Computing $\pi_4(S^3)$

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April 4, 2024

In this exposition, we compute the first not-easily-computable homotopy group of spheres, $\pi_4(S^3)$. First, we recall the definition for higher homotopy groups:

Definition 0.0.1. The nth homotopy group of a pointed topological space (X, x_0), denoted by $\pi_n(X, x_0)$, is the set of homotopy classes of maps $[(S^n, s_0), (X, x_0)]$.

Based on this definition, one might be tempted to think that $\pi_n(S^k)$ is trivial when $n > k$, similar to singular homology/cohomology. However, the famous Hopf fibration $S^1 \hookrightarrow S^3 \to S^2$ gives us a non-trivial element of $\pi_3(S^2)$. It is generally very hard to compute higher homotopy groups, even for spheres except for a certain number of cases.

Without introducing any new tools, we can do at least one case: recall that a covering space $(\tilde{X}, \tilde{x}_0) \to (X, x_0)$ satisfies the homotopy lifting property. In particular, the uniqueness of lifting implies the induced map $\pi_n(\tilde{X}, \tilde{x}_0) \to \pi_n(X, x_0)$ is injective. On the other hand, the lifting criterion states that every map $S^n \to X$ can be lifted to $\tilde{X}$ as S^n is simply-connected for $n \geq 2$, and the induced map is surjective. Therefore, we have

Theorem 0.1. A covering space projection $\tilde{X} \to X$ induces isomorphism on nth homotopy groups for $n \geq 2$.

Corollary 0.1.1. $\pi_n(S^1) = 0$ for $n > 1$

Proof. Take $\mathbb{R}$ to be the universal cover for S^1, which has trivial homotopy groups since it is contractible. $\square$

1 Preliminary Results

Theorem 1.1. (Cellular Approximation Theorem) Any map $f : X \to Y$ of CW-complexes is homotopic a cellular map, i.e the image of the n-skeleton of X is contained in the n-skeleton of Y.

Corollary 1.1.1. $\pi_n(S^k) = 0$ for $k > n$.

Proof. If the image of the map $\phi : S^n \to S^k$ the image misses a point $s_0 \in S^k$, then $S^k - \{s_0\}$ is homotopy equivalent to $\mathbb{R}^k$, and continuous map into $\mathbb{R}^k$ is nullhomotopic. Equip S^n with the CW structure of 2 k-cell in each dimension $k \leq n$. Then, every map $\phi : S^n \to S^k$ is homotopic to a cellular map that is not surjective for $n < k$. $\square$

The next result is very important to our discussion.

Theorem 1.2. (Hurewicz Theorem) A space X is called **n-connected** if $\pi_k(X) = 0$ for all $0 \leq k \leq n$. For $n \geq 2$, if X is $n-1$-connected, then $\tilde{H}_i(X) = 0$ for $i < n$ and $\pi_n(X) \cong \tilde{H}_n(X)$.

An immediate corollary of this result is that we can compute $\pi_n(S^n)$:

Corollary 1.2.1. $\pi_n(S^n) \cong \mathbb{Z}$

Proof. Combine the fact that S^n is $n-1$-connected by Corollary 1.1.1 and the fact that $H_n(S^n) = \mathbb{Z}$. $\square$

2 Serre Fibrations

Recall that a cofibration is a map $A \hookrightarrow X$ satisfying the homotopy extension property. Cofibration plays well with homology/cohomology as it gives us a long exact sequence, and we can then extract homological information of one of $A \to X \to X/A$ from the other two. The dual notion is a fibration, which satisfies the homotopy lifting property.

Definition 2.0.1. A map $E \to B$ is said to satisfy the **homotopy lifting property** (HLP) with respect to a space X if the following diagram commutes

$$\begin{tikzcd} X \times \{0\} \arrow[r, "\tilde{H}_0"] \arrow[d, "i"] & E \arrow[d] \\ X \times [0,1] \arrow[r, "H"] \arrow[ur, dashed, "\exists \tilde{H}"] & B \end{tikzcd}$$

In other words, given a homotopy in $H_t : X \to B$ and a initial lift $\tilde{H}_0 : X \to E$, we can lift the homotopy entirely. Note that in general, the lift is not required to be unique. However for covering spaces, we showed that they satisfies the stronger property that the liftings are in fact unique.

Definition 2.0.2. A map $p : E \to B$ satisfying the HLP with respect to arbitrary X is called a (Hurewicz) **fibration**. A map satisfying the HLP with respect to CW-complexes is called a **Serre fibration**. Assume B is path-connected and based at b_0, the **fiber** of the fibration is $F := p^{-1}(b_0) \subseteq E$. We represent the data of a fibration into the following **fiber sequence**

$$F \to E \to B$$

Example 2.0.1. Covering maps are fibrations with discrete fibers.

Example 2.0.2. Fiber bundles are Serre fibrations. In particular, the Hopf bundle $S^1 \to S^3 \to S^2$ is a Serre fibration.

In practice, Serre fibrations is good enough to give us most of the desired properties/tools. It is fun to know that pathological examples exists (even in CGWH) where a Serre fibration is not a fibration.

From now on we assume the base-space is path-connected and based. The important property that a Serre fibration enjoys is that we have the long exact sequence of homotopy groups:

Theorem 2.1. Given a Serre fibration $p : E \to B$ with fiber F and a choice $x_0 \in F$, we have the following LES:

$$... \longrightarrow \pi_n(F, x_0) \longrightarrow \pi_n(E, x_0) \longrightarrow \pi_n(B, b_0) \longrightarrow \pi_{n-1}(F, x_0) \longrightarrow ...$$

From this theorem, we can already calculated a not so obvious homotopy group:

Corollary 2.1.1. $\pi_3(S^2) \cong \mathbb{Z}$

Proof. We have the exact sequence from the Hopf fibration

$$\pi_3(S^1) = 0 \longrightarrow \pi_3(S^3) \cong \mathbb{Z} \longrightarrow \pi_3(S^2) \longrightarrow \pi_2(S^1) = 0$$

where the triviality of the two groups on the end follows from Corollary 0.1.1; $\pi_3(S^3) \cong \mathbb{Z}$ follows from Corollary 1.2.1 $\square$

Given a CW complex X, we have the following tool to "approximate" the space with successive fibrations.

Theorem 2.2. (Whitehead Tower) Given a CW complex X, there exists the following tower of fibrations, called the **<u>Whitehead tower</u>** of X.

$$\begin{tikzcd}
... & ... \arrow[d] \\
K(\pi_2(X), 1) \arrow[r, hook] & X_2 \arrow[d] \\
K(\pi_1(X), 0) \arrow[r, hook] & X_1 \arrow[d] \\
 & X
\end{tikzcd}$$

where each downward arrow $X_n \to X_{n-1}$ is a fibration with fiber $K(\pi_n(X), n-1)$. Moreover, each X_n is n-connected, and the composed map $X_n \to X$ induces isomorphisms $\pi_i(X_n) \to \pi_i(X)$ for $i > n$.

The construction of Whitehead tower is not difficult once we know the construction of $K(G, n)$ and pathspace fibration. We will skip the details and go straight to the relevant example

Example 2.2.1. Take $X = S^3$. Note that S^3 is 2-connected, so we may take $X_1 = X_2 = S^3$ with identity map as fibrations and the fibers being a point. The first non-trivial fibration in the Whitehead tower of S^3 is the following

$$K(\pi_3(S^3), 2) \to X_3 \to S^3$$

Recall that $\mathbb{CP}^\infty$ is a model for $K(\pi_3(S^3), 2) = K(\mathbb{Z}, 2)$. Therefore, we get a fibration

$$\mathbb{CP}^\infty \to X_3 \to S^3$$

with X_3 being 3-connected. This fibration will be central to our computation of $\pi_4(S^3)$.

From the Hopf fibration, we can see that fibrations do not produce the long exact sequences for homology/cohomology as cofibrations do. However, Serre spectral sequences will save the day by relating the homology/cohomology of the three components of a fibration.

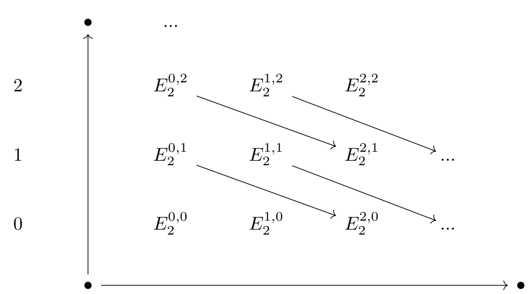
3 Spectral Sequences

There are homological and cohomological versions of spectral sequences, and the general theory between them is not all that diffferent. We opt to introduce the cohomological spectral sequence instead since sometimes it has the additional product structure, which aids computations.

Definition 3.0.1. A **cohomology spectral sequence** in an abelian category A is a family of objects $\{E_r^{p,q} : r \in \mathbb{N};\ (p, q) \in \mathbb{Z}^2\}$, together with **differentials** $d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}$. Moreover, the object $E_{r+1}^{p,q}$ is isomorphic to the homology at $E_r^{p,q}$. Fixing an r, we collect the $\mathbb{Z}^2$ lattice of data of a spectral sequence into what we call the **E_r page**. The **total degree** of $E_r^{p,q}$ is $n := p + q$.

Observe that the differential goes from a total degree n object to a total degree $n+1$ object. In the homology version, the arrows are reversed and the total degree goes down by 1 instead.

Here is an illustration of the E_2 page of a **first quadrant** cohomology spectral sequence, by which we mean $E_r^{p,q} \neq 0$ only if $p, q \geq 0$.



Example 3.0.1. For the first quadrant cohomological spectral sequence above, the differential into and out of $E_2^{0,0}$ are both zero maps, since $E_2^{-2,1}$ and $E_2^{2,-1}$ are both 0. Thus, the homology at $E_2^{0,0}$, which is isomorphic to $E_3^{0,0}$, is still $E_2^{0,0}$. By the same argument, we have $E_r^{0,0} \cong E_2^{0,0}$ for $r \geq 2$.

For a first quadrant (or more generally bounded) cohomological spectral sequence, here is an important observation: it is not hard to see that $E_r^{p,q}$ will eventually stablize when r is large enough, for when $r > p + q$, we see that the differential going in and out of $E_r^{p,q}$ is going to be 0. It is natural to guess that a cohomology spectral sequence is a tool to "approximate" some cohomology.

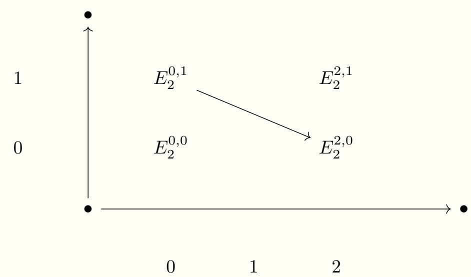
Definition 3.0.2. For a first quadrant cohomological spectral sequence, we denote the stable value at $E_*^{p,q}$ by $E_\infty^{p,q}$. We say the spectral sequence **converges** to H^*, written as $E_r^{p,q} \Longrightarrow H^{p+q}$, if we are given a family of objects $\{H^n : n \in \mathbb{Z}\}$, and each H^n is equipped with a finite filtration

$$0 = F^t H^n \subseteq \ldots \subseteq F^{p+1} H^n \subseteq F^p H^n \subseteq \ldots \subseteq F^s H^n = H^n$$

with $E_\infty^{p,q} \cong F^p H^{p+q} / F^{p+1} H^{p+q}$.

More or less, the data of H^n, in terms of successive quotients of its filtration, is read off from the diagonal of total degree n, that is $\{E_\infty^{p,q} : p + q = n\}$, of the spectral sequence. If there is only one nonzero object in $\{E_\infty^{p,q} : p + q = n\}$, we must have H^n isomorphic to that object.

Example 3.0.2. Given a spectral sequence $E_r^{p,q} \Longrightarrow H^{p+q}$ with the following E_2 page, where everything non-listed are 0:



It is easy to see that $E_2^{2,1} = E_\infty^{2,1}$ and $E_2^{0,0} = E_\infty^{0,0}$. Withoute any explicit information on the differentials we can already see that $H^0 \cong E_2^{0,0}$ and $H^3 \cong E_\infty^{2,1}$, since they are the only non-zero objects with total degree $0, 3$, respectively; moreover, the only non-trivial differential is $d_2^{0,1}$, and we see that $H^1 \cong E_\infty^{0,1} = ker(d_2^{0,1})$ and $H^2 \cong E_\infty^{2,0} = coker(d_2^{0,1})$. Lastly, all other H^n must vanish.

If the objects are all vector spaces, then $H^n \cong \oplus_{p+q=n} E_\infty^{p,q}$ by dimension reasons. However, often in general successive quotients of a filtration of an object does not complete determine the object. For example, if given a filtration $0 \subseteq B \subseteq G$ of abelian groups, the data of the quotients B and G/B gives rise to the short exact sequence

$$0 \longrightarrow B \longrightarrow * \longrightarrow G/B \longrightarrow 0$$

And we run into the good-ol group extension problem.

At this point, it would seem that there are several major problems with spectral sequences: the construction of a spectral sequence converging to a desired cohomology theory seems completely out of hand; the computations of the differentials can get arbitrarily complex; and even when we can compute all the differentials, the data of the spectral sequence does not fully determine the cohomology theory. It turns out that even with these inherent problems, spectral sequences will still serve as a very powerful computation tool.

We refer the readers to Weibel and Hatcher for the general constructions and convergence of spectral sequences. For the remainder of this exposition, we will introduce Serre spectral sequence and start doing some computations.

4 Serre Spectral Sequence

For the following discussions, we will use the assumption that B is simply connected and work over $R = \mathbb{Z}$ to simplify things.

Theorem 4.1. Given a Serre fibration $F \hookrightarrow X \to B$, with fiber F path-connected and base B simply-connected, the **Serre cohomological spectral sequence** is given by the E_2 page

$$E_2^{p,q} = H^p(B; H^q(F;\mathbb{Z})) \Longrightarrow H^{p+q}(X;\mathbb{Z})$$

If the space B is not simply connected, then the coefficient group $H^q(F)$ is actually the local system on B given by the fibers. This reduces to the usual coefficient when the action of $\pi_1(B)$ on the fibers are trivial.

Note that we only have access to the data of the objects on the E_2 page of the spectral sequence. It turns out that this is enough to deduce many cohomologies, as the spectral sequence will often collapse very quickly. Here is a sanity check for $H^n(S^3;\mathbb{Z})$: consider the Hopf fibration $S^1 \to S^3 \to S^2$, which gives us the following E_2 page of the Serre spectral sequence

$$\begin{tikzcd} 1 & H^0(S^2,\mathbb{Z}) \arrow[drr] & & H^2(S^2,\mathbb{Z}) \\ 0 & H^0(S^2,\mathbb{Z}) & & H^2(S^2,\mathbb{Z}) \\ & 0 & 1 & 2 \end{tikzcd}$$

By the abstract analysis given in Example 3.0.2, we can already deduce that $H^3(S^3;\mathbb{Z}) \cong H^2(S^2,\mathbb{Z}) = \mathbb{Z}$, and $H^0(S^3;\mathbb{Z}) \cong H^0(S^2,\mathbb{Z}) = \mathbb{Z}$. On the other hand, say a priori that we know the $H^i(S^3;\mathbb{Z}) = 0$ for $i = 1, 2$, but do not know the cohomology of S^2. Then the spectral sequence tells us the differential $d_2^{0,1} : H^0(S^2;\mathbb{Z}) \to H^2(S^2;\mathbb{Z})$ will be an isomorphism, as no non-trivial cohomology can survive to the next page, where it will stablize.

The major difference between homology and cohomology is the additional product structure:

Theorem 4.2. (Product Structure) The Serre cohomological spectral sequence has a bigraded $\mathbb{Z}$-algebra structure, given by the product

$$E_n^{p,q} \times E_n^{s,t} \to E_n^{p+s,q+t}$$

In particular, the product structure on E_2 page is given by the cup product, and the product on E_n induces the one on E_{n+1}.

Proposition 4.2.1. The differentials $d_n : E_n^{p,q} \to E_n^{p+n,q-n+1}$ is a graded derivation with respect to the product structure. In other words, given $a \in E_n^{p,q}$ and $b \in E_n^{s,t}$, we have

$$d_n(ab) = d_n(a)b + (-1)^{|p+q|} a d_n(b)$$

We are ready to compute $\pi_4(S^3)$.

5 Computations

From Example 2.2.1, we have the following fibration

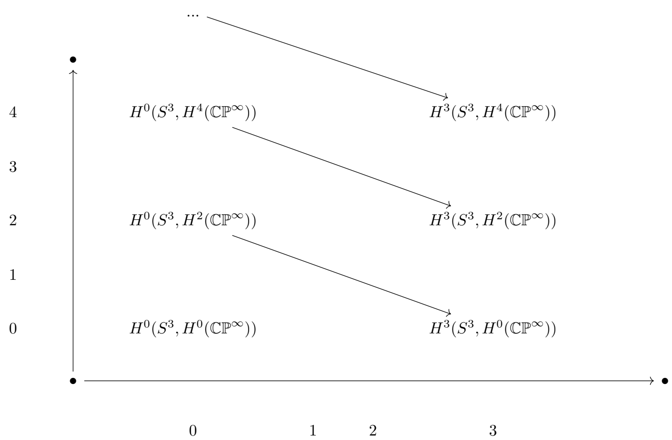
$$\mathbb{CP}^\infty \to X_3 \to S^3$$

with X_3 being 3-connected.

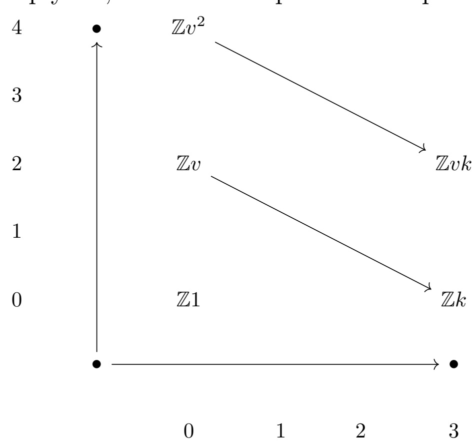
Proposition 5.0.1. $H_4(X_3) \cong \pi_4(X_3) \cong \pi_4(S^3)$; $H_3(X_3) = H_2(X_3) = 0$.

Proof. By construction of the Whitehead tower, we have $\pi_4(X_3) \cong \pi_4(S^3)$. The rest of the proposition follows from an application of Hurewicz Theorem knowing that X_3 is 3-connected. $\square$

Recall that the cohomology ring of $\mathbb{CP}^\infty$ is $\mathbb{Z}[v]$, with $|v| = 2$. The E_2 page of the Serre spectral sequence looks like the following



Note that the coefficients $H^n(\mathbb{CP}^\infty) \cong \mathbb{Z}$ in all degrees, so all the non-trivial cohomologies in the E_2 page are all in fact $\mathbb{Z}$; Let 1 denote the generator for $E_2^{0,0}$ and k denote the generator for $E_2^{3,0}$; the generator for $E_2^{0,2n}$ is v^n as the generator for the coefficient group $H^{2n}(\mathbb{CP}^\infty)$. By the product structure, the generator for $E_2^{3,2}$ is simply vk, for the multiplication map is just multiplication of coefficients.



A quick examination of the spectral sequence above shows that the d_2 differentials are all trivial by degree reasons. Therefore, all cohomologies survive to E_3 page.

Note that a quick application of universal coefficient theorem shows that $H^3(X_3) = 0$. In particular, we see that $d_3 : E_3^{0,2} \to E_3^{3,0}$ must be an isomorphism since both cohomologies cannot survive to the next page. WLOG, we may assume $d_3(v) = k$. Then by the derivation law, we see that $d_3 : E_3^{0,4} \to E_3^{3,2}$ is given by $d_3(v^2) = d_3(v)v + vd_3(v) = 2vk$, so $H^4(X_3) = E_4^{0,4} = 0$. Since there is nothing on $E_3^{6,0}$, we see that $H^5(X_3) = E_4^{3,2} = \mathbb{Z}vk/(2vk) \cong \mathbb{Z}/2\mathbb{Z}$. An application of the following proposition deduced from UCT shows $H^5(X_3) \cong H_4(X_3) = \mathbb{Z}/2$, and we finish.

Proposition 5.0.2. If the relavant homologies and cohomologies are finitely generated, then $\text{rank}(H_n) = \text{rank}(H^n)$ and $\text{Torsion}(H_{n-1}) = \text{Torsion}(H^n)$