MATH 624 HW2

David Zhu

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Homework 2

Problem 1b

Suppose U_f is not empty. Let $W = \{a \in V : k(a)|k \text{ is a finite algebraic extension}\}$, which corresponds to the vanishing locus of maximal ideals of $k[V]$. Clearly $W \subset V(\bar{k})$, so it suffices to show that $W \cap U_f$ is dense in in U_f for every f, which is equivalent to every open U_f containing a point in W. To see this, consider a maximal ideal in $k[V]_f$, which must be the image of a maximal ideal in $k[V]$ under localization: suppose otherwise, then every maximal ideal of $k[V]$ contains f, which implies f is in the Jacobson radical of $k[V]$. However, $k[V]$ has trivial Jacobson radical since $k[X]$ is Jacobson, which implies $f = 0$ and U_f is empty, and contradiction. Then, the locus of the maximal ideal is contyained in $U_f \cap W$.

Problem 2b

A representative of $\tilde{O}_a$ is given by a pair $(W_1, \frac{f_1}{g_1})$, with $g_1 \neq 0$ on W_1, and $(W_1, \frac{f_1}{g_1}) \sim (W_2, \frac{f_2}{g_2})$ iff there exists a open $U_{h'} \subset W_1 \cap W_2$ such that $\frac{f_1}{g_1} = \frac{f_2}{g_2}$ on $U_{h'}$. On the other hand, a representative of $k[V]_{\mathfrak{p}_a}$ is given by some $\frac{f}{g}$, where $g(a) \neq 0$. By continuity, there exists a basic open U_h containing a on which g does not vanish. We define the k-algebra homomorphism:

$$i : k[V]_{\mathfrak{p}_a} \to \tilde{O}_a \quad \frac{f}{g} \mapsto (U_h, \frac{f}{g})$$

Surjectivity is obvious by construction, so there are two things to check: well-definedness (it is clearly that this will be a k-algebra morphism once we check well-definedness) and injectivity.

Well-definedness: suppose $\frac{f}{g} \sim \frac{f'}{g'}$ in $k[V]_{\mathfrak{p}_a}$, which means there exists some $h' \in K[V]$ such that $h'(fg' - f'g) = 0$, which implies $\frac{f}{g} = \frac{f'}{g'}$ on $U_{h'}$. Thus, both will be mapped to the equivalence class $(U_{h'}, \frac{f}{g})$.

Injectivity: suppose $i(\frac{f}{g}) = (U_h, \frac{f}{g})$ represents the 0 element. WLOG, we may assume that f vanishes on U_h, for otherwise we may replace U_h with a smaller basic open. Then, $\frac{f}{g} \sim \frac{0}{1}$ in $k[V]_{\mathfrak{p}_a}$ since $h(f \cdot 1 - g \cdot 0)$ is identically 0 on V.

Problem 3b

By problem 2b, the stalk is isomorphic to $k[V]_{p_a}$, which is always local. In regards to when $k[V]_{p_a}$ is a not a domain, it will be when there exists an $x \in p_a$ such that $\exists y \in p_a$ and $xy = 0$, but $xz \neq 0$ for every non-zero $z \notin p_a$. For example, let $V = V(xy)$. Then, $k[V] = k[x, y]/(xy)$. Take $a = (0, 0)$, then $p_a = (x, y)$, and we have $xy = 0$ but $xz \neq 0$ for every non-zero z not in (x, y).

Note that a reduced Noetherian ring is integral iff it has a unique minimal prime. Another method of detection for integrality is iff p_a contains a unique minimal prime of $k[V]$ (because it is reduced Notherian), which corresponds to a belonging to a unique irreducible component.

Problem 4

(a)

V is irreducible iff $I(V)$ is prime iff $k[V]$ is a domain iff $k(V)$ is a field. The Krull dimension of $k(V)$ and the trascendence degree are the same by Noether normalization.

(b)

Take the finite set of minimal primes $\{p_1, ..., p_n\}$ of $k[V]$, and recall that the union of the minimal primes is precisely the zero-divisors of $k[V]$, and the intersection is the trivial nilradical. Then, localize at $S = k[V] \setminus \cup p_i$, and $S^{-1}k[V]$ has unique maximal primes $S^{-1}p_1, ..., S^{-1}p_n$, which are coprime. By chinese remainder, we have

$$k(V) = S^{-1}k[V]/(0) = S^{-1}k[V]/\cap S^{-1}p_i \cong \prod k(V_i)$$

(c)

(d)

Problem 5

Problem 10

(a)

For $\mathbb{A}^1 \to \mathbb{A}^2$ given by $a \mapsto (a, \frac{1}{a})$, the general situation is discussed in problem 8; for $a \mapsto (a^2, a^3)$, the domain in the entire $\mathbb{A}^1$, and the image is a affine algebraic set given by $V(x^3 - y^2)$. The map is clearly a bijection and a homeomorphism. However, the k-morphism is not an isomorphism, as the coordinate rings $k[t^2, t^3]$ and $k[t]$ are not isomorphic.

(b)

As in part (a), we see that it is possible for the $k-$morphism to not be an isomorphism. However in the case $\mathbb{A}^1 \mapsto \mathbb{A}^3$ given by $a \mapsto (a^1, a^2, a^3)$, the k-morphism is an isomorphism.

Problem 4c

Suppose V is irreducible. Note that $k[V_{k^s}] \cong k[V] \otimes_k k^s$, so $k(V_{k^s}) \cong k(V) \otimes_k k^s$ after taking the field of fractions. Thus, absolute irreducibility of V is equivalent to the integrality of $k(V_{k^s}) \cong k(V) \otimes_k k^s$. Suppose $\overline{k} \cap k(V)$ is not purely inseparable over k, so there exists α algebraic over k, and $k(\alpha) \otimes_k k(\alpha)$ is a subring of $k(V) \otimes_k k^s$, which is not integral. To see this, note, let $p(t)$ be a minimal polynomial of α, then

$$k(\alpha) \otimes_k k[t]/p(t) \cong k(\alpha)[t]/p(t)$$

cleary has $(x - \alpha)$ as a zero-divisor.

Conversely, suppose $k(V) \cap \overline{k}$ is purely inseparable. It suffices to show that $k(V) \otimes_k k[t]/p(t) \cong k(V)[t]/p(t)$ is integral for every irreducible $p(t)$. If there is $q(t) \in k(V)[t]$ that divides $p(t)$, then $q(t)$ is also contained in $k^s[t]$, so $q(t) \in (k^s \cap k(V))[t] = k[t]$, which forces it to be 1 or $p(t)$, and the ring is still integral.

Problem 5

(a)

The correct statement should be $\tilde{O}_x$ is a domain iff x is contained in a unique irreducible component, and the proof is given in problem 3.

(b)

It is a standard point-set topology argument that finite intersection of open dense sets is still open and dense.

(c)

The colimit is the function field of V. The detail proofs are given in HW3 problem 10.

Problem 8

(a)

Clearly the empty set and the whole line is open affine, so the only non-trivial case is the line minus a finite set of points. Let $a_1, ..., a_n$ be a finite number of points, and $\mathbb{A}^n \setminus \{a_1, .., a_n\}$ is isomorphic to the affine algebraic set $V(y(x-a_1)...(x-a_n)-1) \subset \mathbb{A}^{n+1}$ given by the map

$$\varphi : \mathbb{A}^n \setminus \{a_1, .., a_n\} \to V(y(x-a_1)...(x-a_n)-1) \qquad t \mapsto (t, \frac{1}{(t_1-a_1)...(t_n-a_n)})$$

with inverse $\psi : (x, y) \mapsto x$. Both functions are Zariski continuous since they are rational functions. Let T be an open of $\mathbb{A}^n$ and U be an open of $\mathbb{A}^{n+1}$ such that $f(T) \subset U$. Then, given any regular function $\frac{f(x,y)}{g(x,y)}$ on U, the pullback $\frac{f(x,\frac{1}{x})}{g(x,\frac{1}{x})}$ is a regular function on T by multiplying large enough powers of x to the numerator and denominator. The other direction is trivial since the pullback will be the same function on one variable. Thus, φ and ψ are k-isomorphisms.

(b)

The open $U := \mathbb{A}^2 \setminus \{(0,0)\}$ is not affine. Note that U is covered by $U_1 = D_{f(x,y)=x}$ and $U_2 = D_{f(x,y)=y}$, whose ring of regular functions are $k[x,y]_x$ and $k[x,y]_y$. On the overlap, the ring of regular functions is $k[x,y]_{x,y}$. Let f be a regular function on U, which restricts to a regular function of the form p_1/x^m on U_1 and p_2/y^n on U_2. The compatibility condition on $U_1 \cap U_2$ implies that $p_1/x^m = p_2/y^n$, which implies $x^m p_2 = y^n p_1$. Since $k[x,y]$ is a UFD, $x_m | p_1$, and f is in $k[x,y]$. Thus, $O(U) \cong k[x,y] \cong O(\mathbb{A}^2)$. Thus, if U were affine, the inlusion map $i : U \to \mathbb{A}^2$ is an isomorphism, which is false.

Problem 10

(a)

(b)

Homework 3

Problem 1

(a)

First, note that all closed/open immersions $i : Z \to X$ are separated morphisms: the diagonal map to the fiber product $Z \to Z \times_X Z \cong Z$ is an isomorphism.