The Classifying Space of a Small Category

David Zhu

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1 The Simplex Category Δ

Definition 1.0.1. Let Δ be the category of finite, totally ordred sets, where each object is represented by a class $[n] := (0 < 1 < ... < n)$, and the morphisms are non-decreasing functions. The category Δ is also called the **simplex category**.

By elementary combinatorics, we have the following fact:

Proposition 1.0.1. Given $i, n \geq 0$, there are $\binom{n+1+i}{i+1}$ morphisms from $[i]$ to $[n]$.

Similar to cycles in symmetric groups, we can break down each morphism in Δ into compositions of building blocks called face/degeneracy maps.

Proposition 1.0.2. Fix $n \geq 0$. There are $n+1$ injective maps of the form $\epsilon^k : [n-1] \to [n]$, whose image miss k in $[n]$. Similarly, there are $n+1$ surjective maps of the form $\eta^k : [n+1] \to [n]$, with two elements mapping to k in $[n]$. The explicit formular are given by:

$$\epsilon^k(j) = \begin{cases} j & j < k \\ j+1 & j \geq k \end{cases}$$

$$\eta^k(j) = \begin{cases} j & j \leq k \\ j-1 & j > k \end{cases}$$

Definition 1.0.2. The maps ϵ^* are called **coface maps** and η^* are called **codegeneracy maps**.

Lemma 1.1. Every morphism $[n] \to [m]$ can be uniquely decomposed as $\epsilon \circ \eta$, where η and ϵ are compositions of degeneracy maps and face maps, respectively.

Proof. Suppose the image of $[n] \to [m]$ consists of $k+1$ elements, such that $k \leq m, n$. Then, the map will factor as

$$[n] \xrightarrow{\eta} [k] \xrightarrow{\epsilon} [m]$$

and the construction of η and ϵ is obvious. □

It is easy to verify the following composition identities for face maps and degeneracy maps:

Proposition 1.1.1. (Simplicial identities) The following hold:

$$\begin{cases} \epsilon^i\epsilon^j = \epsilon^{j-1}\epsilon^i & i < j \\ \eta^i\eta^j = \eta^{j+1}\eta^i & i \leq j \end{cases}$$

$$\eta^j\epsilon^i = \begin{cases} \mathrm{Id} & i = j, j+1 \\ \epsilon^i\eta^{j-1} & i < j \\ \epsilon^{i-1}\eta^j & i > j+1 \end{cases}$$

The lemma and the proposition shows that the data of the morphisms in Δ can be completely recovered from the face maps and degeneracy maps alone.

2 Simplicial Sets

Definition 2.0.1. Let $\mathcal{C}$ be any category. A **simplicial object** in $\mathcal{C}$ is a functor $X : \Delta^{\text{op}} \to \mathcal{C}$. In particular, a functor $X : \Delta^{\text{op}} \to \mathbf{Set}$ is called a **simplicial set**. The elements in the set $X_n := X[n]$ are called $n-$**simplices**.

If we dualize the above definition, we get the cosimpicial objects/sets, and here is an important example:

Example 2.0.1. (Topological n-simplex) For each $[n]$, we associate the standard topological n-simplex

$$|\Delta^n| := \{(x_0, ..., x_n) \in \mathbb{R}^{n+1} : \sum_{i=0}^{n} x_i = 1, x_i \geq 0\}$$

with vertices $\{v_i\}$ being points with ith coordinate 1 and 0 on all other coordinates. Each morphism $\alpha : [n] \to [m]$ induces a morphism $\alpha_* : |\Delta^n| \to |\Delta^m|$ by first sending vertices $v_i \mapsto v_{\alpha(i)}$, then extending linearly onto all Δ^n. This defines a functor $\Delta \to \mathbf{Top}$, which is a cosimplicial object.
Topological, the coface map ϵ_*

By the discussion of the previous section, we may package the data of a simplicial object in the following form:

Theorem 2.1. The data of a simplicial object in $\mathcal{C}$ is equivalent to a collection of objects X_n for $n \geq 0$, together with **degeneracy maps** $d_i : X_n \to X_{n+1}$ and **face maps** $s_i : X_n \to X_{n-1}$ for $0 \leq i \leq n$ satisfying the composition laws dual to that of proposition 1.1.1.

Example 2.1.1. (The standard n-simplex) We recognize the category of simplicial sets, denoted **sSet**, as the functor category $\mathbf{Set}^{\Delta^{\mathrm{op}}}$. By the Yoneda lemma, the contravariant functor $h : \Delta \to \mathbf{sSet}$ given by $h([n]) :=$ Hom(-,[n]) is full and faithfull, and represents a simplicial set. The object $h([n])$ is called the **standard n-simplex**.

Combinatorially, the k-simplices in the standard n-simplicies are maps in $\mathrm{Hom}([k], [n])$. Geometricially, each morphism $[k] \to [n]$ is understood as the inclusion of the k-dimensional faces into the geometric n-simplex. The face map is precisely taking a k-face to a $k-1$ face by deleting a vertex.

Example 2.1.2. (The nerve of a category) Given a small catgeory $\mathcal{C}$, we define the nerve of $\mathcal{C}$, denoted $N\mathcal{C}$, as the simplicial set consisting of the following data: the objects are

$$N\mathcal{C}_n := \{\text{string of n-composable arrows in } \mathcal{C}\}$$

, where $N\mathcal{C}_0 = \mathrm{Ob}\,\mathcal{C}$ and $N\mathcal{C}_1 = \mathrm{Mor}\,\mathcal{C}$. The face map $s_i : N\mathcal{C}_n \to N\mathcal{C}_{n-1}$ is given by composing the ith and $i+1$th morphism into one if $0 < i < n$, and leaves out the first or last morphism when $i = 0, n$; the degeneracy map $d_i : N\mathcal{C}_n \to N\mathcal{C}_{n+1}$ is inserting the identity map at the ith spot.

3 Total Singular Complex and Geometric Realization

The goal of simplicial sets is to capture topological information categorically: the fundamental groupoid is able to capture π_0 and π_1, but fails to see any higher homotopy groups; the more powerful simplicial set is able to capture all homotopy groups and their interrelations (under mild assumptions). We now describe the two functors, **Sing** and $|*|$ that bridges the topological side and simplicial side.

$$\begin{tikzcd} \mathbf{Top} \arrow[r, bend left, "\mathbf{Sing}"] & \mathbf{sSet} \arrow[l, bend left, "|*|"] \end{tikzcd}$$

Definition 3.0.1. We define the **total singular complex** functor $\mathbf{Sing} : \mathbf{Top} \to \mathbf{sSet}$ as follows: For X a topological space, we associate the simplicial set $\mathbf{Sing}_\bullet(X) : \Delta \to \mathbf{Set}$ defined by

$$\mathbf{Sing}_n(X) := \mathrm{Hom}(|\Delta^n|, X)$$

Given a morphism $\alpha : [n] \to [m]$, we have the induced morphism $\alpha^* : \mathbf{Sing}_m(X) \to \mathbf{Sing}_n(X)$ by precomosition with the map α_* defined in example 2.0.1.