MATH 223

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0.1 Administrativa

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Grading

Assignments	15%	15%
Midterm	25%	0%
Final	60%	85%

The midterm is to be held on February 19th in class.

Chapter 1

Vectors

1.1 Vectors in $\mathbb{R}^n$

$\mathbb{R}^n$ is the set of all n-tuples of real numbers $u = (a_1 ... a_n) \mid a \in \mathbb{R}$ where a are the **components** or **entries**.

Remark 1.1.1. We use the term **scalar** to refer to an element in $\mathbb{R}$.

1.2 Basic Definitions

Definition. Addition
$u, v \in \mathbb{R}^n$
$u = (a_1 ... a_n)$
$v = (b_1 ... b_n)$
$u + v = (a_1 + b_1 ... a_n + b_n)$

Definition. Scalar Multiplication
$k \in \mathbb{R}$
$ku = (ka_1 ... ka_n)$

Definition. Two vectors u and v are said to be **equal** $(u = v)$ if $a_i = b_i \forall i = 1...n$.

Definition. The **zero vector** is defined as $0 = (0...0)$.

Definition. Suppose we are given m vectors $u_1 ... u_m \in \mathbb{R}^n$ and m scalars $k_1 ... k_m \in \mathbb{R}$.
Let $u = k_1 u_1 + ... + k_m u_m$.
Such a vector u is called a **linear combination** of the vectors $u_1 ... u_m$.

Definition. A vector u can be called a **multiple** of v if there is a scalar k such that $u = kv$ with $k \neq 0$. In the case $k > 0$ we say u is in the same direction as v. In the case $k < 0$ we say u is in the opposite direction of v.

1.3 The Dot Product

Definition. Let $u = (a_1...a_n)$ and $v = (b_1...b_n)$. The **dot product** or inner product is given by,

$$u \cdot v = a_1b_1 + ...a_nb_n =$$

Definition. The vectors u and v are **orthogonal** if $u \cdot v = 0$.

1.4 The Vector Norm

Definition. The **norm** or **length** of a vector is given by,

$$\|u\| = \sqrt{a_1^2 + ... + a_n^2}$$

Thus $\|u\| \geq 0$ and $\|u\| = 0$ if and only if (iff) $u = 0$.

Definition. A vector is called a **unit vector** if $\|u\| = 1$.

Definition. For any non-zero vector v, the vector

$$\hat{v} = \frac{1}{\|v\|}v$$

is the only unit vector with the same direction of v. The process of finding $\hat{v}$ is called **normalizing**.

1.5 Theorem: Cauchy-Schwarz Inequality

Theorem 1.5.1. Given any two vectors $u, v \in \mathbb{R}^n$, then,

$$|u \cdot v| \leq \|u\|\|v\|$$

Proof. Let $t \in \mathbb{R}$. So, $\|tu + v\|^2 \geq 0$.

$$\begin{aligned}
\|tu + v\|^2 &= (tu + v)(tu + v) \\
&= (tu \cdot tu) + (tu \cdot v) + (v \cdot tu) + (v \cdot v) \\
&= t^2(u \cdot u) + t(v \cdot u) + t(u \cdot v) + (v \cdot v) \\
&= t^2\|u\|^2 + 2t(u \cdot v) + \|v\|^2
\end{aligned}$$

We can represent this in the form $at^2 + bt + c \geq 0$, so,

$$a = \|u\|^2, b = 2(u \cdot v), c = \|v\|^2$$

Take the Discriminant as $b^2 - 4ac \iff b^2 \leq 4ac$.

$$\begin{gathered}
4(u \cdot v)^2 \leq 4\|u\|^2\|v\|^2 \\
|u \cdot v| \leq \|u\|\|v\|
\end{gathered}$$

□

1.6 Theorem: Minkowski Triangle Inequality

Theorem 1.6.1. Given $u, v \in \mathbb{R}^n$, then $\|u+v\| \leq \|u\| + \|v\|$.

Proof.

$$\begin{aligned} \|u+v\|^2 &= \|u\|^2 + 2(u \cdot v) + \|v\|^2 \\ &\leq \|u\|^2 + 2\|u\|\|v\| + \|v\|^2 \quad \text{by C-S inequality} \\ &= (\|u\| + \|v\|)^2 \end{aligned}$$

So, $\|u+v\|^2 \leq (\|u\| + \|v\|)^2$. Take the square root and we are done. □

1.7 Geometry with Vectors

Definition. The **distance** between vectors $u, v \in \mathbb{R}^n$ is given by,

$$d(u, v) = \|u - v\| = \sqrt{(a_1 - b_1)^2 + ... + (a_n - b_n)^2}$$

Definition. The **angle** between vectors $u, v \in \mathbb{R}^n$ is given by,

$$cos\theta = \frac{u \cdot v}{\|u\|\|v\|} \quad \theta \in [0, \pi]$$

Observe that in the previous definition, the angle is well defined.

$$-\|u\|\|v\| \leq -|u \cdot v| \leq u \cdot v \leq u \cdot v \leq |u \cdot v| \leq \|u\|\|v\|$$

Dividing the entire inequality by $\|u\|\|v\|$ yields,

$$-1 \leq \frac{u \cdot v}{\|u\|\|v\|} \leq 1$$

Definition. A **hyperplane** $\mathcal{H}$ in $\mathbb{R}^n$ is the set of points $(x_1...x_n)$ that satisfy $a_1x_1 + ... + a_nx_n = b$ where $u = [a_1...a_n] \in \mathbb{R}^n$ and $b \in \mathbb{R}$.

Definition. The **line** in $\mathbb{R}^n$ passing through a point $P = (b_1...b_n)$ and in the direction of $v \in \mathbb{R}^n$ with $v \neq 0$.

$$x = P + tu \quad t \in \mathbb{R}, \quad u = [a_1...a_n]$$

$$\begin{cases} x_1 = a_1t + b_1 \\ x_n = a_nt + b_n \end{cases}$$

Chapter 2

Algebra of Matrices

2.1 Introduction

A matrix with n rows and m columns is written as,

$$A_{n\times m} = \begin{bmatrix} a_{11} & \dots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nm} \end{bmatrix}$$

Or,

$$A_{n\times m} = [a_{ij}]$$

Where a_{ij} is the entry in row i and column j.

2.2 Definitions and Properties of Matrices

Definition. Matrix Addition

$$A + B = [a_{ij} + b_{ij}] \quad \forall i = 1...n, j = 1...m$$

Definition. Scalar Multiplication

$$ka = [ka_{ij}] \quad \forall i = 1...n, j = 1...m$$

Definition. Zero Matrix

$$0 = [0]$$

Definition. Given a matrix $A_{m\times p}$ and a matrix $B_{p\times n}$, **matrix multiplication** is defined as,

$$AB = [c_{ij}] \quad c_{ij} = \sum_{k=1}^{p} a_{ik}b_{kj}$$

Definition. Given a matrix A, its **transpose** is $A^T = [a_{ji}]$ where $A = [a_{ij}]$.

Definition. A **square matrix** has the same number of rows as it does columns, i.e. $A_{n\times n}$ is a square matrix.

Definition. Given a matrix $A = [a_{ij}]$ the elements in the **diagonal** are $[a_{11}, ..., a_{nn}]$.

Definition. The **trace** of a matrix A is given by,

$$tr(A) = a_{11} + ... + a_{nn}$$

Definition. The **identity matrix** I_n is the matrix such that for any n-square matrix A,

$$AI = IA = A$$

Definition. The **Kronecker delta** is defined by,

$$\delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

Remark 2.2.1. Given the definitions for the identity matrix and the Kronecker delta, an alternative definition for the identity matrix is as follows,

$$I = [\delta_{ij}]$$

Definition. A matrix A is **invertible** if there is a matrix B such that $AB = BA = I$.

Remark 2.2.2. In general, for any matrices A and B, $AB \neq BA$.

Definition. A matrix D is **diagonal** if all the non-zero entries are in the diagonal.

$$D = diaginal(d_1, ..., d_n)$$

Definition. A matrix A is **upper triangular** if,

$$a_{ij} = 0 \quad \forall i > j$$

2.3 Complex Numbers

The imaginary number i is defined as $i = \sqrt{-1}$ or equivalently, $i^2 = -1$.

Definition. A **complex number** z is given by,

$$z = a + bi \quad a, b \in \mathbb{R}$$

Where a is the real part and b is the imaginary part.

Real numbers are also complex numbers with no imaginary component, i.e. $a + 0i = a$.

Addition for two complex numbers $z = a + bi$ and $w = c + di$ is given by,

$$z + w = (a + c) + (b + d)i$$

Multiplication for the same two complex numbers is given by,

$$\begin{aligned} z \cdot w &= (a + bi)(c + di) \\ &= ac + adi + cbi - bd \\ &= (ac - bd) + (ad + bc)i \end{aligned}$$

Definition. The **conjugate** of $z = a + bi$ is $\bar{z} = a - bi$.

Definition. The **absolute value** or modulus of $z = a + bi$ is $|z| = \sqrt{a^2 + b^2}$.

Example 2.3.1.

$$z^{-1} = \frac{1}{z} = \frac{1}{a + bi} \cdot \frac{a - bi}{a - bi} = \frac{a - bi}{a^2 + b^2}$$

Observe that the following properties are true for conjugates and absolute values,

1. $$z\bar{z} = |z|^2 = a^2 + b^2$$
2. $$\overline{z \pm w} = \bar{z} \pm \bar{w}$$
3. $$\overline{zw} = \bar{z} \cdot \bar{w}$$
4. $$\overline{(\bar{z})} = z$$
5. z is real iff $z = \bar{z}$
6. $$|zw| = |z||w|$$
7. $$|z + w| \leq |z| + |w|$$

Chapter 3

Systems of Linear Equations

3.1 Representing Linear Systems with Matrices

Given a system of linear equations of the form,

$$\begin{cases} a_{11}x_1 + ... + a_{1m}x_n = b_1 \\ \vdots \\ a_{m1}x_1 + ... + a_{mn}x_n = b_m \end{cases}$$

- $x_1, ..., x_m$ are the unknowns, and
- a_{ij} and b_i are the constants.

The system can also be represented by matrices where,

- $A = [a_{ij}]$ is the matrix of coefficients
- $b = [b_i]$ is the column vector of constant
- $M = [A|b]$ is the matrix that represents the system.

Definition. A matrix A is in **echelon form** if

1. all zero rows are at the bottom, and
2. each leading non-zero entry in a row is to the right of the leading non-zero entry in teh preceding row.

Example 3.1.1. This matrix is in echelon form. The boxed entries are the **pivots**.

$$A = \begin{bmatrix} 0 & \boxed{2} & 3 & 4 & 1 & 0 & 6 \\ 0 & 0 & 0 & \boxed{2} & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & \boxed{1} & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Definition. A matrix is said to be in teh **row-reduced echelon form** if it is in the echelon form and,

1. each pivot is equal to 1, and
2. each pivot is the only non-zero entry in its column

Example 3.1.2. This matrix is in row-reduced echelon form. The boxed entries are the **pivots**.

$$A = \begin{bmatrix} 0 & \boxed{1} & 3 & 0 & 0 & 4 \\ 0 & 0 & 0 & \boxed{1} & 0 & -3 \\ 0 & 0 & 0 & 0 & \boxed{1} & 2 \end{bmatrix}$$

3.2 Elementary Row Operations

Suppose that A is a matrix with rows $R_1, ..., R_m$. The elementary row operations that can be performed on A are as follows,

1. Row interchange, $R_i \leftrightarrow R_j$
2. Row scaling, $kR_i \to R_i$
3. Row addition, $kR_i + R_j \to R_j$

The method by which we find the (row-reduced) echelon form of a matrix is using the **Gaussian Elimination** algorithm.

Recall that every matrix is row equivalent to a unique matrix in the row-reduced echelon form.

Definition. The **rank** of a matrix $rank(A)$ is the number of pivots in the row-reduced echelon form. There are many other ways to define rank but they all have the same meaning.

The method by which we find the inverse of a square matrix A is as follows, Let $M = [A \mid I]$. Find the row-reduced echelon form of M. If there is a zero row in the resulting matrix then A is not invertible. Otherwise, $M \sim [I \mid B], \quad A^{-1} = B$.

Theorem 3.2.1. Let A be a square matrix. The following conditions are equivalent,

1. A is invertible
2. the row-reduced echelon for of A is I
3. the only solution to $Ax = 0$ is $x = 0$
4. the system $Ax = b$ has a solution for any choice of column b.

A partial proof is as follows,

Proof. (1) $\Rightarrow$ (3) There is a matrix B such that $AB = I = BA$.
Let x be any solution of $Ax = [0]$.

$$\begin{aligned} BAx &= B[0] \\ Ix &= [0] \\ x &= [0] \end{aligned}$$

(1) $\Rightarrow$ (4) Fix a column b,

$$\begin{aligned} && Ax &= b \\ \Leftrightarrow && A^{-1}Ax &= A^{-1}b \\ && \Leftrightarrow \quad x &= A^{-1}b \end{aligned}$$

$\square$

Definition. A linear system $Ax = b$ is **homogeneous** if $b = 0$. Otherwise, $Ax = b$ is said to be **non-homogeneous**.

Definition. A **particular solution** of $Ax = b$ is a vector x such that $Ax = b$. The set of all particular solutions is called the **general solution** of the solution set.

Definition. A system $Ax = b$ is **consistent** if it has one or more solutions and it is said to be **inconsistent** if it has no solutions.

Theorem 3.2.2. Any system $Ax = b$ has:
(i) an unique solution,
(ii) no solution, or
(iii) an infinite number of solutions.

3.3 Examples

Example 3.3.1. The system,

$$\begin{aligned} x + y + 2z &= 1 \\ 3x - y + z &= -1 \\ -x + 3y + 4z &= 1 \end{aligned}$$

is equivalent to,

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 3 & -1 & 1 & -1 \\ -1 & 3 & 4 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 4 & 5 & 4 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

Example 3.3.2. Back substitution:
$z = -2$
$4y + 5z = 4 \Leftrightarrow y = \frac{7}{2}$
$x + y + 2z = 1 \Leftrightarrow x = \frac{3}{2}$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{3}{2} \\ 0 & 1 & 0 & \frac{7}{2} \\ 0 & 0 & 1 & -2 \end{array}\right]$$

Example 3.3.3.

$$\begin{aligned} -2x + 3y + 3z &= -9 \\ 3x - 4y + z &= 5 \\ -5x + 7y + 2z &= -14 \end{aligned}$$

$$\sim \left[\begin{array}{ccc|c} -2 & 3 & 3 & -9 \\ 3 & -4 & 1 & 5 \\ -5 & 7 & 2 & -14 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & 4 & -4 \\ 0 & 1 & 11 & -17 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

So there are infinitely many solutions.
Set $z = t$ since z is a free variable, then back substitute.

$$y = -17 - 11t \quad x = -21 - 15t \quad t \in \mathbb{R}$$

So the solution space is,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -21 \\ -17 \\ 0 \end{bmatrix} + t \begin{bmatrix} -15 \\ -11 \\ 1 \end{bmatrix}$$

Where $(-21, -17, 0)$ is a particular solution and $(-15, -11, 1)$ is the set of basic solutions of the homogeneous system $Ax = 0$.

Example 3.3.4.

$$\begin{aligned} x + 2y - z &= 2 \\ 2x + 5y - 3z &= 1 \\ x + 4y - 3z &= 3 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -7 & 2 \\ 2 & 5 & -3 & 1 \\ 1 & 4 & -3 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & -1 \end{array}\right]$$

There are no solutions possible for this system.

Chapter 4

Vector Spaces

4.1 Introduction

Adding two vectors in $\mathbb{R}^n$ produces a vector in $\mathbb{R}^n$. Similarly, multiplying by a scalar produces a vector in $\mathbb{R}^n$. These are some properties of a vector space, the following section is a formal list.

4.2 Basic Definitions

Definition. Let V be a non-empty set with two operations,

i Vector addition: this assigns to any $u, v \in V$ the sum $u + v \in V$.

ii Scalar multiplication: this assigns to any $u \in V$ and $k \in K$, a product $ku \in V$ where k is a field.

Then V is called a **vector** space (over the field K) if the following axioms hold for any $u, v, w \in V$.

- A1) $(u + v) + w = u + (v + w)$
- A2) there is a vector in V denoted by 0 called the zero vector such that $v + 0 = v$ for any $v \in V$.
- A3) for each $u \in V$, there is a vector in V denoted $-u$, such that $u + (-u) = 0$. $-u$ is called the negative of u.
- A4) $u + v = v + u$
- A5) $k(u + v) = ku + kv$ for any scalar $k \in K$
- A6) $(a + b)u = au + bu$ for any scalars $a, b \in K$
- A7) $(ab)u = a(bu)$ for any scalars $a, b \in K$

- A8) $1u = u$ for the unit scalar $k \in K$

Remark 4.2.1. A **field** K is a mathematical object with nice properties, with $\mathbb{R}$ and $\mathbb{C}$ being two examples. From now on, we will take it to be $\mathbb{R}$ or $\mathbb{C}$.

4.3 Examples of Vector Spaces

Example 4.3.1. These are some examples of vector spaces,

1. $\mathbb{R}^n$
2. $\mathbb{C}^n$
3. The matrix space: $M_{m\times n}$

   $M_{m\times n}$ denotes the set of all matrices with size m rows, n columns and real entries. $M_{m\times n}(\mathbb{C})$ permits the entries to be complex. The space of the real matrices are a subset of the space of complex matrices.
4. The polynomial space: $P(t)$

   $P(t)$ denotes the set of all polynomials of the form,

   $$P(t) = a_0 + a_1 t + ... + a_n t^n \mid a_i \in \mathbb{R}$$
5. The function space: $F(x)$

   Let X be a non-empty set. Let $F(x)$ denote the set of all functions of X into $\mathbb{R}$. Then $F(x)$ is a vector space (over $\mathbb{R}$) with respect to the following operations,

   i vector addition:

   $$(f+g)(x) = f(x) + g(x) \mid \forall x \in X$$

   ii scalar multiplication: for any $k \in K, f \in F(x)$

   $$(kf)(x) = kf(x) \mid \forall x \in X$$

   iii zero function: $\underline{0}(x) = 0$

Exercise 4.3.1. Consider the set $\mathbb{R}^2$ with the usual scalar multiplication, but with the following vector addition:

$$(a,b) \diamond (c,d) = (a+d, b+c)$$

Is this a vector space?
No because axiom 4 does not hold.

$$(1,2) \diamond (-1,1) = (2,1)$$
$$(-1,1) \diamond (1,2) = (1,2)$$

4.4 Vector Subspaces

Definition. Let V be a vector space and W be a subset of V. Then W is a **subspace** of V if W itself is a vector space with the operations of vector addition and scalar multiplication of V.

Example 4.4.1. $P(t)$ is a subspace of $F(\mathbb{R})$

The next theorem provides a simple criteria to show that a subset W of V is a subspace.

Theorem 4.4.1. Suppose that W is a subset of V, with V being a vector space. Then W is a subspace if the following two conditions hold:

i The zero vector 0 belongs to W.

ii For every two vectors $u, v \in W$ and $k \in R$

- $u + v \in W$ (closed under vector addition)
- $ku \in W$ (closed under scalar multiplication)

Proof. By (i), W is non-empty.
By (ii), the operations of vector addition and scalar multiplication are well defined.
The it remains to prove each of the axioms of a vector space.
A1, 4, 5, 6, 7, and 8 hold in W because they hold in V.
A2 is true by (i).
A3: Let $v \in W$. We know that $-v \in V$ with $v + (-v) = 0$ by A3 for the vector space V. But W is closed under scalar multiplication (by (ii)) and so $v \in W$ and we are done. $\square$

4.5 Examples of Vector Subspaces

Example 4.5.1. These are some examples of vector subspaces,

1. $0, V$ are subspaces of V. These are called the trivial subspaces of V.
2. Subspaces of $\mathbb{R}^3$

   i Line through the origin is a subspace.

   ii Planes through the origin.
3. Subspaces of $P(t)$

   i $P_m(t) = \{p(\cdot) \in P(t); degree(p(\cdot)) \leq m\}$

   ii $Q(t)$ is the set of polynomials with only even powers
4. Subspaces of matrices $M_{m\times n}$

i $W_1 = \{A \in M_{m\times n}; A \text{ is diagonal}\}$

ii $W_2 = \{A \in M + m \times n; A = A^T\}$

5. Subspaces of $F(\mathbb{R})$

i $C(\mathbb{R}) = \{f \in F(\mathbb{R}); f \text{ is continuous}\}$

ii $C'(\mathbb{R}) = \{f \in F(\mathbb{R}); f \text{ is differentiable}\}$

4.6 More on Vector Spaces

Definition. Let $A \in M_{m\times n}$. The nullspace of A is $N(A)$ which is given by,

$$N(A) = \{x \in \mathbb{R}^n \mid Ax = 0\}$$

Proposition 4.6.1. $N(A)$ is a subspace of $\mathbb{R}^n$

Proof. Clearly $N(A)$ is a subset of $\mathbb{R}^n$

i $0 \in N(A)$. True because $A0 = 0$.

ii Let $u, v \in N(A)$, and $a, b \in \mathbb{R}$. We want to show that $au + bv \in N(A)$, which is the same as $A(au + bv) = 0$

$$\begin{aligned} A(au + bv) &= A(au) + a(bv) \\ &= a(Au) + b(Av) \quad \text{Since } u, v \in N(A), \text{ then } Au = 0, Av = 0 \\ &= a0 + b0 \\ &= 0 \end{aligned}$$

□

Remark 4.6.1. The solution set of a non-homogeneous system $\{x \in \mathbb{R}^n \mid Ax = b\}$ where $b \neq 0$ is not a subspace because the zero vector is not present.

Theorem 4.6.2. Let U and W be subspaces of a vector space V. Then $U \cap W$ is also a subspace.

Proof. Since $U \subseteq V$ and $W \subseteq V$ (U and W are subspaces),

$$U \cap W \subseteq V$$

i So, $0 \in V$ and $0 \in W$, therefore $0 \in U \cap W$

ii Let $u, v \in U \cap W$ and $a, b \in \mathbb{R}$

$$\begin{aligned} u, v \in U \cap W &\Rightarrow \begin{cases} u, v \in V \\ u, v \in W \end{cases} \\ &\Rightarrow \begin{cases} au + bv \in V \\ au + bv \in W \end{cases} \quad \text{both } U \text{ and } W \text{are subspaces} \\ &\Rightarrow au + bv \in U \cap W \end{aligned}$$

□

Remark 4.6.2. In general, if U and W are subspaces, $U \cup W$ is **not** a subspace. An example would be two lines through the origin in $\mathbb{R}^3$.

4.7 Linear Combinations

Observe that $au + bv$ is a linear combination.

Definition. Let U be a vector space. A vector $v \in V$ is a **linear combination** of $u_1, ..., u_m$ in V if there exists scalars $a_1, ..., a_m$ so that,

$$v = a_1 u_1 + ... + a_m u_m$$

Example 4.7.1. The following is an example of linear combinations in $\mathbb{R}^3$.
Is $v = (1, 5, 5) \in \mathbb{R}^3$ a linear combination of $u_1 = (1, 2, 3)$, $u_2 = (1, 0, 1)$, $u_3 = (0, 1, 0)$?
That is the same as asking, are there constants $a, b, c \in \mathbb{R}$ such that,

$$v = au_1 + bu_2 + cu_3$$

That is, are there $a, b, c \in \mathbb{R}$ such that,

$$\begin{bmatrix} 1 \\ 5 \\ 5 \end{bmatrix} = a \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + b \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Yes. Take $a = 2, b = -1, c = 1$.

Definition. Let $A \in M_{m \times n}$. The column space of A is $C(A)$ which consists of all linear combinations of the columns of A. Alternatively,

$$C(A) = \{Ax \mid x \in \mathbb{R}^n\}$$

Proposition 4.7.1. The linear system $Ax = b$ is consistent iff $b \in C(A)$.

Example 4.7.2. The following is an example of linear combinations in $P(t)$.
Is the polynomial $P(t) = t^2 + 5t + 5$ a linear combination of the polynomials $P_1(t) = t^2 + 2t + 3, P_2(t) = t^2 + 1, P_3(t) = t$?
Equivalently, are there scalars $a, b, c \in \mathbb{R}$ such that,

$$p(\cdot) = ap_1(\cdot) + bp_2(\cdot) + cp_3(\cdot)$$

There are two ways of solving this,

1. Matching coefficients:

$$t^2 + 5t + 5 = (a + b)t^2 + (2a + c)t + 3a + b$$

$$\begin{cases} 1 = a + b \\ 5 = 2a + c \\ 5 = 3a + b \end{cases} \Leftrightarrow \begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \\ 3 & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 5 \end{bmatrix}$$

2. "Trial approach": We set t in $P(t)$ equal to the three distinct values and each one provides a different equation,

$$\begin{array}{ll} t=0 & 5=3a+b \\ t=1 & 11=6a+2b+c \\ t=-1 & 1=2a+2b-c \end{array}$$

   Then solve for a, b, c.

Example 4.7.3. The following are two examples of subspaces of $\mathbb{R}^3$

1. A line with direction $(1,2,3)$ through the origin,

$$\{t \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \mid t \in \mathbb{R}\}$$

2. A plane through the origin,

$$\begin{aligned} \{(x,y,z) \in \mathbb{R}^3 \mid z=0\} &= \{t(1,0,0)+s(0,1,0) \mid t,s \in \mathbb{R}\} \\ &= \{(t,s,0) \mid t,s \in \mathbb{R}\} \end{aligned}$$

4.8 The Span of a Vector Space

Definition. Let $u_1, ..., u_m$ be vectors in V. The set of all linear combinations of $u_1, ..., u_m$ is called the **span** of $u_1, ..., u_m$ and is denoted by $span\{u_1, ..., u_m\}$.

$$span\{u_1, ..., u_m\} = \{t_1 u_1 + ... + t_m u_m \mid t_1, ..., t_m \in \mathbb{R}\}$$

Definition. The vectors $u_1, ..., u_m \in V$ are said to span V or to form a **spanning set** of V if,

$$span\{u_1, ..., u_m\} = V$$

The following are the properties of spans.

- If $span\{u_1, ..., u_m\} = V$, then for any $v \in V$, $span\{v, u_1, ..., u_m\} = V$.
- If $span\{0, u_1, ..., u_m\} = V$, then $span\{u_1, ..., u_m\} = V$.
- If $span\{u_1, ..., u_m\} = V$ and u_k is a linear combination of $u_1, ..., u_{k-1}, u_{k+1}, ..., u_m$ then $span\{u_1, ..., u_{k-1}, u_{k+1}, ..., u_m\} = V$.

Proposition 4.8.1. Let $u_1, ..., u_m$ be vectors in V. Then $span\{u_1, ..., u_m\}$ is a subspace.

Proof. Clearly $span\{u_1, ..., u_m\} \subseteq V$.
We know $0 \in span\{u_1, ..., u_m\}$ since,

$$0 = 0u_1 + ... + 0u_m \in span\{u_1, ..., u_m\}$$

Take any $u, v \in span\{u_1, ..., u_m\}$ and $a, b \in \mathbb{R}$.

$$u = a_1u_1 + ... + a_mu_m \quad a_1, ..., a_m \in \mathbb{R} \text{ since } u \in span\{u_1, ..., u_m\}$$

Likewise,

$$v = b_1u_1 + ... + b_mu_m \quad b_1, ..., b_m \in \mathbb{R}$$

So,

$$\begin{aligned} au + bv &= aa_1u_1 + ... + aa_mu_m + bb_1u_1 + ... + bb_mu_m \\ &= (aa_1 + bb_1)u_1 + ... + (aa_m + bb_m)u_m \end{aligned}$$

Which shows that $au + bv$ is a linear combination of $u_1, ...u_m$ with scalars $aa_1 + bb_1, ..., aa_m + bb_m$. □

Exercise 4.8.1. $W = \{(x, y, z) \in \mathbb{R}^3 \mid x = 2y = 3z\}$
Clearly, $W \subseteq \mathbb{R}^3$
$(0, 0, 0) \in W$ is true because $0 = 2(0) = 3(0)$.
$u, v \in W \quad a, b \in \mathbb{R}$
$u = (u_1, u_2, u_3)$ with (1) $u_1 = 2u_2$ and (2) $2u_2 = 3u_3$
$v = (v_1, v_2, v_3)$ with (3) $v_1 = 2v_2$ and (4) $2v_2 = 3v_3$

$$\begin{aligned} z = (z_1, z_2, z_3) &= au + bv \\ &= (au_1 + bv_1, au_2 + bv_2, au_3 + bv_3) \end{aligned}$$

We want to show $z \in W$, so,

$$\begin{aligned} z_1 &= 2z_2 = 3z_3 \\ z_1 &= au_1 + bv_1 \\ &= a(2u_2) + b(2v_2) \quad \text{by (1) and (3)} \\ &= 2(au_2 + bv_2) \\ &= 2z_2 \\ &= a(3u_3) + b(3v_3) \\ &= 3(au_3 + bv_3) \\ &= 3z_3 \end{aligned}$$

So, $z_1 = 2z_2 = 3z_3$.
It is also a line through the origin,

$$x = 2y = 3z \Rightarrow$$

$$\{t \begin{bmatrix} 6 \\ 3 \\ 2 \end{bmatrix} \mid t \in \mathbb{R}\} = span\{(6, 3, 2)\}$$

Definition. The **span of a set** S is the set of all linear combinations of vectors in S. If $S \neq 0$, $span(S) = \{0\}$.

Theorem 4.8.2. Let S be a subset of the vector space V. Then,

i) $span(S)$ is a subspace of V.

ii) if W is a subspace of V such that $S \subseteq W$, then $span(S) \subseteq W$.

Proof here

4.9 The Row Space of a Matrix

Definition. Let $A \in M_{m\times n}$. The **row space** of A, written as $rowsp(A)$, is the set of all linear combinations of rows of A.

$$rowsp(A) = col(A^T)$$

The notation for column space can also be $colsp(A^T)$. $A \in M_{m\times n}$. $rowsp(A)$ is a subspace of $\mathbb{R}^n$. $col(A)$ is a subspace of $\mathbb{R}^m$.

Example 4.9.1. Two matrices are row equivalent if you can get from one to the other with only elementary row operations.

$$A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & -1 & 5 \\ 3 & 6 & -2 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = B$$

The following is true for A and B,

- $rowsp(A) = rowsp(B)$
- $rowsp(A) = span(1, 2, -1, 3), (2, 4, -1, 5), (3, 6, -2, 8)$
- $rowsp(B) = span(1, 2, -1, 3), (0, 0, 1, -1)$

Observe that any basis of a subspace is not unique.

Theorem 4.9.1. Row equivalent matrices have the same row space.

4.10 Linear Dependence and Independence

Definition. The vectors $v_1, ..., v_n$ are **linearly independent** if the following condition is satisfied,

$$\text{if } a_1v_1 + ... + a_nv_n = 0 \text{ , then } a_1 = ... = a_n = 0$$

The vectors $v_1, ..., v_n$ are **linearly dependent** if they are not linearly independent.

Remark 4.10.1. Consider the vector equation,

$$x_1v_1 + ... + x_nv_n = 0$$

where $x_1, ..., x_n$ are unknown scalars. If the only solution is $(0, .., 0)$, then the vectors are linearly independent. Otherwise they are linearly dependent.

Example 4.10.1. The following is an example of linear dependence in $\mathbb{R}^3$. Geometrically, linearly dependent vectors run in the same direction.

i) Two vectors in $\mathbb{R}^3$ are linearly dependent if they lie on the same line. i.e., $k \in \mathbb{R}, k \neq 0$
$$v_2 = kv_1 \Leftrightarrow kv_1 - v_2 = 0$$

ii) Three vectors in $\mathbb{R}^3$ are linearly dependent if they lie on the same plane. i.e., $a_1, a_2 \in \mathbb{R}, a_1, a_2 \neq 0$
$$v_3 = a_1v_1 + a_2v_2 \Leftrightarrow a_1v_1 + a_2v_2 - v_3 = 0$$

Definition. An **infinite set of vectors** S is linearly dependent if there exist vectors $v_1, ..., v_n \in S$ that are linearly dependent.

Proposition 4.10.1. Let V be a vector space.

i) If $v \neq 0, \{v\}$ is linearly independent.

ii) No independent set of vectors contains the zero vector. Any non-zero scalar multiplied by the zero vector will still yield the zero vector.

iii) <u>Two</u> vectors are linearly dependent iff one of them is a multiple of the others. Let v_1, v_2 be linearly dependent and $a_1 \neq 0$.
$$\begin{aligned} & a_1v_1 + a_2v_2 = 0 \\ \Rightarrow \quad & v_1 + \frac{a_2}{a_1}v_2 = 0 \\ \Leftrightarrow \quad & v_1 = \frac{-a_2}{a_1}v_2 \end{aligned}$$

iv) No independent set can contain two vectors that are multiples of each other.

Exercise 4.10.1. Show that $1 + t, 3t + t^2, 2 + t - t^2$ is linearly independent in $P_2(t)$.
Suppose,
$$a(1 + t) + b(3t + t^2) + c(2 + t - t^2) = 0 \quad \forall t \in \mathbb{R}$$
We want to show $a = b = c = 0$.
Substitute three different values for t to obtain three equations, then solve.
$$\begin{aligned} t = 0 &\quad a + b(0) + 2(c) = 0 \Leftrightarrow a + 2c = 0 \\ t = -1 &\quad -2b = 0 \Leftrightarrow b = 0 \\ t = 1 &\quad 2a + 0 + 2c = 0 \Leftrightarrow 2a + 2c = 0 \quad \text{zero term from } t = -1 \end{aligned}$$
So, $a + 2c = 0$ (1) and $2a + 2c = 0$ (2). (2)$-$(1) gives $a = 0$.
And we are done.
The alternate method is to match coefficients and solve that system as in the example in 4.7.

Proposition 4.10.2. The vectors $v_1, ..., v_n$ are linearly dependent iff one of them is a linear combination of another.

Proof. ($\Rightarrow$)
The vectors are linearly dependent. Then, $\forall a_1, ..., a_n \exists a_i \neq 0$ such that $a_1v_1 + ... + a_nv_n = 0$.
Say $a_k \neq 0$.

$$\begin{aligned}
& a_1v_1 + ... + a_nv_n = 0 \\
\Leftrightarrow \quad & \frac{a_1}{a_k}v_1 + ... + \frac{a_{k-1}}{a_k} + v_{k-1} + v_k + \frac{a_{k+1}}{a_k}v_{k+1} + ... + \frac{a_n}{a_k}v_n \\
\Leftrightarrow \quad & v_k = -\frac{a_1}{a_k}v_1 - ... - \frac{a_{k-1}}{a_k} - v_{k-1} - \frac{a_{k+1}}{a_k}v_{k+1} - ... - \frac{a_n}{a_k}v_n
\end{aligned}$$

Observe that v_k can be any vector including the zero vector.
($\Leftarrow$)
Say v_i is a linear combination of the other vectors. Then,

$$\begin{aligned}
& v_i = b_1v_1 + ... + b_{i-1}v_{i-1} + b_{i+1}v_{i+1} + ... + b_nv_n \\
\Leftrightarrow \quad & 0 = b_1v_1 + ... + b_{i-1}v_{i-1} - v_i + b_{i+1}v_{i+1} + ... + b_nv_n
\end{aligned}$$

So, the scalar on v_i is -1 which is not zero.
And we are done. $\square$

4.11 Basis

Definition. A set $B = \{u_1, ..., u_n\}$ of vectors in V is a **basis** of V if two conditions are satisfied,

i) B is a linearly independent set

ii) $span(B) = V$

Example 4.11.1. The following are examples of basis,

1) $\mathbb{R}^n$

$$e_1 = (1, 0, ..., 0), e_2 = (0, 1, 0, ..., 0), e_n = (0, ..., 0, 1)$$

$\{e_1, ..., e_n\}$ is a basis for $\mathbb{R}^n$.

$(1, 2, 3) = e_1 + 2e_2 + 3e_3$. This is the canonical or standard basis.

2) $P_m(t)$

$$\{1, t, t^2, ..., t^m\}$$

3) $M_{m\times n}$ For $M_{2\times 3}$ it is,

$$\begin{bmatrix}1 & 0 & 0\\0 & 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1 & 0\\0 & 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0 & 1\\0 & 0 & 0\end{bmatrix}$$

$$\begin{bmatrix}0 & 0 & 0\\1 & 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0 & 0\\0 & 1 & 0\end{bmatrix}, \begin{bmatrix}0 & 0 & 0\\0 & 0 & 1\end{bmatrix}$$

Observe that a basis can also be the minimum span of a vector space.

Example 4.11.2.

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\{R_1, R_2, R_3\}$ is linearly independent as none of the rows are linear combinations of the others.
Observe that R_1, R_2, R_3 spans the rowspace of A.

Theorem 4.11.1. The non-zero rows of a matrix in echelon form are linearly independent and form a basis for the row space.

Proposition 4.11.2. A set $B = \{v_1, ..., v_n\}$ is a basis of V iff every vector $v \in V$ can be uniquely written as a linear combination of $v_1, ..., v_n$.

Proof. $(\Rightarrow)$
Suppose $v = a_1v_1 + ... + a_nv_n \quad a_1, ..., a_n \in \mathbb{R}$ and
$v = b_1v_1 + ... + b_nv_n \quad b_1, ..., b_n \in \mathbb{R}$.
We want to show that $a_i = b_i \quad \forall i = 1...n$.
We have,

$$\begin{aligned} & a_1v_1 + ... + a_nv_n = b_1v_1 + ... + b_nv_n \\ \Leftrightarrow \quad & 0 = (a_1 - b_1)v_1 + ... + (a_n - b_n)v_n \end{aligned}$$

Since $\{v_1, ..., v_n\}$ is linearly independent,

$$\begin{aligned} & a_1 - b_1 = 0, ..., a_n - b_n = 0 \\ \Leftrightarrow \quad & a_1 = b_1, ..., a_n = b_n \end{aligned}$$

$(\Leftarrow)$
Suppose $a_1v_1 + ... + a_nv_n = 0$. We have $a_1, ..., a_n \in \mathbb{R}$. To show that B is linearly independent, we must show $a_1 = ... = a_n = 0$,

$$0 = 0(v_1) + ... + 0(v_n)$$

We assumed the linear combination is unique,

$$a_1 = ... = a_n = 0$$

We assumed every vector can be written as a linear combination. B spans V by the assumption.
And we are done. $\square$

4.12 Coordinates

Definition. Let V be a vector space and $B = \{v_1, ..., v_n\}$ a basis of V. Then for any $v \in V$,

$$v = a_1v_1 + ... + a_nv_n$$

where the $a_1, ..., a_n$ are unique for v.
We call these scalars the **coordinates** of v in the basis B and they form a vector $[a_1, ..., a_n]$ called coordinate vectors of v relative to B and denoted by $[v]_B$.

Example 4.12.1. $t + 1, t - 1, (t - 1)^2$ form a basis of $P_2(t)$. This can be written as :

$$P(t) = (t + 1) - (t - 1) + (t - 1)^2 = t^2 - 2 + 2$$

or as coordinates: $[P(\cdot)]_B = [1, -1, 1]$.

4.13 Dimension

Next we will give a series of auxiliary lemmas and propositions to show that the size of any two basis for a vector space has the same number of vectors.

Proposition 4.13.1. Let V be a vector space and $S = \{v_1, ..., v_n\}$ be a spanning set of V. Then,

i) if $w \in V$, $\{w, v_1, ..., v_n\}$ is linearly dependent and spans V.

ii) if v_i is a linear combination of the other vectors, S without v_i still spans V.

Lemma 4.13.2. Suppose $\{v_1, ..., v_k\}$ is linearly dependent and all the vectors are non-zero. Then one of the vectors is a linear combination of the preceding vectors.

Proof. Then there are scalars, not all zero such that,

$$a_1v_1 + ... + a_nv_n = 0 \quad \text{where } a_1, ..., a_k \in \mathbb{R}$$

Let i be the largest index such that $a_i \neq 0$. We claim $i > 1$. If $i = 1$, then $a_1v_1 = 0$, which is a contradiction since $v_1 \neq 0$.

$$\begin{aligned} &\Rightarrow \quad a_1v_1 + ... + a_iv_i = 0 \\ &\Leftrightarrow \quad v_i = -\frac{a_1}{a_2} - ... - \frac{a_{i-1}}{a_i}v_{i-1} \end{aligned}$$

□

Lemma 4.13.3. <u>"Replacement Lemma".</u>
Suppose $\{v_1, ..., v_n\}$ spans V and $\{w_1, ..., w_m\}$ is linearly independent. Then $m \leq n$ and V is spanned by a set of the form ,

$$\{w_1, ..., w_m, v_{i_1}, ..., v_{i_{n-m}}\}$$

Thus any $n + 1$ or more vectors in V are linearly dependent.

Proof. (General idea) In $span\{v_1, ..., v_n\} = V$, we add w_1.
$span\{w_1, v_1, ..., v_n\} = V$ but it is now linearly dependent. Assume $v_1, ..., v_n$ are all non-zero.
By proposition and lemma from above, we can remove v_i so that

$$span\{w_1, v_1, ..., v_{i-1}, v_{i+1}, ..., v_n\} = V$$

Repeat the above steps when adding w_2 and removing v_k and we get,

$$span\{w_1, w_2, v_1, ..., v_{k-1}, v_{k+1}, ..., v_{i-1}, v_{i+1}, ..., v_n\} = V$$

If $m \leq n$,

$$span\{w_1, ..., w_m, v_{i_1}, ..., v_{i_{n-m}}\} = V$$

Suppose $m > n$,

$$span\{w_1, ..., w_n\} = V$$

If we add w_{n+1}, then $span\{w_1, ..., w_n, w_{n+1}\} = V$ and is linearly dependent. But this is a contradiction, $w_1, ..., w_m$ were all linearly independent. The only possible case is then $m \leq n$. □

Remark 4.13.1. The Replacement Lemma tells us that the size of any spanning set is at least as big as the size of any linearly independent set.

The preceding lemmas and propositions in this section were auxiliary to proving the following theorem.

Theorem 4.13.4. Let $\{u_1, ..., u_m\}$ and $\{v_1, ..., v_m\}$ be basis for V. Then $m = n$.

Proof. The proof is relatively simple as we have the Replacement Lemma.
$\{u_1, ..., u_m\}$ is (1) linearly independent and (2) spans V.
$\{v_1, ..., v_n\}$ (3) spans V and (4) is linearly independent.
Apply the remark on the Replacement Lemma twice to statements (1) with (3) and (2) with (4). Since, $m \leq n$ and $m \geq n$, then $n = m$. □

Definition. A vector space is said to be of **finite dimension** or **n-dimensional**, written $dim(V) = n$ if V has a basis with n elements. The vector space $\{0\}$ has dimension 0. If a vector V does not have a finite basis, then V is said to be of infinite dimension or infinite dimensional.

Example 4.13.1. The following are examples of dimensions,

1) $dim(\mathbb{R}^n) = n$

2) $dim(M_{m \times n}) = mn$

3) $dim(P_n(t)) = n + 1 \quad$ because of $(t^0, t^1, ..., t^n)$

4) $dim(P(t)) = \infty$

To find the basis of a set of vectors, put the vectors into matrix rows and find the echelon form.

Theorem 4.13.5. Let V be a vector space of dimension n. Then,

i) any $n+1$ or more vectors are linearly dependent, and

ii) any linearly independent set $S = \{u_1, ..., u_n\}$ with n elements is a basis of V, and

iii) any spanning set $\{v_1, ..., v_n\}$ is a basis of V.

Proposition 4.13.6. Let V be a vector space and $\{v_1, ..., v_k\}$ a linearly independent set. Suppose that $w \in V$ with $w \notin span\{v_1, ..., v_k\}$, then $\{w, v_1, ..., v_k\}$ is linearly independent.

Theorem 4.13.7. "Basis Extension"
Let V be a vector space of dimension m and a set $\{v_1, ..., v_k\}$ of vectors in V that is linearly independent and $k < n$. Then there exist $n-k$ vectors $w_1, ..., w_{n-k}$ such that $\{v_1, ..., v_k, w_1, ..., w_{n-k}\}$ is a basis of V.

Proof. $\{v_1, ..., v_k\}$ is not a basis since $k < n$ and basis of V have n elements.
So, $\{v_1, ..., v_k\}$ does not span V.
Then there is $w_1 \in V$ such that $w_1 \notin span\{v_1, ..., v_k\}$.
By the proposition, $\{w_1, v_1, ..., v_k\}$ is linearly independent.
If $k+1 = n$, we are done.
If $k+1 < n$, we repeat the argument $n-k$ times to get a set $\{v_1, ..., v_k, w_1, ..., w_{n-k}\}$ that is linearly independent. This set has n elements and is linearly independent, so part (ii) of the above theorem guarantees that it is a basis. □

Theorem 4.13.8. Let W be a subspace of an n-dimensional vector space V. Then,

$$dim(W) \leq n$$

Proof. Suppose we have the basis of W with k elements. By the replacement lemma, $k \leq n$. □

4.14 Application to Matrices

Exercise 4.14.1. Find a basis for the row space, column space, and null space of the matrix A.

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 7 & 7 & 8 \\ 1 & 3 & 2 & 4 & 6 \\ 1 & 2 & 3 & 3 & 4 \end{bmatrix}$$

Row space:

Put the matrix into row-reduced form.

$$A \sim \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & -1 & -2 & -2 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = M$$

$$\begin{aligned} rowsp(A) &= rowsp(M) \\ &= span\{(1,2,3,4,5),(0,1,-1,-1,2),(0,0,0,-1,1)\} \end{aligned}$$

Therefore $\{(1,2,3,4,5),(0,1,-1,-1,2),(0,0,0,-1,1)\}$ is a basis of $rowsp(A)$.

$$dim(rowsp(A)) = 3$$

Column space:

$$col(A) = span\{(1,2,1,1),(2,3,3,2),(4,7,4,3)\}$$

$$M = \begin{bmatrix} \boxed{1} & 2 & 3 & 4 & 5 \\ 0 & \boxed{1} & -1 & -2 & -2 \\ 0 & 0 & 0 & \boxed{-1} & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The boxed entries (the pivots) are the columns of the column space in the original matrix.
Otherwise, we must check whether every column vector in $col(A)$ can be added to the basis (i.e. are linear combinations of the preceding column vectors.

$$dim(col(A)) = 3$$

Null space:

$$\left[\begin{array}{ccccc|c} 1 & 2 & 3 & 4 & 5 & 0 \\ 0 & 1 & -1 & -1 & -2 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$x_3 = t \quad x_5 = s$$

$$\begin{aligned} 0 &= -x_4 + x_5 \\ 0 &= x_2 - x_3 - x_4 - 2x_5 \\ 0 &= x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 \end{aligned}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = t \begin{bmatrix} -5 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 3 \\ 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

$$N(A) = span\{(-5,1,1,0,0),(3,1,0,-1,1)\}$$

So, $\{(-5,1,1,0,0),(3,1,0,-1,1)\}$ is a basis for $N(A)$.

$$dim(N(A)) = 2$$

Theorem 4.14.1. "Fundamental Theorem of Linear Algebra, part I
Let $A \in M_{m\times n}$ with $rank(A) = r$. Then the row space and the column space both have dimension r and the null space has dimension $m - r$.

4.15 Sums and Direct Sums

Definition. Let U and W be subspaces of a vector space V. The **sum** of V and W is given by,

$$U + W = \{v \in V \mid v = u + w, u \in U, w \in W\}$$

Vector space sum is not the same as set union.

Example 4.15.1. The following are examples of the sum of subspaces of $M_{2\times 2}$.

$$U = \left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \mid a, b \in \mathbb{R} \right\} \quad dim(U) = 2$$

$$W = \left\{ \begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix} \mid a, c \in \mathbb{R} \right\} \quad dim(W) = 2$$

$$W = \left\{ \begin{bmatrix} a & b \\ c & 0 \end{bmatrix} \mid a, b, c \in \mathbb{R} \right\} \quad dim(U + W) = 3$$

$$U \cap W = \left\{ \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \mid a \in \mathbb{R} \right\} \quad dim(U \cap W) = 1$$

So we might have,

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \in U + W \text{ but, } \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \notin U \cap W$$

Proposition 4.15.1. Let U and W be subspaces of V. Then,

i) $U + W$ is a subspace;

ii) U and W are contained in $U + W$;

iii) $U + W$ is the smallest subset containing U and W, i.e. if S is a subspace containing U and W then $U + W \subseteq S$

iv) $W + W = W$

Theorem 4.15.2. Suppose that U and W are finite dimensional subspaces of V. Then $U + W$ also has finite dimension and,

$$dim(U + W) = dim(U) + dim(W) - dim(U \cap W)$$

Proof here

Definition. The vector space V is said to be the **direct sum** of its subspaces U and W denoted by $V = U \oplus W$, if every $v \in V$ can be written in one and only one way as $v = u + w$ with $u \in U, w \in W$.

Theorem 4.15.3. The vector space V is the direct sum of U and W iff (i) $V = U + w$ and (ii) $U \cap W = \{0\}$.

Example 4.15.2. The follow are examples of vector space sums.

1)
$$\begin{aligned} U &= \{(a, b, 0) \mid a, b \in \mathbb{R}\} \quad \text{[xy-plane]} \\ W &= \{(0, b, c) \mid b, c \in \mathbb{R}\} \quad \text{[yz-plane]} \\ \mathbb{R}^3 &= U + W \end{aligned}$$

$$\begin{aligned} U \cap W &= \{(0, b, 0) \mid b \in \mathbb{R}\} \quad \text{[y-axis]} \\ &\neq \{0\} \\ (1, 1, 1) &= (1, 1, 0) + (0, 0, 1) \\ &= (1, 0, 0) + (0, 1, 1) \end{aligned}$$

2)
$$\begin{aligned} U &= \{(a, 0, 0) \mid a \in \mathbb{R}\} \quad \text{[x-axis]} \\ W &= \{(0, b, 0) \mid b \in \mathbb{R}\} \quad \text{[y-axis]} \\ &U + W \quad \text{[xy-plane]} \end{aligned}$$

$U \cup W$ is just the union of the axis, it is not a subspace.

Chapter 5

Linear Transformations

5.1 Basic Definitions

A transformation is of the form,

$$v \to Av \quad A \in M_{m\times n}, v \in \mathbb{R}^n, Av \in \mathbb{R}^m$$

It has the following operations,

- $A(v+w) = Av + Aw$
- $k \in \mathbb{R}, A(kv) = k(Av)$

5.2 Introduction to Linear Transformations

Definition. Let V and U be vector spaces. A function $T : V \to U$ is a **linear transformation** or linear mapping if the following two conditions are satisfied:

i) For any vectors $v, w \in V$, $T(v+w) = T(v) + T(w)$. (T preserves vector addition.)

ii) For any $v \in V$ and $k \in \mathbb{R}$, $T(kv) = kT(v)$ (T preserves scalar multiplication.)

Remark 5.2.1. (i) and (ii) can be replaced by the following one. For any $v, w \in V$ and $a, b \in \mathbb{R}$,

$$T(av + bw) = aT(v) + bT(w)$$

Example 5.2.1. The following are examples for linear transformations.

1) Multiplication by matrix $A \in M_{m\times n}$.

$$L_A : \mathbb{R}^n \to \mathbb{R}^m \quad L_A(v) = Av \forall v \in \mathbb{R}^n$$

2) Projection.

$$P : \mathbb{R}^3 \to \mathbb{R}^2$$

This is the projection into the xy-plane.

$$\begin{aligned} P(x, y, z) &= (x, y, 0) \\ P(x, y, z) &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \end{aligned}$$

Let $v, w \in V$ and $a, b \in \mathbb{R}$,

$$\begin{aligned} v &= (v_1, v_2, v_3) \\ w &= (w_1, w_2, w_3) \end{aligned}$$

$$\begin{aligned} P(av + bw) &= P(av_1 + bw_1, av_2 + bw_2, av_3 + bw_3) \\ &= (av_1 + bw_1, av_2 + bw_2, 0) \\ &= (av_1, av_2, 0) + (bw_1, bw_2, 0) \\ &= a(v_1, v_2, 0) + b(w_1, w_2, 0) \\ &= aP(v_1, v_2, v_3) + bP(w_1, w_2, w_3) \end{aligned}$$

3) Reflection.

$$R : \mathbb{R}^2 \to \mathbb{R}^2$$

This is the reflection in the y-axis.

$$\begin{aligned} R(x, y) &= R(-x, y) \\ &= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \end{aligned}$$

4) Derivatives.

$$D : P(t) \to P(t)$$

$$D(p(\cdot)) = \frac{\mathrm{d}P}{\mathrm{d}t}(\cdot)$$

For example, given $P(t) = 1 - t^2, D(P(t)) = 2t$.

5) Integrals.

$$J : P(t) \to \mathbb{R}$$

$$J(D(\cdot)) = \int_0^1 P(t)\,\mathrm{d}t$$

Given $p(\cdot), q(\cdot) \in P(t), a, b \in \mathbb{R}$,

$$\begin{aligned} J(ap(\cdot)+bq(\cdot)) &= \int_0^1 (ap(\cdot)+bq(\cdot)))\,\mathrm{d}t \\ &= \int_0^1 ap(\cdot)\mathrm{d}t + \int_0^1 bq(\cdot)\,\mathrm{d}t \\ &= a\int_0^1 p(\cdot)\mathrm{d}t + b\int_0^1 q(\cdot)\,\mathrm{d}t \\ &= aJ(p(\cdot)) + bJ(q(\cdot)) \end{aligned}$$

6) Zero and Identity Transformations. Given vector spaces V and U,

Zero transformation:	$\bar{0} : V \to U$
	$\bar{0}(v) = 0 \in U$
Identity transformation:	$1_v : V \to V$
	$1_v(v) = v$

Remark 5.2.2. If T is a linear transformation then $T(0) = 0$.

Proof. Take $k = 0$ in (ii) of the definition of linear transformations.

$$\begin{aligned} T(kv) &= kT(v) \\ T(0v) &= 0T(v) \\ T(0) &= 0 \end{aligned}$$

□

Remark 5.2.3. For any scalars $a_i \in \mathbb{R}$ and vectors $v_i \in V$,

$$T(a_1v_1 + ... + a_nv_n) = a_1T(v_1) + ... + a_nT(v_n)$$

Proof. By induction.
Base case ($n = 1$): $T(a_1v_1) = a_1T(v_1)$ [By (ii)]
Induction step ($n = k \Rightarrow n = k + 1$):
Suppose the remark is true for $n = k$. We want to show that the remark is true when $n = k + 1$.

$$\begin{aligned} &T(a_1v_1 + ... + a_{k+1}v_{k+1}) \\ =&T((a_1v_1 + ... + a_kv_k) + a_{k+1}v_{k+1}) \\ =&T(a_1v_1 + ... + a_kv_k) + T(a_{k+1}v_{k+1}) \quad \text{[by (i)]} \\ =&a_1T(v_1) + ... + a_kT(v_k) + T(a_{k+1}v_{k+1}) \quad \text{[Induction hypothesis,]}n = k \\ =&a_1T(v_1) + ... + a_kT(v_k) + a_{k+1}T(v_{k+1}) \quad \text{[by (ii)]} \end{aligned}$$

□

Exercise 5.2.1. How many linear transformations $T : \mathbb{R}^2 \to \mathbb{R}^2$ are there such that $T(1,0) = (-1,1)$ and $T(0,1) = (1,-1)$ exist?

$$\begin{aligned} T(x,y) =& T(x(1,0) + y(0,1)) \\ =& xT(1,0) + yT(0,1) \\ =& x(-1,1) + y(1,-1) \\ =& (-x,x) + (y,-y) \\ =& (-x+y, x-y) \end{aligned}$$

There is one and only one such linear transformation.

Proposition 5.2.1. Let $T : V \to U$ and $S : V \to U$ be two linear transformations. Suppose that $\{v_1, ..., v_n\}$ spans V. If $T(v_i) = S(v_i)\, \forall i = 1, ..., n$, then $T = S$.

Proof. We want to show that,

$$T(v) = S(v)\, \forall v \in V$$

Let $v \in V$. Since $\{v_1, ..., v_n\}$ spans V, these exist scalars $a_1, ..., a_n \in \mathbb{R}$ such that,

$$v = a_1 v_1 + ... + a_n v_n$$

Then,

$$\begin{aligned} T(v) =& a_1 T(v_1) + ... + a_n T(v_n) \\ =& a_1 S(v_1) + ... + a_n S(v_n) \quad \text{[by condition of proposition]} \\ =& S(v) \end{aligned}$$

□

Example 5.2.2. The following is an example of what is not a linear transformation.

$$T : \mathbb{R}^2 \to \mathbb{R}^2$$

Adds the vector $(1,2)$ to any vector $(x,y) \in \mathbb{R}^2$

$$\begin{aligned} T(x,y) =& (x,y) + (1,2) = (x+1, y+2) \\ & \text{Case:} \\ T(0,0) =& (1,2) \neq (0,0) \\ & \text{Case:} \\ T(2,2) =& (3,4) \\ 2T(1,1) =& 2(2,3) = (4,6) \neq (3,4) \end{aligned}$$

Theorem 5.2.2. Let U and V be vector spaces and let $\{v_1, ..., v_n\}$ be a basis of V. Given any vectors $u_1, ..., u_n \in U$, there exists $T : U \to V$, a unique linear transformation such that $T(v_i) = u_i \quad \forall i = 1, ..., n$.

Proof. Step 1: We begin by defining the function $T : V \to U$ such that,

$$T(v_i) = u_i \quad \forall i = 1, ..., n$$

Let $v \in V$. Since $\{v_1, ..., v_n\}$ is a basis then there exists scalars $a_1, ..., a_n$ such that,

$$v = a_1v_1 + ... + a_nv_n$$

We define $T : V \to U$ as,

$$T(v) = a_1u_1 + ... + a_nu_n$$

T is well defined since for each $v \in V$, the scalars a_i are unique.

$$\begin{aligned} T(v_i) &= T(0v_1 + ... + 0v_{i-1} + v_i + 0v_{i+1} + ... + 0v_n) \\ &= u_i \end{aligned}$$

Step 2: Show that T is a linear transformation.
Let $v, w \in V$ and $\alpha, \beta \in \mathbb{R}$. We want to show that,

$$T(\alpha v + \beta w) = \alpha T(v) + \beta T(w)$$

There exist scalars $a_1, ..., a_n, b_1, ..., b_n$ such that,

$$\begin{aligned} v &= a_1v_1 + ... + a_nv_n \\ w &= b_1v_1 + ... + b_nv_n \end{aligned}$$

since $\{v_1, ..., v_n\}$ is a basis of V.

$$\begin{aligned} T(\alpha v + \beta w) &= T((\alpha a_1 + \beta b_1)v_1 + ... + (\alpha a_n + \beta b_n)v_n) \\ &= (\alpha a_1 + \beta b_1)u_1 + ... + (\alpha a_n + \beta b_n)u_n \\ &= \alpha(a_1u_1 + ... + a_nu_n) + \beta(b_1u_1 + ... + b_nu_n) \\ &= \alpha T(v) + \beta T(w) \end{aligned}$$

Step 3: Show that T is unique.
Suppose that $G : V \to U$ is a linear transformation such that $G(v_i) = u_i \quad \forall i = 1, ..., n$. So $G(v_i) = T(v_i) \quad \forall i = 1, ..., n$ where $\{v_1, ..., v_n\}$ is a spanning set (it is actually a basis) and thus by the proposition, $G = T$. $\square$

Definition. Let $T : V \to U$ be a linear transformation. The **kernel** of T is given by,

$$ker(T) = \{v \in V \mid T(v) = 0\}$$

Definition. Let $T : V \to U$ be a linear transformation. The **image** of T is given by,

$$im(T) = \{u \in U \mid \exists v \in V : T(v) = u\}$$

Example 5.2.3. On the relation to matrices.

$$A \in M_{m\times n} \quad L_A : \mathbb{R}^n \to \mathbb{R}^m : L_A(v) = Av$$
$$ker(L_A) = N(A)$$
$$im(L_A) = col(A) = \{Av : v \in \mathbb{R}^n\}$$

Remark 5.2.4. An image can equivalently be given by,

$$\begin{aligned} im(T) &= \{u \in U \mid \exists v \in V : T(v) = u\} \\ &= \{T(v) \mid v \in \mathbb{R}^n\} \end{aligned}$$

Theorem 5.2.3. Let $T : V \to U$ be a linear transformation. Then $ker(T)$ is a subspace of V and $im(T)$ is a subspace of U.

COPY PROOF FROM ASSIGNMENT

Example 5.2.4. 1. Let $P : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation defined as, $P(x, y, z) = (x, y, 0)$. This is the projection onto the xy-plane.

$$\begin{aligned} ker(P) &= \{(x, y, z) \in \mathbb{R}^3 \mid x = y = 0\}\text{z-axis} \\ im(P) &= \{(x, y, z) \in \mathbb{R}^3 \mid z = 0\}\text{xy-axis} \end{aligned}$$

2. Let $D^2 : P(t) \to P(t)$ be a linear transformation defined as, $D^2(p(\cdot)) = \frac{\mathrm{d}^2 p}{\mathrm{d}t^2}(\cdot)$

$$\begin{aligned} p(t) &= a_0 + a_1 t + a_2 t^2 + ... + a_n t^n \\ \frac{\mathrm{d}p}{\mathrm{d}t}(t) &= a_1 + 2a_2 t + 3a_3 t^2 + ... + na_n t^{(n-1)} \\ \frac{\mathrm{d}^2 p}{\mathrm{d}t^2}(t) &= 2a_2 + 3 \cdot 2a_3 t + ... + n(n-1)a_n t^{(n-2)} \end{aligned}$$

$ker(D^2) = P_1(t)$ [degree 1 or less disappears]

$im(D^2) = P(t)$ [all polynomials can result]

Proposition 5.2.4. Suppose that $\{v_1, ..., v_n\}$ is a spanning set of V and let $T : V \to U$ be a linear transformation. Then $\{T(v_1), ..., T(v_n)\}$ spans $im(T)$.

Proof. Let $u \in im(T)$. Then $\exists v \in V$ such that $u = T(v)$. Since $\{v_1, ..., v_n\}$ is a spanning set of V, $\exists$ scalars $a_1, ..., a_n$ such that, $v = a_1 v_1 + ... + a_n v_n$.

$$\begin{aligned} u &= T(v) \\ &= T(a_1 v_1 + ... + a_n v_n) \\ &= a_1 T(v_1) + ... + a_n T(v_n) \end{aligned}$$

□

Example 5.2.5. Let $L_A : \mathbb{R}^4 \to \mathbb{R}^3$ be the linear transformation given by $L_A(x, y, z, t) = (x - y + t, 2x - y + z + 3t, 3x + y + 4z + 7t)$

$$A = \begin{bmatrix} 1 & -1 & 0 & 1 \\ 2 & -1 & 1 & 3 \\ 3 & 1 & 4 & 7 \end{bmatrix}$$

Let $e_1, ..., e_4$ be the canonical basis for $\mathbb{R}^4$.

$$\begin{aligned} L_A(e_1) &= (1, 2, 3) & L_A(e_3) &= (0, 1, 4) \\ L_A(e_2) &= (-1, -1, 1) & L_A(e_4) &= (1, 3, 7) \end{aligned}$$

$$\begin{aligned} im(L_A) &= span\{L_A(e_1), L_A(e_2), L_A(e_3), L_A(e_4)\} \\ &= span\{(1, 2, 3), (-1, -1, 1), (0, 1, 4), (1, 3, 7)\} \\ &= col(A) \end{aligned}$$

Since the image is a subspace, we might want to know properties such as dimension of L_A.

$$A = \begin{bmatrix} 1 & -1 & 0 & 1 \\ 2 & -1 & 1 & 3 \\ 3 & 1 & 4 & 7 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & -1 & 0 & 1 \\ 0 & \boxed{1} & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} im(L_A) &= col(A) = span\{(1, 2, 3), (-1, -1, 1)\} \\ dim(im(L_A)) &= 2 \\ ker(L_A) &= N(A) \end{aligned}$$

z and t are free variables so,

$$= \begin{cases} z = s_1 \\ t = s_2 \\ y + z + t = 0 \\ x - y + t = 0 \end{cases}$$

$$t = s_2 \quad z = s_1 \quad y = -s_1 - s_2 \quad x = -s_1 - 2s_2$$

$$\begin{aligned} ker(L_A) &= N(A) \\ &= span\{(-1, -1, 1, 0), (-2, -1, 0, 1)\} \end{aligned}$$

So given that the two spanning vectors are linearly independent, $dim(ker(L_A)) = 2$.

$$\begin{aligned} dim(N(A)) + dim(col(A)) &= \#\text{ of columns} \\ dim(ker(L_A)) + dim(im(L_A)) &= dim(\mathbb{R}^4) \\ 2 + 2 &= 4 \end{aligned}$$

Definition. Let $T : V \to U$ be a linear transformation. The **rank** of T is the dimension of $im(T)$. The **nullity** of T is the dimension of $ker(T)$.

$$rank(T) = dim(im(T))$$
$$nullity(T) = dim(ker(T))$$

Theorem 5.2.5. Rank-Nullity Theorem.
Let $T : V \to U$ be a linear transformation with V a vector space of dimension n. Then,

$$\begin{aligned} dim(V) &= rank(T) + nullity(T) \\ &= dim(im(T)) + dim(ker(T)) \end{aligned}$$

Lemma 5.2.6. $ker(T)$ is a subspace of V and V has finite dimension. Then $dim(ker(T)) \leq dim(V)$.
COPY PROOF FROM ASSIGNMENT

Proof. By a previous proposition, $dim(im(T)) \leq dim(V)$. We can suppose that $\{u_1, ..., u_s\} \in V$ is a basis for $ker(T)$. $\{w_1, ..., w_r\} \in U$ is a basis for $im(T)$.
By the definition of $im(T)$, there exist vectors $v_1, ..., v_r \in V$ such that $T(v_i) = w_i \quad \forall i = 1, ..., r$.
Consider $B = \{u_1, ..., u_s, v_1, ..., v_r\}$. B has $s + r$ elements. We claim that B is a basis of V. Then,

$$\begin{aligned} dim(V) &= s + r \\ &= dim(ker(T)) + dim(im(T)) \end{aligned}$$

We must prove this claim,

- B spans V.

  Let $v \in V$. Then $T(v) \in im(T)$. Since $\{w_1, ..., w_r\}$ is a basis for the image, $\exists$ scalars $a_1, ..., a_r$ such that,

$$\begin{aligned} T(v) &= a_1w_1 + ... + a_rw_r \\ &= a_1T(v_1) + ... + a_rT(v_r) \quad \text{[from above]} \\ &= T(a_1v_1 + ... + a_rv_r) \\ 0 &= T(v) - T(a_1v_1 + ... + a_rv_r) \\ \Leftrightarrow 0 &= T(v - a_1v_1 + ... + a_rv_r) \end{aligned}$$

$$v - a_1v_1 - ... - a_rv_r \in ker(T)$$

  Since $\{u_1, ..., u_s\}$ is a basis of $ker(T)$, there exist $b_1, ..., b_s$ such that,

$$\begin{aligned} v - a_1v_1 - ... - a_rv_r &= b_1u_1 + ... + b_su_s \\ \Leftrightarrow v &= b_1u_1 + ... + b_su_s + a_1v_1 + ... + a_rv_r \end{aligned}$$

- B is linearly independent.

  Suppose that,

$$b_1u_1 + ... + b_su_s + a_1v_1 + ... + a_rv_r = 0 \quad (*)$$

  where $b_1, ..., b_s, a_1, ..., a_r \in \mathbb{R}$. We have to show that $b_1 = ... = b_s = a_1 = ... = a_r = 0$.

  We have,

$$\begin{aligned}
&T(b_1u_1 + ... + b_su_s + a_1v_1 + ... + a_rv_r) = T(0) \\
\Leftrightarrow &b_1T(u_1) + ... + b_sT(u_s) + a_1T(v_1) + ... + a_rT(v_r) = 0 \quad \text{[by def]} \\
\Leftrightarrow &b_1(0) + ... + b_s(0) + a_1w_1 + ... + a_rw_r = 0 \\
\Leftrightarrow &a_1w_1 + ... + a_rw_r = 0
\end{aligned}$$

  Since $\{w_1, ..., w_r\}$ is a basis, $a_1 = ... = a_r = 0$.

Now we can rewrite $(*)$ as,

$$b_1u_1 + ... + b_su_s = 0$$

Since $\{u_1, ..., u_s\}$ is a basis, then,

$$b_1 = ... = b_s = 0$$

$\square$

Chapter 6

Linear Transformations and Matrices

6.1 Introduction

This chapter will depend greatly on applications of the two previous chapters. Most definitions and theorems will follow an example that demonstrates that result. You have been warned.

Example 6.1.1. $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation given by $T(x, y) = (x + 2y, -x + y)$.

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} \quad L_A = T$$

$$A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + 2y \\ -x + y \end{bmatrix}$$

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Definition. Let $T : V \to V$ be a linear transformation and $S = \{u_1, ..., u_n\}$ is a basis of V. The matrix whose columns are the coordinate vectors of $T(u_1), ..., T(u_n)$ relative to S is denoted by $[T]_s$ and is called the matrix representation of T relative to the basis S.

$$[T]_s = [[T(u_1)]_s, ..., [T(u_m)]_s]$$

Example 6.1.2. Continued from above.

$$V = \mathbb{R}^2 \quad E = \{(1,0),(0,1)\}$$

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} + (-1) \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$[T(u_1)]_E = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$
$$[T(u_2)]_E = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Example 6.1.3. $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation given by $T(x,y) = (x+2y, -x+y)$. We pick a basis $S = \{u_1, u_2\} = \{(1,2),(-1,1)\}$.

$$T(u_1) = T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 5 \\ 1 \end{bmatrix} = x\begin{bmatrix} 1 \\ 2 \end{bmatrix} + y\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

The solution is,

$$\begin{cases} x - y = 5 \\ 2x + y = 1 \end{cases} \Leftrightarrow \begin{cases} x = 2 \\ y = -3 \end{cases} \qquad [T(u_1)]_S = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

Similarly, we have for $T(u_2)$,

$$T(u_2) = T\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = x\begin{bmatrix} 1 \\ 2 \end{bmatrix} + y\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

The solution is,

$$\begin{cases} x = 1 \\ y = 0 \end{cases} \qquad [T(u_2)]_S = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

And thus,

$$[T]_S = \begin{bmatrix} 2 & 1 \\ -3 & 0 \end{bmatrix}$$

Say we take some $v = (0,3)$ under the basis E, we have,

$$T\left(\begin{bmatrix} 0 \\ 3 \end{bmatrix}\right) = Av = [T]_E[v]_E = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$$

To find $[v]_S$, we solve the following,

$$\begin{bmatrix} 0 \\ 3 \end{bmatrix} = x\begin{bmatrix} 1 \\ 2 \end{bmatrix} + y\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

So we have,

$$\begin{cases} x - y = 0 \\ 2x + y = 3 \end{cases} \Leftrightarrow \begin{cases} x = 1 \\ y = 1 \end{cases} \qquad [v]_S = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

To find $[T(v)]_S$, we solve the following,

$$T(v) = \begin{bmatrix} 6 \\ 3 \end{bmatrix} = x \begin{bmatrix} 1 \\ 2 \end{bmatrix} + y \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

So we have,

$$\begin{cases} x - y = 6 \\ 2x + y = 3 \end{cases} \Leftrightarrow \begin{cases} x = 3 \\ y = -3 \end{cases} \qquad [T(v)]_s = \begin{bmatrix} 3 \\ -3 \end{bmatrix}$$

And finally we can verify that the following holds,

$$[T]_S[v]_S = \begin{bmatrix} 2 & 1 \\ -3 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \end{bmatrix} = [T(v)]_S$$

Example 6.1.4. The following is an example of basis of functions.
We pick a basis $S = \{\sin t, \cos t, e^t, te^t\}$. Let $V = span(S)$. Let $D : V \to V$ be a linear transformation given by $D(f(\cdot)) = \frac{\mathrm{d}f}{\mathrm{d}t}(\cdot)$.
Write each of the following in the basis in D,

1. $$D(\sin t) = \cos t \quad [D(\sin t)]_S = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

2. $$D(\cos t) = -\sin t \quad [D(\cos t)]_S = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

3. $$D(e^t) = e^t \quad [D(e^t)]_S = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

4. $$D(te^t) = e^t + te^t \quad [D(te^t)]_S = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

With the above, we can conclude that,

$$[D]_S = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Say we take some $f(t) = \sin t + 2\cos t + 2te^t$, then under the basis S,

$$[f(t)]_S = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 2 \end{bmatrix}$$

$$D(f(t)) = \cos t - 2\sin t + 2e^t + 2te^t$$

$$[D(f(t))]_S = \begin{bmatrix} -2 \\ 1 \\ 2 \\ 2 \end{bmatrix}$$

$$[D]_S[f(t)]_S = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 2 \\ 2 \end{bmatrix}$$

And this is exactly what we expected.

Theorem 6.1.1. Let $T : V \to V$ be a linear transformation and S a finite basis of V. Then, for any vector $v \in V$,

$$[T]_S[v]_S = [T(v)]_S$$