MATH 624 Algebraic Geometry

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September 4, 2024

1 Prevarieties and Varieties

We will assume that $K|k$ a finite extension, K is algebraically closed. We will use $\mathbb{A}^n(K) = K^n = \mathbb{A}^n_K$ to denote the underlying set, not the n-dimensional affine space. Given a point $a = (a_1, ..., a_n) \in \mathbb{A}^n_k$, we will use φ_a to denote the evaluation map $k[X] \to k$. Similarly, given $f \in k[x]$, we have the evalation map $\tilde{f} : \mathbb{A}_k \to k$. This gives a morphism of k-algebras $k[x] \to Maps_k(\mathbb{A}_k, k)$ given by $f \mapsto \tilde{f}$.

Definition 1.0.1. Given $\Sigma \subset k[x]$, define $V(\Sigma) = \{a \in \mathbb{A}_k : f(a) = 0 \text{for every } f \in \Sigma\}$. This is called the **affine k-algebraic set** defined by Σ. If $\Sigma = \{f\}$, then $H_f := V(\Sigma) = V(f)$ defines a **hyperplane** in $\mathbb{A}_k$.

Example 1.0.1. Easy examples

1. $V((0)) = \mathbb{A}_k$.
2. $V((1)) = \emptyset$
3. Let $k = \mathbb{C}$. Then, in $\mathbb{A}^1_k$, $V(x^2 - 1) = \{\pm 1\}$. In $\mathbb{A}^2_k$, $V(x^2 - 1) = \{(\pm 1, n) : n \in k\}$

Definition 1.0.2. Given $V \subset \mathbb{A}_\neg$, defined $I(V) = \{f \in k[x] : f(V) = 0\}$. This is called the **ideal** of V.

Proposition 1.0.1.
1. Let $I_\Sigma \subset k[x]$ be the ideal generated by Σ. Then, $V(\Sigma) = V(I)$.
2. There exists a finite system $f_1, .., f_m$ such that $V(\Sigma) = V(f_1, ..., f_m)$
3. If $\Sigma_1 \subset \Sigma_2$, then $V(\Sigma_1) \supset V(\Sigma_2)$
4. Given $\mathfrak{a}$ an ideal, then $I(V(\mathfrak{a})) = \mathfrak{a}$ iff $\mathfrak{a} = \sqrt{\mathfrak{a}}$.
5. Given ideals $\mathfrak{a}, \mathfrak{b}$, then $V(\mathfrak{a}) = V(\mathfrak{b})$ iff $\sqrt{\mathfrak{a}} = \sqrt{\mathfrak{b}}$.

Definition 1.0.3. Let $\mathcal{A}^n_K := \{V \subset \mathbb{A}^n_K : V \text{affine } k-\text{algebra}\}$. Given $V \in \mathcal{A}^n_K$, let $k[V] := k[x]/I(V)$ be the **affine coordinate ring** generated by V.

Let $Id^{rd}(k[x])$ be the set of reduced ideals of $k[x]$. Let R_n be the set of reduced k-algebras with n-generators.

Theorem 1.1. There is a canonical bijection between the set of reduced affine k-algebras and reduced ideals of $k[x]$, given by the maps

$$R_n \to Id^{re}(k[X]) \to \mathcal{A}_K^k$$

$$k[\underline{x}] \mapsto \mathfrak{a} := ker(k[x] \xrightarrow{f} k) \mapsto V(\mathfrak{a})$$

with f given by $x \mapsto \underline{x}$.

1.1 The Zariski Topology

Given $V \in \mathcal{A}_K^n$, there is a canonical map $K[X] \to K[V]$ given by $f \mapsto f_V$.

Proposition 1.1.1. Let $\Sigma_i \subset k[X]$, and $f \in k[X]$ be given. then

1. $V(\cup_i \Sigma_i) = \cap_i V(\Sigma_i)$
2. $V(\prod \Sigma_i) = \cup V(\Sigma_i)$
3. $V((0)) = \mathbb{A}_k^n$; $V((1)) = \emptyset$

By the proposition above, we can define the Zariski topology on $\mathbb{A}_k^n$

Definition 1.1.1. The Zariski topology on $\mathbb{A}_K^n$ is given by the closed sets $V(\Sigma)$, with $\Sigma \in k[X]$. In particular, the sets $D_f := \mathbb{A}_k^n - H_f$ is an open set and forms a basis for the topology.

Note that the zariski topology on product spaces is not the product of zariski topologies. Moreover, the connectedness/irreducibility is dependent on $K|k$. A point is called a generic point of V if its closure contains V.

Example 1.1.1. If $K|k = \mathbb{C}|\mathbb{Q}$, then $V(x_1^2 - 2x_2^2)$ is connected and irreducible. If $K|k = \mathbb{C}|\mathbb{Q}[\sqrt{2}]$, then $V(x_1^2 - 2x_2^2)$ is connected but not irreducible.

Remark 1.1.1. For a topological space, X, the following are equivalent:

1. Every descending chain of closed subsets is stationary.
2. Every ascending chain of open subsets is stationary.

A topological space satisfiying the above is called **<u>Noetherian</u>**. For example, $Spec(R)$ is Noetherian if R is Noetherian. Note that if X is Noetherian, then it is automatically quasi-compact. Moreover, there are only finitely many irreducible components and connected components of X.