MATH 223

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0.1 Administrativa

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Grading

Assignments	15%	15%
Midterm	25%	0%
Final	60%	85%

The midterm will be scheduled for the 7th week of class.

Chapter 1

Vectors

1.1 Vectors in $\mathbb{R}^n$

$\mathbb{R}^n$ is the set of all n-tuples of real numbers $u = (a_1 ... a_n) \mid a \in \mathbb{R}$ where a are the **components** or **entries**.

Remark 1. We use the term **scalar** to refer to an element in $\mathbb{R}$.

1.2 Basic Definitions

Definition 1. Addition
$u, v \in \mathbb{R}^n$
$u = (a_1 ... a_n)$
$v = (b_1 ... b_n)$
$u + v = (a_1 + b_1 ... a_n + b_n)$

Definition 2. Scalar Multiplication
$k \in \mathbb{R}$
$ku = (ka_1 ... ka_n)$

Definition 3. Two vectors u and v are said to be **equal** ($u = v$) if
$a_i = b_i \forall i = 1...n$.

Definition 4. The **zero vector** is defined as $0 = (0...0)$.

Definition 5. Suppose we are given m vectors $u_1 ... u_m \in \mathbb{R}^n$ and m scalars $k_1 ... k_m \in \mathbb{R}$.
Let $u = k_1 u_1 + ... + k_m u_m$.
Such a vector u is called a **linear combination** of the vectors $u_1 ... u_m$.

Definition 6. A vector u can be called a **multiple** of v if there is a scalar k such that $u = kv$ with $k \neq 0$. In the case $k > 0$ we say u is in the same direction as v. In the case $k < 0$ we say u is in the opposite direction of v.

1.3 The Dot Product

Definition 7. Let $u = (a_1...a_n)$ and $v = (b_1...b_n)$. The **dot product** or inner product is given by,

$$u \cdot v = a_1b_1 + ...a_nb_n =$$

Definition 8. The vectors u and v are **orthogonal** if $u \cdot v = 0$.

1.4 The Vector Norm

Definition 9. The **norm** or **length** of a vector is given by,

$$\|u\| = \sqrt{a_1^2 + ... + a_n^2}$$

Thus $\|u\| \geq 0$ and $\|u\| = 0$ if and only if (iff) $u = 0$.

Definition 10. A vector is called a **unit vector** if $\|u\| = 1$.

Definition 11. For any non-zero vector v, the vector

$$\hat{v} = \frac{1}{\|v\|}v$$

is the only unit vector with the same direction of v. The process of finding $\hat{v}$ is called **normalizing**.

1.5 Theorem: Cauchy-Schwarz Inequality

Theorem 1. *Given any two vectors $u, v \in \mathbb{R}^n$, then,*

$$|u \cdot v| \leq \|u\|\|v\|$$

Proof. Let $t \in \mathbb{R}$. So, $\|tu + v\|^2 \geq 0$.

$$\begin{aligned}
\|tu + v\|^2 &= (tu + v)(tu + v) \\
&= (tu \cdot tu) + (tu \cdot v) + (v \cdot tu) + (v \cdot v) \\
&= t^2(u \cdot u) + t(v \cdot u) + t(u \cdot v) + (v \cdot v) \\
&= t^2\|u\|^2 + 2t(u \cdot v) + \|v\|^2
\end{aligned}$$

We can represent this in the form $at^2 + bt + c \geq 0$, so,

$$a = \|u\|^2, b = 2(u \cdot v), c = \|v\|^2$$

Take the Discriminant as $b^2 - 4ac \iff b^2 \leq 4ac$.

$$\begin{gathered}
4(u \cdot v)^2 \leq 4\|u\|^2\|v\|^2 \\
|u \cdot v| \leq \|u\|\|v\|
\end{gathered}$$

□

1.6 Theorem: Minkowski Triangle Inequality

Theorem 2. *Given* $u, v \in \mathbb{R}^n$*, then* $\|u+v\| \leq \|u\| + \|v\|$*.*

Proof.

$$\begin{aligned} \|u+v\|^2 &= \|u\|^2 + 2(u \cdot v) + \|v\|^2 \\ &\leq \|u\|^2 + 2\|u\|\|v\| + \|v\|^2 \quad \text{by C-S inequality} \\ &= (\|u\| + \|v\|)^2 \end{aligned}$$

So, $\|u+v\|^2 \leq (\|u\| + \|v\|)^2$. Take the square root and we are done. □

1.7 Geometry with Vectors

Definition 12. The **distance** between vectors $u, v \in \mathbb{R}^n$ is given by,

$$d(u,v) = \|u - v\| = \sqrt{(a_1 - b_1)^2 + ... + (a_n - b_n)^2}$$

Definition 13. The **angle** between vectors $u, v \in \mathbb{R}^n$ is given by,

$$cos\theta = \frac{u \cdot v}{\|u\|\|v\|} \quad \theta \in [0, \pi]$$

Observe that in the previous definition, the angle is well defined.

$$-\|u\|\|v\| \leq -|u \cdot v| \leq u \cdot v \leq u \cdot v \leq |u \cdot v| \leq \|u\|\|v\|$$

Dividing the entire inequality by $\|u\|\|v\|$ yields,

$$-1 \leq \frac{u \cdot v}{\|u\|\|v\|} \leq 1$$

Definition 14. A **hyperplane** $\mathcal{H}$ in $\mathbb{R}^n$ is the set of points $(x_1...x_n)$ that satisfy $a_1x_1 + ... + a_nx_n = b$ where $u = [a_1...a_n] \in \mathbb{R}^n$ and $b \in \mathbb{R}$.

Definition 15. The **line** in $\mathbb{R}^n$ passing through a point $P = (b_1...b_n)$ and in the direction of $v \in \mathbb{R}^n$ with $v \neq 0$.

$$x = P + tu \quad t \in \mathbb{R}, \quad u = [a_1...a_n]$$

$$\begin{cases} x_1 = a_1t + b_1 \\ x_n = a_nt + b_n \end{cases}$$

Chapter 2

Algebra of Matrices

2.1 Introduction

A matrix with n rows and m columns is written as,

$$A_{n\times m} = \begin{bmatrix} a_{11} & \dots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nm} \end{bmatrix}$$

Or,

$$A_{n\times m} = [a_{ij}]$$

Where a_{ij} is the entry in row i and column j.

2.2 Definitions and Properties of Matrices

Definition 16. Matrix Addition

$$A + B = [a_{ij} + b_{ij}] \quad \forall i = 1...n, j = 1...m$$

Definition 17. Scalar Multiplication

$$ka = [ka_{ij}] \quad \forall i = 1...n, j = 1...m$$

Definition 18. Zero Matrix

$$0 = [0]$$

Definition 19. Given a matrix $A_{m\times p}$ and a matrix $B_{p\times n}$, **matrix multiplication** is defined as,

$$AB = [c_{ij}] \quad c_{ij} = \sum_{k=1}^{p} a_{ik}b_{kj}$$

Definition 20. Given a matrix A, its **transpose** is $A^T = [a_{ji}]$ where $A = [a_{ij}]$.

Definition 21. A **square matrix** has the same number of rows as it does columns, i.e. $A_{n \times n}$ is a square matrix.

Definition 22. Given a matrix $A = [a_{ij}]$ the elements in the **diagonal** are $[a_{11}, ..., a_{nn}]$.

Definition 23. The **trace** of a matrix A is given by,

$$tr(A) = a_{11} + ... + a_{nn}$$

Definition 24. The **identity matrix** I_n is the matrix such that for any n-square matrix A,

$$AI = IA = A$$

Definition 25. The **Kronecker delta** is defined by,

$$\delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

Remark 2. Given the definitions for the identity matrix and the Kronecker delta, an alternative definition for the identity matrix is as follows,

$$I = [\delta_{ij}]$$

Definition 26. A matrix A is **invertible** if there is a matrix B such that $AB = BA = I$.

Remark 3. In general, for any matrices A and B, $AB \neq BA$.

Definition 27. A matrix D is **diagonal** if all the non-zero entries are in the diagonal.

$$D = diaginal(d_1, ..., d_n)$$

Definition 28. A matrix A is **upper triangular** if,

$$a_{ij} = 0 \quad \forall i > j$$

2.3 Complex Numbers

The imaginary number i is defined as $i = \sqrt{-1}$ or equivalently, $i^2 = -1$.

Definition 29. A **complex number** z is given by,

$$z = a + bi \quad a, b \in \mathbb{R}$$

Where a is the real part and b is the imaginary part.

Real numbers are also complex numbers with no imaginary component, i.e. $a + 0i = a$.

Addition for two complex numbers $z = a + bi$ and $w = c + di$ is given by,

$$z + w = (a + c) + (b + d)i$$

Multiplication for the same two complex numbers is given by,

$$\begin{aligned} z \cdot w &= (a + bi)(c + di) \\ &= ac + adi + cbi - bd \\ &= (ac - bd) + (ad + bc)i \end{aligned}$$

Definition 30. The **conjugate** of $z = a + bi$ is $\bar{z} = a - bi$.

Definition 31. The **absolute value** or modulus of $z = a + bi$ is $|z| = \sqrt{a^2 + b^2}$.

Example 1.

$$z^{-1} = \frac{1}{z} = \frac{1}{a + bi} \cdot \frac{a - bi}{a - bi} = \frac{a - bi}{a^2 + b^2}$$

Observe that the following properties are true for conjugates and absolute values,

1. $$z\bar{z} = |z|^2 = a^2 + b^2$$
2. $$\overline{z \pm w} = \bar{z} \pm \bar{w}$$
3. $$\overline{zw} = \bar{z} \cdot \bar{w}$$
4. $$\overline{(\bar{z})} = z$$
5. z is real iff $z = \bar{z}$
6. $$|zw| = |z||w|$$
7. $$|z + w| \le |z| + |w|$$

Chapter 3

Systems of Linear Equations

3.1 Representing Linear Systems with Matrices

Given a system of linear equations of the form,

$$\begin{cases} a_{11}x_1 + ... + a_{1m}x_n = b_1 \\ \vdots \\ a_{m1}x_1 + ... + a_{mn}x_n = b_m \end{cases}$$

- $x_1, ..., x_m$ are the unknowns, and
- a_{ij} and b_i are the constants.

The system can also be represented by matrices where,

- $A = [a_{ij}]$ is the matrix of coefficients
- $b = [b_i]$ is the column vector of constant
- $M = [A|b]$ is the matrix that represents the system.

Definition 32. A matrix A is in **echelon form** if

1. all zero rows are at the bottom, and
2. each leading non-zero entry in a row is to the right of the leading non-zero entry in teh preceding row.

Example 2. This matrix is in echelon form. The boxed entries are the **pivots**.

$$A = \begin{bmatrix} 0 & \boxed{2} & 3 & 4 & 1 & 0 & 6 \\ 0 & 0 & 0 & \boxed{2} & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & \boxed{1} & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Definition 33. A matrix is said to be in teh **row-reduced echelon form** if it is in the echelon form and,

1. each pivot is equal to 1, and
2. each pivot is the only non-zero entry in its column

Example 3. This matrix is in row-reduced echelon form. The boxed entries are the **pivots**.

$$A = \begin{bmatrix} 0 & \boxed{1} & 3 & 0 & 0 & 4 \\ 0 & 0 & 0 & \boxed{1} & 0 & -3 \\ 0 & 0 & 0 & 0 & \boxed{1} & 2 \end{bmatrix}$$

3.2 Elementary Row Operations

Suppose that A is a matrix with rows $R_1, ..., R_m$. The elementary row operations that can be performed on A are as follows,

1. Row interchange, $R_i \leftrightarrow R_j$
2. Row scaling, $kR_i \to R_i$
3. Row addition, $kR_i + R_j \to R_j$

The method by which we find the (row-reduced) echelon form of a matrix is using the **Gaussian Elimination** algorithm.

Recall that every matrix is row equivalent to a unique matrix in the row-reduced echelon form.

Definition 34. The **rank** of a matrix $rank(A)$ is the number of pivots in the row-reduced echelon form. There are many other ways to define rank but they all have the same meaning.

The method by which we find the inverse of a square matrix A is as follows, Let $M = [A \mid I]$. Find the row-reduced echelon form of M. If there is a zero row in the resulting matrix then A is not invertible. Otherwise,
$M \sim [I \mid B], \quad A^{-1} = B$.

Theorem 3. *Let A be a square matrix. The following conditions are equivalent,*

1. *A is invertible*
2. *the row-reduced echelon for of A is I*
3. *the only solution to $Ax = 0$ is $x = 0$*
4. *the system $Ax = b$ has a solution for any choice of column b.*

A partial proof is as follows,

Proof. (1) $\Rightarrow$ (3) There is a matrix B such that $AB = I = BA$.
Let x be any solution of $Ax = [0]$.

$$\begin{aligned} BAx &= B[0] \\ Ix &= [0] \\ x &= [0] \end{aligned}$$

(1) $\Rightarrow$ (4) Fix a column b,

$$\begin{aligned} Ax &= b \\ \Leftrightarrow \quad A^{-1}Ax &= A^{-1}b \\ \Leftrightarrow \quad x &= A^{-1}b \end{aligned}$$

$\square$

Definition 35. A linear system $Ax = b$ is **homogeneous** if $b = 0$. Otherwise, $Ax = b$ is said to be **non-homogeneous**.

Definition 36. A **particular solution** of $Ax = b$ is a vector x such that $Ax = b$. The set of all particular solutions is called the **general solution** of the solution set.

Definition 37. A system $Ax = b$ is **consistent** if it has one or more solutions and it is said to be **inconsistent** if it has no solutions.

Theorem 4. *Any system $Ax = b$ has:*
(i) an unique solution,
(ii) no solution, or
(iii) an infinite number of solutions.

3.3 Examples

Example 4. The system,

$$\begin{aligned} x + y + 2z &= 1 \\ 3x - y + z &= -1 \\ -x + 3y + 4z &= 1 \end{aligned}$$

is equivalent to,

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 3 & -1 & 1 & -1 \\ -1 & 3 & 4 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 4 & 5 & 4 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

Example 5. Back substitution:
$z = -2$
$4y + 5z = 4 \Leftrightarrow y = \frac{7}{2}$
$x + y + 2z = 1 \Leftrightarrow x = \frac{3}{2}$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{3}{2} \\ 0 & 1 & 0 & \frac{7}{2} \\ 0 & 0 & 1 & -2 \end{array}\right]$$

Example 6.

$$\begin{aligned} -2x + 3y + 3z &= -9 \\ 3x - 4y + z &= 5 \\ -5x + 7y + 2z &= -14 \end{aligned}$$

$$\sim \left[\begin{array}{ccc|c} -2 & 3 & 3 & -9 \\ 3 & -4 & 1 & 5 \\ -5 & 7 & 2 & -14 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & 4 & -4 \\ 0 & 1 & 11 & -17 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

So there are infinitely many solutions.
Set $z = t$ since z is a free variable, then back substitute.

$$y = -17 - 11t \quad x = -21 - 15t \quad t \in \mathbb{R}$$

So the solution space is,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -21 \\ -17 \\ 0 \end{bmatrix} + t \begin{bmatrix} -15 \\ -11 \\ 1 \end{bmatrix}$$

Where $(-21, -17, 0)$ is a particular solution and $(-15, -11, 1)$ is the set of basic solutions of the homogeneous system $Ax = 0$.

Example 7.

$$\begin{aligned} x + 2y - z &= 2 \\ 2x + 5y - 3z &= 1 \\ x + 4y - 3z &= 3 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -7 & 2 \\ 2 & 5 & -3 & 1 \\ 1 & 4 & -3 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & -1 \end{array}\right]$$

There are no solutions possible for this system.

Chapter 4

Vector Spaces

4.1 Introduction

Adding two vectors in $\mathbb{R}^n$ produces a vector in $\mathbb{R}^n$. Similarly, multiplying by a scalar produces a vector in $\mathbb{R}^n$. These are some properties of a vector space, the following section is a formal list.

4.2 Basic Definitions

Definition 38. Let V be a non-empty set with two operations,

i Vector addition: this assigns to any $u, v \in V$ the sum $u + v \in V$.

ii Scalar multiplication: this assigns to any $u \in V$ and $k \in K$, a product $ku \in V$ where k is a field.

Then V is called a **vector** space (over the field K) if the following axioms hold for any $u, v, w \in V$.

- A1) $(u + v) + w = u + (v + w)$
- A2) there is a vector in V denoted by 0 called the zero vector such that $v + 0 = v$ for any $v \in V$.
- A3) for each $u \in V$, there is a vector in V denoted $-u$, such that $u + (-u) = 0$. $-u$ is called the negative of u.
- A4) $u + v = v + u$
- A5) $k(u + v) = ku + kv$ for any scalar $k \in K$
- A6) $(a + b)u = au + bu$ for any scalars $a, b \in K$
- A7) $(ab)u = a(bu)$ for any scalars $a, b \in K$

- A8) $1u = u$ for the unit scalar $k \in K$

Remark 4. A **field** K is a mathematical object with nice properties, with $\mathbb{R}$ and $\mathbb{C}$ being two examples. From now on, we will take it to be $\mathbb{R}$ or $\mathbb{C}$.

4.3 Examples of Vector Spaces

Example 8. These are some examples of vector spaces,

1. $\mathbb{R}^n$
2. $\mathbb{C}^n$
3. The matrix space: $M_{m\times n}$

   $M_{m\times n}$ denotes the set of all matrices with size m rows, n columns and real entries. $M_{m\times n}(\mathbb{C})$ permits the entries to be complex. The space of the real matrices are a subset of the space of complex matrices.
4. The polynomial space: $P(t)$

   $P(t)$ denotes the set of all polynomials of the form,

   $$P(t) = a_0 + a_1 t + ... + a_n t^n \mid a_i \in \mathbb{R}$$
5. The function space: $F(x)$

   Let X be a non-empty set. Let $F(x)$ denote the set of all functions of X into $\mathbb{R}$. Then $F(x)$ is a vector space (over $\mathbb{R}$) with respect to the following operations,

   i vector addition:

   $$(f+g)(x) = f(x) + g(x) \mid \forall x \in X$$

   ii scalar multiplication: for any $k \in K, f \in F(x)$

   $$(kf)(x) = kf(x) \mid \forall x \in X$$

   iii zero function: $\underline{0}(x) = 0$

Exercise 1. Consider the set $\mathbb{R}^2$ with the usual scalar multiplication, but with the following vector addition:

$$(a,b) \diamond (c,d) = (a+d, b+c)$$

Is this a vector space?
No because axiom 4 does not hold.

$$(1,2) \diamond (-1,1) = (2,1)$$
$$(-1,1) \diamond (1,2) = (1,2)$$

4.4 Vector Subspaces

Definition 39. Let V be a vector space and W be a subset of V. Then W is a **subspace** of V if W itself is a vector space with the operations of vector addition and scalar multiplication of V.

Example 9. $P(t)$ is a subspace of $F(\mathbb{R})$

The next theorem provides a simple criteria to show that a subset W of V is a subspace.

Theorem 5. *Suppose that W is a subset of V, with V being a vector space. Then W is a subspace if the following two conditions hold:*

i The zero vector 0 belongs to W.

ii For every two vectors $u, v \in W$ and $k \in R$

- *$u + v \in W$ (closed under vector addition)*
- *$ku \in W$ (closed under scalar multiplication)*

Proof. By (i), W is non-empty.
By (ii), the operations of vector addition and scalar multiplication are well defined.
The it remains to prove each of the axioms of a vector space.
A1, 4, 5, 6, 7, and 8 hold in W because they hold in V.
A2 is true by (i).
A3: Let $v \in W$. We know that $-v \in V$ with $v + (-v) = 0$ by A3 for the vector space V. But W is closed under scalar multiplication (by (ii)) and so $v \in W$ and we are done. □

4.5 Examples of Vector Subspaces

Example 10. These are some examples of vector subspaces,

1. $0, V$ are subspaces of V. These are called the trivial subspaces of V.

2. Subspaces of $\mathbb{R}^3$

   i Line through the origin is a subspace.

   ii Planes through the origin.

3. Subspaces of $P(t)$

   i $P_m(t) = \{p(\cdot) \in P(t); degree(p(\cdot)) \leq m\}$

   ii $Q(t)$ is the set of polynomials with only even powers

4. Subspaces of matrices $M_{m \times n}$

i $W_1 = \{A \in M_{m\times n}; A \text{ is diagonal}\}$

ii $W_2 = \{A \in M + m \times n; A = A^T\}$

5. Subspaces of $F(\mathbb{R})$

   i $C(\mathbb{R}) = \{f \in F(\mathbb{R}); f \text{ is continuous}\}$

   ii $C'(\mathbb{R}) = \{f \in F(\mathbb{R}); f \text{ is differentiable}\}$

4.6 More on Vector Spaces

Definition 40. Let $A \in M_{m\times n}$. The nullspace of A is $N(A)$ which is given by,

$$N(A) = \{x \in \mathbb{R}^n \mid Ax = 0\}$$

Proposition 1. $N(A)$ is a subspace of $\mathbb{R}^n$

Proof. Clearly $N(A)$ is a subset of $\mathbb{R}^n$

i $0 \in N(A)$. True because $A0 = 0$.

ii Let $u, v \in N(A)$, and $a, b \in \mathbb{R}$. We want to show that $au + bv \in N(A)$, which is the same as $A(au + bv) = 0$

$$\begin{aligned} A(au+bv) &= A(au) + a(bv) \\ &= a(Au) + b(Av) \quad \text{Since } u, v \in N(A)\text{, then } Au = 0, Av = 0 \\ &= a0 + b0 \\ &= 0 \end{aligned}$$

□

Remark 5. The solution set of a non-homogeneous system $\{x \in \mathbb{R}^n \mid Ax = b\}$ where $b \neq 0$ is not a subspace because the zero vector is not present.

Theorem 6. *Let U and W be subspaces of a vector space V. Then $U \cap W$ is also a subspace.*

Proof. Since $U \subseteq V$ and $W \subseteq V$ (U and W are subspaces),

$$U \cap W \subseteq V$$

i So, $0 \in V$ and $0 \in W$, therefore $0 \in U \cap W$

ii Let $u, v \in U \cap W$ and $a, b \in \mathbb{R}$

$$\begin{aligned} u, v \in U \cap W &\Rightarrow \begin{cases} u, v \in V \\ u, v \in W \end{cases} \\ &\Rightarrow \begin{cases} au + bv \in V \\ au + bv \in W \end{cases} \quad \text{both } U \text{ and } W \text{are subspaces} \\ &\Rightarrow au + bv \in U \cap W \end{aligned}$$

□

Remark 6. In general, if U and W are subspaces, $U \cup W$ is **not** a subspace. An example would be two lines through the origin in $\mathbb{R}^3$.

4.7 Linear Combinations

Observe that $au + bv$ is a linear combination.

Definition 41. Let U be a vector space. A vector $v \in V$ is a **linear combination** of $u_1, ..., u_m$ in V if there exists scalars $a_1, ..., a_m$ so that,

$$v = a_1u_1 + ... + a_mu_m$$

Example 11. The following is an example of linear combinations in $\mathbb{R}^3$.
Is $v = (1, 5, 5) \in \mathbb{R}^3$ a linear combination of $u_1 = (1, 2, 3)$, $u_2 = (1, 0, 1)$, $u_3 = (0, 1, 0)$?
That is the same as asking, are there constants $a, b, c \in \mathbb{R}$ such that,

$$v = au_1 + bu_2 + cu_3$$

That is, are there $a, b, c \in \mathbb{R}$ such that,

$$\begin{bmatrix} 1 \\ 5 \\ 5 \end{bmatrix} = a \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + b \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Yes. Take $a = 2, b = -1, c = 1$.

Definition 42. Let $A \in M_{m\times n}$. The column space of A is $C(A)$ which consists of all linear combinations of the columns of A. Alternatively,

$$C(A) = \{Ax \mid x \in \mathbb{R}^n\}$$

Proposition 2. The linear system $Ax = b$ is consistent iff $b \in C(A)$.

Example 12. The following is an example of linear combinations in $P(t)$.
Is the polynomial $P(t) = t^2 + 5t + 5$ a linear combination of the polynomials $P_1(t) = t^2 + 2t + 3, P_2(t) = t^2 + 1, P_3(t) = t$?
Equivalently, are there scalars $a, b, c \in \mathbb{R}$ such that,

$$p(\cdot) = ap_1(\cdot) + bp_2(\cdot) + cp_3(\cdot)$$

There are two ways of solving this,

1. Matching coefficients:

$$t^2 + 5t + 5 = (a + b)t^2 + (2a + c)t + 3a + b$$

$$\begin{cases} 1 = a + b \\ 5 = 2a + c \\ 5 = 3a + b \end{cases} \Leftrightarrow \begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \\ 3 & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 5 \end{bmatrix}$$

2. "Trial approach": We set t in $P(t)$ equal to the three distinct values and each one provides a different equation,

$$\begin{array}{ll} t=0 & 5=3a+b \\ t=1 & 11=6a+2b+c \\ t=-1 & 1=2a+2b-c \end{array}$$

   Then solve for a, b, c.

Example 13. The following are two examples of subspaces of $\mathbb{R}^3$

1. A line with direction $(1, 2, 3)$ through the origin,

$$\{t \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \mid t \in \mathbb{R}\}$$

2. A plane through the origin,

$$\begin{aligned} \{(x, y, z) \in \mathbb{R}^3 \mid z = 0\} &= \{t(1, 0, 0) + s(0, 1, 0) \mid t, s \in \mathbb{R}\} \\ &= \{(t, s, 0) \mid t, s \in \mathbb{R}\} \end{aligned}$$

4.8 The Span of a Vector Space

Definition 43. Let $u_1, ..., u_m$ be vectors in V. The set of all linear combinations of $u_1, ..., u_m$ is called the **span** of $u_1, ..., u_m$ and is denoted by $span\{u_1, ..., u_m\}$.

$$span\{u_1, ..., u_m\} = \{t_1u_1 + ... + t_mu_m \mid t_1, ..., t_m \in \mathbb{R}\}$$

Definition 44. The vectors $u_1, ..., u_m \in V$ are said to span V or to form a **spanning set** of V if,

$$span\{u_1, ..., u_m\} = V$$

The following are the properties of spans.

- If $span\{u_1, ..., u_m\} = V$, then for any $v \in V$, $span\{v, u_1, ..., u_m\} = V$.
- If $span\{0, u_1, ..., u_m\} = V$, then $span\{u_1, ..., u_m\} = V$.
- If $span\{u_1, ..., u_m\} = V$ and u_k is a linear combination of $u_1, ..., u_{k-1}, u_{k+1}, ..., u_m$ then $span\{u_1, ..., u_{k-1}, u_{k+1}, ..., u_m\} = V$.

Proposition 3. Let $u_1, ..., u_m$ be vectors in V. Then $span\{u_1, ..., u_m\}$ is a subspace.

Proof. Clearly $span\{u_1, ..., u_m\} \subseteq V$.
We know $0 \in span\{u_1, ..., u_m\}$ since,

$$0 = 0u_1 + ... + 0u_m \in span\{u_1, ..., u_m\}$$

Take any $u, v \in span\{u_1, ..., u_m\}$ and $a, b \in \mathbb{R}$.

$$u = a_1 u_1 + ... + a_m u_m \quad a_1, ..., a_m \in \mathbb{R} \text{ since } u \in span\{u_1, ..., u_m\}$$

Likewise,

$$v = b_1 u_1 + ... + b_m u_m \quad b_1, ..., b_m \in \mathbb{R}$$

So,

$$\begin{aligned} au + bv &= aa_1 u_1 + ... + aa_m u_m + bb_1 u_1 + ... + bb_m u_m \\ &= (aa_1 + bb_1)u_1 + ... + (aa_m + bb_m)u_m \end{aligned}$$

Which shows that $au + bv$ is a linear combination of $u_1, ...u_m$ with scalars $aa_1 + bb_1, ..., aa_m + bb_m$. $\square$

Exercise 2. $W = \{(x, y, z) \in \mathbb{R}^3 \mid x = 2y = 3z\}$
Clearly, $W \subseteq \mathbb{R}^3$
$(0, 0, 0) \in W$ is true because $0 = 2(0) = 3(0)$.
$u, v \in W \quad a, b \in \mathbb{R}$
$u = (u_1, u_2, u_3)$ with (1) $u_1 = 2u_2$ and (2) $2u_2 = 3u_3$
$v = (v_1, v_2, v_3)$ with (3) $v_1 = 2v_2$ and (4) $2v_2 = 3v_3$

$$\begin{aligned} z = (z_1, z_2, z_3) &= au + bv \\ &= (au_1 + bv_1, au_2 + bv_2, au_3 + bv_3) \end{aligned}$$

We want to show $z \in W$, so,

$$\begin{aligned} z_1 &= 2z_2 = 3z_3 \\ z_1 &= au_1 + bv_1 \\ &= a(2u_2) + b(2v_2) \quad \text{by (1) and (3)} \\ &= 2(au_2 + bv_2) \\ &= 2z_2 \\ &= a(3u_3) + b(3v_3) \\ &= 3(au_3 + bv_3) \\ &= 3z_3 \end{aligned}$$

So, $z_1 = 2z_2 = 3z_3$.
It is also a line through the origin,

$$x = 2y = 3z \Rightarrow$$

$$\{t \begin{bmatrix} 6 \\ 3 \\ 2 \end{bmatrix} \mid t \in \mathbb{R}\} = span\{(6, 3, 2)\}$$

Definition 45. The **span of a set** S is the set of all linear combinations of vectors in S. If $S \neq 0$, $span(S) = \{0\}$.

Theorem 7. *Let S be a subset of the vector space V. Then,*

i) $span(S)$ is a subspace of V.

ii) if W is a subspace of V such that $S \subseteq W$, then $span(S) \subseteq W$.

Proof here

Definition 46. Let $A \in M_{m\times n}$. The **row space** of A, written as $rowsp(A)$, is the set of all linear combinations of rows of A.

$$rowsp(A) = col(A^T)$$

The notation for column space can also be $colsp(A^T)$. $A \in M_{m\times n}$. $rowsp(A)$ is a subspace of $\mathbb{R}^n$. $col(A)$ is a subspace of $\mathbb{R}^m$.

Example 14. Two matrices are row equivalent if you can get from one to the other with only elementary row operations.

$$A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & -1 & 5 \\ 3 & 6 & -2 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = B$$

The following is true for A and B,

- $rowsp(A) = rowsp(B)$
- $rowsp(A) = span(1, 2, -1, 3), (2, 4, -1, 5), (3, 6, -2, 8)$
- $rowsp(B) = span(1, 2, -1, 3), (0, 0, 1, -1)$

Observe that any basis of a subspace is not unique.

Theorem 8. *Row equivalent matrices have the same row space.*

4.9 Linear Dependence and Independence

Definition 47. The vectors $v_1, ..., v_n$ are **linearly independent** if the following condition is satisfied,

$$\text{if } a_1v_1 + ... + a_nv_n = 0 \text{ , then } a_1 = ... = a_n = 0$$

The vectors $v_1, ..., v_n$ are **linearly dependent** if they are not linearly independent.

Remark 7. Consider the vector equation,

$$x_1v_1 + ... + x_nv_n = 0$$

where $x_1, ..., x_n$ are unknown scalars. If the only solution is $(0, .., 0)$, then the vectors are linearly independent. Otherwise they are linearly dependent.

Example 15. The following is an example of linear dependence in $\mathbb{R}^3$. Geometrically, linearly dependent vectors run in the same direction.

i) Two vectors in $\mathbb{R}^3$ are linearly dependent if they lie on the same line. i.e., $k \in \mathbb{R}, k \neq 0$
$$v_2 = kv_1 \Leftrightarrow kv_1 - v_2 = 0$$

ii) Three vectors in $\mathbb{R}^3$ are linearly dependent if they lie on the same plane. i.e., $a_1, a_2 \in \mathbb{R}, a_1, a_2 \neq 0$
$$v_3 = a_1v_1 + a_2v_2 \Leftrightarrow a_1v_1 + a_2v_2 - v_3 = 0$$

Definition 48. An **infinite set of vectors** S is linearly dependent if there exist vectors $v_1, ..., v_n \in S$ that are linearly dependent.

Proposition 4. Let V be a vector space.

i) If $v \neq 0, \{v\}$ is linearly independent.

ii) No independent set of vectors contains the zero vector. Any non-zero scalar multiplied by the zero vector will still yield the zero vector.

iii) Two vectors are linearly dependent iff one of them is a multiple of the others. Let v_1, v_2 be linearly dependent and $a_1 \neq 0$.
$$\begin{aligned} &a_1v_1 + a_2v_2 = 0 \\ \Rightarrow &v_1 + \frac{a_2}{a_1}v_2 = 0 \\ \Leftrightarrow &v_1 = \frac{-a_2}{a_1}v_2 \end{aligned}$$

iv) No independent set can contain two vectors that are multiples of each other.

Exercise 3. Show that $1 + t, 3t + t^2, 2 + t - t^2$ is linearly independent in $P_2(t)$.
Suppose,
$$a(1 + t) + b(3t + t^2) + c(2 + t - t^2) = 0 \quad \forall t \in \mathbb{R}$$
We want to show $a = b = c = 0$.
Substitute three different values for t to obtain three equations, then solve.
$$\begin{aligned} t = 0 &\quad a + b(0) + 2(c) = 0 \Leftrightarrow a + 2c = 0 \\ t = -1 &\quad - 2b = 0 \Leftrightarrow b = 0 \\ t = 1 &\quad 2a + 0 + 2c = 0 \Leftrightarrow 2a + 2c = 0 \quad \text{zero term from } t = -1 \end{aligned}$$
So, $a + 2c = 0$ (1) and $2a + 2c = 0$ (2). (2)$-$(1) gives $a = 0$.
And we are done.
The alternate method is to match coefficients and solve that system as in the example in 4.7.

Proposition 5. The vectors $v_1, ..., v_n$ are linearly dependent iff one of them is a linear combination of another.

Proof. ($\Rightarrow$)
The vectors are linearly dependent. Then, $\forall a_1, ..., a_n \exists a_i \neq 0$ such that
$a_1v_1 + ... + a_nv_n = 0$.
Say $a_k \neq 0$.

$$\begin{aligned} & a_1v_1 + ... + a_nv_n = 0 \\ \Leftrightarrow & \frac{a_1}{a_k}v_1 + ... + \frac{a_{k-1}}{a_k} + v_{k-1} + v_k + \frac{a_{k+1}}{a_k}v_{k+1} + ... + \frac{a_n}{a_k}v_n \\ \Leftrightarrow & v_k = -\frac{a_1}{a_k}v_1 - ... - \frac{a_{k-1}}{a_k} - v_{k-1} - \frac{a_{k+1}}{a_k}v_{k+1} - ... - \frac{a_n}{a_k}v_n \end{aligned}$$

Observe that v_k can be any vector including the zero vector.
($\Leftarrow$)
Say v_i is a linear combination of the other vectors. Then,

$$\begin{aligned} v_i &= b_1v_1 + ... + b_{i-1}v_{i-1} + b_{i+1}v_{i+1} + ... + b_nv_n \\ \Leftrightarrow 0 &= b_1v_1 + ... + b_{i-1}v_{i-1} - v_i + b_{i+1}v_{i+1} + ... + b_nv_n \end{aligned}$$

So, the scalar on v_i is -1 which is not zero.
And we are done. □

4.10 Basis

Definition 49. A set $B = \{u_1, ..., u_n\}$ of vectors in V is a **basis** of V if two conditions are satisfied,

i) B is a linearly independent set

ii) $span(B) = V$

Example 16. The following are examples of basis,

1) $\mathbb{R}^n$

$$e_1 = (1, 0, ..., 0), e_2 = (0, 1, 0, ..., 0), e_n = (0, ..., 0, 1)$$

$\{e_1, ..., e_n\}$ is a basis for $\mathbb{R}^n$.

$(1, 2, 3) = e_1 + 2e_2 + 3e_3$. This is the canonical or standard basis.

2) $P_m(t)$

$$\{1, t, t^2, ..., t^m\}$$

3) $M_{m\times n}$ For $M_{2\times 3}$ it is,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$