MATH 624 HW2

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Homework 2

Problem 1b

Suppose U_f is not empty. Let $W = \{a \in V : k(a)|k \text{ is a finite algebraic extension}\}$, which corresponds to the vanishing locus of maximal ideals of $k[V]$. Clearly $W \subset V(\overline{k})$, so it suffices to show that $W \cap U_f$ is dense in in U_f for every f, which is equivalent to every open U_f containing a point in W. To see this, consider a maximal ideal in $k[V]_f$, which must be the image of a maximal ideal in $k[V]$ under localization: suppose otherwise, then every maximal ideal of $k[V]$ contains f, which implies f is in the Jacobson radical of $k[V]$. However, $k[V]$ has trivial Jacobson radical since $k[X]$ is Jacobson, which implies $f = 0$ and U_f is empty, and contradiction. Then, the locus of the maximal ideal is contyained in $U_f \cap W$.

Problem 2b

A representative of $\tilde{O}_a$ is given by a pair $(W_1, \frac{f_1}{g_1})$, with $g_1 \neq 0$ on W_1, and $(W_1, \frac{f_1}{g_1}) \sim (W_2, \frac{f_2}{g_2})$ iff there exists a open $U_{h'} \subset W_1 \cap W_2$ such that $\frac{f_1}{g_1} = \frac{f_2}{g_2}$ on $U_{h'}$. On the other hand, a representative of $k[V]_{\mathfrak{p}_a}$ is given by some $\frac{f}{g}$, where $g(a) \neq 0$. By continuity, there exists a basic open U_h containing a on which g does not vanish. We define the k-algebra homomorphism:

$$i : k[V]_{\mathfrak{p}_a} \to \tilde{O}_a \quad \frac{f}{g} \mapsto (U_h, \frac{f}{g})$$

Surjectivity is obvious by construction, so there are two things to check: well-definedness (it is clearly that this will be a k-algebra morphism once we check well-definedness) and injectivity.

Well-definedness: suppose $\frac{f}{g} \sim \frac{f'}{g'}$ in $k[V]_{\mathfrak{p}_a}$, which means there exists some $h' \in K[V]$ such that $h'(fg' - f'g) = 0$, which implies $\frac{f}{g} = \frac{f'}{g'}$ on $U_{h'}$. Thus, both will be mapped to the equivalence class $(U_{h'}, \frac{f}{g})$.

Injectivity: suppose $i(\frac{f}{g}) = (U_h, \frac{f}{g})$ represents the 0 element. WLOG, we may assume that f vanishes on U_h, for otherwise we may replace U_h with a smaller basic open. Then, $\frac{f}{g} \sim \frac{0}{1}$ in $k[V]_{\mathfrak{p}_a}$ since $h(f \cdot 1 - g \cdot 0)$ is identically 0 on V.

Problem 3b

By problem 2b, the stalk is isomorphic to $k[V]_{p_a}$, which is always local. In regards to when $k[V]_{p_a}$ is a not a domain, it will be when there exists an $x \in p_a$ such that $\exists y \in p_a$ and $xy = 0$, but $xz \neq 0$ for every non-zero $z \notin p_a$. For example, let $V = V(xy)$. Then, $k[V] = k[x, y]/(xy)$. Take $a = (0, 0)$, then $p_a = (x, y)$, and we have $xy = 0$ but $xz \neq 0$ for every non-zero z not in (x, y).

Note that a reduced Noetherian ring is integral iff it has a unique minimal prime. Another method of detection for integrality is iff p_a contains a unique minimal prime of $k[V]$ (because it is reduced Notherian), which corresponds to a belonging to a unique irreducible component.

Problem 4

(a)

V is irreducible iff $I(V)$ is prime iff $k[V]$ is a domain iff $k(V)$ is a field. The Krull dimension of $k(V)$ and the trascendence degree are the same by Noether normalization.

(b)

Take the finite set of minimal primes $\{p_1, ..., p_n\}$ of $k[V]$, and recall that the union of the minimal primes is precisely the zero-divisors of $k[V]$, and the intersection is the trivial nilradical. Then, localize at $S = k[V] \setminus \cup p_i$, and $S^{-1}k[V]$ has unique maximal primes $S^{-1}p_1, ..., S^{-1}p_n$, which are coprime. By chinese remainder, we have

$$k(V) = S^{-1}k[V]/(0) = S^{-1}k[V]/\cap S^{-1}p_i \cong \prod k(V_i)$$

(c)

Suppose V is irreducible. Note that $k[V_{k^s}] \cong k[V] \otimes_k k^s$, so $k(V_{k^s}) \cong k(V) \otimes_k k^s$ after taking the field of fractions. Thus, absolute irreducibility of V is equivalent to the integrality of $k(V_{k^s}) \cong k(V) \otimes_k k^s$. Suppose $\overline{k} \cap k(V)$ is not purely inseparable over k, so there exists α algebraic over k, and $k(\alpha) \otimes_k k(\alpha)$ is a subring of $k(V) \otimes_k k^s$, which is not integral. To see this, note, let $p(t)$ be a minimal polynomial of α, then

$$k(\alpha) \otimes_k k[t]/p(t) \cong k(\alpha)[t]/p(t)$$

cleary has $(x - \alpha)$ as a zero-divisor.

Conversely, suppose $k(V) \cap \overline{k}$ is purely inseparable. It suffices to show that $k(V) \otimes_k k[t]/p(t) \cong k(V)[t]/p(t)$ is integral for every irreducible $p(t)$. If there is $q(t) \in k(V)[t]$ that divides $p(t)$, then $q(t)$ is also contained in $k^s[t]$, so $q(t) \in (k^s \cap k(V))[t] = k[t]$, which forces it to be 1 or $p(t)$, and the ring is still integral.

(d)

Problem 5

(a)

The correct statement should be $\tilde{O}_x$ is a domain iff x is contained in a unique irreducible component, and the proof is given in problem 3.

(b)

It is a standard point-set topology argument that finite intersection of open dense sets is still open and dense.

(c)

The colimit is the function field of V. The detail proofs are given in HW3 problem 10.

Problem 8

(a)

Clearly the empty set and the whole line is open affine, so the only non-trivial case is the line minus a finite set of points. Let $a_1, ..., a_n$ be a finite number of points, and $\mathbb{A}^n \setminus \{a_1, .., a_n\}$ is isomorphic to the affine algebraic set $V(y(x-a_1)...(x-a_n)-1) \subset \mathbb{A}^{n+1}$ given by the map

$$\varphi : \mathbb{A}^n \setminus \{a_1, .., a_n\} \to V(y(x-a_1)...(x-a_n)-1) \qquad t \mapsto (t, \frac{1}{(t_1-a_1)...(t_n-a_n)})$$

with inverse $\psi : (x, y) \mapsto x$. Both functions are Zariski continuous since they are rational functions. Let T be an open of $\mathbb{A}^n$ and U be an open of $\mathbb{A}^{n+1}$ such that $f(T) \subset U$. Then, given any regular function $\frac{f(x,y)}{g(x,y)}$ on U, the pullback $\frac{f(x,\frac{1}{x})}{g(x,\frac{1}{x})}$ is a regular function on T by multiplying large enough powers of x to the numerator and denominator. The other direction is trivial since the pullback will be the same function on one variable. Thus, φ and ψ are k-isomorphisms.

(b)

The open $U := \mathbb{A}^2 \setminus \{(0,0)\}$ is not affine. Note that U is covered by $U_1 = D_{f(x,y)=x}$ and $U_2 = D_{f(x,y)=y}$, whose ring of regular functions are $k[x,y]_x$ and $k[x,y]_y$. On the overlap, the ring of regular functions is $k[x,y]_{x,y}$. Let f be a regular function on U, which restricts to a regular function of the form p_1/x^m on U_1 and p_2/y^n on U_2. The compatibility condition on $U_1 \cap U_2$ implies that $p_1/x^m = p_2/y^n$, which implies $x^m p_2 = y^n p_1$. Since $k[x,y]$ is a UFD, $x_m | p_1$, and f is in $k[x,y]$. Thus, $O(U) \cong k[x,y] \cong O(\mathbb{A}^2)$. Thus, if U were affine, the inlusion map $i : U \to \mathbb{A}^2$ is an isomorphism, which is false.

Homework 3

Problem 1

(a)

Suppose X is separated, and Z is closed in X. Since $Z \to X$ is a closed immersion, $Z \times Y \to X \times Y$ is a closed immersion for all Y. In particular, this implies the composition $Z \times Z \to Z \times X \to X \times X$ is a closed immersion. We then have the commutative triangle

$$\begin{tikzcd} Z \arrow[r, "\Delta_Z"] \arrow[d] & Z \times Z \arrow[d] \\ X \arrow[r] & X \times X \end{tikzcd}$$

where all arrows except Δ_Z is a closed immersion. It follows that Δ_Z is a closed immersion as well. Suppose now X is proper. Let T be the terminal object in our category (Spec(k) for the category of k-prevarieties). We have the pullback squares

$$\begin{tikzcd} W \cong Z \times Y \arrow[r, "f"] \arrow[d] & X \times Y \arrow[r, "g"] \arrow[d] & Y \arrow[d] \\ Z \arrow[r, "i"'] & X \arrow[r] & T \end{tikzcd}$$

where the two smaller squares are pullbacks by definition, and the outer rectangle is also a pullback by general categorical nonsense. Note that closed immersions is stable under pullback, so f is also a closed immersion, and $W \to Y$ is a closed map by composition.

(b)

Following the hint, we have canonical isomorphisms $(X \times X) \times (Y \times Y) \cong (X \times Y) \times (X \times Y)$, which induces an isomorphism $\Delta_X \times \Delta_Y \cong \Delta_{X \times Y}$. We see that $\Delta_{X \times Y}$ is closed iff both Δ_X and Δ_Y are closed, so $X \times Y$ is separated iff X, Y are both separated.

Note that universally closed morphisms are stable under pullbacks by definition, so proper morphisms are stable under pullbacks. Moreover, composition of proper morphisms is also proper. In particular, the product of two proper morphisms is proper since it case be written as the composition of two proper morphisms from pullback.

Problem 2

(c)

Checking R_f^0 is an R_0-algebra is trivial; for the second part, first recall the canonical homeomorphism $D_f \cong Spec(R_f)$. Then, D_f^+ is the subspace of homogeneous primes of $Spec(R_f)$, i.e $Proj(R_f)$. Thus, it suffices to show that $Proj(R_f)$ is homeomorphic to $Spec(R_f^0)$. Consider the map $Proj(R_f) \to Spec(R_f^0)$ given by $\oplus_{d \geq 0} I_d \mapsto I_0$, which is easily see to be well-defined and continuous since it is induced by the inclusion $R_f^0 \to R_f$. We will explicitly construct an inverse $f^{-1} : Spec(R_f^0) \to Proj(R_f)$, given by $p_0 \mapsto \sqrt{\oplus_{d \geq 0} p_0 S_d}$. It is standard to check the image is a homogeneous prime ideal. Let $g = \sum_i g_i$ be an element in R_f, and W_g be a basic open in $Proj(R_f)$. Then, the inverse image of W_g is the finite intersection of basic opens $\cap \tilde{W}_{g_i}$ in $Spec(R_f^0)$, which is open, and we have continuity. The composition $f \circ f^{-1}$ is clearly the identity, and we are left to show that $f^{-1} \circ f(\oplus_{d \geq 0} I_d) = \oplus_{d \geq 0} I_d$. For simplicity, assume $deg(f) = 1$ so we don't have keep track of it. Take $s \in I_d$, then $\frac{s}{f^d} \in I_0$, and it follows that $s \in f^{-1} \circ f(\oplus_{d \geq 0} I_d)$; conversely, suppose $q \in \sqrt{\oplus_{d \geq 0} p_0 S_d}$ where $deg(q) = d$, then $\frac{q}{f^d} = \frac{q'}{f^e}$ for some $q' \in I_e$. We then have

$$f^k(f^e q - f^d q') = 0$$

which implies $q \in \sqrt{\oplus_{d \geq 0} p_0 S_d}$ by primeness as f^{k+e} is not in the prime ideal.

Problem 3b

It is not the coproduct since there are no canonical graded morphism from $R \to R \otimes_A^{gr} S$. Given graded algebras P, Q, then correct coproduct is the graded-algebra

$$P \otimes_A Q := \oplus_{m+n=d} P_m \otimes_A Q_n$$

with coordinate-wise multiplication structure and bilinear A-action, together with canonical inclusions $P \to P \otimes_A Q$ and $Q \to P \otimes_A Q$.

Problem 4

$Hom_k(\mathbb{A}^1_K, \mathbb{A}^1_K)$ is in bijection with $Hom_k(k[x], k[x])$, which is specified by the image of x. Thus,

$$Hom_k(\mathbb{A}^1_K, \mathbb{A}^1_K) \cong k[x]$$

. Automorphisms of $\mathbb{A}^1$ corresponds to automorphisms of $k[x]$, and which correponds to mapping x to a linear polynomial $ax + b$ with $a \neq 0$.

Problem 5

Problem 6

(a)

Since the product of k-prevarieties is the categorical product, it is automatically associative and commutative up to isomorphism by general abstract nonsense.

(b)

The finite product of affine variety $Spec(k[V])$ and $Spec(k[W])$ is isomorphic to $Spec(k[V] \otimes_k k[W])$, which is affine. Note that all affine varieties are separated, since the multiplication map $A \otimes A \to A$ is surjective, so the map $Spec(A) \to Spec(A \otimes A)$ is a closed immersion. The properness of the product follows from the fact that proper morphisms are stable under pullbacks.

(c)

The statement follows from the algebraic fact that

$$dim(k[V]) + dim(k[W]) = dim(k[V] \otimes_k k[W])$$

To see this, use Noether normalization so that the tensor product of coordinate rings is a finite module over tensor product of polynomial rings, which is again a polynoimal ring whose krull dimension is the sum of that of $k[V]$ and $k[W]$.

Problem 7

Problem 8

Problem 10