Deformation Theory

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1 Introduction

In the section, we will introduce the definitions and intuition for first order deformations.

1.1 Intuition

Embedded Deformation: Suppose we have a smooth submanifold X embedded an ambient complex manifold Y. The embedding is equipped with a normal bundle $N_Y X$. By the tubular neighborhood theorem, we have a embedding of its total space:

$$\begin{tikzcd}
X \arrow[rr, hook] \arrow[dr, "\text{0-section}"'] & & Y \\
& N_Y X \arrow[ur, hook, dotted] &
\end{tikzcd}$$

A smooth deformation of Y inside X is then a smooth section of $N_Y X$: at each point $x \in X$, the section give you the normal direction along which to "infinitesimally" deform X inside Y. This definition offers some differential topologically intuition, even though we no longer have an analog of the tubular neighborhood theorem in the holomorphic/algebraic setting.

More generally, we are given the data of

1. A morphism of objects $f : X \to Y$ in some category (e.g. schemes, complex manifolds, etc).
2. An "infinitesimal thickening" of X and Y, which are prescribed injective morphisms $X \to X'$ and $Y \to Y'$.

A deformation of f is then a lift of the morphism $X \to Y$ to a morphism $X' \to Y'$

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & Y \arrow[d] \\
X' \arrow[r, dotted, "?"'] & Y'
\end{tikzcd}$$

Deformation of Complex Structure: Viewing a complex manifold X as a real manifold with an integrable almost complex structure J, a deformation of the complex structure is a family of almost complex structures J_t parametrized by $t \in (-\epsilon, \epsilon)$ such that $J_0 = J$ and each J_t is integrable. Infinitesimally, we can think of this as a perturbation of the almost complex structure J by a small parameter t in the space of endomorphisms of the tangent bundle TX that satisfy $J_t^2 = -\mathrm{Id}$ and the integrability condition (vanishing of the Nijenhuis tensor).

The theory of Kodaira-Spencer describes how such deformations can be understood in terms of certain cohomology groups associated with the manifold. More precisely, first order deformations of the complex structure on X are classified by the cohomology group $H^1(X, T_X)$, where T_X is the holomorphic tangent bundle of X.

Deformation and Moduli Problem In relation to deformation of complex structure, suppose we have concretely constructed the moduli space $\mathcal{M}_g$ parameterizing smooth projective curves of genus g. A point $[C] \in \mathcal{M}_g$ corresponds to an isomorphism class of a smooth projective curve C of genus g. A deformation of the curve C can be thought of as a small perturbation of the complex structure on C, leading to a family of curves C_t parameterized by a small parameter t. Infinitesimally, this corresponds to moving along a tangent vector at the point $[C]$ in the moduli space $\mathcal{M}_g$. The tangent space at $[C]$ can be identified with the first cohomology group $H^1(C, T_C)$, where T_C is the holomorphic tangent bundle of the curve C.

Suppose we have a fine moduli space $\mathcal{M}$ parameterizing certain objects, meaning any family of such objects over a base B is given by a morphism

$$B \to \mathcal{M}$$

Then an infinitesimal deformation of an object X corresponding to a point $[X] \in \mathcal{M}$ is given by a family over the dual numbers $\mathrm{Spec}(\mathbb{C}[\epsilon]/(\epsilon^2))$ (we can think of the dual numbers as a point equipped with a tangent vector), such that the fiber over 0 is isomorphic to X. Then, the classifying map

$$T_0 B \to T_{[X]}\mathcal{M}$$

precisely specifies a tangent vector at $[X]$.

2 First Order Deformation

Definition 2.0.1. The **dual numbers** over a field k is the ring

$$D := k[t]/(t^2)$$

Note that Spec(D) is the one-point space consisting of the prime ideal (t). Its Zariski tangent space is given by

$$T_{(t)}D := \mathrm{Hom}_k((t)/(t^2), k)$$

which is a one-dimensional k-vector space. In contrast, Spec(k) is also a one-point space, but its Zariski tangent space is zero-dimensional. Thus, we can think of Spec(D) as a point equipped with a tangent vector. The dual numbers will be our base space for first order deformations.

Definition 2.0.2. Let Y be a scheme over k, and X be a closed subscheme. A **first order embedded deformation** of X in Y is a closed subscheme $X' \subset Y \times_k \mathrm{Spec}(D)$ such that

1. X' is flat over D
2. The fiber over the point is X, i.e
$$X' \times_{\mathrm{Spec}(D)} \mathrm{Spec}(k) = X$$

We would like to classify all such deformations.

2.1 Computations For Affine Curves

References