Étale Homotopy Theory and Adams' Conjecture

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Definition 0.0.1 (Čech Nerve)**.** Let X be a finite CW complex, and $\mathcal{U} := \{U_i : i \in I\}$ be an open cover of X. Then, we may define a simplicial set call the **Čech Nerve** $N\mathcal{U}$ as follows: we have the assignment on objects $[n] \mapsto \{\text{functions from } [n] \text{ to } I : \cap_{i=1}^{n} U_{f(i)} \neq \emptyset\}$. The face maps and degeneracy maps are defined by deleting and inserting appropriate indices.

Alternatively, we can think of a covering $\mathcal{U}$ as follows: suppose given a covering $X = \cup_{i \in I} U_i$; let $\mathcal{U} = \coprod_{i \in i} U_i$, and the covering is the obvious map $\mathcal{U} \to X$. Note that we have

$$U_i \cap U_j = U_i \times_X U_j$$

so the n-fold fiber product $U \times_X ... \times_X U$ is the disjoint union of n-fold intersections of opens in the cover. Then, the nth simplices of the Čech nerve is $\pi_0(\underbrace{U \times_X ... \times_X U}_{n-fold})$. The face maps are projections, and the degeneracy maps are various diagonal embeddings.

Theorem 0.1. If the covering $\mathcal{U}$ satisfies the property that arbitrary intersections of opens in the cover is either empty or contractible, then th realization $|N\mathcal{U}|$ is weakly equivalent to X.