

# MATH 603 Notes

David Zhu

March 28, 2024

## 1 More on Commutative Rings

Let  $a, b \in R$ . Then  $a|b \iff \exists a' \in R, b = aa'$ ; A semi ring on  $(R, \leq)$  defined by  $a \leq b \iff a|b$ . Note that  $\leq$  is usually not a partial order: let  $b \in R^\times$ , then  $a \leq ab \leq a$ , but  $a \neq ab$ .

**Proposition 1.1.**  $a \sim b$  iff  $a \leq b$  and  $b \leq a$  iff  $(a) = (b)$  is an equivalence relation.

For  $R$  a domain, the induced relation gives a well-defined definition of greatest common divisor.

**Definition 1.1.** The **gcd** of  $a, b$ , denoted by  $gcd(a, b)$ , if exists, is any  $d \in R$  such that  $d|a, b$  and for any other  $d'$  satisfying the condition,  $d'|d$ .

**Definition 1.2.** The **lcm** of  $a, b$ , denoted by  $lcm(a, b)$ , if exists, is any  $d \in R$  such that  $a, b|d$  and for any other  $d'$  satisfying the condition,  $d|d'$ .

**Proposition 1.2.** If  $gcd(a, b)$  exists, then  $gcd(a, b) = \sup\{d : d \leq a, b\}$ . If  $lcm(a, b)$  exists, then  $lcm(a, b) = \inf\{d : a, b \leq d\}$ .

Note that maximal/minimal elements always exists by Zorn's lemma. However, the unique supremum/infimum may not exist. We have our following example:

**Example 1.1.** Take  $R = [\sqrt{-3}]$ . Let  $a = 4 = (1 + \sqrt{-3})(1 - \sqrt{-3})$  and  $b = 2(1 + \sqrt{-3})$ . Then,  $(1 + \sqrt{-3})$  and 2 are both maximal divisors, but they are not comparable since the only divisors of 2 are  $\{\pm 1, \pm 2\}$  by norm reasons, and none divides  $1 + \sqrt{-3}$ .

**Proposition 1.3.** Let  $a, b \in R$  be given. Then the following hold:  $gcd(a, b) = d$  exists iff  $(d)$  is the unique maximal principal ideal such that  $(a) + (b) \subseteq (d)$ . Dually,  $lcm(a, b) = c$  exists iff  $(c) = (a) \cap (b)$ . If both holds, then  $a \cdot b = lcm(a, b) \cdot gcd(a, b)$

*Proof.* Easy exercise. Note that the inclusion can be proper, for example, take  $R = k[x, y]$  and ideals  $(x), (y)$ . Then  $(1)$  is the gcd, but the containment is proper.  $\square$

Recall that  $Id(R)$  is partially ordered by inclusion.

**Definition 1.3.**  $(Id(R), +, \cap, \cdot, \leq)$  is the lattice of ideals of  $R$ .

Note that  $+$ ,  $\cap$  are simply the sums and intersection, but  $\cdot$  is the ideal generated by the products, i.e the set of finite sums of products.

**Theorem 1.1.** Let  $Id^\infty(R)$  be the set of non-finitely generated ideals for  $R$ ; the following are equivalent:

1.  $Id^\infty(R)$  is non-empty;
2. There exists an infinite non-stationary chain of ideals  $(\sigma_i)$ , where  $\sigma_i \in Id(R)$ ;

*Proof.* For  $1 \implies 2$ , let  $I$  be a non-finitely generated ideal of  $R$  and pick  $x_1 \in I$ . Let  $\sigma_1 = (x_1)$ . Because the ideal is non-finitely generated, we can pick  $x_2 \in I$  such that  $x_2 \notin \sigma_1$ . Let  $\sigma_2 = (x_1, x_2)$ . Continue inductively gives us an infinite non-stationary chain of ideals.

For  $2 \implies 1$ , take the union of all the ideals in the infinite non-stationary chain. It is an ideal and it cannot be finitely generated.  $\square$

**Theorem 1.2.** (Cohen's lemma): Let  $Id^\infty(R) \neq \emptyset$ . Then, it has a maximal element and any such maximal element is prime.

Before proving Cohen's lemma, we need the following technical lemma:

**Lemma 1.3.** Let  $I$  be an ideal. Define  $(I : a) := \{b \in R : ab \in I\}$ . If  $I + (x)$  and  $(I : x)$  are both finitely generated, then  $I$  is finitely generated.

*Proof of Lemma 1.3.* By assumption, there is finite set  $\{\alpha_i := a_i + f_i x : a_i \in I, f_i \in R, i = 1, \dots, k\}$  that generate  $I + (x)$ , and a finite set  $\{b_j : j = 1, \dots, l\}$  that generate  $(I : x)$ . We claim that the set  $\{a_i, b_j x : i \in I, j \in J\}$  generate the entire  $I$ : since  $I \subseteq I + (x)$ , we can write any element  $\pi \in I$  as a finite linear combination  $\pi = \sum_{i=1}^k g_i \alpha_i = \sum_{i=1}^k g_i (a_i + f_i x)$ , where  $g_i \in R$ . We note that  $\pi - \sum_{i=1}^k g_i a_i = \sum_{i=1}^k g_i f_i x$  is in  $I$ ; it follows that  $\sum_{i=1}^k g_i f_i \in (I : x)$ , so  $\sum_{i=1}^k g_i f_i x$  is generated by the set  $\{b_j x\}$ , and we are done.  $\square$

With the lemma in hand, now we can prove Theorem 1.2

*Proof of Theorem 1.2.* Zorn's lemma implies  $Id^\infty(R)$  has maximal elements. Let  $I$  one such maximal element, and suppose it is not prime. Then, there exists  $xy \in I$  and WLOG suppose  $x \notin I$ . By maximality,  $I + (x)$  must be finitely generated. By definition, we have  $y \in (I : x)$ . Lemma 1.3 implies  $(I : x)$  is not finitely generated, and in particular,  $I \subseteq (I : x)$ . Applying maximality again, we have  $I = (I : x)$ , which forces  $y \in I$ , a contradiction.  $\square$

## 2 Euclidean Rings

**Definition 2.1.** A Principal Ideal Ring is any ring  $R$  in which every ideal is principally generated. If  $R$  is a domain, then  $R$  is called a PID.

**Definition 2.2.** A **Factorial Ring** is any ring  $R$  in which all units can be written as a finite product of irreducible elements, unique up to a unit. If  $R$  is domain, then it is called a **UFD**.

Note that if the ring  $R$  it is not a domain,  $x|y$  and  $y|x$  does not imply  $x = uy$  for some unit  $u$ . Let us prove that this holds for a domain: suppose  $x = ys$  and  $y = xt$ , and  $x, y \neq 0$  then  $x = xts$ , which implies  $x(1 - ts) = 0$ . This forces  $1 - ts = 0$ , and  $t, s$  are then units. We can concoct counterexamples when  $R$  is not a domain accordingly: let  $R = k[x]/(x^3 - x)$  and take  $a = x, b = x^2$ . Clearly,  $a|b$  and  $b = x^2 \cdot x = x^3$ , so  $b|a$ . However,  $x$  is not a unit.

**Definition 2.3.** A **Noetherian Ring** is any ring  $R$  such that any ideal is finitely generated.

**Definition 2.4.** Let  $R$  be a domain. A **Euclidean norm** on  $R$  is any map  $\phi : R \rightarrow \mathbb{N}$  satisfying  $\phi(x) = 0$  iff  $x = 0$  and for every  $a, b \in R$  with  $b \neq 0$ , then there exists  $q, r \in R$  such that  $a = bq + r$  with  $\phi(r) < \phi(b)$ . A **Euclidean Domain** is any domain equipped with a Euclidean norm.

Example of Euclidean domains include  $\mathbb{Z}, \mathbb{Z}[i]$ . A non-trivial example  $R = F[t]$ , with  $\phi(p(t)) = 2^{\deg(p(t))}$ . A non-example is  $\mathbb{Z}[\sqrt{6}]$  for it is not a PID.

**Proposition 2.1.** Euclidean Domains are PIDs.

*Proof.* By the well-ordering principal, for every ideal  $I$  in a Euclidean domain, there exists an element other than 0 of the smallest norm. It is easy exercise that such element generate the entire ideal.  $\square$

**Proposition 2.2.** (The Euclidean Algorithm): Given  $a, b \in R, b \neq 0$ . Set  $r_0 = a, r_1 = b$ , and continue inductively  $r_{i-1} = r_i \cdot q_i + r_{i+1}$ . Then,  $r_i = 0$  for  $i > \phi(b)$  and if  $r_{i_0} \geq 1$  maximal with  $r_{i_0} \neq 0$ , then  $r_{i_0} = \gcd(a, b)$ .

*Proof.* Note that the remainder is strictly decreasing, so  $r_i$  must become 0 after  $\phi(b)$  steps. Note that once  $r_{i+1} = 0$ , we have  $r_i|r_n$  for all  $n \leq i$ . Conversely, it is clear that any divisor of  $a, b$  divides all  $r_n$  for  $n \leq i$ .  $\square$

### 3 Principal Ideal Domains

**Theorem 3.1.** (Charaterization) For A domain  $R$ , the following are equivalent:

1.  $R$  is a PID.
2. every  $p \in \text{Spec}(R)$  is principal.

*Proof.* One direction is trivial; for the other direction, assume that every prime is principal. Then, Cohen's Lemma implies  $\text{Id}^\infty(R) \neq \emptyset$ ; In particular, every ideal is finitely generated, so the ring is Noetherian. We may apply Zorn's lemma on the set of non-principally generated ideal (since every chain stablizes and has a maximal element), and let  $P$  be a maximal non-principally generated ideal. Suppose it is not prime, and let  $xy \in P$  with  $x \notin P$ . Then,  $P \subset (P : x)$  and  $P \subset P + (x)$  properly. By maximality, we have  $(P : x) = (c)$ , and  $(I : c) = (d)$ . By definition, we have  $cd \in I$ ; moreover, suppose  $x \in I$ , then  $x = cr = cdt$  for some  $r, t \in R$ . Thus,  $I = (cd)$  is principal, a contradiction.  $\square$

**Proposition 3.1.** PIDs are UFDs.

*Proof.* Let  $a \in R$  such that  $a$  is non-zero and not a unit. Then, there exists  $p \in \text{Spec}(R)$  such that  $(a) \subseteq p$ . Hence  $R$  being a PID implies  $\exists \pi \in R$  such that  $p = (\pi)$ . Hence,  $\pi$  must be prime and  $\pi | a$ . Set  $a_1 = a$ ,  $\pi_1 = \pi$ , and let  $a_2$  be the element such that  $\pi_1 a_2 = a_1$ . If  $a_2$  is not a unit, find  $(a_2) \subset (\pi_2)$ , where  $\pi_2$  is prime. Let  $a_3$  be the element such that  $\pi_2 a_3 = a_2$ . Continue inductively until  $a_n$  is a unit. The process must terminate, for otherwise we get an infinite chain of distinct principal ideals  $(a_i)$  that does not stabilize (stabilizing is equivalent to  $(a_n) = (a_{n+1})$  for some  $n$ , which implies they differ by a unit).  $\square$

**Corollary 3.1.1.** Let  $R$  be a PID; let  $P \subset R$  be a set of representatives for the prime elements up to association. For every  $a \in R$ ,  $\exists \epsilon \in R^\times$  and  $e_\pi \in \mathbb{N}$  such that almost all  $e_\pi = 0$ . Then, every  $a \in R$  can be written as  $a = \epsilon \prod_{\pi \in P} \pi^{e_\pi}$ . We proceed to recover  $\gcd$  and  $\text{lcm}$ , up to associates.

Note that the above corollary generalizes to the quotient field by replacing  $\mathbb{N}$  with  $\mathbb{Z}$ .

## 4 Unique Factorization Domains

**Definition 4.1.** The following are equivalent for a domain  $R$ :

1.  $R$  is a UFD.
2. Every minimal prime ideal (prime of height 1) is principal and every non-zero, non-invertible elements in contained in finitely many primes.

*Proof.*  $1 \implies 2$ : For every non-zero prime  $P$ , pick  $x \in P$  has factor. One of the prime factors must be in  $P$ , and it follows by minimality that  $P$  must be generated by such prime factor. For the second part, the finite factorization of the element gives precisely the finite primes that it is contained in.  $2 \implies 1$ : given  $x \in R$ , the finitely many primes containing  $x$  are principally generated by prime elements, which gives a factorization.  $\square$

Remark: we recover the  $\gcd$  and  $\text{lcm}$  definition using the same factorization as Corollary 3.1.1.

**Theorem 4.1.** (Gauss Lemma) Let  $R$  be a UFD; then  $R[t]$  is a UFD.

To prove the theorem, we need the following lemma on contents:

**Definition 4.2.** Let  $f(t) = a_0 + \dots + a_n t^n$  be given. Then, the content of  $f$ , denoted  $C(f)$ , is the GCD of all coefficients. In particular,  $C(f) | a_i$  for all  $i$ , and  $f_0 := f / (C(f))$  has content 1.

**Lemma 4.2.** Let  $R$  be a UFD, then the following hold: (1).  $C(f) : R[t] \rightarrow R$  given by  $f \mapsto C(f)$  is multiplicative; in particular, if  $C(f) = C(g) = 1$ , then  $C(fg) = 1$ .

*Proof of lemma 4.2.* given  $f(t) = a_0 + \dots + a_n t^n$  and  $g(t) = b_0 + \dots + b_m t^m$ . If one of  $f, g$  is constant, then it is easy exercise; suppose neither is constant, then set  $f = f_0 \cdot C(f)$  and  $g = g_0 \cdot C(g)$ . Clearly we have  $C(f) \cdot C(g) | C(fg)$ . Hence it suffices to prove that  $C(f_0 g_0) = 1$ . Equivalently, let  $\pi \in R$  be a prime element, we want to show there exists a coefficient  $c_k \in f_0 g_0$  such that  $\pi$  does not divide  $c_k$ . Suppose

$\pi|_{c_k} = \sum_{i+j=k} a_i b_j$  for all  $k$ . Because  $C(f_0) = C(g_0) = 1$ , then there exists minimal  $a_i, b_j$  such that  $\pi$  does not divide  $a_{i_0}, b_{j_0}$ . Then,  $\pi$  does not divide  $C_{i_0+j_0}$ . □

**Proposition 4.1.** Let  $K := \text{Quot}(R)$ , and  $f \in K[t]$ . Then, let  $d$  be the least common multiple of the denominators of the coefficients of  $f$ . Then,  $f = df/d$ , and  $df \in R[t]$ . Define  $C_K(f) = C(df)/d$ . It is standard to check the analog for lemma 4.2 holds for  $C_K$  as well.

**Proposition 4.2.** Let  $R$  be a UFD. For any irreducible  $f \in R[t]$ , either  $f$  is a constant and thus prime in  $R$ , or  $f$  is primitive, i.e  $C(f) = 1$ .

*Proof.* If  $f$  is a constant, the first part of the proposition is obvious; now suppose  $f$  has degree  $> 0$ ; then  $f$  can be factored into its primitive part and content; if  $C(f) \neq 1$ , we either have a non-trivial factorization of  $f$  or  $f$  will be a constant multiplied by a unit, a contradiction. □

**Theorem 4.3.** Let  $R$  be a UFD. For  $f(t) \in R[t]$ , let  $K := \text{Quot}(R)$ . Then, the following are equivalent:

1.  $f(t)$  is prime
2.  $f(t)$  is irreducible
3. Either  $f$  is an irreducible constant in  $R$  or  $f$  is irreducible in  $K[t]$  and  $C_K(f) = 1$ .

*Proof.*  $1 \implies 2$  holds in every domain: suppose  $a$  is prime and  $a = bc$ . Then by primeness, we have  $a|b$  or  $a|c$ . WLOG, suppose  $a|b$ , such that  $ax = b$  and  $a = axc$ , so  $cx - 1 = 0$ , which implies  $c$  is a unit.

$2 \implies 1$  in UFDs: suppose  $f$  is an irreducible and  $f|gh$ , then we have some  $l$  such that  $fl = gh$ . Because  $g, h, l$  can be uniquely written as a product of irreducibles up to permutation and units, we see that the irreducible  $f$  must appear on the RHS once, i.e  $f|g$  or  $f|h$ .

For  $2 \implies 3$ : If  $f$  is a constant, then it become a unit in the field of fractions; suppose  $\deg(f) > 0$ , so irreducibility implies  $C(f) = 1$ . Suppose by contradiction that  $f$  is reducible over  $K[t]$ , and let  $f = gh$  for  $g, h \in K[t]$  be a factorization in  $K[t]$ . Note that given  $g, h \in K[t]$ , there is some  $x_g, x_h \in K$  such that  $x_g g, x_h h \in R[t]$  and  $C(x_h h) = C(x_g g) = 1$ . Then,  $x_g x_h f = (x_g g)(x_h h) \in R[t]$ . By Proposition 4.2, we have  $C(x_g x_h f) = x_g x_h C(f) = 1$ , which implies  $x_g x_h = 1$  (up to a unit in  $R$ ). However, this implies  $f = (x_g g)(x_h h)$ , a contradiction.

So we are left to prove  $3 \implies 2$ . Suppose  $f$  is not a constant and  $f$  primitive and irreducible. Suppose  $f = gh \in R[x]$ . WLOG  $g$  is a unit in  $K[x]$ , so  $g$  is a nonzero element of  $R$ . Now  $g$  divides all the coefficients of  $f$ , so  $g$  is a unit in  $R$ . □

**Proposition 4.3.**  $R[t_i]_{i \in I}$  is UFD if  $R$  is UFD.

*Proof.* By induction it suffices to show that  $R[t]$  is a UFD. The idea is that  $K[t]$  is PID so it is a UFD. A factorization in  $K[t]$  will correspond to a factorization in  $R[t]$  by the equivalence of 2 and 3 in Theorem 4.3. □

## 5 Noetherian Rings

**Definition 5.1.** A commutative ring  $R$  is called a **Noetherian** ring if every chain of ideals in  $R$  is stationary.

**Proposition 5.1.** The following are equivalent:

1. Every chain of ideals is stationary.
2. All ideals are finitely generated.
3.  $\text{Spec}(R) \subseteq \text{Id}^f(R)$ .

Terminology: the condition 1 is called the ACC (Ascending Chain Condition).

*Proof.* By Cohen's lemma, we deduce  $2 \iff 3$ ;  $1 \iff 2$  is an easy exercise.  $\square$

For non-commutative rings, it is possible that a ring is left Noetherian but not right Noetherian.

**Example 5.1.**  $R = \left\{ \begin{bmatrix} p & q \\ 0 & m \end{bmatrix} : p, q \in \mathbb{Q}; m \in \mathbb{Z} \right\}$  is left Noetherian but not right Noetherian.

**Proposition 5.2.** (Basic Properties) Let  $R$  be a Noetherian ring. The the following hold:

1. If  $\mathfrak{a}$  is an ideal of  $R$ , then  $R/\mathfrak{a}$  is Noetherian if  $R$  is Noetherian.
2. If  $\Sigma \subset R$  is a multiplicative system, then  $R_\Sigma$  is Noetherian.
3. The radical of an ideals  $\mathfrak{a}$ ,  $\text{rad}(\mathfrak{a})$ , has a power contained in  $\mathfrak{a}$ .
4. Let  $\text{Spec}_{\min}(\mathfrak{a}) := \{p \in \text{Spec}(R) : \mathfrak{a} \subseteq p, p \text{ minimal}\}$  is finite.

*Proof.* To 1. Ideals in  $R/\mathfrak{a}$  corresponds bijectively to ideals in  $R$  that contains  $\mathfrak{a}$ . Finite generation of ideals in  $R$  clearly implies the finite generation of ideals in the quotient.

To 2.  $\text{Spec}(R_\Sigma)$  corresponds bijectively to primes in  $\text{Spec}(R)$  with empty intersection with  $\Sigma$ . We also have  $p$  finite generated implies  $p^e$  f.g.

To 3. Suppose  $\text{rad}(\mathfrak{a}) = (r_1, \dots, r_n)$  f.g. For every  $i$ , we have  $r_i^{n_i} \in \mathfrak{a}$  for some  $n_i$ . Take  $n = \sum n_i$  and  $\text{nil}(\mathfrak{a})^n \subset \mathfrak{a}$ .

To 4. The first method to prove this is by contradiction: let  $A = \{\mathfrak{a} : \text{Spec}_{\min}(\mathfrak{a}) \text{ is infinite}\}$ . Then  $A$  has maximal elements. Let  $\mathfrak{a}_0$  be maximal. Note that  $\mathfrak{a}_0$  cannot be prime for it is over itself. Suppose it is not prime, then there exists  $xy \in \mathfrak{a}$  with both  $x$  and  $y$  not in  $\mathfrak{a}$ ; for every prime ideal  $P$  containing  $\mathfrak{a}$ ,  $P$  contains either  $x$  or  $y$ . By pigeonhole, there must be infinite such primes containing either  $\mathfrak{a} + (x)$  or  $\mathfrak{a} + (y)$ , which contradicts maximality.

The second method is using the fact that  $\text{Spec}(R)$  is a Noetherian topological space, which has finitely many irreducible components.  $\square$

The third method is through primary decomposition. An ideal  $I$  is irreducible if  $I = a_1 \cap a_2$  then,  $I = a_1$  or  $I = a_2$ . For principal ideals, this is equivalent to the generator being irreducible.

**Proposition 5.3.** If  $R$  is Noetherian, then every ideal  $I \in R$  is in the finite intersection of irreducible ideals in  $R$ .

*Proof.* By contradiction, let  $X$  be the set of ideals that does not satisfy the proposition. Then,  $X$  is non-empty, and by Noetherian assumption, there is a maximal element  $\mathfrak{a}_0$ . Then,  $\mathfrak{a}_0$  is not irreducible, for it would be the intersection of itself. Therefore, there exists  $I_0, I_1$  such that  $\mathfrak{a}_0 = I_0 \cap I_1$ , where  $\mathfrak{a}_0$  is properly contained in both. By maximality,  $I_0, I_1$  are both finite intersection of irreducibles, and we can decompose  $\mathfrak{a}_0$  based on such, a contradiction.  $\square$

**Definition 5.2.** Let  $R$  be a commutative ring. Then an ideal  $I \subset R$  is primary if for all  $x, y \in R$  we have: if  $xy \in I$ ,  $x \notin I$ , then there exists  $n \in \mathbb{N}$  such that  $y^n \in I$ .

In general, a power of prime ideal is not primary. If  $I = \mathfrak{m}^n$  for some maximal ideal  $\mathfrak{m}$ , then  $I$  is in fact primary.

**Proposition 5.4.** Let  $R$  be Noetherian, and  $\mathfrak{a} \in Id(R)$  be a irreducible ideal. Then,  $\mathfrak{a}$  is primary, and  $nil(\mathfrak{a})$  is prime.

*Proof.* Exercise  $\square$

These two facts imply  $Spec_{min}$  must be finite. In general, quotient of  $UFD$  and  $PID$  are not  $UFD$  or  $PID$ . but this holds for Noetherian rings.

**Theorem 5.1.** Let  $R$  be a Noetherian ring. Then the following hold:

1. (Hilbert Basis Theorem):  $R[t_1, \dots, t_n]$  is Noetherian.
2. Every finitely generated  $R$ -algebra  $S$  is Noetherian.

*Proof.* Note that  $1 \implies 2$  since every finitely generated algebra is a quotient of polynomial rings over finitely many variable. To prove 1, by induction it suffices to show for  $i = 1$ . Let  $\mathfrak{a} \in R[t]$  be an ideal. Claim:  $\mathfrak{a}$  is f.g. Inductively, we may choose elements  $f_i \in I$  with  $deg(f_{i+1})$  being minimal in  $I \setminus (f_1, \dots, f_{i-1})$ . If the process terminates, then we are done; otherwise, let  $a_i$  be the leading coefficient of  $f_i$ , and the chain of ideals  $(I_i := (a_1, \dots, a_i))$  must stabilize by Noetherian assumption on  $R$ . Suppose it stabilizes at step  $N$ , and moreover suppose by contradiction that  $f_1, \dots, f_N$  does not generate  $\mathfrak{a}$ . Then, consider the element  $f_{N+1}$ , which by our argument is not contained in  $(f_1, \dots, f_N)$  and of minimal degree. The leading coefficient of  $f_{N+1}$  is expressed as  $a_{N+1} = \sum_{i=1}^N \mu_i a_i$ . Then, we cook up

$$g = \sum_{i=1}^N \mu_i f_i x^{deg(f_{N+1}) - deg(f_i)}$$

where  $g \in (f_1, \dots, f_N)$  by construction, and  $f_{N+1} - g \notin (f_1, \dots, f_N)$ . However,  $f_{N+1} - g$  has degree strictly less than  $f_N$  since we cancelled the leading term, which is impossible.  $\square$

## 6 Valuation Rings

**Proposition 6.1.** Let  $R$  be a domain. Then the following are equivalent:

1. The ideals in  $R$  are totally ordered by inclusion.
2. The principal ideals in  $R$  is totally ordered by inclusion, i.e  $id(R)$  is a chain
3. For every  $x \in \text{Quot}(R)$ , if  $x \notin R$  then  $x^{-1} \in R$ .

*Proof.* 1  $\implies$  2 is trivial;

□

**Definition 6.1.** A ring  $R$  satisfy one of the conditions above is called a (Krull) **Valuation Ring**.

**Example 6.1.**  $\mathbb{Z}_{(p)} = \{\frac{q}{l} \in \mathbb{Q} : \gcd(l, p) = 1\}$  is a valuation ring with maximal ideal  $(p)$ . The valuation on  $v_p$  is defined by  $v(\frac{q}{l}) = r$  where  $r$  is the maximal natural number such that  $p^r | q$ .

**Proposition 6.2.** (Properties) Let  $R$  be a valuation ring, and  $K$  be its quotient field. The the following hold:

1.  $R$  is local, and  $m = \{x \in R : x^{-1} \notin R\}$ . The maximal ideal is called **valuation ideal** of  $R$ .
2.  $\Gamma_R := K^\times / R^\times$  is totally ordered by  $xR^\times \leq yR^\times$  iff  $yR \subset xR$ . The group is called the **value group** of  $R$ .
3. The natural map  $v_R : K \rightarrow \Gamma_R \cup \{\infty\}$ ,  $v(0) = \infty$  satisfies  $v(xy) = v(x) + v(y)$  and  $v(x + y) \geq \min(v(x), v(y))$ . Such map is called the (canonical) **valuation** of  $R$ .

*Proof.* Exercise.

□

Note  $R$  is the set  $\{x \in K : v_R(x) \geq 0\}$ ;  $\mathfrak{m}$  is the set  $\{x \in K : v_R(x) > 0\}$ ;

**Definition 6.2.** Let  $R$  be a domain, and  $K$  be a field,  $(\Gamma, +, \leq)$  be a totally ordered abelian group. Let  $v : K \rightarrow \Gamma \cup \{\infty\}$  be a map satisfying

1.  $v(x) = \infty$  iff  $x = 0$
2.  $v(xy) = v(x) + v(y)$
3.  $v(x + y) \geq \min(v(x), v(y))$

Then, the map  $v$  is called a **valuation** of  $K$ .

**Proposition 6.3.**  $R_v = \{x \in K : v(x) \geq 0\}$  is a valuation ring. The map  $\tau : \Gamma_{R_v} \rightarrow \Gamma$ , given by  $xR_v^\times \mapsto v(x)$  is an order preserving embedding. Moreover,  $v = \tau \circ v_{R_v} : K \rightarrow \Gamma \cup \{\infty\}$ .

*Proof.* Exercise.

□

Given a valuation ring,  $R \subset K$ , every embedding of totally ordered groups  $\Gamma_R \rightarrow \Gamma$  gives rise to a valuation.



**Definition 6.3.** 1. Two valuations  $v, w$  on  $K$  are equivalent if  $R_v = R_w$ . If  $v : K \rightarrow \Gamma_v \cup \{\infty\}$  and  $w : K \rightarrow \Gamma_w \cup \{\infty\}$ , with embeddings  $\tau_v : \Gamma_{R_v} \rightarrow \Gamma_v$   $\tau_w : \Gamma_{R_w} \rightarrow \Gamma_w$ . There exists an order preserving commutative diagram

$$\begin{array}{ccccc} \Gamma_{R_v} & \longrightarrow & \tau_v(\Gamma_{R_v}) & \longrightarrow & \Gamma_v \\ & & \downarrow \tau_{vw} & & \\ \Gamma_{R_w} & \longrightarrow & \tau_w(\Gamma_{R_w}) & \longrightarrow & \Gamma_w \end{array}$$

2. Two valuations  $v, w$  on  $K$  are equivalent iff  $\mathfrak{m}_v = \mathfrak{m}_w$

A valuation ring  $R$  is called **discrete**, if  $v_R(K) \cong \mathbb{Z}$  as ordered abelian groups. If  $\pi \in R$  has  $v_R(\pi)$  minimal since  $\mathbb{Z}$  has minimal elements, then  $\pi$  is called a uniformizing parameter.

**Example 6.2.**  $\mathbb{Z}_{(p)} \subset \mathbb{Q}$  is a discrete valuation ring. The uniformization parameter is  $p\epsilon$  with  $\epsilon$  a unit.

A valuation ring  $R$  is called rank 1 if  $v_r(K)$  satisfies the Archimedean axiom, i.e for  $\forall \gamma_1, \gamma_2 \in \Gamma_R, \gamma_1 > 0$ ,  $\exists n \in \mathbb{N}$  such that  $\gamma_2 \leq n \cdot \gamma_1$ . A totally ordered group  $\Gamma$  is Archimedean if there is an ordered preserving embedding into the reals. In relation to absolute values,

**Definition 6.4.** An absolute value of a field  $K$  is any map  $|\cdot| : K \rightarrow \mathbb{R}_{\geq 0}^+$  iff it satisfies the norm axioms. An absolute value is called **non-Archimedean** or **ultra-metric** if  $|x + y| \leq \max\{|x|, |y|\}$ .

Let  $|\cdot| : K \rightarrow R$  be a non-Archimedean absolute value. Then  $v_{|\cdot|} := -\log \circ |\cdot| : K \rightarrow \mathbb{R} \cup \{\infty\}$  is rank 1 valuation. Conversely, let  $v : K \rightarrow R \cup \{\infty\}$  be a rank one valuation, then  $|\cdot|_v := e^{-v} : K \rightarrow \mathbb{R}_{\geq 0}$  is a non-Archimedean absolute value.

**Theorem 6.1.** The following facts about possible valuations

1. If  $K|F_p$  algebraic, then no non-trivial valuations.
2. If  $v$  is a valuation of  $F(t)$  such  $v$  is trivial on  $F$ , then  $R_v = F[t]_{p(t)}$ , where  $p(t)$  irreducible or  $R_v = F[\frac{1}{t}]_{(\frac{1}{t})}$ . thus all valuations are discrete.
3. If  $v$  is a non-trivial valuation on  $\mathbb{Q}$ , then  $R_v = \mathbb{Z}_{(p)}$  for some  $p$  prime. Moreover, all non-archimedean absolute values of  $\mathbb{Q}$  corresponds to the valuation above. (Ostrowskis Theorem).

In general, the space of all valuations on  $K$ , denoted  $Val(K)$ , is called the Zariski-Riemann space. Moreover,  $Val(K)$  carries a topology called a patch topology, or construtible topology, that makes the space compact totally disconnected. The space is usually very complicated.

**Theorem 6.2.** (Chevalley's Theorem for Existence of Valuations) Let  $A$  be a domain,  $p \in Spec(a)$  a prime ideal,  $\kappa(p)Quot(A/p) \subset \Omega$ , with  $\Omega$  algebraically closed. Then, there exists a valuation ring  $R$  of  $K = Quot(A)$  such that  $\mathfrak{m}_R \cap A = p$ , and  $R/\mathfrak{m}_R \hookrightarrow \Omega$ .

*Proof.* Set up for Zorn's lemma:  $H = \{(B, q) : A \subset B, q \in Spec(B), q \cap A = p\}$  such that the embedding into the closure  $\Omega$  commutes. Prove that  $H$  has maximal elements  $R$ . If  $R$  is not local, then  $R_{\mathfrak{m}}$  is local and bigger than  $R$ . Thus,  $R$  is local. let  $x \in K$ , we want to show  $x \notin R$  implies  $x^{-1} \in R$ . Claim, if  $m[x] = R[x]$ , then  $m[x^{-1}] \subset R[x^{-1}]$ . If so  $m$  then  $R_x := R[x^{-1}]$  is greater than  $R$ , a contradictin. By contradiction, let  $m[x] = R[x]$ ,  $m[x^{-1}] = R[x^{-1}]$ . Then, there exists coefficients in  $m$  such that polynomials are 1.  $\square$

## 7 Artin Rings

**Definition 7.1.** A commutative ring  $R$  is called Artin, if every descending chain of ideals  $(I_n)$  is stationary.

**Proposition 7.1.** Let  $R$  be Artinian. Then the following hold:

1. If  $\Sigma$  is a multiplicative system, then  $\Sigma^{-1}R$  is also Artinian.
2. If  $I \subset R$  is an ideal. Then,  $R/I$  is Artinian.
3.  $\text{Spec}(R) = \text{Max}(R)$  is finite.

*Proof.* To 1. pull back of ideals respects inclusion. To 2. obvious. To 3, let  $p \in \text{Spec}(R)$ . Then,  $R/p$  is an integral Artinian ring. Then,  $R/p$  must be a field. Thus, all primes are maximal. If  $\mathfrak{m}_1, \dots, \mathfrak{m}_n$ , then  $\mathfrak{m}_1 \subset \mathfrak{m}_1\mathfrak{m}_2 \subset \dots \subset \mathfrak{m}_1\mathfrak{m}_2\mathfrak{m}_3\dots$   $\square$

**Theorem 7.1.** Let  $R$  be an Artinian ring. Then the following hold:

1.  $J(R) = N(R)$  is nilpotent.
2. (Structure Theorem) Let  $\text{Max}(R) = \{m_1, \dots, m_r\}$ . Then,  $R \rightarrow R/(m_1)^n \times \dots \times R/m_r^n$ . Hence,  $R$  is a product of local Artinian rings.

*Proof.* Look at  $J(R) \subset J^2(R) \subset \dots$  becomes stationary. Thus, there exists  $n$  minimal such that  $I = J^n(R)$  such that  $I^k = I$  for all  $k$ . Let  $H$  be the set of ideals in  $R$  such that  $\square$

**Theorem 7.2.** (Relations of Artin Rings and Noether Rings) Let  $R$  be a commutative ring. The following are equivalent:

1.  $R$  is an Artin ring
2.  $R$  is Noether and Krull dimension of  $R$  is 0.

*Proof.* Step one is reduce to  $R$  local as follows:  $R$  Artin ring implies  $R = \prod R_i$ , where  $R_i$  are local Artin rings. In particular, the product is Noetherian iff each  $R_i$  is Noetherian.  $R$  is Noetherian and Krull dimension 0 implies  $\text{Max}(R) = \text{Spec}(R)$ , hence  $\text{rad}(R) = J(R)$ . Thus,  $R$  has only finitely many prime ideals. Hence,  $\text{Max}(R)$  is finite and we reduce to the finite case.

Step two: now assume  $R$  is local. Assume  $(R, \mathfrak{m})$  is a local Artin ring. To prove  $R$  is Noetherian, it suffices to show  $\mathfrak{m}$  is finitely generated. For  $k > 0$ , we have the exact spectral sequences

$$0 \longrightarrow \mathfrak{m}^k/\mathfrak{m}^{k+1} \xrightarrow{f} R/\mathfrak{m}^{k+1} \xrightarrow{g} R/\mathfrak{m}^k \longrightarrow 0$$

Note  $\kappa := R/\mathfrak{m}$  is a field,  $R$  acts on  $\mathfrak{m}^k/\mathfrak{m}^{k+1}$  through the projection  $R \rightarrow \kappa$ . In particular, there is a bijection

$$\{\kappa - \text{vector subspaces of } \mathfrak{m}^k/\mathfrak{m}^{k+1} : \} \iff \{R - \text{ideals } \mathfrak{n} : \mathfrak{m}^{k+1} \subset \mathfrak{n} \subset \mathfrak{m}^k\} = \epsilon$$

Note  $R$  Artinian implies  $R/\mathfrak{m}^{k+1}$  is Artinian. Thus, the set  $\epsilon$  is finite, and  $\mathfrak{m}^k/\mathfrak{m}^{k+1}$  is finite dimensional. Then induction on  $k$ : suppose  $R/\mathfrak{m}^k$  is Noetherian. Then,  $\mathfrak{m}^k/\mathfrak{m}^{k+1}$  is finite  $R$ -module.

For the reverse implication: Given  $(R, \mathfrak{m})$  Noetherian and  $\mathfrak{m}^k = 0$  for  $k$  sufficiently large. If  $k = 0$  then the statement holds; suppose it holds for  $k$ ; consider the exact sequence above and  $R/\mathfrak{m}^k$  is Artin. Let  $(\mathfrak{a}_i)$  be

a descending sequence of  $Id(R/\mathfrak{m}^{k+1})$ . Hence, there exists  $i_0$  such that  $\pi_k(\mathfrak{a}_i)\pi_k(\mathfrak{a}_{i_0})$  for all  $i > i_0$ . Then, look at

$$0 \longrightarrow \mathfrak{a}_i \cap \mathfrak{m}^k / \mathfrak{m}^{k+1} \xrightarrow{f} \mathfrak{a}_i / \mathfrak{m}^{k+1} \xrightarrow{g} \mathfrak{a}_i / \mathfrak{m}^k \longrightarrow 0$$

□

## 8 Krull's Theorem on Noetherian Rings

**Definition 8.1.** Let  $R$  be a commutative ring;  $\mathfrak{a} \subset R$  a proper ideal. Consider  $\mathfrak{a}^n$  and the projection  $p_n : R/\mathfrak{a}^{n+1} \rightarrow R/\mathfrak{a}^n$ . Then,  $(R/\mathfrak{a}^n, p_n)$  is a projective system. The limit  $\hat{R} := \varprojlim R/\mathfrak{a}^n$ , together with  $i : R \rightarrow \hat{R}$  is called  $\mathfrak{a}$ -adic completion of  $R$ .

**Proposition 8.1.** The kernel of the inclusion  $i : R \rightarrow \hat{R}$  is the intersection of all  $\mathfrak{a}^n$ .

**Theorem 8.1.** (The Intersection Theorem) Let  $R$  be a Noetherian ring that is local or integral. Let  $\mathfrak{a} \subset R$  be a proper ideal. Then,  $\cap \mathfrak{a}^n = 0$ . In particular, the inclusion map in the  $\mathfrak{a}$ -adic completion is injection.

*Proof.* Suppose  $R$  is local with maximal ideal  $\mathfrak{m}$ . Let  $\mathfrak{a}_0 = \cap \mathfrak{a}^k$  is a finitely generated ideal. Let  $\mathfrak{m}_0 = \cap \mathfrak{m}^k$  is f.g with  $\mathfrak{a}_0 \subset \mathfrak{m}_0$ . We then have  $\mathfrak{m} \cdot \mathfrak{m}_0 = \mathfrak{m}_0$ , and apply Nakayama's lemma, we get  $\mathfrak{m}_0 = (0)$ .

Now suppose  $\mathfrak{m}$  be a maximal ideal over  $\mathfrak{a}$ , and let  $\phi : R \rightarrow R_{\mathfrak{m}}$  be the localization is injective. □

Much more is true.

**Theorem 8.2.** (Artin-Reese Lemma) Let  $R$  be a Noetherian ring. Given ideals  $\mathfrak{a}, \mathfrak{b}, I$ , there exists  $k$  such that for  $k' > k$ , we have  $\mathfrak{a}^{k'} \mathfrak{b} \cap I = (\mathfrak{a}^{k'-k})$ .

**Theorem 8.3.** If  $R$  is Noetherian, then all  $\mathfrak{a}$ -adic completions of  $R$  is Noetherian.

**Definition 8.2.** Let  $R$  be a ring. For  $r \in R$ , let  $\text{Spec}_{\min}(r) = \{p \in \text{Spec}(R) : (r) \subset p \text{ minimal}\}$ . Definition goes similarly for a set of elements.

**Definition 8.3.** For  $p \in \text{Spec}(R)$ , the height of  $p$  is the krull dimension of  $R_p$ . The coheight is the krull dimension is the krull dimension of  $R/p$ .

**Proposition 8.2.**  $\text{height}(p) + \text{coheight}(p) \leq \text{Krull dimension of } p$ .

**Theorem 8.4.** (Krull's Principal Ideal Theorem/ Hauptideasatz) Let  $R$  be a Noetherian ring. Then, for all non-units  $r \in R$ , one has  $\text{height}(p) \leq 1$  for all  $p \in \text{Spec}_{\min}(r)$ , with equality when  $r$  is not a zero-divisor.

*Proof.* Suppose by contradiction that  $\text{height}(p) > 1$ . Equivalently, there exists a chain  $q_0 \subset q$  prime ideals. Reduction step: can replace  $R$  by  $R/p_0$ , and  $q/q_0$  and  $p/q_0$ . Can replace  $R, q, p$  by  $R_p, p_p, q_p$ . So WLOG,  $R$  is local, and  $p$  is maximal. Hence,  $(r) \subset p$ , and  $\text{rad}(r) = p$ . Conclude  $\bar{R} := R/(r)$  is Artin. Hence,  $q^{(n)}$  gives  $(\overline{q^{(n)}})_n$  a descending sequence, so it must stabilize.  $\square$

**Lemma 8.5.** For  $q \in \text{Spec}(R)$ , let  $q^{(n)} := (q^n R_q)^c$ . Then,  $(q^{(n)})^e = q^n R_q = (q R_q)^n$ .  $q^{(n)}$  is called the symbolic  $n$ -th power of  $q$ .

*Proof.* exercise  $\square$

**Lemma 8.6.**  $q^{(n)}$  is primary if  $ax \in q^{(n)}$ , and  $x \notin q$ , then  $a \in q^{(n)}$ .

*Proof.* exercise  $\square$

**Definition 8.4.** Let  $r = (r_1, \dots, r_n)$  be given. It is called a regular sequence if  $r_i$  is not a zerodivisor in  $R/(r_1, \dots, r_{i-1})$ .

**Theorem 8.7.** (Krull's Dimension Theorem) Let  $R$  be a Noether ring, and  $r = (r_1, \dots, r_m)$  a system. Then  $\text{Spec}_{\min}(r)$  contains prime ideals of height  $\leq m$ , with equality when  $r$  is regular.

*Proof.* Induction:  $n = 1$  is PIT; Given  $r = (r_1, \dots, r_{m+1})$ , and  $p \in \text{Spec}_{\min}(r)$ . Let  $q \subset p$  be a maximal with this property. Claim:  $ht(q) = m$ . Hence  $ht(p) = m + 1$ . By contradiction, let  $ht(q) > m$  and consider  $q_0 \subset q_1 \subset \dots \subset q_m = q$ . Reduction step 1: replace  $R, p, q$  by  $\bar{R} := R/q_0, \bar{p} := p/q_0, \bar{q} := q/q_0$ . WLOG,  $R = \bar{R}$  is a Noetherian domain. Reduction step 2 Replace  $R, p, q$  with their localization at  $p$ . Then,  $R$  is local with  $\text{max}(R) = p$ .

Since  $q$  is properly contained in  $p$ , there exists  $r_i \notin q$  by minimality. WLOG,  $r = m + 1$ . Consider  $\mathfrak{a} = q + (r_{m+1})$ , which implies  $q \subset \mathfrak{a} \subseteq p$ . Then,  $\text{nil}(\mathfrak{a}) = p$  since  $p$  is the only prime ideal containing  $\mathfrak{a}$ . Conclude that  $r_i \in p$  implies  $r_i \in \text{nil}(\mathfrak{a})$ , hence for every  $i = 1, \dots, m$ , there exists  $a_i \in R$  and  $s_i \in q$  such that  $r_i^{m_i} = s_i + a_i r_{m+1}$ . Conclude that  $s = (s_1, \dots, s_m)$  and  $\text{nil}(s) = q$ . In particular,  $q$  is a minimal prime ideal containing elements  $s_1, \dots, s_m$ . Look at  $R \rightarrow R/s$ , and  $\bar{p} = p/s$ . Hence,  $ht(\bar{p})$  is at most 1, and 1  $\bar{r}_{m+1}$  is not a zero divisor.  $\square$

**Corollary 8.7.1.** Let  $R$  be Noether. Then, the following hold:

1. Every descending sequence of prime ideals is stationary.
2. if  $ht(p) = m$ , then there exists a regular system of length  $m$  with  $p$  a minimal prime over it.

*Proof.* exercise  $\square$

## 9 Modules over special classes of rings

### 9.1 Modules over PIDs

**Lemma 9.1.** Let  $C = (\gamma_1, \dots, \gamma_n)$  be an  $R$ -basis of a free  $R$ -module  $P$ , and  $\gamma'_1 = \sum_{i=1}^m r_i \gamma_i$  such that  $\gcd(r_1, \dots, r_m) = 1$ . Then there exists a basis  $C' = (\gamma'_1, \dots, \gamma'_m)$

*Proof.* Induction:  $n = 1$  trivial; let  $x = \sum_{i=1}^{m+1} r_i \gamma_i = \sum_{i=1}^m r_i \gamma_i + r_{m+1} \gamma_{m+1}$ . We can write  $\sum_{i=1}^m r_i \gamma_i$  with respect to the new basis by induction step. since the gcd of the sequence is  $\gcd(d, r_{m+1})$ , which must be 1.  $\square$

**Theorem 9.2.** Let  $R$  be a PID. Then, the following hold:

1. If  $M$  is a free  $R$  module, every  $R$ -submodule of  $M$  is  $R$ -free.
2. A finite torsion free  $R$ -module is  $R$ -free

*Proof.* To 1: let  $A = (\alpha_i)$  be a basis for  $M$ . Suppose  $I$  is well-ordered (otherwise use Zorn's lemma), we make transfinite induction: let  $i_0$  be the minimal element of  $I$ ,  $M_{i_0} = \langle \alpha_{i_0} \rangle$  be the cyclic submodule. Let  $M_{i'} = \langle \alpha_{i'} : i \leq i' \rangle$  and  $N_{i'} = N \cap M_{i'}$ . Note that  $N_{i_0} = N \cap R_{\alpha_{i_0}}$ , and the submodules of a PID is precisely the principal ideals, which are free.

To 2: let  $(x_1, \dots, x_m)$  be a system of generators and let  $R^m \rightarrow M$  be the natural surjection. If the system is linearly independent, we are done. Let  $\sum_{i=1}^k r_i x_i = 0$ . Divide out the gcd, and if there is no torsion, we can apply the lemma.  $\square$

**Theorem 9.3.** (Invariant Factors Theorem) Let  $R$  be a principal ideal domain and  $M$  a free  $R$ -module,  $N \subset M$  a submodule. Then, there exists  $R$ -basis  $A = (\alpha_1, \dots, \alpha_m)$  of  $M$  and  $\delta_1 | \delta_2 | \dots | \delta_n$  in  $R$  such that  $\delta_1 \alpha_1, \dots, \delta_n \alpha_n$  is an  $R$ -basis for  $N$ .

*Proof.* Let  $D = \{d \in R : \exists y \in N, \text{basis}(\beta_1, \dots, \beta_m) \text{ of } M \text{ s.t. } d\beta_1 = y\} = \{d \in R : \exists y = d \sum_{i=1}^k r'_i \alpha'_i \in N, \gcd(r_i) = 1\}$ .  $\square$

**Lemma 9.4.**  $d_1 = \inf D$  exists, where the ordering is by divisibility.

*Proof.* Obviously,  $D$  has minimal elements. Let  $d', d''$  be minimal elements, and  $d = \gcd(d', d'')$ , and show  $d \in D$ .  $\square$