Higher Algebraic K-Theory

David Zhu

August 18, 2024

K-Theory started with Grothendieck's K_0 and projective modules/vector bundles. In this note, we present the various constructions for higher K-groups.

1 Group Completion and $S^{-1}S$-construction

First, we recall Grothendieck's definition of K_0 for an abelian monoid M

Definition 1.0.1. Let M be an abelian monoid. Then, the Grothendieck group $K_0(M)$ is an abelian group K with the inclusion map $i : M \to K$ satisfying the following universal property: given an abelian group A and a monoid morphism $f : M \to A$, we have the factorization

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "i"] & A \\ K \arrow[ur, dashed, "\exists! g"'] & \end{tikzcd}$$

It is an easy exercise to show that K is unique up to isomorphism, which we will denote by $K_0(M)$. Explicitly, we can obtain $K_0(M)$ from the following "group completion" construction:

Proposition 1.0.1. Given an abelian monoid $K_0(M)$ is the abelian group generated by $[m]$ for each $m \in M$, modulo the relation $[x + y] - [x] - [y]$.

Example 1.0.1. (K_0 of a ring) Given a ring R, $K_0(R)$ is defined to be Grothendieck group over the abelian monoid of the isomorphim class of finitely generated projective modules over R, with monoid operation given by direct sum.

If M is a topological monoid, let BM be its classifying space (viewing M as a category with one object). Then, there is a natural map $M \to \Omega BM$, with $\pi_0(\Omega BM) = \pi_1(BM)$ an abelian group. When π_0 is a group, it can be shown that $M \to \Omega BM$ is a homotopy equivalence. Thus the map is referred to as the group completion of a toplogical monoid. When π_0 is not necessarily group, many can still be said: since M is an H-space, $H_*(M)$ is naturally a ring, and $H_0(M) = \mathbb{Z}[\pi_(M)]$. Viewing $\pi_0(M)$ as a multiplicative subset of $H_0(M)$, the induced map $H_*(M) \to H_*(\Omega BM)$ sends $\pi_0(M)$ to units. Mcduff and Segal proved the following result:

Theorem 1.1. If $\pi := \pi_0(M)$ is in the center of $H_*(M)$, then

$$H_*(M)[\pi^{-1}] \cong H_*(\Omega BM)$$

We will outline the idea of the proof following the original paper [MS76], as it provides many insights to the motivate the $S^{-1}S$-construction.

The goal is to find an intermediate space M_∞ with presribed homology $H_*(M_\infty) = H_*(M)[\pi^{-1}]$ and a homology equivalence $M_\infty \to \Omega BM$. The first step uses the Quillen's lemma, given in [Fri01], regarding localization:

Lemma 1.2. Let R be a ring (not necessarily commutative) and S a multiplicative subset. Let C be the category with objects elements of S and morphisms from s_1 to s_2 an element $t \in S$ such that $s_1 t = s_2$. Under the conditions that C is filtered, there is a canonical R-module morphism

$$u : \varinjlim_{C} R \to R[S^{-1}]$$

where the filtered colimit is defined on objects by the inclusion map, and on morphisms by right multiplication by t. If S acts on R by left multiplication bijectively, meaning

1. (injective) Given $r \in R$ and $s \in S$ such that $sr = 0$, then there exists $t \in S$ with $rt = 0$.
2. (bijective)Given $r \in R$ and $s \in S$, there exists $r' \in R$ and $t \in S$ such that $sr' = rt$

Then u is an isomorphism.

A simple example is colimit $\mathbb{Z} \xrightarrow{\times p} \mathbb{Z} \xrightarrow{\times p} ...$ being isomorphic to $\mathbb{Z}[1/p]$ as $\mathbb{Z}$-modules. The hypothesis of the lemma is satisfied when π is in the center of $H_*(M)$, and we can define M_∞ to be the mapping telescope given by $M \xrightarrow{\times m} M \xrightarrow{\times m} ...$, where m is any arbitrary element in the component of $1 \in \pi_0(M)$. Since homology commutes with filtered colimits, M_∞ will have the prescribed homology.

Given a topological group M, we may construct the universal bundle $EM \to BM$ by vieweing M as a topological category. If M acts on a space X, let X_M denote the associated bundle to the universal bundle $X_M \to BM$ with fiber X. The construction still holds when M is only a monoid, and instead of the homotopy equivalent between the fiber and homotopy fiber, Mcduff and Segal recovers the following proposition

Proposition 1.2.1. If M is a topological monoid which acts on a space X, and for each $m \in M$ the map $x \mapsto mx$ from X to itself is a homology equivalence, then the map $X_M \to BM$ is a homology fibration with fibre X, meaning the canonical map between the fiber and homotopy fiber induces isomorphism on homology.

By construction, M acts on M_∞ (which also induces an homology equivalence), and $(M_\infty)_M$ is also contractible since it will be a filtered colimit of the contractible $M_M = EM$. Thus, we have a map $(M_\infty)_M \to BM$ with fiber M_∞ and homotopy fiber ΩBM. The theorem then follows from proposition 1.2.1 and more colimit nonsense on components for the general case.

2 Exact Category

Definition 2.0.1. An **exact category** is an additive category $\mathcal{M}$ equipped with a class $\mathcal{E}$ of short exact sequences of the form

$$M' \xrightarrow{i} M \xrightarrow{j} M''$$

where the first arrow i is denoted an **admissible monomorphism**, and the second arrow j is denoted an **admissible epimorphism**. In addition, the class $\mathcal{E}$ also satisfies the following properties:

1. (closed under trivial extension) For any M', M'' in $\mathrm{ob}(\mathcal{M})$, the SES
$$M' \xrightarrow{(id,0)} M' \oplus M'' \xrightarrow{pr_2} M''$$
is in $\mathcal{E}$.
2. The class of admissible epimorphism is closed under composition. Dually for admissible monomorphisms.
3. (closed under base-change) If $M \to M''$ is an admissible epimorphism and given $N \to M''$, the pullback square exists
$$\begin{tikzcd} N \times_{M''} M \arrow[r] \arrow[d, "p"] & M \arrow[d] \\ N \arrow[r] & M'' \end{tikzcd}$$
and the morphism p is an admissable epimorphism. Dually for admissible monomorphisms.
4. (admissible epimorphism is "epimorphism") Let $M \to M''$ be a map possessing a kernel in $\mathcal{M}$. If there exists a map $N \to M$ in $\mathcal{M}$ such that $N \to M \to M''$ is an admissible epimorphism, then $M \to M''$ is an admissible epimorphism. Dually for admissible monomorphisms.

This is where we want to do K-theory. The motivation for exact catgories is the following scenario: consider any additive category $\mathcal{M}$ embedded as a full subcategory of an abelian category $\mathcal{A}$. Suppose further that $\mathcal{M}$ is closed under taking extensions in $\mathcal{A}$, meaning if

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is exact in $\mathcal{A}$ and A, C is in $\mathcal{M}$, then B is also in $\mathcal{M}$. Then $\mathcal{M}$ can be readily verified to be an exact category, with $\mathcal{E}$ being the class of sequences in $\mathcal{M}$ that is short exact in $\mathcal{A}$. The only non-trivial thing to check is axiom 3, which is a standard theorem regarding pullbacks. We now have a wealth of examples in algebra/algebraic geometry:

Example 2.0.1. The category $\mathbf{P(R)}$ of finitely generated projective modules over a commutative ring R is exact by its embedding in $\mathbf{RMod}$. Note that it is generally not abelian due to lacking of kernel/cokernels. For example $\mathbf{P}(\mathbb{Z})$ is not abelian, since $\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z}$ does not have a cokernel.

Example 2.0.2. The category $\mathbf{VB}(X)$ over a paracompact space X is exact by its embedding in the category of family of vector spaces over X. It is also generally not abelian due to lacking kernels.

Definition 2.0.2. An exact functor $F : M \to M'$ between exact categories is an additive functor preserving exact sequences.

3 The Q-construction and Recovery of K_0

To build K-theory on a exact category such as $\mathbf{P}(\mathbf{R})$, we have to go through an intermediate category, which is called the Q-construction.

Definition 3.0.1. Given an exact category $\mathcal{M}$, let the category $Q\mathcal{M}$ have the same objects as $\mathcal{M}$, and morphisms from M to M' being isomorphisms classes of diagrams of the form

$$M \twoheadleftarrow N \rightarrowtail M'$$

where $\twoheadleftarrow$ signifies an admissible epimorphism and $\rightarrowtail$ an admissible monomorphism in $\mathcal{M}$. An isomorphism between diagrams of the form is one that induces identity on both M and M'. Composition of a morphisms is given by the pullback

$$\begin{tikzcd}
N \times_{M'} N' \arrow[r, tail, "pr_2"] \arrow[d, two heads, "pr_1"] & N' \arrow[r, tail, "i'"] \arrow[d, two heads, "j'"] & M'' \\
N \arrow[r, tail, "i"] \arrow[d, two heads, "i"] & M' & \\
M & &
\end{tikzcd}$$

It is clear that the composition is associative, and we have a well-defined category. Here are a few preliminary observations: 1. the classifying space $BQ\mathcal{M}$ is canonically a CW complex, and it is path-connected by the existence of a zero object, denoted 0, in $\mathcal{M}$. 2. If $i : M' \rightarrowtail M$ is an admissible monomorphism, then it induces a morphism in $Q\mathcal{M}$ denoted by $i_!$ given by

$$M' = M' \rightarrowtail M$$

which will be referred to as **injective maps**. Dually, If $j : M'' \twoheadrightarrow M$ is an admissible epimorphism, then it induces a morphism in $Q\mathcal{M}$ denoted by $j^!$ given by

$$M \twoheadleftarrow M'' = M''$$

which are called **surjective maps**. Note the superscript/subscript follows the contravariant/covariance convention. Then, each morphism u in $Q\mathcal{M}$ is the composition of $i_! \circ j^!$ for some i and j in $\mathcal{M}$ (check the pullback diagram). We will abuse notation onwards and use the same arrows corresponding to admissible monomorphism/epimorphism to denote their induced maps when clear.

As of now, the structure of the intermediate category Q seems murky. We will motivate the definitions by proving it is the universal construction that recovers then well-accepted definition for K_0.

Definition 3.0.2. given an exact category $\mathcal{M}$, $K_0(\mathcal{M})$ is defined to be the abelian group generated by $[C]$, one for each object C in $\mathcal{M}$, subjected to the relations $[B] = [A] + [C]$ whenever there is an short exact sequence $A \to B \to C$.

In some sense, K_0 "breaks up" short exact sequences, forcing them to split. We now examine how the Q-construction break up short exact sequences as well: recall the fact that every short exact sequence $A \rightarrowtail B \twoheadrightarrow C$ is equivalent to the bicartesian diagram

$$\begin{tikzcd}
A \arrow[r, tail] \arrow[d, two heads] & B \arrow[d, two heads] \\
0 \arrow[r, tail] & C
\end{tikzcd}$$

On the other hand, we have the following proposition regarding bicartesian squares in $Q\mathcal{M}$, which occurs in definition for composition.

Proposition 3.0.1. Given a bicartesian square in $\mathcal{M}$

$$\begin{tikzcd} N \arrow[r, tail, "i"] \arrow[d, two heads, "j"] & M' \arrow[d, two heads, "j'"] \\ M \arrow[r, tail, "i'"] & N' \end{tikzcd}$$

we have $i_! j^! = j'^! i'_!$ in $Q\mathcal{M}$

The proof is simply tracing through the definitions. By the proposition, every short exact sequence $A \rightarrowtail B \twoheadrightarrow C$ in $\mathcal{M}$ leads to the equivalence between the morphisms $0 \twoheadleftarrow A \rightarrowtail B$ and $0 \rightarrowtail C \twoheadleftarrow B$ in $Q\mathcal{M}$. We will see how this equivalence is exactly the splitting of exact sequences we desire in the proof of the following theorem:

Theorem 3.1. $\pi_1(BQ\mathcal{M}, 0) \cong K_0(\mathcal{M})$

Before proving the theorem, we first introduce the following lemma regarding trees.

Lemma 3.2. Suppose T is a maximal tree in a small connected category C. Then, $\pi_1(BC)$ is the group generated by $[f]$, one for each morphism not in T, modulo the relations that

1. $[t] = 1$ for every $t \in T$, and $[Id_c] = 1$ for each identity morphism.
2. $[f] \cdot [g] = [f \circ g]$ for every composable f, g.

Proof. Let X be the 1 skeleton of BC, which is a connected graph with maximal tree T. The proof for $\pi_1(X)$ being the free group generated by $X - T$ is given in Hatcher proposition $1A.2$. The lemma is then a direct application of Van-Kampen's Theorem. $\square$

Proof of Theorem 2.1. We construct the isomorphism directly, which follows Weibel and slightly diverts from the original approach given by Quillen. Let T be the set of injective morphisms of the form $0 \rightarrowtail A$ in $Q\mathcal{M}$. Clearly, T contains all vertices (objects) in $Q\mathcal{M}$ and is thus a maximal tree.

Let $B' \rightarrowtail B$ be an injective morphism. Note that its left composition with $0 \rightarrowtail B'$ is the morphism induced by $0 \rightarrowtail B' \rightarrowtail B$, which is in T. Thus by lemma 2.2, all injective morphisms correspond to the identity element; given a surjective morphism $A \twoheadleftarrow A'$, note that its left composition with $0 \twoheadleftarrow A$ induces a surjective map $0 \twoheadleftarrow A'$, so we have the relation $[A \twoheadleftarrow A'] = [0 \twoheadleftarrow A]^{-1}[0 \twoheadleftarrow A']$. By the observation that every morphism in $Q\mathcal{M}$ factors as a surjective morphisms followed by a injective one, we note that $\pi_1(BQ\mathcal{M})$ is generated by classes of the form $[0 \twoheadleftarrow A]$.

By the observation following proposition 2.0.1, each short exact sequence $A \rightarrowtail B \twoheadrightarrow C$ in $\mathcal{M}$ leads to the equivalence between the morphisms $0 \twoheadleftarrow A \rightarrowtail B$ and $0 \rightarrowtail C \twoheadleftarrow B$ in $Q\mathcal{M}$, which in turn induces in the additivity relation in π_1

$$[0 \twoheadleftarrow C][0 \twoheadleftarrow A] = [0 \twoheadleftarrow B] = [0 \twoheadleftarrow A][0 \twoheadleftarrow C]$$

The composition rule follows the additivity relation, and the map $[0 \twoheadleftarrow A] \mapsto [A]$ is the desired isomorphism between $\pi_1(BQ\mathcal{M})$ and $K_0(\mathcal{M})$. $\square$

We can now define K-theory for exact categories.

Definition 3.2.1. For a small exact category $\mathcal{M}$, let $K\mathcal{M}$ be the loop space $\Omega BQ\mathcal{M}$, and set

$$K_i(\mathcal{M}) := \pi_{i+1}(BQ\mathcal{M}, 0) = \pi_i(K\mathcal{M}, 0)$$

For skeletally small categories, we define its K-groups to be those of its skeleton. It is not hard to see that K_i is a functor from the category of small exact categories and exact functors to **Ab**, noting that isomorphic functors induce isomorphism of K-groups by the following proposition

Proposition 3.2.1. A natural transformation $\theta : f \to g$ of functors from C to C' induces an homotopy $BC \times I \to BC'$ between Bf and Bg. In particular, if a functor has a left/right adjoint, then it induces a homotopy equivalence between classifying spaces.

Proof. The key is to realize the data of the triple (f, g, θ) is exactly a functor $C \times 1 \to C$, where 1 is the ordered set $\{0 < 1\}$ with classifying space the unit interval. $\square$

With a bit more effort, one can show that K_i commutes with filtered colimits and products.

References

[Fri01] E. M. 1944- author. (Eric M.) Friedlander. *Appendix Q. On the group completion of a simplicial monoid*. 1st ed. Vol. 110. Providence, Rhode Island, United States : American Mathematical Society, 1994-01-01. ISBN: 0-8218-2591-7.

[MS76] D. McDuff and G. Segal. "Homology Fibrations and the "Group-Completion" Theorem". In: *Inventiones math.* 31 (1976), pp. 279–284. DOI: https://doi.org/10.1007/BF01403148.