Étale Homotopy Theory and Adams Conjecture

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Definition 0.0.1 (Čech Nerve). Let X be a finite CW complex, and $\mathcal{U} := \{U_i : i \in I\}$ be an open cover of X. Then, we may define a simplicial set call the **Čech Nerve** $N\mathcal{U}$ as follows: we have the assignment on objects $[n] \mapsto \{\text{functions from } [n] \text{ to } I : \cap_{i=1}^{n} U_{f(i)} \neq \emptyset\}$. The face maps and degeneracy maps are defined by deleting and inserting appropriate indices.

Alternatively, we can think of a covering $\mathcal{U}$ as follows: suppose given a covering $X = \cup_{i\in I} U_i$; let $\mathcal{U} = \coprod_{i\in i} U_i$, and the covering is the obvious map $\mathcal{U} \to X$. Note that we have

$$U_i \cap U_j = U_i \times_X U_j$$

so the n-fold fiber product $U \times_X ... \times_X U$ is the disjoint union of n-fold intersections of opens in the cover. Then, the nth simplices of the Čech nerve is $\pi_0(\underbrace{U \times_X ... \times_X U}_{n-fold})$. The face maps are projections, and the degeneracy maps are various diagonal embeddings.

Theorem 0.1. If the covering $\mathcal{U}$ satisfies the property that arbitrary intersections of opens in the cover is either empty or contractible, then th realization $|N\mathcal{U}|$ is weakly equivalent to X.

1 Adam's conjecture

Definition 1.0.1. Let X be compact Hausdorff and let $KU(X)$ be Grothendieck group of complex vector bundles over X, and let $\mathcal{SF}(X)$ be the Grothendieck group of sphere bundles over X modulo fiber homotopy equivalence.

Theorem 1.1. The stable sphere bundles over X is classified by the the groups of self-homotopy equivalences of S^n, which we denote by $G(n) := \text{Equiv}(S^n, S^n)$.

Proposition 1.1.1. The complex J-homomorphism $J : KU(X) \to \mathcal{SF}(X)$ is induced by a map between classifying spaces, which we also denote

$$J : BU \to BG := \varinjlim_n BG(n)$$

Definition 1.1.1. The Adams' operation $\psi^k : KU(X) \to KU(X)$ is induced by a map of classifying spaces

$$\psi^k : BU \to BU$$

Theorem 1.2 (The Adams Conjecture)**.** The composite

$$BU \xrightarrow{\psi^k - 1} BU \xrightarrow{J} BG$$

is nullhomotopic up to multiplication by some k^n.

Proposition 1.2.1. The composite $J \circ i : BU(n) \to BU \to BG$, classifyies a sphere bundle over $BU(n)$, and is fiber homotopy equivalent to the fibration

$$BU(n-1) \to BU(n)$$

Fact: the composition $J \circ i : BU(n) \to BG$ classifies the sphere bundle associated to the canonical bundle over $BU(n)$, and it is fiber homotopy equivalent to the fibration

$$BU(n-1) \to BU(n)$$

We can dream of a proof here: if we have unstable adams operations $\psi^k : BU(n) \to BU(n)$, which are homotopy equivalences, with a pullback diagram

$$\begin{tikzcd} BU(n-1) \arrow[r, "\psi^k"] \arrow[d, "i"] & BU(n-1) \arrow[d, "i"] \\ BU(n) \arrow[r, "\psi^k"'] & BU(n) \end{tikzcd}$$

Then, the bundle $J \circ i$ is fiber homotopy equivalent $J \circ \psi^k$. However, unstable Adams operation does not exist on $BU(n)$. However, Sullivan proves that

$$\begin{tikzcd} \widehat{BU(n-1)}_p \arrow[r, "\psi^k"] \arrow[d, "i"] & \widehat{BU(n-1)}_p \arrow[d, "i"] \\ \widehat{BU(n)}_p \arrow[r, "\psi^k"'] & \widehat{BU(n)}_p \end{tikzcd}$$

where ψ^k is an unstable Adams operation on the profinite completion.

2 Algebraic Side

Recall that the classifying space $BU(n)$ is constructed as the direct limit of complex grassmannians $\varinjlim_k Gr_n(k)$. Via Plücker embeddings, the complex grassmannians are naturally affine complex varieties embedded in projective space. Moreover, the defining polynomials also have coefficients in $\mathbb{Q}$.(Example here?)

Thus, an automorphism $Gal(\mathbb{C}|\mathbb{Q})$ gives rise to an automorphism of these varieties, but they are wildly discontinuous in the classical topology. However, the Cech/etale story tells us that the absolute galois group acts on the profinite homotopy type of the grassmannians, preserving the filtration.

By naturality and splitting principal, understanding the action of $Gal(\mathbb{C}|\mathbb{Q})$ on the profinite complex K-theory reduces to understanding the action on $\cup_n \widehat{\mathbb{CP}^n} \cong K(\widehat{\mathbb{Z}}, 2)$, which can be checked to be the composition

$$Gal(\mathbb{C}|\mathbb{Q}) \to \widehat{\mathbb{Z}}^* \to K(\widehat{Z}, 2)$$

Idea of proof