K_0 and Wall's Finiteness Obstruction

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This note will follow the original paper by C.T.C Wall, which discusses the algebraic criteria determining when a CW-complex is homotopy equivalent to one of finite type/dimension.

Definition 0.0.1. A CW-complex is **finite**, or equivalently of **finite type**, if it is constructed with finitely many cells.

Definition 0.0.2. A map $\varphi : K \to X$ is n**-connected** if the relative homotopy group $\pi_i(\varphi) := \pi_i(M_\varphi, K \times 1)$ is trivial for $0 \le i \le n$, where M_φ is the mapping cylinder equivalent to X.

For the following discussion, we will use the notation Λ for the integral group ring on the fundamental group of X.

1 Complexes of Finite Type

For the condition of X being equivalent to a complex with finite n-skeleton, we associate an algerbaic condition F_n, defined as follows:

- F_1 is $\pi_1(X)$ being finitely generated.
- F_2 is $\pi_1(X)$ being finitely presented, and for any finite complex K^2 and a map $\varphi : K^2 \to X$ inducing isomorphism on fundamental groups, $\pi_2(\varphi)$ is a finitely generated Λ-module.
- For $n \ge 3$, F_n stands for F_{n-1} holds and for any finite complex K^{n-1} and an $n-1$ connected map $\varphi : K^{n-1} \to X$, $\pi_n(\varphi)$ is a finitely generated Λ-module.

Theorem 1.1. A CW complex X is equivalent to a complex with finite n-skeleton iff it satisfies F_n.

Proof of Theorem 1.1. $\Rightarrow$ Let us first deal with the case where $n = 1, 2$. Recall that the fundamental group of X is completely determined by its 2-skeleton. The proof using Van-Kampen directly tells us that if the 1-skeleton being finite implies $\pi_1(X)$ is finitely generated, and 2-skeleton being finite implies $\pi_1(X)$ is finitely presented. For the second-part of F_2 and the rest of the theorem, by Hurewicz we have

$$\pi_n(\varphi) \cong \pi_n(X, K) \cong \pi_n(\tilde{X}, \tilde{K}) \cong H_n(\tilde{X}, \tilde{K})$$

By cellular approximation and a lemma of whitehead, we may assume $\tilde{K}$ is the $n-1$ skeleton of $\tilde{X}$. Then, $\cong H_n(\tilde{X}, \tilde{K})$ is a quotient of $\cong H_n(\tilde{X}^n, \tilde{K}) \cong C_n(\tilde{X})$, which is a finite Λ-module.

$\Leftarrow$: Before proving the direction, we make the following observation: for $n \ge 3$, given an $n-1$-connected map $\varphi : K^{n-1} \to X$, the LES of homotopy groups shows that it induces isomorphism on π_i for $i < n-1$ and

$\pi_{n-1}(K^{n-1}) \to \pi_{n-1}(X)$ is a surjection. Then, we may attch n-cells to kill the kernel, each corresponding to a class in $\pi_n(K^{n-1}, X)$. Specifically, consider the LES of the triple (X, K^n, K^{n-1}), where K^n is the complex obtained by attaching n-cells to K^{n-1}:

$$\pi_n(K^n, K^{n-1}) \longrightarrow \pi_n(X, K^{n-1}) \longrightarrow \pi_n(X, K^n)$$

Our goal is to build K^n such that $\pi_n(K^n, X)$ vanishes. Note that $\pi_n(K^n, K^{n-1}) \cong C_n(\tilde{L})$. So by finite generation assumption, only finitely many cells are needed, each corresponding to a generator of $\pi_n(K^{n-1}, X)$ as a Λ-module (The module structure exists when $K^n \to X$ induces isomorphism on fundamental group, which holds for $n \geq 3$).

Now to the proof: If X satisfies F_1, we may start with a finite wedge $K^1 := \bigvee S^1$, each copy corresponding to a generator of $\pi_1(X)$, and a map $K^1 \to X$ inducing surjection on fundamental group. Then, attach cells of dimension ≥ 2 to make it a homotopy equivalence. If X satisfies F_2, by finite presentation of π_1 we can add finitely many 2-cells to K^1 and form a new complex K^2. There is then a map $\varphi : K^2 \to X$ that induces isomorphism of π_1 and thus is 2-connected by LES. We finish by using the construction outlined int he previous paragraph and continue inductively.

□

We also have a stronger result that we will not prove here:

Theorem 1.2. X is equivalent to a complex with finite n-skeleton iff X is a homotopy retract of one.

2 Complexes of Finite Dimension

Similar to the previous section, we associate an algebraic condition D_n to the topological condition that X is equivalent to a CW complex of dimension n.

- D_n: $H_i(\tilde{X}) = 0$ for $i > n$, and $H^{n+1}(X; B) = 0$ for all local coefficient B.

Let us give a quick summary of the construction of the previous section, but without finiteness restriction on the number of cells allowed: given a CW complex X, we may approximate it by inductively building an n-connected map $\varphi_n : K^n \to X$ by attaching n-cells to K^{n-1}. Assuming path-connectedness, we start with 1-cells corresponding to generators of $\pi_1(X)$, 2-cells according to presentation of $\pi_1(X)$, and $n \geq 3$ cells according to generators of $\pi_n(K^{n-1}, X)$. Now, if the module $\pi_n(K^{n-1}, X)$ were free, then by construction and the LES,

$$\pi_{n+1}(X, K^n) \longrightarrow \pi_n(K^n, K^{n-1}) \cong C_n(\tilde{K}^n) \longrightarrow \pi_n(X, K^{n-1}) \longrightarrow \pi_n(X, K^n)$$

we will also get $H_{n+1}(\tilde{X}, \tilde{K}^n) = 0$ and $H_n(\tilde{X}, \tilde{K}^n) = 0$ for free by Hurewicz. This comes in handy when we look at the homology LES of the triple (X, K^n, K^{n-1}):

$$H_i(\tilde{K}^n, \tilde{K}^{n-1}) \longrightarrow H_i(\tilde{X}, \tilde{K}^{n-1}) \longrightarrow H_i(\tilde{X}, \tilde{K}^n)$$

If X satisfies D_n, then $H_i(\tilde{X}, \tilde{K}^{n-1})$ vanishes for $i \neq n$ by LES of the pair; $H_i(\tilde{K}^n, \tilde{K}^{n-1}) = 0$ for $i \neq n$ by cellular homology. Combined with the results in dimension n and $n+1$, we get $H_*(\tilde{X}, \tilde{K}^n) = 0$, which implies $\tilde{K}^n$ is equivalent to X (inductive argument by Hurewicz since by construction $K^2 \to X$ induces isomorphism on fundamental group). What this tells us is that if X satisfies D_n and $\pi_n(X, K^{n-1})$ is free, then we may stop at dimension n and already get a homotopy equivalence. The result of this section is that (for $n \geq 3$) we may alway add $n-1$-cells to K^{n-1} to make $\pi_n(X, K^{n-1})$ free, thus proving part 3 of the following theorem

Theorem 2.1. The following hold:
- X satisfies D_1 iff it is a wedge of circles.
- X satisfies D_2 iff it is equivalent to a 3-dimensional complex.
- X satisfies D_n $(n \geq 3)$ iff it is equivalent to a n-dimensional complex.

The key proposition is the following:

Proposition 2.1.1. For $n \geq 3$, suppose X satisfies D_n. Then, given an $n-1$ connected map of CW complexes $\varphi : K^{n-1} \to X$, we have $\pi_n(\varphi)$ a projective Λ-module.

Proof. We have the usual isomorphism $\pi_n(\varphi) \cong H_n(\tilde{X}, \tilde{K}^{n-1})$. By cellular homology, $H_n(\tilde{X}, \tilde{K}^{n-1})$ is isomorphic $C_n(\tilde{X})/B_n(\tilde{X})$, and we have the commutative diagram

$$\begin{tikzcd} C_{n+2} \arrow[r, "d_{n+2}"] & C_{n+1} \arrow[rr, "d_{n+1}"] \arrow[dr, two heads, "j"] & & C_n \\ & & B_n \arrow[ur, hook, "i"] & \end{tikzcd}$$

where j is the induced surjective map from d_{n+1}. Note B_n is naturally a Λ-module, and let us take B_n to be the local coefficient system. Then, j represents a class in $H^{n+1}(X; B_n)$, which by assumption is trivial. Therefore, j is a coboundary, so there exists a $s : C_n \to B_n$ such that $j = s \circ d_{n+1} = s \circ i \circ j$. Since j is epic, we have $s \circ i = Id$, so the short exact sequence

$$0 \longrightarrow B_n(\tilde{X}) \longrightarrow C_n(\tilde{X}) \longrightarrow \pi_n(\varphi) \longrightarrow 0$$

splits. But $C_n(\tilde{X})$ is free, so $\pi_n(\varphi)$ is projective. $\square$

Here is a nice algebraic fact on projective modules:

Lemma 2.2. (Eilenberg Swindle) Given a projective R module M, there exists a free module N (in general not finitely generated) such that $M \oplus N \cong N$.

Proof. By projectivity, there exists a module L such that $F := M \oplus L$ is free. Then, take

$$N := M \oplus L \oplus M \oplus L...$$

By associativity, N is isomorphic to both $F^{\mathbb{N}}$ and $M \oplus F^{\mathbb{N}}$. $\square$

The topological construction corresponding to Eilenberg Swindle is the following: $\pi_n(\varphi_{n-1}) = \pi_n(K^{n-1}, X)$ measures the kernel of the map $\pi_{n-1}(K^{n-1}) \to \pi_{n-1}(X)$, and our goal is to introduce addition kernel to make $\pi_n(\varphi_{n-1})$ free. Let F be the free Λ-module such that $F \oplus \pi_n(\varphi_{n-1})$ is free by the Eilenberg Swindle. Then, attach $n-1$ cells by the constant boundary map, one for each generator of F, to K^{n-1} (equivalent to wedging $n-1$ spheres). Let K'^{n-1} denote the new complex, and we may extend the map to $\varphi'_{n-1} : K'^{n-1} \to X$ by collapsing the new $n-1$ spheres to the basepoint. It is easily verified that the construction makes $\pi_n(\varphi'_{n-1})$ free.

Now let us finish off Theorem 2.1:

Proof of Theorem 2.1. For D_1: by universal coefficient theorem, D_1 implies all homologies of $\tilde{X}$ vanishes, so X is a $K(\pi, 1)$. In particular, $H^2(X; B) \cong H^2_{\mathrm{Grp}}(\pi; B) = 0$ implies every extension of B by π is split. Now let $B = ker(F \to \pi)$, where $F \to \pi$ is any surjection from a free group F to B. Then, π is realized as a direct summand of a free group and thus free.

For D_2: we construct the 2-connected map $\varphi_2 : K^2 \to X$, it is easily showed that $\pi_3(\varphi_2)$ is projective usuing the same argument as in Proposition 2.1.1 with $\pi_3(\varphi_2) \cong B_2$. We finish by first applying the topological Eilenberg Swindle, and then applying the argument outlined in the beginning paragraphs of the section. □

3 $\tilde{K}_0$ and Obstruction to Finiteness

Note that the construction of the complexes of finite n-skeleton, corresponding to F_n, the construction of n-dimensional complexes, corresponding to D_n, are not "compatible constructions". Namely, when we were turning the projective $\pi_n(\varphi)$ to a free module in the construction of finite dimensionality, we added possibly infinitely many $n-1$ cells. The idea is that this is the only obstruction to X being finite n-dimensional when it satisfies D_n and F_n. Algebraically, the obstruction measures how far the projective $\pi_n(\varphi)$ is from being stably free.

Lemma 3.1. Suppose X satisfies F_n and D_n. Let $\varphi_1 : K_1 \to X$ and $\varphi_2 : K_2 \to X$ be two $n-1$-connected maps, with K_1, K_2 finite n-dimensional. Then, $\pi_n(\varphi_1)$ and $\pi_n(\varphi_2)$ represents the same class in $\tilde{K}_0(\mathbb{Z}[\pi_1(X)])$

Proof. First, show that if X satifies D_n, then both φ_i has a homotopy right inverse by basic obstruction theory. Then, composing one of φ with the right inverse of the other gives a map between K_1 and K_2. It is easy to see that the map is $n-1$-connected. We may view K_1 as subcomplex of a complex equivalent to K_2, with possibly extra cells only in dimension n and $n+1$. Then, we can identify $\pi_n(\varphi_1)$ and $\pi_n(\varphi_2)$ with the nth and $n+1$th homology of $(\tilde{K}_2, \tilde{K}_1)$, and an algebraic argument from there finishes. □

The lemma says that $\pi_n(\varphi)$ is an invariant of X, and determines a class in $\tilde{K}_0(\Lambda)$. From the argument of section 2, we may conclude with the obstruction theorem:

Theorem 3.2. If X satisfies D_n and F_n for $n \geq 3$, then there is a obstruction to finiteness $w(X) := [\pi_n(\varphi)] \in \tilde{K}_0(\Lambda)$. Specifically, X is equivalent to a finite n-dimensional CW complex iff $w(X)$ vanishes.

Proof. If X is equivalent to a finite n-dimensional CW complex iff $w(X)$ vanishes, take $\varphi : X \to X$ to be the identity. For the converse, repeat the construction in section 2 on turning $\pi_n(\varphi)$ free, knowing we only have to attach finitely many $n-1$ cells based on the assumption that $\pi_n(\varphi)$ is stably free. □