Étale Homotopy Theory and Adams Conjecture

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Definition 0.0.1 (Čech Nerve)**.** Let X be a finite CW complex, and $\mathcal{U} := \{U_i : i \in I\}$ be an open cover of X. Then, we may define a simplicial set call the **Čech Nerve** $N\mathcal{U}$ as follows: we have the assignment on objects $[n] \mapsto \{\text{functions from } [n] \text{ to } I : \cap_{i=1}^n U_{f(i)} \neq \emptyset\}$. The face maps and degeneracy maps are defined by deleting and inserting appropriate indices.

Alternatively, we can think of a covering $\mathcal{U}$ as follows: suppose given a covering $X = \cup_{i \in I} U_i$; let $\mathcal{U} = \coprod_{i \in i} U_i$, and the covering is the obvious map $\mathcal{U} \to X$. Note that we have

$$U_i \cap U_j = U_i \times_X U_j$$

so the n-fold fiber product $U \times_X ... \times_X U$ is the disjoint union of n-fold intersections of opens in the cover. Then, the nth simplices of the Čech nerve is $\pi_0(\underbrace{U \times_X ... \times_X U}_{n-fold})$. The face maps are projections, and the degeneracy maps are various diagonal embeddings.

Theorem 0.1. If the covering $\mathcal{U}$ satisfies the property that arbitrary intersections of opens in the cover is either empty or contractible, then th realization $|N\mathcal{U}|$ is weakly equivalent to X.

1 Adam's conjecture

Definition 1.0.1. Let X be compact Hausdorff and let $KU(X)$ be Grothendieck group of complex vector bundles over X, and let $\mathcal{SF}(X)$ be the Grothendieck group of sphere bundles over X modulo fiber homotopy equivalence.

Theorem 1.1. The stable sphere bundles over X is classified by the the groups of self-homotopy equivalences of S^n, which we denote by $G(n) := \text{Equiv}(S^n, S^n)$.

Proposition 1.1.1. The complex J-homomorphism $J : KU(X) \to \mathcal{SF}(X)$ is induced by a map between classifying spaces, which we also denote

$$J : BU \to BG := \varinjlim_n BG(n)$$

Definition 1.1.1. The Adams' operation $\psi^k : KU(X) \to KU(X)$ is induced by a map of classifying spaces

$$\psi^k : BU \to BU$$

Theorem 1.2 (The Adams Conjecture)**.** The composite

$$BU \xrightarrow{\psi^k - 1} BU \xrightarrow{J} BG$$

is nullhomotopic up to multiplication by some k^n.

Proposition 1.2.1. The composite $J \circ i : BU(n) \to BU \to BG$, classifyies a sphere bundle over $BU(n)$, and is fiber homotopy equivalent to the fibration

$$BU(n-1) \to BU(n)$$

Fact: the composition $J \circ i : BU(n) \to BG$ classifies the sphere bundle associated to the canonical bundle over $BU(n)$, and it is fiber homotopy equivalent to the fibration

$$BU(n-1) \to BU(n)$$

We can dream of a proof here: if we have unstable adams operations $\psi^k : BU(n) \to BU(n)$, which are homotopy equivalences, with a pullback diagram

$$\begin{tikzcd} BU(n-1) \arrow[r, "\psi^k"] \arrow[d, "i"] & BU(n-1) \arrow[d, "i"] \\ BU(n) \arrow[r, "\psi^k"'] & BU(n) \end{tikzcd}$$

Then, the bundle $J \circ i$ is fiber homotopy equivalent $J \circ \psi^k$. However, unstable Adams operation does not exist on $BU(n)$. However, Sullivan proves that

$$\begin{tikzcd} \widehat{BU(n-1)}_p \arrow[r, "\psi^k"] \arrow[d, "i"] & \widehat{BU(n-1)}_p \arrow[d, "i"] \\ \widehat{BU(n)}_p \arrow[r, "\psi^k"'] & \widehat{BU(n)}_p \end{tikzcd}$$

where ψ^k is an unstable Adams operation on the profinite completion.

2 Algebraic Side

Idea of proof

<u>**Step 1**</u>: Sullivan proves that the stable fiber homotopy types injects into profinite stable homotopy types. In particular, we have the isomorphism on classifying space level

$$\text{Stable profinite theory}: \ \widehat{B}_\infty \cong B_{SG} \times K(\widehat{\mathbb{Z}}^*, 1)$$

$$\text{Stable theory}: \ BG \cong B_{SG} \times K(\mathbb{Z}/2,1)$$

where $B_{SG} = \varinjlim_n B_{SG(n)}$, and $B_{SG(n)}$ is the classifying space for the component of the identity map in $G(n)$. (Alternatively, it is also the universal cover of $BG(n)$). Thus, the classifying space for the stable theory is a direct factor of the stable profinite theory, and it suffice to formulate and prove the Adams conjecture in the profinite setting.

Step 2: We identify the classical Adams operation in the following way: the classical Adams operation

$$K(X) \xrightarrow{\psi^k} K(X)$$

naturally desends to maps on the profinite completion, which factors as

$$\prod_p \widehat{K(X)}_p \xrightarrow{\widehat{\psi}_p} \prod_p \widehat{K(X)}_p$$

and $\widehat{\psi^k}_p : \widehat{K(X)}_p \to \widehat{K(X)}_p$ is an isomorphism iff k is prime to p. If k is divisible by p, we redefine $\widehat{\psi^k}_p$ to be the identity map. After the redefinition, we obtain

$$\widehat{K(X)} \xrightarrow{\psi^k} \widehat{K(x)}$$

which we call the "isomorphic" part of the Adams operation.

> **Remark 2.0.1.** Before the redefinition, in the case where $k|p$, we note that$widehat\psi^k{}_p$ is topologically nilpotent.

Following this, Sullivan observed that this isomorphic part of the Adams operation is compatible with the natural action of $Gal(\overline{\mathbb{Q}}|\mathbb{Q})$ in the category of profinite homotopy type and maps coming from the algebraic varieties defiend over $\mathbb{Q}$. In particular, there iare homomorphisms

$$Gal(\mathbb{Q}|\mathbb{Q}) \to \widehat{\mathbb{Z}}^* \to \mathrm{Aut}(\widehat{K(X)})$$

by letting G act on the roots of unity. Moreover, for each ψ^k, we note that k defines an element $(k) \in \widehat{\mathbb{Z}}^*$ by giving the automorphism

$$(k)x = \begin{cases} k \cdot x & \text{if } x \in \widehat{\mathbb{Z}_p}, (k,p) = 1 \\ x & \text{if } x \in \widehat{\mathbb{Z}_p}, (k,p) \neq 1 \end{cases}$$

Clearly this is compatible with the Adams operation on profinite K theory. Thus, we have identified the profinite Adams operation with the action of the profinite group $\widetilde{\mathbb{Z}}^*$.

Step 3: There is a natural action of the absolute Galois group $Gal(\overline{\mathbb{Q}}|\mathbb{Q})$ on the profinite classifying space $\widehat{BU}$, and the Adams operation is compactible with such action through the abelianization map.

$$Gal(\overline{\mathbb{Q}|\mathbb{Q}}) \to \widehat{\mathbb{Z}}^*$$

The aboslute Galois group action is how the etale homotopy type theory factors in. Note that the absolute galois group acts algebraically on $\mathbb{C}^n$ and $\mathbb{CP}^n$, but with classical topology this action is wildly discontinuous. However, for every algebraic variety V, we may constrtcut an inverse system of nerves N_α, with natural maps $V \to \{N_\alpha\}$ giving

$$\widehat{\pi_1(V)} \cong \varprojlim_\alpha \pi_1 N_\alpha \text{ and } H^i(V; M) \cong \varinjlim H^i(N_{\alpha;M})$$

for all finite coefficient M.

Sullivan proves that the the above isomorphism imply the profinite completion of V can be constructed from the nerves

$$\widehat{V} \cong \varprojlim_{\alpha} N_\alpha$$

in the sense of compact functors (with the extra assumption that $\pi_i(N_\alpha)$ is finite.)

Since each N_α is constructed using the algebraic structure of V, and each automorphism $\sigma \in Gal(\overline{Q}|\mathbb{Q})$ determines a simplicial automorpihism of N_α, and thus the profinite homotopy type of any complex algebraic variety defined over $\mathbb{Q}$.

Recall that the classifying space $BU(n)$ is constructed as the direct limit of complex grassmannians $\varinjlim_k Gr_n(k)$. Via Plücker embeddings, the complex grassmannians are naturally affine complex varieties embedded in projective space. Moreover, the defining polynomials also have coefficients in $\mathbb{Q}$.(Example here?)

By naturality and splitting principal, understanding the action of $Gal(\mathbb{C}|\mathbb{Q})$ on the profinite complex K-theory reduces to understanding the action on $\cup_n \widehat{\mathbb{CP}^n} \cong K(\widehat{\mathbb{Z}}, 2)$, which can be checked to be the composition

$$Gal(\mathbb{C}|\mathbb{Q}) \xrightarrow{\chi} \widehat{\mathbb{Z}}^* \to K(\widehat{Z}, 2)$$

and thus agrees with the isomorphic part of he Adams operations discussed above.

3 The Adams Conjecture-Proof

Again, we note that by choosing $\sigma \in Gal(\overline{\mathbb{Q}}|\mathbb{Q})$ such that $\chi(\sigma) = k^{-1}$, we have the commutative square

$$\begin{tikzcd} \widehat{BU(n)} \arrow[r] \arrow[d, "\sigma"] & \widehat{BU} \arrow[d, "\psi^k"'] \\ \widehat{BU(n)} \arrow[r] & \widehat{BU} \end{tikzcd}$$

<u>**Final Step**</u>: We note that the inclusion map $\widehat{BU(n)} \to \widehat{BU}$ is the tautological spherical fibration. The pullback of the fibration is also the bundle classfied by the map $\psi^k \circ i$. In other words, we have the homotopy cartesian square

$$\begin{tikzcd} B\widehat{U(n-1)} & B\widehat{U(n-1)} \arrow[l, "\psi^k"'] \arrow[d, "i"'] \\ \widehat{BU(n)} \arrow[from=u, "i"] & \widehat{BU(n)} \arrow[l, "\psi^k"'] \end{tikzcd}$$

where we are pulling back along a homotopy equivalence, so the two fibrations are fiber homotopically equivalent, and we finish.