Math 223 Assignment 2

Due in class: February 3, 2015

Instructions: Submit a hard copy of your solution with your name and student number. Late assignments will not be graded and will receive a grade of zero.

1. (10 points) Show that the set $\{e^x, xe^x\}$ is linearly independent in $F(\mathbb{R})$, where $F(\mathbb{R})$ denotes the vector space of functions from $\mathbb{R}$ to $\mathbb{R}$.

2. (10 points) Let V be a vector space and suppose $\{u, v, w\}$ is linearly independent. Show that $\{u + w + v, u - w + v, w + v\}$ is linearly independent.

3. (10 points) Let V be a vector space and let $S = \{v_1, \dots, v_n\}$ be a spanning set of V. Given any vector $u \in V$, show that the set $\{v_1, \dots, v_n, u\}$ is linearly dependent.

4. (10 points) Let V be a vector space and let $\{v_1, \dots, v_k\}$ be a linearly independent subset of V. Suppose that $w \in V$ with $w \notin \operatorname{span}\{v_1, \dots, v_k\}$. Show that $\{v_1, \dots, v_k, w\}$ is linearly independent.

5. (15 points) Let V be a vector space of dimension n and let $\{u_1, \dots, u_n\}$ be a linearly independent set in V. Show that $B = \{u_1, \dots, u_n\}$ is a basis of V.

   Hint: Try a proof by contradiction and recall the following result seen in class: any set of $n + 1$ vectors in an n-dimensional vector space is linearly dependent.

6. (15 points) Let V be a space of dimension n and let $\{v_1, \dots, v_n\}$ be a spanning set for V. Show that $B = \{v_1, \dots, v_n\}$ is a basis for V.

7. Let X denote the set of matrices $\{A \in M_{2\times 2} : A = A^T\}$, i.e., the set of all symmetric matrices.

   (a) (5 points) Show that X is a subspace of $M_{2\times 2}$.

   (b) (5 points) Find a basis for X and compute its dimension.

8. Let W be the subspace of $\mathbb{R}^4$ spanned by the vectors

   $$u_1 = (1, 2, 1, 1), \quad u_2 = (-1, -1, 1, 2), \quad u_3 = (2, 2, 1, -1).$$

   (a) (5 points) Find a basis and dimension of W.

   (b) (5 points) Extend the basis of W to a basis of $\mathbb{R}^4$.

9. (10 points) Let W be the subspace of $P_2(t)$ spanned by the polynomials

   $$p_1(t) = 1 + t + t^2, \quad p_2(t) = 2 + 2t + 2t^2, \quad p_3(t) = -t, \quad p_4(t) = 1 + t^2.$$

   Find a subset of the polynomials that form a basis of W.