

# Étale Homotopy Theory and Adams Conjecture

David Zhu

October 30, 2024

**Definition 0.0.1** (Čech Nerve ). Let  $X$  be a finite CW complex, and  $\mathcal{U} := \{U_i : i \in I\}$  be an open cover of  $X$ . Then, we may define a simplicial set call the **Čech Nerve**  $N\mathcal{U}$  as follows: we have the assignment on objects  $[n] \mapsto \{\text{functions from } [n] \text{ to } I : \cap_{i=1}^n U_{f(i)} \neq \emptyset\}$ . The face maps and degeneracy maps are defined by deleting and inserting appropriate indices.

Alternatively, we can think of a covering  $\mathcal{U}$  as follows: suppose given a covering  $X = \cup_{i \in I} U_i$ ; let  $\mathcal{U} = \coprod_{i \in I} U_i$ , and the covering is the obvious map  $\mathcal{U} \rightarrow X$ . Note that we have

$$U_i \cap U_j = U_i \times_X U_j$$

so the  $n$ -fold fiber product  $U \times_X \dots \times_X U$  is the disjoint union of  $n$ -fold intersections of opens in the cover. Then, the  $n$ th simplices of the Čech nerve is  $\pi_0(\underbrace{U \times_X \dots \times_X U}_{n\text{-fold}})$ . The face maps are projections, and the degeneracy maps are various diagonal embeddings.

**Theorem 0.1.** If the covering  $\mathcal{U}$  satisfies the property that arbitrary intersections of opens in the cover is either empty or contractible, then th realization  $|N\mathcal{U}|$  is weakly equivalent to  $X$ .

## 1 Adam's conjecture

**Definition 1.0.1.** Let  $X$  be compact Hausdorff and let  $KU(X)$  be Grothendieck group of complex vector bundles over  $X$ , and let  $\mathcal{SF}(X)$  be the Grothendieck group of sphere bundles over  $X$  modulo fiber homotopy equivalence.

**Theorem 1.1.** The stable sphere bundles over  $X$  is classified by the the groups of self-homotopy equivalences of  $S^n$ , which we denote by  $G(n) := \text{Equiv}(S^n, S^n)$ .

**Proposition 1.1.1.** The complex  $J$ -homomorphism  $J : KU(X) \rightarrow \mathcal{SF}(X)$  is induced by a map between classifying spaces, which we also denote

$$J : BU \rightarrow BG := \varinjlim_n BG(n)$$

**Definition 1.1.1.** The Adams' operation  $\psi^k : KU(X) \rightarrow KU(X)$  is induced by a map of classifying spaces

$$\psi^k : BU \rightarrow BU$$

**Theorem 1.2** (The Adams Conjecture). The composite

$$BU \xrightarrow{\psi^k - 1} BU \xrightarrow{J} BG$$

is nullhomotopic up to multiplication by some  $k^n$ .

**Proposition 1.2.1.** The composite  $J \circ i : BU(n) \rightarrow BU \rightarrow BG$ , classifies a sphere bundle over  $BU(n)$ , and is fiber homotopy equivalent to the fibration

$$BU(n-1) \rightarrow BU(n)$$