MATH 624 Algebraic Geometry

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1 Prevarieties and Varieties

We will assume that $K|k$ a finite extension, K is algebraically closed. We will use $\mathbb{A}^n(K) = K^n = \mathbb{A}^n_K$ to denote the underlying set, not the n-dimensional affine space. Given a point $a = (a_1, ..., a_n) \in \mathbb{A}^n_k$, we will use φ_a to denote the evaluation map $k[X] \to k$. Similarly, given $f \in k[x]$, we have the evalation map $\tilde{f} : \mathbb{A}_k \to k$. This gives a morphism of k-algebras $k[x] \to Maps_k(\mathbb{A}_k, k)$ given by $f \mapsto \tilde{f}$.

Definition 1.0.1. Given $\Sigma \subset k[x]$, define $V(\Sigma) = \{a \in \mathbb{A}_k : f(a) = 0 \text{for every } f \in \Sigma\}$. This is called the **affine k-algebraic set** defined by Σ. If $\Sigma = \{f\}$, then $H_f := V(\Sigma) = V(f)$ defines a **hyperplane** in $\mathbb{A}_k$.

Example 1.0.1. Easy examples

1. $V((0)) = \mathbb{A}_k$.
2. $V((1)) = \emptyset$
3. Let $k = \mathbb{C}$. Then, in $\mathbb{A}^1_k$, $V(x^2 - 1) = \{\pm 1\}$. In $\mathbb{A}^2_k$, $V(x^2 - 1) = \{(\pm 1, n) : n \in k\}$

Definition 1.0.2. Given $V \subset \mathbb{A}_\neg$, defined $I(V) = \{f \in k[x] : f(V) = 0\}$. This is called the **ideal** of V.

Proposition 1.0.1.
1. Let $I_\Sigma \subset k[x]$ be the ideal generated by Σ. Then, $V(\Sigma) = V(I)$.
2. There exists a finite system $f_1, .., f_m$ such that $V(\Sigma) = V(f_1, ..., f_m)$
3. If $\Sigma_1 \subset \Sigma_2$, then $V(\Sigma_1) \supset V(\Sigma_2)$
4. Given $\mathfrak{a}$ an ideal, then $I(V(\mathfrak{a})) = \mathfrak{a}$ iff $\mathfrak{a} = \sqrt{\mathfrak{a}}$.
5. Given ideals $\mathfrak{a}, \mathfrak{b}$, then $V(\mathfrak{a}) = V(\mathfrak{b})$ iff $\sqrt{\mathfrak{a}} = \sqrt{\mathfrak{b}}$.

Definition 1.0.3. Let $\mathcal{A}^n_K := \{V \subset \mathbb{A}^n_K : V \text{ affine } k-\text{algebraic sets}\}$. Given $V \in \mathcal{A}^n_K$, let $k[V] := k[x]/I(V)$ be the **affine coordinate ring** generated by V.

Let $Id^{rd}(k[x])$ be the set of reduced ideals of $k[x]$. Let R_n be the set of reduced k-algebras with n-generators.

Theorem 1.1. There is a canonical bijection between the set of reduced affine k-algebras and reduced ideals of $k[x]$, given by the maps

$$R_n \to Id^{re}(k[X]) \to \mathcal{A}_K^k$$

$$k[\underline{x}] \mapsto \mathfrak{a} := ker(k[x] \xrightarrow{f} k) \mapsto V(\mathfrak{a})$$

with f given by $x \mapsto \underline{x}$.

1.1 The Zariski Topology

Given $V \in \mathcal{A}_K^n$, there is a canonical map $K[X] \to K[V]$ given by $f \mapsto f_V$.

Proposition 1.1.1. Let $\Sigma_i \subset k[X]$, and $f \in k[X]$ be given. then
1. $V(\cup_i \Sigma_i) = \cap_i V(\Sigma_i)$
2. $V(\prod \Sigma_i) = \cup V(\Sigma_i)$
3. $V((0)) = \mathbb{A}_k^n$; $V((1)) = \emptyset$

By the proposition above, we can define the Zariski topology on $\mathbb{A}_k^n$

Definition 1.1.1. The Zariski topology on $\mathbb{A}_K^n$ is given by the closed sets $V(\Sigma)$, with $\Sigma \in k[X]$. In particular, the sets $D_f := \mathbb{A}_k^n - H_f$ is an open set and forms a basis for the topology.

Note that the zariski topology on product spaces is not the product of zariski topologies. Moreover, the connectedness/irreducibility is dependent on $K|k$. A point is called a generic point of V if its closure contains V.

Example 1.1.1. If $K|k = \mathbb{C}|\mathbb{Q}$, then $V(x_1^2 - 2x_2^2)$ is connected and irreducible. If $K|k = \mathbb{C}|\mathbb{Q}[\sqrt{2}]$, then $V(x_1^2 - 2x_2^2)$ is connected but not irreducible.

Remark 1.1.1. For a topological space, X, the following are equivalent:
1. Every descending chain of closed subsets is stationary.
2. Every ascending chain of open subsets is stationary.

A topological space satisfiying the above is called **Noetherian**. For example, $Spec(R)$ is Noetherian if R is Noetherian. Note that if X is Noetherian, then it is automatically quasi-compact. Moreover, there are only finitely many irreducible components and connected components of X.

Proposition 1.1.2. The following hold:
1. The Zariski topology is Noetherian on $\mathbb{A}_K$, therefore also on any $V \in \mathcal{A}_K^n$.
2. For every $V \in \mathcal{A}_K$, there are only finitely many irreducible components and connected components.
3. $V \in \mathcal{A}_K$ is irreducible iff $I(V)$ is a prime ideal.
4. Given $V_0 \subset V$, V_0 is irreducible iff $I_V(V_0) := I(V_0)/I(V) \in Spec(k(V))$ is minimal.
5. The connected components in $V \in \mathcal{A}_K$ correspond bijectively to the indecomposable idempotents of $k[V]$.
6. For $V \in \mathcal{A}_K$, $a \in V$ is a generic point iff the evaluation map $k[V] \to k[a]$ is an isomorphism of k-algebras.

Definition 1.1.2. Let T be a topological space, and let $V \subset T$.

1. dim(V):=sup { chain of irreducible components ending in V: }
2. codim(V):=sup { chain of irreducible components starting with V and ending in T: }

Note that if $V = \cup V_\alpha$, then $dim(V) = supdim(V_\alpha)$, and similarly for codimensions. Moreover, $dim(V) = dim(\overline{V})$.

Proposition 1.1.3. (Notions of dimension) Let $V \in \mathcal{A}_K$ be irreducible. Then, the dimension of V is the same as the krull dimension of $K[V]$.

Proposition 1.1.4. Suppose irreducible $W \subset V \in \mathcal{A}_K$. Then,

$$dim(W) + codim_V(W) = dim(V)$$

Proposition 1.1.5. $V \in \mathcal{A}_K$ has generic points a iff $td(K|k) \geq dim(V) = td(k(V))$.

1.2 Base change and Rational Points

Definition 1.1.3. Suppose there is an embedding

$$\begin{tikzcd} K \arrow[r] & L \\ k \arrow[u] \arrow[r] & l \arrow[u] \end{tikzcd}$$

Then, there is a natural morphism $k[x] \to l[x]$, which induces a pushforward of ideals and a map $\mathcal{A}_K \to \mathcal{A}_L$. Take the vanishing locus of the pushforward of $I(V)$ gives the base change of V.

Remark 1.1.2. Base change does not preserve connectedness or irreducibility.

Definition 1.1.4. $V \in \mathcal{A}_K$ is called **absolutely (geometrically) irreducible** if V_l is irreducible for all field extension $l|k$. It is **geometrically connected** is V_l is connected for all $l|k$.

Proposition 1.1.6. Let $V \in \mathcal{A}_K$ be affine k-algebraic set. Then the following are equivalent:

1. V is absolutely irreducible.
2. V_{k^s} is irreducible.
3. $V_{\overline{k}}$ irreducible.

The key observation is that $K^s[x] \to \overline{k}[X]$ is an integral extensions of domains. Therefore, we have going up and going down, and it straightforward to show that $Spec(k^s[X]) \to Spec(\overline{k}[X])$ is a homeomorphism. Thus, we have (2) $\implies$ (3).

To (3) $\implies$ (1), apply the following:

Lemma 1.2. For every $V \in \mathcal{A}_K$, one has $V(\overline{k})$ is zariski dense in V. Therefore, $V_{\overline{k}}$ irreducible implies V irreducible.

The proof is exercise. The key point is that if there exists f with k-coefficients such that f vanishes on all of A

Proposition 1.2.1. Let $V \in \mathcal{A}_K$ be affine k-algebraic set. Then the following are equivalent:

1. V is geometrically connected.
2. V_{K^s} is connected.
3. $V_{\overline{k}}$ is connected.

2 The category of quasi-affine k-algebraic sets

Definition 2.0.1. A **quasi-affine** k-algebrac set is any zariski open subset $U \subset V$ for $V \in \mathcal{A}_K$.

The complement of hyperplanes is a basis of quasi-affine k-algebraic sets. Let $V \in \mathcal{A}_K$ be non-empty, $f \in K[V]$. Then, the evaluation map $f : V \to \mathcal{A}_K$ is continuous. Moreover, $\varphi = (f_1, ..., f_n)$ is also continuous.

Definition 2.0.2. Let $V \in \mathcal{A}_K$ and $\mathcal{V} \subset V$ be zariski dense. Then, a functions $\varphi : \mathcal{V} \to \mathcal{A}_K$ is called **regular** at $x \in V$ if there exists $f_x, g_x \in k[x]$ and $\S \subset V$ such that $g_x \neq 0$ everywhere on $\mathcal{U}_x$ and $\varphi = \frac{f_x}{g_x}$. A fucntion $\varphi : \mathcal{V} \to \mathcal{A}_K$ is **regular** if it is regular at every point in V. Let $\mathcal{O}_x := \{\varphi \in Maps(\mathcal{V}, K) : \varphi \text{ regular at } x\}$. Define an equivalence relation on $\mathcal{O}_x$ by equivalence on any open neiborhood around x. $\mathcal{O}(V)$ is the set of regular functions on V.

Proposition 2.0.1. (rings of regular functions) We have the following:

1. $k[V] \to \tilde{\mathcal{O}}(V)$ is an isomorphism of k-algebra.
2. $k[V]_f \to \tilde{O}(U_f)$ is an isomorphism of k-algebra.

It is helpful to remember that Zariski open sets are dense. Thus, it suffices to show that a function is zero on a basic open U_f to deduce it is globally zero.

3 Presheaves and Sheaves

Definition 3.0.1. Let $\mathcal{C}$ be a concrete category such as **Top, Set, Ab**. Let X be a topological space with topology τ_X. Then, τ_X is naturally poset category where morphisms are inclusions. A **presheaf** is a contravariant functor $\mathcal{P} : \tau_X \to \mathcal{C}$.

Explicitly, $\mathcal{P}$ is given by two data: 1.$\mathcal{P}(U) \in Obj(\mathcal{C})$ for every $U \in \tau_X$. 2.$\rho_{u',u''} : \mathcal{P}(U'') \to \mathcal{P}(U')$ for every $U' \subset U''$. The elements in the set $P(U)$ are called **sections** above U. The image of a section under ρ is called the **restriction**.

Definition 3.0.2. A presheaf is a **sheaf** if it has the covering preperty: given an open cover of an open set $U = \cup_i U_i$, with $U + i, j := U_i \cap U_j$ with $s_i \in \mathcal{P}(U_i)$ such that $\rho_{U_i, U_{i,j}}(s_i) = \rho_{U_j, U_{i,j}}(s_i)$, then there exists $s \in \mathcal{P}(U)$ such that $s_i \in \rho_{U, U_i}(s)$ for every U_i.

Definition 3.0.3. Suppose that limits exists in $\mathcal{C}$. Then $\mathcal{P}_x := \mathcal{P}(U_x)$ is called the **stalk** of $\mathcal{P}$ at x.

Proposition 3.0.1. $\mathcal{P}$ is a sheaf iff for every $U \in \tau_X$, the map $\varphi_U : U \to \coprod_{x \in U} \mathcal{P}_x$ is injective.

Proposition 3.0.2. For every presheaf $\mathcal{P}$, there is a sheafification functor $\mathcal{P} \to \mathcal{F}$ that induces isomorphism on stalks.

Definition 3.0.4. Let $f : X \to Y$ be a continuous map of topological spaces. Then,

1. Given a (pre)sheaf $\mathcal{P}$ on X, then the **direct image** (pre)sheaf $f_*\mathcal{P}$ on Y is defined by $f_*\mathcal{V} := \mathcal{P}(f^{-1}(V))$ for all $V \in \tau_Y$. In particular, the direct image sheaf is also a sheaf.
2. Given a presheaf $\mathcal{P}$ on Y. There is an **inverse image** sheaf $f^{-1}\mathcal{P}$ on X defined by the limit:

$$f^{-1}\mathcal{P}(U) := \varinjlim_{U \subset U'} \mathcal{P}(f(U'))$$

where $U \subset U'$ and $f(U')$ is open.

Remark 3.0.1. Note that the preimage sheaf is always a preseeaf, but not necessarily a sheaf.

Definition 3.0.5. A (locally) **ringed space** is a pair $(X, \mathcal{F})$, where X is a topological space and $\mathcal{F}$ a sheaf of rings on X such that the stalks at each point is a local ring.

Definition 3.0.6. Given locally ringed spaces $(X, \mathcal{F})$, $(Y, \mathcal{G})$, a morphism of locally ringed space is a pair $(f, f^\sharp)$ such that $f : X \to Y$ is continuous and $f^\sharp : \mathcal{G} \to f_*\mathcal{F}$ a morphism of sheaves.

4 Back to Varieties

Proposition 4.0.1. Let V be an affine k-algebraic set, $U \subset V$ zariski open.

1. The assignmenet τ_U, $U' \mapsto \tilde{O}(U')$ defined a locally ringed space on U.
2. A morphism of quasi-affine algebraic set $T \to U$ is any morphism of locally ringed spaces $(f, f^\sharp) : (T, \mathcal{O}_T) \to (U, \mathcal{O}_U)$

The checks are fullfilled by proposition 2.0.1.

Proposition 4.0.2. Let $(T, \mathcal{O}_T), (U, \mathcal{O}_U)$, and $\Phi : T \to U$ continuous. Then,

1. Φ defined a morphism of locally ringed spaces iff $\mathcal{O}_U \circ \varphi \subset \mathcal{O}_T$, i.e for every U and T' open such that $\Phi(T') \subset U'$ and $\varphi \in \mathcal{O}_U(U')$, then $\varphi \circ \Phi \in \mathcal{O}_T(T')$.
2. Suppose Φ defines such a morphism, and let $U \subset \mathbb{A}_K$, $p : \mathbb{A}_K^n \to K$ the ith projection, then $p_i|_U \circ \Phi$ completely determines Φ.

Remark 4.0.1. Let $U_f := \{x \in V | f(x) \neq 0 :\}$ be a basic open. Consider $W_f \subset \mathbb{A}_K^n$ defined by $W_f := \{(a, b) | a \in \mathbb{A}_K^n, b \in \mathbb{A}_K^1 : f(a)b - 1 = 0\}$ is an algebraic set in $\mathbb{A}_K^{mn}$. Prove that $\Phi : W_f \to U_f$ given by $(a, b) \mapsto a$ is an isomorphism of quasi affine k-algebraic sets. Then inverse is given by $\psi : U_f \to W_f$ given by $a \mapsto (a, \frac{1}{f(a)})$.

Proposition 4.0.3. Every quasi-affine k-algebraic set contains a n zariski dense k-algebraic set.

Definition 4.0.1. A quasi-affine k-algebraic set is called **affine** if it is isomorphic as a locally ringed space to an affine k-algebraic set.

Theorem 4.1. The following hold:

1. The catgeory of K-valued affine k-algebraic sets, $\mathcal{A}_k$, is anti-equivalent to the category of reduced k-algebras of finite type. In particular, a k-algebraic set $V \subset \mathcal{A}_K$ is mapped to $k[V]$. Note that the projection maps $V \to W \to \mathcal{A}_k$ defined a regular function on V, and by proposition 4.0.2 determined the morphism of the algebraic set. There is a canonical map from the ring of regular functions on V to the coordinate ring $k[V]$ by proposition 2.0.1.
2. Let U be a quasi-affine k-algebraic set, W and affine k-algebraic set. Then, a morphism $\Phi : U \to W$ is determined by a map $\Phi^* : k[W] \to \tilde{O}(U)$.

Definition 4.1.1. $\mathcal{A}_k^n := (\mathcal{A}_K^n, \tilde{O}_{\mathcal{A}_K^n})$ is called the **n-dimensional affine sapce.**

Definition 4.1.2. An **open immersion** of quasi-affine k-algebriac setd $j : U \to T$ is any k-morphism which is a zariksi open immersion and $\tilde{O}_U = \tilde{O}_T \circ j$

Definition 4.1.3. A **closed immersion** of quasi-affine k-algebraic sets $i : U \to T$ is a topological closed immersion and $i_* O_U$ is a factor sheaf of $\mathcal{O}_T$. In other words, the map $\Phi* : \tilde{\mathcal{O}}_T(T') \to \tilde{\mathcal{O}}_U(U')$ is surjective.

Definition 4.1.4. A k-**prevariety** is any quasi-compact locally ringed space X that is locally isomorphic to K-valued affine k- algebraic sets. Locally isomorphic here means that there exists an finite open cover $X = \cup X_\alpha$ and isomorphism of locally ringed spaces $\varphi_\alpha : X_\alpha \to V_\alpha$, where V_α is affine k-algebraic set. Moreover, the transition maps are isomorphisms of quasi-affine k-algebraic sets.

Remark 4.1.1. A **k-morphism** of k-prevarieties is a morphism of locally ringed spaces, such that there exists $X = \cup X_\alpha, Y = \cup Y_\alpha$ and $f(X_\alpha) \subset Y_\alpha$, and the structure maps induce a map of affine k-algebraic sets.

Definition 4.1.5. Let $f : X \to Y$ be a k-morphism of k-prevarieties. Then,

1. f is an open immersion iff f induced structure maps is an open immersions of affine k-algebraic sets.
2. f is an closed immersion iff f induced structure maps is a closed immersions of affine k-algebraic sets.
3. X is called affine if it is isomorphic as a k-prevariety to an affine k-algebraic set.
4. X is called quasi-affine if there is an open immersion into a affine k-prevariety.

Proposition 4.1.1. (Gluing datat for k-prevarieties and k-morphisms)

1. (X_i) be a finite set of k-prevarieties.
2. $X_{ij} \subset X_i$ open for every i, j
3. $\varphi_{ij} : X_{ij} \to X_{ji}$ a $k-$isomorphism such that $\varphi_{ii} = Id$, $\varphi_{ij} = \varphi_{ji}^{-1}$ and $\varphi_{ij} \circ \varphi_{jk} = \varphi_{ik}$.
4. A solution is $X = \cup X'_i$ and k-isomorphisms $X'_i \to X_i$

Remark 4.1.2. The solution is unique up to k-isomorphism.

Proposition 4.1.2. (Gluing morphisms)Suppose $f_\alpha : X_\alpha \to Y_\alpha$ such that

$$\begin{tikzcd} X_{\alpha\beta} \arrow[r, "f_\alpha"] \arrow[d, "\varphi_{\alpha\beta}"'] & Y_{\alpha\beta} \arrow[d, "\psi_{\beta\alpha}"] \\ X_{\beta\alpha} \arrow[r, "f_\beta"'] & Y_{\beta\alpha} \end{tikzcd}$$

The there exists a unique k-morphism $X \to Y$ compactible with the gluing data.

The idea of proof for 4.1.1 and 4.1.2 is to take the disjoint union of the topological spaces first and define an equivalence relation accordingly. Then, define the structure sheaf on the quotient as the unique sheaf of k-algebras such that $\mathcal{O}_X|_{i_\alpha(X_\alpha)} = (i_\alpha)_* \mathcal{O}_{X_\alpha}$. We can check that $\mathcal{O}_X$ is well-defined. The glued morphism is as expectedly induced by morphisms from the glued components.

Example 4.1.1. (Line with two origins)Let $X_1, X_2 = \mathbb{A}^1$, and $U_{12} = U_{21} = \mathbb{A}^1 - \{0\}$, and let $\varphi : U_{12} \to U_{21}$ be the identity.

Example 4.1.2. (The projective line)Let $X_1, X_2 = \mathbb{A}^1$, and $U_{12} = U_{21} = \mathbb{A}^1 - \{0\}$, and let $\varphi : U_{12} \to U_{21}$ be $\varphi(x) = \frac{1}{x}$.

Theorem 4.2. Let X be a k-prevariety, and V be an affine k-prevariety. Then, one has a canonical bijection

$$Hom_k(k[V], \mathcal{O}_X(X)) \to Mor_K(X, V)$$

Proof uses Theorem 4.1 part 2. Break up the morphisms by $X = \cup X_\alpha$, where each X_α is affine. Then use the gluing theorems to glue back.

Theorem 4.3. Finite products and coproducts exists in the catgeory of affine k-algebraic sets. The coproduct correspond to the product of the affine rings. The product correspond to the reduced tensor product of affine k-algebras.

Remark 4.3.1. Note that the k-tensor algebra of two reduced k-algebras might no longer be reduced. Therefore we need to take the quotient by the nilradical.

Theorem 4.4. The category of k-prevarieties has finite products and coproducts.

Proof. Let T be the category of locally ringed spaces in k-algebras. Let T_0 be a subcategory that is

1. closed under open immersions.
2. closed under finite products.
3. Every object in T can be glued from thatof T_0.

Then, products exists in T. Take T to be category of k-prevarieties and T_0 be subcategory quasi-affine k-prevarieties. The hard part is to show that T_0 has all fintie products. □

Proposition 4.4.1. Let $f : X' \to X$. $g : Y' \to Y$ morphisms of k-prevarieties. Then, TFH
1. There exists a canonical k-morphisms $f \times g : X' \times Y' \to X \times Y$
2. f, g are open/closed immersions iff $f \times g$ is.
3. The diagonal morphism is a closed immersion of k-varities iff it is a topological closed immersion.

Definition 4.4.1. Let $\mathcal{T}$ be a category of topological spaces in which finite products exist. Then,
1. an object T is called **separated** if the diagonal map is $T \to T \times T$ is a closed immersion.
2. A k-prevariety is called **k-variety** if it is separated.

Proposition 4.4.2. Let $f : Y \to X$ be a morphism of k-prevarieties. Then,
1. Suppose X is a k-variety, and f a closed/open immersion, then Y is a k-variety.
2. $X \times Y$ is a k-variety iff X, Y are k-varieties.

Theorem 4.5. The following hold:
1. Affine k-prevarieties are actually k-varieties.
2. Let X be a k-prevariety such that for every $x, y \in X$, there is $V \subset X$ affine k-subprevariety such that $x, y \in V$. Then, X is a k-variety.

Proof. To 1: $\mathbb{A}^n_K$ is separated. Then, X is affine iff $\exists$ a closed immersion $X \to \mathbb{A}^n_K$. Deduce that the diagonal map is a closde immersion. □

Example 4.5.1. The line with two origins is not separated. The diagonal morphism is not closed.

Example 4.5.2. The projective line is a k-variety.

Definition 4.5.1. Let $\mathcal{T}$ be a category of topological spaces in which finite product exists. Then,

1. $T \in \mathcal{T}$ is called **universally closed** if for every object Y and $Y \times T$ if the projection of T onto Y is closed.
2. X is called proper if separated and universally closed.

Proposition 4.5.1. If X is universally closed/proper, $Y \to X$ a closed immersion. Then, Y is universally closed/proper.
$X \times Y$ is universally closed/proper if X, Y are so.

Definition 4.5.2. (Graded rings) $R = \oplus_{d \geq 0} R_d$ a ring such that

1. R_d is a subgroup of R, and
2. $R_d \cdot R_q \subset R_{d+q}$.

R_d is called the **graded piece of degree d**.

Definition 4.5.3. I is a **graded ideal** if $I = \oplus I_d$ and $I_d = I \cap R_d$. If so, $R/I = \oplus R_d/I_d$.

Definition 4.5.4. **Proj(R)**$:= \{p \in spec(R) : p \text{ graded}, p \neq \oplus_{d>0} R_d\}$

Proposition 4.5.2. $p \in Proj(R)$ iff for every a, b homogeneous, $ab \in p$ implies a or b in p.

Proposition 4.5.3. For $a \in R$ homogeneous, $deg(a) > 0$, $D_a^+ = \{g \in Proj(R) | a \notin g\}$ define the open basis for Zariski topology on $Proj$.

Localization works the same way as in non-graded case.

Definition 4.5.5. Let $\Sigma \subset R$ be a multiplicatively closed set of homogeneous elements. Define $\Sigma^{-1} R := \{\frac{f}{g} : g \in \Sigma, deg(g) = deg(f)\}$

Definition 4.5.6. The **ith dehomogenization** $D^i : \cup_i R_d \to k[\underline{y'}]$, where $y' = (\frac{x_1}{x_i}, ...\frac{\hat{x_i}}{x_i}, ..., \frac{x_n}{x_i})$.

Proposition 4.5.4. D^i gives a bijection from $D_+^i \to Spec(k[\underline{y'}])$

Definition 4.5.7. The **ith homogenization** $H^i : k[y'] \to k[x]$ given by $f(y') \mapsto x_i^{deg(f)} f(y')$

Note that both homogenization and dehomogenization are multiplicative, however they are not inverse to each other. In general $H^i D^i(f) = f_0 x_i^n$.

Definition 4.5.8. A **cone** in $\mathbb{A}_K^{n+1}$ is any subset T such that $x \in T$ implies $\lambda x \in T$ for all $\lambda \in K$. The **projectivization** of T, denoted $\mathbb{P}(T) = T^*/\sim$, where the equivalence relstion is given by $x \sim y$ iff $x = \lambda y$ for some λ. Denote the projectivization of $\mathbb{A}_K^n$ as $\mathbb{P}^n(K)$, and the K-rational points of the n-th dimensional projective space.

Definition 4.5.9. (Zariski topology on projective space)Given a homogeneous polynomial $f(x) \in k[X]$, define $D_f^+ = \{x \in \mathbb{P}_K^n : f(x) \neq 0\}$. Then, D_f^+ is a basis for a topology on $\mathbb{P}_K^n$.

Definition 4.5.10. The **standard open covering** for $\mathbb{P}_K^n$ is the set $D_{f_i}^+$ where $f_i = x_i$. Note $\mathbb{A}_K^n \to D_{f_i}^+$ is a homeomorphism.

Remark 4.5.1. $\mathbb{P}_K^n$ admits the union of n-copies of $\mathbb{A}_K^n$ as the standard open covering.

Corollary 4.5.1. The Zariski topology on $\mathbb{P}_K^n$ is Noetherian. In particular, one may define when V is irreducible/connected, etc.

Definition 4.5.11 (projective k-algebraic sets)**.** Let $\Sigma \subset k[X]$ be a set of homogeneous elements. Then $V(\Sigma) = \{x \in \mathbb{P}^n(K) : f(x) = 0 \ \forall f \in \Sigma\}$ is called a **projective k-algebraic set** in $\mathbb{P}^n$.

Definition 4.5.12. Given $V' \in \mathbb{P}^n$, define $I(V) := \{f \in k[X] : f(V) = 0\}$ is a homogeneous ideal in $k[X]$.

Proposition 4.5.5. If V is a projective k-algebraic set, then $V(I(V)) = V$. Moreover, $I(V)$ is reduced.

Proposition 4.5.6. $k[V] := k[X]/I(V)$ is canonically a graded reduced k-algebra. More precisely, $I(V) = \oplus_{d \geq 0} I(V)_d$, and $k[V] = \oplus_{d \geq 0} R_d/I(V)_d$

Proposition 4.5.7. The projective k-algebraic subsets $V \subset \mathbb{P}^n$ are closed, and any closed subset is a projective k-algebraic subset.

Proposition 4.5.8. Given a projective k-algebraic set V, the standard covering of $\mathbb{P}^n$ induces a covering of V, and the parts $D_{x_i}^+ \cap V_i$ is a closed affine k-algebraic set. To see this, consider the ith dehomogenization of f.

Theorem 4.6. In the above context, let $R_{n=1}^{gr,red}$ be the graded reduced algebra over k generated by $n+1$ variables, and P_K^n be the set of projective k-algebraic sets.. Then, one has bijections

$$R_{n+1}^{gr,red} \xleftarrow{g} Id^{gr,red}(k[X]) \xleftarrow{f} P_K^n$$

where $f(V(\Sigma)) = (\Sigma)$, and $g((\Sigma)) = \frac{k[X]}{(\Sigma)}$

Proposition 4.6.1. The following holds:
1. V is irreducible iff $I(V)$ is a prime ideal in $Proj(k[X])$ iff $R[V] := k[X]/I(V)$ is a domain.
2. The irreducible components of V is in bijections with the minimal projective prime ideals of $k[X]$.

Definition 4.6.1 (The ring of regular functions of projective algebraic sets)**.** let $V \subset \mathbb{P}_K^n$ be any non-empty subset. Then,
1. A function $\varphi : V \to K$ is called **regular at** $a \in V$, if there exists neighborhood U of a, and $p, q \in k[X]$ homogeneous and of the same degree, q non-vanishing on U, such that $\varphi = \frac{p}{q}$ on U.
2. A function $\varphi : V \to K$ is called **regular** if it is regular at all points of V.
3. Let $\mathbb{O}_a := \{\varphi : V' \to K : \text{regular at } a\}$ modulo the relation of agreement on a neighborhood around a.

Proposition 4.6.2. In the above context, the set of regular functions $\varphi : V' \to K$ regular at a is a k-subalgebra of $Maps(V', K)$. Hence, the set of regular functions on V' is also a k-subalgebra.

Proposition 4.6.3. If V is projective k-algebraic set, then $U \mapsto \mathbb{O}_U$ defines a sheaf of k-algebras on V. Thus, V is naturally a ringed space. Moreover, $\mathbb{O}'_x$ is the stalk of $\mathbb{O}'_{V'}$ if V' is a projective k-algebraic set.

Definition 4.6.2. Given a projective algebraic set V, an open subset $U \subset V$ is called a **quasi-projective** set. In particular, U is canonically a ringed space by restriction from V.

Proposition 4.6.4. The inclusion map $i : U \to V$ from a quasi-projective algebraic set to a projective algebraic set induces an open immersion of ringed spaces.

Definition 4.6.3. A k-prevariety is called **projective** k-variety if X is isomorphic as k-prevarieties to $(V, \tilde{O}_V)$ for some projective algebraic set V.

Theorem 4.7. The following hold:
1. Every quasi-projective k-algebraic set $U \subset V$ is a k-variety.
2. Every quasi-projective k-algebraic set V is a proper k-variety.

Proof. step 1: show that $\mathbb{P}_K^n$ endowed with the sheaf of regular functions is a k-prevariety. Moreover, the intersection of a projective k-algebraic set with any standard affine open is affine open, and the covering forms a k-subvariety. The situation is the same when we take V a closed projective subset instead of $\mathbb{P}^n_K$.

Step 2: show that $\mathbb{P}^n_K$ is separated, and so are all quasi-projective sub-prevarieties since there is an open-immersion into $\mathbb{P}^n_K$. Recall that a k-prevariety X is separated if for every x, y, there exists an open affinne set $U \subset X$ such that $x, y \in U$. A useful fact here is that $GL(k)$ defines an automorphism of $\mathbb{P}^n_K$ and takes (separated, affine)k-prevarieties to (separated, affine) k-prevarieties. Then, x, y both live in the affine open $D_{a_i + a_j}$, where a_i, a_j are the two non-zero coordinates of the two points.

Step 3: The reduction steps is assume V is $\mathbb{P}^n$ by closed immersion reflects properness. To check universally closed property of $\mathbb{P}^n_K$, it suffices to show that the projection from $\mathbb{P}^n_K \times_k X \to X$ is closed for X affine by choosing affine covers in the general case of k-prevarieties. The final reduction step reduces X affine to $X = \mathbb{A}^n$, since $X \times_k \mathbb{P}^n \to \mathbb{A}^n \times_k \mathbb{P}^n$ is a closed immersion.

$\square$

Theorem 4.8 (Fundamental Theorem of Elimination Theory)**.** Let V be $V(I) \subset \mathbb{A}^n_K \times_k \mathbb{P}^n_K$ a closed subset. Then the projection $pr_{\mathbb{A}^n_K}(V) = V(J)$, where J is the set of all polynomials $J := \{b(\underline{y}) : \exists N > 0 \text{with } x_i^N b(\underline{y}) \in I \text{ for every } i\}$

Remark 4.8.1. Apparently this is equivalent to the statement in Model theory, which roughly states that an algebraically-closed field has elimination of quantifiers.

Definition 4.8.1. Let $R = \oplus_d R_d$ and $S = \oplus_d S_d$ be graded A-algebras. Then, a **morphism of graded-A algebras of degree k** is an A-algebra homomorphism $\phi : R \to S$ such that $\phi(R_d) \subset S_{kd}$.

Example 4.8.1. Let $V \subset \mathbb{P}^n_K$ be a projective k-variety. Then, $k[V] = k[\underline{X}]/I(V)$. Then, the projection $p : k[\underline{X}] \to k[V]$ is a graded morphism of degree 1.

Recall the category of affine k-varieties and k-morphisms is anti-equivalent to the category of reduced k-algebras of finite type and k-morphisms. We want a similar statement for projective k-varieties, but the answer is we do not really know what happens in general.

Remark 4.8.2. Starting with $\phi^\sharp : k[V] \to k[W]$ surjective, we have a map $\phi : W \to V$ a k-morphism of projective k-varieties. However, different $\phi^\sharp, \phi'^\sharp$ may induce the same map on projective k-varieties; if $\phi^\sharp$ is not surjective, then more may go wrong.

We do have the special case:

Proposition 4.8.1. Let R, S be reduced graded k-algebras, and $\phi^\sharp : R \to S$ a graded morphisms of degree > 0. Then, let $b = \phi^0(a) \neq 0$. Then, $\phi^\sharp$ gives rise to $\phi^\sharp_a : R^0_a \to S^0_b$, where R_a and S_b are dehomogenization are R, S with respect to a, b. Let $I = ker(k[\underline{X}] \to R)$ given by mapping $x_1, ..., x_n$ to the generators of R_1 and $J = ker(k[\underline{Y}] \to R)$ given by mapping $y_1, ..., y_m$ to the generators of S_1. Let $V = V(I) \subset \mathbb{P}^n_k$ and $W = V(J)$. Let $V^+_a := V \cap D^+_a$ and $W^+_b = W \cap D^+_b$, then $\phi^\sharp_a : R^0_a \to S^0_b$ define $\phi_a : W^+_b \to V^+_a$ a k-morphism, hence $k[V^+_a] = R^0_a$ and $k[W^+_b] = S^0_b$. We may conlcude that if $\phi^\sharp : R \to S$ is a surjection, then for every $b_i \in S_1$, there exists $q_i \in R_1$ such that $\phi^\sharp(a_i) = b_i$, hence $\phi_{a_i} : W^+_{a_i} \to V^+_{a_i}$ can be glued to a map $\phi : W \to V$ given by $\phi^\sharp : k[V] \to k[W]$.

Remark 4.8.3. We will resolve this issue in later discussion on projective schemes.

4.1 Product of projective Varieties

Theorem 4.9 (Segre Embedding)**.** The product of projective k-varieties in the category of k-prevarieties is a projective k-variety. The product is called the **Segre Embedding.**

Proof. Let $V \subset \mathbb{P}^m_K$ and $W \subset \mathbb{P}^n_K$ be projective varieties. Note $V \times_k W \to \mathbb{P}^m_K \times_k \mathbb{P}^n_K$ is a closed immersion. Thus, it suffices to show that $\mathbb{P}^m_K \times_k \mathbb{P}^n_K$ is a projective variety for all m, n.

The construction is as follows: let $N = (m+1)(n+1)-1 = mn+m+n$. Then, define $\Phi : \mathbb{P}^m \times_k \mathbb{P}^n \to \mathbb{P}^N$ by $(a_0 : ... : a_m), (b_0 : ....b_n) \mapsto (a_i b_j)$ where $(a_i b_j)$ is ordered lexicographically. This is a topological embedding. Consider $\underline{Z} = (Z_{ij})$ a set of projective variables. Then,

$$im(\Phi) = V(Z_{ij}Zkl - Z_{il}Z_{kj})_{i,j,k,l}$$

$\square$

Remark 4.9.1. The embedding is an example to the following problem: given morphisms of projective varieties, there is no canonical unique morphisms of ring of regular functions, as we can embed the varieties to a projective space of higher dimension.

Remark 4.9.2. Given $\mathbb{P}^n, \mathbb{P}^m$ with $m \leq n$. Let $x = (x_0 : ...x_n)$ and $y = (y_0 : ...y_n)$. Then, $k[\underline{y}] \to k[\underline{x}]$ given by $y_i \mapsto x_i$ for $i \leq m$ and $y_i \mapsto 0$ if $i > m$ defines a k-embedding $\mathbb{P}^m \to \mathbb{P}^n$ defines a k-embedding. The image is $V(y_{m+1}, ..., y_n)$.

Theorem 4.10. (Chow's Lemma) Let X be a proper k-variety. Then, there exists projective k-varieties $\tilde{X}$ together with a surjective morphism $\tilde{f} : \tilde{X} \to X$ satisfying $\tilde{f}$ is an isomorphism on an open affine subset $U \subset X$.

Proof. If X is reducible, decompose X into irreducible components $\cup X_\alpha$. Let $U_\alpha \subset X_\alpha$ be affine open dense such that $U_\alpha \cap U_\beta = \emptyset$. (every open will be dense on irreducible susbet, and point-set topology argument on disjointness). Do the thing on each irreducible component, and then take the disjoint union.

Thus, we consider X irreducible. Let $X = \cup U_\beta$ be an affine open covering. Pick closed immersions $U_\beta \to \mathbb{A}^n_\beta$, and take $U = \cup U_\beta$. $\square$

Theorem 4.11 (Nagata's Theorem)**.** Let X be a k-variety. Then, there is a proper k-variety $\widehat{X}$ and an open embedding $i : X \to \widehat{X}$ with $i(X)$ dense.

5 Schemes and Varieties in Mordern Sense

Recall that $Spec(A)$ has the zariski topology, and comes equipped with the structure sheaf

Definition 5.0.1. The structure sheaf $\mathcal{O}_A$ associated to $Spec(A)$ is defined by the following: let $U \subset Spec(A)$, then

$$\mathcal{O}_A(U) = \{f : U \to \coprod_{p \in U} A_p : \text{if } p \in U, \text{then } f(p) \in A_p \text{ and } f \text{is locally a constant fraction}\}$$

. One may check that this defines a sheaf on $Spec(A)$.

Note that the global sections $\mathcal{O}_A(Spec(A))$ is canonically isomorphic to A.

Definition 5.0.2 (Affine Schemes)**.** An **affine scheme** is a locally ringed space isomorphic to $Spec(A)$ for some commutative ring A.

Theorem 5.1. There is a canonical bijection: $\mathrm{Hom}_{\text{Aff Scheme}}(B, A) \cong \mathrm{Hom}_{LRS}(Spec(A), Spec(B))$.

Definition 5.1.1. A **scheme** is a locally ringed space locally isomorphism to an affine scheme. Alternatively, one may intepret a scheme as a glueing of affine schemes.
A morphism of schemes $(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is any morphism of locally ringed spaces.

Concretely, let $(X, \mathcal{O}_X)$ be the gluing of affine schemes $(X_\alpha, \mathcal{O}_{X_\alpha}) \cong (Spec(A_\alpha), \mathcal{O}_{A_\alpha})$. Similarly, let $(Y, \mathcal{O}_Y)$ be the gluing of affine schemes $(Y_\beta, \mathcal{O}_{Y_\beta}) \cong (Spec(B_\beta), \mathcal{O}_{B_\beta})$. Then, $f(X_\alpha) \subset Y_\beta$ and $f^\sharp$ locally is the morphsim of affine schemes. Thus, a morphism is equivalent to gluing morphisms of affine schemes.

Definition 5.1.2 (Relative Morphisms)**.** Let S be a fixed scheme. An **S-scheme** is a morphism $\varphi : X \to S$. A morphism of S-schemes $f : X \to Y$ is a commutative triangle

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\varphi_x"'] & & Y \arrow[dl, "\varphi_y"] \\ & S & \end{tikzcd}$$

Note that this is simply the slice catgeory Sch/S.

Theorem 5.2. The product exists in the category of schemes over S. This is equivalent to saying fiber products exists in the category of schemes.

Proof. The first step is to do it for affine schemes. Let S be affine. Then, the product of affine S-schemes exists, and is realized as the spec of the tensor S-algebra. This is in fact the product in the catgeory of schemes, not only in the category of affine schemes.

Step 2 is to show the product of quasi-affine schemes exists. Step 3 Break up general schemes X, Y to affine charts, and use gluing schemes and morphism. Step 4 Break up S into affine charts and glue. $\square$

Definition 5.2.1. $\mathbb{A}^n := \mathbb{A}^n_{\mathbb{Z}}$ by definition is $Spec(\mathbb{Z}[x_1, ..., x_n])$. In general, $\mathbb{A}^n_R$ for any commutative ring R is $Spec(R[x_1, ..., x_n]) \cong Spec(R) \otimes_{\mathbb{Z}} \mathbb{A}^n_{\mathbb{Z}}$. For S a scheme, we define $\mathbb{A}^n_S := S \times_{\mathbb{Z}} \mathbb{A}^n$.

Definition 5.2.2 (Base change)**.** Let $T \to S$ be a morphism fixed schemes. Then, the we have the base change functor from Sch_S to Sch_T given by $X \mapsto X \times_S T$.

Definition 5.2.3. Let $s \in S$ be a point. Then, $\mathcal{O}_s$ is a local ring, and let $\kappa(s) = \mathcal{O}_s/m_s$ be the residue field at s. Given a field K, $spec(K)$ has the data of the zero prime ideal, and the global section isomorphisc K. Given a morphism $Spec(K) \to S$ is precisely a point on S and a field embedding from $\kappa(s) \to K$.

Definition 5.2.4. Fiber of a S-scheme: X $\varphi_X : X \to S$ at $s \in S$ is $X \times_S s$ is the fiber of φ_X at s.

Definition 5.2.5. A Scheme X is irreducible if the underlying topological space is irreducible. A scheme is reduced if every stalk is reduced, equivalently covered by affine schemes corresponding to reduced rings. A scheme is integral if X is irreducible and reduced.

Definition 5.2.6. A morphism of schemes $f : X \to Y$ is called **dominant** is $f(X)$ is dense in Y.

Definition 5.2.7. A scheme is called **normal** if each stalk is integrally closed.

Example 5.2.1. If $X = spec(R)$, then X is normal iff R is normal, i.e every localization at prime is integrally closed.

Theorem 5.3 (Normalization and Generic Fiber)**.** Given an integral scheme X, let $K = \kappa(X)$ be its function field and $L|K$ be an algebraic extension. Then, there exists a scheme X_L such that $L := \kappa(X_L)$ and there exists a dominant morphism $\varphi_L : X_L \to X$ satisfiying the universal property: Given a integral scheme Y and a dominant morphism $f : Y \to X$, there is a commutative diagram of fields

$$\begin{tikzcd} L \arrow[dr, dashed, "f_L"] & & K = \kappa(X) \arrow[ll, "\varphi_L*"'] \arrow[dl, "f^*"'] \\ & \kappa(Y) & \end{tikzcd}$$

and f_L is the generic fiber. In the case of $L = K$, $X_L \xrightarrow{\varphi} X$ is called the normalization of X.

Corollary 5.3.1. Let X be a separated scheme of finite type over a field k. Then, the normalization of X in any field extension $L|\kappa(X)$ is again a k-variety, and $\varphi; X_L \to X$ is a finite morphism. Moreover, of X is a projective k-variety, then X_L is projective as well.

Theorem 5.4. Let X be a noetherian integral normal scheme, $L|\kappa(X)$ a finite field extension. Then, X_L is Noetherian and $X_L \to X$ is a finite morphism.

Definition 5.4.1. Let $f : X \to Y$ be a morphism in Sch_S. Then,

1. f is **affine** is it can be factored as

$$X \xrightarrow{\text{cl. imm}} \mathbb{A}^n_Y \to Y$$

2. f is **projective** is it can be factored as

$$X \xrightarrow{\text{cl. imm}} \mathbb{P}^n_Y \to Y$$

Proposition 5.4.1. Affine morphism are separated and affine schemes are separated.

Theorem 5.5. Projective schemes are proper and projective morphisms are proper.