MATH 618 Algebraic Topology

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1 The Correct Category

Let $T =$ compactly generated weakly Hausdorff. Let $T_2 =$ pairs of spaces (X, A), $A \subseteq X$, and

$$T_2((X, A), (Y, B)) = \{f \in T(X, Y) : f(A) \subseteq B\}$$

We define $(X, A) \otimes (Y, B) = (X \times Y, X \times B \cup Y \times A)$. (Think about product of boundaries). We want tot understand the analogue of $T(X \times Y, Z) \cong T(X, T(Y, Z))$.

Theorem 1.1. Let $(X, A), (Y, B), (Z, C) \in T_2$, then

$$T_2((X, A) \otimes (Y, B), (Z, C)) \cong T_2((X, A), T_2(T_2((Y, B), (Z, C))), T(Y, C))$$

Let T_* be the full subcategory of T_2 consisting of pairs $(X, *)$. There exists a pair of functors $T_2 \to T_*$ defined by $(X, A) \mapsto (X/A, A/A = *)$.

Proposition 1.1.1. $q : X \to X/A$, we get

$$q* : T_*(X/A, Y) \to T_2((X, A), (Y, *))$$

an isomorphism.

We want a producgt in T_* which works well with function spaces:

Definition 1.1.1. Given $X, Y \in T_*$, defined $X \wedge Y = (X \times Y/X \vee Y, * = X \vee Y)$ called the **smash product**.

Note that the smash product is not the categorical product here.(The categorical product is simply the cartesian product carring the canonical basepoint).

Definition 1.1.2. The **reduced suspension** $\Sigma X := S^1 \wedge X$.

Theorem 1.2. The catgegory of based spaces T_* has the following properties

1. $T_*(X,Y) \in T_*$, with basedpoint the constant map to basepoint.
2. $T_*(X,Y) \wedge X \to Y$ is continuous.
3. $T_*(X,Y) \wedge T_*(Y,Z) \to T_*(X,Z)$ is continuous.
4. $T_*(X \wedge Y, Z) \cong T_*(X, T_*(Y,Z))$
5. Small limits and colimits exists in T_*.
6. The forgetful functor $T_* \to T$ preserves limits.

Definition 1.2.1. The **reduced cone** is a functor $C : T_* \to T_*$ defined by $X \mapsto IX$, with the basepoint of I being 1.

Proposition 1.2.1. There exists a pushout

$$\begin{tikzcd} X \arrow[r] \arrow[d] & CX \arrow[d] \\ CX \arrow[r] & \Sigma X \end{tikzcd}$$

Definition 1.2.2. The **loop space** is defined to be $T_*(S^1, X)$.

Theorem 1.3. (Eckmann-Hilton duality)

$$T_*(X, \Omega Y) \cong T_*(\Sigma X, Y)$$

There exists a functor $T \to T_*$ given by $X \mapsto X \coprod \{x\}$, which is left adjoint to the forgetful functor. If $X \in T_*$, then $\Omega X \in T_*$, and note that $\pi_0(\Omega X) = [S^1, X]_* = \pi_1(X)$.

Definition 1.3.1. A functor $F : T_* \to C$ for some catgeory C is called a **homotopy functor** if $F(f) = F(g)$ when $f \cong g$

Definition 1.3.2. Define $\pi_n(X) := [S^n, X]_*$. Then, π_n is a homotopy functor from T_* to set.

Proposition 1.3.1. π_n is a group for $n \geq 1$ and abelian when $n \geq 2$.

Proof. The group structure is given by: suppose we have $\varphi, \psi : I^n \to X$. Then, $\varphi + \psi : I^n \to X$ is explicitly given by

$$\varphi + \psi(t_1, ..., t_n) = \begin{cases} \varphi(2t_1, t_2, ..., t_n) & t \in [0, \frac{1}{2}] \\ \psi(2t_1 - 1, t_2, ..., t_n) & t \in [\frac{1}{2}, 1] \end{cases}$$

$\square$

Definition 1.3.3. Define H-space be a topological space X with homotopy associative map

$$X \wedge X \to X$$

Proposition 1.3.2. If (Y, y_0) is an H-space, then $[X, Y]_*$ has a group structure.