MATH 618 Algebraic Topology

David Zhu

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1 The Correct Category

Let T = compactly generated weakly Hausdorff. Let T_2 = pairs of spaces (X, A), $A \subseteq X$, and

$$T_2((X,A),(Y,B)) = \{f \in T(X,Y) : f(A) \subseteq B\}$$

We define $(X,A) \otimes (Y,B) = (X \times Y, X \times B \cup Y \times A)$. (Think about product of boundaries). We want tot understand the analogue of $T(X \times Y, Z) \cong T(X, T(Y,Z))$.

Theorem 1.1. Let $(X,A),(Y,B),(Z,C) \in T_2$, then

$$T_2((X,A)\otimes(Y,B),(Z,C)) \cong T_2((X,A), T_2(T_2((Y,B),(Z,C))), T(Y,C))$$

Let T_* be the full subcategory of T_2 consisting of pairs $(X,*)$. There exists a pair of functors $T_2 \to T_*$ defined by $(X,A) \mapsto (X/A, A/A = *)$.

Proposition 1.1.1. $q : X \to X/A$, we get

$$q* : T_*(X/A, Y) \to T_2((X,A),(Y,*))$$

an isomorphism.

We want a producgt in T_* which works well with function spaces:

Definition 1.1.1. Given $X, Y \in T_*$, defined $X \wedge Y = (X \times Y / X \vee Y, * = X \vee Y)$ called the **smash product**.

Note that the smash product is not the categorical product here.(The categorical product is simply the cartesian product carring the canonical basepoint).

Definition 1.1.2. The **reduced suspension** $\Sigma X := S^1 \wedge X$.

Theorem 1.2. The catgegory of based spaces T_* has the following properties

1. $T_*(X, Y) \in T_*$, with basedpoint the constant map to basepoint.
2. $T_*(X, Y) \wedge X \to Y$ is continuous.
3. $T_*(X, Y) \wedge T_*(Y, Z) \to T_*(X, Z)$ is continuous.
4. $T_*(X \wedge Y, Z) \cong T_*(X, T_*(Y, Z))$
5. Small limits and colimits exists in T_*.
6. The forgetful functor $T_* \to T$ preserves limits.

Definition 1.2.1. The **reduced cone** is a functor $C : T_* \to T_*$ defined by $X \mapsto IX$, with the basepoint of I being 1.

Proposition 1.2.1. There exists a pushout

$$\begin{tikzcd} X \arrow[r] \arrow[d] & CX \arrow[d] \\ CX \arrow[r] & \Sigma X \end{tikzcd}$$

Definition 1.2.2. The **loop space** is defined to be $T_*(S^1, X)$.

Theorem 1.3. (Eckmann-Hilton duality)

$$T_*(X, \Omega Y) \cong T_*(\Sigma X, Y)$$

There exists a functor $T \to T_*$ given by $X \mapsto X \coprod \{x\}$, which is left adjoint to the forgetful functor. If $X \in T_*$, then $\Omega X \in T_*$, and note that $\pi_0(\Omega X) = [S^1, X]_* = \pi_1(X)$.

Definition 1.3.1. A functor $F : T_* \to C$ for some catgeory C is called a **homotopy functor** if $F(f) = F(g)$ when $f \cong g$

Definition 1.3.2. Define $\pi_n(X) := [S^n, X]_*$. Then, π_n is a homotopy functor from T_* to set.

Proposition 1.3.1. π_n is a group for $n \geq 1$ and abelian when $n \geq 2$.

Proof. The group structure is given by: suppose we have $\varphi, \psi : I^n \to X$. Then, $\varphi + \psi : I^n \to X$ is explicitly given by

$$\varphi + \psi(t_1, ..., t_n) = \begin{cases} \varphi(2t_1, t_2, ..., t_n) & t \in [0, \frac{1}{2}] \\ \psi(2t_1 - 1, t_2, ..., t_n) & t \in [\frac{1}{2}, 1] \end{cases}$$

$\square$

Definition 1.3.3. Define H-space be a topological space X with homotopy associative map

$$X \wedge X \to X$$

Proposition 1.3.2. If (Y, y_0) is an H-space, then $[X, Y]_*$ has a group structure.

Lemma 1.4. Let $C = (C^{*,*}, d, \delta)$, be a double complex, (C^*, d) a complex with $\epsilon : C^* \to C^{*,0}$ such that $d\epsilon + \epsilon d = 0$. Furthermore, assume that the columns are exact(in particular, this implies that $H^q(C^{p,*}, \delta) \cong C^p$ for $q = 0$). Then, $\epsilon : H^p(C, d) \to H^p(\mathrm{Tot}(C), D)$ is an isomorphism.

Proof. standard augmentation/resolution lemma. Mattie you can do this. □

Let M be a smooth manifold. Suppose M has a good cover $U = \{U_\alpha\}$ such that each intersection $U_{\alpha_1,\ldots,\alpha_n}$ is either empty or contractible. (Note that you can always do this for a Riemannian manifold). We define a double complex as follows: Let

$$C^{p,q} = \prod_{\alpha_0,\ldots,\alpha_q} \Omega^q(U_{\alpha_0,\ldots,\alpha_q})$$

The horizontal differentials $d : C^{p,q} \to C^{p+1,q}$ is $d(\omega \in \Omega^p(u_{\alpha_0,\ldots,\alpha_q})) = d\omega \in \Omega^{p+1}(U_{\alpha_0,\ldots,\alpha_q})$, which is just the De Rham differentials.

The vertical differential $\delta : C^{p,q} \to C^{p,q+1}$ is defined to by: let $\omega \in \prod_{\alpha_0,\ldots,\alpha_q} \Omega^p(U_{\alpha_0,\ldots,\alpha_q})$

$$\delta(\omega)_{\alpha_0,\ldots,\alpha_{q+1}} = \sum_{j=0}^{q+1} (-1)^j \omega_{\alpha_0,\ldots,\widehat{\alpha_j},\ldots,\alpha_{q+1}}|_{U_{\alpha_0,\ldots,\widehat{\alpha_j},\ldots,\alpha_{q+1}}}$$

which is the Cech differential.

It is easy to check that $\delta^2 = 0$ and $d^2 = 0$ and $d\delta = \delta d$.

We have an agumentation of the double complex by the De Rham complex, and apply the previous lemma. The trick to checking the columns are exact is to let λ_α be a partition of unity for U_α, and define a chain contraction $s : C^{p,q} \to C^{p,q-1}$ defined by $(s\omega)_{\alpha_0,\ldots,\alpha_{q-1}} = \sum \lambda_{\alpha_0} \omega_{\alpha_0,\ldots,\alpha_{q-1}}$. Thus, we get that $H^p(\Omega M) \cong H^p(\mathrm{Tot}(C), D)$.

Similarly, there is an augmentation of the Cech comomplex $C^{\vee,q}(U) = \prod_{\alpha_0,\ldots,\alpha_q} \mathbb{R}$, with $\delta : C^{\vee q} \to C^{\vee q+1}$ defined by $\delta(c)_{\alpha_0,\ldots,\alpha_{q+1}} := \sum (-1)^j c_{\alpha_0,\ldots,\widehat{\alpha_j},\ldots,\alpha_{q+1}}$. We may check that the rows are exact directly this time, and that the cohomology of the complex is isomorphic to the cech cohomology $H^*(C^{\vee,*}, \delta)$ by the row analog of the lemma. In the end, we have

Theorem 1.5 (De Rham Theorem)**.**

$$H^*_{DR}(M) \cong H^*(\Omega^* M) \cong H^{\vee,*}(M)$$

Definition 1.5.1. A <u>**spectral sequence**</u> is a sequence of abelian groups and differentials (A_n, d_n), such that $d_n : A_n \to \overline{A_n}$, such that $A_n \cong H^*(A_n, d_n)$ (Block's notation A_n is the page). A morphism of spectral sequences is a sequence of map $A_n \to E_n$ that commutes with all differentials and turning the page.

Theorem 1.6 (Serre Spectral Sequence)**.** Let $\pi : Y \to X$ be a Serre fibration, with fiber F. Then, there exists a spectral sequence $(E_r^{p,q}, d_r)$, with the differentials going the way

$$d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$$

If X is simply connected, then the E_2 page is given by

$$E_2^{p,q} = H^p(X; H^q(F)) \Rightarrow H^{p+q}(Y)$$

Example 1.6.1. Consider the fibration $S^1 \to S^{2n+1} \to \mathbb{CP}^n$. Then, $E_2^{p,q} = H^p(\mathbb{CP}^n, H^q(S^1))$. You can draw the E_2 page of the spectral sequence, and find that the non-trivial terms concentrate on row $0, 1$ by our knowledge of the cohomology of S^1. Moreover, we know the cohomology of $\mathbb{CP}^n$ is only in even degrees, so we only get non-zero terms in the even columns. Combining the information, the E_2 page is pretty clear to see. You can work out when the spectral sequence collapses and how it converges.