How to Compute $\pi_4(S^3)$

David Zhu

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In this exposition, we compute the first not-easily-computable homotopy group of spheres, $\pi_4(S^3)$. First we recall the definition for higher homotopy groups

Definition 0.1. For $n > 0$, the nth homotopy group of a pointed topological space (X, x_0), denoted by $\pi_n(X, x_0)$, is the group of homotopy classes of maps from $(I^n, \partial I^n) \to (X, x_0)$. Equivalently, it is also the group of homotopy classes of maps from $(S^n, s_0) \to (X, x_0)$.

Note that when $n = 0$, the homotopy classes of maps no longer form a group, but it is still well-defined as a set. Based on this definition, one might be tempted to think that $\pi_n(S^k)$ is trivial when $n > k$. However, the Hopf fibration $S^1 \hookrightarrow S^3 \to S^2$ gives us a non-trivial element of $\pi_3(S^2)$. It is generally very hard to compute higher homotopy groups, even for spheres except for a certain number of cases.

Without introducing any new tools, we can do at least one case: recall that a covering space $(\tilde{X}, \tilde{x}_0) \to (X, x_0)$ satisfies the homotopy lifting property. In particular, the induced map $\pi_n(\tilde{X}, \tilde{x}_0) \to \pi_n(X, x_0)$ is injective. On the other hand, the lifting criterion states that every map $S^n \to X$ can be lifted to $\tilde{X}$ as S^n is simply-connected for $n \geq 2$. Therefore, we have

Theorem 0.1. A covering space projection $\tilde{X} \to X$ induces isomorphism on nth homotopy groups for $n \geq 2$.

Corollary 0.1.1. $\pi_n(S^1) = 0$ for $n > 1$

Proof. Take $\mathbb{R}$ to be the universal cover for S^1, which has trivial homotopy groups since it is contractible. $\square$

1 Prelimary Results

Theorem 1.1. (Cellular Approximation Theorem) Any map $f : X \to Y$ of CW-complexes is homotopic a cellular map, i.e the image of the n-skeleton of X is contained in the n-skeleton of Y.

Corollary 1.1.1. $\pi_n(S^k) = 0$ for $k > n$.

Proof. If the image of the map $\phi : S^n \to S^k$ the image misses a point $s_0 \in S^k$, then $S^k - \{s_0\}$ is homotopy equivalent to $\mathbb{R}^k$, and continuous map into $\mathbb{R}^k$ is nullhomotopic. Equip S^n with the CW structure of 2 k-cell in each dimension k. Then, every map $\phi : S^n \to S^k$ is homotopic to a cellular map that is not surjective. $\square$

The next result is very important to our discussion.

Theorem 1.2. (Hurewicz Theorem) A space X is called **n-connected** if $\pi_k(X) = 0$ for all $0 \leq k \leq n$. For $n \geq 2$, if X is n-connected, then $\pi_n(X) \cong \tilde{H}_n(X)$.

An immediate corollary of this result is that we can compute $\pi_n(S^n)$, which is generated by the degree map, as one might expect.

Corollary 1.2.1. $\pi_n(S^n) \cong \mathbb{Z}$

Proof. Combine the fact that S^n is $n-1$-connected by Corollary 1.1.1 and the fact that $H_n(S^n) = \mathbb{Z}$. $\square$

Recall that a cofibration is a map $A \hookrightarrow X$ satisfying the homotopy extension property. Cofibration plays well with homology/cohomology as it gives us a long exact sequence, and we can then extract homological information from one space from the other. The dual notion is a fibration, which satisfies the homotopy lifiting property.

Definition 1.1. A map $E \to B$ is said to satisfy the **homotopy lifting property** (HLP) with respect to a space X if the following diagram commutes

$$\begin{tikzcd} X \times \{0\} \arrow[r, "\tilde{H}_0"] \arrow[d, "i"] & E \arrow[d] \\ X \times [0,1] \arrow[r, "H"] \arrow[ur, dashed, "\tilde{H}"] & B \end{tikzcd}$$

In other words, given a homotopy in $H_t : X \to B$ and a initial lift $\tilde{H}_0 : X \to E$, we can lift the homotopy entirely.

Definition 1.2. A map $p : E \to B$ satisfying the HLP with respect to arbitrary X is called a (Hurewicz) **fibration**. A map satisfying the HLP with respect to CW-complexes is called a **Serre fibration**. Assume B is path-connected and based at b_0, the **fiber** of the fibration is $F = p^{-1}(b_0) \subseteq E$. We organize the data of a fibration into the following **Fiber Sequence**

$$F \to E \to B$$

Note that covering maps are fibrations with discrete fibers. In practice, Serre fibrations is good enough to give us most of the desired properties/tools. It is fun to know that pathological examples exists (even in CGWH) where a Serre fibration is not a fibration.

From now on we assume the base-space is path-connected.

Theorem 1.3. Given a Serre fibration $p : E \to B$ with fiber F and a choice $x_0 \in F$, we have the following LES:

$$... \longrightarrow \pi_n(F, x_0) \longrightarrow \pi_n(E, x_0) \longrightarrow \pi_n(B, b_0) \longrightarrow \pi_{n-1}(F, x_0) \longrightarrow ...$$

From this theorem, we can already calculated a not so obvious homotopy group

Corollary 1.3.1. $\pi_3(S^2) \cong \mathbb{Z}$

Proof. We have the exact sequence from the hopf fibration

$$\pi_3(S^1) = 0 \longrightarrow \pi_3(S^3) \cong \mathbb{Z} \longrightarrow \pi_3(S^2) \longrightarrow \pi_2(S^1) = 0$$

where the triviality of the two groups on the end follows from Corollary 0.1.1; $\pi_3(S^3) \cong \mathbb{Z}$ follows from Corollary 1.2.1 $\square$

Theorem 1.4. (Fiber replacement)Every map $f : X \to Y$ can be turned into a fibration in the following sense: there exists a space E_f in

Theorem 1.5. (Puppe Sequence) Given a fibration $F \to E \to B$, we have the following sequence where any two consecutive maps form a fibration

$$... \longrightarrow \Omega^2 B \longrightarrow \Omega F \longrightarrow \Omega E \longrightarrow \Omega B \longrightarrow F \longrightarrow E \longrightarrow B$$

where continuing to the left is applying the loop space functor.

Theorem 1.6. (Universal Coefficient Theorem) Given a coefficient group G, We have a split short exact sequence

$$0 \longrightarrow \mathbf{Ext}^1_{\mathbb{Z}}(H_{n-1}(X;\mathbb{Z}), G) \longrightarrow H^n(X;G) \longrightarrow \mathbf{Hom}(H_n(X;\mathbb{Z}), G) \longrightarrow 0$$

Corollary 1.6.1. $\text{rank}(H_n) = \text{rank}(H^n)$ and $\text{Torsion}(H_{n-1}) = \text{Torsion}(H^n)$

2 Serre Spectral Sequence

For the following discussions, we will use the assumption that B is simply connected and work over $R = \mathbb{Z}$ to simplify things.

Definition 2.1. Given a Serre fibration $F \hookrightarrow X \to B$, with fiber F path-connected and base B simply-connected. Then, the **Serre cohomological spectral sequence** is given by the E_2 page

$$E_2^{p,q} = H^p(B; H^q(F;\mathbb{Z})) \Longrightarrow H^{p+q}(X;\mathbb{Z})$$

If the space B is not simply connected, then the coefficient group $H^q(F)$ is actually the local system on B given by the fibers. This reduces to the integral cohomology when the action of $\pi_1(B)$ on the fibers are trivial.

Theorem 2.1. (Product Structure) The Serre cohomological spectral sequence has a bigraded $\mathbb{Z}$-algebra structure, given by the product

$$E_n^{p,q} \times E_n^{s,t} \to E_n^{p+s,q+t}$$

In particular, the product structure on E_2 page is given by the cup product, and the product on E_n induces the one on E_{n+1}.

Proposition 2.1. The differentials $d_n : E_n^{p,q} \to E_n^{p+n,q-n+1}$ is a graded derivation with respect to the product structure. In other words, given $a \in E_n^{p,q}$ and $b \in E_n^{s,t}$, we have

$$d_n(ab) = d_n(a)b + (-1)^{|p+q|} a d_n(b)$$

We are ready to compute $\pi_4(S^3)$.

3 Computations

Theorem 3.1. $\pi_4(S^3) \cong \mathbb{Z}/2$.

We know $\mathbb{Z} = H^3(S^3; \mathbb{Z}) = [S^3, K(\mathbb{Z}, 3)] = \pi_3(K(\mathbb{Z}, 3))$. In particular, we may choose a map $f : S^3 \to K(\mathbb{Z}, 3)$ representing the generator of the group. Note that by construction, $f_* : \pi_3(S^3) \to \pi_3(K(\mathbb{Z}, 3))$ is an isomorphism. Let F_f be the homotopy fiber of f,

Proposition 3.1. $H_4(F_f) \cong \pi_4(S^3)$; $H_3(F) = H_2(F) = 0$.

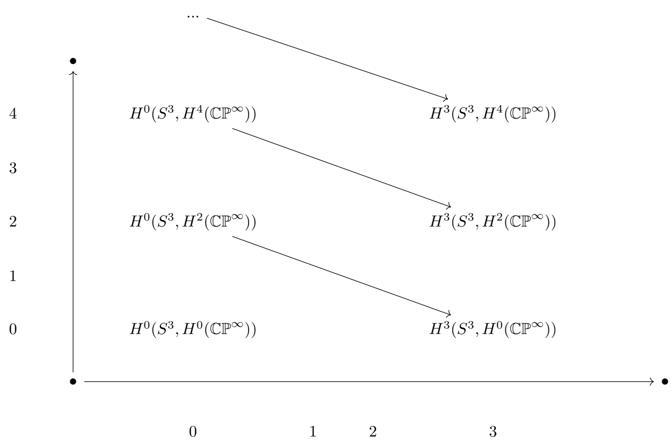
Proof. We have the the long exact sequence

$$...\pi_{n+1}(K(\mathbb{Z}, 3)) \to \pi_n(F_f) \to \pi_n(S^3) \to \pi_n(K(\mathbb{Z}, 3)) \to ...$$

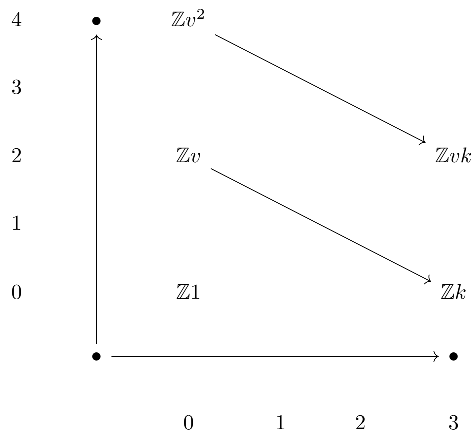
For $n = 3$, we note $\pi_4(K(\mathbb{Z}, 3))$ is trivial, and $\pi_n(S^3) \to \pi_n(K(\mathbb{Z}, 3))$ is an isomorphism, so $\pi_3(F_f)$ must be trivial; similar argument shows $\pi_n(F_f)$ is trivial for $0 < n \leq 3$ by corollary 1.1.1 and the fact that $\pi_k(K(\mathbb{Z}, 3)) \neq 0$ iff $k = 3$. Thus, F_f is 3-connected. Note that the long exact sequence at degree 4 also gives us the isomorphism $\pi_4(F_f) \cong \pi_4(S^3)$. Apply Hurewicz Theorem gives us the desired result. $\square$

We may extend the fiber sequence one step to the left, which is the next step in the Puppe sequence $\Omega K(\mathbb{Z}, 3) = K(\mathbb{Z}, 2) \to F_f \to S^3$. Our goal now is to use the spectral sequence to calculate the cohomology of F_f using the the cohomology of S^3 and $K(\mathbb{Z}, 2)$, which is realized as $\mathbb{CP}^\infty$.

Recall that the cohomology ring of $\mathbb{CP}^\infty$ is $\mathbb{Z}[v]$, with $|v| = 2$. The E_2 page of the Serre spectral sequence looks like the following



Note that the coefficients $H^n(\mathbb{CP}^\infty) \cong \mathbb{Z}$ in all degrees, so all the non-trivial cohomologies in the E_2 page are all in fact $\mathbb{Z}$; Let 1 denote the generator for $E_2^{0,0}$ and k denote the generator for $E_2^{3,0}$; the generator for $E_2^{0,2n}$ is v^n as the generator for the coefficient group $H^4(\mathbb{CP}^\infty)$. By the product structure, the generator for $E_2^{3,2}$ is simply vk, for the multiplication map is just multiplication of coefficients.



A quick examination of the spectral seqeunce above shows that the d_2 differentials are all trivial by degree reasons. Therefore, all cohomologies survive to E_3 page.

Note that a quick application of UCT and Proposition 3.1 shows that $H^3(F_f) = 0$. In particular, we see that $d_3 : E_3^{0,2} \to E_3^{3,0}$ must be an isomorphism since both cohomology cannot survive to the next page. WLOG, we may assume $d_3(v) = k$. Then by the derivation law, we see that $d_3 : E_3^{0,4} \to E_3^{3,2}$ is given by $d_3(v^2) = d_3(v)v + vd_3(v) = 2vk$, so $H^4(F_f) = E_4^{0,4} = 0$. Since there is nothing on $E_3^{6,0}$, we see that $H^5(F_f) = E_4^{3,2} = \mathbb{Z}vk/(2vk) \cong \mathbb{Z}/2\mathbb{Z}$. An application of Corollary 1.6.1 finishes.