MATH 624 HW2

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Homework 2

Problem 1b

Suppose U_f is not empty. Let $W = \{a \in V : k(a)|k \text{ is a finite algebraic extension}\}$, which corresponds to the vanishing locus of maximal ideals of $k[V]$. Clearly $W \subset V(\bar{k})$, so it suffices to show that $W \cap U_f$ is dense in in U_f for every f, which is equivalent to every open U_f containing a point in W. To see this, consider a maximal ideal in $k[V]_f$, which must be the image of a maximal ideal in $k[V]$ under localization: suppose otherwise, then every maximal ideal of $k[V]$ contains f, which implies f is in the Jacobson radical of $k[V]$. However, $k[V]$ has trivial Jacobson radical since $k[X]$ is Jacobson, which implies $f = 0$ and U_f is empty, and contradiction. Then, the locus of the maximal ideal is contyained in $U_f \cap W$.

Problem 2b

A representative of $\tilde{O}_a$ is given by a pair $(W_1, \frac{f_1}{g_1})$, with $g_1 \neq 0$ on W_1, and $(W_1, \frac{f_1}{g_1}) \sim (W_2, \frac{f_2}{g_2})$ iff there exists a open $U_{h'} \subset W_1 \cap W_2$ such that $\frac{f_1}{g_1} = \frac{f_2}{g_2}$ on $U_{h'}$. On the other hand, a representative of $k[V]_{\mathfrak{p}_a}$ is given by some $\frac{f}{g}$, where $g(a) \neq 0$. By continuity, there exists a basic open U_h containing a on which g does not vanish. We define the k-algebra homomorphism:

$$i : k[V]_{\mathfrak{p}_a} \to \tilde{O}_a \quad \frac{f}{g} \mapsto (U_h, \frac{f}{g})$$

Surjectivity is obvious by construction, so there are two things to check: well-definedness (it is clearly that this will be a k-algebra morphism once we check well-definedness) and injectivity.

Well-definedness: suppose $\frac{f}{g} \sim \frac{f'}{g'}$ in $k[V]_{\mathfrak{p}_a}$, which means there exists some $h' \in K[V]$ such that $h'(fg' - f'g) = 0$, which implies $\frac{f}{g} = \frac{f'}{g'}$ on $U_{h'}$. Thus, both will be mapped to the equivalence class $(U_{h'}, \frac{f}{g})$.

Injectivity: suppose $i(\frac{f}{g}) = (U_h, \frac{f}{g})$ represents the 0 element. WLOG, we may assume that f vanishes on U_h, for otherwise we may replace U_h with a smaller basic open. Then, $\frac{f}{g} \sim \frac{0}{1}$ in $k[V]_{\mathfrak{p}_a}$ since $h(f \cdot 1 - g \cdot 0)$ is identically 0 on V.

Problem 3b

By problem 2b, the stalk is isomorphic to $k[V]_{p_a}$, which is always local. In regards to when $k[V]_{p_a}$ is a not a domain, it will be when there exists an $x \in p_a$ such that $\exists y \in p_a$ and $xy = 0$, but $xz \neq 0$ for every non-zero $z \notin p_a$. For example, let $V = V(xy)$. Then, $k[V] = k[x, y]/(xy)$. Take $a = (0, 0)$, then $p_a = (x, y)$, and we have $xy = 0$ but $xz \neq 0$ for every non-zero z not in (x, y).

Note that a reduced Noetherian ring is integral iff it has a unique minimal prime. Another method of detection for integrality is iff p_a contains a unique minimal prime of $k[V]$ (because it is reduced Notherian), which corresponds to a belonging to a unique irreducible component.

Problem 4

(a)

V is irreducible iff $I(V)$ is prime iff $k[V]$ is a domain iff $k(V)$ is a field. The Krull dimension of $k(V)$ and the trascendence degree are the same by Noether normalization.

(b)

Take the finite set of minimal primes $\{p_1, ..., p_n\}$ of $k[V]$, and recall that the union of the minimal primes is precisely the zero-divisors of $k[V]$, and the intersection is the trivial nilradical. Then, localize at $S = k[V] \setminus \cup p_i$, and $S^{-1}k[V]$ has unique maximal primes $S^{-1}p_1, ..., S^{-1}p_n$, which are coprime. By chinese remainder, we have

$$k(V) = S^{-1}k[V]/(0) = S^{-1}k[V]/\cap S^{-1}p_i \cong \prod k(V_i)$$

(c)

(d)

Problem 5

Problem 10

(a)

For $\mathbb{A}^1 \to \mathbb{A}^2$ given by $a \mapsto (a, \frac{1}{a})$, the general situation is discussed in problem 8; for $a \mapsto (a^2, a^3)$, the domain in the entire $\mathbb{A}^1$, and the image is a affine algebraic set given by $V(x^3 - y^2)$. The map is clearly a bijection and a homeomorphism. However, the k-morphism is not an isomorphism, as the coordinate rings $k[t^2, t^3]$ and $k[t]$ are not isomorphic.

(b)

As in part (a), we see that it is possible for the $k-$morphism to not be an isomorphism. However in the case $\mathbb{A}^1 \mapsto \mathbb{A}^3$ given by $a \mapsto (a^1, a^2, a^3)$, the k-morphism is an isomorphism.

Homework 3

Problem 1

(a)

First, note that all closed/open immersions $i : Z \to X$ are separated morphisms: the diagonal map to the fiber product $Z \to Z \times_X Z \cong Z$ is an isomorphism.