Algebraic K-Theory and Motivic Filtrations

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This is a colloquim talk given by Mathew Morrow on March 27.

Algebraic K-theory is basically a cohomology theory for rings. We associate a given ring with a collection of abelian groups. The goal is to understand these groups and understand their relations with other invariants. In principal, they see information about many areas/subjects in mathematics.

1 K_0

Definition 1.1. $K_0(R)$ is the free abelian group generated by the finitely generated projective modules over R under the relations of direct sums.

Example 1.1. Over a field F, $K_0(F) = \mathbb{Z}$ by dimension.

Example 1.2. $D =$ ring of integers of number field. Any ideal of D is projective. Class group of the number field is embedded in $K_0(D)$.

Example 1.3. X compact Hausdorff space, and $C(X)$ be the set of continuous functions. The section of any locally trivial vector bundles of a projective $C(X)$-module. This is topological K-theory $K_0(C(X))$.

2 K_1

Definition 2.1. $GL_\infty(R)/[GL_\infty(R), GL_\infty(R)]$, where $GL_\infty(R)$ iscolimit of $GL_n(R)$.

Lemma 2.1. $M, M' \in GL_\infty(R)$ iff they are equivalent up to row and column operations.

Example 2.1. If F is a field, then we have Gaussian elimination, and reduce the invariant to the determinant. $K_1(F) = F^\times$

Example 2.2. D integers of number field: if it is euclidean domains have analog of Gaussian elimination, which gives us $k_1(D) = D^\times$. If not, the result still holds but proof is more involved by Bass-Milnor-Serre.

Example 2.3. X is a smooth compact manifold of dimension ≥ 5. Consider the $G := \mathbb{Z}\pi_1(X)$ as the group ring over the fundamental group. Then it is the $s-$ cobordism theorem $K_1(G)/(\pm 1 \oplus \pi_1)$ classifiied $h-$ cobordism of X.

In the 60s, more relations between k_0 and K_1 are discovered.

Theorem 2.2. (Fundamental theorem of K_0, Bass)$K_0(R)$ is the cokernel of the map $K_1(R[t] \oplus K_1(R[t^{-1}])) \to K_1(R[t, t^{-1}])$

The slogan is K_1 is a refinement of K_0. Thus, it is natural to continue to look for higher K_n.

3 K_n

Quillen's idea of to derive the construction of K_1 is as follows: we start with $R \to GL_\infty(R)$. Consider its classifying space $BGL_\infty(R)$ with $\pi_1(BGL_\infty(R)) = GL_\infty(R)$. Then, we have to modify the spaces using the plus construction to kill the maximal perfect subgroup (which turns out to be the commutator subgroups) of π_1, which makes the K_1 abelian.

Theorem 3.1. (Quillen) $K_n(R) := \pi_n(BGL_\infty(R)^+)$

4 Techniques to Understand K-theory(for algebrac/arithmetic geometry)

The idea is to replace K-theory with something easier to understand: Motivic cohomology, cylic homology and a common ground, motivic filtrations.

For Motivic cohomology, R is a smooth algebra over a field. $K(R)$ admits a filtration/stratification/ decomposition with buiding bricks motivic cohomoogy groups $H^i_{mot}(R, \mathbb{Z}(j))$. Encode Blcok-Kato conjecture $H_{Gal}(F, \mu_t^{\otimes n}) \cong K_n^M(F)$. The decomposition of $K(R)$ is the first example of a "motivic" filtration. This only works for smooth algebras over a field.

Definition 4.1. R a fring, its Hochschild homology groups are the homology of the complex

$$R^{\otimes n} \to R^{\otimes n-1};\ a \otimes b \mapsto ab - ba$$

Refinements: Cylic hmoology.

The above is an analog of De Rham cohomology, and it works well with rings of characteristic 0. A better approach is Crystalline/prismatic cohomology proposed by Nikolas-Scholze. We get Topological cyclic cohomology $TC_n(R)$.

Theorem 4.1. There exists delogaritmn maps $K_n(R) \to TC_n(R) \to HH_n(R) = \Omega^1(R)$, where the last equality holds when the ring is commutative.

Theorem 4.2. The failure of dlog to be an somorphism has remarkably nice properties. Example: $R_1 \to R_2$ nilpotent surjection implies 3 out of 4 properties: $K(R_1), K(R_2), TC(R_1), TC(R_2)$, which means we get a MV sequence relating all four cohomologies.

The question is: for which ring R is K-theory an homotopy invariant: $K_n(R[t]) = K_n(R)$ for all n?

Theorem 4.3. (Fundamental Theorem, Quillen) R is regular Noetherian implies it is an invariant..

Theorem 4.4. (Cortinas)R commutative C^*-algebras, probe $R = C(x)$ by blowing up affine algebraic varieties.

Theorem 4.5. (Mathhew-Antieu-M) R perfectoid. Tilt, blow-up, homological properties of valuation rings.

The conjecture is for non-regular Noetherian rings, it is not invariant. Classical motivic cohomology: geometric in flavor, smooth algebra over a field; cyclic homology: very homological algebra. What is the common ground?

Theorem 4.6. (Bhatt-Scholze) R ring with mild singularities. Then, the cyclic homologies of R admit motivic filtrations with building blocks simpler/more geometric cohomologies(De Rham, crystalline, etale, prismatic).

Example 4.1. $K(\mathbb{Z}/p^{\mathbb{Z}}) \to TC(\mathbb{Z}/p^m\mathbb{Z})$ with building blocks syntomic cohomology, which leads to new computations of $K(\mathbb{Z}/p\mathbb{Z})$ by Antieau.

Theorem 4.7. The K-theory of any ring containing a field admits a motivic filtration with building blocks given by a new theory of motivic cohomology that recovers motivic cohomology when R is smooth.

The idea of proof is approximate K-theory orthogonally by TC and KH. Equip the latter two thoeries with motivic filtration. Finally glue. A corollary is intersection theory on singular algebraic varieties. Hot current questions: which invariants admmit motivic filtrations, and in what generality?