MATH 603 Notes

David Zhu

March 17, 2024

1 More on Commutative Rings

Let $a, b \in R$. Then $a|b \iff \exists a' \in R, b = aa'$; A semi ring on $(R, \leq)$ defined by $a \leq b \iff a|b$. Note that $\leq$ is usally not a partial order: let $b \in R^{\times}$, then $a \leq ab \leq a$, but $a \neq ab$.

Proposition 1.1. $a \sim b$ iff $a \leq b$ and $b \leq a$ iff $(a) = (b)$ is an equivalence relation.

For R a domain, the induced relation gives a well-defined definition of greatest common divisor.

Definition 1.1. The **gcd** of a, b, denoted by $gcd(a, b)$, if exists, is any $d \in R$ such that $d|a, b$ and for any other d' satisfying the condition, $d'|d$.

Definition 1.2. The **lcm** of a, b, denoted by $lcm(a, b)$, if exists, is any $d \in R$ such that $a, b|d$ and for any other d' satisfying the condition, $d|d'$.

Proposition 1.2. If $gcd(a, b)$ exists, then $gcd(a, b) = sup\{d : d \leq a, b\}$. If $lcm(a, b)$ exists, then $lcm(a, b) = \inf\{d : a, b \leq d\}$.

Note that maximal/minimal elements always exists by Zorn's lemma. However, the unique supremum/ infimum may not exist. We have our following example:

Example 1.1. Take $R = [\sqrt{-3}]$. Let $a = 4 = (1 + \sqrt{-3})(1 - \sqrt{-3})$ and $b = 2(1 + \sqrt{-3})$. Then, $(1 + \sqrt{-3})$ and 2 are both maximal divisors, but they are not comparable since the only divisors of 2 are $\{\pm 1, \pm 2\}$ by norm reasons, and none divides $1 + \sqrt{-3}$.

Proposition 1.3. Let $a, b \in R$ be given. Then the following hold: $gcd(a, b) = d$ exists iff (d) is the unique maximal prinipal ideal such that $(a) + (b) \subseteq (d)$. Dually, $lcm(a, b) = c$ exists iff $(c) = (a) \cap (b)$. If both holds, then $a \cdot b = lcm(a, b) \cdot gcd(a, b)$

Proof. Easy exercise. Note that the inclusion can be proper, for example, take $R = k[x, y]$ and ideals $(x), (y)$. Then (1) is the gcd, but the containment is proper. $\square$

Recall that $Id(R)$ is partially ordered by inclusion.

Definition 1.3. $(Id(R), +, \cap, \cdot, \leq)$ is the lattice of ideals of R.

Note that $+, , \cap$ are simply the sums and intersection, but $\cdot$ is the ideal generated by the products, i.e the set of finite sums of products.

Theorem 1.1. Let $Id^{\infty}(R)$ be the set of non-finitely generated ideals for R; the following are equivalent:

1. $Id^{\infty}(R)$ is non-empty;
2. There exists an infinite non-stationary chain of ideals (σ_i), where $\sigma_i \in Id(R)$;

Proof. For $1 \implies 2$, let I be a non-finitely generated ideal of R and pick $x_1 \in\in I$. Let $\sigma_1 = (x_1)$. Because the ideal is non-finitely generated, we can pick $x_2 \in I$ such that $x_2 \notin \sigma_1$. Let $\sigma_2 = (x_1, x_2)$. Continue inductively gives us an infinite non-stationary chain of ideals.

For $2 \implies 1$, take the union of all the ideals in the infinite non-stationary chain. It is an ideal and it cannot be finitely generated. $\square$

Theorem 1.2. (Cohen's lemma): Let $Id^{\infty}(R) \neq \emptyset$. Then, it has a maximal element and any such maximal element is prime.

Before proving Cohen's lemma, we need the following technical lemma:

Lemma 1.3. Let I be an ideal. Define $(I : a) := \{b \in R : ab \in I\}$. If $I + (x)$ and $(I : x)$ are both finitely generated, then I is finitely generated.

Proof of Lemma 1.3. By assumption, there is finite set $\{\alpha_i := a_i + f_i x : a_i \in I, f_i \in R, i = 1, ..., k\}$ that generate $I + (x)$, and a finite set $\{b_j : j = 1, ..., l\}$ that generate $(I : x)$. We claim that the set $\{a_i, b_j x : i \in I, j \in J\}$ generate the entire I: since $I \subseteq I + (x)$, we can write any element $\pi \in I$ as a finite linear combination $\pi = \sum_{i=1}^{k} g_i \alpha_i = \sum_{i=1}^{k} g_i(a_i + f_i x)$, where $g_i \in R$. We note that $\pi - \sum_{i=1}^{k} g_i a_i = \sum_{i=1}^{k} g_i f_i x$ is in I; it follows that $\sum_{i=1}^{k} g_i f_i \in (I : x)$, so $\sum_{i=1}^{k} g_i f_i x$ is generated by the set $\{b_j x\}$, and we are done. $\square$

With the lemma in hand, now we can prove Theorem 1.2

Proof of Theorem 1.2. Zorn's lemma implies $Id^{\infty}(R)$ has maximal elements. Let I one such maximal element, and suppose it is not prime. Then, there exists $xy \in I$ and WLOG suppose $x \notin I$. By maximality, $I + (x)$ must be finitely generated. By definition, we have $y \in (I : x)$. Lemma 1.3 implies $(I : x)$ is not finitely generated, and in particular, $I \subseteq (I : x)$. Applying maximality again, we have $I = (I : x)$, which forces $y \in I$, a contradiction. $\square$

2 Euclidean Rings

Definition 2.1. A **Principal Ideal Ring** is any ring R i which every ideal is principally generated. If R is a domain, then R is called a **PID**.

Definition 2.2. A **Factorial Ring** is any ring R in which all units can be written as a finite product of irreducible elements, unique up to a unit. If R is domain, then it is called a **UFD**.

Note that if the ring R it is not a domain, $x|y$ and $y|x$ does not imply $x = uy$ for some unit u. Let us prove that this holds for a domain: suppose $x = ys$ and $y = xt$, and $x, y \neq 0$ then $x = xts$, which implies $x(1 - ts) = 0$. This forces $1 - ts = 0$, and t, s are then units. We can concoct counterexamples when R is not a domain accordingly: let $R = k[x]/(x^3 - x)$ and take $a = x$, $b = x^2$. Clearly, $a|b$ and $b = x^2 \cdot x = x^3$, so $b|a$. However, x is not a unit.

Definition 2.3. A **Noetherian Ring** is any ring R such that any ideal is finitely generated.

Definition 2.4. Let R be a domain. A **Euclidean norm** on R is any map $\phi : R \to \mathbb{N}$ satisfying $\phi(x) = 0$ iff $x = 0$ and for every $a, b \in R$ with $b \neq 0$, then there exists $q, r \in R$ such that $a = bq + r$ with $\phi(r) < \phi(b)$. A **Euclidean Domain** is any domain equipped with a Euclidean norm.

Example of Euclidean domains include $\mathbb{Z}, \mathbb{Z}[i]$. A non-trivial example $R = F[t]$, with $\phi(p(t)) = 2^{deg(p(t))}$. A non-example is $\mathbb{Z}[\sqrt{6}]$ for it is not a PID.

Proposition 2.1. Eucldiean Domains are PIDs.

Proof. By the well-ordering principal, for every ideal I in a Euclidean domain, there exists an element other than 0 of the smallest norm. It is easy exercise that such element generate the entire ideal. ☐

Proposition 2.2. (The Euclidean Algorithm): Given $a, b \in R$, $b \neq 0$. Set $r_0 = a, r_1 = b$, and continue inductively $r_{i-1} = r_i \cdot q_i + r_{i+1}$. Then, $r_i = 0$ for $i > \phi(b)$ and if $r_{i_0} \geq 1$ maximal with $r_{i_0} \neq 0$, then $r_{i_0} = gcd(a, b)$.

Proof. Note that the remainder is strictly decreasing, so r_i must become 0 after $\phi(b)$ steps. Note that once $r_{i+1} = 0$, we have $r_i|r_n$ for all $n \leq i$. Coversely, it is clear that any divisor of a, b divdes all r_n for $n \leq i$. ☐

3 Principal Ideal Domains

Theorem 3.1. (Charaterization) For A domain R, the following are equivalent:
1. R is a PID.
2. every $p \in Spec(R)$ is principal.

Proof. One direction is trivial; for the other direction, assume that every prime is principal. Then, Cohen's Lemma implies $Id^\infty(R) \neq \emptyset$; In particular, every ideal is finitely generated, so the ring is Noetherian. We may apply Zorn's lemma on the set of non-principally generated ideal (since every chain stablizes and has a maximal element), and let P be a maximal non-principally generated ideal. Suppose it is not prime, and let $xy \in P$ with $x \notin P$. Then, $P \subset (P : x)$ and $P \subset P + (x)$ properly. By maximality, we have $(P : x) = (c)$, and $(I : c) = (d)$. By definition, we have $cd \in I$; moreover, suppose $x \in I$, then $x = cr = cdt$ for some $r, t \in R$. Thus, $I = (cd)$ is principal, a contradiction. ☐

Proposition 3.1. PIDs are UFDs.

Proof. Let $a \in R$ such that a is non-zero and not a unit. Then, there exists $p \in Spec(R)$ such that $(a) \subseteq p$. Hence R being a PID implies $\exists \pi \in R$ such that $p = (\pi)$. Hence, π must be prime and $\pi | a$. Set $a_1 = a$, $\pi_1 = \pi$, and let a_2 be the element such that $\pi_1 a_2 = a_1$. If a_2 is not a unit, find $(a_2) \subset (\pi_2)$, where π_2 is prime. Let a_3 be the element such that $\pi_2 a_3 = a_2$. Continue inductively until a_n is a unit. The process must terminate, for otherwise we get an infinite chain of distinct principal ideals (a_i) that does not stablize(stablizing is equivalent to $(a_n) = (a_{n+1})$ for some n, which implies they differ by a unit). $\square$

Corollary 3.1.1. Let R be a PID; let $P \subset R$ be a set of representatives for the prime elements up to association. For every $a \in R$, $\exists \epsilon \in R^\times$ and $e_\pi \in \mathbb{N}$ such that almost all $e_\pi = 0$. Then, every $a \in R$ can be written as $a = \epsilon \prod_{\pi \in P} \pi^{e_\pi}$. We proceed to recover gcd and lcm, up to associates.

Note that the above corollary generalizes to the quotient field by replacing $\mathbb{N}$ with $\mathbb{Z}$.

4 Unique Factorization Domains

Definition 4.1. The following are equivalent for a domain R:

1. R is a UFD.
2. Every minimal prime ideal (prime of height 1) is principal and every non-zero, non-invertible elements in contained in finitely many primes.

Proof. $1 \implies 2$: For every non-zero prime P, pick $x \in P$ has factor. One of the prime factors must be in P, and it follows by minimality that P must be generated by such prime factor. For the second part, the finite factorization of the element gives precisely the finite primes that it is contained in. $2 \implies 1$:given $x \in R$, the finitely many primes containing x are principally generated by prime elements, which gives a factorization.

$\square$

Remark: we recover the gcd and lcm definition using the same factorization as Corollary 3.1.1.

Theorem 4.1. (Gauss Lemma)Let R be a UFD; then $R[t]$ is a UFD.

To prove the theorem, we need the following lemma on contents:

Definition 4.2. Let $f(t) = a_0 + ... + a_n t^n$ be given. Then, the **content** of f, denoted $C(f)$, is the GCD of all coefficients. In particular, $C(f) | a_i$ for all i, and $f_0 := f/(C(f))$ has content 1.

Lemma 4.2. Let R be a UFD, then the following hold: (1). $C(f) : R[t] \to R$ given by $f \mapsto C(f)$ is multiplicative; in particular, if $C(f) = C(g) = 1$, then $C(fg) = 1$.

Proof of lemma 4.2. given $f(t) = a_0 + ... + a_n t^n$ and $g(t) = b_0 + ... + b_m t^m$. If one of f, g is constant, then it is easy exercise; suppose neither is constant, then set $f = f_0 \cdot C(f)$ and $g = g_0 \cdot C(g)$. Clearly we have $C(f) \cdot C(g) | C(fg)$. Hence it suffices to prove that $C(f_0 g_0) = 1$. Equivalently, let $\pi \in R$ be a prime element, we want to show there exists a coefficient $c_k \in f_0 g_0$ such that π does not divide c_k. Suppose

$\pi | c_k = \sum_{i+j=k} a_i b_j$ for all k. Because $C(f_0) = C(g_0) = 1$, then there exists minimal a_i, b_j such that π does not divide a_{i_0}, b_{j_0}. Then, π does not divede $C_{i_0+j_0}$.

□

Note that proof goes similarly for quotient fields.

Theorem 4.3. Let R be a UFD. For $f(t) \in R[t]$, the following are equivalent:

1. $f(t)$ is prime
2. $f(t)$ is irreducible
3. If $f = a_0 \in R$ and a_0 is prime or $C(f) = 1$ and f is irreducible.