Equivariant Stable Homotopy Notes

David Zhu

September 12, 2024

For the entire note, we will assume a group G to be a compact Lie group, and subgroups $H \subset G$ are always closed. [Blu17]

1 Unstable Equivariant Homotopy Theory

1.1 G-CW Complexes

Fix a compact Lie group G acting on a space X. Similar to CW-complexes, we want to deconstruct X into cells, but this time with the addditional data of the G-action along with each cell. The idea is that cells are of the form of a product $G/H \times D^n$, where G acts trivially on D^n, and G/H "represents" the orbits of D^n. To make this work, H must be the isotropy group of D^n.

Definition 1.0.1. A **G-CW complex** is the sequential colimit of spaces X_n, where X_{n+1} is a pushout:

$$\begin{tikzcd} \coprod G/H \times S^n \arrow[r] \arrow[d] & X_n \arrow[d] \\ \coprod G/H \times D^{n+1} \arrow[r] & X_{n+1} \end{tikzcd}$$

We will denote $G/H \times D^n$ as an **n-cell**.

Remark 1.0.1. Note that the topological dimension of an n-cell in a G-CW complex might be greater than n. For example, a 0-cell $S^1/e \times *$ is one dimensional.

Example 1.0.1. Let $G = C_2$ acting on S^2 by rotation by π along the Z-axis. It has a G-CW structure given by the following cells: 2 zero-cells $C_2/C_2 \times *$, which are the poles corresponding to the fixed points of the C_2 action. 1 one-cell $C_2/e \times D^1$, which are the two great circles joining the poles; 1 two-cell $C_2/C_2 \times D^2$, which are the two hemispheres.

Example 1.0.2. Let $G = C_2$ acting on S^2 by the antipodal map. It has a G-CW structure given by the following cells: 1 zero-cells $C_2/e \times *$, which are the poles; 1 one-cell $C_2/e \times D^1$, which are the two great circles joining the poles; 1 two-cell $C_2/C_2 \times D^2$, which are the two hemispheres.

Definition 1.0.2. Let H be a subgroup of G. Define $\pi_n^H(X) := \pi_n(X^H)$. A map $f : X \to Y$ of G-spaces is a **weak equivalence** if for all subgroups $H \subset G$,

$$f_* : \pi_n^H(X) \to \pi_n^H(Y)$$

is an isomorphism.

Let **GTop** be the category of G-spaces and G-maps. There is a cofibrantly-generated model structure that we can put on **GTop**:

Theorem 1.1. There is a cofibrantly-generated model structure on **GTop**, given by

1. A G-map $f : X \to Y$ is a fibration iff for all $H \subset G$, $f^H : X^H \to Y^H$ is a fibration.
2. A G-map $f : X \to Y$ is a weak equivalence iff for all $H \subset G$, $f^H : X^H \to Y^H$ is a weak equivalence.

An immediate consequence of the model category structure is the equivariant Whitehead's Theorem

Corollary 1.1.1. Let $f : X \to Y$ be a weak equivalence of cofibrant-fibrant objects in a model category. Then, f is a homotopy equivalence. In particular, every object in **GTop** is fibrant, and G-CW complexes are cofibrant.

1.2 Elmendorf's Theorem