MATH 223

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0.1 Administrativa

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Grading

Assignments	15%	15%
Midterm	25%	0%
Final	60%	85%

The midterm will be scheduled for the 7th week of class.

Chapter 1

Vectors

1.1 Vectors in $\mathbb{R}^n$

$\mathbb{R}^n$ is the set of all n-tuples of real numbers $u = (a_1...a_n) \mid a \in \mathbb{R}$ where a are the **components** or **entries**.

Remark 1. We use the term **scalar** to refer to an element in $\mathbb{R}$.

1.2 Basic Definitions

Definition 1. Addition
$u, v \in \mathbb{R}^n$
$u = (a_1...a_n)$
$v = (b_1...b_n)$
$u + v = (a_1 + b_1...a_n + b_n)$

Definition 2. Scalar Multiplication
$k \in \mathbb{R}$
$ku = (ka_1...ka_n)$

Definition 3. Two vectors u and v are said to be **equal** ($u = v$) if
$a_i = b_i \forall i = 1...n$.

Definition 4. The **zero vector** is defined as $0 = (0...0)$.

Definition 5. Suppose we are given m vectors $u_1...u_m \in \mathbb{R}^n$ and m scalars
$k_1...k_m \in \mathbb{R}$.
Let $u = k_1u_1 + ... + k_mu_m$.
Such a vector u is called a **linear combination** of the vectors $u_1...u_m$.

Definition 6. A vector u can be called a **multiple** of v if there is a scalar k such that $u = kv$ with $k \neq 0$. In the case $k > 0$ we say u is in the same direction as v. In the case $k < 0$ we say u is in the opposite direction of v.

1.3 The Dot Product

Definition 7. Let $u = (a_1 ... a_n)$ and $v = (b_1 ... b_n)$. The **dot product** or inner product is given by,

$$u \cdot v = a_1 b_1 + ... a_n b_n =$$

Definition 8. The vectors u and v are **orthogonal** if $u \cdot v = 0$.

1.4 The Vector Norm

Definition 9. The **norm** or **length** of a vector is given by,

$$\|u\| = \sqrt{a_1^2 + ... + a_n^2}$$

Thus $\|u\| \geq 0$ and $\|u\| = 0$ if and only if (iff) $u = 0$.

Definition 10. A vector is called a **unit vector** if $\|u\| = 1$.

Definition 11. For any non-zero vector v, the vector

$$\hat{v} = \frac{1}{\|v\|} v$$

is the only unit vector with the same direction of v. The process of finding $\hat{v}$ is called **normalizing**.

1.5 Theorem: Cauchy-Schwarz Inequality

Theorem 1. *Given any two vectors $u, v \in \mathbb{R}^n$, then,*

$$|u \cdot v| \leq \|u\| \|v\|$$

Proof. Let $t \in \mathbb{R}$. So, $\|tu + v\|^2 \geq 0$.

$$\begin{aligned} \|tu + v\|^2 &= (tu + v)(tu + v) \\ &= (tu \cdot tu) + (tu \cdot v) + (v \cdot tu) + (v \cdot v) \\ &= t^2 (u \cdot u) + t(v \cdot u) + t(u \cdot v) + (v \cdot v) \\ &= t^2 \|u\|^2 + 2t(u \cdot v) + \|v\|^2 \end{aligned}$$

We can represent this in the form $at^2 + bt + c \geq 0$, so,

$$a = \|u\|^2, b = 2(u \cdot v), c = \|v\|^2$$

Take the Discriminant as $b^2 - 4ac \iff b^2 \leq 4ac$.

$$\begin{aligned} 4(u \cdot v)^2 &\leq 4\|u\|^2 \|v\|^2 \\ |u \cdot v| &\leq \|u\| \|v\| \end{aligned}$$

$\square$

1.6 Theorem: Minkowski Triangle Inequality

Theorem 2. *Given $u, v \in \mathbb{R}^n$, then $\|u+v\| \leq \|u\| + \|v\|$.*

Proof.

$$\begin{aligned} \|u+v\|^2 &= \|u\|^2 + 2(u \cdot v) + \|v\|^2 \\ &\leq \|u\|^2 + 2\|u\|\|v\| + \|v\|^2 \quad \text{by C-S inequality} \\ &= (\|u\| + \|v\|)^2 \end{aligned}$$

So, $\|u+v\|^2 \leq (\|u\| + \|v\|)^2$. Take the square root and we are done. □

1.7 Geometry with Vectors

Definition 12. The **distance** between vectors $u, v \in \mathbb{R}^n$ is given by,

$$d(u,v) = \|u - v\| = \sqrt{(a_1 - b_1)^2 + ... + (a_n - b_n)^2}$$

Definition 13. The **angle** between vectors $u, v \in \mathbb{R}^n$ is given by,

$$cos\theta = \frac{u \cdot v}{\|u\|\|v\|} \quad \theta \in [0, \pi]$$

Observe that in the previous definition, the angle is well defined.

$$-\|u\|\|v\| \leq -|u \cdot v| \leq u \cdot v \leq u \cdot v \leq |u \cdot v| \leq \|u\|\|v\|$$

Dividing the entire inequality by $\|u\|\|v\|$ yields,

$$-1 \leq \frac{u \cdot v}{\|u\|\|v\|} \leq 1$$

Definition 14. A **hyperplane** $\mathcal{H}$ in $\mathbb{R}^n$ is the set of points $(x_1...x_n)$ that satisfy $a_1x_1 + ... + a_nx_n = b$ where $u = [a_1...a_n] \in \mathbb{R}^n$ and $b \in \mathbb{R}$.

Definition 15. The **line** in $\mathbb{R}^n$ passing through a point $P = (b_1...b_n)$ and in the direction of $v \in \mathbb{R}^n$ with $v \neq 0$.

$$x = P + tu \quad t \in \mathbb{R}, \quad u = [a_1...a_n]$$

$$\begin{cases} x_1 = a_1t + b_1 \\ x_n = a_nt + b_n \end{cases}$$

Chapter 2

Algebra of Matrices

2.1 Introduction

A matrix with n rows and m columns is written as,

$$A_{n\times m} = \begin{bmatrix} a_{11} & \dots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nm} \end{bmatrix}$$

Or,

$$A_{n\times m} = [a_{ij}]$$

Where a_{ij} is the entry in row i and column j.

2.2 Definitions and Properties of Matrices

Definition 16. Matrix Addition

$$A + B = [a_{ij} + b_{ij}] \quad \forall i = 1...n, j = 1...m$$

Definition 17. Scalar Multiplication

$$ka = [ka_{ij}] \quad \forall i = 1...n, j = 1...m$$

Definition 18. Zero Matrix

$$0 = [0]$$

Definition 19. Given a matrix $A_{m\times p}$ and a matrix $B_{p\times n}$, **matrix multiplication** is defined as,

$$AB = [c_{ij}] \quad c_{ij} = \sum_{k=1}^{p} a_{ik}b_{kj}$$

Definition 20. Given a matrix A, its **transpose** is $A^T = [a_{ji}]$ where $A = [a_{ij}]$.

Definition 21. A **square matrix** has the same number of rows as it does columns, i.e. $A_{n \times n}$ is a square matrix.

Definition 22. Given a matrix $A = [a_{ij}]$ the elements in the **diagonal** are $[a_{11}, ..., a_{nn}]$.

Definition 23. The **trace** of a matrix A is given by,

$$tr(A) = a_{11} + ... + a_{nn}$$

Definition 24. The **identity matrix** I_n is the matrix such that for any n-square matrix A,

$$AI = IA = A$$

Definition 25. The **Kronecker delta** is defined by,

$$\delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

Remark 2. Given the definitions for the identity matrix and the Kronecker delta, an alternative definition for the identity matrix is as follows,

$$I = [\delta_{ij}]$$

Definition 26. A matrix A is **invertible** if there is a matrix B such that $AB = BA = I$.

Remark 3. In general, for any matrices A and B, $AB \neq BA$.

Definition 27. A matrix D is **diagonal** if all the non-zero entries are in the diagonal.

$$D = diaginal(d_1, ..., d_n)$$

Definition 28. A matrix A is **upper triangular** if,

$$a_{ij} = 0 \quad \forall i > j$$

2.3 Complex Numbers

The imaginary number i is defined as $i = \sqrt{-1}$ or equivalently, $i^2 = -1$.

Definition 29. A **complex number** z is given by,

$$z = a + bi \quad a, b \in \mathbb{R}$$

Where a is the real part and b is the imaginary part.

Real numbers are also complex numbers with no imaginary component, i.e. $a + 0i = a$.

Addition for two complex numbers $z = a + bi$ and $w = c + di$ is given by,

$$z + w = (a + c) + (b + d)i$$

Multiplication for the same two complex numbers is given by,

$$\begin{aligned} z \cdot w &= (a + bi)(c + di) \\ &= ac + adi + cbi - bd \\ &= (ac - bd) + (ad + bc)i \end{aligned}$$

Definition 30. The **conjugate** of $z = a + bi$ is $\bar{z} = a - bi$.

Definition 31. The **absolute value** or modulus of $z = a + bi$ is $|z| = \sqrt{a^2 + b^2}$.

Example 1.

$$z^{-1} = \frac{1}{z} = \frac{1}{a + bi} \cdot \frac{a - bi}{a - bi} = \frac{a - bi}{a^2 + b^2}$$

Observe that the following properties are true for conjugates and absolute values,

1. $$z\bar{z} = |z|^2 = a^2 + b^2$$

2. $$\overline{z \pm w} = \bar{z} \pm \bar{w}$$

3. $$\overline{zw} = \bar{z} \cdot \bar{w}$$

4. $$\overline{(\bar{z})} = z$$

5. z is real iff $z = \bar{z}$

6. $$|zw| = |z||w|$$

7. $$|z + w| \leq |z| + |w|$$

Chapter 3

Systems of Linear Equations

3.1 Representing Linear Systems with Matrices

Given a system of linear equations of the form,

$$\begin{cases} a_{11}x_1 + ... + a_{1m}x_n = b_1 \\ \vdots \\ a_{m1}x_1 + ... + a_{mn}x_n = b_m \end{cases}$$

- $x_1, ..., x_m$ are the unknowns, and
- a_{ij} and b_i are the constants.

The system can also be represented by matrices where,

- $A = [a_{ij}]$ is the matrix of coefficients
- $b = [b_i]$ is the column vector of constant
- $M = [A|b]$ is the matrix that represents the system.

Definition 32. A matrix A is in **echelon form** if

1. all zero rows are at the bottom, and
2. each leading non-zero entry in a row is to the right of the leading non-zero entry in teh preceding row.

Example 2. This matrix is in echelon form. The boxed entries are the **pivots**.

$$A = \begin{bmatrix} 0 & \boxed{2} & 3 & 4 & 1 & 0 & 6 \\ 0 & 0 & 0 & \boxed{2} & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & \boxed{1} & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Definition 33. A matrix is said to be in teh **row-reduced echelon form** if it is in the echelon form and,

1. each pivot is equal to 1, and
2. each pivot is the only non-zero entry in its column

Example 3. This matrix is in row-reduced echelon form. The boxed entries are the **pivots**.

$$A = \begin{bmatrix} 0 & \boxed{1} & 3 & 0 & 0 & 4 \\ 0 & 0 & 0 & \boxed{1} & 0 & -3 \\ 0 & 0 & 0 & 0 & \boxed{1} & 2 \end{bmatrix}$$

3.2 Elementary Row Operations

Suppose that A is a matrix with rows $R_1, ..., R_m$. The elementary row operations that can be performed on A are as follows,

1. Row interchange, $R_i \leftrightarrow R_j$
2. Row scaling, $kR_i \to R_i$
3. Row addition, $kR_i + R_j \to R_j$

The method by which we find the (row-reduced) echelon form of a matrix is using the **Gaussian Elimination** algorithm.

Recall that every matrix is row equivalent to a unique matrix in the row-reduced echelon form.

Definition 34. The **rank** of a matrix $rank(A)$ is the number of pivots in the row-reduced echelon form. There are many other ways to define rank but they all have the same meaning.

The method by which we find the inverse of a square matrix A is as follows, Let $M = [A \mid I]$. Find the row-reduced echelon form of M. If there is a zero row in the resulting matrix then A is not invertible. Otherwise, $M \sim [I \mid B], \quad A^{-1} = B$.

Theorem 3. *Let A be a square matrix. The following conditions are equivalent,*

1. *A is invertible*
2. *the row-reduced echelon for of A is I*
3. *the only solution to $Ax = 0$ is $x = 0$*
4. *the system $Ax = b$ has a solution for any choice of column b.*

A partial proof is as follows,

Proof. (1) $\Rightarrow$ (3) There is a matrix B such that $AB = I = BA$.
Let x be any solution of $Ax = [0]$.

$$\begin{aligned} BAx &= B[0] \\ Ix &= [0] \\ x &= [0] \end{aligned}$$

(1) $\Rightarrow$ (4) Fix a column b,

$$\begin{aligned} Ax &= b \\ \Leftrightarrow \quad A^{-1}Ax &= A^{-1}b \\ \Leftrightarrow \quad x &= A^{-1}b \end{aligned}$$

$\square$

Definition 35. A linear system $Ax = b$ is **homogeneous** if $b = 0$. Otherwise, $Ax = b$ is said to be **non-homogeneous**.

Definition 36. A **particular solution** of $Ax = b$ is a vector x such that $Ax = b$. The set of all particular solutions is called the **general solution** of the solution set.

Definition 37. A system $Ax = b$ is **consistent** if it has one or more solutions and it is said to be **inconsistent** if it has no solutions.

Theorem 4. *Any system $Ax = b$ has:*
(i) an unique solution,
(ii) no solution, or
(iii) an infinite number of solutions.

3.3 Examples

Example 4. The system,

$$\begin{aligned} x + y + 2z &= 1 \\ 3x - y + z &= -1 \\ -x + 3y + 4z &= 1 \end{aligned}$$

is equivalent to,

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 3 & -1 & 1 & -1 \\ -1 & 3 & 4 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 4 & 5 & 4 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

Example 5. Back substitution:
$z = -2$
$4y + 5z = 4 \Leftrightarrow y = \frac{7}{2}$
$x + y + 2z = 1 \Leftrightarrow x = \frac{3}{2}$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{3}{2} \\ 0 & 1 & 0 & \frac{7}{2} \\ 0 & 0 & 1 & -2 \end{array}\right]$$

Example 6.

$$-2x + 3y + 3z = -9$$
$$3x - 4y + z = 5$$
$$-5x + 7y + 2z = -14$$

$$\sim \left[\begin{array}{ccc|c} -2 & 3 & 3 & -9 \\ 3 & -4 & 1 & 5 \\ -5 & 7 & 2 & -14 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & 4 & -4 \\ 0 & 1 & 11 & -17 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

So there are infinitely many solutions.
Set $z = t$ since z is a free variable, then back substitute.

$$y = -17 - 11t \quad x = -21 - 15t \quad t \in \mathbb{R}$$

So the solution space is,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -21 \\ -17 \\ 0 \end{bmatrix} + t \begin{bmatrix} -15 \\ -11 \\ 1 \end{bmatrix}$$

Where $(-21, -17, 0)$ is a particular solution and $(-15, -11, 1)$ is the set of basic solutions of the homogeneous system $Ax = 0$.

Example 7.

$$x + 2y - z = 2$$
$$2x + 5y - 3z = 1$$
$$x + 4y - 3z = 3$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -7 & 2 \\ 2 & 5 & -3 & 1 \\ 1 & 4 & -3 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & -1 \end{array}\right]$$

There are no solutions possible for this system.

Chapter 4

Vector Spaces

4.1 Introduction

Adding two vectors in $\mathbb{R}^n$ produces a vector in $\mathbb{R}^n$. Similarly, multiplying by a scalar produces a vector in $\mathbb{R}^n$. These are some properties of a vector space, the following section is a formal list.

4.2 Basic Definitions