Deformation Theory

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December 11, 2025

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1 Introduction

In the section, we will introduce the definitions and intuition for first order deformations.

1.1 Intuition

Embedded Deformation: Suppose we have a smooth submanifold X embedded an ambient complex manifold Y. The embedding is equipped with a normal bundle $N_Y X$. By the tubular neighborhood theorem, we have a embedding of its total space:

$$\begin{tikzcd}
X \arrow[rr, hook] \arrow[dr, "\text{0-section}"'] & & Y \\
& N_Y X \arrow[ur, hook, dotted] &
\end{tikzcd}$$

A smooth deformation of Y inside X is then a smooth section of $N_Y X$: at each point $x \in X$, the section give you the normal direction along which to "infinitesimally" deform X inside Y. This definition offers some differential topologically intuition, even though we no longer have an analog of the tubular neighborhood theorem in the holomorphic/algebraic setting.

More generally, we are given the data of

1. A morphism of objects $f : X \to Y$ in some category (e.g. schemes, complex manifolds, etc).
2. An "infinitesimal thickening" of X and Y, which are prescribed injective morphisms $X \to X'$ and $Y \to Y'$.

A deformation of f is then a lift of the morphism $X \to Y$ to a morphism $X' \to Y'$

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & Y \arrow[d] \\
X' \arrow[r, dotted, "?"'] & Y'
\end{tikzcd}$$

Deformation of Complex Structure: Viewing a complex manifold X as a real manifold with an integrable almost complex structure J, a deformation of the complex structure is a family of almost complex structures J_t parametrized by $t \in (-\epsilon, \epsilon)$ such that $J_0 = J$ and each J_t is integrable. Infinitesimally, we can think of this as a perturbation of the almost complex structure J by a small parameter t in the space of endomorphisms of the tangent bundle TX that satisfy $J_t^2 = -\mathrm{Id}$ and the integrability condition (vanishing of the Nijenhuis tensor).

The theory of Kodaira-Spencer describes how such deformations can be understood in terms of certain cohomology groups associated with the manifold. More precisely, first order deformations of the complex structure on X are classified by the cohomology group $H^1(X, T_X)$, where T_X is the holomorphic tangent bundle of X.

Deformation and Moduli Problem In relation to deformation of complex structure, suppose we have concretely constructed the moduli space $\mathcal{M}_g$ parameterizing smooth projective curves of genus g. A point $[C] \in \mathcal{M}_g$ corresponds to an isomorphism class of a smooth projective curve C of genus g. A deformation of the curve C can be thought of as a small perturbation of the complex structure on C, leading to a family of curves C_t parameterized by a small parameter t. Infinitesimally, this corresponds to moving along a tangent vector at the point $[C]$ in the moduli space $\mathcal{M}_g$. The tangent space at $[C]$ can be identified with the first cohomology group $H^1(C, T_C)$, where T_C is the holomorphic tangent bundle of the curve C.

Suppose we have a fine moduli space $\mathcal{M}$ parameterizing certain objects, meaning any family of such objects over a base B is given by a morphism

$$B \to \mathcal{M}$$

Then an infinitesimal deformation of an object X corresponding to a point $[X] \in \mathcal{M}$ is given by a family over the dual numbers $\mathrm{Spec}(\mathbb{C}[\epsilon]/(\epsilon^2))$ (we can think of the dual numbers as a point equipped with a tangent vector), such that the fiber over 0 is isomorphic to X. Then, the classifying map

$$T_0 B \to T_{[X]}\mathcal{M}$$

precisely specifies a tangent vector at $[X]$.

2 First Order Deformation

Definition 2.0.1. The **dual numbers** over a field k is the ring

$$D := k[t]/(t^2)$$

Note that $\mathrm{Spec}(D)$ is the one-point space consisting of the prime ideal (t). Its Zariski tangent space is given by

$$T_{(t)}D := \mathrm{Hom}_k((t)/(t^2), k)$$

which is a one-dimensional k-vector space. In contrast, $\mathrm{Spec}(k)$ is also a one-point space, but its Zariski tangent space is zero-dimensional. Thus, we can think of $\mathrm{Spec}(D)$ as a point equipped with a tangent vector. The dual numbers will be our base space for first order deformations.

Definition 2.0.2. Let Y be a scheme over k, and X be a closed subscheme. A **first order embedded deformation** of X in Y is a closed subscheme $X' \subset Y \times_k \mathrm{Spec}(D)$ such that

1. X' is flat over D
2. The fiber over the point is X, i.e

$$X' \times_{\mathrm{Spec}(D)} \mathrm{Spec}(k) = X$$

We would like to classify all such deformations.

2.1 Embedded Deformation of Affine Schemes

2.2 An Easy Example

We first inspect the following example: first-order embedded deformations of a curve C in $\mathbb{A}^2_k$. Translating Definition 2.0.2 into algebra, we have the following setup:

1. We are given curve C carved out by $f(x, y)$, embedded in $\mathbb{A}^2$ via the canonical map

$$k[x, y] \to k[x, y]/(f)$$

2. An embedded deformation $\mathcal{A}$ is a quotient of

$$k[x, y] \otimes_k D \cong D[x, y]$$

i.e of the form $\mathcal{A} = D[x, y]/I$, where $I = (\phi)$ why principal?, such that $\mathcal{A}$ is a flat D-algebra, and an isomorphism

$$D[x, y]/(\phi) \to k[x, y]/(f)$$

induced by sending the generator $t \in D$ to 0.

Intuitively, the defining equation for the curve can be expressed as a finite sum

$$f(x, y) = \sum_{i,j} a_{ij} x^i y^j$$

where $a_{ij} \in k$. The deformation, defined by the vanishing of $\phi(x, y)$, is of the form

$$\phi(x, y) = \sum_{i,j} (a_{ij} + b_{ij} t) x^i y^j$$

so the polynomial $g(x, y) = \sum_{i,j} b_{ij} x^i y^j$ determines the deformation, and can be thought of as the "deformed" f. However, writing out explicit defining equations as above requires choices up to a unit, i.e $(\phi) = (u\phi)$ as ideals for any unit u.

Lemma 2.1. The units in $k[x, y]$ are k^*. The units in $D[x, y]$ are of the form $c + ht$, where $c \in k^*$, and $h \in k[x, y]$.

Proof. The first statement is easy by Gauss lemma. For the second statement, suppose we have $f = \sum_{i,j}(a_{ij} + b_{ij}t)x^i y^j$ and $g = \sum_{i,j}(a'_{ij} + b'_{ij}t)x^i y^j$ such that $fg = 1$. By inductively looking at coefficients of the expanded product, we see we must have $a_{00} = a_{00}^{-1}$, and all other $a_{ij} = 0$. $\square$

Proposition 2.1.1. The set of first order embedded deformations of C in $\mathbb{A}^2$ is in bijection with elements in $k[x, y]/(f)$.

Proof. We already see that the deformation is determined by a choice of g and let $\phi = f + gt$. Suppose another choice g' yields the same deformation, i.e

$$f + g't = u\phi = cf + (fh + cg)t$$

we see $c = 1$, and $g' = fh + g$, so g and g' define the same element in $k[x, y]/(f)$. $\square$

2.3 General Results

Suppose $X = \mathrm{Spec}(B)$ is affine, and Y is a closed subscheme defined by an ideal $I \subset B$. We see that first-order embedded deformations of Y in X correspond to ideals $J \subset B' := B[t]/(t^2)$ such that

1. B'/J is flat over D
2. The image of J under the canonical collapse

$$B[t]/(t^2) \to B$$

is I.

Proposition 2.1.2. There is a bijection between the set of first-order embedded deformations of Y in X and $\mathrm{Hom}_B(I, B/I)$.

To prove the proposition, we first need a characterization of flatness.

Lemma 2.2. Let $A' \to A$ be a surjective homomorphism of Noetherian rings with kernel J such that $J^2 = 0$. An A'-module M' is flat over A' if and only if the following are satisfied:

1. $M := M' \otimes_{A'} A$ is flat over A
2. the canonical map $M \otimes_A J \to M'$ is injective.

Proof. Note that since $J^2 = 0$, the ideal J is automatically an $A \cong A'/J$ module, and

$$M \otimes_A J \cong (M' \otimes_{A'} A) \otimes_A J \cong M' \otimes_{A'} J$$

so the second criterion makes sense. The forward direction is clear: the first criterion follows from the fact that flatness is preserved under base change; the second criterion follows from the exactness of the functor $- \otimes_{A'} M'$, applied to the short exact sequence

$$0 \to J \to A' \to A \to 0$$

$\square$

References