

Math 622: Complex Algebraic Geometry

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This is notes taken for Math 622: Complex Algebraic Geometry in Fall 2025, taught by Professor Ron Donagi at UPenn.

1 Overview of Hodge Theory

1.1 Wed 27

The first part of the course will be a primer on complex algebraic geometry, in particular Hodge theory, without too much detailed proofs.

Definition 1.0.1. A **Kahler manifold** is a complex manifold with a Hermitian metric whose imaginary part is closed.

Example 1.0.1. The complex projective spaces \mathbb{CP}^n with Fubini-Study metric are Kahler manifolds. Submanifolds of projective space inherit Kahler structures.

A reference for this is [2] 3.1.9.

Theorem 1.1 (Kodaira Embedding). A compact complex manifold admits a holomorphic embedding into projective space iff it has a Kahler metric whose Kahler form is integral.

The upshot is that Hodge and Lefshetz decompositions work for compact Kahler manifolds.

Review of complex differential forms

Definition 1.1.1 (Complex Differential Forms).

Definition 1.1.2. For a complex manifold X , we have a decomposition of differential forms

$$\Omega_{X,\mathbb{C}}^k := \bigoplus_{p+q=k} \Omega_X^{p,q}$$

Theorem 1.2 (Hodge Decomposition). For a compact Kahler manifold

$$H^k(X, \mathbb{C}) \cong \bigoplus_{p+q=k} H^{p,q}(X)$$

and

$$H^{p,q}(X) \cong \overline{H^{p,q}(X)}$$

The left hand side will be the DeRham cohomology, and the left hand side is the Dolbeaut cohomology.

Example 1.2.1 (Hodge Diamond of a Riemann Surface). For a compact Riemann surface X of genus g , the Hodge numbers are

$$h^{0,0} = 1, h^{1,0} = g, h^{0,1} = g, h^{1,1} = 1$$

1

g g

1

Example 1.2.2. For a compact complex surface X , the Hodge numbers are

$$h^{0,0}$$

$$h^{1,0} \quad h^{0,1}$$

$$h^{2,0} \quad h^{1,1} \quad h^{0,2}$$

$$h^{2,1} \quad h^{1,2}$$

$$h^{2,2}$$

Definition 1.2.1. Given the Kahler form ω . define the Lefshetz operator

$$L : H^k(X, \mathbb{R}) \rightarrow H^{k+2}(X, \mathbb{R})$$

given by

$$\alpha \mapsto \omega \wedge \alpha$$

Theorem 1.3 (Hard Lefshetz). For $n = \dim X$, and $k \leq n$, we have

$$L^{n-k} : H^k(X, \mathbb{R}) \cong H^{2n-k}(X, \mathbb{R})$$

Definition 1.3.1 (Primitive Cohomology).

$$H^k(X, \mathbb{R})_{\text{prim}} := \ker(L^{n-k+1})$$

1.2 Harmonic Forms

1. Equip X with a Kahler metric.
2. Define the adjoint d^* of d , and the Laplacian

$$\Delta_d = dd^* + d^*d$$

3. The harmonic forms are

$$\mathcal{H}^k X := \ker \Delta_d$$

Corollary 1.3.1 (Harmonic Decomposition). If α is harmonic, then each (p, q) -component of α is harmonic., hnece

$$\mathcal{H}(X) \cong \bigoplus_{p+q=k} \mathcal{H}^{p,q}(X)$$

The Frolicher spectra seauence is obtained from the Hodge filtratiokn on the de Rham complex.

Theorem 1.4. For a compact Kahler manifold, the Frolicher spectral sequence degenerates at E_1 .

The theorem implies the Hodge filtration on cohomology

$$F^p H^k(X, \mathbb{C}) = \bigoplus_{r \geq p} H^{r, k-r}(X)$$

Definition 1.4.1. A **Hodge structure** of **weight** k is a decomposition

$$H_{\mathbb{C}} = \bigoplus_{p+q=k} H^{p,q}$$

$$\text{with } H^{p,q} = \overline{H^{q,p}}$$

A story is that there is a dense open subset in \mathbb{P}^9 that parameterizes cubic curves in \mathbb{P}^3 . Replacing the curves with their cohomology gives you a vector bundle over the base space.

Definition 1.4.2. An **analytic cycle** of codimension k is a linear combination of irreducible analytic subset of codimension k .

The Hodge conjecture states that if

$$\alpha \in H^{2k}(X, \mathbb{Q}) \cap H^{k,k}(X)$$

, then α is an algebraic cycle.

2 Wed, Sep 3

Did not attend LOL. But here is what I think is covered:

We begin with the definition of a complex manifold.

Definition 2.0.1. A **holomorphic atlas** on a smooth manifold is an atlas $\{U_i, \varphi_i\}$ of the form

$$\varphi_i : U_i \cong \varphi_i(U_i) \subset \mathbb{C}^n$$

where the \cong sign denotes a homeomorphism. Two atlases are considered equivalent if for every (U_i, φ_i) and (U'_j, φ'_j) , the maps

$$\varphi_i \circ \varphi_j'^{-1} : \varphi_j'(U_i \cap U'_j) \rightarrow \varphi_i(U_i \cap U'_j)$$

are holomorphic.

Definition 2.0.2. A **complex manifold of dimension n** is a smooth manifold of dimension $2n$ endowed with an equivalence class of holomorphic atlas.

One difference between smooth manifold and complex manifolds is: every open subset of \mathbb{R}^n contains an open ball diffeomorphic to \mathbb{R}^n . Thus, every smooth manifold can be covered by charts diffeomorphic to \mathbb{R}^n . However, in general a complex manifold cannot be covered by open subsets biholomorphic to \mathbb{C}^n . This is because \mathbb{C} is not biholomorphic to any of its proper open subsets. Considering the basis for the topology, one can say complex manifolds are modelled after the complex disks \mathbb{D}^n .

Definition 2.0.3. A **holomorphic function** on complex manifold X is a function

$$f : X \rightarrow \mathbb{C}$$

such that for all charts $\varphi : U_i \rightarrow \mathbb{C}$, the maps

$$f \circ \varphi_i^{-1} : \varphi_i(U_i) \rightarrow \mathbb{C}$$

is holomorphic.

From the maximum modulus principle for several variables, one can deduce the following version of Liouville's theorem

Proposition 2.0.1. Any global holomorphic function on a connected compact complex manifold X is constant.

In particular, this means there is no “partition of unity” for complex manifolds, where we would dream of gluing holomorphic functions together like we could for smooth manifolds. Thus, complex manifolds are much more rigid.

Definition 2.0.4. Let X be a complex manifold. The **sheaf of holomorphic functions** on X , denoted by \mathcal{O}_X , is defined by

$$\mathcal{O}_X(U) := \{f : U \rightarrow \mathbb{C} : f \text{ holomorphic}\}$$

The sheaf conditions are easy to check. By the previous proposition, it is immediate that if X is compact and connected,

$$\Gamma(X, \mathcal{O}_X) = \mathbb{C}$$

Here is a famous theorem in complex analysis showing the compactness can be slightly weakened

Theorem 2.1 (Hartog's Theorem). Let G be an open subset of \mathbb{C}^n , where $n \geq 2$. Let $K \subset G$ be a compact subset. If $G \setminus K$ is connected, then any holomorphic function on $G \setminus K$ can be extended over to the entire G .

In particular, for X a complex manifold of dimension at least 2, we have

$$\Gamma(X, \mathcal{O}_X) \cong \Gamma(X - \{x\}, \mathcal{O}_X)$$

Remark 2.1.1. Here is an example of the failure of Hartog's theorem in dimension 1: $f(z) = \frac{1}{z}$ is holomorphic on the punctured plane, but cannot be extended to the entire \mathbb{C} .

Definition 2.1.1. A continuous map $f : X \rightarrow Y$ between complex manifolds is **holomorphic** if the corresponding atlas maps are holomorphic.

Definition 2.1.2. A **meromorphic function** on a complex manifold is a map

$$f : X \rightarrow \coprod_{x \in X} \text{Quot}(\mathcal{O}_{X,x})$$

which associates every point $x \in X$ an element $f_x \in \text{Quot}(\mathcal{O}_{X,x})$. Moreover, f_x are required to be locally a quotient of holomorphic functions. The **sheaf of meromorphic functions** is then denoted by \mathcal{K}_X , and its global sections $K(X)$.

Here is a theorem that bounded the size of meromorphic functions on a complex manifold.

Theorem 2.2 (Siegel's Theorem). Let X be a compact connected complex manifold of dimension n . Then

$$\text{Trdeg}_{\mathbb{C}} K(X) \leq n$$

Definition 2.2.1. The **algebraic dimension** of a compact connect complex manifold X is

$$a(X) := \text{Trdeg}_{\mathbb{C}}(X)$$

Note that the definition works properly only for compact manifold. For example, if we consider the function field of \mathbb{C} , it is much larger than its algebraic counter part, since

$$f(z) = \frac{1}{\sin(z)}$$

is a meromorphic function on \mathbb{C} .

Example 2.2.1. The algebraic dimension of \mathbb{P}^n is n .

Proof. For \mathbb{P}^2 : we claim the algebraic dimension for \mathbb{P}^2 is 2. From Siegel's theorem, it suffices to show $a(\mathbb{P}^2) \geq 2$. Define

$$f_i : \mathbb{P}^2 \rightarrow \coprod_{x \in \mathbb{P}^2} Q(\mathcal{O}_{\mathbb{P}^2,x})$$

by sending a point p to the germ of the rational function $\frac{z_i}{z_0} \in Q(\mathcal{O}_{\mathbb{P}^2, p})$ for $i = 1, 2$. Locally on $U_0 \cong \mathbb{A}^2 = \{(u, v) := (\frac{z_1}{z_0}, \frac{z_2}{z_0})\}$, the assignment f_i are just the coordinate functions; on $U_1 \cong \mathbb{A}^2 = \{(u, v) := (\frac{z_0}{z_1}, \frac{z_2}{z_1})\}$, we have $f_1 = \frac{1}{u}$ and $f_2 = \frac{v}{u}$. The change of coordinates with U_2 is similar. Clearly, $\frac{z_1}{z_0}$ and $\frac{z_2}{z_0}$ do not satisfy any algebraic relations by degree reasons, so the function field is of trascendence degree at least 2.

□

2.1 Examples of Complex Manifolds

Of course, \mathbb{C}^n and \mathbb{CP}^n are well-known examples of complex manifolds. Their complex structure is very easy to check.

Example 2.2.2 (Quotients). Let X be a complex manifold and G a complex Lie group acting properly discontinuously on X . Then the quotient X/G is a complex manifold such that the quotient map

$$\pi : X \rightarrow X/G$$

is holomorphic.

By acting properly discontinuously, we ensure X/G is Hausdorff and admits holomorphic charts: we may find an open covering $\{U_i\}$ of X such that $g(U_i) \cap U_i = \emptyset$ for $g \neq 1$. The image of the U_i under the projection map gives us charts on X/G , since

$$U_i \rightarrow \pi(U_i)$$

is a homeomorphism.

An important class of complex manifolds arising from such quotients is complex tori.

Example 2.2.3. Let $V = \mathbb{C}^n$ and $\Gamma \subset V$ be a discrete subgroup under addition that generates V . Then, Γ must be freely generated by a real basis of V , which is of order $2n$. Then, Γ acts properly discontinuously on V by translation. The quotient

$$\mathbb{C}^n/\Gamma$$

is an n -dimensional **complex torus**. Topologically they are homeomorphic to a real torus of dimension $2n$. However, two different lattices Γ_1 and Γ_2 can induce non-isomorphic complex manifolds. The problem is that we must have a \mathbb{C} -linear automorphism on \mathbb{C}^n that takes Γ_1 to Γ_2 to induce an isomorphism.

A one-dimensional complex torus is called an **elliptic curve**. By a change of coordinates, we may assume that the defining lattice is of the form $\mathbb{Z} + \tau\mathbb{Z}$ for some $\tau \in \mathbb{H}$. It turns out that τ_1, τ_2 define isomorphic elliptic curves iff there is a matrix in $\mathrm{SL}(2, \mathbb{Z})$ that takes one to the other. cn

Example 2.2.4. Let $f : \mathbb{C}^n \rightarrow \mathbb{C}$ be a holomorphic function, and 0 is a regular value. Then, the locus

$$V(f) \subset \mathbb{C}^n$$

is called an **affine hypersurface**, and by the implicit function theorem we get that $V(f)$ can be endowed with a complex structure.

Similarly, **projective hypersurfaces**, which are vanishing sets of homogeneous polynomials in \mathbb{P}^n , are complex manifolds. More generally, suppose $f_1, \dots, f_k : \mathbb{C}^n \rightarrow \mathbb{C}$ are holomorphic functions, and 0 is a regular value of the function

$$(f_1, \dots, f_k) : \mathbb{C}^n \rightarrow \mathbb{C}^k$$

then the intersection $V(f_1) \cap \dots \cap V(f_k)$ is a complex manifold of dimension $n - k$. Manifolds arising this way are called **complete intersections**.

We spend the rest of the section discussing Grassmannians carefully.

Definition 2.2.2. For $1 \leq k \leq n$, the **Grassmannian** $\text{Gr}(n, k)$ is the set of k -dimensional linear subspaces of \mathbb{C}^n .

One way to define a smooth manifold structure on the Grassmannians is via identification with projection operators. We now introduce the Plucker embedding to show that the Grassmannians are actually projective varieties.

For any k -plane $V \subset \mathbb{C}^n$, we may choose any basis v_1, \dots, v_k of V

Definition 2.2.3. The **Plucker embedding** is the map

$$i : \text{Gr}(k, n) \rightarrow \mathbb{P}(\bigwedge^k \mathbb{C}^n)$$

given by $V \mapsto [v_1 \wedge \dots \wedge v_k]$.

Note that the wedge product of a different choice of basis vectors differs by the determinant of change of basis matrix, so the map is well-defined and injective. Suppose we are given by a basis v_1, \dots, v_k of a k -plane V , which forms a $n \times k$ matrix M_V ; the standard basis of $\bigwedge^k \mathbb{C}^n$ are given by basis $\{e_{i_1} \wedge \dots \wedge e_{i_k} : 1 \leq i_1 < \dots < i_k \leq n\}$. Thus, the coordinate of $v_1 \wedge \dots \wedge v_k$, with respect to the standard basis vector indexed by I , is given by the I th $k \times k$ minors of M_V .

The image of the Plucker embedding has to satisfy the Plucker relations, which is a set of quadratic polynomials relating the coordinates. It turns out that these quadratics precisely cut out the image of the Plucker embedding. A proof of the fact can be found in [1] Page 210.

3 Mon, Sep 8

Review of complex manifolds and sheaves.

Proposition 3.0.1. There is an equivalence between vector bundles and locally free \mathcal{O}_X -modules through sheaf of sections.

Definition 3.0.1. The **Picard group** of X is the set of isomorphism classes of line bundles under tensor product.

$$\mathrm{Pic}(X) \cong H^1(X, \mathcal{O}_X^*)$$

The exponential sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X^* \longrightarrow 0$$

induces the map

$$c_1 : \mathrm{Pic}(X) \rightarrow H^2$$

Definition 3.0.2 (Canonical ring). We define

$$R(X) = \bigoplus_{m \geq 0} H^0(X, K_X^{\otimes m})$$

where K_X is the canonical bundle.

Definition 3.0.3. A sheaf F is **flasque** if every restriction map

$$F(X) \rightarrow F(U)$$

is surjective.

Remark 3.0.1. Quote from Ron: A flasque sheaf is almost never “nice”: for example, the sheaf of continuous functions of a punctured disk is not flasque.

Proposition 3.0.2. Every sheaf has a flasque resolution.

Definition 3.0.4. The **sheaf cohomology** of a space with a sheaf (M, F) is the cohomology of the chain complex obtained from taking global sections of any flasque resolution.

As usual, it is independent of the chosen resolution, and a short exact sequence of sheaves induces a long exact sequence on cohomology.

Definition 3.0.5. A sheaf is **acyclic** if all higher cohomologies are 0.

Theorem 3.1. For paracompact spaces, the Čech cohomology and sheaf cohomology are naturally isomorphic.

Proposition 3.1.1. For smooth line bundles,

$$c_1 : \mathrm{Pic}(X) \rightarrow H^2(X; \mathbb{Z})$$

is an isomorphism, for \mathcal{O}_X^* is **soft**, thus acyclic.

For holomorphic case, this is **NOT** true.

Proposition 3.1.2 (Global Sections of $\mathcal{O}(k)$). For $k > 0$, there is an isomorphism

$$H^0(\mathbb{P}^n, \mathcal{O}(k)) \cong \mathbb{C}[z_0, \dots, z_n]_k \cong \text{Sym}^k * (V^\vee)$$

Let $T_{\mathbb{P}^n}$ and $\Omega_{\mathbb{P}^n}^1$ be the holomorphic tangent and cotangent bundles.

Proposition 3.1.3 (Euler Sequence). On \mathbb{P}^n there is a natural short exact sequence of holomorphic vector bundles

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus(n+1)} \longrightarrow T_{\mathbb{P}^n} \longrightarrow 0$$

Alternatively, one may dualize and get

$$0 \longrightarrow \Omega_{\mathbb{P}^n}^1 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus(n+1)} \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow 0$$

Proposition 3.1.4. The canonical bundle of \mathbb{P}^n is

$$K_{\mathbb{P}^n} \cong \mathcal{O}_{\mathbb{P}^n}(-n - 1)$$

4 Line Bundles and Divisors

There is a bijection between vector bundles and locally free sheaves of rank n on a complex manifold/variety. We outline the proof for the rank 1 case:

Theorem 4.1. There is a bijection between

$$\{\text{line bundles}\} \leftrightarrow \{\text{locally free sheaves of rank 1}\}$$

Proof. Given a line bundle $L \rightarrow X$, trivialized by $\{U_i\}$ and (holomorphic) transition functions $g_{ij} : U_i \cap U_j \rightarrow \mathbb{C}^*$, we construct its associated sheaf of sections: define the sheaf F_L given by

$$F_L(U) := \{s : U \rightarrow L : s \text{ a local (holomorphic) section of } L\}$$

Note that a local section on U_i amounts to a (holomorphic) function

$$s : U_i \rightarrow \mathbb{C}$$

so the restriction of F_L to U_i is isomorphic to the sheaf $(\mathcal{O}_X)|_{U_i}$. Hence F_L is the gluing of these locally free sheaves of rank 1 via the transition functions g_{ij} : an isomorphism

$$\varphi_{ij} : (\mathcal{O}_X)|_{U_i \cap U_j} \rightarrow (\mathcal{O}_X)|_{U_j \cap U_i}$$

given by

$$s_i \mapsto g_{ij} s_i$$

and the cocycle condition is automatic.

Given a locally free sheaf \mathcal{G} of rank 1, find locally trivializations U_i and isomorphisms

$$\mathcal{G}|_{U_i} \xrightarrow{\varphi_i} (\mathcal{O}_X)|_{U_i}$$

This gives rise to transition isomorphisms

$$\phi_{ij} := \varphi_j \circ \varphi_i^{-1} : \varphi_{ij} : (\mathcal{O}_X)|_{U_i \cap U_j} \rightarrow (\mathcal{O}_X)|_{U_j \cap U_i}$$

The isomorphism comes from an automorphism in $\mathcal{O}(U_{ij})^*$, which gives us a transition function to build a line bundle.

Example 4.1.1. We want to understand the (holomorphic) line bundle/invertible sheaf $\mathcal{O}(-1)$ on \mathbb{P}^n . In term of line bundle, it is the subbundle

$$\mathcal{O}(-1) := \{(x, v) \in \mathbb{P}^n \times \mathbb{C}^{n+1} : x \in \mathbb{P}^n, v \in [x]\}$$

Its local trivialization is given by affine charts on \mathbb{P}^n : on U_i , a point $x \in U_i$ is expressed as

$$x = [\frac{x_0}{x_i}, \frac{x_1}{x_i}, \dots, 1, \frac{x_{i+1}}{x_i}, \dots, \frac{x_n}{x_i}]$$

and a vector on the line spanned by x is of the form

$$v = \lambda_i(\frac{x_0}{x_i}, \frac{x_1}{x_i}, \dots, 1, \frac{x_{i+1}}{x_i}, \dots, \frac{x_n}{x_i})$$

and this gives us local trivialization. The transition function from U_i to U_j is then multiplication by $\frac{x_j}{x_i}$, since we have

$$\lambda_i/x_i = \lambda_j/x_j$$

Locally $\frac{x_j}{x_i}$ is always holomorphic, so the line bundle is holomorphic. Suppose

$$s : \mathbb{P}^n \rightarrow \mathcal{O}(-1)$$

is a global section, then it restricts to local sections

$$s_i : U_i = \mathbb{C}^n \rightarrow \mathbb{C}$$

and satisfies

$$s_i = \frac{x_i}{x_j} s_j$$

But as we approach the hyperplane at infinity $x_i = 0$, the section s_i either has a pole, or it would be a constant by Liouville. But the right hand side is bounded when approaching $x_i = 0$ by holomorphicity, therefore s_i must be a constant. Doing this for other charts, we see that the local sections must all be constant, and the compatibility with transition functions says that they do not change when multiplied by x_j/x_i , which forces them to be 0.

Example 4.1.2. We want to understand the dual bundle $\mathcal{O}(1)$ on \mathbb{P}^n . We may define it to be the dual bundle of $\mathcal{O}(-1)$:

$$\mathcal{O}(1) := \text{Hom}(\mathcal{O}(-1), \mathcal{O}_X)$$

Geometrically, a fiber over $x \in \mathbb{P}^n$ is a linear functional on the line $[x]$. The transition function is the transpose of the inverse, which in our case, is $\frac{x_j}{x_i}$. If we repeat the analysis of local sections above

$$s_i = \frac{x_j}{x_i} s_j$$

we see no obstructions as before. In fact, for any linear homogeneous function $f(x_0, \dots, x_n)$, we see that $s_i = f/x_i$ defines a local holomorphic section on U_i , compatible with the transition functions. Conversely, if we are given a global holomorphic section s , we may lift this to a holomorphic function

$$\tilde{s} : \mathbb{C}^{n+1} - \{0\} \rightarrow \mathbb{C}$$

defined by

$$\tilde{s}(x_0, \dots, x_n) = x_i s_i(x_0 : \dots : x_n)$$

By Hartog's lemma, the function extends holomorphically over to \mathbb{C}^{n+1} , and is homogeneous of degree 1, therefore must be a homogeneous polynomial of degree 1.

□

4.1 Divisors

Definition 4.1.1. A **hypersurface** on a complex manifold X is a subset $Y \subset X$ such that for every point $p \in Y$, there exists an open neighborhood $U \subset X$, and a regular function $f \in \mathcal{O}_X(U)$ such that

$$Y \cap U = V(f)$$

In words, a hypersurface is a subset of X locally cut out by a single holomorphic function. In particular, such hypersurface is not necessarily smooth. On the other hand, every codimension 1 submanifold is locally cut out by the vanishing of a single local coordinate, so it is always a hypersurface.

Definition 4.1.2. A **Weil divisor** is a formal \mathbb{Z} -linear combination of codimension 1 irreducible hypersurfaces.

If Y is a hypersurface with irreducible components Y_i , we can associate a Weil divisor to it

$$[Y] = \sum [Y_i]$$

Given a irreducible hypersurface Y , we have the data of a covering (U_i, f_i) , where f_i is a choice of irreducible local equation cutting out Y . This also gives us a compatibility condition on overlaps: if $U_i \cap U_j \cap Y$ is non-empty, then f_i, f_j cut out the same hypersurface, they differ by a unit in $\mathcal{O}_X(U_i \cap U_j)$. (This is not so trivial to prove complex-analytically, maybe a DVR argument the stalk is easier). This is precisely the data of a Cartier divisor

Definition 4.1.3. A **Cartier divisor** on X is the data of an open cover U_i and a non-zero meromorphic function $f_i \in \mathcal{K}_X(U_i)$ such that on overlaps $U_i \cap U_j \neq \emptyset$,

$$f_i = u_{ij} f_j$$

where $u_{ui} \in \mathcal{O}_X^*(U_i \cap U_j)$. Equivalently, this is an element in $H^0(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$. A Cartier divisor is **effective** if $f_i \in \mathcal{O}_X(U_i)$ for all i .

In the definition, we see that we are also interested in the case where the local equation has poles instead of zeros.

Proposition 4.1.1. There is a natural isomorphism

$$\text{Cart}(X) \cong \text{Div}(X)$$

The isomorphism is roughly as follows: given the data of a Cartier divisor (U_i, f_i) , each f_i has a well-defined **order** on any hypersurface Y . Precisely, we define $\text{ord}_{Y,x}(f_i)$ as the integer such that

$$f_i = g^{\text{ord}_Y(f_i)} \cdot h$$

where $h \in \mathcal{O}_{X,x}^*$. It is the order of vanishing/pole of the function along Y . Conversely, given a Weil divisor $\sum n[Y_i]$, we know Y_i is locally cut off by some irreducible g on U_i so this gives us the data $(U_i, \prod g_i^n)$ of a Cartier divisor.

5 Line bundle to Divisor

Note that given the data of a Cartier divisor $D = (U_i, f_i)$, we see that

$$g_{ij} = \frac{f_i}{f_j}$$

are all non-vanishing holomorphic functions on $U_i \cap U_j$, which gives rise to a transition function. Thus, we may define

Definition 5.0.1. Given a Cartier divisor $D = (U_i, f_i)$ on X , its associate line bundle $\mathcal{O}_X(D)$ is given by the transition function

$$g_{ij} = \frac{f_i}{f_j}$$

Example 5.0.1. Consider the hyperplane $H_0 := \{x_0 = 0\}$ on \mathbb{P}^n . Its associated Cartier divisor is (U_i, f_i) , where U_i are the standard affines. The hyperplane does not intersect H_0 , so we may take $f_0 = 1$. On other U_i , we see that $f_i = \frac{x_0}{x_i}$. Thus, the transition functions are

$$g_{ij} = \frac{x_j}{x_i}$$

which we see is precisely the transition functions of $\mathcal{O}(1)$. Conversely, H_0 is cut out by the global section x_0 of $\mathcal{O}(1)$ glued from the local holomorphic sections f_i . If we think a little bit more regarding the example above, we see that there is nothing special with H_0 : in fact taking any other hyperplane H_i yield the same line bundle.

Let us describe the line bundle $\mathcal{O}_X(D)$ in terms of its local sections: on U_i , a local section is a holomorphic function $s_i \in \mathcal{O}_X(U_i)$, and on intersections we have

$$\frac{f_i}{f_j} s_i = s_j$$

By taking $g_i = \frac{s_i}{f_i}$, we see this is equivalent to the data of meromorphic function g such that gf_i is holomorphic on each U_i , i.e a meromorphic function allowed to have poles controlled by D .

Remark 5.0.1. This confuses me: if D is effective, some global section of the line bundle $\mathcal{O}_X(D)$ should cut out D when it is effective; but a general global section is allowed to have poles no worse than each f_i on D .

If D is an effective divisor, there is a short exact sequence of \mathcal{O}_X -modules,

$$0 \rightarrow \mathcal{O}_X \xrightarrow{i^*} \mathcal{O}_X(D)$$

where on each U_i , i is given by multiplication of the defining equation of D . The image of i is the ideal sheaf I_D defining D , so the cokernel is

$$\mathcal{O}_D(D) := \mathcal{O}_X(D)/I_D$$

The notation suggests the sheaf $\mathcal{O}_D(D)$ on X is supported on D , which is the case, and the pullback of $\mathcal{O}_D(D)$ along the inclusion $D \rightarrow X$ is the structure sheaf of D . The stalk of $\mathcal{O}_D(D)_x$ is then isomorphic to $\mathcal{O}_{X,x}/(f)$ if $x \in D$, and trivial otherwise.

5.1 Normal Bundle

Given a codimension 1 submanifold $i : D \subset X$, we can define its normal bundle

$$0 \rightarrow TD \rightarrow i^*TX \rightarrow N_D X \rightarrow 0$$

as the cokernel. Its dual is called the **conormal bundle**. Heuristically, a conormal vector is a form that vanishes on tangent vectors on Y . Locally, suppose we have a chart with coordinates (z_0, \dots, z_n) on which D is cut out by the vanishing of z_0 . A tangent vectors on D linear combinations of $\{\frac{\partial}{\partial z_k} : k > 0\}$. Thus, a one form vanishing on all these tangent vectors is of the form

$$\alpha = g \cdot dz_0$$

More generally, the conormal sheaf should be the \mathcal{O}_X -module $I_D/(I_D)^2$, where I_D is the ideal sheaf of functions vanishing on D . Locally, I_D is generated by the defining function of D . The bijection of conormal vector and a function vanishing on D is given by

$$[f] \mapsto df$$

This is well-defined since $d(g^2) = 2gdg$ vanishes identically on D .

We then have the dual **conormal exact sequence**

$$0 \rightarrow I_D/(I_D)^2 \rightarrow i^*(\Omega_X) \rightarrow \Omega_D \rightarrow 0$$

The line bundle $I_D/(I_D)^2$ has transition functions $\frac{f_j}{f_i}$, so it is dual to the bundle $\mathcal{O}_D(D)$.

References

- [1] Phillip Griffiths and Joseph Harris. *Principles of Algebraic Geometry*. Wiley Classics Library, John Wiley & Sons, New York, 1994.
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