MATH 223, Winter 2015

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Chapter 1

Introduction

1.1 Administrativa

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Office: BH1036
Office Hours: W4:15-5:45PM, F3:00-4:30PM

Grading

Assignments	15%	15%
Midterm	25%	0%
Final	60%	85%

The midterm will be scheduled for the 7th week of class.

1.2 A Review of Vectors

1.2.1 Vectors in $\mathbb{R}^n$

$\mathbb{R}^n$ is the set of all n-tuples of real numbers $u = (a_1 ... a_n) \mid a \in \mathbb{R}$ where a are the **components** or **entries**.

Remark 1. We use the term **scalar** to refer to an element in $\mathbb{R}$.

1.2.2 Basic Definitions

Addition
$u, v \in \mathbb{R}^n$
$u = (a_1 ... a_n)$
$v = (b_1 ... b_n)$
$u + v = (a_1 + b_1 ... a_n + b_n)$

Scalar Multiplication
$k \in \mathbb{R}$
$ku = (ka_1 ... ka_n)$

Equality
Two vectors u and v are said to be equal $(u = v)$ if $a_i = b_i \forall i = 1...n$.

Zero Vector
The zero vector is defined as $0 = (0...0)$.

Linear Combination
Suppose we are given m vectors $u_1...u_m \in \mathbb{R}^n$ and m scalars $k_1...k_m \in \mathbb{R}$.
Let $u = k_1 u_1 + ... + k_m u_m$.
Such a vector u is called a linear combination of the vectors $u_1...u_m$.

Vector Multiple
A vector u can be called a multiple of v if there is a scalar k such that $u = kv$ with $k \neq 0$. In the case $k > 0$ we say u is in the same direction as v. In the case $k < 0$ we say u is in the opposite direction of v.

1.2.3 The Dot Product

Definition 1. Let $u = (a_1...a_n)$ and $v = (b_1...b_n)$. The **dot product** or inner product is given by,

$$u \cdot v = a_1 b_1 + ... a_n b_n =$$

Definition 2. The vectors u and v are **orthogonal** if $u \cdot v = 0$.

1.2.4 The Vector Norm

Definition 3. The **norm** or **length** of a vector is given by,

$$\|u\| = \sqrt{a_1^2 + ... + a_n^2}$$

Thus $\|u\| \geq 0$ and $\|u\| = 0$ if and only if (iff) $u = 0$.

Definition 4. A vector is called a **unit vector** if $\|u\| = 1$.

For any non-zero vector v, the vector

$$\hat{v} = \frac{1}{\|v\|} v$$

is the only unit vector with the same direction of v. The process of finding $\hat{v}$ is called **normalizing**.

1.2.5 Theorem: Cauchy-Schwarz Inequality

Theorem 1. *Given any two vectors $u, v \in \mathbb{R}^n$, then,*

$$|u \cdot v| \leq \|u\|\|v\|$$

Proof. Let $t \in \mathbb{R}$. So, $\|tu + v\|^2 \geq 0$.

$$\begin{aligned}\|tu + v\|^2 &= (tu + v)(tu + v) \\ &= (tu \cdot tu) + (tu \cdot v) + (v \cdot tu) + (v \cdot v) \\ &= t^2(u \cdot u) + t(v \cdot u) + t(u \cdot v) + (v \cdot v) \\ &= t^2\|u\|^2 + 2t(u \cdot v) + \|v\|^2\end{aligned}$$

We can represent this in the form $at^2 + bt + c \geq 0$, so,

$$a = \|u\|^2, b = 2(u \cdot v), c = \|v\|^2$$

Take the Discriminant as $b^2 - 4ac \iff b^2 \leq 4ac$.

$$\begin{aligned}4(u \cdot v)^2 &\leq 4\|u\|^2\|v\|^2 \\ |u \cdot v| &\leq \|u\|\|v\|\end{aligned}$$

□

1.2.6 Theorem: Minkowski Triangle Inequality

Theorem 2. *Given $u, v \in \mathbb{R}^n$, then $\|u + v\| \leq \|u\| + \|v\|$.*

Proof.

$$\begin{aligned}\|u + v\|^2 &= \|u\|^2 + 2(u \cdot v) + \|v\|^2 \\ &\leq \|u\|^2 + 2\|u\|\|v\| + \|v\|^2 \text{ by C-S inequality} \\ &= (\|u\| + \|v\|)^2\end{aligned}$$

So, $\|u + v\|^2 \leq (\|u\| + \|v\|)^2$. Take the square root and we are done. □

1.2.7 Geometry with Vectors

Definition 5. The **distance** between vectors $u, v \in \mathbb{R}^n$ is given by,

$$d(u, v) = \|u - v\| = \sqrt{(a_1 - b_1)^2 + ... + (a_n - b_n)^2}$$

Definition 6. The **angle** between vectors $u, v \in \mathbb{R}^n$ is given by,

$$cos\theta = \frac{u \cdot v}{\|u\|\|v\|} \quad \theta \in [0, \pi]$$

Observe that in the previous definition, the angle is well defined.

$$-\|u\|\|v\| \leq -|u \cdot v| \leq u \cdot v \leq u \cdot v \leq |u \cdot v| \leq \|u\|\|v\|$$

Dividing the entire inequality by $\|u\|\|v\|$ yields,

$$-1 \leq \frac{u \cdot v}{\|u\|\|v\|} \leq 1$$

Definition 7. A **hyperplane** $\mathcal{H}$ in $\mathbb{R}^n$ is the set of points $(x_1...x_n)$ that satisfy $a_1x_1 + ... + a_nx_n = b$ where $u = [a_1...a_n] \in \mathbb{R}^n$ and $b \in \mathbb{R}$.

Definition 8. The **line** in $\mathbb{R}^n$ passing through a point $P = (b_1...b_n)$ and in the direction of $v \in \mathbb{R}^n$ with $v \neq 0$.

$$x = P + tu \quad t \in \mathbb{R}, \quad u = [a_1...a_n]$$

$$\begin{cases} x_1 = a_1t + b_1 \\ x_n = a_nt + b_n \end{cases}$$

1.3 Algebra of Matrices