K Theory of Fields

David Zhu

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Background

Conjecture (Quillen-Lichtenbaum, circa 1972)

Let F be an algebraically closed field of characteristic exponent p. Then for $i \geq 1$, $K_{2i}(F)$ is a divisible torsion-free abelian group, and $K_{2i-1}(F)$ is a divisible group whose torsion subgroup is isomorphic to $\mathbb{Q}/\mathbb{Z}[\frac{1}{p}]$

- For $F = \overline{\mathbb{F}}_p$ computation of $K_*(\mathbb{F}_p)$, and the a colimit argument

Background

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- For $F = \overline{\mathbb{F}}_p$ computation of $K_*(\mathbb{F}_p)$, and the a colimit argument
- For K_1, $K_1(F) \cong F^\times$
- For K_2, a theorem by Tate.
- Then conjectured relationship between K-theory and special values of zeta functions.

Some Progress

Theorem (Harris-Segal, 1975)

Let R be the ring of integers of some number field F. Then, $K_{2i-1}(R)$ contains an explicit cyclic summand that maps isomorphically to $K_{2i-1}(F)$.

The explicit cyclic summand is determined as follows: Harris-Segal showed that for $R = \mathcal{O}_F$, one may construct homomorphism

$$\phi : R \to \mathbb{F}_q$$

such that the induced homomorphism

$$\phi_* : K_{2i-1}(R) \to K_{2i-1}(\mathbb{F}_q)$$

becomes a split surjection on l primary parts. The splitting arises from choosing a subgroup in $GL_i(\mathbb{F}_q)$ and some representation theoretical tools.

e-invariant

For a field F, let $\mu = \mu(F)$ be the group of all roots of unity in its closure.

Definition

For each i, we define the ith **Tate twist** of μ as the $G := \mathrm{Gal}(\overline{F}/F)$-module structure on μ by

$$g \cdot \zeta = g^i(\zeta)$$

denoted by $\mu(i)$. We let $w_i := |\mu(i)^G|$ and $w_i^{(l)^G} = |\mu(i)_{(l)}|$

e-invariant

From the known $K_{2i-1}(\mathbb{F}_q)$, one determines the explicit cyclic summand to be $\mathbb{Z}/w_i$. Such summand is detected by the natural map

$$e : K_{2i-1}(F)_{\mathrm{tor}} \to K_{2i-1}(\overline{F})^G_{\mathrm{tor}} \cong \mu(i)^G$$

which is called the **e-invariant**.

Fact

The classical Adams e-invariant detects a summand in the stable homotopy groups of sphere.

Homotopy Groups with Coefficients

Recall the Moore space $M(G, k)$ is characterized by having its reduced homology concentrated in degree k and $\tilde{H}^k(M(G, k), \mathbb{Z}) = G$. For G cyclic of order n, $M(G, k)$ can be constructed by gluing a k-cell to S^{k-1} by a degree n map.

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Definition

Let $M(\mathbb{Z}/n, m)$ be the Moore space of $\mathbb{Z}/n$. The mod n homotopy group of a space X is defined by

$$\pi_m(X;\mathbb{Z}/n) := [M(\mathbb{Z}/n,m), X]$$

K-theory with finite coefficients

For a general space X, $\pi_m(X, \mathbb{Z}/n)$ is only a group when $m \geq 2$. For infinite loop spaces, these homotopy groups with coefficients are all abelian groups.

Definition

We define the K-theory of a ring R with with $\mathbb{Z}/n$ coefficients as

$$K_m(R; \mathbb{Z}/n) := \pi_m(K(R); \mathbb{Z}/n)$$

Universal Coefficient Theorem

The mod n homotopy groups satisfy the following exact sequence

$$0 \longrightarrow \pi_n(X) \otimes \mathbb{Z}/p \longrightarrow \pi_n(X; \mathbb{Z}/p) \longrightarrow \mathrm{Tor}_1(\pi_{n-1}X, \mathbb{Z}/p) \longrightarrow 0$$

- The group $\pi_n(X) \otimes \mathbb{Z}/p$ measures how far $\pi_n(X)$ is from being p-divisible.
- The group $\mathrm{Tor}_1(\pi_{n-1}X, \mathbb{Z}/p)$ detects p-torsion in $\pi_{n-1}X$.

Suslin Ridigity

Theorem (Suslin, 83)

Let $i : k \to L$ be an extension of algebraically closed fields and n an integer such that $gcd(n, char(k)) = 1$. Then,

$$i_* : K_*(k, \mathbb{Z}/n) \to K_*(L; \mathbb{Z}/n)$$

is an isomorphism.

From Quillen's computation of $K_*(\mathbb{F}_p)$, the conjecture of Quillen-Lichtenbaum is solved for positive characteristic.

Proof of Suslin Rigidity

Preliminary Reduction

A Zorn's lemma argument allows reduction to the case where $\mathrm{Trdeg}_k(L) = 1$.

Given such an L, we can realize L as a colimit

$$L \cong \varinjlim_{A \subset L} A$$

where A is a finitely generated, integrally closed k-subalgebra of L.

Smooth Curves

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$$K(L; \mathbb{Z}/n) \cong \varinjlim_{A \subset L} K(A; \mathbb{Z}/n)$$

We may view A as a smooth k-curve. A point $x \in Spec(A)$ gives rise to a splitting

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is injective. We may deduce

$$K_*(k; \mathbb{Z}/n) \to K_*(L; \mathbb{Z}/n)$$

is injective after passage to colimit.

Further Reduction

Let $\alpha \in K_*(L; \mathbb{Z}/n)$ be an arbitrary element. By the colimit characterization, α is in the image of some

$$K_*(A; \mathbb{Z}/n) \to K_*(L; \mathbb{Z}/n)$$

for some subalgebra A. The next "trick" helps us to reduce the problem to the following proprosition:

Proposition

If A is a finitely generated smooth k-algebra (of $\dim 1$), then two k-points x, y of $Spec(A)$ induces the same map

$$x_* = y_* : K_*(A; \mathbb{Z}/n) \to K_*(k; \mathbb{Z}/n)$$

The Trick

The inclusion $A \xrightarrow{i} L$ factors as

$$A \xrightarrow{Id \otimes 1} A \otimes_k L \xrightarrow{i_0} L$$

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$$A \xrightarrow{/x} k \xrightarrow{i} L$$

which induces another morphism

$$A \otimes_k L \xrightarrow{i_x} L$$

The Trick

Assuming the Proposition, the two maps

$$A \otimes_k L \underset{i_x}{\overset{i_0}{\rightrightarrows}} L$$

induce the same map on K-theory, so we have two equivalent compositions.

$$K_*(A; \mathbb{Z}/n) \longrightarrow K_*(A \otimes_k L; \mathbb{Z}/n) \underset{(i_x)_*}{\overset{(i_0)_*}{\rightrightarrows}} K(L; \mathbb{Z}/n)$$

The top composition is just the map induced by the inclusion $A \hookrightarrow L$;

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The top composition is just the map induced by the inclusion $A \hookrightarrow L$;the second map also factors as

$$K(A; \mathbb{Z}/n) \xrightarrow{(/x)_*} K(k; \mathbb{Z}/n) \xrightarrow{(i)_*} K(L; \mathbb{Z}/n)$$

therefore α is in the image of i_*, and we have surjectivity.

Transfer

Definition

If $f : R \to S$ is a ring map such that S becomes a finitely generated R-module, then the forgetful exact functor

$$M(S) \to M(R)$$

induce the **transfer map**

$$f^* : G(S) \to G(R)$$

Recall for regular Noetherian rings, the G-theory and K-theory agree, as a consequence of the resolution theorem. Therefore, we have a transfer map between K-theory as well.

More on Transfer

Now let R be a Dedekind domain (Noetherian, integrally closed, Krull dimension 1), with $F := Frac(R)$. Localizing at the non-zero elements gives us the LES

$$K_{n+1}(F) \xrightarrow{\partial} \oplus_{\mathfrak{p}} K(R/\mathfrak{p}) \xrightarrow{\oplus \mathfrak{p}^*} K_n(R) \longrightarrow K_n(F)$$

and the middle map is the sum of the individual transfer maps from the projection $R \to R/\mathfrak{p}$.

Specialization

Now let $C = Spec(A)$ be a smooth k-curve, and x a closed point on C. Each stalk $\mathcal{O}_x$ is a then a DVR. Let π_x be a choice of uniformizing parameter, which induces a residue map $\pi_s : \mathcal{O}_x \to k$. The localization sequence for $\mathcal{O}_x$ looks like

$$G(k)/n \longrightarrow G(\mathcal{O}_x)/n \longrightarrow G(F)/n$$

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$$G(k)/n \longrightarrow G(\mathcal{O}_x)/n \longrightarrow G(F)/n$$

This is a sequence of $K(\mathcal{O}_x)$-modules. We will investigate how the module structure interact with the product structure.

Specialization

The module structure gives us pairings

$$K_p(\mathcal{O}_x) \otimes K_q(-;\mathbb{Z}/n) \to K_{p+q}(-;\mathbb{Z}/n)$$

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We are interested in the following composition

$$\begin{tikzcd}
K_1(\mathcal{O}_x) \otimes K_q(F;\mathbb{Z}/n) \arrow[r, "i_* \otimes 1"] & K_1(F) \otimes K_q(F;\mathbb{Z}/n) \arrow[r, "\smile"] & K_{1+q}(F;\mathbb{Z}/n) \arrow[d, "\partial"] \\
& & K_q(k;\mathbb{Z}/n)
\end{tikzcd}$$

where ∂ comes from the localization.

Product Formula

Let $a \in K_1(\mathcal{O}_x)$ and $b \in K_q(F; \mathbb{Z}/n)$. We have the product formula

$$\partial(i_*(a) \smile b) = (\pi_s)_*(a) \smile \partial b$$

This follows from everything being a $K(\mathcal{O}_x)$ module homomorphism, and the Leibniz rule.

Critial Observation: $(\pi_s)_*(a)$ lives in $K_1(k) \cong k^*$, which is n-divisible by assumption on characteristic. Thus, the composition must be 0!

Specialization

Now a choice of uniformizing parameter gives us a splitting of

$$0 \longrightarrow \mathcal{O}_x^* \longrightarrow F^* \longrightarrow \mathbb{Z} \longrightarrow 0$$

which is an isomorphism $K_1(F) \cong \mathbb{Z} \oplus K_1(\mathcal{O}_x)$.

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Thus, there is a **specialization map**

$$s_x : K_*(F; \mathbb{Z}/n) \to K_*(k; \mathbb{Z}/n)$$

that makes

$$\begin{tikzcd}
K_1(F) \otimes K_q(F; \mathbb{Z}/n) \arrow[r] \arrow[d] & \mathbb{Z} \otimes K_q(F; \mathbb{Z}/n) \arrow[d, dashed, "s_x"] \\
K_{1+q}(F; \mathbb{Z}/n) \arrow[r] & K_q(k; \mathbb{Z}/n)
\end{tikzcd}$$

commute.

Properties of Specialization

The diagram

$$\begin{tikzcd} K_1(F) \otimes K_q(F;\mathbb{Z}/n) \arrow[r] \arrow[d, "\smile"] & \mathbb{Z} \otimes K_q(F;\mathbb{Z}/n) \arrow[d, dashed, "s_x"] \\ K_{1+q}(F;\mathbb{Z}/n) \arrow[r, "\partial"] & K_q(k;\mathbb{Z}/n) \end{tikzcd}$$

gives us the formula for specialization

$$s_x(a) = \partial(\pi_x \smile a)$$

Properties of Specialization

Lemma

The diagram

$$\begin{tikzcd} K_q(A;\mathbb{Z}/n) \arrow[d] \arrow[dr, "(-1)^q x_*"] & \\ K_q(F;\mathbb{Z}/n) \arrow[r, "s_x"'] & K_q(k;\mathbb{Z}/n) \end{tikzcd}$$

commutes.

1. We know x_* is a split surjection, so each specialization map s_x is a surjection as well.

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Final Reduction

We look at the localization sequence again:

$$K_{q+1}(F;\mathbb{Z}/n) \xrightarrow{\partial} K_q(k;\mathbb{Z}/n) \xrightarrow{(\pi_x)^*} K_q(\mathcal{O}_x;\mathbb{Z}/n) \longrightarrow K_q(F;\mathbb{Z}/n)$$

The boundary differential being a surjection implies the transfer $(\pi_x)^*$ is the zero map; we may further deduce the inclusion homomorphism

$$K_q(\mathcal{O}_x;\mathbb{Z}/n) \to K_q(F;\mathbb{Z}/n)$$

is an injection for all $x \in A$. It is clear then the inclusion homomorphism

$$K_*(A;\mathbb{Z}/n) \to K_*(F;\mathbb{Z}/n)$$

is an injection as well.

Reduction to Birational

$$\begin{tikzcd} K_q(A;\mathbb{Z}/n) \arrow[d, "inject"] \arrow[dr, "(-1)^q x_*"] & \\ K_q(F;\mathbb{Z}/n) \arrow[r, "s_x"'] & K_q(k;\mathbb{Z}/n) \end{tikzcd}$$

We may further reduce Proposition to the following

Main Theorem

Let C be a smooth curve over k, and let $F := k(C)$ be its function field. If x, y are two closed k-points on the curve, then the **specialization maps**

$$s_x, x_y : K_*(F;\mathbb{Z}/n) \to K_*(k;\mathbb{Z}/n)$$

are equal.

Proving the Main Theorem

First, we may complete the smooth curve C to a smooth projective curve, which does not alter the function field. And here some algebraic geometry comes in.

Recall the set of cartier divisors $\mathrm{Cart}(C)$ is the free abelian group of closed points on C. Consider the homomorphism

$$\lambda : \mathrm{Cart}(C) \to \mathrm{Hom}(K_*(F;\mathbb{Z}/n), K_*(k;\mathbb{Z}/n))$$

by sending $[c]$ to the specialization map s_c.

Proving the Main Theorem

We have the exact sequence

$$0 \to F^* \xrightarrow{\mathrm{div}} \mathrm{Cart}(C) \to \mathrm{Pic}(C) \to 0$$

where $\mathrm{div}(f) = \sum_{f(p)=0} e_c[p] - \sum_{f(p)=\infty} e_c[p]$.

Proposition

The composition $\lambda \circ \mathrm{div}$ is 0 on F^*.

Deducing Ridigity

Since the composition $\lambda \circ \operatorname{div} = 0$, we know λ factors through the cokernel of div, which is $\operatorname{Pic}(C)$. Moreover, classes of the form $[c_0] - [c_1]$ all live in the Jacobian $J(C)$, which is the kernel the degree map

$$J(C) \to \operatorname{Pic}(C) \xrightarrow{\deg} \mathbb{Z}$$

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Fact (Abelian Varieties)

The group $J(C)$ is the k-point of the **Jacobian variety**, which is an abelian variety. All abelian varieties are divisible.

However, we see the target group $\mathrm{Hom}(K_*(F;\mathbb{Z}/n), K_*(k;\mathbb{Z}/n))$ is of exponent n, so λ is 0 on $J(C)$, which implies $s_{c_0} = s_{c_1}$ for all c_0, c_1.

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Overview and Preview

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3. This action is trivial as the Jacobian is divisible.

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2. A certain action of finite correspondences on $K(-;\mathbb{Z}/n)$ factors through the Jacobian of a smooth projective curve
3. This action is trivial as the Jacobian is divisible.

The same ingredients also appeared in many subsequent rigidity-type results in K-theory. Later on, motivic homotopy theorists generalized these results to sheaves with transfers.

More Ridigity

Theorem (Gillet-Thomason, 84)

Let k be a separably closed field, R a strictly Henselian regular k-algebra of geometrical type, and n be an integer invertible in k. Then,

$$K(k; \mathbb{Z}/n) \to K(R; \mathbb{Z}/n)$$

is an isomorphism.

Theorem (Gabber, 92)

Let R be a Henselian local ring with residue field k and assume n is an integer invertible in R. Then

$$K(R; \mathbb{Z}/n) \to K(k; \mathbb{Z}/n)$$

is an isomorphism.

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Bott Element

Suppose R contans a primitive pth root of unity ζ.

$$0 \longrightarrow K_2(R) \otimes \mathbb{Z}/p \longrightarrow K_2(R; \mathbb{Z}/p) \longrightarrow \mathrm{Tor}_1(K_1(R), \mathbb{Z}/p) \longrightarrow 0$$

Definition

An element in $K_2(R; \mathbb{Z}/p)$ that maps to $\zeta \in \mathrm{Tor}_1(K_1(R), \mathbb{Z}/p)$ is called a **Bott element**.

Finite Field

Note the mod 2 Moore spectrum is not a ring spectrum, but Browder still proves a product structure on K theory with coefficients.

Theorem (Browder, 78)

The K-theory with coefficients $K_*(\mathbb{F}_p; \mathbb{Z}/n)$ admits a graded ring structure for all n.

Finite Field

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Theorem (Browder, 78)

The K-theory with coefficients $K_*(\mathbb{F}_p; \mathbb{Z}/n)$ admits a graded ring structure for all n.

So from the ring structure, we have a canonical homomorphism

$$\mathbb{Z}/n[\beta, \zeta] \to K_*(\mathbb{F}_p; \mathbb{Z}/n)$$

Example

Theorem (Browder, 78)

For $l \neq p$, we have

$$K_*(\mathbb{F}_p; \mathbb{Z}/l) \cong \mathbb{Z}/l[\beta, \zeta]/\zeta^2$$

as a graded ring. Passing to the colimit gives

$$K_*(\overline{\mathbb{F}}_p; \mathbb{Z}/l) \cong \mathbb{Z}/l[\beta]$$

Field of positive characteristic

Corollary

Let F be an algebraically closed field of positive characteristic p. Then,

1. If n is even and $n > 0$, $K_n(F)$ is uniquely divisible.
2. If $n = 2i - 1$ is odd, $K_{2i-1}(F)$ is the direct sum of a uniquely divisible group and the torsion group $\mathbb{Q}/\mathbb{Z}[\frac{1}{p}]$
3. If $gcd(n, p) = 1$, a choice of a Bott element $\beta \in K_2(F; \mathbb{Z}/n)$ determines a graded ring isomorphism

$$K_*(F; \mathbb{Z}/n) \cong \mathbb{Z}/n[\beta]$$

Characteristic 0

Proposition

If F is an algebraically closed field of characteristic 0 then for every $N > 0$ the choice of a Bott element $\beta \in K_2(F; \mathbb{Z}/n)$ determines a graded ring isomorphism

$$K_*(F; \mathbb{Z}/n) \cong \mathbb{Z}/n[\beta]$$

By Suslin Ridigity, it suffices to demonstrate one such example. The idea is to consider $F = \overline{\mathbb{Q}}_p$. One may write $F \cong \varinjlim E_{p^r}$, where E_{p^r} is the maximal algebraic extension of $\mathbb{Q}_p$ in F with residue field $\mathbb{F}_{p^r}$. One then applies Gabber Ridigity to show that

$$K_*(E_{p^r}; \mathbb{Z}/n) \cong \mathbb{Z}/n[\beta]$$

and take the colimit.

Final Thoughts

- K-theory of $\mathbb{R}()$
- K-theory of local/global fields.
- Relation with étale/motivic cohomology.