Étale Homotopy Theory and Adams Conjecture

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We only prove the Adams conjecture for complex vector bundles, but the same strategy also applies for the real case, albeit needing some modifications.

1 The Adams Conjecture

Definition 1.0.1. Let X be compact Hausdorff and let $KU(X)$ be Grothendieck group of complex vector bundles over X, and let $\mathcal{SF}(X)$ be the Grothendieck group of stable sphere bundles over X modulo fiber homotopy equivalence. The **complex J-homomorphism** can be viewed as the homomorphism

$$J : KU(X) \to \mathcal{SF}(X)$$

by sending a complex vector bundle to its fiberwise one-point compactification.

Theorem 1.1. The stable sphere bundles over X is classified by the the groups of self-homotopy equivalences of S^n, which we denote by $G(n) := \mathrm{Equiv}(S^n, S^n)$.

Proposition 1.1.1. The complex J-homomorphism $J : KU(X) \to \mathcal{SF}(X)$ is induced by a map between respective classifying spaces, which we also denote

$$J : BU \to BG := \varinjlim_n BG(n)$$

Definition 1.1.1. **The k-th Adams Operation** $\psi^k : KU(X) \to KU(X)$ is defined to be a ring homomorphism that is natural in X, and satisfies

$$\psi^k(L) = L^{\otimes k}$$

where L is any line bundle over X.

Note that ψ^k is unique by the splitting principal.

Theorem 1.2 (**The Adams Conjecture**)**.** The composite

$$BU \xrightarrow{\psi^k - 1} BU \xrightarrow{J} BG$$

is nullhomotopic up to multiplication by some k^n.

Remark 1.2.1. Historically, Adams realized Whithead's J-homomorphism $J : \pi_i(SO) \to \pi_i^s$ can be used to understand the stable homotopy groups of spheres, since $\pi_i(SO)$ is known by Bott periodicity. The Adams conjecture then serves a central role in the identification of the image of the J-homomorphism.

Multiple proofs of the Adams conjecture were completed around 1970. Notably, Quillen's (second) proof using Brauer lifting and by computing the cohomology of $BGL(\mathbb{F}_q)$ led to his later construction of higher algebraic K-theory. In this talk, we present Sullivan's proof using profinite completion and étale homotopy theory, which turns the Adams conjecture into a beautiful case of Galois symmetry of algebraic varieties.

2 Preview of Proof

We are trying to show that the map

$$BU \xrightarrow{id} BU \xrightarrow{J} BG$$

and

$$BU \xrightarrow{\psi^k} BU \xrightarrow{J} BG$$

are homotopic modulo k. The first map classifies the spherical bundle associated to the tautological bundle γ over BU, and the second map is the spherical bundle associated to the pullback $\psi^{k*}\gamma$, so it suffices to show that these sphere bundles are fiber homotopy equivalent. To make a further reduction, the tautological bundle over BU are constructed as the limit of the tautological bundles over $BU(n)$, so it suffices to prove that the "unstable version" of the Adams conjecture is true for all n.

It turns out that the sphere bundle associated to the tautological bundle over $BU(n)$ is fiber homotopy equivalent to $BU(n-1)$. Thus, we will be done if we have unstable Adams operations $\psi^k : BU(n) \to BU(n)$, which are homotopy equivalences, with a pullback diagram

$$\begin{tikzcd} BU(n-1) \arrow[r, "\psi^k"] \arrow[d, "i"] & BU(n-1) \arrow[d, "i"] \\ BU(n) \arrow[r, "\psi^k"'] & BU(n) \end{tikzcd}$$

It follows that the bundle $J \circ i$ is fiber homotopy equivalent $J \circ \psi^k$. However, unstable Adams operation does not exist on $BU(n)$.

Sullivan's idea is to produce an unstable Adams operation on the pro-p completion $BU(n)$, where one naturally has the diagram

$$\begin{tikzcd} \widehat{BU(n-1)}_p \arrow[r, "\psi^k"] \arrow[d, "i"] & \widehat{BU(n-1)}_p \arrow[d, "i"] \\ \widehat{BU(n)}_p \arrow[r, "\psi^k"'] & \widehat{BU(n)}_p \end{tikzcd}$$

And then he showed that it sufficed to prove the profinite version of Adams conjecture.

3 Profinite Completion

We recall some facts about profinte completion in algebra.

Definition 3.0.1. A **profinite group** is the projective limit of finite discrete groups.

Example 3.0.1. The Galois group of an infinite field extension $K|F$ is the profinite limit of the Galois groups $Gal(L|F)$, where L ranges from all finite Galois extensions of F.

Let $\mathbb{Q}^{ab}$ denote the maximal abelian extension of $\mathbb{Q}$. From class field theory, one know that any abelian extension of $\mathbb{Q}$ is contained in some cyclotomic extension (meaning joining some root of unity). Thus,

$$Gal(\mathbb{Q}^{ab}|\mathbb{Q}) = \varprojlim Gal(\mathbb{Q}(\zeta_n)|\mathbb{Q}) = \varprojlim_n (\mathbb{Z}/n)^\times$$

So here is Sullivan's idea of profinite completion of spaces, inspired by Artin-Mazur's étale homotopy theory. We start by the following example:

Example 3.0.2. Fix a space F with finite homotopy groups. For every CW complex Y, the set $[Y, F]$ carries a compact topology since we have

$$[Y, F] = \varprojlim_\alpha [Y_\alpha, F]$$

where Y_α ranges from all finite subcomplexes of Y and as $[Y_\alpha, F]$ is finite by obstruction theory.

Now fix a CW complex X. Let $X/\mathcal{F}$ denote the category whose objects are maps $X \to F$ for some F with finite homotopy groups, and morphisms diagrams of the form

$$\begin{tikzcd} & X \arrow[dl] \arrow[dr] & \\ F' \arrow[rr] & & F \end{tikzcd}$$

Artin-Mazur showed that this category is equivalent to a small filtering category.

Theorem 3.1. The functor $Y \mapsto \varprojlim_{X/\mathcal{F}} [Y, F]$ satisfies the hypothesis of Brown Representability, therefore is represented by some CW complex, which we call $\widehat{X}$, the **profinite completion of X**. Note that there is a natural map $X \to \widehat{X}$ given by the structure maps $X \to F$ in $X/\mathcal{F}$.

Sullivan also proved the following equivalence:

Theorem 3.2. For a CW complex X, the profinite completion $X \to \widehat{X}$, and the inverse system of spaces F in $X/\mathcal{F}$, the following hold:

1. $\widehat{X} = \varprojlim_{X/\mathcal{F}} F$
2. $\widehat{\pi_1(X)} \cong \varprojlim_{X/\mathcal{F}} \pi_1(F)$.
3. $H^*(X; M) \cong \varprojlim_{X/\mathcal{F}} H^*(F; M)$ for all finite coefficient M.

The converse is also true: let F_α be any inverse system of spaces with finite homotopy groups, together with maps $X \to F_\alpha$. If The system satisfies 2 and 3 listed above, then $\widehat{X} \cong \varprojlim_\alpha F_\alpha$.

If the reader if familiar, criteria 2 and 3 corresponds to the comparison theorems of étale fundamental group and étale cohomology respectively. The general construction of this is motivated by the theory of étale homotopy type, which tried to capture the theory of étale fundamental group and cohomology into one unifying ordinary homotopy type.

By developing the theory of profinite completion, Sullivan introduced profinite K-theory, defined by

$$\widehat{K}(X) = \varprojlim_n K(X) \otimes \mathbb{Z}/n$$

and showed that it is classified by $\widehat{BU}$, which is also built from the limit of $\widehat{BU}(n)$. Moreover, Sullivan developed the theory of localized and completed spherical fibrations, with the main theorem being that the stable fiber homotopy types injects into profinite stable homotopy types. In particular, we have the isomorphism on classifying space level

$$\text{Stable profinite theory}: \ \widehat{B}_\infty \cong B_{SG} \times K(\widehat{\mathbb{Z}^*}, 1)$$

$$\text{Stable theory}: \ BG \cong B_{SG} \times K(\mathbb{Z}/2, 1)$$

where $B_{SG} = \varinjlim_n B_{SG(n)}$, and $B_{SG(n)}$ is the classifying space for the component of the identity map in $G(n)$. (Alternatively, it is also the universal cover of $BG(n)$). Thus, the classifying space for the stable theory is a direct factor of the stable profinite theory, and it suffice to formulate and prove the Adams conjecture in the profinite setting.

4 Étale Homotopy Theory

Suppose we have a smooth complex variety X. Its complex points has a complex analytic topology, which we denote X^{an}. The Zariski topology is much more coarse than the analytic topology, so how can we extract classical algebraic invariants such as homotopy groups and cohomology groups of X^{an} from algebraic data of X? On one hand, one has the étale fundamental group, which classifies the finite étale covers over a scheme X. For a smooth variety over $\mathbb{C}$, we have the comparison theorem

Theorem 4.1. Let X be a smooth variety over $\mathbb{C}$, then

$$\pi_1^{et}(X, \overline{x}) \cong \pi_1\widehat{(X^{an}, x)}$$

On the other hand, Grothendieck developed the theory of étale topology and étale cohomology, which yields the following comparison theorem

Theorem 4.2. Let X be a smooth variety over $\mathbb{C}$, then

$$H^*(X_{et}; M) \cong H^*(X^{an}; M)$$

for all finite coefficients M.

4.1 Étale morphisms

The motivation is to have an analog of covering space theory in algebraic geometry, and the correct notion for that is a finite étale morphism.

Definition 4.2.1. A morphism of schemes $f : X \to Y$ is **étale** if it is flat and unramified.

Proposition 4.2.1. Suppose Y is a connected variety/scheme. A finite étale morphism $\varphi : X \to Y$ is open, closed, and thus surjective. Moreover, for each point $y \in Y$, there is an étale neighborhood $(U, u) \to (Y, y)$ such that $X \times_Y U$ is a disjoint union of open subvarieties/subschemes, each mapping isomorphically down to U.

From this we see that finite étale morphisms are indeed natural analog of a finite covering spaces. However, there is no direct algebraic analog for the "universal cover," when it is often an infinite sheeted covering.

Example 4.2.1. The universal cover of $\mathbb{C} - \{0\}$ is given by

$$\mathbb{C} \xrightarrow{exp} \mathbb{C} - \{0\}$$

which is a $\mathbb{Z}$-sheeted cover.

Example 4.2.2. There is no algebraic analog for the universal covering space as in example 4.1.1: any morphism

$$\mathbb{A}^1_{\mathbb{C}} \to \mathbb{A}^1_{\mathbb{C}} - \{0\}$$

must be a constant map, hence not even étale. The finite étale covers of $\mathbb{A}^1_{\mathbb{C}} - \{0\}$ are of the form

$$\mathbb{A}^1_{\mathbb{C}} - \{0\} \xrightarrow{z \mapsto z^n} \mathbb{A}^1_{\mathbb{C}} - \{0\}$$

for some n.

It is not hard to check by hand that these morphisms are étale. The fact that these are essentially the only ones is given by Riemann existence theorem.

Theorem 4.3 (Riemann Existence Theorem)**.** Let X be a nonsingular variety over $\mathbb{C}$. Then, a finite étale morphism $Y \to X$ lifts to a finite covering space of analytifications $Y^{an} \to X^{an}$. Moreover, there is a equivalence of categories between finite etale covers of X and finite coverings of X^{an}.

Remark 4.3.1. The solution to the lack of universal cover is to consider a projective system of finite covers $\{(X_i, \pi_i)\}$ that **pro-represents** the functor taking the fiber over a chosen geometric point of a finite étale morphism. This leads to the definition of the étale fundamental group

$$\pi_1^{et}(X, \overline{x}) = \varprojlim_i \mathrm{Aut}_X(X_i)$$

It follows from the Riemann existence theorem that

$$\pi_1^{et}(X, \overline{x}) \cong \pi_1(\widehat{X^{an}, x})$$

4.2 Étale Topology

Recall in topology, we have the Leray nerve theorem, which says given a "good" open covering of X (good meaning each finite intersection of opens sets in the cover is either empty of contractible), then the associated nerve complex is weak homotopy equivalent to X.

Example 4.3.1. Let $\{U_1, .., U_n\}$ be a Zariski covering of an irreducible variety X. Then, the associated nerve complex is Δ^{n-1}.

This is because any intersection of non-empty open set in the Zariski topology is non-empty. Grothendieck's idea is then to replace Zariski open sets with étale opens.

Definition 4.3.1. A morphism between schemes $f : X \to Y$ is **étale** if it is flat and unramified.

In particular, an étale morphism is open; moreover, an étale morphism between smooth affine varieties over an algebraically closed field is precisely a regular map that induces an isomorphism of Zariski tangent spaces. Thus, one can think of étale morphism as a generalization for local diffeomorphisms. The work of Verdier on hypercoverings, and independently the work of Lubkin on punctually finite coverings, made possible to associate any locally notherian scheme X a pro-object $\{N_\alpha\}$ in the homotopy category of simplicial sets. This is what Artin-Mazur referred to as the **étale homotopy type** of X. The homotopy groups/homology groups of the etale homotopy type are then defined to be the projective/injective limit of homotopy groups/homology groups of N_α.

We now explain more in detail Sullivan's slight variation of Lubkin's method: for a complex algebraic variety V, he considers an étale covering $\mathcal{U}_\alpha := \{(U_i, \varphi_i)\}$ of V to be a collection of étale morphisms $\varphi : U_i \to V$ such that $\cup_i \varphi_i(U_i) = V$. Then, consider $\mathcal{U}_\alpha$ as a full category of schemes over V, so that morphism $U_i \to U_j$ are V-morphisms. Let N_α denote the realization of U_α, and we get an inverse system of complexes $\{N_\alpha\}$ indexed by étale covers of V and filtered by refinement. The work of Artin-Mazur yield the following comparison theorem

Theorem 4.4 (Comparison Theorem)**.** If V is normal, then

$$\widehat{\pi_1(V^{an})} \cong \varprojlim \pi_1 N_\alpha$$

and

$$H^*(V^{an}; M) \cong \varinjlim H^*(N_\alpha; M)$$

for finite coefficients.

so indeed the étale homotopy type $\varprojlim N_\alpha$ does give the profinite completion of V^{an}, through a purely algebraic method.

5 The Case of $\mathbb{CP}^1$

We now discuss the main example to the construction above. Let $\mathcal{U}_n$ denote the following étale cover of $\mathbb{P}^1_{\mathbb{C}}$: take $V_0 = \mathbb{A}^1$ and $V_1 = \mathbb{A}^1$ be the standard affine opens of $\mathbb{P}^1$ and $V_{01} := V_1 \cap V_2 \cong \mathbb{A}^1 - \{0\}$. Then, we may étale cover $\mathbb{P}^1$ by

$$U_0 = \mathbb{A}_1 \xrightarrow{\cong} V_0$$

$$U_1 = \mathbb{A}_1 \xrightarrow{\cong} V_0$$

$$U_2 = \mathbb{A}_1 - \{0\} \xrightarrow{\deg\ n} V_{01}$$

The category $\mathcal{U}_n$ has the following morphisms: there is a unique morphism from U_2 to U_0 and to U_1, and U_2 has automorphisms group $\mathbb{Z}/n$. We may represent $\mathcal{U}_n$ as

$$\bullet \longleftarrow \mathbb{Z}/n \longrightarrow \bullet$$

and the nerve of the category is $\Sigma K(\mathbb{Z}/n, 1)$. One may see this as attaching two cones over $K(\mathbb{Z}/n, 1)$.

We may consider the inverse system of nerves $\{N\mathcal{U}_n\} = \{\Sigma K(\mathbb{Z}/n, 1)\}$ partially ordered by divisiblility of n.The system is homotopically cofinal in the inverse system of all nerves of etale coverings, so it represents the étale homotopy type of $\mathbb{P}^1$. Let $\widehat{X} = \varprojlim_n \Sigma K(\mathbb{Z}/n, 1)$. One easily see that $\pi_1(\widehat{X}) = 0 = \pi_1(\mathbb{CP}^1)^\wedge$, and

$$H^*(\widehat{X}; M) = \varinjlim_n H^*(\Sigma K(\mathbb{Z}/n, 1)) = H^*(\mathbb{CP}^1; M)$$

for all finite coefficient system M. Thus, we have constructed the profintie completion of $\mathbb{CP}^1$ purely from the etale coverings of $\mathbb{P}^1$.

6 Galois symmetry

Note that $BU(n)$ is the inductive limit of grassmannians, which are naturally complex varieties through plucker embedding. In fact, they are varieties over $\mathbb{Q}$, and there is a natural absolute Galois group $Gal(\mathbb{Q}^{sep}|\mathbb{Q})$ action on the grassmannians. The action also induces an automorphism of the etale covers, as we have the pullback

$$\begin{tikzcd}
\sigma^* U \arrow[r] \arrow[d] & U \arrow[d] \\
V \arrow[r, "\sigma"] & V
\end{tikzcd}$$

It is easy to verify for the example

$$\mathbb{A}^1 - \{0\} \xrightarrow{z \mapsto z^n} \mathbb{A}^1 - \{0\}$$

which has covering automoprhism $\mathbb{Z}/n$, the aboslute galois group acts by moving the nth root of unity around. Thus, there is an action on each nerve $\Sigma K(\mathbb{Z}, 1)$, and the one may check

Theorem 6.1. The absolute Galois $Gal(\mathbb{Q}^{sep}|\mathbb{Q})$ acts via acts via the cyclotomic character through $\widehat{\mathbb{Z}}^*$ on $H^2(\widehat{P}^1; \widehat{\mathbb{Z}})$, and the action is given by sending the generator $x \to ax$, where $a \in \widehat{\mathbb{Z}}^*$.

Corollary 6.1.1. The absolute Galois $Gal(\mathbb{Q}^{sep}|\mathbb{Q})$ acts via the cyclotomic character through $\widehat{\mathbb{Z}}^*$ on $H^*(\widehat{BU}(n);\widehat{\mathbb{Z}}) = \widehat{\mathbb{Z}}[c_1,...,c_n]$, and the action is given by map on the ith chern classs $c_i \to a^i c_i$, where $a \in \widehat{\mathbb{Z}}^*$.

The first Chern class classifies isomorphism class of line bundles. So for any k, we may choose a class $\sigma \in Gal(\mathbb{C}|\mathbb{Q})$ such that $\chi(\sigma) = k \in \widehat{\mathbb{Z}}^*$. It follows from the profinite theory of characteristic classes, the homotopy equivalence

$$BU \xrightarrow{\sigma} BU$$

sends lines bundles to their kth power, as desired.

7 Proof

Definition 7.0.1 (Čech Nerve)**.** Let X be a finite CW complex, and $\mathcal{U} := \{U_i : i \in I\}$ be an open cover of X. Then, we may define a simplicial set call the **Čech Nerve** $N\mathcal{U}$ as follows: we have the assignment on objects $[n] \mapsto \{\text{functions from } [n] \text{ to } I : \cap_{i=1}^{n} U_{f(i)} \neq \emptyset\}$. The face maps and degeneracy maps are defined by deleting and inserting appropriate indices.

Alternatively, we can think of a covering $\mathcal{U}$ as follows: suppose given a covering $X = \cup_{i\in I} U_i$; let $\mathcal{U} = \coprod_{i\in i} U_i$, and the covering is the obvious map $\mathcal{U} \to X$. Note that we have

$$U_i \cap U_j = U_i \times_X U_j$$

so the n-fold fiber product $U \times_X ... \times_X U$ is the disjoint union of n-fold intersections of opens in the cover. Then, the nth simplices of the Čech nerve is $\pi_0(\underbrace{U \times_X ... \times_X U}_{n-fold})$. The face maps are projections, and the degeneracy maps are various diagonal embeddings.

Theorem 7.1. If the covering $\mathcal{U}$ satisfies the property that arbitrary intersections of opens in the cover is either empty or contractible, then th realization $|N\mathcal{U}|$ is weakly equivalent to X.

8 Adam's conjecture

9 Algebraic Side

Idea of proof

Step 2: We identify the classical Adams operation in the following way: the classical Adams operation

$$K(X) \xrightarrow{\psi^k} K(X)$$

naturally desends to maps on the profinite completion, which factors as

$$\prod_p \widehat{K(X)}_p \xrightarrow{\widehat{\psi}_p} \prod_p \widehat{K(X)}_p$$

and $\widehat{\psi^k}_p : \widehat{K(X)}_p \to \widehat{K(X)}_p$ is an isomorphism iff k is prime to p. If k is divisible by p, we redefine $\widehat{\psi^k}_p$ to be the identity map. After the redefinition, we obtain

$$\widehat{K(X)} \xrightarrow{\psi^k} \widehat{K(x)}$$

which we call the "isomorphic" part of the Adams operation.

Remark 9.0.1. Before the redefinition, in the case where $k|p$, we note that$widehat\psi^k{}_p$ is topologically nilpotent.

Following this, Sullivan observed that this isomorphic part of the Adams operation is compatible with the natural action of $Gal(\overline{\mathbb{Q}}|\mathbb{Q})$ in the category of profinite homotopy type and maps coming from the algebraic varieties defiend over $\mathbb{Q}$. In particular, there iare homomorphisms

$$Gal(\mathbb{Q}|\mathbb{Q}) \to \widehat{\mathbb{Z}}^* \to \mathrm{Aut}(\widehat{K(X)})$$

by letting G act on the roots of unity. Moreover, for each ψ^k, we note that k defines an element $(k) \in \widehat{\mathbb{Z}}^*$ by giving the automorphism

$$(k)x = \begin{cases} k \cdot x & \text{if } x \in \widehat{\mathbb{Z}_p}, (k,p) = 1 \\ x & \text{if } x \in \widehat{\mathbb{Z}_p}, (k,p) \neq 1 \end{cases}$$

Clearly this is compatible with the Adams operation on profinite K theory. Thus, we have identified the profinite Adams operation with the action of the profinite group $\widetilde{\mathbb{Z}}^*$.

<u>**Step 3**</u>: There is a natural action of the absolute Galois group $Gal(\overline{\mathbb{Q}}|\mathbb{Q})$ on the profinite classifying space $\widehat{BU}$, and the Adams operation is compactible with such action through the abelianization map.

$$Gal(\overline{\mathbb{Q}|\mathbb{Q}}) \to \widehat{\mathbb{Z}}^*$$

The aboslute Galois group action is how the etale homotopy type theory factors in. Note that the absolute galois group acts algebraically on $\mathbb{C}^n$ and $\mathbb{CP}^n$, but with classical topology this action is wildly discontinuous. However, for every algebraic variety V, we may constrtcut an inverse system of nerves N_α, with natural maps $V \to \{N_\alpha\}$ giving

$$\widehat{\pi_1(V)} \cong \varprojlim_{\alpha} \pi_1 N_\alpha \text{ and } H^i(V; M) \cong \varinjlim H^i(N_{\alpha;M})$$

for all finite coefficient M.

Sullivan proves that the the above isomorphism imply the profinite completion of V can be constructed from the nerves

$$\widehat{V} \cong \varprojlim_{\alpha} N_\alpha$$

in the sense of compact functors (with the extra assumption that $\pi_i(N_\alpha)$ is finite.)

Since each N_α is constructed using the algebraic structure of V, and each automorphism $\sigma \in Gal(\overline{Q}|\mathbb{Q})$ determines a simplicial automorpihism of N_α, and thus the profinite homotopy type of any complex algebraic variety defined over $\mathbb{Q}$.

Recall that the classifying space $BU(n)$ is constructed as the direct limit of complex grassmannians $\varinjlim_k Gr_n(k)$. Via Plücker embeddings, the complex grassmannians are naturally affine complex varieties embedded in projective space. Moreover, the defining polynomials also have coefficients in $\mathbb{Q}$.(Example here?)

By naturality and splitting principal, understanding the action of $Gal(\mathbb{C}|\mathbb{Q})$ on the profinite complex K-theory reduces to understanding the action on $\cup_n \widehat{\mathbb{CP}^n} \cong K(\widehat{\mathbb{Z}}, 2)$, which can be checked to be the composition

$$Gal(\mathbb{C}|\mathbb{Q}) \xrightarrow{\chi} \widehat{\mathbb{Z}}^* \to K(\widehat{Z}, 2)$$

and thus agrees with the isomorphic part of he Adams operations discussed above.

10 The Adams Conjecture-Proof

Again, we note that by choosing $\sigma \in Gal(\overline{\mathbb{Q}}|\mathbb{Q})$ such that $\chi(\sigma) = k^{-1}$, we have the commutative square

$$\begin{tikzcd}
\widehat{BU(n)} \arrow[r] \arrow[d, "\sigma"] & \widehat{BU} \arrow[d, "\psi^k"'] \\
\widehat{BU(n)} \arrow[r] & \widehat{BU}
\end{tikzcd}$$

<u>**Final Step**</u>: We note that the inclusion map $\widehat{BU(n)} \to \widehat{BU}$ is the tautological spherical fibration. The pullback of the fibration is also the bundle classfied by the map $\psi^k \circ i$. In other words, we have the homotopy cartesian square

$$\begin{tikzcd}
B\widehat{U(n-1)} \arrow[d, "i"] & B\widehat{U(n-1)} \arrow[l, "\psi^k"'] \arrow[d, "i"'] \\
\widehat{BU(n)} & \widehat{BU(n)} \arrow[l, "\psi^k"']
\end{tikzcd}$$

where we are pulling back along a homotopy equivalence, so the two fibrations are fiber homotopically equivalent, and we finish.