Deformation Theory

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1   Introduction

In the section, we will introduce the definitions and intuition for first order deformations.

1.1   Intuition

Embedded Deformation: Suppose we have a smooth submanifold X embedded an ambient complex manifold Y. The embedding is equipped with a normal bundle $N_Y X$. By the tubular neighborhood theorem, we have a embedding of its total space:

$$X \hookrightarrow Y$$

0-section

$$N_Y X$$

A smooth deformation of Y inside X is then a smooth section of $N_Y X$: at each point $x \in X$, the section give you the normal direction along which to "infinitesimally" deform X inside Y. This definition offers some differential topologically intuition, even though we no longer have an analog of the tubular neighborhood theorem in the holomorphic/algebraic setting.

More generally, we are given the data of

1. A morphism of objects $f : X \to Y$ in some category (e.g. schemes, complex manifolds, etc).

2. An "infinitesimal thickening" of X and Y, which are prescribed injective morphisms $X \to X'$ and $Y \to Y'$.

A deformation of f is then a lift of the morphism $X \to Y$ to a morphism $X' \to Y'$

$$
\begin{array}{ccc}
X & \longrightarrow & Y \\
\downarrow & & \downarrow \\
X' & \dashrightarrow{?} & Y'
\end{array}
$$

Deformation of Complex Structure: Viewing a complex manifold X as a real manifold with an integrable almost complex structure J, a deformation of the complex structure is a family of almost complex structures J_t parametrized by $t \in (-\epsilon, \epsilon)$ such that $J_0 = J$ and each J_t is integrable. Infinitesimally, we can think of this as a perturbation of the almost complex structure J by a small parameter t in the space of endomorphisms of the tangent bundle TX that satisfy $J_t^2 = -\mathrm{Id}$ and the integrability condition (vanishing of the Nijenhuis tensor).

The theory of Kodaira-Spencer describes how such deformations can be understood in terms of certain cohomology groups associated with the manifold. More precisely, first order deformations of the complex structure on X are classified by the cohomology group $H^1(X, T_X)$, where T_X is the holomorphic tangent bundle of X.

Deformation and Moduli Problem In relation to deformation of complex structure, suppose we have concretely constructed the moduli space $\mathcal{M}_g$ parameterizing smooth projective curves of genus g. A point $[C] \in \mathcal{M}_g$ corresponds to an isomorphism class of a smooth projective curve C of genus g. A deformation of the curve C can be thought of as a small perturbation of the complex structure on C, leading to a family of curves C_t parameterized by a small parameter t. Infinitesimally, this corresponds to moving along a tangent vector at the point $[C]$ in the moduli space $\mathcal{M}_g$. The tangent space at $[C]$ can be identified with the first cohomology group $H^1(C, T_C)$, where T_C is the holomorphic tangent bundle of the curve C.

Suppose we have a fine moduli space $\mathcal{M}$ parameterizing certain objects, meaning any family of such objects over a base B is given by a morphism

$$B \to \mathcal{M}$$

Then an infinitesimal deformation of an object X corresponding to a point $[X] \in \mathcal{M}$ is given by a family over the dual numbers $\mathrm{Spec}(\mathbb{C}[\epsilon]/(\epsilon^2))$ (we can think of the dual numbers as a point equipped with a tangent vector), such that the fiber over 0 is isomorphic to X. Then, the classifying map

$$T_0 B \to T_{[X]} \mathcal{M}$$

precisely specifies a tangent vector at $[X]$.

2 First Order Deformation

Definition 2.0.1. The **dual numbers** over a field k is the ring

$$D := k[t]/(t^2)$$

Note that $\mathrm{Spec}(D)$ is the one-point space consisting of the prime ideal (t). Its Zariski tangent space is given by

$$T_{(t)}D := \mathrm{Hom}_k((t)/(t^2), k)$$

which is a one-dimensional k-vector space. In contrast, $\mathrm{Spec}(k)$ is also a one-point space, but its Zariski tangent space is zero-dimensional. Thus, we can think of $\mathrm{Spec}(D)$ as a point equipped with a tangent vector. The dual numbers will be our base space for first order deformations.

Definition 2.0.2. Let Y be a scheme over k, and X be a closed subscheme. A **first order embedded deformation** of X in Y is a closed subscheme $X' \subset Y \times_k \mathrm{Spec}(D)$ such that

1. X' is flat over D

2. The fiber over the point is X, i.e

$$X' \times_{\mathrm{Spec}(D)} \mathrm{Spec}(k) = X$$

We would like to classify all such deformations. It is useful to characterize flatness over the dual numbers first:

Lemma 2.1. A module M over a commutative ring R is flat iff for every ideal $I \subset R$, the module homomorphism

$$I \otimes_R M \to M$$

is injective.

Proof. Follows directly from the fact that flatness is equivalent to the vanishing of $\mathrm{Tor}_1^R(R/I, M)$ for all ideals $I \subset R$. □

2.1 Embedded Deformation of Schemes

We first deal with the affine case. Suppose $X = \mathrm{Spec}(B)$ is affine, and Y is a closed subscheme defined by an ideal $I \subset B$. We see that first-order embedded deformations of Y in X correspond to ideals $J \subset B' := B[t]/(t^2)$ such that

1. B'/J is flat over D

2. The image of J under the quotient map

$$B[t]/(t^2) \to B$$

is I.

Proposition 2.1.1. There is a bijection between the set of first-order embedded deformations of Y in X and $\mathrm{Hom}_B(I, B/I)$.

Proof. Suppose we are given an ideal $J \subset B'$ such that the $J \bmod t = I$. By Lemma 2.1, flatness of B'/J over D is equivalent to the exactness of

$$0 \to B'/J \otimes_D (t) \to B'/J \to B'/J \otimes_D k \to 0$$

Viewing (t) as a k-module, we have

$$B/I \otimes_k (t) \cong (B'/J \otimes_D k) \otimes_k (t) \cong B'/J \otimes_D (t)$$

So we may rewrite the exact sequence as

$$0 \to B/I \xrightarrow{\cdot t} B'/J \to B/I \to 0$$

Since $B' := B[t]/(t^2)$ splits as $B \oplus tB$ as B-modules, for every $f \in I$, there exist lifts of the form $f + tg \in J$, where $f, g \in B$. By exactness of the sequence above, different lifts differ by an element in tI, so we may then define a function

$$\varphi : I \to B/I$$

by $\varphi(f) = \bar{g}$. It is easy to check this is well-defined and a B-module homomorphism. Conversely, given a B-module homomorphism $\phi : I \to B/I$, we may define an ideal:

$$J := \{f + tg \mid f \in I, g \in B, g \bmod I = \phi(f)\}$$

Flatness of B'/J over D can be checked by by the same exact sequence as above: an element tb is in the ideal J iff $b \in I$. The two constructions are inverse to each other, and we have the desired bijection.

$\square$

Definition 2.1.1. The first order embedded deformations of Y in X is **trivial** if they correspond to the zero homomorphism in $\mathrm{Hom}_B(I, B/I)$, which corresponds to the ideal $J = I \oplus tI \subset B[t]/(t^2)$.

Example 2.1.1. We are given curve C carved out by $f(x, y)$, embedded in $\mathbb{A}^2$ via the canonical map

$$k[x, y] \to k[x, y]/(f)$$

By our above analysis, first order embedded deformations of C in $\mathbb{A}^2$ correspond to elements of

$$\mathrm{Hom}_{k[x,y]/(f)}((f), k[x, y]/(f)) \cong k[x, y]/(f)$$

In particular, each $g \in k[x, y]/(f)$ gives rise to a first order deformation defined by the principal ideal $J = (f + tg)$. We can think of this as perturbing f by g infinitesimally.

Recall that for a closed embedding of schemes $X \to Y$ defined by an ideal sheaf $\mathcal{I} \subset \mathcal{O}_Y$, the normal sheaf $\mathcal{N}_{X/Y}$ is defined as the dual of the conormal sheaf $\mathcal{I}/\mathcal{I}^2$. Moreover,

$$\mathrm{Hom}_Y(\mathcal{I}, \mathcal{O}_X) \cong \mathrm{Hom}_X(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_X)$$

Since flatness is a local property, and our construction above is natural and compatible with localization, we may globalize Proposition 2.1.1 to obtain:

Theorem 2.2. Let Y be a scheme over a field k, and let X be a closed subscheme of Y. Then the deformations of X over D in Y are in natural one-to-one correspondence with elements of $H^0(X, N_{X/Y})$, with the zero element corresponding to the trivial deformation.

2.2 Deformation of Coherent Sheaves

Let X be a scheme over a field k, and $\mathcal{F}$ be a coherent sheaf on X. A first order deformation of $\mathcal{F}$ is a coherent sheaf $\mathcal{F}'$ on $X' := X \times_k \mathrm{Spec}(D)$ that is

1. flat over D;

2. equipped with a homomorphism $\mathcal{F}' \to \mathcal{F}$ such that the induced map $\mathcal{F}' \otimes_D k \to \mathcal{F}$ is an isomorphism.

Theorem 2.3. Let X be a scheme over a field k, and $\mathcal{F}$ be a coherent sheaf on X. Then the set of first order deformations of $\mathcal{F}$ is in natural one-to-one correspondence with elements of $\mathrm{Ext}_X^1(\mathcal{F}, \mathcal{F})$, with the zero element corresponding to the trivial deformation.

Proof. By Lemma 2.1, a first order deformation $\mathcal{F}'$ of $\mathcal{F}$ fits into a short exact sequence of D-modules

$$0 \to \mathcal{F} \xrightarrow{t} \mathcal{F}' \to \mathcal{F} \to 0$$

$\square$

References