Higher Algebraic K-Theory

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K-Theory started with Grothendieck's K_0 and projective modules/vector bundles. In this note, we present the various constructions for higher K-groups.

1 The Group Completion Theorem

First, we recall Grothendieck's definition of K_0 for an abelian monoid M

Definition 1.0.1. Let M be an abelian monoid. Then, the Grothendieck group $K_0(M)$ is an abelian group K with the inclusion map $i : M \to K$ satisfying the following universal property: given an abelian group A and a monoid morphism $f : M \to A$, we have the factorization

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "i"] & A \\ K \arrow[ur, dashed, "\exists! g"'] & \end{tikzcd}$$

It is an easy exercise to show that K is unique up to isomorphism, which we will denote by $K_0(M)$. Explicitly, we can obtain $K_0(M)$ from the following "group completion" construction:

Proposition 1.0.1. Given an abelian monoid $K_0(M)$ is the abelian group generated by $[m]$ for each $m \in M$, modulo the relation $[x + y] - [x] - [y]$.

Example 1.0.1. (K_0 of a ring) Given a ring R, $K_0(R)$ is defined to be Grothendieck group over the abelian monoid of the isomorphim class of finitely generated projective modules over R, with monoid operation given by direct sum.

If M is a topological monoid, let BM be its classifying space (viewing M as a category with one object). Then, there is a natural map $M \to \Omega BM$, with $\pi_0(\Omega BM) = \pi_1(BM)$ an abelian group. When π_0 is a group, it can be shown that $M \to \Omega BM$ is a homotopy equivalence. Thus the map is referred to as the group completion of a toplogical monoid. When π_0 is not necessarily group, many can still be said: since M is an H-space, $H_*(M)$ is naturally a ring, and $H_0(M) = \mathbb{Z}[\pi_(M)]$. Viewing $\pi_0(M)$ as a multiplicative subset of $H_0(M)$, the induced map $H_*(M) \to H_*(\Omega BM)$ sends $\pi_0(M)$ to units. Mcduff and Segal proved the following result:

Theorem 1.1. If $\pi := \pi_0(M)$ is in the center of $H_*(M)$, then

$$H_*(M)[\pi^{-1}] \cong H_*(\Omega BM)$$

We will outline the idea of the proof following the original paper [MS76], as it provides many insights to the motivate the $S^{-1}S$-construction.

The goal is to find an intermediate space M_∞ with presribed homology $H_*(M_\infty) = H_*(M)[\pi^{-1}]$ and a homology equivalence $M_\infty \to \Omega BM$. The first step uses the Quillen's lemma, given in [Fri94], regarding localization:

Lemma 1.2. Let R be a ring (not necessarily commutative) and S a multiplicative subset. Let C be the category with objects elements of S and morphisms from s_1 to s_2 an element $t \in S$ such that $s_1 t = s_2$. Under the conditions that C is filtered, there is a canonical R-module morphism

$$u : \varinjlim_{C} R \to R[S^{-1}]$$

where the filtered colimit is defined on objects by the inclusion map, and on morphisms by right multiplication by t. If S acts on R by left multiplication bijectively, meaning

1. (injective) Given $r \in R$ and $s \in S$ such that $sr = 0$, then there exists $t \in S$ with $rt = 0$.
2. (bijective)Given $r \in R$ and $s \in S$, there exists $r' \in R$ and $t \in S$ such that $sr' = rt$

Then u is an isomorphism.

A simple example is colimit $\mathbb{Z} \xrightarrow{\times p} \mathbb{Z} \xrightarrow{\times p} ...$ being isomorphic to $\mathbb{Z}[1/p]$ as $\mathbb{Z}$-modules. The hypothesis of the lemma is satisfied when π is in the center of $H_*(M)$, and we can define M_∞ to be the mapping telescope given by $M \xrightarrow{\times m} M \xrightarrow{\times m} ...$, where m is any arbitrary element in the component of $1 \in \pi_0(M)$. Since homology commutes with filtered colimits, M_∞ will have the prescribed homology.

Given a topological group M, we may construct the universal bundle $EM \to BM$ by vieweing M as a topological category. If M acts on a space X, let X_M denote the associated bundle to the universal bundle $X_M \to BM$ with fiber X. The construction still holds when M is only a monoid, and instead of the homotopy equivalent between the fiber and homotopy fiber, Mcduff and Segal recovers the following proposition

Proposition 1.2.1. If M is a topological monoid which acts on a space X, and for each $m \in M$ the map $x \mapsto mx$ from X to itself is a homology equivalence, then the map $X_M \to BM$ is a homology fibration with fibre X, meaning the canonical map between the fiber and homotopy fiber induces isomorphism on homology.

By construction, M acts on M_∞ (which also induces an homology equivalence), and $(M_\infty)_M$ is also contractible since it will be a filtered colimit of the contractible $M_M = EM$. Thus, we have a map $(M_\infty)_M \to BM$ with fiber M_∞ and homotopy fiber ΩBM. The theorem then follows from proposition 1.2.1 and more colimit nonsense on components for the general case.

We now define group completion for general homotopy commutative, homotopy associative H-spaces.

Definition 1.2.1. Let X be a homotopy commutative, homotopy associative H-space. A **group completion** of X is an H-space map $f : X \to Y$ such that f induces the group completion on $\pi := \pi_0$, and the isomorphism

$$H_*(X)[\pi^{-1}] \cong H_*(Y)$$

2 The $S^{-1}S$-construction

Given a symmetric monoidal category S, Quillen constructs the category $S^{-1}S$ such that $B(S^{-1}S)$ is the group completion of BS in the sense of definition 1.2.1. The group completion recovers the plus construction, and gives another more natural definition for higher K-theory. We shall note that the construction below is closely related to the group completion construction in chapter 1.

A left action of a monoidal category S on a category X is a functor $+ : S \times X \to X$ with natural isomorphisms $A+(B+F) \cong (A+B)+F$ and $0+F \cong F$, where $A, B \in S$ and $F \in X$, satisfying the associativity coherence conditions. An action is call invertible if each translation $F \mapsto A + F$ is a homotopy equivalence.

Definition 2.0.1. If S acts on X the category $\langle S, X\rangle$ has the same objects as X. A morphisms is represented by an isomorphisms class of the tuple $(s, sx \xrightarrow{\phi} t)$, where ϕ is a morphism in X. An isomorphism of tuples is given by an isomorphism $s \cong s'$, which induces the commutative diagram

$$\begin{tikzcd}
s + x \arrow[rr, "\cong"] \arrow[dr] & & s' + x \arrow[dl] \\
& t &
\end{tikzcd}$$

We let $S^{-1}S := \langle S, S \times S\rangle$, where S acts on both facts of $S \times S$ by the monoidal operation. We also define an action of S on $S^{-1}S$ by $s + (s_1, s_2) = (s_1, s + s_2)$. Note that this action is invertible: the translation $(s_1, s_2) \mapsto (s_1, s + s_2)$ has homotopy inverse $(s_1, s_2) \mapsto (s + s_1, s_2)$, since there is the natural transformation $(s_1, s_2) \mapsto (s + s_1, s + s_2)$. Quillen then proves

Theorem 2.1. If every map in S is an isomorphism, and the translations are faithful in S, then the functor $S \to S^{-1}S$ given by $x \mapsto (0, x)$ induces isomorphism

$$H_*(S)[\pi^{-1}] \cong H_*(S^{-1}S)$$

In particular, $BS^{-1}S$ is the group completion of BS.

Proof. It is not hard to show that the projection functor $S^{-1}S \to \langle S, S\rangle$ onto the second factor is cofibered with fiber S. Then, there is a spectral sequence [Gra76]

$$E^2_{p,q} = H_p(\langle S, S\rangle, H_q(S)) \Rightarrow H_{p+q}(S^{-1}S)$$

Localizating at $\pi_0(S)$ and noting the contractibility of $\langle S, S\rangle$ gives the desired degeneration. $\square$

We now show that the plus construction for $BGL(R)$ is a group completion.

Theorem 2.2. Let $S = \coprod GL_n(R)$ be the category of free R-modules. Then, $BS^{-1}S$ is the group completion of $BS = \coprod BGL_n(R)$, and

$$BS^{-1}S \cong \mathbb{Z} \times BGL(R)^+$$

Proof. We will construct a map $f : BGL(R) \to Y_S$, where Y_S is the connected component of $BS^{-1}S$ containing the basepoint, such that f induces isomorphism on integral homology. The universal property of the plus contruction then implies a homotopy equivalence $BGL(R)^+ \to Y_S$.

□

3 Exact Category

Definition 3.0.1. An **exact category** is an additive category $\mathcal{M}$ equipped with a class $\mathcal{E}$ of short exact sequences of the form

$$M' \xrightarrow{i} M \xrightarrow{j} M''$$

where the first arrow i is denoted an **admissible monomorphism**, and the second arrow j is denoted an **admissible epimorphism**. In addition, the class $\mathcal{E}$ also satisfies the following properties:

1. (closed under trivial extension) For any M', M'' in ob($\mathcal{M}$), the SES

$$M' \xrightarrow{(id,0)} M' \oplus M'' \xrightarrow{pr_2} M''$$

   is in $\mathcal{E}$.
2. The class of admissible epimorphism is closed under composition. Dually for admissible monomorphisms.
3. (closed under base-change) If $M \to M''$ is an admissible epimorphism and given $N \to M''$, the pullback square exists

$$\begin{tikzcd} N \times_{M''} M \arrow[r] \arrow[d, "p"] & M \arrow[d] \\ N \arrow[r] & M'' \end{tikzcd}$$

   and the morphism p is an admissable epimorphism. Dually for admissible monomorphisms.
4. (admissible epimorphism is "epimorphism") Let $M \to M''$ be a map possessing a kernel in $\mathcal{M}$. If there exists a map $N \to M$ in $\mathcal{M}$ such that $N \to M \to M''$ is an admissible epimorphism, then $M \to M''$ is an admissible epimorphism. Dually for admissible monomorphisms.

This is where we want to do K-theory. The motivation for exact catgories is the following scenario: consider any additive category $\mathcal{M}$ embedded as a full subcategory of an abelian category $\mathcal{A}$. Suppose further that $\mathcal{M}$ is closed under taking extensions in $\mathcal{A}$, meaning if

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is exact in $\mathcal{A}$ and A, C is in $\mathcal{M}$, then B is also in $\mathcal{M}$. Then $\mathcal{M}$ can be readily verified to be an exact category, with $\mathcal{E}$ being the class of sequences in $\mathcal{M}$ that is short exact in $\mathcal{A}$. The only non-trivial thing to check is axiom 3, which is a standard theorem regarding pullbacks. We now have a wealth of examples in algebra/algebraic geometry:

Example 3.0.1. The category $\mathbf{P(R)}$ of finitely generated projective modules over a commutative ring R is exact by its embedding in **RMod**. Note that it is generally not abelian due to lacking of kernel/cokernels. For example $\mathbf{P}(\mathbb{Z})$ is not abelian, since $\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z}$ does not have a cokernel.

Example 3.0.2. The category $\mathbf{VB}(X)$ over a paracompact space X is exact by its embedding in the category of family of vector spaces over X. It is also generally not abelian due to lacking kernels.

Definition 3.0.2. An exact functor $F : M \to M'$ between exact categories is an additive functor preserving exact sequences.

4 The Q-construction and Recovery of K_0

To build K-theory on a exact category such as $\mathbf{P}(\mathbf{R})$, we have to go through an intermediate category, which is called the Q-construction.

Definition 4.0.1. Given an exact category $\mathcal{M}$, let the category $Q\mathcal{M}$ have the same objects as $\mathcal{M}$, and morphisms from M to M' being isomorphisms classes of diagrams of the form

$$M \twoheadleftarrow N \rightarrowtail M'$$

where $\twoheadleftarrow$ signifies an admissible epimorphism and $\rightarrowtail$ an admissible monomorphism in $\mathcal{M}$. An isomorphism between diagrams of the form is one that induces identity on both M and M'. Composition of a morphisms is given by the pullback

$$\begin{tikzcd}
N \times_{M'} N' \arrow[r, tail, "pr_2"] \arrow[d, two heads, "pr_1"] & N' \arrow[r, tail, "i'"] \arrow[d, two heads, "j'"] & M'' \\
N \arrow[r, tail, "i"] \arrow[d, two heads, "i"] & M' & \\
M & &
\end{tikzcd}$$

It is clear that the composition is associative, and we have a well-defined category. Here are a few preliminary observations: 1. the classifying space $BQ\mathcal{M}$ is canonically a CW complex, and it is path-connected by the existence of a zero object, denoted 0, in $\mathcal{M}$. 2. If $i : M' \rightarrowtail M$ is an admissible monomorphism, then it induces a morphism in $Q\mathcal{M}$ denoted by $i_!$ given by

$$M' = M' \rightarrowtail M$$

which will be referred to as **injective maps**. Dually, If $j : M'' \twoheadrightarrow M$ is an admissible epimorphism, then it induces a morphism in $Q\mathcal{M}$ denoted by $j^!$ given by

$$M \twoheadleftarrow M'' = M''$$

which are called **surjective maps**. Note the superscript/subscript follows the contravariant/covariance convention. Then, each morphism u in $Q\mathcal{M}$ is the composition of $i_! \circ j^!$ for some i and j in $\mathcal{M}$ (check the pullback diagram). We will abuse notation onwards and use the same arrows corresponding to admissible monomorphism/epimorphism to denote their induced maps when clear.

As of now, the structure of the intermediate category Q seems murky. We will motivate the definitions by proving it is the universal construction that recovers then well-accepted definition for K_0.

Definition 4.0.2. given an exact category $\mathcal{M}$, $K_0(\mathcal{M})$ is defined to be the abelian group generated by $[C]$, one for each object C in $\mathcal{M}$, subjected to the relations $[B] = [A] + [C]$ whenever there is an short exact sequence $A \to B \to C$.

In some sense, K_0 "breaks up" short exact sequences, forcing them to split. We now examine how the Q-construction break up short exact sequences as well: recall the fact that every short exact sequence $A \rightarrowtail B \twoheadrightarrow C$ is equivalent to the bicartesian diagram

$$\begin{tikzcd} A \arrow[r, tail] \arrow[d, two heads] & B \arrow[d, two heads] \\ 0 \arrow[r, tail] & C \end{tikzcd}$$

On the other hand, we have the following proposition regarding bicartesian squares in $Q\mathcal{M}$, which occurs in definition for composition.

Proposition 4.0.1. Given a bicartesian square in $\mathcal{M}$

$$\begin{tikzcd} N \arrow[r, tail, "i"] \arrow[d, two heads, "j"] & M' \arrow[d, two heads, "j'"] \\ M \arrow[r, tail, "i'"] & N' \end{tikzcd}$$

we have $i_! j^! = j'^! i'_!$ in $Q\mathcal{M}$

The proof is simply tracing through the definitions. By the proposition, every short exact sequence $A \rightarrowtail B \twoheadrightarrow C$ in $\mathcal{M}$ leads to the equivalence between the morphisms $0 \twoheadleftarrow A \rightarrowtail B$ and $0 \rightarrowtail C \twoheadleftarrow B$ in $Q\mathcal{M}$. We will see how this equivalence is exactly the splitting of exact sequences we desire in the proof of the following theorem:

Theorem 4.1. $\pi_1(BQ\mathcal{M}, 0) \cong K_0(\mathcal{M})$

Before proving the theorem, we first introduce the following lemma regarding trees.

Lemma 4.2. Suppose T is a maximal tree in a small connected category C. Then, $\pi_1(BC)$ is the group generated by $[f]$, one for each morphism not in T, modulo the relations that

1. $[t] = 1$ for every $t \in T$, and $[Id_c] = 1$ for each identity morphism.
2. $[f] \cdot [g] = [f \circ g]$ for every composable f, g.

Proof. Let X be the 1 skeleton of BC, which is a connected graph with maximal tree T. The proof for $\pi_1(X)$ being the free group generated by $X - T$ is given in Hatcher proposition $1A.2$. The lemma is then a direct application of Van-Kampen's Theorem. □

Proof of Theorem 2.1. We construct the isomorphism directly, which follows Weibel and slightly diverts from the original approach given by Quillen. Let T be the set of injective morphisms of the form $0 \rightarrowtail A$ in $Q\mathcal{M}$. Clearly, T contains all vertices (objects) in $Q\mathcal{M}$ and is thus a maximal tree.

Let $B' \rightarrowtail B$ be an injective morphism. Note that its left composition with $0 \rightarrowtail B'$ is the morphism induced by $0 \rightarrowtail B' \rightarrowtail B$, which is in T. Thus by lemma 2.2, all injective morphisms correspond to the identity element; given a surjective morphism $A \twoheadleftarrow A'$, note that its left composition with $0 \twoheadleftarrow A$ induces a surjective map $0 \twoheadleftarrow A'$, so we have the relation $[A \twoheadleftarrow A'] = [0 \twoheadleftarrow A]^{-1}[0 \twoheadleftarrow A']$. By the observation that every morphism in $Q\mathcal{M}$ factors as a surjective morphisms followed by a injective one, we note that $\pi_1(BQ\mathcal{M})$ is generated by classes of the form $[0 \twoheadleftarrow A]$.

By the observation following proposition 2.0.1, each short exact sequence $A \rightarrowtail B \twoheadrightarrow C$ in $\mathcal{M}$ leads to the equivalence between the morphisms $0 \twoheadleftarrow A \rightarrowtail B$ and $0 \rightarrowtail C \twoheadleftarrow B$ in $Q\mathcal{M}$, which in turn induces in the additivity relation in π_1

$$[0 \twoheadleftarrow C][0 \twoheadleftarrow A] = [0 \twoheadleftarrow B] = [0 \twoheadleftarrow A][0 \twoheadleftarrow C]$$

The composition rule follows the additivity relation, and the map $[0 \twoheadleftarrow A] \mapsto [A]$ is the desired isomorphism between $\pi_1(BQ\mathcal{M})$ and $K_0(\mathcal{M})$. □

We can now define K-theory for exact categories.

Definition 4.2.1. For a small exact category $\mathcal{M}$, let $K\mathcal{M}$ be the loop space $\Omega BQ\mathcal{M}$, and set

$$K_i(\mathcal{M}) := \pi_{i+1}(BQ\mathcal{M}, 0) = \pi_i(K\mathcal{M}, 0)$$

For skeletally small categories, we define its K-groups to be those of its skeleton. It is not hard to see that K_i is a functor from the category of small exact categories and exact functors to **Ab**, noting that isomorphic functors induce isomorphism of K-groups by the following proposition

Proposition 4.2.1. A natural transformation $\theta : f \to g$ of functors from C to C' induces an homotopy $BC \times I \to BC'$ between Bf and Bg. In particular, if a functor has a left/right adjoint, then it induces a homotopy equivalence between classifying spaces.

Proof. The key is to realize the data of the triple (f, g, θ) is exactly a functor $C \times 1 \to C$, where 1 is the ordered set $\{0 < 1\}$ with classifying space the unit interval. □

With a bit more effort, one can show that K_i commutes with filtered colimits and products.

References

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