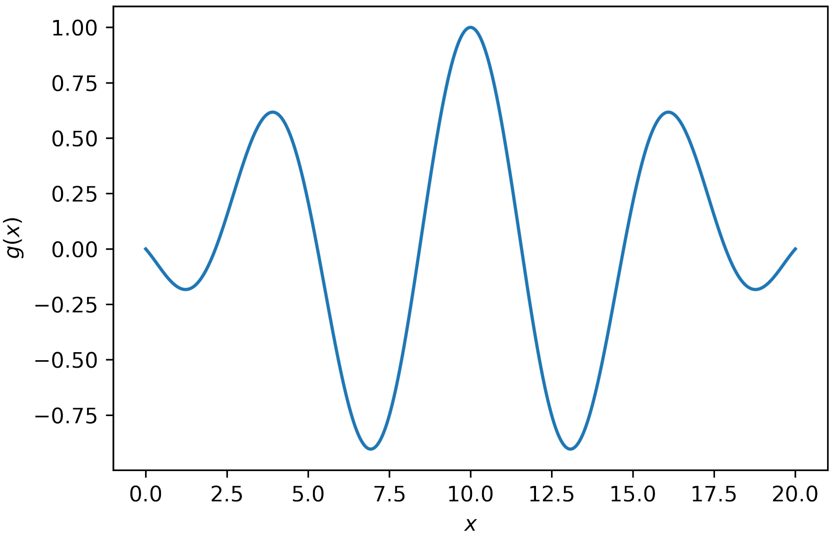

Figure 2.1: Test function g for real saddle point with global and local maxima

The function has a local and global maximum at $x = 10$ and other local maxima which are "not much smaller" roughly spoken.

However, if we plot $f(x) = e^{\lambda g(x)}$ we see that even for smaller lambda, the maxima of g get very dominant:

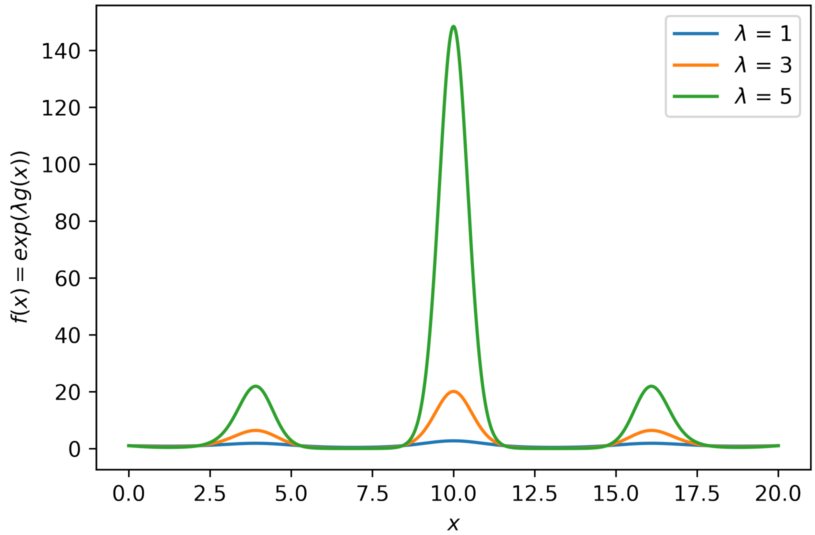

Figure 2.2: Visualization real saddle point, $f = \exp(\lambda g)$ for $\lambda = 1, 3, 5$

If we rescale the functions by dividing by their mean $\langle f \rangle = \frac{1}{20} \int_0^{20} f$, their integrals all produce the same result and we get

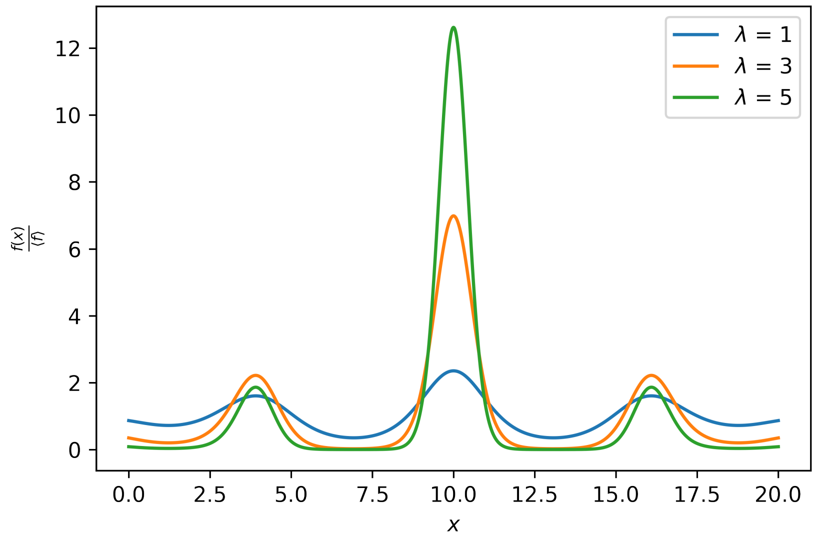

Figure 2.3: Visualization real saddle point, $\tilde{f} = \frac{f}{\langle f \rangle}$ for $\lambda = 1, 3, 5$

We quickly see that a narrow peak dominates the integral over f for large λ and the entire contribution of the smaller local maxima almost vanishes in f even though g is not significantly smaller in them.

2.3.2 *Complex Exponent*

For complex exponents, the situation is a bit different but yields a similar result.

We now take a function $f(x) = e^{\lambda j(x)} = e^{g(x)+i\lambda h(x)} = e^{g(x)} e^{i\lambda h(x)}$.
Note that this is not the rigorous mathematical proof but rather a reduced version of the argument. We take our function j to be defined like $j : \mathbb{R} \to \mathbb{C}$ and not like $j : \mathbb{C} \to \mathbb{C}$. The proper mathematical way to prove the saddle point approximation can be found in [4], chapter 7.

The complex exponential is oscillating with changing h. The faster h changes, the faster the oscillation and the more of $e^{g(x)}$ cancels for continuous g.
If we want to find the areas/points that contribute significantly to the integral over f, we thus look for slow changes in h, i.e. parts where $h' = 0$. As h is the imaginary part of j, these critical points of h may be saddle points of j. Again, see [4] for a further analysis.

This can be easily visualized as well:
We first plot two example functions g and h. Again, they do not necessarily form a holomorphic function or a saddle point together. The intuition here will just be that a critical point of h, the imaginary part of j, is giving the main contribution for the integral while the rest oscillates and cancels in the limit $\lambda \to \infty$.

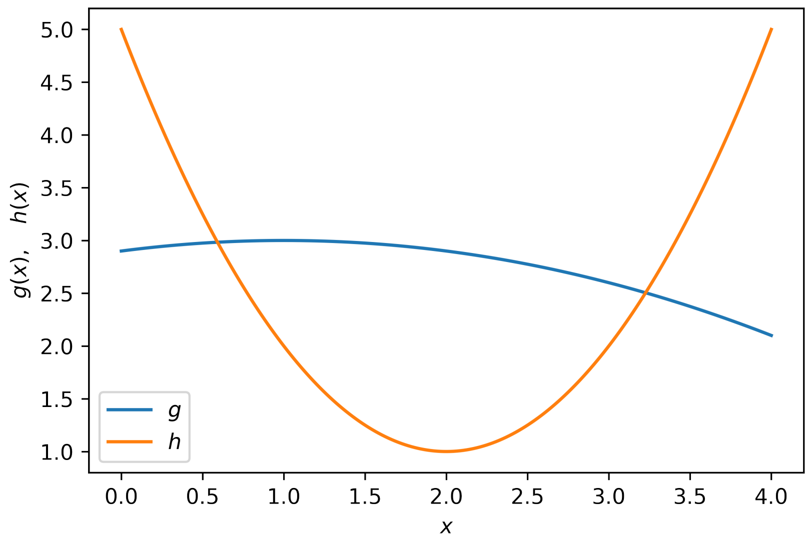

Figure 2.4: Test functions for complex saddle point

where g is some arbitrary smooth function and h has a saddle point.

We now look at the plot of only $e^{g(x)}$:

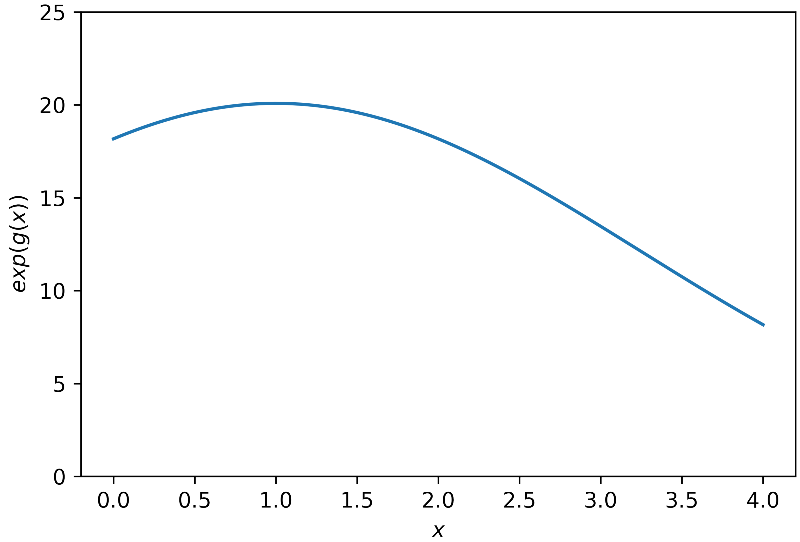

Figure 2.5: Exponentiated test function $\exp(g(x))$ without oscillation

If we now introduce the complex oscillation $e^{i\lambda h(x)}$ we see the effect of growing λ. For simplicity we plot the real part, so $e^{g(x)}\cos(\lambda h(x))$:

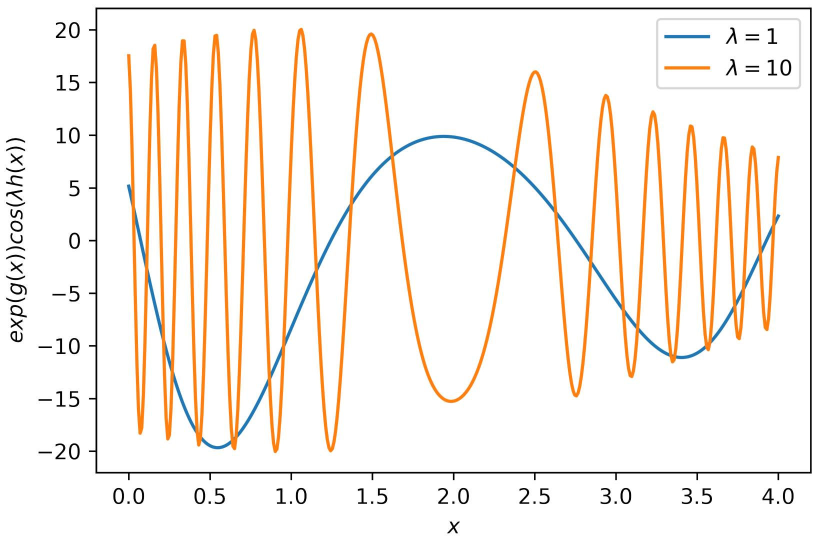

Figure 2.6: Exponentiated test function with faster oscillations, $\exp(g(x))\cos(\lambda h(x))$ for $\lambda = 1, 10$

We can already see that the oscillation becomes faster resulting in the cancellation of the positive and negative parts of $Re(f)$ away from the critical point of h.

Even more extrem is the case $\lambda = 100$:

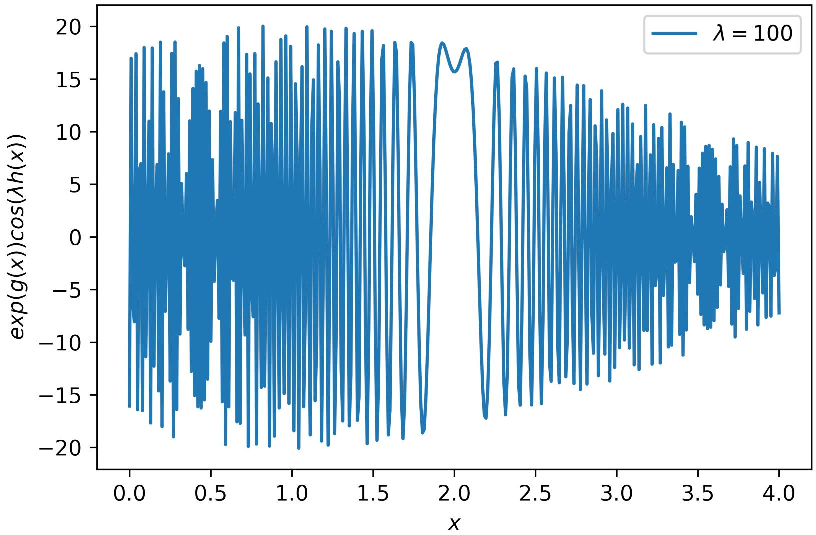

Figure 2.7: Exponentiated test function with fast oscillations, $\exp(g(x))\cos(\lambda h(x))$ for $\lambda = 100$

Clearly, only the saddle point near $x = 2$ contributes significantly to the integral and the oscillations cancel each other.

This is the basic intuition of the saddle point approximation that we will use in the following. See [4] for the correct and complete mathematical proof.

2.4 OUR SADDLE POINT

If we take a look at our object of interest 2.7 and 2.6, we can combine the two to get

$$P(k,t) = \int \mathrm{d}^3\mathbf{q}_1 \mathrm{d}^3\mathbf{q}_2 \mathrm{d}^3\mathbf{p}_1 \mathrm{d}^3\mathbf{p}_2 \frac{V^{-2}}{\sqrt{(2\pi)^6 \det C^2_{pp}}} \cdot e^{-\{p\}^T (C^2_{pp})^{-1}\{p\}/2} e^{i\mathbf{k}(\mathbf{x}_1(t)-\mathbf{x}_2(t))} \tag{2.8}$$

$$= \int \mathrm{d}^3\mathbf{q}_1 \mathrm{d}^3\mathbf{q}_2 \mathrm{d}^3\mathbf{p}_1 \mathrm{d}^3\mathbf{p}_2 \frac{V^{-2}}{\sqrt{(2\pi)^6 \det C^2_{pp}}} e^{-(\mathbf{p}_{21}^T \Sigma^{-1} \mathbf{p}_{21} - 2\mathbf{a}\mathbf{p}_{21} - 2B)/2} \cdot e^{i\mathbf{k}(\mathbf{q}_{21} + g_{qp}\mathbf{p}_{21} + \vec{\Psi})} \tag{2.9}$$

where we have expanded the relative trajectory into initial position $\mathbf{q}_{21}$, initial momentum $\mathbf{p}_{21}$ and a displacement $\vec{\Psi}$, that is caused by gravitational particle interaction and will be defined precisely in 5. The correlation matrix is split into components quadratic, linear and constant in $\mathbf{p}_{21}$. The latter will be thoroughly defined in 3.

In alignment with 2.3, we see that we have an exponent with $\mathbf{k}$ a parameter that goes to infinity in the small scale limit, like λ in 2.3. We want to find saddle points. These will also be critical points of the exponent

$$\Phi = -\frac{1}{2}\mathbf{p}_{21}^T \Sigma^{-1} \mathbf{p}_{21} + \mathbf{a}\cdot\mathbf{p}_{21} + B + i\mathbf{q}_{21}\cdot\mathbf{k} + ig_{qp}\mathbf{p}_{21}\cdot\mathbf{k} + i\mathbf{k}\cdot\vec{\Psi}$$

with respect to $\mathbf{p}_{21}$ and $\mathbf{q}_{21}$ [4]. We get the defining equations

$$\frac{d\Phi}{d\mathbf{p}_{21}} = -\Sigma^{-1}\mathbf{p}_{21} + \mathbf{a} + ig_{qp}\mathbf{k} + i\mathbf{k}\Psi_p = 0 \tag{2.10}$$

$$\frac{d\Phi}{d\mathbf{q}_{21}} = -\frac{1}{2}p^i \frac{\partial\Sigma_{ij}^{-1}}{\partial\mathbf{q}_{21}} p^j + p_i \frac{\partial a^i}{\partial\mathbf{q}_{21}} + \frac{\partial B}{\partial\mathbf{q}_{21}} + i\mathbf{k} + i\mathbf{k}\Psi_q = 0 \tag{2.11}$$

After some further calculations and logical arguments that can be found in [5], we can see that we have a movable saddle point at

$$\mathbf{p}_{21} = k^{-1}\mathbf{c} \quad \text{and} \quad \mathbf{q}_{21} = k^{-1}\mathbf{d} \tag{2.12}$$

where $\mathbf{c}$ and $\mathbf{d}$ are constants. We remember that we are looking for the power spectrum in the small-scale limit, i.e. $k \to \infty$. This means that we can approximate $\vec{\Psi}$ around $(\mathbf{q}_{21}, \mathbf{p}_{21}) = (0, 0)$:

$$\vec{\Psi} = \Psi_q \cdot \mathbf{q}_{21} + \Psi_p \cdot \mathbf{p}_{21} \tag{2.13}$$

with the natural definitions $(\Psi_q)_{ij} = \frac{\partial\Psi_i}{\partial q_j}$ and $(\Psi_p)_{ij} = \frac{\partial\Psi_i}{\partial p_j}$. We bear in mind that the value of $\vec{\Psi}(q = 0, p = 0)$ at the saddle point vanishes. The saddle point $(q, p) = (0, 0)$ initially sets the particles at the same place. Thus, the two particles experience exactly the same force field and the difference in their displacement from this field is zero and only the first order of approximation near the saddle point contributes.

This has an important implication: The fact that we are interested in small scale structures, i.e. $k \to \infty$, and that our saddle point is given by 2.12 means that we have to consider the behaviour of particles close to each other, i.e. $\mathbf{q}_{21}, \mathbf{p}_{21} \to 0$.

This may sound intuitively clear and the only natural solution but it is not trivial in the first place, however very useful as we will see in the following chapters.

3 THE CORRELATION MATRIX C_{pp}^N

If we remember the rather abstract expression for the power spectrum 2.7, we see that the initial distribution $\mathcal{P}(\{q\},\{p\})$ plays a significant role. Looking at the formula for this initial distribution 2.6, we see that it includes an exponential function which becomes a Gaussian if the matrix C_{pp}^N is symmetric and positive definite.
This matrix is called the correlation matrix and expresses the correlation between the momenta of the particles (like the name suggests) and we will need to be able to work with it.

3.1 GENERAL FORM AND DEFINITION

The correlation matrix is defined via its components in terms of the initial particle momenta $\mathbf{p}_i$ as

$$C_{\mathbf{p}_i\mathbf{p}_j} := \langle \mathbf{p}_i \otimes \mathbf{p}_j \rangle \tag{3.1}$$
$$C_{pp}^N := C_{\mathbf{p}_i\mathbf{p}_j} \otimes E_{ij} \tag{3.2}$$

where E_{ij} is the $N \times N$ matrix with a 1 at position (i,j) and 0 elsewhere ([1], chapter 4). In the following, we denote the components of the correlation matrix as $C_{\mathbf{p}_i\mathbf{p}_j}^N =: C_{ij}$.

From [1], chapter 4, we know that

$$C_{ij} = -(a_1(r_{ij})\mathbb{1}_3 + a_2(r_{ij})\pi_{\|,ij}) \tag{3.3}$$

with $\mathbf{r}_{ij}$ the distance vector between the two particles,

$$a_1(r_{ij}) = -\frac{1}{2\pi}\int_0^\infty \mathrm{d}k\; P_\delta(k)\frac{j_1(kr_{ij})}{kr_{ij}} \tag{3.4}$$

$$a_2(r_{ij}) = -\frac{1}{2\pi}\int_0^\infty \mathrm{d}k\; P_\delta(k) j_2(kr_{ij}) \tag{3.5}$$

where j_1 and j_2 are spherical Bessel-functions, P_δ is the power spectrum of the density fluctuations ([3]) and

$$\pi_{\|,ij} = \hat{\mathbf{r}}_{ij} \otimes \hat{\mathbf{r}}_{ij} \tag{3.6}$$

is the projector onto the line $\mathbf{r}_{ij}$ connecting the two particles. Suppose we have a vector $\mathbf{v}$ with components parallel and perpendicular to $\mathbf{r}_{ij}$, so $\mathbf{v} = a\,\hat{\mathbf{r}}_{ij} + \mathbf{v}_\perp$. Then

$$\pi_{\|,ij}\mathbf{v} = \hat{\mathbf{r}}_{ij} \otimes \hat{\mathbf{r}}_{ij}(a\,\hat{\mathbf{r}}_{ij} + \mathbf{v}_\perp) = a\,\hat{\mathbf{r}}_{ij} \tag{3.7}$$

and

$$(\mathbb{1} - \pi_{\parallel,ij})\mathbf{v} = \mathbf{v}_\perp \tag{3.8}$$

As all the components a_1, a_2 and $\pi_{\parallel,ij}$ either depend only on the distance between the two particles or are quadratic in $\hat{\mathbf{r}}_{ij}$ like 3.6, we immediately see that the correlation matrix is symmetric, i.e. $C_{ij} = C_{ji}$. Furthermore, we see that $\mathbf{r}_{ii} = 0$ and thus

$$C_{ii} = C_{jj} = -a_1(0)\mathbb{1}_3$$

The following subchapters will appear very tedious and not particularly interesting but will prove necessary for the later calculations.

3.2 GENERAL FORM IN TERMS OF $\mathbf{p}$,$\mathbf{P}$

We first define the relative momentum $\mathbf{p} = \mathbf{p}_1 - \mathbf{p}_2$ and the momentum $\mathbf{P} = \frac{1}{2}(\mathbf{p}_1 + \mathbf{p}_2)$. This change of basis is useful because in the formula for the power spectrum 2.7 we consider the relative trajectory of particles 1 and 2, which are arbitrarily chosen, and we would like to integrate over relative momentum and position later on. These two momenta (and positions with the same definitions) are also orthogonal to each other, not only linearly independent.

We can write the correlation matrix in terms of these new coordinates as:

$$\begin{aligned} C_{\mathbf{pp}} &= \langle \mathbf{p} \otimes \mathbf{p} \rangle = \langle (\mathbf{p}_1 - \mathbf{p}_2) \otimes (\mathbf{p}_1 - \mathbf{p}_2) \rangle \\ &= \langle \mathbf{p}_1 \otimes \mathbf{p}_1 \rangle + \langle \mathbf{p}_2 \otimes \mathbf{p}_2 \rangle - \langle \mathbf{p}_1 \otimes \mathbf{p}_2 \rangle - \langle \mathbf{p}_2 \otimes \mathbf{p}_1 \rangle \\ &= C_{11} + C_{22} - C_{12} - C_{21} \\ &= 2(C_{11} - C_{12}) \end{aligned} \tag{3.9}$$

$$\begin{aligned} C_{\mathbf{pP}} &= \langle \mathbf{p} \otimes \mathbf{P} \rangle = \left\langle (\mathbf{p}_1 - \mathbf{p}_2) \otimes \frac{1}{2}(\mathbf{p}_1 + \mathbf{p}_2) \right\rangle \\ &= \frac{1}{2}(\langle \mathbf{p}_1 \otimes \mathbf{p}_1 \rangle + \langle \mathbf{p}_2 \otimes \mathbf{p}_2 \rangle - \langle \mathbf{p}_1 \otimes \mathbf{p}_2 \rangle - \langle \mathbf{p}_2 \otimes \mathbf{p}_1 \rangle) \\ &= \frac{1}{2}(C_{11} - C_{22}) \\ &= 0 \end{aligned} \tag{3.10}$$

$$\begin{aligned} C_{\mathbf{PP}} &= \langle \mathbf{P} \otimes \mathbf{P} \rangle = \left\langle \frac{1}{2}(\mathbf{p}_1 + \mathbf{p}_2) \otimes \frac{1}{2}(\mathbf{p}_1 + \mathbf{p}_2) \right\rangle \\ &= \frac{1}{4}(\langle \mathbf{p}_1 \otimes \mathbf{p}_1 \rangle + \langle \mathbf{p}_2 \otimes \mathbf{p}_2 \rangle + \langle \mathbf{p}_1 \otimes \mathbf{p}_2 \rangle + \langle \mathbf{p}_2 \otimes \mathbf{p}_1 \rangle) \\ &= \frac{1}{4}(C_{11} + C_{22} + C_{12} + C_{21}) \\ &= \frac{1}{2}(C_{11} + C_{12}) \end{aligned} \tag{3.11}$$

$$\begin{aligned} C_{\mathbf{p}n} &= \langle \mathbf{p} \otimes \mathbf{p}_n \rangle = \langle (\mathbf{p}_1 - \mathbf{p}_2) \otimes \mathbf{p}_n \rangle \\ &= C_{1n} - C_{2n} \end{aligned} \tag{3.12}$$

$$\begin{aligned} C_{\mathbf{P}n} &= \langle \mathbf{P} \otimes \mathbf{p}_n \rangle = \left\langle \frac{1}{2}(\mathbf{p}_1 + \mathbf{p}_2) \otimes \mathbf{p}_n \right\rangle \\ &= \frac{1}{2}(C_{1n} + C_{2n}) \end{aligned} \tag{3.13}$$

We thus get

$$C_{pp}^N = \begin{pmatrix} 2(C_{11} - C_{12}) & 0 & C_{1m} - C_{2m} \\ 0 & \frac{1}{2}(C_{11} + C_{12}) & \frac{1}{2}(C_{1m} + C_{2m}) \\ C_{1n} - C_{2n} & \frac{1}{2}(C_{1n} + C_{2n}) & C_{nm} \end{pmatrix} \tag{3.14}$$

with respect to the basis $(\mathbf{p}, \mathbf{P}, \mathbf{p}_3, \ldots, \mathbf{p}_N)$.

3.3 $(C_{pp}^N)^{-1}$ IN TERMS OF $\mathbf{p}$,$\mathbf{P}$

We can write the inverse of the correlation matrix in terms of the relative momentum $\mathbf{p}$ and the momentum $\mathbf{P}$:

Note that $2\mathbf{p}_1 = 2\mathbf{P} + \mathbf{p}$ and $2\mathbf{p}_2 = 2\mathbf{P} - \mathbf{p}$.

For simpler notation, we define $F := (C_{pp}^N)^{-1}$ and denote the 3×3-matrices that form the components of F as F_{ij}. This allows us to rewrite

$$\begin{aligned} \{p\}^T F \{p\} &= \mathbf{p}_1^T F_{11} \mathbf{p}_1 + \mathbf{p}_2^T F_{22} \mathbf{p}_2 + 2\mathbf{p}_1^T F_{12} \mathbf{p}_2 \\ &\quad + 2\mathbf{p}_1^T F_{1n} \mathbf{p}_n + 2\mathbf{p}_2^T F_{2n} \mathbf{p}_n + 2\mathbf{p}_n^T F_{nm} \mathbf{p}_m \\ &= \left(\mathbf{P} + \frac{\mathbf{p}}{2}\right)^T F_{11} \left(\mathbf{P} + \frac{\mathbf{p}}{2}\right) + \left(\mathbf{P} - \frac{\mathbf{p}}{2}\right)^T F_{22} \left(\mathbf{P} - \frac{\mathbf{p}}{2}\right) \\ &\quad + 2\left(\mathbf{P} + \frac{\mathbf{p}}{2}\right)^T F_{12} \left(\mathbf{P} - \frac{\mathbf{p}}{2}\right) \\ &\quad + 2\left(\mathbf{P} + \frac{\mathbf{p}}{2}\right)^T F_{1n} \mathbf{p}_n + 2\left(\mathbf{P} - \frac{\mathbf{p}}{2}\right)^T F_{2n} \mathbf{p}_n + 2\mathbf{p}_n^T F_{nm} \mathbf{p}_m \\ &= \mathbf{P}^T (F_{11} + F_{22} + 2F_{12}) \mathbf{P} + \frac{1}{4} \mathbf{p}^T (F_{11} + F_{22} - 2F_{12}) \mathbf{p} \\ &\quad + \mathbf{p}^T (F_{11} - F_{22}) \mathbf{P} \\ &\quad + 2\mathbf{P}^T (F_{1n} + F_{2n}) \mathbf{p}_n + \mathbf{p}^T (F_{1n} - F_{2n}) \mathbf{p}_n + \mathbf{p}_n^T F_{nm} \mathbf{p}_m \\ &= \mathbf{p}^T (\frac{1}{4}(F_{11} + F_{22} - 2F_{12})) \mathbf{p} \\ &\quad + \left(\mathbf{P}^T (F_{11} - F_{22}) + \mathbf{p}_n^T (F_{1n} - F_{2n})\right) \mathbf{p} \qquad (3.15) \\ &\quad + \mathbf{P}^T (F_{11} + F_{22} + 2F_{12}) \mathbf{P} + 2\mathbf{P}^T (F_{1n} + F_{2n}) \mathbf{p}_n + \mathbf{p}_n^T F_{nm} \mathbf{p}_m \\ &= \mathbf{p}^T \Sigma^{-1} \mathbf{p} - 2\mathbf{a}^T \mathbf{p} - 2B \qquad (3.16) \end{aligned}$$

This dissection into terms quadratic in, linear in and independent of $\mathbf{p}$ will prove important later on.

We now have

$$(C^N_{pp})^{-1} = \begin{pmatrix} \frac{1}{4}(F_{11}+F_{22}-2F_{12}) & \frac{1}{2}(F_{11}-F_{22}) & \frac{1}{2}(F_{1m}-F_{2m}) \\ \frac{1}{2}(F_{11}-F_{22}) & F_{11}+F_{22}+2F_{12} & F_{1m}+F_{2m} \\ \frac{1}{2}(F_{1n}-F_{2n}) & F_{1n}+F_{2n} & F_{nm} \end{pmatrix} \tag{3.17}$$

with respect to the basis $(\mathbf{p}, \mathbf{P}, \mathbf{p}_3, ..., \mathbf{p}_N)$.

3.4 THE MATRIX Σ

As mentioned earlier, the matrix Σ is defined via its inverse Σ^{-1}.

Σ^{-1} is defined as the matrix that appears in the term quadratic in $\mathbf{p}$ of the expansion of $\{p\}^T (C^N_{pp})^{-1}\{p\}$. The preceding calculations thus give us two ways to compute Σ^{-1}:

$$\begin{aligned} \Sigma^{-1} :&= (C^N_{pp})^{-1}_{\mathbf{pp}} = \frac{1}{4}(F_{11}+F_{22}-2F_{12}) \\ &= \frac{1}{4}\Big((C^N_{pp})^{-1}_{11} + (C^N_{pp})^{-1}_{22} - 2(C^N_{pp})^{-1}_{21}\Big) \end{aligned} \tag{3.18}$$

or in terms of C^N_{pp} by the inversion rules of block matrices A.2 and 3.14: Using the notation of A.2 we have

$$C^N_{pp} = \begin{pmatrix} 2(C_{11}-C_{12}) & 0 & C_{1m}-C_{2m} \\ 0 & \frac{1}{2}(C_{11}+C_{12}) & \frac{1}{2}(C_{1m}+C_{2m}) \\ C_{1n}-C_{2n} & \frac{1}{2}(C_{1n}+C_{2n}) & C_{nm} \end{pmatrix} = \begin{pmatrix} A & 0 & B_1 \\ 0 & D_{11} & D_{12} \\ C_1 & D_{21} & D_{22} \end{pmatrix} \tag{3.19}$$

so that we can use the inversion rule A.9 to get

$$\begin{aligned} \Sigma^{-1} &= \left(A - B_1\left(D_{22} - D_{21}D_{11}^{-1}D_{12}\right)^{-1} C_1\right)^{-1} \\ &= \Big(2(C_{11}-C_{12}) - (C_{1m}-C_{2m}) \\ &\qquad \cdot \left[C_{nm} - \frac{1}{2}(C_{1n}+C_{2n})(C_{11}+C_{12})^{-1}(C_{1m}+C_{2m})\right]^{-1} \\ &\qquad \cdot (C_{1n}-C_{2n})\Big)^{-1} \end{aligned} \tag{3.20}$$

where $n, m \leq 3$ as usual.

The latter form has the advantage that we can easily compute Σ simply by carrying out the inversion:

$$\begin{aligned} \Sigma &= 2(C_{11}-C_{12}) - (C_{1m}-C_{2m}) \\ &\quad \cdot \left(C_{nm} - \frac{1}{2}(C_{1n}+C_{2n})(C_{11}+C_{12})^{-1}(C_{1m}+C_{2m})\right)^{-1} (C_{1n}-C_{2n}) \end{aligned} \tag{3.21}$$

3.5 SIMPLIFICATIONS FOR TWO PARTICLES

The original paper by Ginat et al. [5] starts with the assumption that the dark matter is modelled as a cloud of N particles. All of these particles have some initial phase-space position $(\mathbf{q}_i, \mathbf{p}_i)$ and they are all correlated with each other. In the formula 2.7 we see that we select two of those particles (which are of course arbitrary) and consider their relative trajectory.

Because our goal is not only to derive the proportionality of the power spectrum but also to compute the amplitude explicitly, we will approach the problem by assuming that these two particles are correlated with each other but not with the $N-2$ others. This is a simplification that enables us to approach the problem analytically and not numerically and it will allow us to get a better understanding of the general case.

But let us see how the components simplify under this assumption.

3.5.1 C^2_{pp}

We remember the definition of the correlation matrix C^N_{pp} from 3.1. For $N = 2$ we have

$$C^2_{pp} = \begin{pmatrix} C_{\mathbf{p}_1\mathbf{p}_1} & C_{\mathbf{p}_1\mathbf{p}_2} \\ C_{\mathbf{p}_1\mathbf{p}_2} & C_{\mathbf{p}_2\mathbf{p}_2} \end{pmatrix} \tag{3.22}$$

with $C_{\mathbf{p}_1\mathbf{p}_1} = C_{\mathbf{p}_2\mathbf{p}_2} = -a_1(0)\mathbb{1}_3$ and $C_{\mathbf{p}_1\mathbf{p}_2} = C_{\mathbf{p}_2\mathbf{p}_1} = -(a_1(q)\mathbb{1} + a_2(q)\pi_{\parallel,12})$.

From [7] we know that in the small scale limit, that results in our saddle point being $(\mathbf{q}_{21}, \mathbf{p}_{21}) \to (0, 0)$ (see 2.12), we get

$$a_1(q) = -\frac{\sigma_1^2}{3} + q^2\frac{\sigma_2^2}{30} \tag{3.23}$$

$$a_2(q) = q^2\frac{\sigma_2^2}{15} \tag{3.24}$$

Here σ_1^2 and σ_2^2 are known constants, the statistical moments of the initial density-fluctuation power spectrum (see [1], Chapter 3)

$$\sigma_n^2 = \frac{1}{2\pi^2}\int_0^\infty \mathrm{d}k\, k^{2n-2}P_\delta(k) \tag{3.25}$$

As a result,

$$C_{\mathbf{p}_i\mathbf{p}_i} = \mathrm{diag}\left(\frac{\sigma_1^2}{3}, \frac{\sigma_1^2}{3}, \frac{\sigma_1^2}{3}\right) \tag{3.26}$$

and

$$C_{\mathbf{p}_1\mathbf{p}_2} = C_{\mathbf{p}_2\mathbf{p}_1} = \mathrm{diag}\left(\frac{\sigma_1^2}{3} - q^2\frac{\sigma_2^2}{30}, \frac{\sigma_1^2}{3} - q^2\frac{\sigma_2^2}{30}, \frac{\sigma_1^2}{3} - q^2\frac{\sigma_2^2}{30}\right) - q^2\frac{\sigma_2^2}{15}\pi_{\parallel,12} \tag{3.27}$$

3.5.2 $(C^2_{pp})^{-1}$

For $N = 2$ the inverse correlation matrix 3.17 reduces to

$$(C^2_{pp})^{-1} = \begin{pmatrix} \frac{1}{4}(F_{11} + F_{22} - 2F_{12}) & \frac{1}{2}(F_{11} - F_{22}) \\ \frac{1}{2}(F_{11} - F_{22}) & F_{11} + F_{22} + 2F_{12} \end{pmatrix} \tag{3.28}$$

3.5.3 Σ

From 3.5.1 we know that

$$C^2_{pp} = \begin{pmatrix} A & B \\ B & A \end{pmatrix} \tag{3.29}$$

with

$$A = \operatorname{diag}\left(\frac{\sigma_1^2}{3}, \frac{\sigma_1^2}{3}, \frac{\sigma_1^2}{3}\right) \tag{3.30}$$

and

$$B = \operatorname{diag}\left(\frac{\sigma_1^2}{3} - q^2\frac{\sigma_2^2}{30}, \frac{\sigma_1^2}{3} - q^2\frac{\sigma_2^2}{30}, \frac{\sigma_1^2}{3} - q^2\frac{\sigma_2^2}{30}\right) - q^2\frac{\sigma_2^2}{15}\pi_{\parallel,12} \tag{3.31}$$

With 3.21, can now compute Σ:

$$\begin{aligned} \Sigma &= 2(C^2_{pp})_{11} - 2(C^2_{pp})_{12} \\ &= 2A - 2B \\ &= \frac{2q^2\sigma_2^2}{15}\left(\operatorname{diag}\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) + \frac{\mathbf{q} \otimes \mathbf{q}}{q^2}\right) \end{aligned} \tag{3.32}$$

where we again used the definition of $\pi_{\parallel,12}$ from 3.6.

In addition, from 3.15 we read off that $\mathbf{a} = 0$ for $N = 2$.

As C^2_{pp} only depends on the relative positions of the particles, and not on the position of the centre of mass, we can also see that e^B is just a Gaussian in $\mathbf{P}$ for a given $\mathbf{q}$.

4 THE POWER SPECTRUM WITHOUT INTERACTION

After the cumbersome calculations of the previous chapter, we take a step back and summarize, where we are. From KFT we have the formula for the power spectrum of the density correlation 2.7:

$$P(k,t) \propto \prod_{n=1}^{N} \int \mathrm{d}^3 q_n \mathrm{d}^3 p_n \mathcal{P}(\{q\},\{p\}) e^{i\mathbf{k}[\mathbf{x}_1(t)-\mathbf{x}_2(t)]} \tag{4.1}$$

with $\mathcal{P}(\{q\},\{p\})$ the initial distribution of the particles given by 2.6

$$\mathcal{P}(\{q\},\{p\}) = \frac{V^{-N}}{\sqrt{(2\pi)^{3N} \det C_{pp}^N}} e^{-\{p\}^T (C_{pp}^N)^{-1} \{p\}/2} \tag{4.2}$$

We are working under the assumption that the two particles 1 and 2 are correlated with each other but not with any other particle in the system. This means that we can directly leave out the integral over the particles 3 to N because they simply give us a Gaussian integral cancelled by the normalization in the initial distribution. Furthermore, we will now use the simplifications of the correlation matrix in its explicit form from the previous chapter.

We are left with the following expression for the power spectrum:

$$P(k,t) \propto \prod_{n=1}^{2} \int \mathrm{d}^3 q_n \mathrm{d}^3 p_n \frac{V^{-2}}{\sqrt{(2\pi)^6 \det C_{pp}^2}} e^{-\{p\}^T (C_{pp}^2)^{-1} \{p\}/2} e^{i\mathbf{k}[\mathbf{x}_1(t)-\mathbf{x}_2(t)]} \tag{4.3}$$

Even though it is not the actual goal of this work and is already well-known we will calculate the power spectrum for the case of no interaction between the particles. Mathematically, this means that the particle trajectories are given by

$$\mathbf{x}_i(t) = \mathbf{q}_i + g_{qp} \mathbf{p}_i \tag{4.4}$$

with g_{qp} a propagator, something like the generalized time as explained in 2.1.2. This ansatz of constant momentum is known as the Zel'dovich approximation [9, 11], named after Yakov Zel'dovich.

Calculating the power spectrum in this much simpler case will be both a good exercise to see how the calculations work and also a sanity check for our procedure.

After all these explanations we can now start with the actual calculation of the power spectrum in the case of no interaction between the particles:

$$
\begin{aligned}
P(k,t) &= \int \mathrm{d}^3\mathbf{q}_1 \mathrm{d}^3\mathbf{q}_2 \mathrm{d}^3\mathbf{p}_1 \mathrm{d}^3\mathbf{p}_2 \frac{V^{-2}}{\sqrt{(2\pi)^6 \det C_{pp}^2}} e^{-\{p\}^T (C_{pp}^2)^{-1}\{p\}/2} \\
&\quad \cdot e^{i\mathbf{k}(\mathbf{x}_1(t)-\mathbf{x}_2(t))} \\
&= \int \mathrm{d}^3\mathbf{q}_1 \mathrm{d}^3\mathbf{q}_2 \mathrm{d}^3\mathbf{p}_1 \mathrm{d}^3\mathbf{p}_2 \frac{V^{-2}}{\sqrt{(2\pi)^6 \det C_{pp}^2}} e^{-\{p\}^T (C_{pp}^2)^{-1}\{p\}/2} \\
&\quad \cdot e^{i(\mathbf{q}_{21}+g_{qp}\mathbf{p}_{21})\mathbf{k}} \\
&\overset{(1)}{=} \int \mathrm{d}^3\mathbf{q}_1 \mathrm{d}^3\mathbf{q}_2 e^{i\mathbf{q}_{21}\mathbf{k}} \\
&\quad \cdot \int \mathrm{d}^3\mathbf{P} \mathrm{d}^3\mathbf{p}_{21} \frac{V^{-2}}{\sqrt{(2\pi)^6 \det\left(C_{pp}^2\right)}} e^{-(\mathbf{p}_{21}^T\Sigma^{-1}\mathbf{p}_{21} - 2\mathbf{a}\mathbf{p}_{21} - 2B)/2} e^{ig_{qp}\mathbf{k}\mathbf{p}_{21}} \\
&\overset{(2)}{=} \int \mathrm{d}^3\mathbf{q}_1 \mathrm{d}^3\mathbf{q}_2 e^{i\mathbf{q}_{21}\mathbf{k}} V^{-2} \\
&\quad \cdot \int \mathrm{d}^3\mathbf{P} \mathrm{d}^3\mathbf{p}_{21} \frac{1}{\sqrt{(2\pi)^6 \det\left(C_{pp}^2\right)}} e^{-\frac{1}{2}(\mathbf{p}_{21}^T\Sigma^{-1}\mathbf{p}_{21})} e^{ig_{qp}\mathbf{k}\mathbf{p}_{21}} e^{B} \\
&\overset{(3)}{=} \int \mathrm{d}^3\mathbf{q}_1 \mathrm{d}^3\mathbf{q}_2 \; e^{i\mathbf{q}_{21}\mathbf{k}} \; V^{-2} \int d^3\mathbf{P} \; e^{B} \frac{1}{\sqrt{(2\pi)^3 \alpha}} e^{\frac{1}{2}(ig_{qp}\mathbf{k})^T \Sigma (ig_{qp}\mathbf{k})} \\
&\overset{(4)}{=} \int \mathrm{d}^3\mathbf{q}_1 \mathrm{d}^3\mathbf{q}_2 \; e^{i\mathbf{q}_{21}\mathbf{k}} \; V^{-2} \; e^{-\frac{k^2}{2} g_{qp}^2 \hat{k}^T \Sigma \hat{k}} \\
&\overset{(5)}{=} \int \mathrm{d}^3\mathbf{q}_{21} \int \mathrm{d}^3\mathbf{Q} \; V^{-2} e^{i\mathbf{q}_{21}\mathbf{k}} e^{-\frac{k^2}{2} g_{qp}^2 \hat{k}^T \Sigma \hat{k}} \\
&\overset{(6)}{=} V^{-2} \int \mathrm{d}^3\mathbf{q}_{21} \; e^{i\mathbf{q}_{21}\mathbf{k}} e^{-\frac{k^2 g_{qp}^2 \sigma_2^2}{15} \hat{k}^T \left(q^2 \mathrm{diag}\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right) + \mathbf{q}\otimes\mathbf{q}\right)\hat{k}} \\
&\overset{(7)}{=} V^{-2} \int \mathrm{d}^3\mathbf{q}_{21} \; e^{i\mathbf{q}_{21}\mathbf{k}} e^{-\frac{k^2 g_{qp}^2 \sigma_2^2}{15} \mathbf{q}_{21}^T \left(\mathrm{diag}\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right) + \hat{k}\otimes\hat{k}\right)\mathbf{q}_{21}} \\
&\overset{(8)}{=} V^{-2} \frac{1}{\sqrt{3}} \left(\frac{2\pi \cdot 15}{k^2 g_{qp}^2 \sigma_2^2}\right)^{\frac{3}{2}} \\
&\quad \cdot \exp\left(-\frac{15}{4k^2 g_{qp}^2 \sigma_2^2} \mathbf{k}^T \left(\mathrm{diag}\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right) + \hat{k}\otimes\hat{k}\right)^{-1} \mathbf{k}\right) \\
&\overset{(9)}{=} V^{-2} \frac{1}{\sqrt{3}} \left(\frac{2\pi \cdot 15}{k^2 g_{qp}^2 \sigma_2^2}\right)^{\frac{3}{2}} \exp\left(-\frac{5}{2 g_{qp}^2 \sigma_2^2}\right)
\end{aligned}
\tag{4.5}
$$

This is exactly the result that was worked out by Konrad [6] for the Zel'dovich approximation.

Now to the explanations for the individual steps:

- In (1) we change basis from $(\mathbf{p}_1, \mathbf{p}_2)$ to $(\mathbf{P}, \mathbf{p}_{21})$, express $(C_{pp}^2)^{-1}$ in this basis and dissect $\{p\}^T (C_{pp}^2)^{-1}\{p\}$ into terms quadratic in, linear in and independent of $\mathbf{p}_{21}$ like in 3.16:

$$\{p\}^T (C_{pp}^2)^{-1}\{p\} = \mathbf{p}_{21}^T \Sigma^{-1} \mathbf{p}_{21} - 2\mathbf{a}\mathbf{p}_{21} - 2B$$

- In (2) we use that $\mathbf{a} = 0$ for $N = 2$.
- In (3) we compute the Gaussian integral over $\mathbf{p}_{21}$, which leaves the normalization factor with an $\alpha = \frac{det(C^2_{pp})}{det(\Sigma)}$.
- In (4) we make use of the fact, that e^B is just a Gaussian in $\mathbf{P}$ 3.5.3 with the correct normalization $\frac{1}{\sqrt{(\pi)^3\alpha}}$. We can integrate that out.
- In (5) we change our integration variables from $\mathbf{q}_1, \mathbf{q}_2$ to $\mathbf{Q}, \mathbf{q}_{21}$ and set our coordinate system to be comoving with $\mathbf{Q}$ giving us a delta function. Thus the integral vanishes.
- In (6) we use the explicit form of the matrix Σ which was easy to compute for only two particles 3.32.
- In (7) we swap $\hat{k}$ with $\mathbf{q}_{21}$. We do that in the following way:

$$\hat{k}^T \left(q^2 \text{diag}\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)\right) \hat{k} = \frac{1}{2} q^2 \hat{k}^T \hat{k} = \frac{1}{2} q^2 = \mathbf{q}_{21}^T \text{diag}\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) \mathbf{q}_{21} \quad (4.6)$$

  And

$$\mathbf{a}^T(\mathbf{b} \otimes \mathbf{b})\mathbf{a} = a_i(b_i b_j)a_j = (a_i b_i)(a_j b_j) = b_i(a_i a_j)b_j = \mathbf{b}^T(\mathbf{a} \otimes \mathbf{a})\mathbf{b} \quad (4.7)$$

  which lets us swap $\hat{k}$ with $\mathbf{q}_{21}$ in the second term as well.
- In (8) we compute the Gaussian integral over $\mathbf{q}_{21}$. We should take a look at the determinant of the matrix in the exponential that drops out as the normalization factor. The matrix in question is

$$\frac{k^2 g_{qp}^2 \sigma_2^2}{15} \left[\text{diag}\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) + \hat{k} \otimes \hat{k} \right]$$

  We remind ourselves that the determinant stays invariant under the rotation of the matrix. Thus we can rotate our coordinate system such that the x-axis is aligned with $\hat{k}$. This will make the matrix diagonal with the entries

$$\frac{k^2 g_{qp}^2 \sigma_2^2}{15} \text{diag}\left(\frac{3}{2}, \frac{1}{2}, \frac{1}{2}\right)$$

  Now one can easily compute the determinant to be $3\left(\frac{k^2 g_{qp}^2 \sigma_2^2}{2 \cdot 15}\right)^3$. This explains the prefactor in (8).
- In (9) we simplify the result by calculating

$$\mathbf{k}^T \left[\text{diag}\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) + \hat{k} \otimes \hat{k} \right]^{-1} \mathbf{k}$$

In order to do so we remember that this is a scalar and thus invariant under rotation of the matrix. We can thus rotate our coordinate system in such a way that the x-axis is aligned with $\hat{k}$ again. We have the easy form

$$\begin{pmatrix} k & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{3}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}^{-1} \begin{pmatrix} k \\ 0 \\ 0 \end{pmatrix} = k^2 \frac{2}{3} \tag{4.8}$$

which simplifies the result to the final form in (9).

This result is, as mentioned earlier, nothing new but a simpler example for the calculations we will do now and at least the justification to hope that we are on the right path.

5 THE POWER SPECTRUM INCLUDING INTERACTION

Now that the result for the non-interacting case has given us confidence, we include the interaction term into the particle trajectory. We will use the same notation as in the previous section and the calculations will mostly be similar. The main difference is that now the particles' relative trajectory is given by

$$i\mathbf{k}(\mathbf{x}_1(t) - \mathbf{x}_2(t)) = i\mathbf{k}(\mathbf{q}_{21} + g_{qp}\mathbf{p}_{21} + \vec{\Psi}) \tag{5.1}$$

where $\vec{\Psi}$ is the displacement caused by interaction: As one can read in [1], chapter 2, the phase space trajectory of a test particle in an expanding space-time is given by

$$X(t) = G(t,0)X^{(i)} - \int_0^t dt' G(t,t') \begin{pmatrix} 0 \\ m\vec{\nabla}\phi \end{pmatrix} \tag{5.2}$$

where $G(t,t')$ is given by

$$G(t,t') = \begin{pmatrix} \mathbb{1}_3 & g_{qp}\mathbb{1}_3 \\ 0 & \mathbb{1}_3 \end{pmatrix} \tag{5.3}$$

$X = \begin{pmatrix} x \\ \dot{x} \end{pmatrix}$ and we have a potential ϕ satisfying the Poisson equation

$$\Delta\phi = 4\pi G\rho \tag{5.4}$$

This gives us the trajectory for particle j

$$\mathbf{x}_j(t) = \mathbf{q}_j + g_{qp}\mathbf{p}_j - \int_0^t \mathrm{d}t' m g_{qp} \vec{\nabla}\phi(\mathbf{x}_j(t')) =: \mathbf{q}_j + g_{qp}\mathbf{p}_j + \vec{\Psi}_j \tag{5.5}$$

The difference $\vec{\Psi}_2 - \vec{\Psi}_1$ gives us the $\vec{\Psi}$ in 5.1.
This results in the slightly modified formula

$$\begin{aligned} P(k,t) &= \int \mathrm{d}^3\mathbf{q}_1 \mathrm{d}^3\mathbf{q}_2 \mathrm{d}^3\mathbf{p}_1 \mathrm{d}^3\mathbf{p}_2 \frac{V^{-2}}{\sqrt{(2\pi)^6 \det C_{pp}^2}} e^{-\{p\}^T (C_{pp}^2)^{-1}\{p\}/2} \\ &\quad \cdot e^{i\mathbf{k}(\mathbf{x}_1(t) - \mathbf{x}_2(t))} \\ &= \int \mathrm{d}^3\mathbf{q}_1 \mathrm{d}^3\mathbf{q}_2 \mathrm{d}^3\mathbf{p}_1 \mathrm{d}^3\mathbf{p}_2 \frac{V^{-2}}{\sqrt{(2\pi)^6 \det C_{pp}^2}} e^{-(\mathbf{p}_{21}^T \Sigma^{-1}\mathbf{p}_{21} - 2\mathbf{a}\mathbf{p}_{21} - 2B)/2} \\ &\quad \cdot e^{i\mathbf{k}(\mathbf{q}_{21} + g_{qp}\mathbf{p}_{21} + \vec{\Psi})} \end{aligned} \tag{5.6}$$

5.1 THE POWER SPECTRUM

With this approach, we can solve the integral in the small-scale limit.

$$
\begin{aligned}
P(k,t) &= \int \mathrm{d}^3\mathbf{q}_1\mathrm{d}^3\mathbf{q}_2\mathrm{d}^3\mathbf{p}_1\mathrm{d}^3\mathbf{p}_2 \frac{V^{-2}}{\sqrt{(2\pi)^6 \det C_{pp}^2}} e^{-\{p\}^T (C_{pp}^2)^{-1}\{p\}/2} \\
&\quad \cdot e^{i\mathbf{k}(\mathbf{x}_1(t)-\mathbf{x}_2(t))} \\
&= \int \mathrm{d}^3\mathbf{q}_1\mathrm{d}^3\mathbf{q}_2\mathrm{d}^3\mathbf{p}_1\mathrm{d}^3\mathbf{p}_2 \frac{V^{-2}}{\sqrt{(2\pi)^6 \det C_{pp}^2}} e^{-(\mathbf{p}_{21}^T\Sigma^{-1}\mathbf{p}_{21}-2\mathbf{a}\mathbf{p}_{21}-2B)/2} \\
&\quad \cdot e^{i\mathbf{k}(\mathbf{q}_{21}+g_{qp}\mathbf{p}_{21}+\vec{\Psi})} \\
&\overset{(1)}{=} \int \mathrm{d}^3\mathbf{q}_1\mathrm{d}^3\mathbf{q}_2\mathrm{d}^3\mathbf{p}_1\mathrm{d}^3\mathbf{p}_2 \frac{V^{-2}}{\sqrt{(2\pi)^6 \det C_{pp}^2}} e^{-(\mathbf{p}_{21}^T\Sigma^{-1}\mathbf{p}_{21}-2B)/2} \\
&\quad \cdot e^{i\mathbf{k}(\mathbf{q}_{21}+g_{qp}\mathbf{p}_{21}+(\Psi_p\mathbf{p}_{21}+\Psi_q\mathbf{q}_{21}))} \qquad (5.7) \\
&\overset{(2)}{=} V^{-2}\int \mathrm{d}^3\mathbf{q}_1\mathrm{d}^3\mathbf{q}_2\; e^{i\mathbf{k}^T(\mathbb{1}+\Psi_q)\mathbf{q}_{21}} \int d^3\mathbf{P} d^3\mathbf{p}_{21} \frac{1}{\sqrt{(2\pi)^6 \det\left(C_{pp}^2\right)}} \\
&\quad \cdot e^{-\frac{1}{2}\mathbf{p}_{21}^T\Sigma^{-1}\mathbf{p}_{21}} e^{i\mathbf{k}^T(g_{qp}\mathbb{1}+\Psi_p)\mathbf{p}_{21}} e^B \\
&\overset{(3)}{=} V^{-2}\int \mathrm{d}^3\mathbf{q}_1\mathrm{d}^3\mathbf{q}_2\; e^{i\mathbf{k}^T(\mathbb{1}+\Psi_q)\mathbf{q}_{21}} \int d^3\mathbf{P} \frac{1}{\sqrt{(2\pi)^3\alpha}} \\
&\quad \cdot e^{-\frac{1}{2}\left(\mathbf{k}^T(g_{qp}\mathbb{1}+\Psi_p)\Sigma(g_{qp}\mathbb{1}+\Psi_p)^T\mathbf{k}\right)} e^B \\
&\overset{(4)}{=} V^{-2}\int \mathrm{d}^3\mathbf{q}_1\mathrm{d}^3\mathbf{q}_2\; e^{i\mathbf{k}^T(\mathbb{1}+\Psi_q)\mathbf{q}_{21}} e^{-\frac{1}{2}\left(\mathbf{k}^T\left(g_{qp}\mathbb{1}+\Psi_p\right)\Sigma\left(g_{qp}\mathbb{1}+\Psi_p\right)^T\mathbf{k}\right)} \\
&\overset{(5)}{=} V^{-2}\int \mathrm{d}^3\mathbf{Q}\mathrm{d}^3\mathbf{q}_{21}\; e^{i\mathbf{k}^T(\mathbb{1}+\Psi_q)\mathbf{q}_{21}} \\
&\quad \cdot e^{-\frac{2k^2\sigma_2^2}{15}\left(\hat{k}^T\left(g_{qp}\mathbb{1}+\Psi_p\right)\left(\frac{q^2}{2}\mathbb{1}+\mathbf{q}_{21}\otimes\mathbf{q}_{21}\right)\left(g_{qp}\mathbb{1}+\Psi_p\right)^T\hat{k}\right)} \\
&\overset{(6)}{=} V^{-2}\int \mathrm{d}^3\mathbf{q}_{21}\; e^{i\mathbf{k}^T(\mathbb{1}+\Psi_q)\mathbf{q}_{21}} e^{-\frac{1}{2}\left(\frac{2k^2\sigma_2^2}{15}\vec{\lambda}^T\left(\frac{q^2}{2}\mathbb{1}+\mathbf{q}_{21}\otimes\mathbf{q}_{21}\right)\vec{\lambda}\right)} \\
&\overset{(7)}{=} V^{-2}\int \mathrm{d}^3\mathbf{q}_{21}\; e^{i\mathbf{k}^T(\mathbb{1}+\Psi_q)\mathbf{q}_{21}} e^{-\frac{1}{2}\left(\frac{2k^2\sigma_2^2}{15}\mathbf{q}_{21}^T\left(\frac{\lambda^2}{2}\mathbb{1}+\vec{\lambda}\otimes\vec{\lambda}\right)\mathbf{q}_{21}\right)} \\
&\overset{(8)}{=} V^{-2}\sqrt{2\pi}^3\left(\frac{15\cdot 2}{2k^2\sigma_2^2\lambda^2}\right)^{\frac{3}{2}}\frac{1}{\sqrt{3}} \\
&\quad \cdot \exp\left(-\frac{1}{2}\frac{15}{2k^2\sigma_2^2}\mathbf{k}^T\left(\mathbb{1}+\Psi_q\right)\left(\frac{\lambda^2}{2}\mathbb{1}+\vec{\lambda}\otimes\vec{\lambda}\right)^{-1}\left(\mathbb{1}+\Psi_q\right)^T\mathbf{k}\right) \\
&\overset{(9)}{=} V^{-2}\frac{\sqrt{2\pi}^3}{\sqrt{3}}\left(\frac{15}{\sigma_2^2}\right)^{\frac{3}{2}}\frac{1}{k^3\lambda^3} \qquad (5.8) \\
&\quad \cdot \exp\left(-\frac{15}{4k^2\sigma_2^2}\mathbf{k}^T\left(\mathbb{1}+\Psi_q\right)\left[\frac{2}{\lambda^2}\mathbb{1}-\frac{4}{\lambda^4}\frac{\vec{\lambda}\otimes\vec{\lambda}}{1+\vec{\lambda}^T\vec{\lambda}\frac{2}{\lambda^2}}\right](\mathbb{1}+\Psi_q)^T\mathbf{k}\right)
\end{aligned}
$$

$$
\begin{aligned}
P(k,t) &= V^{-2}\frac{\sqrt{2\pi}^3}{\sqrt{3}}\left(\frac{15}{\sigma_2^2}\right)^{\frac{3}{2}}\frac{1}{k^3\lambda^3} \\
&\quad\cdot\exp\Bigg(-\frac{15}{2\sigma_2^2\lambda^2} \\
&\qquad\cdot\left[\left|\left(\mathbb{1}+\Psi_q^T\right)\hat{k}\right|^2-\frac{2}{3\lambda^2}\hat{k}^T\left(\mathbb{1}+\Psi_q\right)\vec{\lambda}\otimes\vec{\lambda}\left(\mathbb{1}+\Psi_q\right)^T\hat{k}\right]\Bigg) \\
&\overset{(10)}{=} V^{-2}\frac{\sqrt{2\pi}^3}{\sqrt{3}}\left(\frac{15}{\sigma_2^2}\right)^{\frac{3}{2}}\frac{1}{k^3\lambda^3} \\
&\quad\cdot\exp\left(-\frac{15}{2\sigma_2^2\lambda^2}\left[\left|\left(\mathbb{1}+\Psi_q^T\right)\hat{k}\right|^2-\frac{2}{3\lambda^2}\left(\hat{k}^T\left(\mathbb{1}+\Psi_q\right)\cdot\vec{\lambda}\right)^2\right]\right) \\
&\overset{(11)}{=} V^{-2}\frac{\sqrt{2\pi}^3}{\sqrt{3}}\left(\frac{15}{\sigma_2^2}\right)^{\frac{3}{2}}\frac{1}{k^3\lambda^3} \\
&\quad\cdot\exp\left(-\frac{15}{2\sigma_2^2\lambda^2}\left[\left|\left(\mathbb{1}+\Psi_q^T\right)\hat{k}\right|^2-\frac{2}{3}\left|\left(\mathbb{1}+\Psi_q^T\right)\hat{k}\right|^2\cos^2(\gamma)\right]\right) \\
&\overset{(12)}{=} V^{-2}\frac{\sqrt{2\pi}^3}{\sqrt{3}}\left(\frac{15}{\sigma_2^2}\right)^{\frac{3}{2}}\frac{1}{k^3}\frac{1}{\left|\left(g_{qp}\mathbb{1}+\Psi_p^T\right)\hat{k}\right|^3} \\
&\quad\cdot\exp\left(-\frac{15}{2\sigma_2^2}\frac{\left|\left(\mathbb{1}+\Psi_q^T\right)\hat{k}\right|^2}{\left|\left(g_{qp}\mathbb{1}+\Psi_p^T\right)\hat{k}\right|^2}\left[1-\frac{2}{3}\cos^2(\gamma)\right]\right)
\end{aligned}
\tag{5.9}
$$

We can first analyse the result briefly: If we set $\Psi_q = 0$ and $\Psi_p = 0$ we get exactly the result from the non-interacting case 4.5. This is a good sign.
The result can also be observed concerning dependences on the parameters.

$$
P(k,t) = a\frac{1}{k^3}\frac{1}{\left|(g_{qp}\mathbb{1}+\Psi_p^T)\hat{k}\right|^3}exp\left(-b\cdot c\left(\hat{k},g_{qp},\Psi_q,\Psi_p\right)\frac{\left|\left(\mathbb{1}+\Psi_q^T\right)\hat{k}\right|^2}{\left|\left(g_{qp}\mathbb{1}+\Psi_p^T\right)\hat{k}\right|^2}\right) \tag{5.10}
$$

where a and b are constants > 0 and c is bounded by $\frac{1}{3} \leq c \leq 1$. More on that later.

Now to the explanations of the individual steps:

- In (1) we use that $\mathbf{a} = 0$ for $N = 2$ and expand the term $\vec{\Psi}$ in a Taylor series as stated in 2.13.

- In (2) we change the integration variables from $(\mathbf{p}_1, \mathbf{p}_2)$ to $(\mathbf{P}, \mathbf{p}_{21})$ and reorganize the exponential terms.

- In (3) we compute the Gaussian integral over $\mathbf{p}_{21}$, which leaves the normalization factor with an $\alpha = \frac{\det\left(C_{pp}^2\right)}{\det(\Sigma)}$.

- In (4) we integrate out the Gaussian in $\mathbf{P}$.

- In (5) we change our integration variables from $\mathbf{q}_1, \mathbf{q}_2$ to $\mathbf{Q}, \mathbf{q}_{21}$ and set our coordinate system to be comoving with $\mathbf{Q}$ giving us a delta function. Thus, the integral vanishes. We furthermore use the explicit form of Σ for two particles 3.32.

- In (6) we leave out the integral over $\mathbf{Q}$ because of the delta function. We also simplify the matrix in the exponential by defining $\vec{\lambda} = \left(g_{qp}\mathbb{1} + \Psi_p\right)^T \hat{k}$.

- In (7) we swap $\vec{\lambda}$ with $\mathbf{q}_{21}$ in the exponential like explained in 4.7.

- In (8) we compute the Gaussian integral over $\mathbf{q}_{21}$. The determinant is computed in the same way as in the non-interacting case 4 where the main argument is that the determinant stays invariant under rotation of the matrix. Thus we can take $\vec{\lambda}$ to be aligned with the x-axis (but solely for the calculation of the determinant) and the determinant can be read off easily as $3\left(\frac{2k^2\sigma_2^2\lambda^2}{15\cdot 2}\right)^3$.

- In (9) we reorder the prefactors and use the Sherman-Morrison formula A.3 to compute the inverse of the matrix in the exponential.

- In (10) we essentially use 4.7 to get to the scalar product in the exponential.

- In (11) we use the definition of the scalar product $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\left(\sphericalangle(\vec{a},\vec{b})\right)$ to simplify the exponential. We name the ocurring angle $\sphericalangle\left(\vec{\lambda}, \left(\mathbb{1} + \Psi_q^T\right)\hat{k}\right) =: \gamma$

- In (12) we just reorder to get a clear picture of the result.

5.2 CLOSER EXAMINATION OF THE RESULT AND DISCUSSION

Now that we have spent all this work preparing the calculation and computing it explicitly, we should also take our time to examine the result. We have

$$\begin{aligned} P(k,t) =& V^{-2}\frac{\sqrt{2\pi}^3}{\sqrt{3}}\left(\frac{15}{\sigma_2^2}\right)^{\frac{3}{2}}\frac{1}{k^3}\frac{1}{|(g_{qp}\mathbb{1}+\Psi_p^T)\hat{k}|^3} \\ &\cdot\exp\left(-\frac{15}{2\sigma_2^2}\frac{|(\mathbb{1}+\Psi_q^T)\hat{k}|^2}{|(g_{qp}\mathbb{1}+\Psi_p^T)\hat{k}|^2}\left[1-\frac{2}{3}\cos^2(\gamma)\right]\right) \end{aligned} \tag{5.11}$$

with $\gamma = \sphericalangle\left(\left(g_{qp}\mathbb{1} + \Psi_p^T\right)\hat{k}, \left(\mathbb{1} + \Psi_q^T\right)\hat{k}\right)$.

If we set the interaction term (and with that its derivatives Ψ_q and Ψ_p) to zero,

$$\gamma = 0 \quad \Longrightarrow \quad \cos^2(\gamma) = 1 \tag{5.12}$$

so

$$P(k,t) = V^{-2}\frac{\sqrt{2\pi}^3}{\sqrt{3}}\left(\frac{15}{\sigma_2^2}\right)^{\frac{3}{2}}\frac{1}{k^3}\frac{1}{g_{qp}^3}\exp\left(-\frac{5}{2\sigma_2^2}\right) \tag{5.13}$$

which is exactly the result from the non-interacting case 4.5. This is a good sanity check and gives us confidence in our calculation but we finally want to know what the interaction does to the power spectrum.

As briefly mentioned before, we can analyse the result for dependences on the parameters. To make things easier, we write our result in the form

$$\begin{aligned} P(k,t) = a\frac{1}{k^3}&\frac{1}{\left|\left(g_{qp}\mathbb{1} + \Psi_p^T\right)\hat{k}\right|^3} \\ &\cdot \exp\left(-b \cdot c\left(\hat{k}, g_{qp}, \Psi_q, \Psi_p\right)\frac{\left|\left(\mathbb{1} + \Psi_q^T\right)\hat{k}\right|^2}{\left|\left(g_{qp}\mathbb{1} + \Psi_p^T\right)\hat{k}\right|^2}\right) \end{aligned} \tag{5.14}$$

where a and b are constants > 0 and c is bounded by $\frac{1}{3} \leq c \leq 1$. In that way, we have the parameters in a more isolated form.

The prefactor $\frac{1}{k^3}$, the only dependence on k, is exactly what we expect after [5, 6].

What is truly interesting and worth observing is the behaviour over time. We remember that g_{qp} can be imagined as "something like time" and is at least growing monotonically with time. We expect the power spectrum not to go to zero for $g_{qp} \to \infty$ but to stay constant or even increase in that limit.
In the fraction $\frac{1}{\left|\left(g_{qp}\mathbb{1}+\Psi_p^T\right)\hat{k}\right|^3}$ in front of the exponential function, we see that there already is a part of the denominator growing with g_{qp}. The exponential function itself is bound by 1, so it will not prevent the power spectrum from going to zero for large times.

If we take a step back, we see that the $g_{qp}\mathbb{1}$ in the denominator is the interaction-free Zel'dovich part in a sense. Without interaction, two particles just follow their initial momentum from their initial position, resulting in a linear growth of the distance between them, letting them contribute to smaller wavenumbers. The power spectrum on small scales should thus decrease with time in the non-interacting case.

Here we do not want any behaviour like that. If we let the result of the interaction, Ψ_p, be dependent on time in any other way than linear, the fraction $\frac{1}{\left|\left(g_{qp}\mathbb{1}+\Psi_p^T\right)\hat{k}\right|^3}$ will go to zero for large times, for some cases slower and for some faster than the free case. Also, even for linear growth of the interaction, the denominator will grow with time in most cases, just with different rates. However, if we want the denominator to not go to zero, we need to have the term Ψ_p grow with time in a way that it cancels out the growth of $g_{qp}\mathbb{1}$, i.e. $\Psi_p = -g_{qp}\mathbb{1}$.

This may seem like a very strict condition at first sight but it is not. We remember that we are currently in the regime of a saddle point approach, where we almost neglected the exact definition of the saddle point throughout our calculations so far and just used

$$\mathbf{p}_{21} = k^{-1}\mathbf{c} \quad \text{and} \quad \mathbf{q}_{21} = k^{-1}\mathbf{d} \overset{k\to\infty}{\Longrightarrow} \mathbf{p}_{21}, \mathbf{q}_{21} \to 0 \tag{5.15}$$

If we take a look at our defining equations for the saddle point 2.10 and 2.11, they give us

$$\frac{\mathrm{d}\phi}{\mathrm{d}\mathbf{p}_{21}} = -\Sigma^{-1}\mathbf{p}_{21} + \mathbf{a} + ig_{qp}\mathbf{k} + i\mathbf{k}\Psi_p = 0 \tag{5.16}$$

and

$$\frac{\mathrm{d}\phi}{\mathrm{d}\mathbf{q}_{21}} = -\frac{1}{2}p^i \frac{\partial \Sigma^{-1}_{ij}}{\partial \mathbf{q}_{21}} p^j + p_i \frac{\partial a^i}{\partial \mathbf{q}_{21}} + \frac{\partial B}{\partial \mathbf{q}_{21}} + i\mathbf{k} + i\mathbf{k}\Psi_q = 0 \tag{5.17}$$

We remember from the definition of Σ, a and B that they are all real, so the imaginary parts of the equations yield

$$\Psi_p\mathbf{k} + g_{qp}\mathbf{k} = 0 \quad \text{and} \quad \Psi_q\mathbf{k} + \mathbf{k} = 0 \tag{5.18}$$

at the saddle point.

This means that the unwanted part in the denominator does indeed get cancelled out by the linear approximation of the interaction term near the saddle point.

If we plug these equations with some perturbations ϵ_p and ϵ_q for each of the interaction terms in 5.14 in order to prevent the expression from diverging, we get

$$P(k,t) = a\frac{1}{k^3}\frac{1}{\epsilon_p^3}\exp\left(-bc\frac{\epsilon_q^2}{\epsilon_p^2}\right) \tag{5.19}$$

This is a remarkable result. We have taken care of the unwanted growth of our structure which would have resulted in the power spectrum going to zero for late times. By including the interaction term and analysing which part of it contributes the most, we have found a way to prevent the unphysical behaviour of the interaction-free case.

However, we are still not able to predict the behaviour of the power spectrum because of these yet unknown perturbations ϵ_p and ϵ_q. As we only deal with asymptotic behaviour $q, p \to 0$, but not $q, p = 0$, we cannot assume the condition on Ψ_p, Ψ_q from 2.10, 2.11 and 5.18 to be exact. We must approximate these derivatives by their derivative again, giving us the second-order approximation of the term caused by interaction that determines the behaviour of our expression for the power spectrum.

5.3 OUTLOOK

As we just found out, the second derivative of the interaction term in the trajectory will give us the behaviour and exact amplitude of our power spectrum. The next step would be to calculate this second derivative. We can try to give an idea of how one might do this: Remember the form of this term in the trajectory 5.5:

$$\vec{\Psi}_j(t) = -\int_0^t \mathrm{d}t' m g_{qp} \vec{\nabla}\Phi(\mathbf{x}_j(t')) \tag{5.20}$$

We tried two approaches to calculate the second derivative of this term or even just a more practical form for it which both did not lead to satisfying results:
To calculate the second derivative of the term 5.20 with respect to position turns out to be quite complicated in the general case, not even mentioning the difference between two of these terms for particles 1 and 2. The expression also depends on a potential that depends on the position of all the particles around. Including multiple sources of gravitational force in the potential makes everything even more complicated.
An idea used in KFT before is the mean-field approximation [2]. This may enable us to find something like an averaged gravitational field that can be used to make better statements about our displacement term caused by this field. However, it is unclear how one would distinguish between the different forces on the two particles if they experience the same mean-field.

One could try a different approach: If we want to take the second derivative of the difference between two of those terms with respect to position, this leaves the integral unchanged and we get (within an integral over time with the propagator)

$$\left(\vec{\nabla} \otimes \vec{\nabla}\right) \vec{\Psi} = \left(\vec{\nabla} \otimes \vec{\nabla} \otimes \vec{\nabla}\right) \Phi \tag{5.21}$$

In Fourier space, this gives us

$$\mathcal{F}\left[\left(\vec{\nabla} \otimes \vec{\nabla}\right) \vec{\Psi}\right] = \mathcal{F}\left[\left(\vec{\nabla} \otimes \vec{\nabla} \otimes \vec{\nabla}\right) \Phi\right] = -i\left(\mathbf{k} \otimes \mathbf{k} \otimes \mathbf{k}\right) \tilde{\Phi} \tag{5.22}$$

On the other hand, Φ fulfils the Poisson equation 5.4 which gives us in Fourier space

$$\vec{k}^2 \tilde{\Phi} = 4\pi G \rho \quad \Longrightarrow \quad \tilde{\Phi} = \frac{4\pi G \tilde{\rho}}{k^2} \tag{5.23}$$

This leaves us with the result

$$\mathcal{F}\left[\left(\vec{\nabla} \otimes \vec{\nabla}\right) \vec{\Psi}\right] = 4\pi G k \tilde{\rho} \left(\hat{k} \otimes \hat{k} \otimes \hat{k}\right) \tag{5.24}$$

We want a mean value of this term over all configurations and this would vanish because of the isotropy of the universe. Instead, we can calculate something like that:

$$\sqrt{\left\langle \left(\frac{\partial^2 \vec{\Psi}}{\partial x_i \partial x_j}\right)^2 \right\rangle} \approx \frac{\partial^2 \vec{\Psi}}{\partial x_i \partial x_j} \tag{5.25}$$

The square in real space turns to a convolution in Fourier space, so we get

$$\begin{aligned}
\sqrt{\left\langle\left(\frac{\partial^2\vec{\Psi}}{\partial x_i\partial x_j}\right)^2\right\rangle} &= \sqrt{\mathcal{F}^{-1}\left[\left\langle\mathcal{F}\left[\left(\frac{\partial^2\vec{\Psi}}{\partial x_i\partial x_j}\right)^2\right]\right\rangle\right]} \\
&= \sqrt{\int\frac{\mathrm{d}^3\mathbf{k}}{2\pi}\,k^2\left(\hat{k}_i\hat{k}_j\hat{k}_l\right)^2 16\pi^2G^2\left\langle\tilde{\rho}\tilde{\rho}\right\rangle(k)e^{i\mathbf{kx}}} \\
&= 4\pi G\sqrt{\int\frac{\mathrm{d}^3\mathbf{k}}{2\pi}\,k^2\left(\hat{k}_i\hat{k}_j\hat{k}_l\right)^2 P_\delta(k)e^{i\mathbf{kx}}}
\end{aligned} \tag{5.26}$$

In this expression, one could now insert for example either the power spectrum from the non-interacting case or just the approach $P_\delta(k) = P^{(0)}\frac{1}{k^3}$ to compute the Fourier transform numerically. The latter approach might result in a self-consistency equation for the amplitude of the power spectrum later on.

This expression could get us the components of $\frac{\partial^2\vec{\Psi}}{\partial x_i\partial x_j}$. Note that what we do want here is actually $\frac{\partial^2\vec{\Psi}}{\partial q_i\partial q_j}$, but with the approach of the Zel'dovich approximation $\mathbf{x} = \mathbf{q} + g_{qp}\mathbf{p}$, we get $\frac{\partial}{\partial q_i} = \frac{\partial}{\partial x_i} = \frac{1}{g_{qp}}\frac{\partial}{\partial p_i}$.

The result we get for the second derivatives and the cancellation of the first derivatives with $\mathbb{1}$ and $\mathbb{1}g_{qp}$ respectively, we would ideally use in 5.7, i.e.

$$\begin{aligned}
P(k,t) &= \int\mathrm{d}^3\mathbf{q}_1\mathrm{d}^3\mathbf{q}_2\mathrm{d}^3\mathbf{p}_1\mathrm{d}^3\mathbf{p}_2\frac{V^{-2}}{\sqrt{(2\pi)^6\det C_{pp}^2}}e^{-\left(\mathbf{p}_{21}^T\Sigma^{-1}\mathbf{p}_{21}-2B\right)/2} \\
&\quad\cdot\exp\left(i\mathbf{k}\left[\mathbf{q}_{21}+g_{qp}\mathbf{p}_{21}+\left(\Psi_p\mathbf{p}_{21}+\Psi_q\mathbf{q}_{21}\right)\right]\right) \\
&= \int\mathrm{d}^3\mathbf{q}_1\mathrm{d}^3\mathbf{q}_2\mathrm{d}^3\mathbf{p}_1\mathrm{d}^3\mathbf{p}_2\frac{V^{-2}}{\sqrt{(2\pi)^6\det C_{pp}^2}}e^{-\left(\mathbf{p}_{21}^T\Sigma^{-1}\mathbf{p}_{21}-2B\right)/2} \\
&\quad\cdot\exp\left(i\mathbf{k}\left[q_i\frac{\partial^2\vec{\Psi}}{\partial q_i\partial q_j}q_j+2q_i\frac{\partial^2\vec{\Psi}}{\partial q_i\partial p_j}p_j+p_i\frac{\partial^2\vec{\Psi}}{\partial p_i\partial p_j}p_j\right]\right) \\
&= \int\mathrm{d}^3\mathbf{q}_1\mathrm{d}^3\mathbf{q}_2\mathrm{d}^3\mathbf{p}_1\mathrm{d}^3\mathbf{p}_2\frac{V^{-2}}{\sqrt{(2\pi)^6\det C_{pp}^2}}e^{-\left(\mathbf{p}_{21}^T\Sigma^{-1}\mathbf{p}_{21}-2B\right)/2} \\
&\quad\cdot\exp\left(i\mathbf{k}\left[q_i\frac{\partial^2\vec{\Psi}}{\partial x_i\partial x_j}q_j+2q_i\frac{\partial^2\vec{\Psi}}{\partial x_i\partial x_j}g_{qp}p_j+g_{qp}^2p_i\frac{\partial^2\vec{\Psi}}{\partial x_i\partial x_j}p_j\right]\right) \qquad (5.27) \\
&= \int\mathrm{d}^3\mathbf{q}_1\mathrm{d}^3\mathbf{q}_2\mathrm{d}^3\mathbf{p}_1\mathrm{d}^3\mathbf{p}_2\frac{V^{-2}}{\sqrt{(2\pi)^6\det C_{pp}^2}}e^{-\left(\mathbf{p}_{21}^T\Sigma^{-1}\mathbf{p}_{21}-2B\right)/2} \\
&\quad\cdot\exp\left(i\mathbf{k}\left[\left(q_i+g_{qp}p_i\right)\frac{\partial^2\vec{\Psi}}{\partial x_i\partial x_j}\left(q_j+g_{qp}p_j\right)\right]\right) \qquad (5.28)
\end{aligned}$$

where in 5.27 we used the relation $\frac{\partial}{\partial q_i} = \frac{1}{g_{qp}}\frac{\partial}{\partial p_i}$.

These steps are only a vague direction and it would break the boundaries of this thesis to go into detail here. This is a possible approach one could pursue in future work, but the assumptions that are made here, i.e. that all the integrals converge or have good asymptotic behaviour and can at least be approximated, that we can exchange the derivatives with respect to **x** with those with respect to **q** and **p** and get a good approximation of the power spectrum in the small-scale limit, that the combination of square, square root, Fourier transforming back and forth and a mean value in between produces a meaningful result, are bold statements, should raise doubt and thus be checked carefully. We also have not included the time integral in 5.20 and have no justification for assuming that the final integral in 5.28 can be calculated analytically or at least numerically.

As soon as one has some approach for $\frac{\partial^2 \vec{\Psi}}{\partial q_i \partial q_j}$ and the other second derivatives, one could also plug them into the expression for the power spectrum 5.14 and see what happens and how the power spectrum would behave.

Obviously, all of this is to be taken with a grain of salt and should be seen only as one possible direction to go after in future work.

6 CONCLUSION

Let us take a step back and try to grasp the broader picture of what we have done and gained from that.

We started with the wish for a better understanding or even a specific amplitude of the power spectrum in the limit for small scales. Following the path of kinetic field theory and the work of Ginat et al. [5], we began with a very general formula for this power spectrum,

$$P(k,t) \propto \prod_{n=1}^{N} \int \mathrm{d}^3 q_n \mathrm{d}^3 p_n \frac{V^{-N}\mathcal{C}(\{q\},\{p\})}{\sqrt{(2\pi)^{3N}\det C_{pp}^N}} e^{-\{p\}^T (C_{pp}^N)^{-1}\{p\}/2} e^{i\mathbf{k}[\mathbf{x}_1(t)-\mathbf{x}_2(t)]} \tag{6.1}$$

and made the first major simplification by using the saddle point approximation. This gave us the mathematical confirmation that we can focus on the limit $\mathbf{q}, \mathbf{p} \to 0$.

The formula still turned out to be rather complicated because the prominent correlation matrix C_{pp}^N depends on the relative initial positions of all particles with respect to each other. We tried to simplify this further by assuming that only the two arbitrary ones, whose relative trajectory we are interested in, are correlated. This new assumption allowed us to simplify the components of the correlation matrix extremely. They now only depended on the one relative position of particles 1 and 2 and we got explicit formulas for them for our saddle point approximation. Those were very handy as they essentially gave integrable Gaussian integrals or contributions like $\mathbf{q}^T\mathbf{q}$ and $\mathbf{q} \otimes \mathbf{q}$.

We also could throw away $N-2$ integrals over $\mathbf{q}$ and $\mathbf{p}$ as they were just normalized Gaussian integrals for the momenta or cancelled by $V^{-(N-2)}$ for the positions $\mathbf{q}_i$.

With this simplification, we could perform the integrals explicitly, which we tested for the non-interacting case. We there got the expected result which gave us confidence in this approach.

We then tried to generalize this to the interacting case. Using the saddle point approach, we approximated the possibly complicated dependence of the interaction on $\mathbf{q}$ and $\mathbf{p}$ by simple linear terms. Again, this allowed us to perform similar calculations as for the non-interacting case, leaving us with a more explicit formula in the beginning, but naturally also a much more complicated one than for the non-interacting case.

We analysed this result, primarily its behaviour over time and found that the linear approximation of the interaction must behave in a certain way to ensure the physically correct solution. Looking back at our assumptions and the saddle point approach, we found that the linear approximation does behave in the desired way but that we also need another order of the expansion to make qualified statements about the behaviour of the power spectrum.

There are two main results of this work:
The first one is the justification of the assumption that we can approach the power spectrum by two correlated particles. The fact that the next order of expansion for our "integrated force term" transporting the interaction, so the expansion of the tidal force, mainly determines the behaviour of the power spectrum, is the second one.

This second order of the expansion $\frac{\partial^2 \vec{\Psi}}{\partial q_i \partial q_j}$ is the next step to take.

We have not yet reached the final goal of calculating the power spectrum in the limit for small scales, but we have made a significant step towards it and can give somewhat promising directions for future work.

APPENDIX

A MATHEMATICAL TOOLS

A.1 Gaussian integrals

We will need to compute Gaussian integrals of the form

$$\int d^3\mathbf{x}\ \exp\left(-\frac{1}{2}\mathbf{x}^T A\mathbf{x} + \mathbf{b}^T\mathbf{x}\right) \tag{A.1}$$

with $A \in \mathbb{R}^{3\times 3}$ symmetric and positive definite and $\mathbf{b} \in \mathbb{R}^3$.
To the reader of this thesis, their result will be well known. Nevertheless, I find it helpful to have the derivation and exact result at hand.
We can first diagonalize A by an orthogonal transformation $S \in \mathrm{SO}(3)$:

$$A = S^T D S \tag{A.2}$$

with $D = \mathrm{diag}(d_1, ..., d_i)$ diagonal and non-singular.
We can now perform the change of variables $\mathbf{y} = S\mathbf{x}$:

$$\begin{aligned}
&\int d^3\mathbf{x}\ \exp\left(-\frac{1}{2}\mathbf{x}^T A\mathbf{x} + \mathbf{b}^T\mathbf{x}\right)\\
=&\int d^3\mathbf{y}\ \exp\left(-\frac{1}{2}\mathbf{y}^T D\mathbf{y} + \mathbf{b}^T S^{-1}\mathbf{y}\right)\\
=&\int d^3\mathbf{y}\ \exp\left(-\frac{1}{2}\sum_{i=1}^{3} d_i y_i^2 + \sum_{i=1}^{3} b_j S_{ji} y_i\right)\\
=&\prod_{i=1}^{3}\int dy_i\ \exp\left(-\frac{1}{2}d_i y_i^2 + b_j S_{ji} y_i\right)\\
=&\prod_{i=1}^{3}\int dy_i\ \exp\left(-\frac{1}{2}d_i y_i^2 + b_j S_{ji} y_i - \frac{1}{2}\left(b_j S_{ji}\right)^2\frac{1}{d_i}\right)\exp\left(\frac{1}{2}(b_j S_{ji})^2\frac{1}{d_i}\right)\\
=&\exp\left(\frac{1}{2}\mathbf{b}^T S^T D^{-1} S\mathbf{b}\right)\prod_{i=1}^{3}\int dy_i\ \exp\left(-\frac{1}{2}\left(\sqrt{d_i}y_i - \frac{b_j S_{ji}}{\sqrt{d_i}}\right)^2\right)\\
=&\exp\left(\frac{1}{2}\mathbf{b}^T A^{-1}\mathbf{b}\right)\prod_{i=1}^{3}\int dz_i \frac{1}{\sqrt{d_i}}\exp\left(-\frac{1}{2}z_i^2\right)\\
=&\sqrt{\frac{(2\pi)^3}{\det(A)}}\exp\left(\frac{1}{2}\mathbf{b}^T A^{-1}\mathbf{b}\right)
\end{aligned} \tag{A.3}$$

where we use the Einstein summation convention for the index j.

A.2 BLOCK MATRIX INVERSION

Another tool we will need is the inversion of block matrices [8]. Given a block matrix of the form

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{A.4}$$

where both A and D are invertible square matrices, the inverse is given by

$$\begin{pmatrix} (A-BD^{-1}C)^{-1} & -(A-BD^{-1}C)^{-1}BD^{-1} \\ -(D-CA^{-1}B)^{-1}CA^{-1} & (D-CA^{-1}B)^{-1} \end{pmatrix} \tag{A.5}$$

We can either believe Lu et al. [8] or easily verify this by multiplying the matrix with its inverse:

$$\begin{aligned} &\begin{pmatrix} (A-BD^{-1}C)^{-1} & -(A-BD^{-1}C)^{-1}BD^{-1} \\ -(D-CA^{-1}B)^{-1}CA^{-1} & (D-CA^{-1}B)^{-1} \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \\ &= \begin{pmatrix} (A-BD^{-1}Ct)^{-1}(A-BD^{-1}C) & (A-BD^{-1}C)^{-1}(B-B) \\ (D-CA^{-1}B)^{-1}(-C+C) & (D-CA^{-1})^{-1}(-CA^{-1}B+D) \end{pmatrix} \\ &= \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} \end{aligned} \tag{A.6}$$

But this is not enough. We will need to invert a block matrix of the form

$$P = \begin{pmatrix} A & 0 & B_1 \\ 0 & D_{11} & D_{12} \\ C_1 & D_{21} & D_{22} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{A.7}$$

so we use the inversion rule first for $B = (0, B_1)$, $C = (0, C_1)^T$ and $D = \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix}$, assuming, of course, that D_{11} and D_{22} are both invertible.

As we will be interested in the top left block of the inverse, we specifically want to compute $BD^{-1}C$. We use the inversion rule 3.17 for the block matrix D and are only interested in the bottom right block of the inverse. The other parts will be cancelled by the zeroes in B and C.

$$BD^{-1}C = B_1 \left(D_{22} - D_{21}D_{11}^{-1}D_{12}\right)^{-1} C_1 \tag{A.8}$$

This leaves us with the top left block of the inverse of P:

$$\left(A - B_1 \left(D_{22} - D_{21}D_{11}^{-1}D_{12}\right)^{-1} C_1\right)^{-1} \tag{A.9}$$

A.3 SHERMAN-MORRISON FORMULA

The Sherman-Morrison formula [10] is a useful method to compute the inverse of a matrix of the following form:

$$\left(A - uv^T\right)^{-1} = A^{-1} + \frac{A^{-1}uv^TA^{-1}}{\mathbb{1} - v^TA^{-1}u} \tag{A.10}$$

Where $A \in Gl_n(\mathbb{R})$ and $u, v \in \mathbb{R}^n$. This can again be verified by multiplying the matrix with its inverse.

$$\begin{aligned}
&\left(A - uv^T\right)\left(A^{-1} + \frac{A^{-1}uv^TA^{-1}}{1 - v^TA^{-1}u}\right) \\
&= \mathbb{1} - uv^TA^{-1} + \frac{uv^TA^{-1} - uv^TA^{-1}uv^TA^{-1}}{1 - v^TA^{-1}u} \\
&= \mathbb{1} - uv^TA^{-1} + u\frac{1 - v^TA^{-1}u}{1 - v^TA^{-1}u}v^TA^{-1} \\
&= \mathbb{1} - uv^TA^{-1} + uv^TA^{-1} \\
&= \mathbb{1}
\end{aligned} \tag{A.11}$$

BIBLIOGRAPHY

[1] Matthias Bartelmann. *Cosmic Structure Formation*. https://heibox.uni-heidelberg.de/f/85e148765540472d8f7d/. Accessed: 2025-02-10. 2023.

[2] Matthias Bartelmann, Johannes Dombrowski, Sara Konrad, Elena Kozlikin, Robert Lilow, Carsten Littek, Christophe Pixius, and Felix Fabis. "Kinetic field theory: Non-linear cosmic power spectra in the mean-field approximation." In: *SciPost Physics* 10.6 (June 2021). ISSN: 2542-4653. DOI: 10.21468/scipostphys.10.6.153. URL: http://dx.doi.org/10.21468/SciPostPhys.10.6.153.

[3] Matthias Bartelmann, Elena Kozlikin, Robert Lilow, Carsten Littek, Felix Fabis, Ivan Kostyuk, Celia Viermann, Lavinia Heisenberg, Sara Konrad, and Daniel Geiss. "Cosmic Structure Formation with Kinetic Field Theory." In: *Annalen der Physik* 531.11 (Sept. 2019). ISSN: 1521-3889. DOI: 10.1002/andp.201800446. URL: http://dx.doi.org/10.1002/andp.201800446.

[4] N. Bleistein and R.A. Handelsman. *Asymptotic Expansions of Integrals*. Dover Books on Mathematics Series. Dover Publications, 1986. ISBN: 9780486650821. URL: https://books.google.de/books?id=3GZf-bCLFxcC.

[5] Yonadav Barry Ginat, Michael L. Nastac, Robert J. Ewart, Sara Konrad, Matthias Bartelmann, and Alexander A. Schekochihin. *Gravitational Turbulence: the Small-Scale Limit of the Cold-Dark-Matter Power Spectrum*. 2025. arXiv: 2501.01524 [astro-ph.CO]. URL: https://arxiv.org/abs/2501.01524.

[6] Sara Konrad. "Cosmic structure formation in the limit of small scales within Kinetic Field Theory." PhD thesis. 2020.

[7] Sara Konrad and Matthias Bartelmann. "Kinetic field theory for cosmic structure formation." In: *La Rivista del Nuovo Cimento* 45.11 (Aug. 2022), 737–799. ISSN: 1826-9850. DOI: 10.1007/s40766-022-00037-y. URL: http://dx.doi.org/10.1007/s40766-022-00037-y.

[8] Tzon-Tzer Lu and Sheng-Hua Shiou. "Inverses of 2×2 block matrices." In: *Computers and Mathematics with Applications* 43.1 (2002), pp. 119–129. ISSN: 0898-1221. DOI: https://doi.org/10.1016/S0898-1221(01)00278-4. URL: https://www.sciencedirect.com/science/article/pii/S0898122101002784.

[9] C. Pixius, S. Celik, and M. Bartelmann. "Kinetic field theory: perturbation theory beyond first order." In: *Journal of Cosmology and Astroparticle Physics* 2022.12 (Dec. 2022), p. 030. ISSN: 1475-7516. DOI: 10.1088/1475-7516/2022/12/030. URL: http://dx.doi.org/10.1088/1475-7516/2022/12/030.

[10] Jack Sherman and Winifred J. Morrison. "Adjustment of an Inverse Matrix Corresponding to a Change in One Element of a Given Matrix." In: *The Annals of Mathematical Statistics* 21.1 (1950), pp. 124 –127. DOI: 10.1214/aoms/1177729893. URL: https://doi.org/10.1214/aoms/1177729893.

[11] Martin White. "The Zel'dovich approximation." In: *Monthly Notices of the Royal Astronomical Society* 439.4 (Feb. 2014), 3630–3640. ISSN: 0035-8711. DOI: 10.1093/mnras/stu209. URL: http://dx.doi.org/10.1093/mnras/stu209.

Erklärung

Ich versichere, dass ich diese Arbeit selbstständig verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe.

D.Barth

Heidelberg, den 20.02.2025, David Zacharias Barth

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