Assignment 4

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Problem 1. *Computing an inner product.*

Solution.

$$\begin{aligned}
&\langle u_1 - 3u_2 + 2u_3, -u_1 + u_2 - 3u_3\rangle \\
=&\langle u_1, -u_1 + u_2 - 3u_3\rangle - 3\langle u_2, -u_1 + u_2 - 3u_3\rangle \\
&\quad + 2\langle u_3, -u_1 + u_2 - 3u_3\rangle \\
=&-\langle u_1, u_1\rangle + \langle u_1, u_2\rangle - 3\langle u_1, u_3\rangle \\
&\quad - 3(-\langle u_2, u_1\rangle + \langle u_2, u_2\rangle - 3\langle u_2, u_3\rangle) \\
&\quad + 2(-\langle u_3, u_1\rangle + \langle u_3, u_2\rangle - 3\langle u_3, u_3\rangle) \\
=&-\langle u_1, u_1\rangle + \langle u_1, u_2\rangle - 3\langle u_1, u_3\rangle \\
&\quad + 3\langle u_1, u_2\rangle - 3\langle u_2, u_2\rangle + 9\langle u_2, u_3\rangle) \\
&\quad - 2\langle u_1, u_3\rangle + 2\langle u_2, u_3\rangle - 6\langle u_3, u_3\rangle) \\
=&-\langle u_1, u_1\rangle + 4\langle u_1, u_2\rangle - 5\langle u_1, u_3\rangle \\
&\quad - 3\langle u_2, u_2\rangle + 11\langle u_2, u_3\rangle - 6\langle u_3, u_3\rangle) \\
=&-(1) + 4(-2) - 5(1) - 3(2) + 11(-1) - 6(3) \\
=&-1 - 8 - 5 - 6 - 11 - 18 \\
=&\boxed{-49}
\end{aligned}$$

Problem 2. *Proof of an inequality.*

Solution.

Claim.

$$\frac{(a_1 + ... + a_n)^2}{n} \leq a_1^2 + ... + a_n^2$$

Proof. Let $v = (1, ..., 1) \in \mathbb{R}^n$. For any $u = (a_1, ..., a_n) \in \mathbb{R}^n$,

$$u \cdot v = (1 \cdot a_1, ..., 1 \cdot a_n) = a_1 + ... + a_n$$

So we apply the Cauchy-Schwartz inequality to u and v,

$$
\begin{aligned}
(u \cdot v)^2 &\leq \|u\|^2 \|v\|^2 \\
\Leftrightarrow (a_1 + ... + a_n)^2 &\leq \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} 1^2 \\
\Leftrightarrow (a_1 + ... + a_n)^2 &\leq (a_1^2 + ... + a_n^2)n \\
\Leftrightarrow \frac{(a_1 + ... + a_n)^2}{n} &\leq a_1^2 + ... + a_n^2
\end{aligned}
$$

□

Problem 3. *Proof of a remark on orthogonal sets of vectors.*

Solution.

Claim.

$$\|u_1 + ... + u_r\|^2 = \|u_1\|^2 + ... + \|u_r\|^2$$

Proof. It is given that $\forall u_i, u_j \in S \quad \langle u_i, u_j \rangle = 0$ where $i \neq j$.

$$
\begin{aligned}
&\|u_1 + ... + u_r\|^2 \\
=&\left(\sqrt{\langle u_1 + ... + u_r, u_1 + ... + u_r \rangle}\right)^2 \\
=&\langle u_1 + ... + u_r, u_1 + ... + u_r \rangle \\
=&\sum_{i=1}^{r} \sum_{i=1}^{r} \langle u_i, u_j \rangle \quad \text{[by definition]} \\
=&\sum_{i=1}^{r} \langle u_i, u_i \rangle \quad \text{[from given, all other terms become 0]} \\
=&\sum_{i=1}^{r} \|u_i\|^2 \quad \text{[by definition]}
\end{aligned}
$$

So finally we have the result that,

$$\|u_1 + ... + u_r\|^2 = \|u_1\|^2 + ... + \|u_r\|^2$$

□

Problem 4. *Proof of a property of inner product spaces.*

Solution.

Claim. Let S be a subset of an inner product space V, then $S^{\perp}$ is a subspace of V.

Proof. By definition, $S \subseteq V$ and $S^{\perp} = \{v \in V : \langle v, w \rangle = 0 \quad \forall w \in S\}$.

The zero vector is in $S^{\perp}$, $0 \in S^{\perp}$, because the zero vector is orthogonal to every vector in the inner product space V.

Take any $u, v \in S^{\perp}$ and scalars $a, b \in \mathbb{R}$.

$$\begin{aligned}
&\langle au + bv, w \rangle \\
=&a\langle u, w \rangle + b\langle v, w \rangle \\
=&a \cdot 0 + b \cdot \\
=&0
\end{aligned}$$

Thus, $au + bv \in S^{\perp}$ and therefore $S^{\perp}$ is a subspace of V. □

Problem 5. *Matrix representation of an inner product.*

Solution. It is given that a symmetric matrix $A \in M_{n\times n}$ is positive definite if $u^T Au > 0 \quad \forall u \in \mathbb{R}^n$.

(a)

Claim. Let $\langle , \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be defined by $\langle u, v \rangle = u^T Av$ where $A \in M_{n\times n}$ and A is positive definite. $\langle , \rangle$ is an inner product in $\mathbb{R}^n$.

Proof. Take any $u_1, u_2, v \in \mathbb{R}^n$ and scalars $a, b \in \mathbb{R}$,

$$\begin{aligned}
&\langle au_1 + bu_2, v \rangle \\
=&(au_1 + bu_2)^T Av \\
=&((au_1)^T + (bu_2)^T)Av \\
=&(au_1^T + bu_2^T)Av \\
=&au_1^T Av + bu_2^T Av \\
=&a\langle u_1, v \rangle + b\langle u_2, v \rangle
\end{aligned}$$

Thus, $\langle , \rangle$ satisfies axiom 1 of inner products.

Take any $u, v \in \mathbb{R}^n$, we have that $u^T Av \in \mathbb{R}$ and so $(u^T Av)^T = u^T Av$ because the transpose of a scalar is equal to itself.

$$\begin{aligned}
\langle u, v \rangle &= u^T Av \\
&= (u^T Av)^T \\
&= v^T A^T u^{TT} \quad \text{[property of matrix transpose]} \\
&= v^T Au \quad \text{by definition, } A = A^T \\
&= \langle v, u \rangle
\end{aligned}$$

Thus, $\langle , \rangle$ satisfies axiom 2 of inner products.

By definition of positive definite matrix A, for any non-zero $u \in \mathbb{R}^n$, and $\langle u, u \rangle > 0$. If $u = 0$, then $\langle 0, 0 \rangle = 0^T A 0 = 0$. Thus $\langle , \rangle$ satisfies axiom 3 of inner products.

And thus, $\langle , \rangle$ satisfies all three axioms and is an inner product. $\square$

(b)

Claim. Let $\langle , \rangle : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be defined by $\langle u, v \rangle = u^T A v$ where,

$$A = \begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}$$

$\langle , \rangle$ is an inner product in $\mathbb{R}^2$.

The claim is false. The null space is as follows,

$$\begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}$$

And so,

$$N(A) = span\left(\begin{bmatrix} -3 \\ 1 \end{bmatrix}\right)$$

Since the null space is not empty, then there are might be vectors that are not orthogonal whose product is 0.

In $\langle u, u \rangle$, if $u = (-3, 1)$ then $\langle u, u \rangle = 0$. Thus $\langle , \rangle$ violates axiom 3 as u is not the zero vector and therefore $\langle , \rangle$ is not an inner product.

Problem 6. *Orthogonal sets and orthogonal basis.*

Solution. It is given that,

$$S = \{v_1, v_2, v_3\} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} \right\}$$

(a)

Claim. S is an orthogonal basis of $\mathbb{R}^3$

Proof.

$$\begin{aligned} v_1 \cdot v_2 &= -2 + 0 + 2 = 0 \\ v_2 \cdot v_3 &= 2 + 0 - 2 = 0 \\ v_3 \cdot v_1 &= -1 + 2 - 1 = 0 \end{aligned}$$

Thus S is an orthogonal set and therefore S is linearly independent. Any 3 linearly independent vectors in $\mathbb{R}^3$ forms a basis for $\mathbb{R}^3$. Thus, S is an orthogonal basis of $\mathbb{R}^3$. $\square$

(b) I will use the Fourier co-efficients. The vector v is given by $v = (3, 4, -1)$.

$$[v]_S = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}^T$$

$$\begin{aligned}
a_1 &= \frac{\langle v, v_1 \rangle}{\langle v_1, v_1 \rangle} = \frac{3+4-1}{1+1+1} = \frac{6}{3} = 2 \\
a_2 &= \frac{\langle v, v_2 \rangle}{\langle v_2, v_2 \rangle} = \frac{-6+0-2}{4+0+4} = \frac{-8}{8} = -1 \\
a_3 &= \frac{\langle v, v_3 \rangle}{\langle v_3, v_3 \rangle} = \frac{-3+8+1}{1+4+1} = \frac{6}{6} = 1\frac{-x+2y-z}{6}
\end{aligned}$$

$$\boxed{[v]_S = \begin{bmatrix} 2 & -1 & 1 \end{bmatrix}^T}$$

(c) I will use the Fourier co-efficients again. The vector v is given by $v = (x, y, z)$.

$$[v]_S = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}^T$$

$$\begin{aligned}
a_1 &= \frac{\langle v, v_1 \rangle}{\langle v_1, v_1 \rangle} = \frac{x+y+z}{1+1+1} = \frac{x+y+z}{3} \\
a_2 &= \frac{\langle v, v_2 \rangle}{\langle v_2, v_2 \rangle} = \frac{-2x+2z}{4+0+4} = \frac{-x+z}{4} \\
a_3 &= \frac{\langle v, v_3 \rangle}{\langle v_3, v_3 \rangle} = \frac{-x+2y-z}{1+4+1} = \frac{-x+2y-z}{6}
\end{aligned}$$

$$\boxed{[v]_S = \begin{bmatrix} \frac{x+y+z}{3} & \frac{-x+z}{4} & \frac{-x+2y-z}{6} \end{bmatrix}^T}$$

Problem 7. *Inner product of a polynomial space.*

Solution. The work for this problem is attached.

The non-normalized orthogonal basis I found after applying Gram-Schmidt is as follows,

$$\begin{aligned}
w_1 &= 1 \\
w_2 &= t - \frac{1}{2} \\
w_3 &= t^2 - t + \frac{1}{6} \\
w_4 &= t^3 - \frac{3}{2}t^2 - \frac{3}{5}t - \frac{1}{20}
\end{aligned}$$

After normalizing, the orthonormal basis I found after normalizing each vector,

$$\begin{aligned}
\hat{w}_1 &= 1 \\
\hat{w}_2 &= \sqrt{3}(2t-1) \\
\hat{w}_3 &= \sqrt{5}(6t^2-6t+1) \\
\hat{w}_4 &= \frac{\sqrt{35}}{\sqrt{269}}\left(4t^3-6t^2-\frac{12}{5}t-\frac{1}{20}\right)
\end{aligned}$$