# MATH 624 Algebraic Geometry

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## 1 Prevarieties and Varieties

We will assume that $K|k$ a finite extension, $K$ is algebraically closed. We will use $\mathbb{A}^n(K) = K^n = \mathbb{A}^n_K$ to denote the underlying set, not the $n$-dimensional affine space. Given a point $a = (a_1, ..., a_n) \in \mathbb{A}^n_k$, we will use $\varphi_a$ to denote the evaluation map $k[X] \to k$. Similarly, given $f \in k[x]$, we have the evalation map $\tilde{f} : \mathbb{A}_k \to k$. This gives a morphism of $k$-algebras $k[x] \to Maps_k(\mathbb{A}_k, k)$ given by $f \mapsto \tilde{f}$.

**Definition 1.0.1.** Given $\Sigma \subset k[x]$, define $V(\Sigma) = \{a \in \mathbb{A}_k : f(a) = 0 \text{for every } f \in \Sigma\}$. This is called the **affine k-algebraic set** defined by $\Sigma$. If $\Sigma = \{f\}$, then $H_f := V(\Sigma) = V(f)$ defines a **hyperplane** in $\mathbb{A}_k$.

**Example 1.0.1.** Easy examples
1. $V((0)) = \mathbb{A}_k$.
2. $V((1)) = \emptyset$
3. Let $k = \mathbb{C}$. Then, in $\mathbb{A}^1_k$, $V(x^2 - 1) = \{\pm 1\}$. In $\mathbb{A}^2_k$, $V(x^2 - 1) = \{(\pm 1, n) : n \in k\}$

**Definition 1.0.2.** Given $V \subset \mathbb{A}_\neg$, defined $I(V) = \{f \in k[x] : f(V) = 0\}$. This is called the **ideal** of $V$.

**Proposition 1.0.1.**
1. Let $I_\Sigma \subset k[x]$ be the ideal generated by $\Sigma$. Then, $V(\Sigma) = V(I)$.
2. There exists a finite system $f_1, .., f_m$ such that $V(\Sigma) = V(f_1, ..., f_m)$
3. If $\Sigma_1 \subset \Sigma_2$, then $V(\Sigma_1) \supset V(\Sigma_2)$
4. Given $\mathfrak{a}$ an ideal, then $I(V(\mathfrak{a})) = \mathfrak{a}$ iff $\mathfrak{a} = \sqrt{\mathfrak{a}}$.
5. Given ideals $\mathfrak{a}, \mathfrak{b}$, then $V(\mathfrak{a}) = V(\mathfrak{b})$ iff $\sqrt{\mathfrak{a}} = \sqrt{\mathfrak{b}}$.

**Definition 1.0.3.** Let $\mathcal{A}^n_K := \{V \subset \mathbb{A}^n_K : V \text{ affine } k-\text{algebraic sets}\}$. Given $V \in \mathcal{A}^n_K$, let $k[V] := k[x]/I(V)$ be the **affine coordinate ring** generated by $V$.

Let $Id^{rd}(k[x])$ be the set of reduced ideals of $k[x]$. Let $R_n$ be the set of reduced $k$-algebras with $n$-generators.

**Theorem 1.1.** There is a canonical bijection between the set of reduced affine $k$-algebras and reduced ideals of $k[x]$, given by the maps

$$R_n \to Id^{re}(k[X]) \to \mathcal{A}_K^k$$

$$k[\underline{x}] \mapsto \mathfrak{a} := ker(k[x] \xrightarrow{f} k) \mapsto V(\mathfrak{a})$$

with $f$ given by $x \mapsto \underline{x}$.

## 1.1 The Zariski Topology

Given $V \in \mathcal{A}_K^n$, there is a canonical map $K[X] \to K[V]$ given by $f \mapsto f_V$.

**Proposition 1.1.1.** Let $\Sigma_i \subset k[X]$, and $f \in k[X]$ be given. then
1. $V(\cup_i \Sigma_i) = \cap_i V(\Sigma_i)$
2. $V(\prod \Sigma_i) = \cup V(\Sigma_i)$
3. $V((0)) = \mathbb{A}_k^n$; $V((1)) = \emptyset$

By the proposition above, we can define the Zariski topology on $\mathbb{A}_k^n$

**Definition 1.1.1.** The Zariski topology on $\mathbb{A}_K^n$ is given by the closed sets $V(\Sigma)$, with $\Sigma \in k[X]$. In particular, the sets $D_f := \mathbb{A}_k^n - H_f$ is an open set and forms a basis for the topology.

Note that the zariski topology on product spaces is not the product of zariski topologies. Moreover, the connectedness/irreducibility is dependent on $K|k$. A point is called a generic point of $V$ if its closure contains $V$.

**Example 1.1.1.** If $K|k = \mathbb{C}|\mathbb{Q}$, then $V(x_1^2 - 2x_2^2)$ is connected and irreducible. If $K|k = \mathbb{C}|\mathbb{Q}[\sqrt{2}]$, then $V(x_1^2 - 2x_2^2)$ is connected but not irreducible.

**Remark 1.1.1.** For a topological space, $X$, the following are equivalent:
1. Every descending chain of closed subsets is stationary.
2. Every ascending chain of open subsets is stationary.

A topological space satisfiying the above is called **Noetherian**. For example, $Spec(R)$ is Noetherian if $R$ is Noetherian. Note that if $X$ is Noetherian, then it is automatically quasi-compact. Moreover, there are only finitely many irreducible components and connected components of $X$.

**Proposition 1.1.2.** The following hold:
1. The Zariski topology is Noetherian on $\mathbb{A}_K$, therefore also on any $V \in \mathcal{A}_K^n$.
2. For every $V \in \mathcal{A}_K$, there are only finitely many irreducible components and connected components.
3. $V \in \mathcal{A}_K$ is irreducible iff $I(V)$ is a prime ideal.
4. Given $V_0 \subset V$, $V_0$ is irreducible iff $I_V(V_0) := I(V_0)/I(V) \in Spec(k(V))$ is minimal.
5. The connected components in $V \in \mathcal{A}_K$ correspond bijectively to the indecomposable idempotents of $k[V]$.
6. For $V \in \mathcal{A}_K$, $a \in V$ is a generic point iff the evaluation map $k[V] \to k[a]$ is an isomorphism of $k$-algebras.

**Definition 1.1.2.** Let $T$ be a topological space, and let $V \subset T$.

1. dim(V):=sup { chain of irreducible components ending in V: }
2. codim(V):=sup { chain of irreducible components starting with $V$ and ending in $T$: }

Note that if $V = \cup V_\alpha$, then $dim(V) = supdim(V_\alpha)$, and similarly for codimensions. Moreover, $dim(V) = dim(\overline{V})$.

**Proposition 1.1.3.** (Notions of dimension) Let $V \in \mathcal{A}_K$ be irreducible. Then, the dimension of $V$ is the same as the krull dimension of $K[V]$.

**Proposition 1.1.4.** Suppose irreducible $W \subset V \in \mathcal{A}_K$. Then,

$$dim(W) + codim_V(W) = dim(V)$$

**Proposition 1.1.5.** $V \in \mathcal{A}_K$ has generic points $a$ iff $td(K|k) \geq dim(V) = td(k(V))$.

## 1.2 Base change and Rational Points

**Definition 1.1.3.** Suppose there is an embedding

$$\begin{array}{ccc} K & \longrightarrow & L \\ \uparrow & & \uparrow \\ k & \longrightarrow & l \end{array}$$

Then, there is a natural morphism $k[x] \to l[x]$, which induces a pushforward of ideals and a map $\mathcal{A}_K \to \mathcal{A}_L$. Take the vanishing locus of the pushforward of $I(V)$ gives the base change of $V$.

**Remark 1.1.2.** Base change does not preserve connectedness or irreducibility.

**Definition 1.1.4.** $V \in \mathcal{A}_K$ is called **absolutely (geometrically) irreducible** if $V_l$ is irreducible for all field extension $l|k$. It is **geometrically connected** is $V_l$ is connected for all $l|k$.

**Proposition 1.1.6.** Let $V \in \mathcal{A}_K$ be affine $k$-algebraic set. Then the following are equivalent:

1. $V$ is absolutely irreducible.
2. $V_{k^s}$ is irreducible.
3. $V_{\overline{k}}$ irreducible.

The key observation is that $K^s[x] \to \overline{k}[X]$ is an integral extensions of domains. Therefore, we have going up and going down, and it straightforward to show that $Spec(k^s[X]) \to Spec(\overline{k}[X])$ is a homeomorphism. Thus, we have (2) $\implies$ (3).

To (3) $\implies$ (1), apply the following:

**Lemma 1.2.** For every $V \in \mathcal{A}_K$, one has $V(\overline{k})$ is zariski dense in $V$. Therefore, $V_{\overline{k}}$ irreducible implies $V$ irreducible.

The proof is exercise. The key point is that if there exists $f$ with $k$-coefficients such that $f$ vanishes on all of $A$

**Proposition 1.2.1.** Let $V \in \mathcal{A}_K$ be affine $k$-algebraic set. Then the following are equivalent:

1. $V$ is geometrically connected.
2. $V_{K^s}$ is connected.
3. $V_{\overline{k}}$ is connected.

## 2 The category of quasi-affine $k$-algebraic sets

**Definition 2.0.1.** A **quasi-affine** k-algebrac set is any zariski open subset $U \subset V$ for $V \in \mathcal{A}_K$.

The complement of hyperplanes is a basis of quasi-affine $k$-algebraic sets. Let $V \in \mathcal{A}_K$ be non-empty, $f \in K[V]$. Then, the evaluation map $f : V \to \mathcal{A}_K$ is continuous. Moreover, $\varphi = (f_1, ..., f_n)$ is also continuous.

**Definition 2.0.2.** Let $V \in \mathcal{A}_K$ and $\mathcal{V} \subset V$ be zariski dense. Then, a functions $\varphi : \mathcal{V} \to \mathcal{A}_K$ is called **regular** at $x \in V$ if there exists $f_x, g_x \in k[x]$ and $\S \subset V$ such that $g_x \neq 0$ everywhere on $\mathcal{U}_x$ and $\varphi = \frac{f_x}{g_x}$. A fucntion $\varphi : \mathcal{V} \to \mathcal{A}_K$ is **regular** if it is regular at every point in $V$. Let $\mathcal{O}_x := \{\varphi \in Maps(\mathcal{V}, K) : \varphi \text{ regular at } x\}$. Define an equivalence relation on $\mathcal{O}_x$ by equivalence on any open neiborhood around $x$. $\mathcal{O}(V)$ is the set of regular functions on $V$.

**Proposition 2.0.1.** (rings of regular functions) We have the following:

1. $k[V] \to \tilde{\mathcal{O}}(V)$ is an isomorphism of $k$-algebra.
2. $k[V]_f \to \tilde{O}(U_f)$ is an isomorphism of $k$-algebra.

It is helpful to remember that Zariski open sets are dense. Thus, it suffices to show that a function is zero on a basic open $U_f$ to deduce it is globally zero.

## 3 Presheaves and Sheaves

**Definition 3.0.1.** Let $\mathcal{C}$ be a concrete category such as **Top, Set, Ab**. Let $X$ be a topological space with topology $\tau_X$. Then, $\tau_X$ is naturally poset category where morphisms are inclusions. A **presheaf** is a contravariant functor $\mathcal{P} : \tau_X \to \mathcal{C}$.

Explicitly, $\mathcal{P}$ is given by two data: 1.$\mathcal{P}(U) \in Obj(\mathcal{C})$ for every $U \in \tau_X$. 2.$\rho_{u',u''} : \mathcal{P}(U'') \to \mathcal{P}(U')$ for every $U' \subset U''$. The elements in the set $P(U)$ are called **sections** above $U$. The image of a section under $\rho$ is called the **restriction**.

**Definition 3.0.2.** A presheaf is a **sheaf** if it has the covering property: given an open cover of an open set $U = \cup_i U_i$, with $U + i, j := U_i \cap U_j$ with $s_i \in \mathcal{P}(U_i)$ such that $\rho_{U_i, U_{i,j}}(s_i) = \rho_{U_j, U_{i,j}}(s_i)$, then there exists $s \in \mathcal{P}(U)$ such that $s_i \in \rho_{U, U_i}(s)$ for every $U_i$.

**Definition 3.0.3.** Suppose that limits exists in $\mathcal{C}$. Then $\mathcal{P}_x := \mathcal{P}(U_x)$ is called the **stalk** of $\mathcal{P}$ at $x$.

**Proposition 3.0.1.** $\mathcal{P}$ is a sheaf iff for every $U \in \tau_X$, the map $\varphi_U : U \to \coprod_{x \in U} \mathcal{P}_x$ is injective.

**Proposition 3.0.2.** For every presheaf $\mathcal{P}$, there is a sheafification functor $\mathcal{P} \to \mathcal{F}$ that induces isomorphism on stalks.

**Definition 3.0.4.** Let $f : X \to Y$ be a continuous map of topological spaces. Then,

1. Given a (pre)sheaf $\mathcal{P}$ on $X$, then the **direct image** (pre)sheaf $f_*\mathcal{P}$ on $Y$ is defined by $f_*\mathcal{V} := \mathcal{P}(f^{-1}(V))$ for all $V \in \tau_Y$. In particular, the direct image sheaf is also a sheaf.
2. Given a presheaf $\mathcal{P}$ on $Y$. There is an **inverse image** sheaf $f^{-1}\mathcal{P}$ on $X$ defined by the limit:

$$f^{-1}\mathcal{P}(U) := \varinjlim_{U \subset U'} \mathcal{P}(f(U'))$$

where $U \subset U'$ and $f(U')$ is open.

**Remark 3.0.1.** Note that the preimage sheaf is always a preseeaf, but not necessarily a sheaf.

**Definition 3.0.5.** A (locally) **ringed space** is a pair $(X, \mathcal{F})$, where $X$ is a topological space and $\mathcal{F}$ a sheaf of rings on $X$ such that the stalks at each point is a local ring.

**Definition 3.0.6.** Given locally ringed spaces $(X, \mathcal{F})$, $(Y, \mathcal{G})$, a morphism of locally ringed space is a pair $(f, f^\sharp)$ such that $f : X \to Y$ is continuous and $f^\sharp : \mathcal{G} \to f_*\mathcal{F}$ a morphism of sheaves.

# 4 Back to Varieties

**Proposition 4.0.1.** Let $V$ be an affine $k$-algebraic set, $U \subset V$ zariski open.

1. The assignmenet $\tau_U$, $U' \mapsto \tilde{O}(U')$ defined a locally ringed space on $U$.
2. A morphism of quasi-affine algebraic set $T \to U$ is any morphism of locally ringed spaces $(f, f^\sharp) : (T, \mathcal{O}_T) \to (U, \mathcal{O}_U)$

The checks are fullfilled by proposition 2.0.1.

**Proposition 4.0.2.** Let $(T, \mathcal{O}_T), (U, \mathcal{O}_U)$, and $\Phi : T \to U$ continuous. Then,

1. $\Phi$ defined a morphism of locally ringed spaces iff $\mathcal{O}_U \circ \varphi \subset \mathcal{O}_T$, i.e for every $U$ and $T'$ open such that $\Phi(T') \subset U'$ and $\varphi \in \mathcal{O}_U(U')$, then $\varphi \circ \Phi \in \mathcal{O}_T(T')$.
2. Suppose $\Phi$ defines such a morphism, and let $U \subset \mathbb{A}_K$, $p : \mathbb{A}_K^n \to K$ the $i$th projection, then $p_i|_U \circ \Phi$ completely determines $\Phi$.

**Remark 4.0.1.** Let $U_f := \{x \in V | f(x) \neq 0 :\}$ be a basic open. Consider $W_f \subset \mathbb{A}_K^n$ defined by $W_f := \{(a,b) | a \in \mathbb{A}_K^n, b \in \mathbb{A}_K^1 : f(a)b - 1 = 0\}$ is an algebraic set in $\mathbb{A}_K^{mn}$. Prove that $\Phi : W_f \to U_f$ given by $(a,b) \mapsto a$ is an isomorphism of quasi affine $k$-algebraic sets. Then inverse is given by $\psi : U_f \to W_f$ given by $a \mapsto (a, \frac{1}{f(a)})$.

**Definition 4.0.1.** A quasi-affine $k$-algebraic set is called **affine** if it is isomorphic as a locally ringed space to an affine $k$-algebraic set.