Assignment 2

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Problem 1. *Linear Independence.*

Solution. Suppose,
$$a(e^x) + b(xe^x) = 0 \quad \forall x \in \mathbb{R}$$
We want to show $a = b = 0$. Take $x = 0$ and $x = 1$,

$$\begin{array}{lll} x = 0 & a(e^0) + b(0 \cdot e^0) = 0 & \\ & a = 0 & \\ x = 1 & a(e^1) + b(e^1) = 0 & \text{From above, } a = 0 \\ & b(e) = 0 & \\ & b = 0 & \end{array}$$

So $a = b = 0$ and we are done.

Problem 2. *More Linear Independence.*

Solution. Suppose,

$$a(u + w + v) + b(u - w + v) + c(w + v) = 0$$

We want to show $a = b = c = 0$.

$$\begin{aligned} a(u + w + v) + b(u - w + v) + c(w + v) &= 0 \\ au + aw + av + bu - bw + bv + cw + cv &= 0 \\ au + bu + av + bv + cv + aw - bw + cw &= 0 \\ u(a + b) + v(a + b + c) + w(a - b + c) &= 0 \end{aligned}$$

It is given that u, v, w are linearly independent so,

$$\begin{array}{lrcll} (1) & a + b = 0 & \Leftrightarrow & a = -b & (4) \\ (2) & a + b + c = 0 & \Leftrightarrow & c = -a - b & (5) \\ (3) & a - b + c = 0 & & & \end{array}$$

Substitute (4) and (5) in (3),

$$
\begin{aligned}
(-b) - b + (-a - b) &= 0 \\
-a - 3b &= 0 \quad - (6)
\end{aligned}
$$

Substitute (4) in (6),

$$
\begin{aligned}
-(-b) - 3b &= 0 \\
-2b &= 0 \\
b &= 0
\end{aligned}
$$

Substitute $b = 0$ in (1),

$$
\begin{aligned}
a + 0 &= 0 \\
a &= 0
\end{aligned}
$$

Substitute $a = 0, b = 0$ in (2),

$$
\begin{aligned}
0 + 0 + c &= 0 \\
c &= 0
\end{aligned}
$$

And so $a = b = c = 0$ and we are done.

Problem 3. *Spanning Sets.*

Solution. S spans V so every vector $u \in V$ is a linear combination of some vectors in S.

So either $u \in S$ or $u \in V - S$.

Case $u \in S$: Suppose $v_i \in S$ and $v_i = u$. So, $u = 1 \cdot v_i$. Therefore,

$$\{v_1, ..., v_i, ..., v_n, u\}$$

is linearly dependent because u is a multiple of v.

Case $u \in V - S$: S spans V so suppose,

$$u = a_i v_i + ... + a_j v_j$$

But then,

$$\{v_1, ..., v_i, ..., v_j, ..., v_n, u\}$$

is linearly dependent because w is a linear combination of $v_i, ..., v_j$. And we are done.

Problem 4. *More Spanning Sets.*

Solution. It is given that $w \notin span\{v_1, ..., v_k\}$. So w is not a linear combination of $v_1, ..., v_k$ by definition of *span*. It is given that $\{v_1, ..., v_k\}$ is linearly independent. From lectures we have that if w is not a linear combination of $v_1, ..., v_k$ then $\forall v_i$, v_i is not a linear combination of $v_1, ..., v_{i-1}, v_{i+1}, ..., v_k$. Therefore $\{v_1, ..., v_k, w\}$ is also linearly independent.

Problem 5. *Dimension and Basis.*

Solution. *Proof.* Proof by contradiction.

Assume B is not a basis for V.

It is given that B is linearly independent so for it not to be a basis, it must not span V. Suppose $u_i \in V$ must be added to B to make it a basis. But when v_i is added to B, it will have $n+1$ vectors and $dim(V) = n$ so it must be linearly dependent. But then $B' = \{u_1, ..., u_n, v_i\}$ cannot be a basis. Contradiction. B must be a basis.

This argument holds for any number of vectors $v_i, ..., v_j$ that might be added from B to make B' a basis. □

Problem 6. *More Dimensions and Basis.*

Solution. *Proof.* Proof by contradiction.

Assume B is not a basis for V.

It is given that B spans V so for it not to be a basis, it must not be linearly independent. Suppose $v_i \in B$ is a linear combination of some other vectors in B, so

$$B' = \{v_1, ..., v_{i-1}, v_{i+1}, ..., v_n\}$$

is linearly independent and spans V. But this would make B' a basis for V which is a contradiction because $dim(V) = n$ and B' has $n-1$ elements. B must be a basis.

This argument holds for any number of vectors $v_i, ..., v_j$ that might be removed from B to make B' a basis. □

Problem 7. *Subspace of a Matrix.*

Solution. (a)

Clearly $X \subseteq M_{2\times 2}$.

i) The zero vector is in X.

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}^T = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

ii) Take two matrices in X,

$$\begin{bmatrix} a & c \\ c & b \end{bmatrix}^T = \begin{bmatrix} a & c \\ c & b \end{bmatrix} \text{ and } \begin{bmatrix} d & f \\ f & e \end{bmatrix}^T = \begin{bmatrix} d & f \\ f & e \end{bmatrix}$$

For any two scalars $k_1, k_2 \in \mathbb{R}$,

$$\begin{aligned}
&k_1 \begin{bmatrix} a & c \\ c & b \end{bmatrix} + k_2 \begin{bmatrix} d & f \\ f & e \end{bmatrix} \\
&= \begin{bmatrix} k_1 a & k_1 c \\ k_1 c & k_1 b \end{bmatrix} + \begin{bmatrix} k_2 d & k_2 f \\ k_2 f & k_2 e \end{bmatrix} \\
&= \begin{bmatrix} k_1 a + k_2 d & k_1 c + k_2 f \\ k_1 c + k_2 f & k_1 b + k_2 e \end{bmatrix} \\
&= \begin{bmatrix} k_1 a + k_2 d & k_1 c + k_2 f \\ k_1 c + k_2 f & k_1 b + k_2 e \end{bmatrix}^T
\end{aligned}$$

So X is closed under scalar multiplication and vector addition.

(b) This is a basis for X,

$$\left\{ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

It is linearly independent because any row-column entry has only one component where it is non-zero. Clearly any linear combination of these three matrices is in X. The dimension is 3 by definition of the dimension of a vector space.

Problem 8. *Finding a Basis.*

Solution. (a) Use the Row-Space Algorithm.

$$\begin{aligned}
&\begin{bmatrix} 1 & 2 & 1 & 1 \\ -1 & -1 & 1 & 2 \\ 2 & 2 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 2 & 2 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & -2 & -1 & -3 \end{bmatrix} \\
\sim &\begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 3 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \\
\sim &\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}
\end{aligned}$$

Thus,

$$\{(1,0,0,-2), (0,1,0,1), (0,0,1,1)\}$$

is a basis. The dimension is 3 as the basis has 3 vectors.

(b) Add the vector $(0,0,0,1)$, so,

$$\{(1,0,0,-2), (0,1,0,1), (0,0,1,1), (0,0,0,1)\}$$

Clearly no vector is a linear combination of the others because for the first 3 vectors, there is only one element with a non-zero entry in each of the first 3 components.

Problem 9. *Finding Another Basis.*

Solution. Use the Casting-Out Algorithm.

$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 1 & 2 & -1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & -1 & -1 \\ 1 & 2 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} \boxed{1} & 2 & 0 & 1 \\ 0 & 0 & \boxed{1} & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So, $p_1(t) = 1 + t + t^2$, $p_3(t) = -t$ form a basis of W.