Math 223 Assignment 4

Due in class: March 17, 2015

Instructions: Submit a hard copy of your solution with your name and student number. Late assignments will not be graded and will receive a grade of zero.

1. Let u_1, u_2, u_3 be vectors in V and consider the matrix
$$H = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$
where the ij-entry of H is equal to $\langle u_i, u_j \rangle$. Compute $\langle u_1 - 3u_2 + 2u_3, -u_1 + u_2 - 3u_3 \rangle$.

2. Let $a_1, \ldots, a_n$ be real numbers. Show that
$$\frac{(a_1 + \ldots + a_n)^2}{n} \leq a_1^2 + \ldots + a_n^2.$$
Hint: use Cauchy-Schwartz inequality.

3. Suppose $\{u_1, \ldots, u_r\}$ is an orthogonal set of vectors. Show that
$$\|u_1 + \ldots + u_r\|^2 = \|u_1\|^2 + \ldots + \|u_r\|^2 .$$

4. Let S be a subset of an inner product space V. Show that $S^\perp$ is a subspace of V.

5. Let $A \in M_{n\times n}$ be a symmetric matrix, i.e., $A = A^T$. Then A is said to be positive definite if $u^T A u > 0$ for every nonzero vector $u \in \mathbb{R}^n$.
   (a) Let $\langle\ ,\ \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be defined by $\langle u, v\rangle = u^T A v$ where $A \in M_{n\times n}$ is positive definite and $u, v \in \mathbb{R}^n$. Show that $\langle\ ,\ \rangle$ is an inner product in $\mathbb{R}^n$.
   (b) Let now
$$A = \begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}$$
and $\langle\ ,\ \rangle : \mathbb{R}^2 \to \mathbb{R}$ be defined by $\langle u, v\rangle = u^T A v$. Is $\langle\ ,\ \rangle$ an inner product? If so explain why. It not, prove that it is not an inner product.
Hint: find the nullspace of A.

6. Let
$$S = \{v_1, v_2, v_3\} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} \right\}.$$
   (a) Show that S is an orthogonal basis of $\mathbb{R}^3$.
   (b) Find the coordinates of the vector
$$v = \begin{bmatrix} 3 \\ 4 \\ -1 \end{bmatrix}$$
relative to the basis S, i.e., find $[v]_S$.

(c) Given an arbitrary vector

$$v = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

find its coordinates under the basis S, i.e., find $[v]_S$.

7. Consider the vector space $P_3(t)$ the space of polynomials with degree less or equal 3 with the inner product defined by

$$\langle f, g \rangle = \int_0^1 f(t)g(t)\, dt$$

Apply the Gram-Schmidt process to $\{1, 1+t, t+t^2, t^3\}$ to obtain an orthonormal basis V.

Hint: use Gram-Schmidt algorithm to produce an orthogonal basis and then normalize each vector.