

MATH 618 Algebraic Topology

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1 The Correct Category

Let T = compactly generated weakly Hausdorff. Let T_2 = pairs of spaces (X, A) , $A \subseteq X$, and

$$T_2((X, A), (Y, B)) = \{f \in T(X, Y) : f(A) \subseteq B\}$$

We define $(X, A) \otimes (Y, B) = (X \times Y, X \times B \cup Y \times A)$. (Think about product of boundaries). We want to understand the analogue of $T(X \times Y, Z) \cong T(X, T(Y, Z))$.

Theorem 1.1. Let $(X, A), (Y, B), (Z, C) \in T_2$, then

$$T_2((X, A) \otimes (Y, B), (Z, C)) \cong T_2((X, A), T_2(T_2((Y, B), (Z, C))), T(Y, C))$$

Let T_* be the full subcategory of T_2 consisting of pairs $(X, *)$. There exists a pair of functors $T_2 \rightarrow T_*$ defined by $(X, A) \mapsto (X/A, A/A = *)$.

Proposition 1.1.1. $q : X \rightarrow X/A$, we get

$$q_* : T_*(X/A, Y) \rightarrow T_2((X, A), (Y, *))$$

an isomorphism.

We want a product in T_* which works well with function spaces:

Definition 1.1.1. Given $X, Y \in T_*$, defined $X \wedge Y = (X \times Y / X \vee Y, * = X \vee Y)$ called the smash product.

Note that the smash product is not the categorical product here. (The categorical product is simply the cartesian product carrying the canonical basepoint).

Definition 1.1.2. The reduced suspension $\Sigma X := S^1 \wedge X$.

Theorem 1.2. The category of based spaces T_* has the following properties

1. $T_*(X, Y) \in T_*$, with basedpoint the constant map to basepoint.
2. $T_*(X, Y) \wedge X \rightarrow Y$ is continuous.
3. $T_*(X, Y) \wedge T_*(Y, Z) \rightarrow T_*(X, Z)$ is continuous.
4. $T_*(X \wedge Y, Z) \cong T_*(X, T_*(Y, Z))$
5. Small limits and colimits exists in T_* .
6. The forgetful functor $T_* \rightarrow T$ preserves limits.

Definition 1.2.1. The reduced cone is a functor $C : T_* \rightarrow T_*$ defined by $X \mapsto IX$, with the basepoint of I being 1.

Proposition 1.2.1. There exists a pushout

$$\begin{array}{ccc} X & \longrightarrow & CX \\ \downarrow & & \downarrow \\ CX & \longrightarrow & \Sigma X \end{array}$$

Definition 1.2.2. The loop space is defined to be $T_*(S^1, X)$.

Theorem 1.3. (Eckmann-Hilton duality)

$$T_*(X, \Omega Y) \cong T_*(\Sigma X, Y)$$

There exists a functor $T \rightarrow T_*$ given by $X \mapsto X \coprod \{x\}$, which is left adjoint to the forgetful functor. If $X \in T_*$, then $\Omega X \in T_*$, and note that $\pi_0(\Omega X) = [S^1, X]_* = \pi_1(X)$.

Definition 1.3.1. A functor $F : T_* \rightarrow C$ for some category C is called a homotopy functor if $F(f) = F(g)$ when $f \cong g$

Definition 1.3.2. Define $\pi_n(X) := [S^n, X]_*$. Then, π_n is a homotopy functor from T_* to set.

Proposition 1.3.1. π_n is a group for $n \geq 1$ and abelian when $n \geq 2$.

Proof. The group structure is given by: suppose we have $\varphi, \psi : I^n \rightarrow X$. Then, $\varphi + \psi : I^n \rightarrow X$ is explicitly given by

$$\varphi + \psi(t_1, \dots, t_n) = \begin{cases} \varphi(2t_1, t_2, \dots, t_n) & t \in [0, \frac{1}{2}] \\ \psi(2t_1 - 1, t_2, \dots, t_n) & t \in [\frac{1}{2}, 1] \end{cases}$$

□

Definition 1.3.3. Define H -space be a topological space X with homotopy associative map

$$X \wedge X \rightarrow X$$

Proposition 1.3.2. If (Y, y_0) is an H -space, then $[X, Y]_*$ has a group structure.

Lemma 1.4. Let $C = (C^{*,*}, d, \delta)$, be a double complex, (C^*, d) a complex with $\epsilon : C^* \rightarrow C^{*,0}$ such that $d\epsilon + \epsilon d = 0$. Furthermore, assume that the columns are exact (in particular, this implies that $H^q(C^{p,*}, \delta) \cong C^p$ for $q = 0$). Then, $\epsilon : H^p(C, d) \rightarrow H^p(\text{Tot}(C), D)$ is an isomorphism.

Proof. standard augmentation/resolution lemma. Mattie you can do this. \square

Let M be a smooth manifold. Suppose M has a good cover $U = \{U_\alpha\}$ such that each intersection $U_{\alpha_1, \dots, \alpha_n}$ is either empty or contractible. (Note that you can always do this for a Riemannian manifold). We define a double complex as follows: Let

$$C^{p,q} = \prod_{\alpha_0, \dots, \alpha_q} \Omega^q(U_{\alpha_0, \dots, \alpha_q})$$

The horizontal differentials $d : C^{p,q} \rightarrow C^{p+1,q}$ is $d(\omega \in \Omega^p(u_{\alpha_0, \dots, \alpha_q})) = d\omega \in \Omega^{p+1}(U_{\alpha_0, \dots, \alpha_q})$, which is just the De Rham differentials.

The vertical differential $\delta : C^{p,q} \rightarrow C^{p,q+1}$ is defined to be: let $\omega \in \prod_{\alpha_0, \dots, \alpha_q} \Omega^p(U_{\alpha_0, \dots, \alpha_q})$

$$\delta(\omega)_{\alpha_0, \dots, \alpha_{q+1}} = \sum_{j=0}^{q+1} (-1)^j \omega_{\alpha_0, \dots, \widehat{\alpha_j}, \dots, \alpha_{q+1}} |_{U_{\alpha_0, \dots, \widehat{\alpha_j}, \dots, \alpha_{q+1}}}$$

which is the Cech differential.

It is easy to check that $\delta^2 = 0$ and $d^2 = 0$ and $d\delta = \delta d$.

We have an augmentation of the double complex by the De Rham complex, and apply the previous lemma. The trick to checking the columns are exact is to let λ_α be a partition of unity for U_α , and define a chain contraction $s : C^{p,q} \rightarrow C^{p,q-1}$ defined by $(s\omega)_{\alpha_0, \dots, \alpha_{q-1}} = \sum \lambda_{\alpha_0} \omega_{\alpha_0, \dots, \alpha_{q-1}}$. Thus, we get that $H^p(\Omega M) \cong H^p(\text{Tot}(C), D)$.

Similarly, there is an augmentation of the Cech comcomplex $C^{\vee,q}(U) = \prod_{\alpha_0, \dots, \alpha_q} \mathbb{R}$, with $\delta : C^{\vee,q} \rightarrow C^{\vee,q+1}$ defined by $\delta(c)_{\alpha_0, \dots, \alpha_{q+1}} := \sum (-1)^j c_{\alpha_0, \dots, \widehat{\alpha_j}, \dots, \alpha_{q+1}}$. We may check that the rows are exact directly this time, and that the cohomology of the complex is isomorphic to the cech cohomology $H^*(C^{\vee,*}, \delta)$ by the row analog of the lemma. In the end, we have

Theorem 1.5 (De Rham Theorem).

$$H_{DR}^*(M) \cong H^*(\Omega^* M) \cong H^{\vee,*}(M)$$

Definition 1.5.1. A **spectral sequence** is a sequence of abelian groups and differentials (A_n, d_n) , such that $d_n : A_n \rightarrow A_n$, such that $A_n \cong H^*(A_n, d_n)$ (Block's notation A_n is the page). A morphism of spectral sequences is a sequence of map $A_n \rightarrow E_n$ that commutes with all differentials and turning the page.

Theorem 1.6 (Serre Spectral Sequence). Let $\pi : Y \rightarrow X$ be a Serre fibration, with fiber F . Then, there exists a spectral sequence $(E_r^{p,q}, d_r)$, with the differentials going the way

$$d_r : E_r^{p,q} \rightarrow E_r^{p+r, q-r+1}$$

If X is simply connected, then the E_2 page is given by

$$E_2^{p,q} = H^p(X; H^q(F)) \Rightarrow H^{p+q}(Y)$$

Example 1.6.1. Consider the fibration $S^1 \rightarrow S^{2n+1} \rightarrow \mathbb{CP}^n$. Then, $E_2^{p,q} = H^p(\mathbb{CP}^n, H^q(S^1))$. You can draw the E_2 page of the spectral sequence, and find that the non-trivial terms concentrate on row 0, 1 by our knowledge of the cohomology of S^1 . Moreover, we know the cohomology of \mathbb{CP}^n is only in even degrees, so we only get non-zero terms in the even columns. Combining the information, the E_2 page is pretty clear to see. You can work out when the spectral sequence collapses and how it converges.