Extensive Derivation of the ADM-Formalism

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August 2024

Abstract

The three pysicists R. Arnowitt, S. Deser and C. Misner introduced a formalism to split the relativistic spacetime into a timelike direction and spacelike "weaves". With that is is possible to apply the processes of Quantum Mechanics and Quantum Field Theory, where space and time are clearly seperated, to the relativistic spacetime. This laid the foundation for Canonical Quantum Gravity, also known as Loop Quantum Gravity. In this review, the derivation of this formalism shall be laid out in detail.

Acknowledgements

I was confronted with the ADM-formalism during my internship on Loop Quantum Gravity at the University of Lethbridge with my supervisor Prof. Arundhati Dasgupta. She introduced me to and guided me through this fairly abstract topic and without her patient answers to my numerous questions, I would have never been able to understand this topic in the way I do now.
The second big help on my journey in that area was unknowingly Thomas Thiemann, himself a pioneer in Loop Quantum Gravity, whose book "Introduction to Modern Canonical Quantum General Relativity", 2001 [2] led the structure for me learning this topic. I have worked a lot with that book but found it unsatisfyingly short on some calculations I didn't understand in the first place. Motivated by Prof. Dasgupta, I performed these calculations myself in greater detail and wanted to provide this more extensive and detailed derivation of the ADM formalism and action to a curious reader who wants to learn the topic themselves and like me appreciates more detailed and hopefully understandable derivations.

1 Introduction

The Lagrangian and later the Hamiltonian formalism impose a structural approach, first to mechanical problems and later to all variants of problems in physics. In its classical form, the Hamiltonian is defined as

$$H(\mathbf{p}, \mathbf{q}, t) := \sum_{i=1}^{n} p_i \dot{q}^i - \mathcal{L}(\dot{\mathbf{q}}, \dot{\mathbf{q}}, t) \tag{1}$$

as a function depending on generalized coordinates $\mathbf{q}$ and $\mathbf{p}$. The Hamiltonian equations

$$\frac{d\mathbf{q}}{dt} = \frac{\partial H}{\partial \mathbf{p}}, \quad \frac{d\mathbf{p}}{dt} = -\frac{\partial H}{\partial \mathbf{q}} \tag{2}$$

tell the time evolution of the system.

In quantum mechanics, this gets extended. Instead of a Hamiltonian function H, we have a Hamiltonian operator $\hat{H}$. Its form depends on the setup of the physical system and it imposes the time evolution of our state function $\Psi(\mathbf{x}, t)$ as

$$\hat{H} |\Psi\rangle = i\hbar \frac{\partial}{\partial t} |\Psi\rangle \tag{3}$$

Both ways require certain information about the system and give information about the change of this system over time.

However, in special and general relativity time is, like space, relative and there is no such thing as one unique correct time. That time that we experience in our day to day life is only one of many coordinates and has no highlighted significance.
If one were to follow the "canonical way" of fitting the theory of gravity into the framework of quantum mechanics, one clearly needs a time coordinate.

2 The Slicing of Spacetime

Arnowitt, Deser and Misner adress this issue by assuming that the 4-dimensional spacetime manifold $\mathcal{M}$ has the structure

$$\mathcal{M} = \mathbb{R} \times \sigma$$

with σ a 3-dimensional, potentially curved manifold. We call the local coordinates of σ x^1, x^2, x^3 and fix for each $t \in \mathbb{R}$ an embedding

$$X_t : \quad \sigma \to \mathcal{M}$$

defined by

$$X_t(x) := X(t, x)$$

X^μ are the coordinates of $\mathcal{M}$ This allows to foliate spacetime into hypersurfaces

$$\Sigma_t := X_t(\sigma)$$

.

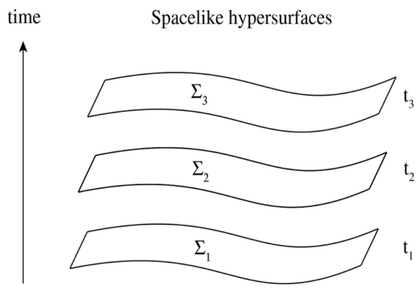

Figure 1: Space time slices [1]

Using the $(-, +, +, +)$ convention, we set up a direction of time n^μ orthogonal to the Σ_t and with $n^\mu n^\nu g_{\mu\nu} = -1$ to assure, that we have indeed used a suitable time coordinate.

We express the deformation of the hypersurfaces by

$$T^\mu(X) := \left(\frac{\partial X^\mu(t,x)}{\partial t}\right)\Bigg|_{X=X(x,t)} = N(x)n^\mu(X) + N^\mu(X) \tag{4}$$

where n^μ is orthogonal like discussed earlier and N^μ is tangential and called the "shift vector field". N is the "lapse function", which we force to be positive everywhere. We see that

$$f(X) = t = const.$$

defines Σ_t and deduce

$$0 = lim_{\epsilon\to 0}\frac{f(X_t(x+\epsilon b)) - f(X_t(x))}{\epsilon} = b^a X^\mu_{,a} f_{,\mu}$$

$\forall b$ tangetial vectors in x. It follows that n^μ is proportional to an exact one form:

$$n_\mu = F f_{,\mu}$$

or

$$n = n_\mu dX^\mu = F df$$

We will use this result later in 8

3 New Tensors

Now we are able to set up a new metric:

$$q_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu \tag{5}$$

This metric has the important property to give our time direction n^μ length zero:

$$\begin{aligned} n^\mu q_{\mu\nu} &= n^\mu g_{\mu\nu} + n^\mu n_\mu n_\nu \\ &= n_\nu - n_\nu \\ &= 0 \end{aligned} \tag{6}$$

We also define a new curvature

$$K_{\mu\nu} := q^\sigma_\mu q^\rho_\nu \nabla_\sigma n_\rho \tag{7}$$

which, because of the qs in the definition also gives zero when contracted with n^μ. It is furthermore symmetric:

$$\begin{aligned} K_{[\mu\nu]} &= q^\rho_\mu q^\sigma_\nu \nabla_{[\rho} n_{\sigma]} \\ &= q^\rho_\mu q^\sigma_\nu \nabla_{[\rho} F f_{,\sigma]} \\ &= q^\rho_\mu q^\sigma_\nu ((\nabla_{[\rho} F) f_{,\sigma]} + F(\nabla_{[\rho}\nabla_{\sigma]} f)) \\ &= q^\rho_\mu q^\sigma_\nu ((\nabla_{[\rho} e^{ln(F)}) f_{,\sigma]} + 0 \\ &= q^\rho_\mu q^\sigma_\nu (\nabla_{[\rho} ln(F)) e^{ln(F)} f_{,\sigma]} \\ &= q^\rho_\mu q^\sigma_\nu n_{[\sigma} \nabla_{\rho]} ln(F) \\ &= 0 \end{aligned} \tag{8}$$

where in the last step we made use of the fact that n^μ and $q_{\mu\nu}$ contract to zero 6. We can thus use a formulation of the Lie-derivative for a metric compatible covariant derivative ∇ and see that

$$2K_{\mu\nu} = 2K_{(\mu\nu)} = (\nabla_\mu n_\nu) + (\nabla_\nu n_\mu) = (\mathcal{L}_n q)_{\mu\nu} = \frac{1}{N}(\mathcal{L}_{T-N} q)_{\mu\nu} \tag{9}$$

4 New Derivative

The metric $q_\mu \nu$ is non-degenerate as a bijection between purely spacial tensors. We would like to have a covariant differential, that means
1) $\nabla g = 0$ (metric compatibility)
2) $\nabla_{[\mu}\nabla_{\nu]} f = 0 \quad \forall f \in C^\infty$ (torsion free)
These conditions fix a unique Christoffel-connection with

$$\nabla_\mu u_\nu = \partial_\mu u_\nu - \Gamma^\rho_{\mu\nu} u_\rho$$

We now would like to get covariant differential for metric tensors that is compatible with q, so
1) $D_\mu q_{\nu\rho} = 0$
2) $D_{[\mu} D_{\nu]} f = 0$ for scalars f
We can easily see that the following differential fulfils these properties:

$$\begin{aligned} D_\mu f &:= q^\nu_\mu \nabla_\nu \tilde{f} \\ D_\mu u_\nu &:= q^\rho_\mu q^\sigma_\nu \nabla_\rho \tilde{u}_\sigma \quad \text{for} \quad u_\mu n^\mu = 0 \end{aligned} \tag{10}$$

where $\tilde{f}$ and $\tilde{u}$ are smooth extensions of f and u on $\mathcal{M}$. These extensions can have an arbitrary form as q projects everything not tangential to Σ_t to zero. We will therefore leave the tilde out again.

5 Riemann Tensor and Codacci Equation

Now that we have a proper derivative, we would now like to know what the Riemann curvature tensor $R^{(3)\ \sigma}_{\mu\nu\rho}$ will look like in 3 dimensions. In order to get that, we will have to do some work beforehand:

$$\begin{aligned} D_\mu D_\nu u_\rho &= q^{\mu'}_\mu q^{\nu'}_\nu q^{\rho'}_\rho \nabla_{\mu'} D_{\nu'} u_{\rho'} \\ &= q^{\mu'}_\mu q^{\nu'}_\nu q^{\rho'}_\rho \nabla_{\mu'} q^{\nu''}_{\nu'} q^{\rho''}_{\rho'} \nabla_{\nu''} u_{\rho''} \end{aligned} \tag{11}$$

In a second we will continue this calculation, but need to realize a few facts before that. Since we want the 3-dimensional curvature tensor now, u_ρ is purely spacial, i.e.

$$u_\rho n^\rho = 0 \tag{12}$$

we also keep in mind that, every time n^μ contracts with a q^ν_μ, we get zero as well. The terms that vanish instantly according to one of these two rules will be marked ~~like this~~ when they appear and be left out on the next step.

Another fact that should be kept in mind is the covariant derivative ∇_μ being metric-compatible with respect to g so

$$\begin{aligned}\nabla_\rho q_{\mu\nu} &= \nabla_\rho(g_{\mu\nu} + n_\mu n\nu)\\ &= 0 + \nabla_\rho(n_\mu n_\nu)\\ &= n_\nu(\nabla_\rho n_\mu) + n_\mu(\nabla_\rho n_\nu)\end{aligned} \tag{13}$$

The following will be a little cumbersome and tedious but I found the explanations in [2] very unsatisfying and did not want to believe them until I had seen the explicit details at least once. So here you are.

Let us now start with the calculations:

$$\begin{aligned}D_\mu D_\nu u_\rho &= q_\mu^{\mu'} q_\nu^{\nu'} q_\rho^{\rho'} \nabla_{\nu'}(q_{\nu'}^{\nu''} q_{\rho'}^{\rho''} \nabla_{\nu''} u_{\rho''})\\ &= q_\mu^{\mu'} q_\nu^{\nu'} q_\rho^{\rho'} \nabla_{\nu'}(q_{\nu'}^{\nu''}(\nabla_{\nu''} u_{\rho'}) - q_{\nu'}^{\nu''} u_{\rho''} \nabla_{\nu''} q_{\rho'}^{\rho''})\\ &= q_\mu^{\mu'} q_\nu^{\nu'} q_\rho^{\rho'} \Big((\nabla_{\mu'} q_{\nu'}^{\nu''})(\nabla_{\nu''} u_{\rho'}) + q_{\nu'}^{\nu''}(\nabla_{\mu'}\nabla_{\nu''} u_{\rho'}) - u_{\rho''}(\nabla_{\mu'} q_{\nu'}^{\nu''})(\nabla_{\nu''} q_{\rho'}^{\rho''})\\ &\qquad - q_{\nu'}^{\nu''}(\nabla_{\mu'} u_{\rho''})(\nabla_{\nu''} q_{\rho'}^{\rho''}) - q_{\nu'}^{\nu''} u_{\rho''}(\nabla_{\mu'}\nabla_{\nu''} q_{\rho'}^{\rho''}) \Big)\\ &= q_\mu^{\mu'} q_\nu^{\nu'} q_\rho^{\rho'} \Big(\big(\cancel{n_{\nu'}(\nabla_{\mu'} n^{\nu''})} + n^{\nu''}(\nabla_{\mu'} n_{\nu'})\big)(\nabla_{\nu''} u_{\rho'}) + q_{\nu'}^{\nu''}(\nabla_{\mu'}\nabla_{\nu''} u_{\rho'})\\ &\qquad - u_{\rho''}\big(\cancel{n_{\nu'}(\nabla_{\mu'} n^{\nu''})} + n^{\nu''}(\nabla_{\mu'} n_{\nu'})\big)\big(\cancel{n_{\rho'}(\nabla_{\nu''} n^{\rho''})} + \cancel{n^{\rho''}(\nabla_{\nu''} n_{\rho'})}\big)\\ &\qquad - q_{\nu'}^{\nu''}\big((\nabla_{\mu'} u_{\rho''})(\cancel{n_{\rho'}(\nabla_{\nu''} n^{\rho''})} + n^{\rho''}(\nabla_{\nu''} n_{\rho'}))\big)\\ &\qquad - q_{\nu'}^{\nu''} u_{\rho''}\big(\cancel{n_{\rho'}(\nabla_{\mu'}\nabla_{\nu''} n^{\rho''})} + \cancel{n^{\rho''}(\nabla_{\mu'}\nabla_{\nu''} n_{\rho'})}\\ &\qquad\qquad + (\nabla_{\mu'} n_{\rho'})(\nabla_{\nu''} n^{\rho''}) + (\nabla_{\mu'} n^{\rho''})(\nabla_{\nu''} n_{\rho'})\big)\Big)\\ &= q_\mu^{\mu'} q_\nu^{\nu'} q_\rho^{\rho'} \Big((\nabla_{\mu'} n_{\nu'})(\nabla_n u_{\rho'}) + q_{\nu'}^{\nu''}(\nabla_{\mu'}\nabla_{\mu'} u_{\rho'}) - q_{\nu'}^{\nu''}(\nabla_{\mu'} u_{\rho''}) n^{\rho''}(\nabla_{\nu''} n_{\rho'})\\ &\qquad - q_{\nu'}^{\nu''} u_{\rho''}\big((\nabla_{\mu'} n_{\rho'})(\nabla_{\nu''} n^{\rho''}) + (\nabla_{\mu'} n^{\rho''})(\nabla_{\nu''} n_{\rho'})\big)\Big)\end{aligned} \tag{14}$$

$\nabla_n u_\rho$ is a notation for $n^\nu \nabla_\nu u_\rho$ in case this questions arises.

We will now antisymmetrize this result in μ and ν to get the desired curvature tensor. For the very last line, we see that it is symmetrical in ν'' and μ' this symmetry gets carried to μ and ν by the metrics q so the term will disappear instantly when we perform the antisymmetrization.

We again make use of the fact 12 by taking the derivative, which still needs to be 0:

$$\begin{aligned}\nabla_\mu(u_\rho n^\rho) = (\nabla_\mu u_\rho) n^\rho + (\nabla_\mu n^\rho) u_\rho = 0\\ \Rightarrow n^\rho(\nabla_\mu u_\rho) = -u_\rho(\nabla_\mu n^\mu)\end{aligned} \tag{15}$$

Apart from that we will use the definition of the extrinsic curvature 7 and will make use of the fact that it is symmetrical 8.

$$
\begin{aligned}
R^{(3)\ \sigma}_{\mu\nu\rho} u_\sigma &= 2D_{[\mu} D_{\nu]} u_\rho \\
&= 2 q^{\mu'}_{[\mu} q^{\nu'}_{\nu]} q^{\rho'}_\rho \left((\nabla_{\mu'} n_{\nu'})(\nabla_n u_{\rho'}) + q^{\nu''}_{\nu'} (\nabla_{\mu'} \nabla_{\nu''} u_{\rho'}) + q^{\nu''}_{\nu'} u_{\rho''} (\nabla_{\mu'} n^{\rho''})(\nabla_{\nu''} n_{\rho'}) \right) \\
&= 2 \left(K_{[\mu\nu]} q^{\rho'}_\rho (\nabla_n u_{\rho'}) + q^{\mu'}_{[\mu} q^{\nu''}_{\nu]} (\nabla_{\mu'} \nabla_{\nu''} u_{\rho'}) q^{\rho'}_\rho + q^{\mu'}_{[\mu} q^{\nu''}_{\nu]} u_\sigma q^\sigma_{\rho''} (\nabla_{\mu'} n^{\rho''}) K_{\nu'\rho} \right) \\
&= 0 + q^{\mu'}_\mu q^{\nu'}_\nu q^{\rho'}_\rho (2\nabla_{[\mu'} \nabla_{\nu']} u_{\rho'}) + 2 u_\sigma q^\sigma_{\rho''} (\nabla_{\mu'} n^{\rho''}) q^{\mu'}_{[\mu} K_{\nu]\rho} \\
&= (q^{\mu'}_\mu q^{\nu'}_\nu q^{\rho'}_\rho q^\sigma_{\sigma'} R^{(4)\ \sigma'}_{\mu'\nu'\rho'} - 2K_{\rho[\mu} K^{\ \sigma}_{\nu]}) u_\sigma
\end{aligned} \tag{16}
$$

This formula is called the **Gauss equation**.
Alternative form:

$$
R^{(3)}_{\mu\nu\rho\sigma} = -2K_{\rho[\mu} K_{\nu]\sigma} + q^{\mu'}_\mu q^{\nu'}_\nu q^{\rho'}_\rho q^{\sigma'}_\sigma R^{(4)}_{\mu'\nu'\rho'\sigma'} \tag{17}
$$

We further calculate

$$
\begin{aligned}
R^{(3)} &= R^{(3)}_{\mu\nu\rho\sigma} q^{\mu\rho} q^{\nu\rho} \\
&= -(K_{\rho\mu} K_{\mu\sigma} q^{\mu\rho} q^{\nu\rho} - K_{\rho\nu} K_{\mu\sigma} q^{\mu\rho} q^{\nu\rho}) + q^{\mu\rho} q^{\nu\rho} R^{(4)}_{\mu\nu\rho\sigma} \\
&= -(K^2 - K^{\mu\sigma} K_{\mu\sigma}) + q^{\mu\rho} q^{\nu\rho} R^{(4)}_{\mu\nu\rho\sigma}
\end{aligned} \tag{18}
$$

where $K = K^\mu_\mu$.
We further remember the definition of $q_{\mu\nu}$ 5 and also $R\mu\nu\rho\sigma^{(4)} n^\sigma = 2\nabla_{[\mu} \nabla_{\nu]} n_\rho$ to get

$$
\begin{aligned}
R^{(4)} &= R^{(4)}_{\mu\nu\rho\sigma} g^{\mu\rho} g^{\nu\sigma} \\
&= R^{(4)}_{\mu\nu\rho\sigma} (q^{\mu\rho} q^{\nu\sigma} - q^{\mu\rho} n^\nu n^\sigma - q^{\nu\sigma} n^\rho n^\mu + n^\rho n^\mu n^\nu n^\sigma) \\
&= R^{(4)}_{\mu\nu\rho\sigma} q^{\mu\rho} q^{\nu\sigma} - 2q^{\mu\rho} n^\nu \nabla_{[\mu} \nabla_{\nu]} n_\rho + \cancel{2n^\rho n^\mu n^\nu \nabla_{[\mu} \nabla_{\nu]} n_\rho} \\
&= R^{(4)}_{\mu\nu\rho\sigma} q^{\mu\rho} q^{\nu\sigma} - 2(g^{\rho\mu} n^\nu + \cancel{n^\rho n^\mu n^\nu}) \nabla_{[\mu} \nabla_{\nu]} n_\rho \\
&= R^{(4)}_{\mu\nu\rho\sigma} q^{\mu\rho} q^{\nu\sigma} - 2n^\nu \nabla_{[\mu} \nabla_{\nu]} n^\mu
\end{aligned} \tag{19}
$$

The crossed out terms vanish because we have an antisymmetrization of a symmetric expression but apart from that we just plug in the identities mentioned above.
We can put 18 in 19 and get

$$
R^{(4)} = R^{(3)} + (K^2 - K^{\mu\sigma} K_{\mu\sigma}) - 2n^\nu \nabla_{[\mu} \nabla_{\nu]} n^\mu \tag{20}
$$

To deal with the last term, we have some more work to do:

$$
\begin{aligned}
0 &= \nabla_\mu \nabla_\nu (n^\mu n^\nu) - \nabla_\nu \nabla_\mu (n^\mu n^\nu) \\
&= \nabla_\mu (n^\nu \nabla_\nu n^\nu + n^\mu \nabla_\nu n^\nu) - \nabla_\nu (n^\mu \nabla_\mu n^\nu + n^\nu \nabla_\mu n^\mu) \\
&= \nabla_\mu (n^\nu \nabla_\nu n^\mu) + (\nabla_\mu n^\mu)(\nabla_\nu n^\nu) + n^\mu \nabla_\mu \nabla_\nu n^\nu \\
&\quad - \nabla_\nu (n^\nu \nabla_\mu n^\mu) - (\nabla_\nu n^\mu)(\nabla_\mu n^\nu) - n^\mu \nabla_\nu \nabla_\mu n^\nu \\
&= \nabla_\mu (n^\nu \nabla_\nu n^\mu - n^\mu \nabla_\nu n^\nu) + (\nabla_\mu n^\mu)^2 - (\nabla_\nu n^\mu)(\nabla_\mu n^\nu) \\
&\quad - n^\nu (\nabla_{[\mu} \nabla_{\nu]} n^\mu)
\end{aligned} \tag{21}
$$

$$
\Rightarrow \qquad n^\nu \nabla_{[\mu} \nabla_{\nu]} n^\mu = \nabla_\mu (n^\nu \nabla_\nu n^\mu - n^\mu \nabla_\nu n^\nu) + (\nabla_\mu n^\mu)^2 - (\nabla_\nu n^\mu)(\nabla_\mu n^\nu) \tag{22}
$$

Almost done, we also need

$$
\begin{aligned}
0 &= \nabla_\mu(-1) \\
&= \nabla_\mu(n_\nu n^\nu) \\
&= n_\nu(\nabla_\mu n^\nu) + n^\nu(\nabla_\mu n_\nu) \\
&= n_\nu(\nabla_\mu n^\nu) + n^\nu(\nabla_\mu g_{\nu\sigma} n^\sigma) \\
&= n_\nu(\nabla_\mu n^\nu) + n_\sigma(\nabla_\mu n^\sigma) \\
&= 2n_\nu(\nabla_\mu n^\nu)
\end{aligned}
\tag{23}
$$

to identify

$$
\begin{aligned}
\nabla_\mu n^\mu &= g_\mu^\nu \nabla_\nu n^\mu + 0 \\
&= g_\mu^\nu \nabla_\nu n^\mu + n_\mu n^\nu \nabla_\nu n^\mu \\
&= q_\mu^\nu \nabla_\nu n^\mu \\
&= q_\mu^\rho q_\rho^\nu \nabla_\nu n^\mu \\
&= K_\rho^\rho \\
&= K
\end{aligned}
\tag{24}
$$

and in a similar manner

$$
(\nabla_\mu n^\nu)(\nabla_\nu n^\mu) = q^{\nu\sigma} q^{\rho\mu}(\nabla_\mu n_\sigma)(\nabla_\nu n_\rho) = K_{\mu\nu}K^{\mu\nu} \tag{25}
$$

Now we are ready to write down our final result: We remember 20, plug in 22 and 24 and 25 and get:

$$
\begin{aligned}
R^{(4)} &= R^{(3)} + (K^2 - K^{\mu\sigma}K_{\mu\sigma}) - 2n^\nu \nabla_{[\mu}\nabla_{\nu]} n^\mu \\
&= R^{(3)} + (K^2 - K^{\mu\sigma}K_{\mu\sigma}) - 2(\nabla_\mu(n^\nu \nabla_\nu n^\mu - n^\mu \nabla_\nu n^\nu) + (\nabla_\mu n^\mu)^2 - (\nabla_\nu n^\mu)(\nabla_\mu n^\nu) \\
&= R^{(3)} + (K^2 - K^{\mu\sigma}K_{\mu\sigma}) - 2\big(\nabla_\mu(n^\nu \nabla_\nu n^\mu - n^\mu \nabla_\nu n^\nu) + K^2 - K_{\mu\nu}K^{\mu\nu}\big) \\
&= R^{(3)} + (K_{\mu\nu}K^{\mu\nu} - K^2) + \nabla_\mu(n^\nu \nabla_\nu n^\mu - n^\mu \nabla_\nu n^\nu)
\end{aligned}
\tag{26}
$$

This is the famous **Codacci equation**.

But why did we have to put in all this work? The Ricci scalar is a central part of the **Einstein-Hilbert action**

$$
S = \frac{1}{2\kappa}\int \sqrt{-g}R \; d^4x \tag{27}
$$

whose variation gives us the Einstein field equations

$$
R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu} \tag{28}
$$

defining the theory of General Relativity.

6 Pull-back to 3-dimensional Space

The next step is to pull back our quantities to our 3-dimensional slice σ. We will do so with 3-dimensional spacial vector fields on Σ_t. They are defined by

$$
X_a^\mu(X) := X_{,a}^\mu(x,t)\big|_{X(x,t)=X} \tag{29}
$$

Because these vector fields are spatial, $n_\mu X^\mu_a = 0$. We can therefore canonically define:

$$q_{ab}(t,x) := (X^\mu_{,a} X^\nu_{,b} q_{\mu\nu})\big(X(x,t)\big) = g_{\mu\nu}\big(X(x,t)\big) X^\mu_{,a}(t,x) X^\nu_{,b}(t,x) \tag{30}$$

and

$$K_{ab}(t,x) := (X^\mu_{,a} X^\nu_{,b} K_{\mu\nu})\big(X(x,t)\big) = (X^\mu_{,a} X^\nu_{,b} \nabla_\mu n_\nu)(t,x) \tag{31}$$

The latter can be expressed in a different more convenient way. It turns out that actually

$$K_{ab} = \frac{1}{2N}\big(\dot{q}_{ab} - (\mathcal{L}_{\vec{N}} q)_{ab}\big)(x,t) \tag{32}$$

However, to prove this we need some more tools and identities, so here we can only refer to 42.

We furthermore express

$$\begin{aligned} q^{ab} &= \epsilon^{aa_1a_2}\epsilon^{bb_1b_2}\frac{q_{a_1b_1}q_{a_2b_2}}{2det(q)} \\ q^{\mu\nu}(X) &= \big(q^{ab}(x,t) X^\mu_{,a} X^\nu_{,b}\big)(x,t) \\ q^\nu_\mu(X) &= g_{\mu\rho}(X) q^{\rho\nu}(X) \\ q_{\mu\nu}(X) &= g_{\nu\rho} q^\rho_\mu \end{aligned} \tag{33}$$

and define

$$N(x,t) := N(X(x,t)), \qquad \vec{N}^a(x,t) := q^{ab}(x,t)(X^\mu_b g_{\mu\nu} N^\nu)\big(X(x,t)\big) \tag{34}$$

which is all really nothing conceptually new and about what we would expect. We essentially try to express all our important quantities in terms of our space coordinates in one slice at a given time.

We can use 33 to see that we can write

$$\begin{aligned} K(x,t) &= (q^{\mu\nu} K_{\mu\nu})\big(X(x,t)\big) = (q^{ab} X^\mu_{,a} X^\nu_{,b} K_{\mu\nu})(x,t) = (q^{ab} K_{ab})(x,t) \\ (K_{\mu\nu} K^{\mu\nu})(x,t) &= (K_{\mu\nu} K_{\rho\sigma} q^{\mu\rho} q^{\nu\sigma})\big(X(x,t)\big) = (K_{ab} K_{cd} q^{ac} q^{bd})(x,t) \end{aligned} \tag{35}$$

We can also pull back the 3D curvature tensor and scalar

$$\begin{aligned} R^{(3)}_{abcd}(x,t) &= (X^\mu_a X^\nu_b X^\rho_c X^\sigma_d R_{\mu\nu\rho\sigma})\big(X(x,t)\big) \\ R^{(3)}(x,t) &= (R^{(3)}_{\mu\nu\rho\sigma} q^{\mu\rho} q^{\nu\sigma})\big(X(x,t)\big) = (R^{(3)}_{abcd} q^{ac} q^{bd})(x,t) \end{aligned} \tag{36}$$

Which is consistent if we define

$$\begin{aligned} (D_a f)(x,t) := \partial_a f\big(X(x,t)\big) &= \left(\frac{\partial X^\mu}{\partial a}\frac{\partial f}{\partial X^\mu}\right)\big(X(x,t)\big) = (X^\mu_a D_\mu f)\big(X(x,t)\big) \\ u_a(x,t) &:= (X^\mu_a u_\mu)\big(X(x,t)\big) \end{aligned} \tag{37}$$

All these definitions seem a little tedious but enable us to work with the metric q_{ab} just as we are used to and do anything in 3 dimensions with latin indices.

7 Explicit form of the metrics

We have the definition 5 for the metric but it is kind of abstract. We wish to be able to go straight from one to the other and maybe even get N and $\vec{N}$ on the go. Therefore we calculate, keeping in mind 4 and 30

$$\begin{aligned} ds^2 &= g_{\mu\nu} dX^\mu \otimes dX^\nu \\ &= g_{\mu\nu}[X^\mu_{,t}dt + X^\mu_{,a}dx^a] \otimes [X^\nu_{,t}dt + X^\nu_{,b}dx^b] \\ &= g_{\mu\nu}[Nn^\mu dt + X^\mu_{,a}(dx^a + N^a dt)] \otimes [Nn^\nu dt + X^\nu_{,b}(dx^b + N^b dt] \\ &= (-N^2 + q_{ab}N^aN^b)dt \otimes dt + q_{ab}N^b(dt \otimes dx^a + dx^a \otimes dt) + q_{ab}dx^a \otimes dx^b \end{aligned} \tag{38}$$

Thus, we can write

$$(g_{\mu\nu}) = \begin{pmatrix} -N^2 + N_kN^k & N_j \\ N_i & q_{ij} \end{pmatrix} \tag{39}$$

$$(g^{\mu\nu}) = \begin{pmatrix} -\frac{1}{N^2} & \frac{N^j}{N^2} \\ \frac{N^i}{N^2} & q^{ij} - \frac{N^iN^j}{N^2} \end{pmatrix} \tag{40}$$

and from 38 we clearly see

$$\begin{aligned} (\partial_t)^\mu &= Nn^\mu + N^\mu \\ \Rightarrow n^\mu &= \frac{(\partial_t - N)^\mu}{N} = \left(\frac{1}{N}, -\frac{N^i}{N}\right) \\ n_\mu &= \left(-\frac{N^2 + N_kN^k}{N} - \frac{N_jN^j}{N}, \frac{N_i}{N} - \frac{N_i}{N}\right) = (-N, \vec{0}) \end{aligned} \tag{41}$$

We use this to finally proof 32:

$$\begin{aligned} K_{ab} :&= X^\mu_{,a}X^\nu_{,b}K_{\mu\nu} \\ &= X^\mu_{,a}X^\nu_{,b}\nabla_\mu n_\nu \\ &= \nabla_a n_b \\ &= \partial_a n_b - \Gamma^\mu_{ab}n_\mu \\ &= -\Gamma^\mu_{ab}n_\mu \\ &= N\Gamma^0_{ab} \\ &= \frac{N}{2}g^{0\mu}(\partial_a g_{b\mu} + \partial_b g_{\mu a} - \partial_\mu g_{ab} \\ &= \frac{N}{2}\left(g^{00}(\partial_a g_{b0} + \partial_b g_{0a} - \partial_t g_{ab}) + g^{0k}(\partial_a g_{bk} + \partial_b g_{ka} - \partial_k g_{ab})\right) \\ &= \frac{N}{2}\left(-\frac{1}{N^2}(\partial_a N_b + \partial_b N_a - \partial_t q_{ab}) + \frac{N^k}{N^2}(\partial_a q_{bk} + \partial_b q_{ka} - \partial_k q_{ab})\right) \\ &= \frac{1}{2N}\left(\dot{q}_{ab} - \partial_a N_b - \partial_b N_a + N_l q^{lk}(\partial_a q_{bk} + \partial_b q_{ka} - \partial_k q_{ab})\right) \\ &= \frac{1}{2N}(\dot{q}_{ab} - \partial_a N_b - \partial_b N_a + 2N_l\Gamma^l_{ab}{}^{(3)}) \\ &= \frac{1}{2N}(\dot{q}_{ab} - \nabla_a N_b - \nabla_b N_a) \\ &= \frac{1}{2N}\left(\dot{q}_{ab} - (\mathcal{L}_{\vec{N}}q)_{ab}\right) \end{aligned} \tag{42}$$

Another useful relation is the one between $\det(q)$ and $\det(g)$. We remember

$$(g_{\mu\nu}) = \begin{pmatrix} -N^2 + N_k N^k & N_j \\ N_i & q_{ij} \end{pmatrix} \tag{43}$$

and use Laplace expansion to calculate the determinant:

$$\begin{aligned} \det(g) &= (-N^2 + N_k N^k)\det(q) + \epsilon^{i\mu\nu\rho} N_i g_{1\mu} g_{2\nu} g_{3\rho} \\ &\overset{(1)}{=} (-N^2 + N_k N^k)\det(q) - N^i N_i \det(q) \\ &= -N^2 \det(q) \end{aligned} \tag{44}$$

We will show what happened at (1) by calculating the second term of the sum for $i = 3$. We will again just use Laplace expansion

$$\begin{aligned} &\epsilon^{3\mu\nu\rho} N_3 g_{1\mu} g_{2\nu} g_{3\rho} \\ &= -N_3\big(N_1(q_{21}q_{32} - q_{31}q_{22}) - N_2(q_{11}q_{32} - q_{31}q_{12}) + N_3(q_{11}q_{22} - q_{21}q_{12}\big) \\ &= -N_3\big((N^1 q_{11} + N^2 q_{21} + N^3 q_{31})(q_{21}q_{32} - q_{31}q_{22}) \\ &\qquad - (N^1 q_{12} + N^2 q_{22} + N^3 q_{32})(q_{11}q_{32} - q_{31}q_{12}) \\ &\qquad + (N^1 q_{13} + N^2 q_{23} + N^3 q_{33})(q_{11}q_{22} - q_{21}q_{12})\big) \\ &\overset{(2)}{=} -N_3\big(N^1(q_{11}q_{21}q_{32} - q_{12}q_{11}q_{32} - q_{11}q_{31}q_{22} + q_{13}q_{11}q_{22} + q_{12}q_{31}q_{12} - q_{13}q_{21}q_{12}) \\ &\qquad + N^2(q_{21}q_{21}q_{32} - q_{23}q_{21}q_{12} - q_{21}q_{31}q_{22} + q_{22}q_{31}q_{12} - q_{22}q_{11}q_{32} + q_{23}q_{11}q_{22}) \\ &\qquad + N^3\big(q_{13}(q_{21}q_{32} - q_{31}q_{22}) - q_{23}(q_{11}q_{32} - q_{31}q_{12}) + q_{33}(q_{11}q_{22} - q_{21}q_{12})\big) \\ &\overset{(3)}{=} -N_3 N^3 \det(q) \end{aligned} \tag{45}$$

where in (2) we just do a convenient reordering and in (3) most of the terms just cancel (keep in mind that q is symmetrical) and exactly the terms with N^3 stay and are equal to the determinant (again Laplace expansion).
The same can be done with $i = 1, 2$ and so we get 44.

These relations for the metric and curvature will be crucial for some calculations when we dive deeper into Loop Quantum Gravity.

8 The ADM-Action

As mentioned earlier, General Relativity can be derived entirely from the Einstein-Hilbert action 27

$$S = \frac{1}{2\kappa} \int \sqrt{-g} R \, d^4x$$

We will now use the Codacci equation 26 in terms of 35 and 44 to express that in terms of our new variables:

$$S(q_{ab}, N, N^a) = \frac{1}{2\kappa} \int_{\mathbb{R}} dt \int_{\sigma} d^3x \sqrt{det(q)} |N| \big(R^{(3)} + (K_{ab}K^{ab} - (K^a_a)^2)\big) \tag{46}$$

The cautious reader might have observed that the Codacci equation also contains a term $\nabla_\mu(n^\nu \nabla_\nu n^\mu - n^\mu \nabla_\nu n^\nu)$ which we have left out. This is because the variation of the action is always invariant under

addition of derivatives. (The integral can be executed, cancels with the derivative, and we just get the function $(n^\nu \nabla_\nu n^\mu - n^\mu \nabla_\nu n^\nu)$ evaluated at the endpoints, which is constant and vanishes under variation.)
We vary this action with respect to the time derivatives of our variables (Remember 32!) and get the momenta

$$\begin{aligned} \frac{1}{\kappa} P^{ab} &= \frac{\delta S}{\delta \dot{q}_{ab}} = \sqrt{\det(q)} |N| \left(\frac{1}{N} K^{ab} - \frac{1}{N} q^{ab} K^c_c \right) \\ &= \sqrt{\det(q)} \frac{|N|}{NK} (K^{ab} - q^{ab} K^c_c) \\ \Pi &= \frac{\delta S}{\delta \dot{N}} = 0 \\ \Pi_a &= \frac{\delta S}{\delta \dot{N}^a} = 0 \end{aligned} \tag{47}$$

Here we get the first constraints

$$C(t,x) := \Pi(t,x) = 0 \qquad C^a(t,x) := \Pi^a(t,x) = 0 \tag{48}$$

These and some other constraints will play a bigger role in Loop Quantum Gravity.

Before we continue, there is some more work to do. We remember 32 and the momenta we just calculated and compute the quantities

$$\begin{aligned} \dot{q}_{ab} &= 2NK_{ab} + (\mathcal{L}_{\vec{N}} q)_{ab} \\ \dot{q}_{ab} P^{ab} &= (\mathcal{L}_{\vec{N}} q)_{ab} P^{ab} + 2\sqrt{\det(q)} |N| \big(K_{ab} K^{ab} - (K^a_a)^2 \big) \\ P_{ab} P^{ab} &= \det(q) \big(K_{ab} K^{ab} + (K^a_a)^2 \big) \\ P^2 &= (P^a_a)^2 = 4 \det(q) K^2 \end{aligned} \tag{49}$$

which we use to rewrite our action 46

$$
\begin{aligned}
2\kappa S(q_{ab}, N, N^a) &= \int_{\mathbb{R}} dt \int_\sigma d^3x \ \sqrt{\det(q)}|N|\big(R^{(3)} + (K_{ab}K^{ab} - (K^a_a)^2)\big) \\
&= \int_{\mathbb{R}} dt \int_\sigma d^3x \ \Big[\dot{q}_{ab}P^{ab} + \dot{N}\Pi + \dot{N}^a\Pi_a \\
&\qquad - \Big(\dot{q}_{ab}P^{ab} + \lambda C + \lambda^a C_a - \sqrt{\det(q)}|N|\big(R + (K_{ab}K^{ab} - K^2)\big)\Big)\Big] \\
&= \int_{\mathbb{R}} dt \int_\sigma d^3x \ \Big[\dot{q}_{ab}P^{ab} + \dot{N}\Pi + \dot{N}^a\Pi_a \\
&\qquad - \Big((\mathcal{L}_{\vec{N}}q)_{ab}P^{ab} + \lambda C + \lambda^a C_a - \sqrt{\det(q)}|N|\big(R - (K_{ab}K^{ab} - K^2)\big)\Big)\Big] \\
&= \int_{\mathbb{R}} dt \int_\sigma d^3x \ \Big[\dot{q}P^{ab} - \dot{N}\Pi + \dot{N}^a\Pi_a \\
&\qquad - \Big(\lambda C + \lambda^a C_a - 2N^a q_{ac}D_bP^{bc} \\
&\qquad + |N|\big(-\frac{1}{\sqrt{\det(q)}}(P_{ab}P^{ab} - \frac{1}{2}P^2) - \sqrt{\det(q)}R\big)\Big)\Big] \\
&= \int_{\mathbb{R}} dt \int_\sigma d^3x \ \big(\dot{q}P^{ab} - \dot{N}\Pi + \dot{N}^a\Pi_a - (\lambda C + \lambda^a C_a + N^a H_a + |N|H)\big)
\end{aligned}
\tag{50}
$$

with the obvious identifications

$$
H_a = -2q_{ac}D_bP^{bc} \qquad H = \frac{1}{\sqrt{\det(q)}}(P^{ab}P_{ab} - \frac{1}{2}P^2) - \sqrt{\det(q)}R \tag{51}
$$

These two are called the **Diffeomorphism constraint** and the **Hamiltonian constraint** and will define the physics of our setup.

Since Π, Π_a, C and C_a will be zero anyway, we drop the primary constraints C, C_a and also $\dot{N}$ and $\dot{N}^a$ and get the **ADM-action**

$$
S = \frac{1}{2\kappa}\int_{\mathbb{R}} dt \int_\sigma d^3x\big(\dot{q}_{ab}P^{ab} - (N^a H_a + |N|H)\big) \tag{52}
$$

named after Arnowitt, Deser and Misner.

References

[1] Brandon Mattingly. *Curvature Invariants for Wormholes and Warped Spacetimes.* Mar. 2021.

[2] Thomas Thiemann. *Introduction to Modern Canonical Quantum General Relativity.* 2001. arXiv: `gr-qc/0110034 [gr-qc]`. URL: `https://arxiv.org/abs/gr-qc/0110034`.