MATH 624 Algebraic Geometry

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September 18, 2024

1 Prevarieties and Varieties

We will assume that $K|k$ a finite extension, K is algebraically closed. We will use $\mathbb{A}^n(K) = K^n = \mathbb{A}^n_K$ to denote the underlying set, not the n-dimensional affine space. Given a point $a = (a_1, ..., a_n) \in \mathbb{A}^n_k$, we will use φ_a to denote the evaluation map $k[X] \to k$. Similarly, given $f \in k[x]$, we have the evalation map $\tilde{f} : \mathbb{A}_k \to k$. This gives a morphism of k-algebras $k[x] \to Maps_k(\mathbb{A}_k, k)$ given by $f \mapsto \tilde{f}$.

Definition 1.0.1. Given $\Sigma \subset k[x]$, define $V(\Sigma) = \{a \in \mathbb{A}_k : f(a) = 0 \text{for every } f \in \Sigma\}$. This is called the **affine k-algebraic set** defined by Σ. If $\Sigma = \{f\}$, then $H_f := V(\Sigma) = V(f)$ defines a **hyperplane** in $\mathbb{A}_k$.

Example 1.0.1. Easy examples

1. $V((0)) = \mathbb{A}_k$.
2. $V((1)) = \emptyset$
3. Let $k = \mathbb{C}$. Then, in $\mathbb{A}^1_k$, $V(x^2 - 1) = \{\pm 1\}$. In $\mathbb{A}^2_k$, $V(x^2 - 1) = \{(\pm 1, n) : n \in k\}$

Definition 1.0.2. Given $V \subset \mathbb{A}_\neg$, defined $I(V) = \{f \in k[x] : f(V) = 0\}$. This is called the **ideal** of V.

Proposition 1.0.1. 1. Let $I_\Sigma \subset k[x]$ be the ideal generated by Σ. Then, $V(\Sigma) = V(I)$.

2. There exists a finite system $f_1, .., f_m$ such that $V(\Sigma) = V(f_1, ..., f_m)$
3. If $\Sigma_1 \subset \Sigma_2$, then $V(\Sigma_1) \supset V(\Sigma_2)$
4. Given $\mathfrak{a}$ an ideal, then $I(V(\mathfrak{a})) = \mathfrak{a}$ iff $\mathfrak{a} = \sqrt{\mathfrak{a}}$.
5. Given ideals $\mathfrak{a}, \mathfrak{b}$, then $V(\mathfrak{a}) = V(\mathfrak{b})$ iff $\sqrt{\mathfrak{a}} = \sqrt{\mathfrak{b}}$.

Definition 1.0.3. Let $\mathcal{A}^n_K := \{V \subset \mathbb{A}^n_K : V \text{ affine } k-\text{algebraic sets}\}$. Given $V \in \mathcal{A}^n_K$, let $k[V] := k[x]/I(V)$ be the **affine coordinate ring** generated by V.

Let $Id^{rd}(k[x])$ be the set of reduced ideals of $k[x]$. Let R_n be the set of reduced k-algebras with n-generators.

Theorem 1.1. There is a canonical bijection between the set of reduced affine k-algebras and reduced ideals of $k[x]$, given by the maps

$$R_n \to Id^{re}(k[X]) \to \mathcal{A}_K^k$$

$$k[\underline{x}] \mapsto \mathfrak{a} := ker(k[x] \xrightarrow{f} k) \mapsto V(\mathfrak{a})$$

with f given by $x \mapsto \underline{x}$.

1.1 The Zariski Topology

Given $V \in \mathcal{A}_K^n$, there is a canonical map $K[X] \to K[V]$ given by $f \mapsto f_V$.

Proposition 1.1.1. Let $\Sigma_i \subset k[X]$, and $f \in k[X]$ be given. then
1. $V(\cup_i \Sigma_i) = \cap_i V(\Sigma_i)$
2. $V(\prod \Sigma_i) = \cup V(\Sigma_i)$
3. $V((0)) = \mathbb{A}_k^n$; $V((1)) = \emptyset$

By the proposition above, we can define the Zariski topology on $\mathbb{A}_k^n$

Definition 1.1.1. The Zariski topology on $\mathbb{A}_K^n$ is given by the closed sets $V(\Sigma)$, with $\Sigma \in k[X]$. In particular, the sets $D_f := \mathbb{A}_k^n - H_f$ is an open set and forms a basis for the topology.

Note that the zariski topology on product spaces is not the product of zariski topologies. Moreover, the connectedness/irreducibility is dependent on $K|k$. A point is called a generic point of V if its closure contains V.

Example 1.1.1. If $K|k = \mathbb{C}|\mathbb{Q}$, then $V(x_1^2 - 2x_2^2)$ is connected and irreducible. If $K|k = \mathbb{C}|\mathbb{Q}[\sqrt{2}]$, then $V(x_1^2 - 2x_2^2)$ is connected but not irreducible.

Remark 1.1.1. For a topological space, X, the following are equivalent:
1. Every descending chain of closed subsets is stationary.
2. Every ascending chain of open subsets is stationary.

A topological space satisfiying the above is called **Noetherian**. For example, $Spec(R)$ is Noetherian if R is Noetherian. Note that if X is Noetherian, then it is automatically quasi-compact. Moreover, there are only finitely many irreducible components and connected components of X.

Proposition 1.1.2. The following hold:
1. The Zariski topology is Noetherian on $\mathbb{A}_K$, therefore also on any $V \in \mathcal{A}_K^n$.
2. For every $V \in \mathcal{A}_K$, there are only finitely many irreducible components and connected components.
3. $V \in \mathcal{A}_K$ is irreducible iff $I(V)$ is a prime ideal.
4. Given $V_0 \subset V$, V_0 is irreducible iff $I_V(V_0) := I(V_0)/I(V) \in Spec(k(V))$ is minimal.
5. The connected components in $V \in \mathcal{A}_K$ correspond bijectively to the indecomposable idempotents of $k[V]$.
6. For $V \in \mathcal{A}_K$, $a \in V$ is a generic point iff the evaluation map $k[V] \to k[a]$ is an isomorphism of k-algebras.

Definition 1.1.2. Let T be a topological space, and let $V \subset T$.

1. dim(V):=sup { chain of irreducible components ending in V: }
2. codim(V):=sup { chain of irreducible components starting with V and ending in T: }

Note that if $V = \cup V_\alpha$, then $dim(V) = supdim(V_\alpha)$, and similarly for codimensions. Moreover, $dim(V) = dim(\overline{V})$.

Proposition 1.1.3. (Notions of dimension) Let $V \in \mathcal{A}_K$ be irreducible. Then, the dimension of V is the same as the krull dimension of $K[V]$.

Proposition 1.1.4. Suppose irreducible $W \subset V \in \mathcal{A}_K$. Then,

$$dim(W) + codim_V(W) = dim(V)$$

Proposition 1.1.5. $V \in \mathcal{A}_K$ has generic points a iff $td(K|k) \geq dim(V) = td(k(V))$.

1.2 Base change and Rational Points

Definition 1.1.3. Suppose there is an embedding

$$\begin{array}{ccc} K & \longrightarrow & L \\ \uparrow & & \uparrow \\ k & \longrightarrow & l \end{array}$$

Then, there is a natural morphism $k[x] \to l[x]$, which induces a pushforward of ideals and a map $\mathcal{A}_K \to \mathcal{A}_L$. Take the vanishing locus of the pushforward of $I(V)$ gives the base change of V.

Remark 1.1.2. Base change does not preserve connectedness or irreducibility.

Definition 1.1.4. $V \in \mathcal{A}_K$ is called **absolutely (geometrically) irreducible** if V_l is irreducible for all field extension $l|k$. It is **geometrically connected** is V_l is connected for all $l|k$.

Proposition 1.1.6. Let $V \in \mathcal{A}_K$ be affine k-algebraic set. Then the following are equivalent:

1. V is absolutely irreducible.
2. V_{k^s} is irreducible.
3. $V_{\overline{k}}$ irreducible.

The key observation is that $K^s[x] \to \overline{k}[X]$ is an integral extensions of domains. Therefore, we have going up and going down, and it straightforward to show that $Spec(k^s[X]) \to Spec(\overline{k}[X])$ is a homeomorphism. Thus, we have (2) $\implies$ (3).

To (3) $\implies$ (1), apply the following:

Lemma 1.2. For every $V \in \mathcal{A}_K$, one has $V(\overline{k})$ is zariski dense in V. Therefore, $V_{\overline{k}}$ irreducible implies V irreducible.

The proof is exercise. The key point is that if there exists f with k-coefficients such that f vanishes on all of A

Proposition 1.2.1. Let $V \in \mathcal{A}_K$ be affine k-algebraic set. Then the following are equivalent:
1. V is geometrically connected.
2. V_{K^s} is connected.
3. $V_{\overline{k}}$ is connected.

2 The category of quasi-affine k-algebraic sets

Definition 2.0.1. A **quasi-affine** k-algebrac set is any zariski open subset $U \subset V$ for $V \in \mathcal{A}_K$.

The complement of hyperplanes is a basis of quasi-affine k-algebraic sets. Let $V \in \mathcal{A}_K$ be non-empty, $f \in K[V]$. Then, the evaluation map $f : V \to \mathcal{A}_K$ is continuous. Moreover, $\varphi = (f_1, ..., f_n)$ is also continuous.

Definition 2.0.2. Let $V \in \mathcal{A}_K$ and $\mathcal{V} \subset V$ be zariski dense. Then, a functions $\varphi : \mathcal{V} \to \mathcal{A}_K$ is called **regular** at $x \in V$ if there exists $f_x, g_x \in k[x]$ and $\S \subset V$ such that $g_x \neq 0$ everywhere on $\mathcal{U}_x$ and $\varphi = \frac{f_x}{g_x}$. A fucntion $\varphi : \mathcal{V} \to \mathcal{A}_K$ is **regular** if it is regular at every point in V. Let $\mathcal{O}_x := \{\varphi \in Maps(\mathcal{V}, K) : \varphi \text{ regular at } x\}$. Define an equivalence relation on $\mathcal{O}_x$ by equivalence on any open neiborhood around x. $\mathcal{O}(V)$ is the set of regular functions on V.

Proposition 2.0.1. (rings of regular functions) We have the following:
1. $k[V] \to \tilde{\mathcal{O}}(V)$ is an isomorphism of k-algebra.
2. $k[V]_f \to \tilde{O}(U_f)$ is an isomorphism of k-algebra.

It is helpful to remember that Zariski open sets are dense. Thus, it suffices to show that a function is zero on a basic open U_f to deduce it is globally zero.

3 Presheaves and Sheaves

Definition 3.0.1. Let $\mathcal{C}$ be a concrete category such as **Top, Set, Ab**. Let X be a topological space with topology τ_X. Then, τ_X is naturally poset category where morphisms are inclusions. A **presheaf** is a contravariant functor $\mathcal{P} : \tau_X \to \mathcal{C}$.

Explicitly, $\mathcal{P}$ is given by two data: 1.$\mathcal{P}(U) \in Obj(\mathcal{C})$ for every $U \in \tau_X$. 2.$\rho_{u',u''} : \mathcal{P}(U'') \to \mathcal{P}(U')$ for every $U' \subset U''$. The elements in the set $P(U)$ are called **sections** above U. The image of a section under ρ is called the **restriction**.

Definition 3.0.2. A presheaf is a **sheaf** if it has the covering property: given an open cover of an open set $U = \cup_i U_i$, with $U + i, j := U_i \cap U_j$ with $s_i \in \mathcal{P}(U_i)$ such that $\rho_{U_i, U_{i,j}}(s_i) = \rho_{U_j, U_{i,j}}(s_i)$, then there exists $s \in \mathcal{P}(U)$ such that $s_i \in \rho_{U,U_i}(s)$ for every U_i.

Definition 3.0.3. Suppose that limits exists in $\mathcal{C}$. Then $\mathcal{P}_x := \mathcal{P}(U_x)$ is called the **stalk** of $\mathcal{P}$ at x.

Proposition 3.0.1. $\mathcal{P}$ is a sheaf iff for every $U \in \tau_X$, the map $\varphi_U : U \to \coprod_{x \in U} \mathcal{P}_x$ is injective.

Proposition 3.0.2. For every presheaf $\mathcal{P}$, there is a sheafification functor $\mathcal{P} \to \mathcal{F}$ that induces isomorphism on stalks.

Definition 3.0.4. Let $f : X \to Y$ be a continuous map of topological spaces. Then,

1. Given a (pre)sheaf $\mathcal{P}$ on X, then the **direct image** (pre)sheaf $f_*\mathcal{P}$ on Y is defined by $f_*\mathcal{V} := \mathcal{P}(f^{-1}(V))$ for all $V \in \tau_Y$. In particular, the direct image sheaf is also a sheaf.
2. Given a presheaf $\mathcal{P}$ on Y. There is an **inverse image** sheaf $f^{-1}\mathcal{P}$ on X defined by the limit:

$$f^{-1}\mathcal{P}(U) := \varinjlim_{U \subset U'} \mathcal{P}(f(U'))$$

   where $U \subset U'$ and $f(U')$ is open.

Remark 3.0.1. Note that the preimage sheaf is always a preseeaf, but not necessarily a sheaf.

Definition 3.0.5. A (locally) **ringed space** is a pair $(X, \mathcal{F})$, where X is a topological space and $\mathcal{F}$ a sheaf of rings on X such that the stalks at each point is a local ring.

Definition 3.0.6. Given locally ringed spaces $(X, \mathcal{F})$, $(Y, \mathcal{G})$, a morphism of locally ringed space is a pair $(f, f^\sharp)$ such that $f : X \to Y$ is continuous and $f^\sharp : \mathcal{G} \to f_*\mathcal{F}$ a morphism of sheaves.

4 Back to Varieties

Proposition 4.0.1. Let V be an affine k-algebraic set, $U \subset V$ zariski open.

1. The assignmenet τ_U, $U' \mapsto \tilde{O}(U')$ defined a locally ringed space on U.
2. A morphism of quasi-affine algebraic set $T \to U$ is any morphism of locally ringed spaces $(f, f^\sharp) : (T, \mathcal{O}_T) \to (U, \mathcal{O}_U)$

The checks are fullfilled by proposition 2.0.1.

Proposition 4.0.2. Let $(T, \mathcal{O}_T), (U, \mathcal{O}_U)$, and $\Phi : T \to U$ continuous. Then,

1. Φ defined a morphism of locally ringed spaces iff $\mathcal{O}_U \circ \varphi \subset \mathcal{O}_T$, i.e for every U and T' open such that $\Phi(T') \subset U'$ and $\varphi \in \mathcal{O}_U(U')$, then $\varphi \circ \Phi \in \mathcal{O}_T(T')$.
2. Suppose Φ defines such a morphism, and let $U \subset \mathbb{A}_K$, $p : \mathbb{A}^n_K \to K$ the ith projection, then $p_i|_U \circ \Phi$ completely determines Φ.

Remark 4.0.1. Let $U_f := \{x \in V | f(x) \neq 0 :\}$ be a basic open. Consider $W_f \subset \mathbb{A}^n_K$ defined by $W_f := \{(a,b) | a \in \mathbb{A}^n_K, b \in \mathbb{A}^1_K : f(a)b - 1 = 0\}$ is an algebraic set in $\mathbb{A}^{mn}_K$. Prove that $\Phi : W_f \to U_f$ given by $(a,b) \mapsto a$ is an isomorphism of quasi affine k-algebraic sets. Then inverse is given by $\psi : U_f \to W_f$ given by $a \mapsto (a, \frac{1}{f(a)})$.

Proposition 4.0.3. Every quasi-affine k-algebraic set contains a n zariski dense k-algebraic set.

Definition 4.0.1. A quasi-affine k-algebraic set is called **affine** if it is isomorphic as a locally ringed space to an affine k-algebraic set.

Theorem 4.1. The following hold:

1. The catgeory of K-valued affine k-algebraic sets, $\mathcal{A}_k$, is anti-equivalent to the category of reduced k-algebras of finite type. In particular, a k-algebraic set $V \subset \mathcal{A}_K$ is mapped to $k[V]$. Note that the projection maps $V \to W \to \mathcal{A}_k$ defined a regular function on V, and by proposition 4.0.2 determined the morphism of the algebraic set. There is a canonical map from the ring of regular functions on V to the coordinate ring $k[V]$ by proposition 2.0.1.
2. Let U be a quasi-affine k-algebraic set, W and affine k-algebraic set. Then, a morphism $\Phi :$ $U \to W$ is determined by a map $\Phi^* : k[W] \to \tilde{O}(U)$.

Definition 4.1.1. $\mathcal{A}^n_k := (\mathcal{A}^n_K, \tilde{O}_{\mathcal{A}^n_K})$ is called the **n-dimensional affine sapce.**

Definition 4.1.2. An **open immersion** of quasi-affine k-algebriac setd $j : U \to T$ is any k-morphism which is a zariksi open immersion and $\tilde{O}_U = \tilde{O}_T \circ j$

Definition 4.1.3. A **closed immersion** of quasi-affine k-algebraic sets $i : U \to T$ is a topological closed immersion and $i_* O_U$ is a factor sheaf of $\mathcal{O}_T$. In other words, the map $\Phi* : \tilde{\mathcal{O}}_T(T') \to \tilde{\mathcal{O}}_U(U')$ is surjective.

Definition 4.1.4. A k-**prevariety** is any quasi-compact locally ringed space X that is locally isomorphic to K-valued affine k- algebraic sets. Locally isomorphic here means that there exists an finite open cover $X = \cup X_\alpha$ and isomorphism of locally ringed spaces $\varphi_\alpha : X_\alpha \to V_\alpha$, where V_α is affine k-algebraic set. Moreover, the transition maps are isomorphisms of quasi-affine k-algebraic sets.

Remark 4.1.1. A **k-morphism** of k-prevarieties is a morphism of locally ringed spaces, such that there exists $X = \cup X_\alpha, Y = \cup Y_\alpha$ and $f(X_\alpha) \subset Y_\alpha$, and the structure maps induce a map of affine k-algebraic sets.

Definition 4.1.5. Let $f : X \to Y$ be a k-morphism of k-prevarieties. Then,

1. f is an open immersion iff f induced structure maps is an open immersions of affine k-algebraic sets.
2. f is an closed immersion iff f induced structure maps is a closed immersions of affine k-algebraic sets.
3. X is called affine if it is isomorphic as a k-prevariety to an affine k-algebraic set.
4. X is called quasi-affine if there is an open immersion into a affine k-prevariety.

Proposition 4.1.1. (Glueing datat for k-prevarieties and k-morphisms)

1. (X_i) be a finite set of k-prevarieties.
2. $X_{ij} \subset X_i$ open for every i, j
3. $\varphi_{ij} : X_{ij} \to X_{ji}$ a $k-$isomorphism such that $\varphi_{ii} = Id$, $\varphi_{ij} = \varphi_{ji}^{-1}$ and $\varphi_{ij} \circ \varphi_{jk} = \varphi_{ik}$.
4. A solution is $X = \cup X_i'$ and k-isomorphisms $X_i' \to X_i$

Remark 4.1.2. The solution is unique up to k-isomorphism.