Assignment 6

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Problem 1. *Proving a remark of eigenvectors and eigenvalues*

Solution. (a) To proceed with this proof I first need to prove the following fact:

Lemma. Suppose A is a square matrix with an eigenvalue λ and eigenvector v, then,

$$A^m v = \lambda^m v \quad \forall m > 0$$

Proof.

$$\begin{aligned}
A^k v =& A^{k-1} A v \\
=& A^{k-1} \lambda v \quad \text{[Usual definition of eigen vector]} \\
=& \lambda (A^{k-1} v) \quad \text{[An eigenvalue is over a field]} \\
=& \lambda (A^{k-2} A v) \\
=& \lambda (A^{k-2} \lambda v) \\
=& \lambda^2 (A^{k-2} v) \\
& \vdots \quad \text{[We repeat the above argument a total of k times]} \\
=& \lambda^k v
\end{aligned}$$

□

Claim. Suppose A is a square matrix and v be an eigenvector with eigenvalue λ. If $p(t)$ is a polynomial then,

$$p(A)v = p(\lambda)v$$

Proof. $p(t)$ is a polynomial so it can defined as follows,

$$p(t) = a_m t^m + ... + a_1 t + a_0$$

$$
\begin{aligned}
p(A)v &= (a_m A^m + ... + a_1 A + a_0 I)v \\
&= a_m A^m v + ... + a_1 A v + a_0 I v \\
&= a_m \lambda^m v + ... + a_1 \lambda v + a_0 v \quad \text{[Applying the above lemma]} \\
&= (a_m \lambda^m + ... + a_1 \lambda + a_0)v \\
&= p(\lambda)v
\end{aligned}
$$

$\square$

(b)

The following remark was given in lecture.

Lemma. The characteristic polynomial and the minimum polynomial have the same roots.

Suppose $m(t)$ is the minimum polynomial of A. Then a consequence of the above remark is that,

$$m(A) = 0$$

We apply the claim proven in part (a).

$$
\begin{aligned}
m(\lambda)v &= m(A)v \\
&= 0
\end{aligned}
$$

But $v \neq 0$ by definition of eigenvectors so in $m(\lambda)v = 0$, it must be the $m(\lambda)$ term that is zero. Therefore $m(\lambda) = 0$.

Problem 2. *Computing minimum polynomials and Jordan canonical forms*

Solution. (a)

A is triangular, so,

$$
\begin{aligned}
\Delta(t) &= (t-1)(t-1)(t-1)(t-2) \\
&= (t-1)^3(t-2)
\end{aligned}
$$

Let $m(t)$ be the minimum polynomial of A. $m(t)$ must divide $\Delta(t)$, so $m(t)$ is either

- $$f(t) = (t-1)(t-2)$$

- $$g(t) = (t-1)^2(t-2)$$
- $$h(t) = (t-1)^3(t-2)$$

To be certain we must test $f(t)$ and $g(t)$.

$$\begin{aligned}
f(A) &= \begin{bmatrix} 0 & -1 & 1 & -3 \\ -1 & 0 & 4 & 1 \\ -1 & -1 & 0 & -1 \\ -1 & -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} -1 & -2 & 0 & -4 \\ -2 & -1 & 3 & 0 \\ -2 & -2 & -1 & -2 \\ -2 & -2 & -2 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 6 & 5 & 2 & -2 \\ -9 & -8 & -6 & -4 \\ 5 & 5 & -1 & 4 \\ 3 & 3 & -4 & 6 \end{bmatrix}
\end{aligned}$$

So, $m(t) \neq f(t)$.

$$\begin{aligned}
g(A) &= \begin{bmatrix} 0 & -1 & 1 & -3 \\ -1 & 0 & 4 & 1 \\ -1 & -1 & 0 & -1 \\ -1 & -1 & -1 & -1 \end{bmatrix}^2 \begin{bmatrix} -1 & -2 & 0 & -4 \\ -2 & -1 & 3 & 0 \\ -2 & -2 & -1 & -2 \\ -2 & -2 & -2 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 3 & 2 & -1 & -5 \\ -5 & -4 & -2 & 0 \\ 2 & 2 & -4 & 1 \\ 1 & 1 & -6 & 4 \end{bmatrix} \begin{bmatrix} -1 & -2 & 0 & -4 \\ -2 & -1 & 3 & 0 \\ -2 & -2 & -1 & -2 \\ -2 & -2 & -2 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 11 & 4 & 17 & -10 \\ 7 & 18 & -10 & 24 \\ 4 & 0 & 8 & 0 \\ 3 & 1 & 1 & 8 \end{bmatrix}
\end{aligned}$$

So, $m(t) \neq g(t)$.

Therefore $m(t) = h(t) = \Delta(t) = (t-1)^3(t-2)$.

(b)

In part (a) we determined that $\Delta(t) = m(t) = (t-1)^3(t-2)$.

Therefore the Jordan canonical form for A is,

$$J = diag\left(\begin{bmatrix}1 & 1 & \\ & 1 & 1 \\ & & 1\end{bmatrix}, [2]\right)$$

Problem 3. *Linear independence of a set of matrix-vector products*

Solution.

Let $S = \{v, Av, A^2v, ..., A^{k-1}\}$. Given that $\forall v \in \mathbb{R}^n$ we have $A^kv = 0$ and $A^{k-1}v \neq 0$.

Claim. S is linearly independent.

Proof. Let $a_0, ..., a_{k-1} \in \mathbb{R}$. If S is linearly independent, then in,

$$a_0v + a_1Av + a_2A^2v + ... + a_{k-1}A^{k-1} = 0$$

it is sufficient to show that $a_0 = ... = a_{k-1} = 0$.

A consequence of the fact that $A^kv = 0$ is that $A^mv = 0$ if $m > k$ because A^mv can be rewritten as $A^{m-k}A^kv = 0$.

So we begin by multiplying the above equation by A^{k-1} to obtain,

$$\begin{aligned} a_0A^{k-1}v + a_1A^kv + a_2A^{k+1}v + ... + a_{k-1}A^{2k-2} &= 0 \\ a_0A^{k-1}v &= 0 \quad \text{[above consequence]} \end{aligned}$$

Since it is given that $A^{k-1} \neq 0$ then clearly $a_0 = 0$ as we obtained the result that $a_0A^{k-1}v = 0$.

Now we have the fact that $a_0 = 0$ and we repeat the above argument but we multiply across with A^{k-2} instead to obtain,

$$\begin{aligned} a_0A^{k-2}v + a_1A^{k-1}v + a_2A^kv + ... + a_{k-1}A^{2k-3} &= 0 \\ a_0A^{k-2}v + a_1A^{k-1}v &= 0 \quad \text{[above consequence]} \\ a_1A^{k-1}v &= 0 \quad [a_0 = 0] \end{aligned}$$

Again, because $A^{k-1} \neq 0$ so $a_1 = 0$ as we obtained the result that $a_1A^{k-1}v = 0$.

If we repeat the above argument a total of $k - 2$ times then we will obtain the result that $a_0 = a_1 = ... = a_k - 1 = 0$ and therefore the set S is linearly independent.

$\square$