# Equivariant Stable Homotopy Notes

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For the entire note, we will assume a group $G$ to be a compact Lie group, and subgroups $H \subset G$ are always closed. [Blu17]

# 1 Unstable Equivariant Homotopy Theory

## 1.1 G-CW Complexes

Fix a compact Lie group $G$ acting on a space $X$. Similar to $CW$-complexes, we want to deconstruct $X$ into cells, but this time with the addditional data of the $G$-action along with each cell. The idea is that cells are of the form of a product $G/H \times D^n$, where $G$ acts trivially on $D^n$, and $G/H$ "represents" the orbits of $D^n$. To make this work, $H$ must be the isotropy group of $D^n$.

**Definition 1.0.1.** A **G-CW complex** is the sequential colimit of spaces $X_n$, where $X_{n+1}$ is a pushout:

$$\begin{tikzcd} \coprod G/H \times S^n \arrow[r] \arrow[d] & X_n \arrow[d] \\ \coprod G/H \times D^{n+1} \arrow[r] & X_{n+1} \end{tikzcd}$$

We will denote $G/H \times D^n$ as an **n-cell**.

**Remark 1.0.1.** Note that the topological dimension of an $n$-cell in a $G$-CW complex might be greater than $n$. For example, a 0-cell $S^1/e \times *$ is one dimensional.

**Example 1.0.1.** Let $G = C_2$ acting on $S^2$ by rotation by $\pi$ along the Z-axis. It has a $G$-CW structure given by the following cells: 2 zero-cells $C_2/C_2 \times *$, which are the poles corresponding to the fixed points of the $C_2$ action. 1 one-cell $C_2/e \times D^1$, which are the two great circles joining the poles; 1 two-cell $C_2/C_2 \times D^2$, which are the two hemispheres.

**Example 1.0.2.** Let $G = C_2$ acting on $S^2$ by the antipodal map. It has a $G$-CW structure given by the following cells: 1 zero-cells $C_2/e \times *$, which are the poles; 1 one-cell $C_2/e \times D^1$, which are the two great circles joining the poles; 1 two-cell $C_2/C_2 \times D^2$, which are the two hemispheres.

**Definition 1.0.2.** Let $H$ be a subgroup of $G$. Define $\pi_n^H(X) := \pi_n(X^H)$. A map $f : X \to Y$ of $G$-spaces is a **weak equivalence** if for all subgroups $H \subset G$,

$$f_* : \pi_n^H(X) \to \pi_n^H(Y)$$

is an isomorphism.

Let **GTop** be the category of $G$-spaces and $G$-maps. There is a cofibrantly-generated model structure that we can put on **GTop**:

**Theorem 1.1.** There is a cofibrantly-generated model structure on **GTop**, given by

1. A $G$-map $f : X \to Y$ is a fibration iff for all $H \subset G$, $f^H : X^H \to Y^H$ is a fibration.
2. A $G$-map $f : X \to Y$ is a weak equivalence iff for all $H \subset G$, $f^H : X^H \to Y^H$ is a weak equivalence.

An immediate consequence of the model category structure is the equivariant Whitehead's Theorem

**Corollary 1.1.1.** Let $f : X \to Y$ be a weak equivalence of cofibrant-fibrant objects in a model category. Then, $f$ is a homotopy equivalence. In particular, every object in **GTop** is fibrant, and $G$-CW complexes are cofibrant.

## 1.2 Elmendorf's Theorem

From the model structure given in Theorem 1.1, we have a vague sense of the following "equivalence":

$$\text{G-Homotopy Type of } X \Leftrightarrow \{\text{ordinary homotopy type of } X^H : H \subset G\}$$

And Elmendorf's Theorem will make the equivalence precise. We start by introducing the orbit category:

**Definition 1.1.1.** The **orbit category** $\mathcal{O}_G$ is the full subcategory of GTop on the objects $\{G/H : H \subset G\}$.

The following lemma will make the structure of $\mathcal{O}_G$ clearer.

**Lemma 1.2.** $\mathrm{Map}^G(G/H, G/K) \cong (G/K)^H$

*Proof.* Note that there exists a $G$-equivariant maps $\varphi : G/H \to G/K$, determined by $\varphi(H) = gK$ iff $gHg^{-1} \subseteq K$ iff $h(gK) = gK$ for all $h \in H$. □

Let $\mathrm{Fun}(\mathcal{O}_G^o p, \mathrm{Top})$ be the functor category. We have the following fact on the model structure on functor categories:

**Theorem 1.3.** Let $\mathcal{D}$ be a model category and $\mathcal{C}$ be a cofibrantly generated model category. Then, $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ admits a model structure.

It is useful to know that the weak equivalences in $\mathrm{Fun}(\mathcal{O}_G^{op}, \mathrm{Top})$ is given pointwise: a natural transformation $\eta : \mathcal{F} \to \mathcal{G}$ is a weak equivalence iff $\eta_{G/H} : \mathcal{F}(G/H) \to \mathcal{G}(G/H)$ is a weak equivalence.

**Definition 1.3.1.** There is a functor $\psi : \mathrm{GTop} \to \mathrm{Fun}(\mathcal{O}_G^{op}, \mathrm{Top})$ given by

$$X \to (G/H \mapsto X^H)$$

It is easy to check the functoriality. Note that if we restrict $\psi$ to $\mathcal{O}_G$, the functor is just the Yoneda embedding: $\mathrm{Map}^G(G/H, G/K) \cong (G/K)^H$.

**Proposition 1.3.1.** There is a funcor $\theta : \mathrm{Fun}(\mathcal{O}_G^{op}, \mathrm{Top}) \to \mathrm{GTop}$ given by $X \mapsto X(G/e)$, where $X(G/e)$ is equipped with the following $G$-action: note that every $g \in G$ defines an $G$-map $G/e \to G/e$, which we denote by $R_g$.

$$g \cdot x = X(R_g)(x)$$

It is easy to check that $(\theta, \psi)$ is an adjoint pair. In fact, more can be said:

**Theorem 1.4.** (Elmendorf's Theorem) $\mathrm{Fun}(\mathcal{O}_G^{op}, \mathrm{Top})$ and GTop have the same homotopy category.

The original proof due to Elmendorf constructs the equivalence explicitly using the Bar construction to obtain a homotopy inverse to the embedding $\psi$. The theorem can now be put into a more modern framework:

**Theorem 1.5.** $(\theta, \psi)$ is an Quillen equivalence. $\psi$ is an equivalence of $(\infty, 1)$ categories.

However, there is no hope that $\psi$ itself is an equivalence