Assignment 1

Yang David Zhou, ID 260517397

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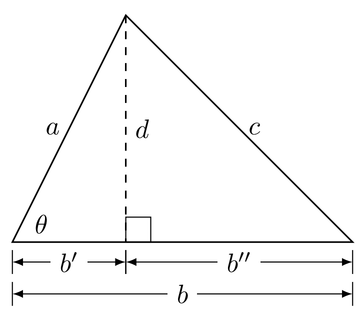
Problem 1. *Proofs.*

Solution. (a)

Proof. The Pythagorean theorem states that in a right triangle where c is the length of the hypotenuse and a, b are the lengths of the other two sides, the formula $c^2 = a^2 + b^2$ is true.

In general, any non-right triangle can be divided into two right angled triangles by dropping a vertical from a vertex that is perpendicular to the opposite edge, like in the diagram below.



Define $b = b' + b''$ (1).

Which can be rearranged, $b'' = (b - b')$ (2).

By trigonometry we have $b' = a\cos\theta$ (3).

By the Pythagorean we have $a^2 = d^2 + (b')^2$ (4) and $c^2 = d^2 + (b'')^2$ (5).

Rearrange (4) and (5) for d^2 then set them equal,

$$\begin{aligned}
c^2 - (b'')^2 &= a^2 - (b')^2 \\
c^2 &= a^2 - (b')^2 + (b'')^2 \\
&= a^2 - (b')^2 + (b - b')^2 \quad \text{By (2)} \\
&= a^2 - (a\cos\theta)^2 + (b - a\cos\theta)^2 \quad \text{By (3)} \\
&= a^2 - a^2\cos^2\theta + b^2 - 2ab\cos\theta + a^2\cos^2\theta \\
c^2 &= a^2 + b^2 - 2ab\cos\theta
\end{aligned}$$

And we are done. □

(b)

Proof. Geometrically, v and w form a plane and $u = v - w$ is a vector on that plane. Together, u, v, w form a triangle. So we apply the Law of Cosines.

Define θ to be the angle opposite $u = v - w$. By definition of the angle between two vectors, $\cos\theta = \frac{v \cdot w}{\|v\|\|w\|}$.

$$\begin{aligned}
\|v - w\|^2 &= \|v\|^2 + \|w\|^2 - 2\|v\|\|w\|\cos\theta \\
\|v - w\|^2 &= \|v\|^2 + \|w\|^2 - 2\|v\|\|w\|\left(\frac{v \cdot w}{\|v\|\|w\|}\right) \quad \text{By definition} \\
\|v - w\|^2 &= \|v\|^2 + \|w\|^2 - 2v \cdot w \\
2v \cdot w &= \|v\|^2 + \|w\|^2 - \|v - w\|^2 \\
2v \cdot w &= \|v\|^2 + \|w\|^2 - (\|v\|^2 + \|w\|^2 - 2\|v\|\|w\|\cos\theta) \\
2v \cdot w &= 2\|v\|\|w\|\cos\theta \\
v \cdot w &= \|v\|\|w\|\cos\theta
\end{aligned}$$

And we are done. □

Problem 2. *Complex numbers.*

Solution. (a)

$$\begin{aligned}
\frac{1}{2}(z + \tilde{z}) &= \frac{1}{2}((a + bi) + (a - bi)) \\
&= \frac{1}{2}(a + a + bi - bi) \\
&= \frac{1}{2}(2a) \\
&= a
\end{aligned}$$

(b)

$$\begin{aligned}
\frac{1}{2i}(z-\tilde{z}) &= \frac{1}{2i}((a+bi)-(a-bi)) \\
&= \frac{1}{2i}(a+bi-a+bi) \\
&= \frac{1}{2i}(2bi) \\
&= b
\end{aligned}$$

(c)

Case $z = 0$:

$$\begin{aligned}
zw &= 0 \\
z &= \frac{0}{(a+bi)} \\
&= 0
\end{aligned}$$

Case $w = 0$:

$$\begin{aligned}
zw &= 0 \\
w &= \frac{0}{(a+bi)} \\
&= 0
\end{aligned}$$

Problem 3. *Matrix multiplication.*

Solution. In order for the product of two matrices $M = AB$ to be valid, the number of columns in A must equal the number of rows in B. Otherwise the product is not well defined.

Only ABC, BCA, CAB will yield valid products.

$$\begin{aligned}
ABC &= \begin{bmatrix} 1 & 0 & -i \\ -1 & i & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} i & 0 \\ 0 & -i \\ 1 & 0 \end{bmatrix} C = \begin{bmatrix} (i+0-i) & (0+0+0) \\ (-i+0+0) & (0+1+0) \\ (0+0+1) & (0+0+0) \end{bmatrix} C \\
&= \begin{bmatrix} 0 & 0 \\ -i & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 1 & 2 \\ i & 2i & i \end{bmatrix} = \begin{bmatrix} (0+0) & (0+0) & (0+0) \\ (-i+i) & (i+2i) & (2i+i) \\ (-1+0) & (1+0) & (2+0) \end{bmatrix} \\
&= \boxed{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 3i & 3i \\ -1 & 1 & 2 \end{bmatrix}}
\end{aligned}$$

$$BCA = \begin{bmatrix} i & 0 \\ 0 & -i \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 1 & 2 \\ i & 2i & i \end{bmatrix} A = \begin{bmatrix} (-i+0) & (i+0) & (2i+0) \\ (0+1) & (0+2) & (0+1) \\ (-1+0) & (1+0) & (2+0) \end{bmatrix} A$$

$$= \begin{bmatrix} -i & i & 2i \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & -i \\ -1 & i & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} (-i-i+0) & (0-1+0) & (-1+0+2i) \\ (1-2+0) & (0+2i+0) & (-i+0+1) \\ (-1-1+0) & (0+i+0) & (i+0+2) \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} -2i & -1 & (-1+2i) \\ -1 & 2i & (1-i) \\ -2 & i & (2+i) \end{bmatrix}}$$

$$CAB = \begin{bmatrix} -1 & 1 & 2 \\ i & 2i & i \end{bmatrix} \begin{bmatrix} 1 & 0 & -i \\ -1 & i & 0 \\ 0 & 0 & 1 \end{bmatrix} B = \begin{bmatrix} (-1-1+0) & (0+i+0) & (i+0+2) \\ (i-2i+0) & (0-2+0) & (1+0+i) \end{bmatrix} B$$

$$= \begin{bmatrix} -2 & i & (2+i) \\ -i & -2 & (1+i) \end{bmatrix} \begin{bmatrix} i & 0 \\ 0 & -i \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} (-2i+0+(2+i)) & (0+1+0) \\ (1+0+(1+i)) & (0+2i+0) \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} (2-i) & 1 \\ (2+i) & 2i \end{bmatrix}}$$

Problem 4. *Row-reduced echelon form.*

Solution. My work for this is attached. Shown below is only the final result.

$$A = \begin{bmatrix} 1 & 2 & 3 & -1 \\ 2 & 3 & 0 & 1 \\ 1 & 1 & -3 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -9 & 5 \\ 0 & 1 & 6 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 1 & 3 \\ 2 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Problem 5. *Inverting matrices.*

Solution. My work for this is attached. Shown below is only the final result.

$$A^{-1} = \begin{bmatrix} 0 & -1 & 0 & 1 \\ 1 & -4 & -3 & 4 \\ 0 & 1 & 1 & -1 \\ 0 & -2 & -3 & 3 \end{bmatrix}$$

Problem 6. *Linear combinations in a vector space.*

Solution. $M_{2,2}$ is clearly a vector space and a subspace of a vector space is also a vector space. This subspace must then be closed under addition and multiplication so a linear combination of any two vectors in the space is also in this space.

I^2 can be shown to be a linear combination as follows,

$$\begin{aligned} I^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} &= -2 \begin{bmatrix} 2 & 10 \\ 10 & 12 \end{bmatrix} + 5 \begin{bmatrix} 1 & 4 \\ 4 & 5 \end{bmatrix} \\ &= \begin{bmatrix} -4 & -20 \\ -20 & -24 \end{bmatrix} + \begin{bmatrix} 5 & 20 \\ 20 & 25 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

I arrived at this solution by guessing and checking multiples of 10 and 4 that add up to 0.

Problem 7. *Validity of subspace.*

Solution. W is clearly a subset of $\mathbb{R}^3$.

The zero vector $(0, 0, 0)$ is in W. So, $a = 0, b = 0, c = 0$.

a) $0 = 3(0)$

b) $0 \leq 0 \leq 0$

c) $(0)(0) = 0$

d) $0 = (0)^2$

$u, v \in W \quad a, b \in \mathbb{R}$

$u = (u_1, u_2, u_3)$ with

a) $u_1 = 3u_2$

b) $u_1 \leq u_2 \leq u_3$

c) $u_1 u_2 = 0$

d) $u_2 = u_1^2$

$v = (v_1, v_2, v_3)$ with

a) $v_1 = 3v_2$

b) $v_1 \leq v_2 \leq v_3$

c) $v_1 v_2 = 0$

d) $v_2 = v_1^2$

$$\begin{aligned} z = (z_1, z_2, z_3) &= au + bv \\ &= (au_1 + bv_1, au_2 + bv_2, au_3 + bv_3) \end{aligned}$$

(a) is a valid vector space constraint. The argument for (a) is that both $u_1 = 3u_2$ and $v_1 = 3v_2$. So for any $u, v, z \in W$,

$$\begin{aligned} z_1 &= au_1 + bv_1 \\ &= a(3u_2) + b(3v_2) \\ &= 3(au_2 + bv_2) \\ &= 3z_2 \end{aligned}$$

And so (a) holds for addition and scalar multiplication.

(b) is not a valid vector space constraint.

The argument for (b) is that if $u_1 \leq u_2$ then $au_1 \leq au_2$.

Similarly, $bv_1 \leq bv_2$.

But if $a < 0$ and $b \geq 0$ then $|a|u_1 \geq |a|u_2$.

Then there is no guarantee that $au_1 + bv_2 \leq au_2 + bv_2$ is true.

The case where $b < 0$ and $a \geq 0$ is true by symmetry.

A simple counter example would be $(0, 0, 0) + (-1)(0, 0, 1) = (0, 0, -1)$. Clearly $(0, 0, -1) \notin W$.

And so W is not closed under addition and scalar multiplication. Therefore W is not a vector space.

Problem 8. *The union of two subspaces.*

Solution. *Proof.* Take the contrapositive to show that if neither $U \subseteq W$ nor $W \subseteq U$ then $U \cup W$ is not a subspace.

Choose two vectors u, w such that $u \in U, u \notin W$ and $w \in W, w \notin U$. So, $u, w \in U \cup W$. Assume $U \cup W$ is a subspace. So, $z = au + bw, z \in U \cup W$ which means $z \in U$ or $z \in W$.

Case $z \in U$:

U is a vector space so if $u \in U$, then $(-a)u \in U$. So, $z + (-a)u$ should also be in U.

$$\begin{aligned} z + (-a)u &= au + bw - au \\ &= bw \end{aligned}$$

But $w \notin U$ so $bw \notin U$. Contradiction.

The case where $z \in W$ is true by symmetry. $\square$