Math 622: Complex Algebraic Geomtry

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This is notes taken for Math 622: Complex Algebraic Geometry in Fall 2025, taught by Professor Ron Donagi at UPenn.

1 Overview of Hodge Theory

1.1 Wed 27

The first part of the course will be a primer on complex algebraic geometry, in particular Hodge theory, without too much detailed proofs.

Definition 1.0.1. A **Kahler manifold** is a complex maniolfd with a Hermitian metric whose imaginary part is closed.

Example 1.0.1. The complex projective spaces $\mathbb{CP}^n$ with Fubini-Study metric are Kahler manifolds. Submanifolds of projective space inherit Kahler structures.

A reference for this is [1] 3.1.9.

Theorem 1.1 (Kodaira Embedding)**.** A compact complex maniolf dadmits a holomorphic embedding into projective space iff it has a Kahler metric whose Kahler form is integral.

The upshot is that Hodge and Lefshetz decomositions work for compact Kahler manifolds.

Review of complex differential forms

Definition 1.1.1 (Complex Differential Forms)**.**

Definition 1.1.2. For a complex manifold X, we have a decomposition of differential forms

$$\Omega^k_{X,\mathbb{C}} := \oplus_{p+q=k}\Omega^{p,q}_X$$

Theorem 1.2 (Hodge Decomposition)**.** For a compact Kahler manifold

$$H^k(X, \mathbb{C}) \cong \oplus_{p+q=k} H^{p,q}(X)$$

and

$$H^{p,q}(X) \cong \overline{H^{p,q}(X)}$$

The left hand side will be the DeRham cohomology, and the left hand side is the Dolbeaut cohomoology.

Example 1.2.1 (Hodge Diamond of a Riemann Surface)**.** For a compact Riemann surafce X of genus g, the Hodge numbers are

$$h^{0,0} = 1, h^{1,0} = g, h^{0,1} = g, h^{1,1} = 1$$

$$\begin{array}{ccc} & 1 & \\ g & & g \\ & 1 & \end{array}$$

Example 1.2.2. For a compact complex surface X, the Hodge numbers are

$$\begin{array}{ccccc} & & h^{0,0} & & \\ & h^{1,0} & & h^{0,1} & \\ h^{2,0} & & h^{1,1} & & h^{0,2} \\ & h^{2,1} & & h^{1,2} & \\ & & h^{2,2} & & \end{array}$$

Definition 1.2.1. Given the Kahler form ω. define the Lefshetz operator

$$L : H^k(X, \mathbb{R}) \to H^{k+2}(X, \mathbb{R})$$

given by

$$\alpha \mapsto \omega \wedge \alpha$$

Theorem 1.3 (Hard Lefshetz)**.** For $n = dimX$, and $k \leq n$, we have

$$L^{n-k} : H^k(X, \mathbb{R}) \cong H^{2n-k}(X, \mathbb{R})$$

Definition 1.3.1 (Primitive Cohomology)**.**

$$H^k(X, \mathbb{R})_{\text{prim}} := ker(L^{n-k+1})$$

1.2 Harmonic Forms

1. Equip X with a Kahler metric.
2. Define the adjoint d^* of d, and the Laplacian

$$\Delta_d = dd^* + d^*d$$

3. The harmonic forms are

$$\mathcal{H}^k X := ker \Delta_d$$

Corollary 1.3.1 (Harmonic Decomposition)**.** If α is harmonic, then each (p, q)-component of α is harmonic,. hnece

$$\mathcal{H}(X) \cong \oplus_{p+q=k} \mathcal{H}^{p,q}(X)$$

The Frolicher spectra seauence is obtained from the Hodge filtratiokn on the de Rham complex.

Theorem 1.4. For a compact Kahler manifold, the Frolicher spectral sequence degenerates at E_1.

The theorem implies the Hodge filtration on cohomology

$$F^p H^k(X, \mathbb{C}) = \oplus_{r \geq p} H^{r,k-r}(X)$$

Definition 1.4.1. A **Hodge structure** of **weight** k is a decomposition

$$H_{\mathbb{C}} = \oplus_{p+1=k} H^{p,q}$$

with $H^{p,q} = \overline{H^{p,q}}$

A story is that there is a dense open subset in $\mathbb{P}^9$ that parameterizes cubic curves in $\mathbb{P}^3$. Replacing the curves with their cohomology gives you a vector bundle over the base space.

Definition 1.4.2. An **analytic cycle** of codimension k is a linear combination of irreducible analytic subset of codimension k.

The Hodge conjecture states that if

$$\alpha \in H^{2k}(X, \mathbb{Q}) \cap H^{k,k}(X)$$

, then α is an algebraic cycle.

2 Wed, Sep 3

Did not attend lol.

3 Mon, Sep 8

Review of complex manifolds and sheaves.

Proposition 3.0.1. There is an equivalence between vector bundles and locally free $\mathcal{O}_X$-modules through sheaf of sections.

Definition 3.0.1. The **Picard group** of X is the set of isomorphism classes of line bundles under tensor product.

$$\mathrm{Pic}(X) \cong H^1(X, \mathcal{O}_X^*)$$

The exponential sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X^* \longrightarrow 0$$

induces the map

$$c_1 : \mathrm{Pic}(X) \to H^2$$

Definition 3.0.2 (Canonical ring). We define

$$R(X) = \oplus_{m \geq 0} H^0(X, K_X^{\otimes m})$$

where K_X is the canonical bundle.

Definition 3.0.3. A sheaf F is **flasque** is every restriction map

$$F(X) \to F(U)$$

is surjective.

Remark 3.0.1. Quote from Ron: A flasque sheaf is almost never "nice": for example, the sheaf of continuous functions of a punctured disk is not flasque.

Proposition 3.0.2. Every sheaf has a flasque resolution.

Definition 3.0.4. The **sheaf cohomology** of a space with a sheaf (M, F) is the cohomology of the chain complex obtained from takning global sections of any flasque resolution.

As usual, it is independent of the chosen resolution, and a short exact sequence of sheaves induces a long exact sequence on cohomology.

Definition 3.0.5. A sheaf is **acyclic** if all higher cohomologies are 0.

Theorem 3.1. For paracompact spaces, the Cech cohomology and sheaf cohomology are naturally isomorphic.

Proposition 3.1.1. For smooth line bundles,

$$c_1 : \mathrm{Pic}(X) \to H^2(X; \mathbb{Z})$$

is an isomorphism, for $\mathcal{O}_X^*$ is **soft**, thus acyclic.

For holomorphic case, this is **NOT** true.

Proposition 3.1.2 (Global Sections of $\mathcal{O}(k)$)**.** For $k > 0$, there is an isomorphism

$$H^0(\mathbb{P}^n, \mathcal{O}(k)) \cong \mathbb{C}[z_0, ..., z_n]_k \cong \mathrm{Sym}^k * (V^\vee)$$

Let $T_{\mathbb{P}^n}$ and $\Omega^1_{\mathbb{P}^n}$ be the holomorphic tangent and cotangent bundles.

Proposition 3.1.3 (Euler Sequence)**.** On $\mathbb{P}^n$ there is a natural short exact sequence of holomorphic vector bundles

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus(n+1)} \longrightarrow T_{\mathbb{P}^n} \longrightarrow 0$$

Alternatively, one may dualize and get

$$0 \longrightarrow \Omega^1_{\mathbb{P}^n} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus(n+1)} \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow 0$$

Proposition 3.1.4. The **canonical bundle** of $\mathbb{P}^n$ is

$$K_{\mathbb{P}^n} \cong \mathcal{O}_{\mathbb{P}^n}(-n-1)$$

References

[1] Daniel Huybrechts. *Complex Geometry: An Introduction.* Universitext. Springer, Berlin, Heidelberg, 2005.