MATH 624 Algebraic Geometry

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1 Prevarieties and Varieties

We will assume that $K|k$ a finite extension, K is algebraically closed. We will use $\mathbb{A}^n(K) = K^n = \mathbb{A}^n_K$ to denote the underlying set, not the n-dimensional affine space. Given a point $a = (a_1, ..., a_n) \in \mathbb{A}^n_k$, we will use φ_a to denote the evaluation map $k[X] \to k$. Similarly, given $f \in k[x]$, we have the evalation map $\tilde{f} : \mathbb{A}_k \to k$. This gives a morphism of k-algebras $k[x] \to Maps_k(\mathbb{A}_k, k)$ given by $f \mapsto \tilde{f}$.

Definition 1.0.1. Given $\Sigma \subset k[x]$, define $V(\Sigma) = \{a \in \mathbb{A}_k : f(a) = 0 \text{for every } f \in \Sigma\}$. This is called the **affine k-algebraic set** defined by Σ. If $\Sigma = \{f\}$, then $H_f := V(\Sigma) = V(f)$ defines a **hyperplane** in $\mathbb{A}_k$.

Example 1.0.1. Easy examples

1. $V((0)) = \mathbb{A}_k$.
2. $V((1)) = \emptyset$
3. Let $k = \mathbb{C}$. Then, in $\mathbb{A}^1_k$, $V(x^2 - 1) = \{\pm 1\}$. In $\mathbb{A}^2_k$, $V(x^2 - 1) = \{(\pm 1, n) : n \in k\}$

Definition 1.0.2. Given $V \subset \mathbb{A}_\neg$, defined $I(V) = \{f \in k[x] : f(V) = 0\}$. This is called the **ideal** of V.

Proposition 1.0.1.
1. Let $I_\Sigma \subset k[x]$ be the ideal generated by Σ. Then, $V(\Sigma) = V(I)$.
2. There exists a finite system $f_1, .., f_m$ such that $V(\Sigma) = V(f_1, ..., f_m)$
3. If $\Sigma_1 \subset \Sigma_2$, then $V(\Sigma_1) \supset V(\Sigma_2)$
4. Given $\mathfrak{a}$ an ideal, then $I(V(\mathfrak{a})) = \mathfrak{a}$ iff $\mathfrak{a} = \sqrt{\mathfrak{a}}$.
5. Given ideals $\mathfrak{a}, \mathfrak{b}$, then $V(\mathfrak{a}) = V(\mathfrak{b})$ iff $\sqrt{\mathfrak{a}} = \sqrt{\mathfrak{b}}$.

Definition 1.0.3. Let $\mathcal{A}^n_K := \{V \subset \mathbb{A}^n_K : V \text{ affine } k-\text{algebraic sets}\}$. Given $V \in \mathcal{A}^n_K$, let $k[V] := k[x]/I(V)$ be the **affine coordinate ring** generated by V.

Let $Id^{rd}(k[x])$ be the set of reduced ideals of $k[x]$. Let R_n be the set of reduced k-algebras with n-generators.

Theorem 1.1. There is a canonical bijection between the set of reduced affine k-algebras and reduced ideals of $k[x]$, given by the maps

$$R_n \to Id^{re}(k[X]) \to \mathcal{A}_K^k$$

$$k[\underline{x}] \mapsto \mathfrak{a} := ker(k[x] \xrightarrow{f} k) \mapsto V(\mathfrak{a})$$

with f given by $x \mapsto \underline{x}$.

1.1 The Zariski Topology

Given $V \in \mathcal{A}_K^n$, there is a canonical map $K[X] \to K[V]$ given by $f \mapsto f_V$.

Proposition 1.1.1. Let $\Sigma_i \subset k[X]$, and $f \in k[X]$ be given. then

1. $V(\cup_i \Sigma_i) = \cap_i V(\Sigma_i)$
2. $V(\prod \Sigma_i) = \cup V(\Sigma_i)$
3. $V((0)) = \mathbb{A}_k^n$; $V((1)) = \emptyset$

By the proposition above, we can define the Zariski topology on $\mathbb{A}_k^n$

Definition 1.1.1. The Zariski topology on $\mathbb{A}_K^n$ is given by the closed sets $V(\Sigma)$, with $\Sigma \in k[X]$. In particular, the sets $D_f := \mathbb{A}_k^n - H_f$ is an open set and forms a basis for the topology.

Note that the zariski topology on product spaces is not the product of zariski topologies. Moreover, the connectedness/irreducibility is dependent on $K|k$. A point is called a generic point of V if its closure contains V.

Example 1.1.1. If $K|k = \mathbb{C}|\mathbb{Q}$, then $V(x_1^2 - 2x_2^2)$ is connected and irreducible. If $K|k = \mathbb{C}|\mathbb{Q}[\sqrt{2}]$, then $V(x_1^2 - 2x_2^2)$ is connected but not irreducible.

Remark 1.1.1. For a topological space, X, the following are equivalent:

1. Every descending chain of closed subsets is stationary.
2. Every ascending chain of open subsets is stationary.

A topological space satisfiying the above is called **Noetherian**. For example, $Spec(R)$ is Noetherian if R is Noetherian. Note that if X is Noetherian, then it is automatically quasi-compact. Moreover, there are only finitely many irreducible components and connected components of X.

Proposition 1.1.2. The following hold:

1. The Zariski topology is Noetherian on $\mathbb{A}_K$, therefore also on any $V \in \mathcal{A}_K^n$.
2. For every $V \in \mathcal{A}_K$, there are only finitely many irreducible components and connected components.
3. $V \in \mathcal{A}_K$ is irreducible iff $I(V)$ is a prime ideal.
4. Given $V_0 \subset V$, V_0 is irreducible iff $I_V(V_0) := I(V_0)/I(V) \in Spec(k(V))$ is minimal.
5. The connected components in $V \in \mathcal{A}_K$ correspond bijectively to the indecomposable idempotents of $k[V]$.
6. For $V \in \mathcal{A}_K$, $a \in V$ is a generic point iff the evaluation map $k[V] \to k[a]$ is an isomorphism of k-algebras.

Definition 1.1.2. Let T be a topological space, and let $V \subset T$.

1. dim(V):=sup { chain of irreducible components ending in V: }
2. codim(V):=sup { chain of irreducible components starting with V and ending in T: }

Note that if $V = \cup V_\alpha$, then $dim(V) = supdim(V_\alpha)$, and similarly for codimensions. Moreover, $dim(V) = dim(\overline{V})$.

Proposition 1.1.3. (Notions of dimension) Let $V \in \mathcal{A}_K$ be irreducible. Then, the dimension of V is the same as the krull dimension of $K[V]$.

Proposition 1.1.4. Suppose irreducible $W \subset V \in \mathcal{A}_K$. Then,

$$dim(W) + codim_V(W) = dim(V)$$

Proposition 1.1.5. $V \in \mathcal{A}_K$ has generic points a iff $td(K|k) \geq dim(V) = td(k(V))$.

1.2 Base change and Rational Points

Definition 1.1.3. Suppose there is an embedding

$$\begin{tikzcd} K \arrow[r] & L \\ k \arrow[u] \arrow[r] & l \arrow[u] \end{tikzcd}$$

Then, there is a natural morphism $k[x] \to l[x]$, which induces a pushforward of ideals and a map $\mathcal{A}_K \to \mathcal{A}_L$. Take the vanishing locus of the pushforward of $I(V)$ gives the base change of V.

Remark 1.1.2. Base change does not preserve connectedness or irreducibility.

Definition 1.1.4. $V \in \mathcal{A}_K$ is called **absolutely (geometrically) irreducible** if V_l is irreducible for all field extension $l|k$. It is **geometrically connected** is V_l is connected for all $l|k$.

Proposition 1.1.6. Let $V \in \mathcal{A}_K$ be affine k-algebraic set. Then the following are equivalent:

1. V is absolutely irreducible.
2. V_{k^s} is irreducible.
3. $V_{\overline{k}}$ irreducible.

The key observation is that $K^s[x] \to \overline{k}[X]$ is an integral extensions of domains. Therefore, we have going up and going down, and it straightforward to show that $Spec(k^s[X]) \to Spec(\overline{k}[X])$ is a homeomorphism. Thus, we have (2) $\implies$ (3).

To (3) $\implies$ (1), apply the following:

Lemma 1.2. For every $V \in \mathcal{A}_K$, one has $V(\overline{k})$ is zariski dense in V. Therefore, $V_{\overline{k}}$ irreducible implies V irreducible.

The proof is exercise. The key point is that if there exists f with k-coefficients such that f vanishes on all of A

Proposition 1.2.1. Let $V \in \mathcal{A}_K$ be affine k-algebraic set. Then the following are equivalent:

1. V is geometrically connected.
2. V_{K^s} is connected.
3. $V_{\overline{k}}$ is connected.

2 The category of quasi-affine k-algebraic sets

Definition 2.0.1. A **quasi-affine** k-algebrac set is any zariski open subset $U \subset V$ for $V \in \mathcal{A}_K$.

The complement of hyperplanes is a basis of quasi-affine k-algebraic sets. Let $V \in \mathcal{A}_K$ be non-empty, $f \in K[V]$. Then, the evaluation map $f : V \to \mathcal{A}_K$ is continuous. Moreover, $\varphi = (f_1, ..., f_n)$ is also continuous.

Definition 2.0.2. Let $V \in \mathcal{A}_K$ and $\mathcal{V} \subset V$ be zariski dense. Then, a functions $\varphi : \mathcal{V} \to \mathcal{A}_K$ is called **regular** at $x \in V$ if there exists $f_x, g_x \in k[x]$ and $\S \subset V$ such that $g_x \neq 0$ everywhere on $\mathcal{U}_x$ and $\varphi = \frac{f_x}{g_x}$. A fucntion $\varphi : \mathcal{V} \to \mathcal{A}_K$ is **regular** if it is regular at every point in V. Let $\mathcal{O}_x := \{\varphi \in Maps(\mathcal{V}, K) : \varphi \text{ regular at } x\}$. Define an equivalence relation on $\mathcal{O}_x$ by equivalence on any open neiborhood around x. $\mathcal{O}(V)$ is the set of regular functions on V.

Proposition 2.0.1. (rings of regular functions) We have the following:

1. $k[V] \to \tilde{\mathcal{O}}(V)$ is an isomorphism of k-algebra.
2. $k[V]_f \to \tilde{O}(U_f)$ is an isomorphism of k-algebra.

It is helpful to remember that Zariski open sets are dense. Thus, it suffices to show that a function is zero on a basic open U_f to deduce it is globally zero.

3 Presheaves and Sheaves

Definition 3.0.1. Let $\mathcal{C}$ be a concrete category such as **Top, Set, Ab**. Let X be a topological space with topology τ_X. Then, τ_X is naturally poset category where morphisms are inclusions. A **presheaf** is a contravariant functor $\mathcal{P} : \tau_X \to \mathcal{C}$.

Explicitly, $\mathcal{P}$ is given by two data: 1.$\mathcal{P}(U) \in Obj(\mathcal{C})$ for every $U \in \tau_X$. 2.$\rho_{u',u''} : \mathcal{P}(U'') \to \mathcal{P}(U')$ for every $U' \subset U''$. The elements in the set $P(U)$ are called **sections** above U. The image of a section under ρ is called the **restriction**.

Definition 3.0.2. A presheaf is a **sheaf** if it has the covering property: given an open cover of an open set $U = \cup_i U_i$, with $U + i, j := U_i \cap U_j$ with $s_i \in \mathcal{P}(U_i)$ such that $\rho_{U_i,U_{i,j}}(s_i) = \rho_{U_j,U_{i,j}}(s_i)$, then there exists $s \in \mathcal{P}(U)$ such that $s_i \in \rho_{U,U_i}(s)$ for every U_i.

Definition 3.0.3. Suppose that limits exists in $\mathcal{C}$. Then $\mathcal{P}_x := \mathcal{P}(U_x)$ is called the **stalk** of $\mathcal{P}$ at x.

Proposition 3.0.1. $\mathcal{P}$ is a sheaf iff for every $U \in \tau_X$, the map $\varphi_U : U \to \coprod_{x\in U} \mathcal{P}_x$ is injective.

Proposition 3.0.2. For every presheaf $\mathcal{P}$, there is a sheafification functor $\mathcal{P} \to \mathcal{F}$ that induces isomorphism on stalks.

Definition 3.0.4. Let $f : X \to Y$ be a continuous map of topological spaces. Then,

1. Given a (pre)sheaf $\mathcal{P}$ on X, then the **direct image** (pre)sheaf $f_*\mathcal{P}$ on Y is defined by $f_*\mathcal{V} := \mathcal{P}(f^{-1}(V))$ for all $V \in \tau_Y$. In particular, the direct image sheaf is also a sheaf.
2. Given a presheaf $\mathcal{P}$ on Y. There is an **inverse image** sheaf $f^{-1}\mathcal{P}$ on X defined by the limit:

$$f^{-1}\mathcal{P}(U) := \varinjlim_{U\subset U'} \mathcal{P}(f(U'))$$

   where $U \subset U'$ and $f(U')$ is open.

Remark 3.0.1. Note that the preimage sheaf is always a preseeaf, but not necessarily a sheaf.

Definition 3.0.5. A (locally) **ringed space** is a pair $(X, \mathcal{F})$, where X is a topological space and $\mathcal{F}$ a sheaf of rings on X such that the stalks at each point is a local ring.

Definition 3.0.6. Given locally ringed spaces $(X, \mathcal{F})$, $(Y, \mathcal{G})$, a morphism of locally ringed space is a pair $(f, f^\sharp)$ such that $f : X \to Y$ is continuous and $f^\sharp : \mathcal{G} \to f_*\mathcal{F}$ a morphism of sheaves.

4 Back to Varieties

Proposition 4.0.1. Let V be an affine k-algebraic set, $U \subset V$ zariski open.

1. The assignmenet τ_U, $U' \mapsto \tilde{O}(U')$ defined a locally ringed space on U.
2. A morphism of quasi-affine algebraic set $T \to U$ is any morphism of locally ringed spaces $(f, f^\sharp) : (T, \mathcal{O}_T) \to (U, \mathcal{O}_U)$

The checks are fullfilled by proposition 2.0.1.

Proposition 4.0.2. Let $(T, \mathcal{O}_T), (U, \mathcal{O}_U)$, and $\Phi : T \to U$ continuous. Then,

1. Φ defined a morphism of locally ringed spaces iff $\mathcal{O}_U \circ \varphi \subset \mathcal{O}_T$, i.e for every U and T' open such that $\Phi(T') \subset U'$ and $\varphi \in \mathcal{O}_U(U')$, then $\varphi \circ \Phi \in \mathcal{O}_T(T')$.
2. Suppose Φ defines such a morphism, and let $U \subset \mathbb{A}_K$, $p : \mathbb{A}_K^n \to K$ the ith projection, then $p_i|_U \circ \Phi$ completely determines Φ.

Remark 4.0.1. Let $U_f := \{x \in V | f(x) \neq 0 :\}$ be a basic open. Consider $W_f \subset \mathbb{A}_K^n$ defined by $W_f := \{(a, b) | a \in \mathbb{A}_K^n, b \in \mathbb{A}_K^1 : f(a)b - 1 = 0\}$ is an algebraic set in $\mathbb{A}_K^{mn}$. Prove that $\Phi : W_f \to U_f$ given by $(a, b) \mapsto a$ is an isomorphism of quasi affine k-algebraic sets. Then inverse is given by $\psi : U_f \to W_f$ given by $a \mapsto (a, \frac{1}{f(a)})$.

Proposition 4.0.3. Every quasi-affine k-algebraic set contains a n zariski dense k-algebraic set.

Definition 4.0.1. A quasi-affine k-algebraic set is called **affine** if it is isomorphic as a locally ringed space to an affine k-algebraic set.

Theorem 4.1. The following hold:

1. The catgeory of K-valued affine k-algebraic sets, $\mathcal{A}_k$, is anti-equivalent to the category of reduced k-algebras of finite type. In particular, a k-algebraic set $V \subset \mathcal{A}_K$ is mapped to $k[V]$. Note that the projection maps $V \to W \to \mathcal{A}_k$ defined a regular function on V, and by proposition 4.0.2 determined the morphism of the algebraic set. There is a canonical map from the ring of regular functions on V to the coordinate ring $k[V]$ by proposition 2.0.1.
2. Let U be a quasi-affine k-algebraic set, W and affine k-algebraic set. Then, a morphism $\Phi : U \to W$ is determined by a map $\Phi^* : k[W] \to \tilde{O}(U)$.

Definition 4.1.1. $\mathcal{A}_k^n := (\mathcal{A}_K^n, \tilde{O}_{\mathcal{A}_K^n})$ is called the **n-dimensional affine sapce.**

Definition 4.1.2. An **open immersion** of quasi-affine k-algebriac setd $j : U \to T$ is any k-morphism which is a zariksi open immersion and $\tilde{O}_U = \tilde{O}_T \circ j$

Definition 4.1.3. A **closed immersion** of quasi-affine k-algebraic sets $i : U \to T$ is a topological closed immersion and $i_* O_U$ is a factor sheaf of $\mathcal{O}_T$. In other words, the map $\Phi* : \tilde{\mathcal{O}}_T(T') \to \tilde{\mathcal{O}}_U(U')$ is surjective.

Definition 4.1.4. A k-**prevariety** is any quasi-compact locally ringed space X that is locally isomorphic to K-valued affine k- algebraic sets. Locally isomorphic here means that there exists an finite open cover $X = \cup X_\alpha$ and isomorphism of locally ringed spaces $\varphi_\alpha : X_\alpha \to V_\alpha$, where V_α is affine k-algebraic set. Moreover, the transition maps are isomorphisms of quasi-affine k-algebraic sets.

Remark 4.1.1. A k-**morphism** of k-prevarieties is a morphism of locally ringed spaces, such that there exists $X = \cup X_\alpha, Y = \cup Y_\alpha$ and $f(X_\alpha) \subset Y_\alpha$, and the structure maps induce a map of affine k-algebraic sets.

Definition 4.1.5. Let $f : X \to Y$ be a k-morphism of k-prevarieties. Then,

1. f is an open immersion iff f induced structure maps is an open immersions of affine k-algebraic sets.
2. f is an closed immersion iff f induced structure maps is a closed immersions of affine k-algebraic sets.
3. X is called affine if it is isomorphic as a k-prevariety to an affine k-algebraic set.
4. X is called quasi-affine if there is an open immersion into a affine k-prevariety.

Proposition 4.1.1. (Gluing datat for k-prevarieties and k-morphisms)

1. (X_i) be a finite set of k-prevarieties.
2. $X_{ij} \subset X_i$ open for every i, j
3. $\varphi_{ij} : X_{ij} \to X_{ji}$ a $k-$isomorphism such that $\varphi_{ii} = Id$, $\varphi_{ij} = \varphi_{ji}^{-1}$ and $\varphi_{ij} \circ \varphi_{jk} = \varphi_{ik}$.
4. A solution is $X = \cup X_i'$ and k-isomorphisms $X_i' \to X_i$

Remark 4.1.2. The solution is unique up to k-isomorphism.

Proposition 4.1.2. (Gluing morphisms)Suppose $f_\alpha : X_\alpha \to Y_\alpha$ such that

$$\begin{tikzcd} X_{\alpha\beta} \arrow[r, "f_\alpha"] \arrow[d, "\varphi_{\alpha\beta}"'] & Y_{\alpha\beta} \arrow[d, "\psi_{\beta\alpha}"] \\ X_{\beta\alpha} \arrow[r, "f_\beta"'] & Y_{\beta\alpha} \end{tikzcd}$$

The there exists a unique k-morphism $X \to Y$ compactible with the gluing data.

The idea of proof for 4.1.1 and 4.1.2 is to take the disjoint union of the topological spaces first and define an equivalence relation accordingly. Then, define the structure sheaf on the quotient as the unique sheaf of k-algebras such that $\mathcal{O}_X|_{i_\alpha(X_\alpha)} = (i_\alpha)_* \mathcal{O}_{X_\alpha}$. We can check that $\mathcal{O}_X$ is well-defined. The glued morphism is as expectedly induced by morphisms from the glued components.

Example 4.1.1. (Line with two origins)Let $X_1, X_2 = \mathbb{A}^1$, and $U_{12} = U_{21} = \mathbb{A}^1 - \{0\}$, and let $\varphi : U_{12} \to U_{21}$ be the identity.

Example 4.1.2. (The projective line)Let $X_1, X_2 = \mathbb{A}^1$, and $U_{12} = U_{21} = \mathbb{A}^1 - \{0\}$, and let $\varphi : U_{12} \to U_{21}$ be $\varphi(x) = \frac{1}{x}$.

Theorem 4.2. Let X be a k-prevariety, and V be an affine k-prevariety. Then, one has a canonical bijection

$$Hom_k(k[V], \mathcal{O}_X(X)) \to Mor_K(X, V)$$

Proof uses Theorem 4.1 part 2. Break up the morphisms by $X = \cup X_\alpha$, where each X_α is affine. Then use the gluing theorems to glue back.

Theorem 4.3. Finite products and coproducts exists in the catgeory of affine k-algebraic sets. The coproduct correspond to the product of the affine rings. The product correspond to the reduced tensor product of affine k-algebras.

Remark 4.3.1. Note that the k-tensor algebra of two reduced k-algebras might no longer be reduced. Therefore we need to take the quotient by the nilradical.

Theorem 4.4. The category of k-prevarieties has finite products and coproducts.

Proof. Let T be the category of locally ringed spaces in k-algebras. Let T_0 be a subcategory that is

1. closed under open immersions.
2. closed under finite products.
3. Every object in T can be glued from thatof T_0.

Then, products exists in T. Take T to be category of k-prevarieties and T_0 be subcategory quasi-affine k-prevarieties. The hard part is to show that T_0 has all fintie products. $\square$

Problem 5

(a)

$(a) \implies (b)$ Take a nullhomotopy $H : S^n \times I \to X$, where $H_0 = f$ and H_1 is a constant map. Then, H factors through a map $H' : CS^n = D^{n+1} \to X$. The boundary of the cone is precisely $S^n \times \{0\}$, so H' is the desired extension.

$(b) \implies (c)$ Every representative $\varphi \in \pi_n(X, x_0)$ is a map $\varphi : (S^n, s_0) \to (X, x_0)$, which is nullhomotopic since it factors through (D^{n+1}, s_0) which deformation retracts onto s_0.

$(c) \implies (a)$ Obvious by definition.

(b)

Clearly $(a) \implies (b)$, $(b) \implies (c)$ is also obvious since D^n is contractible, so we may first homotope the map $f : (D^n, \partial D^n) \to (X, A)$ to a map $D^n \to A$ through such maps, and then homotope it to a constant map by a deformation retraction of D^n. $(c) \implies (d)$ by definition of the relative homotopy groups; $(d) \implies (a)$ by the compression lemma.