Equivariant Stable Homotopy Notes

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September 30, 2024

For the entire note, we will assume a group G to be a compact Lie group, and subgroups $H \subset G$ are always closed.

1 Unstable Equivariant Homotopy Theory

1.1 G-CW Complexes

Fix a compact Lie group G acting on a space X. Similar to CW-complexes, we want to deconstruct X into cells, but this time with the addditional data of the G-action along with each cell. The idea is that cells are of the form of a product $G/H \times D^n$, where G acts trivially on D^n, and G/H "represents" the orbits of D^n. To make this work, H must be the isotropy group of D^n.

Definition 1.0.1. A **G-CW complex** is the sequential colimit of spaces X_n, where X_{n+1} is a pushout:

$$\begin{tikzcd} \coprod G/H \times S^n \arrow[r] \arrow[d] & X_n \arrow[d] \\ \coprod G/H \times D^{n+1} \arrow[r] & X_{n+1} \end{tikzcd}$$

We will denote $G/H \times D^n$ as an **n-cell**.

Remark 1.0.1. Note that the topological dimension of an n-cell in a G-CW complex might be greater than n. For example, a 0-cell $S^1/e \times *$ is one dimensional.

Example 1.0.1. Let $G = C_2$ acting on S^2 by rotation by π along the Z-axis. It has a G-CW structure given by the following cells: 2 zero-cells $C_2/C_2 \times *$, which are the poles corresponding to the fixed points of the C_2 action. 1 one-cell $C_2/e \times D^1$, which are the two great circles joining the poles; 1 two-cell $C_2/C_2 \times D^2$, which are the two hemispheres.

Example 1.0.2. Let $G = C_2$ acting on S^2 by the antipodal map. It has a G-CW structure given by the following cells: 1 zero-cells $C_2/e \times *$, which are the poles; 1 one-cell $C_2/e \times D^1$, which are the two great circles joining the poles; 1 two-cell $C_2/C_2 \times D^2$, which are the two hemispheres.

Definition 1.0.2. Let H be a subgroup of G. Define $\pi_n^H(X) := \pi_n(X^H)$. A map $f : X \to Y$ of G-spaces is a **weak equivalence** if for all subgroups $H \subset G$,

$$f_* : \pi_n^H(X) \to \pi_n^H(Y)$$

is an isomorphism.

Let **GTop** be the category of G-spaces and G-maps. There is a cofibrantly-generated model structure that we can put on **GTop**:

Theorem 1.1. There is a cofibrantly-generated model structure on **GTop**, given by
1. A G-map $f : X \to Y$ is a fibration iff for all $H \subset G$, $f^H : X^H \to Y^H$ is a fibration.
2. A G-map $f : X \to Y$ is a weak equivalence iff for all $H \subset G$, $f^H : X^H \to Y^H$ is a weak equivalence.

An immediate consequence of the model category structure is the equivariant Whitehead's Theorem

Corollary 1.1.1. Let $f : X \to Y$ be a weak equivalence of cofibrant-fibrant objects in a model category. Then, f is a homotopy equivalence. In particular, every object in **GTop** is fibrant, and G-CW complexes are cofibrant.

1.2 Elmendorf's Theorem

From the model structure given in Theorem 1.1, we have a vague sense of the following "equivalence":

$$\text{G-Homotopy Type of } X \Leftrightarrow \{\text{ordinary homotopy type of } X^H : H \subset G\}$$

And Elmendorf's Theorem will make the equivalence precise. We start by introducing the orbit category:

Definition 1.1.1. The **orbit category** $\mathcal{O}_G$ is the full subcategory of GTop on the objects $\{G/H : H \subset G\}$.

The following lemma will make the structure of $\mathcal{O}_G$ clearer.

Lemma 1.2. $\mathrm{Map}^G(G/H, G/K) \cong (G/K)^H$

Proof. Note that there exists a G-equivariant maps $\varphi : G/H \to G/K$, determined by $\varphi(H) = gK$ iff $gHg^{-1} \subseteq K$ iff $h(gK) = gK$ for all $h \in H$. □

Let $\mathrm{Fun}(\mathcal{O}_G^op, \mathrm{Top})$ be the functor category. We have the following fact on the model structure on functor categories:

Theorem 1.3. Let $\mathcal{D}$ be a model category and $\mathcal{C}$ be a cofibrantly generated model category. Then, $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ admits a model structure.

It is useful to know that the weak equivalences in $\mathrm{Fun}(\mathcal{O}_G^{op}, \mathrm{Top})$ is given pointwise: a natural transformation $\eta : \mathcal{F} \to \mathcal{G}$ is a weak equivalence iff $\eta_{G/H} : \mathcal{F}(G/H) \to \mathcal{G}(G/H)$ is a weak equivalence.

Definition 1.3.1. There is a functor $\psi : \text{GTop} \to \text{Fun}(\mathcal{O}_G^{op}, \text{Top})$ given by

$$X \to (G/H \mapsto X^H)$$

It is easy to check the functoriality. Note that if we restrict ψ to $\mathcal{O}_G$, the functor is just the Yoneda embedding: $\text{Map}^G(G/H, G/K) \cong (G/K)^H$.

Proposition 1.3.1. There is a funcor $\theta : \text{Fun}(\mathcal{O}_G^{op}, \text{Top}) \to \text{GTop}$ given by $X \mapsto X(G/e)$, where $X(G/e)$ is equipped with the following G-action: note that every $g \in G$ defines an G-map $G/e \to G/e$, which we denote by R_g.

$$g \cdot x = X(R_g)(x)$$

It is easy to check that (θ, ψ) is an adjoint pair. In fact, more can be said:

Theorem 1.4. (Elmendorf's Theorem) $\text{Fun}(\mathcal{O}_G^{op}, \text{Top})$ and GTop have the same homotopy category.

The original proof due to Elmendorf constructs the equivalence explicitly using the Bar construction to obtain a homotopy inverse to the embedding ψ. The theorem can now be put into a more modern framework:

Theorem 1.5. (θ, ψ) is an Quillen equivalence. ψ is an equivalence of $(\infty, 1)$ categories.

1.3 Bredon Cohomology

The goal is to construct a cohomology theory satisfying the Eilenberg-Steenrod axioms under the equivariant setting.

Definition 1.5.1. (Equivariant reduced generalized cohomology) Let $G\text{CW}_*$ be the category of pointed G-CW complexes with equivariant maps. Then, a generalized cohomology theory on $G\text{CW}_*$ is a sequence of contravariant functors

$$\tilde{H}^n := G\text{CW}_* \to \mathbf{Ab}$$

satisfying the following:

1. if f, g are equivariantly homotopic, then $\tilde{H}^n(f) = \tilde{H}^n(g)$.
2. There exists a sequence of natural isomorphisms

   $$\tilde{H}^n(X) \to \tilde{H}^{n+1}(S^1 \wedge X)$$

   where G acts trivially on S^1 in the smash.
3. The sequence

   $$\tilde{H}^n(X/A) \to \tilde{H}^n(X) \to \tilde{H}^n(A)$$

   is exact.

Remark 1.5.1. The above axioms is built upon pointed "single" spaces. It is in fact equivalent to the usual theory built upon pairs, and is justified in

For a non-equivariant reduced generalized cohomology theory $\tilde{h}^*$, the Atiyah-Hirzebruch spectral sequence tells us that knowing $\tilde{h}^*(pt)$ basically determines the cohomology theory on CW complexes. Heuristically, the cohomology is determined by the building blocks, which are contractible open cells. However, in the equivariant setting, the building blocks are more complicated: the building blocks have become orbits of the form G/H. We are lead to the following definitions:

Definition 1.5.2. A **coefficient system** is a contravariant functor $\mathcal{F} : \mathcal{O}_G \to \mathrm{Ab}$.

Recall that if a reduced cohomology theory is called **ordinary** if it satisfies the dimension axiom, i.e the zeroth reduced cohomology group (a.k.a the coeffcient system) is trivial on a point. Our goal now is to construct such a theory in the equivariant setting, which is called Bredon cohomology.

Note that by the general theory of abelian categories, the functor category of coefficient systems $\mathcal{CS} := \mathrm{Fun}(\mathcal{O}_G, \mathrm{Ab})$ is abelian. It is now possible to define Bredon cohomology on a G-CW complex by explicitly defining the cochain complexes on cells, for example see . However, we may package the cochains into the following form

Definition 1.5.3. For each n, we may define a coefficient system $C_n(-)$, given by

$$G/H \mapsto H_n((X^H)_n, (X^H)_{n-1}; \mathbb{Z}) = C_n^{CW}(X^H)$$

The differential of the CW chain complex induces a chain complex of coefficient systems $C.(-)$.

It is easy to check that $Hom_{\mathcal{CS}}(C.(-), M)$ is a cochain complex whose differentials is induced by those of $C.()$.

Definition 1.5.4. The **Bredon cohomology** of X with coefficients in a system M is defined by

$$H^n_G(X; M) := H^n(Hom_{\mathcal{CS}}(C.(X), M))$$

proof of the axioms and computation of examples.

1.4 Classical Application

homotopy fixed points and actual fixed points: coarse vs fine Here are two questions we may ask: given a finite p-group G acting on X, can we recover the cohomology of X^G based on the cohomology of X, while somehow incoporating the G-action? The second point of interest is the Sullivan conjecture: in The study of étale homotopy theory allows one to determines algebraically the profinite completion of a variety given only its étale homotopy type. The same situation occurs as before: given a variety V over the complex numbers, there is a natural $\mathbb{Z}/2$-action on the variety, with $V^G = V(\mathbb{R})$. How much can we say about $V(\mathbb{R})$ if we are given the étale homotopy type of V and the G-action? IN this case, the fixed points is not a homotopy invariant, so we should consider the homotopy fixed points instead, and the sullivan conjecture states that

Theorem 1.6. (Sullivan Conjecture) $V(\mathbb{C})^{\mathbb{Z}/2} \to V(\mathbb{C})^{h\mathbb{Z}/2}$ becomes an isomorphism after 2-adic completion. More generally, let G be a finite p-group, then $X^G \to X^{hG}$ becomes an equivalence after p-adic completion.

One possible application is a quick proof of a theorem by Smith:

Theorem 1.7. Let G be a finite p-group and X be a finite G-CW complex such that (the underlying topological space of) X is an p-cohomology sphere.16 Then, X^G is either empty or an p-cohomology sphere of smaller dimension.

2 Spectra and the Stable Category

2.1 Motivations

The first motivation is Brown Reresentability Theorem, which is about represeting reduced cohomology theories

Theorem 2.1 (Brown Representability Theorem)**.** Let $\tilde{h}^*$ be a reduced cohomology theory on pointed CW-complexes. Then, for each $n \in \mathbb{Z}$, there exists a connected pointed CW complex K_n such that

$$\tilde{h}^n(X) \cong [X, K_n]$$

for all n. Moreover, the K_n are determined up to homotopy equivalence.

In fact, there are more structure to the set $\{K_n\}$: using the suspension axiom and loop-suspension adjunction, we see that there must be an isomorphism

$$[X, K_n] \cong \tilde{h}^n(X) \cong \tilde{h}^{n+1}(\Sigma X) \cong [\Sigma X, K_n] \cong [X, \Omega K_n]$$

Taking X to be K_n, we note that the indetitiy map from the LHS corresponds uniquely to a map $\alpha_n : K_n \to \Omega K_n$, which we will call the **structure map.**. By naturality and taking $A = S^k$, the structure map induces a weak equivalence

$$K_n \cong \Omega K_n$$

. This motivates the definintion for Ω-Spectra

Definition 2.1.1. A **spectrum** is a sequence of pointed topological spaces $\{X_n\}$ with **structure maps**

$$\Sigma X_n \to X_{n+1}$$

. An **Ω spectrum** is a spectrum whose adjoint structure maps

$$X_n \to \Omega X_{n+1}$$

are weak equivalences.

So we see that a reduced cohomology theory corresponds to an Ω-spectrum. In fact, it is not hard to show that an Ω-spectrum defines a reduced cohomology theory as well. We can then present Brown Representability Theorem in the following way:

Theorem 2.2 (Brown Representability Theorem)**.** Every reduced cohomology theory on the category of basepointed CW complexes has the form $\tilde{h}^n(X) = [X, K_n]$ for some Ω-spectrum $\{K_n\}$

Example 2.2.1 (Eilenberg-Maclane Spectrum)**.** The Ω-spectrum that represents reduced ordinary cohomology with coefficients in $\mathbb{Z}$ is given by the Ω-spectra that is the Eilenberg-Maclane spaces $HG := \{K(G, n)\}$.

Example 2.2.2. The Ω-spectrum that represents reduced complex topological K-theory is given by $\{KU_n\}$, where

$$KU_n = \begin{cases} BU \times \mathbb{Z} & \text{n}-\text{even} \\ \Omega BU & \text{n}-\text{odd} \end{cases}$$

In particular, Bott-periodicity shows that the structure maps are weak-equivalences.

Here is a natural question: how do we design a category of spectra, such that an "equivalence" if spectra will give equivalent reduced cohomology theories?

The second motivation is stable homotopy groups and stable maps:

Definition 2.2.1. Let X and Y be pointed CW-complexes. The set of **stable homotopy classes of maps** from X to Y is defined to be

$$[X, Y]^s := \varinjlim_k [\Sigma^k X, \Sigma^k Y]$$

One form of the Freudenthal suspension theorem says that if Y is n-connected and X has dimension less than $2n+1$, then the suspension map $[X, Y] \to [\Sigma X, \Sigma Y]$ is bijective. In this case, we see that the colimit actual stablizes after at a finite stage.

Definition 2.2.2. For a pointed CW-complex, the n-th **stable homotopy group** is defined to be

$$\pi_n^{st}(X) := \varinjlim_k \pi_{n+k}(\Sigma^k X) = \varinjlim_k [\Sigma^k S^n, \Sigma^k X] = [S^n, X]^s$$

By definition, we see that the stable homotopy group should be the homotopy group in some "stable category". Moreover, stable homotopy groups actually defines a reduced homology theory: the two axioms that are not trivial are the LES and the wedge axiom. The key point is that Blaker's Massey Theorem plus the LES of homotopy groups of a pair will give us the LES of stable homotopy groups of a pair; the wedge axiom follows from that $\Sigma^i X \wedge \Sigma^i Y$ is the $2i-1$ skeleton of $\Sigma^i X \times \Sigma^i Y$. Generalizing this, we have

Theorem 2.3. Let K be a CW-complex. The sequence $h_i(X) = \pi_i^s(X \wedge K)$ forms a reduced homology theory on the category of basepointed CW-complexes and basepoint-preserving maps.

We can also define the homotopy groups of a spectrum as a generalization:

Definition 2.3.1 (Homotopy groups of a spectrum)**.** Suppose $K = \{K_i\}$ is a spectrum. Then,

$$\pi_n(K) := \varinjlim \pi_{n+i}(K_i)$$

where the inductive limit is induced by the suspension structure maps.

Example 2.3.1. Given a topological space X, we have the associated **suspension spectrum** $\Sigma^\infty X = \{\Sigma^k X\}$. Then, the stable homotopy groups of X is given by the homotopy groups of its associated suspension spectrum.

Definition 2.3.2. Given a spectrum $K = \{K_n\}$ and a CW complex X, there is a associated smash spectrum $K \wedge X$, where $(K \wedge X)_n = K_n \wedge X$. The structure maps is given by

$$\Sigma(K \wedge X)_n = \Sigma K_n \wedge X \to K_{n+1} \wedge X = \Sigma(K \wedge X)_{n+1}$$

Proposition 2.3.1. Let K be a spectrum. Given a CW complex X, the sequence

$$h_n := \pi_n(X \wedge K)$$

is a reduced homology theory.

The proof mostly follows from the same tactics in proving the stable homotopy groups being a reduced homology theory.

Example 2.3.2 (Singular homology)**.** If $K = HG$ is the Eilenberg-Maclane spectrum, then $h_i(X) := \pi_i(X \wedge K)$ is isomorphic to singular homology with coefficients in G. We only have to check the dimension axiom

$$h_i(S^0) := \varinjlim_n \pi_{n+i}(K(G,n))$$

which is trivial except $i = 0$.

Thus, we have seen that the singular homology is recovered from a spectrum; moreover, a spectrum gives a reduced homology theory by proposition 2.3.1.

2.2 The Desired Stable Homotopy Category

Building upon the motivations in the previous section, we want to build a category $\mathcal{S}$ that captures the stable phenomena and cohomology theories/homology theories. In particular, we want it to satisfy the following properties (not necessarily axioms):

1. There is a functor $\Sigma^\infty : \mathrm{HoTop}_* \to \mathrm{Sp}$, together with an adjoint $\Omega^\infty : \mathcal{S} \to \mathrm{HoTop}_*$.
   The motivation for this is the suspension spectrum.

2. Let A, B be CW complexes, where A has finite dimension. Then, there is a natural isomorphism

   $$[\Sigma^\infty A, \Sigma^\infty B] \cong [A, B]^s$$

   The motivation for this is to capture stable morphisms.

3. $\mathcal{S}$ has a closed symmetric monoidal structure, with $\wedge$ as the tensor product, and $F(-,-)$ as internal hom. $\Sigma^\infty S^0$ is the identity object.

4. The morphisms of $\mathcal{S}$ has the structure of graded abelian groups, with bilinear composition. We denote that graded components as $[-,-]_*$. Moreover, given a reduced cohomology theory E^* there exists an object K in Sp such that

   $$E^*(A) \cong [\Sigma^\infty A, K]_{-*}$$

   The motivation for this is to represent cohomology theories.

5. For every object K in $\mathcal{S}$, one defines a reduced homology theory via

   $$E_n(A) := \pi(K \wedge A)$$

The motivation for this is to deterime homology theories.

6. $\mathcal{S}$ has a closed symmetric monoidal structure, with $\wedge$ as the tensor product and $F(-,-)$ as internal hom. The object $\Sigma^\infty S^0$ is the identity object.
   The motivation for this is that smashing with the suspension funcor induces isomorphism of stable homotopy groups, so it should induce an equivalence of spectra. The tensor-hom adjunction is going to help set up duality statements similar to Spanier-Whitehead duality and Poincaré duality..

In particular, property 6 was the central one in the sense that the entire theory began with the work of Spanier and Whitehead with their search for a category where one has the desired dualities in homotopy theory. It is also the structure that seems to be the hardest to obtain.

2.3 Attempts

2.3.1 The Spanier-Whitehead Category

The first attempt to construct such a category is the Spanier-Whitehead category, where the objects are finite CW complexes, and the morphisms are stable maps. The catgeory was originally set up to perform Spanier-Whitehead duality (chronologically it was), but the objects "almost" represents cohomology theories. We use the word almost because since the objects are only finite complexes, we do not have arbitrary products and coproduts, and the wedge axiom do not hold.

2.3.2 An Attempt for Sp

By Brown representability, it should be natural to consider a category with objects spectra, and the natural choice for morphisms between spectra is a collection of level-wise maps that is compatible with the structure maps. A natural candidate for homotopies in this category is a map $H : X \wedge I_+ \to Y$ However, it turns out there are not enough maps in this category, such that non-homotopy equivalent spectra might represent isomorphic cohomology theories. Adams then in his 74 paper resolves the issue by consider equivalences classes of "cofinal spectra," which turns out to be quite a complicated a construction.

2.3.3 Ho(Sp) and Structured Spectra

The first construction in line with modern treatment of the subject is Bousfield and Friedlander's construction, which puts a model structure on the naive category of spectra, and considers its homotopy category. This category turns out satisfies all the properties that people wanted. However, people still searched for a point-set model that does so without going through the homotopy category. Lewis in 91 showed that it is impossible to put a symmetric monoidal structure on the naive category of spectra that also satifies a list of axioms. The problem laid in the missing of an appropriate commutative smash product. People later constructed more structured spectra, such as orthogonal spectra and symmetric spectra, whose additional structure then provided a fix to such a problem.