Deformation Theory

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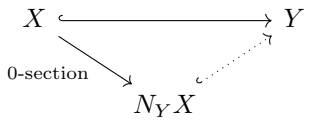
1 Introduction

In the section, we will introduce the definitions and intuition for first order deformations.

1.1 Intuition

Embedded Deformation: Suppose we have a smooth submanifold X embedded an ambient complex manifold Y. The embedding is equipped with a normal bundle $N_Y X$. By the tubular neighborhood theorem, we have a embedding of its total space:

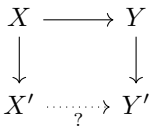


A smooth deformation of Y inside X is then a smooth section of $N_Y X$: at each point $x \in X$, the section give you the normal direction along which to "infinitesimally" deform X inside Y. This definition offers some differential topologically intuition, even though we no longer have an analog of the tubular neighborhood theorem in the holomorphic/algebraic setting.

More generally, we are given the data of

1. A morphism of objects $f : X \to Y$ in some category (e.g. schemes, complex manifolds, etc).
2. An "infinitesimal thickening" of X and Y, which are prescribed injective morphisms $X \to X'$ and $Y \to Y'$.

A deformation of f is then a lift of the morphism $X \to Y$ to a morphism $X' \to Y'$



Deformation of Complex Structure:

Deformation as a Family Given an object X in some category, a deformation of X can also be thought of as a family of objects $\mathcal{X} \to S$ over some parameterizing base S, such that the fiber over a distinguished point $s_0 \in S$ is isomorphic to X.

References