Equivariant Stable Homotopy Notes

David Zhu

April 16, 2025

These notes aim to survey major topics in equivariant stable homotopy theory, a field revitalized by the groundbreaking work of Hill, Hopkins, and Ravenel on the Kervaire invariant one problem. The primary references are HHR's original paper [HHR16] and their later book [HHR21]. Additional classical results are drawn from [LMM86] and [BR20]. The structure of the thesis is inspired by the excellent notes [Blu17].

Throughout the notes, G will be a finite discrete group, although many theorems can be generalized to the case where G is a compact Lie group. Chapter 1 introduces unstable equivariant homotopy theory, including some concrete computations of Bredon cohomology. Chapter 2 establishes the equivariant stable category and discusses key constructions following [HHR16]. Chapter 3 focuses on two explicit computations of $RO(C_2)$-graded cohomology, which are elementary in principle but intricate to perform. Chapter 4 presents the basics of the slice spectral sequence, the main computational tool used in [HHR16] for the Gap theorem. The thesis assumes readers' familiarity with the basics of unstable homotopy theory, enriched category theory, and model category theory. An appendix on nonequivariant stable homotopy theory is included.

Contents

1 Unstable Equivariant Homotopy Theory **3**
- 1.1 Topological G-spaces . . . 3
- 1.2 Homotopy Theory and Elmendorf's Theorem . . . 4
- 1.3 G-CW Complexes . . . 7
- 1.4 Bredon Cohomology . . . 10
- 1.5 Computations . . . 12
  - 1.5.1 Constant Coefficients . . . 12
  - 1.5.2 A general example . . . 16
  - 1.5.3 Free Action . . . 17

2 The Equivariant Stable Homotopy Category **18**
- 2.1 Orthogonal G-spectra . . . 18
- 2.2 Homotopical Properties of $\mathbf{Sp}^G$ . . . 22
- 2.3 Transfer and Mackey functor . . . 24
- 2.4 Fixed Points and tom Dieck Splitting . . . 31

3 $RO(G)$-graded cohomology Theories **33**
- 3.1 Representability . . . 33
- 3.2 The $RO(C_2)$ cohomology of a point . . . 35
  - 3.2.1 Computing $H^{p,q}_{C_2}(*;\underline{\mathbb{Z}})$ . . . 36
  - 3.2.2 Computing $H^{p,q}_{C_2}(*;A_{C_2})$ . . . 39

4 The Slice Spectral Sequence **44**
- 4.1 Localizing Subcategory and Slice Connectivity . . . 45

4.2 The Slice Tower and the Slice Spectral Sequence 48
4.3 Identifying the Slices 50
4.4 The Case of $K\mathbb{R}$ 53

A Nonequivariant Stable Homotopy Theory **57**
A.1 Motivations 57
A.2 Duality 61
A.3 The Desired Stable Homotopy Category 62
A.4 The Stable Model Structure of Sequential Spectra 63
A.5 Triangulated Structure and Brown Representability 67
A.6 Orthogonal Spectra 68

1 Unstable Equivariant Homotopy Theory

1.1 Topological G-spaces

A **G-space** is a compactly generated, weak Hausdorff topological space with a G-action. A **G-equivariant map** (or G-map) $f\colon X \to Y$ of G-spaces X and Y is a continuous map such that $f(g \cdot x) = g \cdot f(x)$. The category **GTop** has objects G-spaces and morphisms G-maps, and $\mathbf{GTop}_*$ is the category of pointed G-spaces and based G-maps.

Given a G-space X and a subgroup $H \leq G$, there is a functor

$$\mathrm{Res}^G_H\colon G\mathbf{Top} \to H\mathbf{Top}$$

that sends a G-space to H-space by restriction of the G-action. The functor has a left adjoint, which is called **induction**.

Definition 1.1. Given a subgroup $H \leq G$ and an H-space X, the **balanced product** $G \times_H X$ is the G-space defined to be the quotient $G \times X/\sim$, where the equivalence relation is given by $(gh, x) \sim (g, hx)$. The pointed version is the balanced smash product $G_+ \wedge_H X$.

Proposition 1.2. The balanced product $G \times_H -$ is left adjoint to Res^G_H. Specifically, we have the isomorphism

$$\mathrm{Hom}_H(X, \mathrm{Res}^G_H Y) \cong \mathrm{Hom}_G(G \times_H X, Y)$$

A similar adjunction holds for the pointed version.

Proof. The isomorphism

$$\mathrm{Hom}_H(X, \mathrm{Res}^G_H Y) \cong \mathrm{Hom}_G(G \times_H X, Y)$$

is given by sending an H-map $f\colon X \to Y$ to the G-map

$$\tilde{f}(g, x) := g \cdot f(x)$$

The map is clearly well-defined and G-equivariant. The inverse is given by sending a G-map $\varphi\colon G \times_H X \to Y$ to the H map

$$\overline{\varphi}(x) := \varphi(e, x)$$

□

The restriction also has a right adjoint: viewing G as an H-space by right multiplication, the space of maps $\mathrm{Maps}_H(G, -)$ is naturally equipped with a G-action, given by $g \cdot f(g') := f(g'g)$. This is called **coinduction**.

Proposition 1.3. The functor $\mathrm{Maps}_H(G, -)$ is right adjoint to Res^G_H. Specifically, we have the isomorphism

$$\mathrm{Hom}_H(\mathrm{Res}^G_H X, Y) \cong \mathrm{Hom}_G(X, \mathrm{Maps}_H(G, Y))$$

A similar adjunction holds for the pointed version.

Proof. Given an H-map $f\colon X \to Y$, we assign it to the G-map that sends $x \in X$ to the map

$$f_x(g) := f(gx)$$

The map f_x is indeed H-equivariant, as $h \cdot f_x(g) = f_x(gh) = f(ghx) = f_x(h \cdot g)$. The assignment is also G-equivariant, as $f_{gx}(g') = f(g'gx) = g \cdot f_x(g')$. Thus, we have a well-defined map

$$\mathrm{Hom}_H(\mathrm{Res}_H^G X, Y) \to \mathrm{Hom}_G(X, \mathrm{Maps}_H(G, Y))$$

and the inverse is given by sending a G-map $g(x) = g_x$ to the H-map $g' : X \to Y$ defined by $g'(x) = g_x(e)$. $\square$

Taking H to be the trivial group, the restriction functor works out to be monadic, therefore creates all limits and colimits in $G\mathbf{Top}$.

Corollary 1.4. The categories $G\mathbf{Top}$ and $G\mathbf{Top}_*$ are complete and cocomplete.

In particular, **product** of G spaces is the cartesian product of the underlying topological spaces equipped with the diagonal G-action; **coproduct** of G-spaces is the disjoint union of the underlying topological spaces with the obvious G-action. The **product**, **wedge**, and **smash product** are defined in the same fashion for pointed G-spaces.

Recall that the category $\mathbf{Top}_*$ of pointed compactly generated, weak Hausdorff spaces is topologically enriched and closed symmetric monoidal. Its tensor product is given by the smash product and internal hom is $\mathbf{Top}_*(-,-)$. Similarly, the homset $G\mathbf{Top}_*(X,Y)$ naturally inherits a topology as a subspace of $\mathbf{Top}_*(X,Y)$, which makes $G\mathbf{Top}_*$ a topologically enriched category. However, $G\mathbf{Top}_*(X,Y)$ is only equipped with a trivial G-action, so we do not expect the internal hom object, if it exists, to be $G\mathbf{Top}_*(-,-)$.

Instead, we consider the category $\mathbf{Top}_G^*$, with objects pointed G-spaces and morphisms continuous maps. The homset $\mathbf{Top}_G^*(X,Y)$ is naturally equipped with a G-action, given by

$$(g \cdot f)(x) := g \cdot f(g^{-1} \cdot x)$$

and the smash-hom adjunction holds

$$\mathbf{Top}_G^*(X \wedge Y, Z) \cong \mathbf{Top}_G^*(X, \mathbf{Top}_G^*(Y, Z))$$

Proposition 1.5. The categories $G\mathbf{Top}$ and $G\mathbf{Top}_*$ are closed symmetric monoidal, with internal hom object given by $\mathbf{Top}_G(-,-)$ and $\mathbf{Top}_G^*(-,-)$, respectively.

Proof. We will treat the pointed case. First, we may identify $G\mathbf{Top}_*(X,Y)$ with $\mathbf{Top}_G^*(X,Y)^G$. It follows that

$$\begin{aligned} G\mathbf{Top}_*(X \wedge Y, Z) &\cong \mathbf{Top}_G^*(X \wedge Y, Z)^G \\ &\cong \mathbf{Top}_G^*(X, \mathbf{Top}_G^*(Y, Z))^G \\ &\cong G\mathbf{Top}_*(X, \mathbf{Top}_G^*(Y, Z)) \end{aligned}$$

and we have our desired adjunction. $\square$

1.2 Homotopy Theory and Elmendorf's Theorem

Definition 1.6. Given G-maps $f, g\colon X \to Y$, a G-**homotopy** between f, g is a G-equivariant map

$$H\colon X \times I \to Y$$

where G acts trivially on I, such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$. A G-**homotopy equivalence** between G-spaces X, Y is a pair of G-maps $f\colon X \to Y$ and $g\colon Y \to X$ and G-homotopies $f \circ g \sim Id_Y$ and $g \circ f \sim Id_X$.

Definition 1.7. Let H be a subgroup of G. Define $\pi_n^H(X) := \pi_n(X^H)$. A map $f\colon X \to Y$ of G-spaces is a **G-weak equivalence** if for all subgroups $H \leq G$,

$$f_*\colon \pi_n^H(X) \to \pi_n^H(Y)$$

is an isomorphism.

There is a cofibrantly-generated model structure that we can put on **GTop**:

Theorem 1.8 (Bredon Model Structure)**.** There is a cofibrantly-generated model structure on **GTop**, given as follows:

1. A G-map $f\colon X \to Y$ is a fibration iff for all $H \leq G$, $f^H\colon X^H \to Y^H$ is a fibration.
2. A G-map $f\colon X \to Y$ is a weak equivalence iff for all $H \leq G$, $f^H\colon X^H \to Y^H$ is a weak equivalence. Equivalently, the weak equivalences are the G-weak equivalences.

The generating sets are given as follows:

$$I_G = \{G/H \times S^n \to G/H \times D^{n+1} : H \leq G, n \geq 0\}$$

$$J_G = \{G/H \times I^n \to G/H \times (I^n \times I) : H \leq G, n \geq 0\}$$

A reference of the theorem is Chapter 8 of [HHR21]. An immediate consequence of the model category structure is the equivariant Whitehead's Theorem:

Corollary 1.9. Let $f\colon X \to Y$ be a weak equivalence of cofibrant-fibrant objects in a model category. Then, f is a homotopy equivalence. In particular, every object in **GTop** is fibrant, and G-CW complexes are cofibrant.

From the model structure given in Theorem 1.8, we have a vague sense of the following "equivalence":

$$G-\text{homotopy type of } X \Leftrightarrow \{\text{ordinary homotopy type of } X^H\colon H \leq G\}$$

And Elmendorf's Theorem will make the equivalence precise. We start by introducing the orbit category:

Definition 1.10. The **orbit category** $\mathcal{O}_G$ is the full subcategory of **GTop** on the objects $\{G/H\colon H \leq G\}$.

The morphisms in $\mathcal{O}_G$ are described by the following proposition:

Corollary 1.11. We have

$$\mathrm{Map}^G(G/H, G/K) \cong (G/K)^H$$

As a consequence, every morphism of finite G-sets $G/H \to G/K$ factors as a composition of a conjugation map and a projection map:

$$G/H \xrightarrow{c_g} G/g^{-1}Hg \xrightarrow{\pi} G/K$$

Proof. For the first part, note that there exists a G-equivariant maps $\varphi\colon G/H \to G/K$, determined by $\varphi(H) = gK$ iff $g^{-1}Hg \subseteq K$ iff $h(gK) = gK$ for all $h \in H$. For the second part, the conjugation map c_g is given by $c_g(H) = g(g^{-1}Hg)$, and the projection map π is induced by inclusion. $\square$

The conjugation induces a Weyl group action on each object in the orbit category, and other morphisms are projection maps.

Example 1.12. The orbit category for the cyclic group C_p is given by

$$\begin{tikzcd} C_p/C_p \\ C_p/e \arrow[u, "\mathrm{proj}"'] \arrow[loop below, "C_p"] \end{tikzcd}$$

Example 1.13. The orbit category for the cyclic group C_4 is given by

$$\begin{tikzcd} C_4/C_4 \\ C_4/C_2 \arrow[u, "\mathrm{proj}"'] \arrow[loop left, "W(C_2)"] \\ C_4/e \arrow[u, "\mathrm{proj}"'] \arrow[loop right, "C_4"] \end{tikzcd}$$

where $W(C_2) := N(C_2)/C_2$ is the Weyl group of C_2.

Let $\mathrm{Fun}(\mathcal{O}_G^{op}, \mathbf{Top})$ be the functor category of contravariant functors from $\mathcal{O}_G$ to the category of topological spaces. We have the following fact about the model structures on functor categories:

Theorem 1.14 (Bredon Model Structure)**.** Let $\mathcal{D}$ be a model category and $\mathcal{C}$ be a cofibrantly generated model category. Then, $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ admits a model structure.

It is useful to know that the weak equivalences in $\mathrm{Fun}(\mathcal{O}_G^{op}, \mathbf{Top})$ are given pointwise: a natural transformation $\eta : \mathcal{F} \to \mathcal{G}$ is a weak equivalence iff $\eta_{G/H} : \mathcal{F}(G/H) \to \mathcal{G}(G/H)$ is a weak equivalence.

Definition 1.15. There is a functor $\psi\colon \mathbf{GTop} \to \mathrm{Fun}(\mathcal{O}_G^{op}, \mathbf{Top})$ given by

$$X \to (G/H \mapsto X^H)$$

It is easy to check the functoriality. Note that if we restrict ψ to the subcategory $\mathcal{O}_G$, the functor is just the Yoneda embedding

$$\mathrm{Hom}_{\mathcal{O}_G}(G/-, G/K) = \mathrm{Map}_G(G/-, G/K) \cong (G/K)^-$$

as in Proposition 1.11.

Proposition 1.16. There is a functor $\theta : \mathrm{Fun}(\mathcal{O}_G^{op}, \mathbf{Top}) \to \mathbf{GTop}$, which is left adjoint to the functor ψ in Definition 1.15.

The functor θ is given by $X \mapsto X(G/e)$, where $X(G/e)$ is equipped with the G-action induced by the map $G/e \xrightarrow{g\cdot} G/e$. It is easy to check that (θ, ψ) is an adjoint pair. In fact, more can be said:

Theorem 1.17. $\text{Fun}(\mathcal{O}_G^{op}, \mathbf{Top})$ and **GTop** have isomorphic homotopy categories.

The original proof due to Elmendorf constructs the equivalence explicitly using the Bar construction to obtain a homotopy inverse to the embedding ψ. The theorem can also be put into a more modern framework with the development of model categories (or underlying ∞-category).

Theorem 1.18. (θ, ψ) is an Quillen equivalence between model categories.

1.3 G-CW Complexes

Similar to how we build CW-complexes, we want to construct a G-space X from cells, but this time with the additional data of the G-action along with each cell. We want:

1. an underlying CW structure on which G acts cellularly;
2. the fixed points X^H is a subcomplex for all subgroups $H \leq G$.

Definition 1.19. A **G-CW complex** is the sequential colimit of spaces X_n, where X_{n+1} is a pushout:

$$\begin{tikzcd}
\coprod G/H \times S^n \arrow[r] \arrow[d] & X_n \arrow[d] \\
\coprod G/H \times D^{n+1} \arrow[r] & X_{n+1}
\end{tikzcd}$$

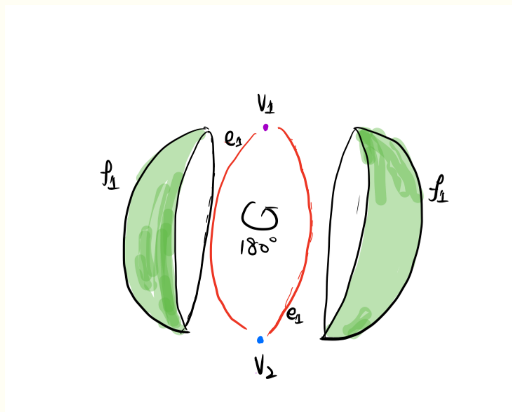
We will call $G/H \times D^n$ as an **G-cell** of dimension n. The pointed version replaces the disjoint union with wedge sums and adjoin an additional base point to the cosets, which we will write as G/H_+.

We see that G/H is the orbit of the ordinary cell D^n, and H is the isotropy group of D^n.

Remark 1.20. Note that the topological dimension of an n-cell in a G-CW complex might be greater than n if G is not finite. For example, a 0-cell $S^1/e \times *$ is one-dimensional.

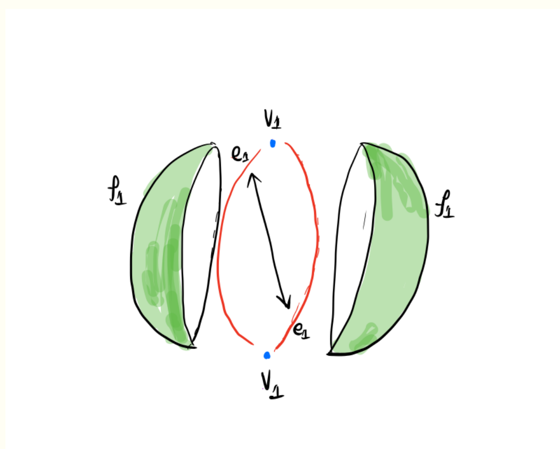
Example 1.21. Let $G = C_2$ act on S^2 by rotation by π along the Z-axis, given by the figure below. It has a G-CW structure given by the following cells:

- 2 zero-cells v_1, v_2 of the form $C_2/C_2 \times *$, which are the poles corresponding to the fixed points of the C_2 action;
- 1 one-cell e_1 of the form $C_2/e \times D^1$, which are the two great circles e_1 joining the poles;
- 1 two-cell f_1 of the form $C_2/e \times D^2$, which are the two hemispheres.



Example 1.22. Let $G = C_2$ act on S^2 by the antipodal map, given by the figure below. It has a G-CW structure given by the following cells:

- 1 zero-cells v_1 of the form $C_2/e \times *$, which are the poles corresponding to one orbit of the C_2 action;
- 1 one-cell e_1 of the form $C_2/e \times D^1$, which are the two great circles e_1 joining the poles;
- 1 two-cell f_1 of the form $C_2/e \times D^2$, which are the two hemispheres.



An important class of G-CW complexes consists of the representation spheres: Let V be a k -dimensional real representation G, which can also be interpreted as an $\mathbb{R}[G]$ -module. We may one-point compactify the underlying Euclidean space and get a k-dimensional sphere, which inherits a G-action from V. We denote the resulting G-space as S^V, which is called the **representation sphere** associated with V. Here are a few elementary facts about the representation spheres:

Proposition 1.23. For finite G representations V, W, the following hold:

1. $S^V \wedge S^W$ is equivariantly homeomorphic to $S^{V\oplus W}$.
2. S^V is equivariantly homeomorphic to $S(1\oplus V)$, the unit sphere is the representation $1 \oplus V$.

Proof. First, one notes that an equivariant bijection $f\colon X \to Y$ automatically has an equivariant inverse: we have

$$f(g \cdot f^{-1}(y)) = g \cdot f(f^{-1}y) = g \cdot y = f(f^{-1}(g \cdot y))$$

so by injectivity we get $g \cdot f^{-1}(y) = f^{-1}(g \cdot y)$. Therefore to show the two statements, it suffices to produce for each a homeomorphism that is also equivariant.

For 1, consider the composition

$$V \oplus W \xrightarrow{i} S^V \times S^W \xrightarrow{q} S^V \times S^W/(S^V \vee S^W)$$

The inclusion map i and the projection map q are open G-maps by construction. In particular, $q \circ i\colon V \oplus W \to S^V \wedge S^W$ is an open G-embedding that misses only the point at infinity. By universal property of one-point compactification, $q \circ i$ extends to a homeomorphism between $S^{V\oplus W}$ and $S^V \wedge S^W$. Since the point at infinity is a fixed point, the homeomorphism is also a G-map.

For 2, consider the stereographical projection

$$S(1 \oplus V) - \{(1, 0, 0, 0, ..)\} \to V$$

$$(x_1, ..., x_n) \mapsto \frac{1}{1 - x_1}(x_2, ..., x_n)$$

which is compatible with the G-action, and we know it is a homeomorphism. $\square$

Example 1.24. If $G = C_2$, then a finite dimensional G-representation is the direct sum of copies of the trivial representation, denoted by 1, and the sign representation, denoted by σ. For example, the representation 2σ is given coordinate-wise as $(x, y) \to (-x, -y)$, and S^σ is the G-space in Example 1.21.

For a finite group G, the **(real) regular representation** of G, denoted by ρ_G, is the $\mathbb{R}[G]$-module $\mathbb{R}[G]$ itself. In other words, it is the real vector space with a basis identified with elements of G, and G acts on the basis canonically on the left. The **reduced regular representation**, denoted by $\overline{\rho}_G$, is the vector subspace of ρ_G where the sum of coordinates is 0. By letting 1 denote the trivial subrepresentation generated by $(1, 1, ..., 1)$, we see that $\rho_G = \overline{\rho}_G \oplus 1$. An important fact is that the regular representation contains a copy of all irreducible representations.

Understanding the regular representation spheres is one of the important aspects of [HHR16]. The first step is to find a G-CW structure for $S^{\overline{\rho}_G}$.

Proposition 1.25. There is a G-CW structure on $S^{\overline{\rho}_G}$, with cells whose dimension range from 0 to $|G| - 1$. Similarly, a G-CW structure on S^{ρ_G} has cells whose dimension range from 0 to $|G|$.

Proof. By Proposition 1.23, $S^{\overline{\rho}_G}$ is equivariantly homeomorphic to the unit sphere in ρ_G, which is also homeomorphic to the boundary of the standard simplex in ρ_G. The ordinary cell decomposition of the standard simplex does not produce directly a G-CW structure: the top cell with vertices the elements of the group is neither fixed nor being permuted to another cell by the G-action. For example, in Figure 1, the ordinary 1-cell eg should break down to an orbit of two ordinary 1-cells.

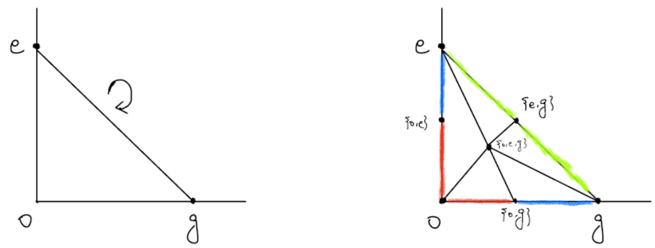


Figure 1: G-CW structure of $S^{\bar{\rho}_{C_2}}$

However, after barycentric subdividing the standard simplex, all the new cells are being permuted, which will give us a G-CW structure. For S^{ρ_G}, we know it is the ordinary suspension of $S^{\bar{\rho}_G}$, and every cell's dimension gets shifted up by one degree.

$\square$

1.4 Bredon Cohomology

The goal is to construct a cohomology theory satisfying the Eilenberg-Steenrod axioms in the equivariant setting. Let $G\mathbf{CW}$ be the category of G-CW complexes and equivariant maps. Then, a generalized cohomology theory on $G\mathbf{CW}$ is a sequence of contravariant functors

$$E^n : G\mathbf{CW} \to \mathbf{Ab}$$

satisfying the following:

1. If f, g are equivariantly homotopic, then $E^n(f) = E^n(g)$;
2. Given a G-CW pair $A \subseteq X$, there is the associated long exact sequence;
3. E^n satisfies excision.
4. E^n takes coproducts to products;

In addition, E^* is called **ordinary** if it satisfies the dimension axiom, i.e. $E^n(pt) = 0$ for $n \neq 0$. Our goal now is to construct such a theory in the equivariant setting, which is called Bredon cohomology. The original construction of this is due to G. Bredon in [Bre67].

For a non-equivariant generalized cohomology theory E^*, the Atiyah-Hirzebruch spectral sequence tells us that the coefficient groups $E^*(pt)$ basically determine the cohomology theory on CW complexes. Heuristically, the cohomology is determined by the building blocks, which are contractible open cells. However, in the equivariant setting, the building blocks are more complicated: they are orbits of ordinary cells of the form $G/H \times D^n$. Thus, it makes sense to understand $E^*(G/H)$ for all H, and they will be our new "coefficients." We are led to the following definition:

Definition 1.26. A **coefficient system** is a functor $\mathcal{F} : \mathcal{O}_G^{op} \to \mathbf{Ab}$.

Example 1.27. Fix an abelian group N. A **constant coefficient system** $\mathcal{O}_G^{op} \to \mathbf{Ab}$ is given by mapping all objects to N and all morphisms to the identity.

The first thing is to correctly construct the analog of cellular cochains. If we take X to be a point, then Proposition 1.2 reduces to the isomorphism $Y^H \cong \mathrm{Hom}_G(G/H, Y)$. In fact, this isomorphism promotes to an isomorphism of the topologically enriched homsets and is in fact a homeomorphism. In the meantime, a G-CW structure on Y naturally induces a CW structure on all fixed point spaces Y^H, for a G-CW cell breaks up into an orbit of ordinary CW cells in a way that respects glueing.

Example 1.28. Fix a G-space X, Proposition 1.2 gives us that the fixed point functor $G/H \mapsto X^H$ is a functor $\mathcal{O}_G^{op} \to \mathbf{Top}$. Then we have the composition

$$\mathcal{O}_G^{op} \xrightarrow{\text{fixed point}} \mathbf{Top} \xrightarrow{\text{cellular n-chain}} \mathbf{Ab}$$

$$G/H \mapsto X^H \to C_n^{\mathrm{CW}}(X^H)$$

which is a coefficient system.

Note that by the general theory of abelian categories, the functor category of coefficient systems $\mathcal{CS} := \mathrm{Fun}(\mathcal{O}_G^{op}, \mathbf{Ab})$ is abelian. It is now possible to define Bredon cohomology on a G-CW complex by explicitly defining the cochain complexes on cells, similar to CW cohomology and done in [Bre67]. However, we may package the cochains into the following form:

Definition 1.29. Fix an G-CW complex X. For each n, we may define a coefficient system $C_n(X)$ given by

$$G/H \mapsto H_n((X^H)_n, (X^H)_{n-1}; \mathbb{Z}) = C_n^{\mathrm{CW}}(X^H)$$

The differential of the CW chain complex induces a chain complex of coefficient systems $C.(X)$.

Definition 1.30. The **Bredon cohomology** of X with coefficient system M is defined by

$$H^n_{\mathrm{Bredon}}(X; M) := H^n(\mathrm{Hom}_{\mathcal{CS}}(C.(X), M))$$

Similarly, we may define Bredon homology with coefficients: for singular homology, we are tensoring the chain complex with some abelian group; for coefficient systems, we have the analog of "tensoring," which is a coend. We first recall the following construction

Remark 1.31. The tensor product of a right R-module A and a left R-module B is the coequalizer of the diagram

$$\alpha_A : \alpha_B : A \otimes R \otimes B \to A \otimes B$$

where the the tensor product is given in $\mathbf{Ab}$ and the two maps are given by the R action on A, B, respectively.

To distinguish a covariant coefficient system from a contravariant one, we give it another name

Definition 1.32. A covariant functor $N : \mathcal{O}_G \to \mathbf{Ab}$ is called an $\mathcal{O}_G$**-module**.

Definition 1.33. The **Bredon homology** with coefficients an $\mathcal{O}_G$-module N is defined to be the homology of the chain complex

$$C_*(X) \otimes_{\mathcal{O}_G} N$$

where each degree $C_n(X) \otimes_{\mathcal{O}_G} N$ is defined to be the coequalizer of the following

$$\bigoplus_{H,K \leq G} \bigoplus_{\mathrm{Hom}_{\mathcal{O}_G}(G/H, G/K)} C_n(X)(G/K) \otimes N(G/H) \rightrightarrows \bigoplus_{H \leq G} C_n(X)(G/H) \otimes N(G/H)$$

where the two arrows are given by the contravariant functoriality of C_n and covariant functoriality of N. The differentials are induced by the cellular differentials of $C_*(X)$.

Remark 1.34. The above construction is the **coend**

$$C_n(X) \otimes_{\mathcal{O}_G} N = \int^{G/H \in \mathcal{O}_G} C_n(X)(G/H) \otimes_{\mathcal{O}_G} N(G/H)$$

One may unravel the definition of coequalizer and note that

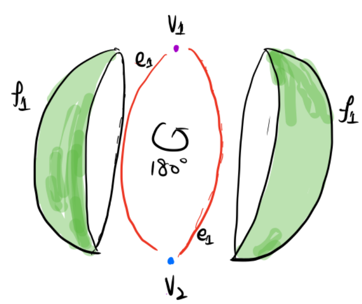
$$C_n(X) \otimes_{\mathcal{O}_G} N \cong \bigoplus_{H \leq G} C_n(X)(G/H) \otimes N(G/H)/\sim$$

where the equivalence relation is given by: for every $f \in \mathrm{Hom}_{\mathcal{O}_G}(G/H, G/K)$, we have $f^*x \otimes y \sim x \otimes f_*y$ for every $x \otimes y \in C_n(X)(G/K) \otimes N(G/H)$.

1.5 Computations

1.5.1 Constant Coefficients

The representation sphere $S^{2\sigma}$, which was introduced in Example 1.21, has the following G-CW decomposition



and we shall compute its Bredon cohomology with constant coefficient system $\underline{\mathbb{Z}}$, defined in Example 1.27.

The orbit category of C_2 is easy to describe:

$$\begin{tikzcd}
C_2/C_2 \\
C_2/e \arrow[u, "\mathrm{pr}"'] \arrow[loop below, "\mathrm{swap}"]
\end{tikzcd}$$

Then, the nth group in our chain complex, which is $\mathrm{Hom}_{\mathcal{CS}}(C_n(X), \underline{\mathbb{Z}})$, is computed as follows: the zero cells are simply the two poles, which are fixed under the G-action. Thus, for $n = 0$, we are looking for morphisms of the following diagram

$$\begin{tikzcd}
C_2/C_2 : & \mathbb{Z}v_1 \oplus \mathbb{Z}v_2 \arrow[r, dashed] \arrow[d, "Id"'] & \mathbb{Z} \arrow[d, "Id"'] \\
C_2/e : & \mathbb{Z}v_1 \oplus \mathbb{Z}v_2 \arrow[r, dashed] \arrow[loop below, "Id"] & \mathbb{Z} \arrow[loop below, "Id"]
\end{tikzcd}$$

It is clear that the morphism is determined by the what happens at the C_2/e spot, and we have total freedom to send v_1, v_2 to whichever element through some φ_1, φ_2. Thus, $\mathrm{Hom}_{\mathcal{CS}}(C_0(X), \underline{\mathbb{Z}}) \cong \mathbb{Z}\varphi_1 \oplus \mathbb{Z}\varphi_2$.

For $n = 1$, the story is a bit different: we have two 1-cells, and the G-action swaps these two. The morphisms we are looking for morphisms of the following diagram

$$\begin{tikzcd}
C_2/C_2 : & 0 \arrow[r, dashed] \arrow[d] & \mathbb{Z} \arrow[d, "Id"] \\
C_2/e : & \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \arrow[r, dashed] \arrow[loop below, "swap"] & \mathbb{Z} \arrow[loop below, "Id"]
\end{tikzcd}$$

To be compatible with the swap map, we see that e_1, e_2 must be mapped to the same element through some f, thus $\mathrm{Hom}_{\mathcal{CS}}(C_1(X), \underline{\mathbb{Z}}) \cong \mathbb{Z}f$. The story with $n = 2$ is the same as $n = 1$, and the Bredon cochain complex is then

$$0 \to \mathbb{Z}\varphi_1 \oplus \mathbb{Z}\varphi_2 \xrightarrow{d^0} \mathbb{Z}f \xrightarrow{d^1} \mathbb{Z}g \to 0$$

The cellular differentials are $d(e_1) = v_1 - v_2 = d(e_2)$ and $d(f_1) = e_1 - e_2 = -d(f_2)$. It then follows that the induced differentials $d_0((\varphi_1, 0)) = f = -d_0((0, \varphi_2))$, and d_1 is the 0-map. Thus, we have

$$H^n_{\mathrm{Bredon}}(S^{2\sigma}; \underline{\mathbb{Z}}) = \begin{cases} \mathbb{Z} & n = 0, 2 \\ 0 & \text{otherwise} \end{cases}$$

This procedure can be generalized: note that since we are taking the constant coefficient system, each $\mathrm{Hom}_{\mathcal{CS}}(C_n(X), \underline{\mathbb{Z}}) \cong \mathbb{Z}$ is determined by what happens at the G/e level; the cell complex at G/e level are the n-cells under the G-action, which corresponds to morphisms

$$\begin{tikzcd}
\bigoplus_{\text{n-cells}} \mathbb{Z} \arrow[r] \arrow[loop above, "G"] & \mathbb{Z}
\end{tikzcd}$$

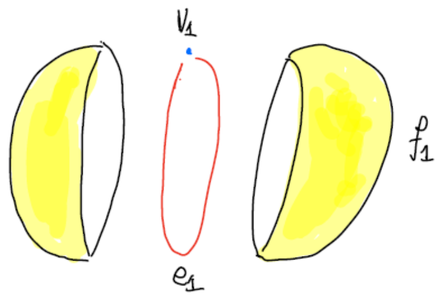


Figure 2: A G-CW structure for $S^{\rho_{C_2}}$

which is determined by the values on each G-orbit of n-cells. We then have the canonical identification

Theorem 1.35. We have the canonical isomorphism

$$\mathrm{Hom}_{\mathcal{CS}}(C_n(X), \underline{\mathbb{Z}}) \cong \mathrm{Hom}_{\mathbf{Ab}}(C_n(X)(G/e)/G, \mathbb{Z})$$

As a corollary, we also have the isomorphism:

$$H^*_{\mathrm{Bredon}}(X; \underline{\mathbb{Z}}) \cong H^*_{\mathrm{CW}}(X/G; \mathbb{Z})$$

For the second example, consider the representation $1 \oplus \sigma$ on $\mathbb{R}^2$, which acts on coordinates by $(x, y) \mapsto (x, -y)$. Taking the one-point compactification gives us the representation sphere $S^{1+\sigma}$ with the C_2-action. It has a G-CW decomposition of one fixed 0-cell, one fixed 1-cell, and one orbit of 2-cell, as depicted in Figure 2.

For $n = 0$, we look for

$$\begin{tikzcd}
C_2/C_2 : & \mathbb{Z}v_1 \arrow[r, dashed] \arrow[d, "Id"] & \mathbb{Z} \arrow[d, "Id"] \\
C_2/e : & \mathbb{Z}v_1 \arrow[r, dashed] \arrow[loop below, "Id"] & \mathbb{Z} \arrow[loop below, "Id"]
\end{tikzcd}$$

Thus, $\mathrm{Hom}_{\mathcal{CS}}(C_0(X), \underline{\mathbb{Z}}) \cong \mathbb{Z}\varphi$.

For $n = 1$, the diagram is the same as above and we have $\mathrm{Hom}_{\mathcal{CS}}(C_1(X), \underline{\mathbb{Z}}) \cong \mathbb{Z}f$. For $n = 2$, we are looking for morphisms of the following diagram

$$\begin{tikzcd}
C_2/C_2 : & 0 \arrow[r, dashed] \arrow[d] & \mathbb{Z} \arrow[d, "Id"] \\
C_2/e : & \mathbb{Z}g_1 \oplus \mathbb{Z}g_2 \arrow[r, dashed] \arrow[loop below, "Swap"] & \mathbb{Z} \arrow[loop below, "Id"]
\end{tikzcd}$$

so $\mathrm{Hom}_{\mathcal{CS}}(C_2(X), \underline{\mathbb{Z}}) \cong \mathbb{Z}g$ and the Bredon cochain complex is

$$0 \to \mathbb{Z}\varphi \xrightarrow{d^0} \mathbb{Z}f \xrightarrow{d^1} \mathbb{Z}g \to 0$$

The cellular differential implies d_1 is an isomorphism, and d_0 is the zero map. Thus, we computed that

$$H^n_{\text{Bredon}}(S^{1+\sigma};\underline{\mathbb{Z}}) = \begin{cases} \mathbb{Z} & n=0 \\ 0 & \text{otherwise} \end{cases}$$

In particular, this example implies that $H^1_{\text{Bredon}}(S^\sigma;\underline{\mathbb{Z}}) \cong H^2_{\text{Bredon}}(S^{1+\sigma};\underline{\mathbb{Z}}) \cong 0$, so the cohomology is really different from that of the underlying sphere. This matches Theorem 1.35, since S^σ/C_2 is contractible.

There is a similar statement for homology:

Theorem 1.36. We have the canonical isomorphism

$$C_n(X) \otimes_{\mathcal{O}_G} \underline{\mathbb{Z}} \cong C_n(X)(G/e)/G \otimes \mathbb{Z}$$

As a corollary, we also have the isomorphism:

$$H_*^{\text{Bredon}}(X;\underline{\mathbb{Z}}) \cong H_*^{CW}(X/G;\mathbb{Z})$$

Example 1.37. Consider the Bredon homology of $S^{2\sigma}$ with coefficients $\underline{\mathbb{Z}}$. By Theorem 1.36, this reduces to the computation of the cellular homology of $S^{2\sigma}/C_2$. Note that in the representation sphere $S^{2\sigma}$, the C_2-action on every latitude circle S^1 is the antipodal action, given by $(x,y) \mapsto (-x,-y)$ on the plane. In other words, $S^{2\sigma}/C_2$ is the suspension of the orbit space of the unit circle $S(2\sigma)$. This gives us the homotopy equivalence

$$S^{2\sigma}/C_2 \cong \Sigma\mathbb{RP}^1$$

Similarly, we have $S^{n\sigma}/C_2 \cong \Sigma\mathbb{RP}^{n-1}$, and

$$\tilde{H}_k^{\text{Bredon}}(S^{n\sigma}) \cong \tilde{H}_{k-1}(\mathbb{RP}^{n-1};\mathbb{Z})$$

by suspension isomorphism.

With Proposition 1.35, we can show that there is a gap in Bredon cohomology of regular representation spheres. This leads to the Gap theorem of [HHR16].

Corollary 1.38 (The Cell Lemma)**.** Let G be any finite group other than C_3, and $n \geq 0$. The groups

$$\tilde{H}^i_{\text{Bredon}}(S^{n\rho_G};\underline{\mathbb{Z}}) = 0$$

for $0 < i < 4$.

Proof. By Proposition 1.35, we have

$$\tilde{H}^1_{\text{Bredon}}(S^{n(\rho_G-1)};\underline{\mathbb{Z}}) \cong \tilde{H}^1(S^{n(\rho_G-1)}/G;\mathbb{Z})$$

Note that $S^{n(\rho_G-1)}$ is the suspension of the unit sphere $S(n(\rho_G-1))$, which is connected for $n>0$ and $G \neq C_2$. Thus, the orbit $S(n(\rho_G-1))/G$ is connected. In the exceptional case $n=1$ and $G=C_2$, $S(\rho_{C_2}-1) \cong C_2$ and the orbit space is still connected. Thus, its suspension $S^{n(\rho_G-1)}/G$ is simply connected, and $\tilde{H}^1(S^{n(\rho_G-1)}/G;\mathbb{Z}) \cong 0$. In particular, this implies

$$\tilde{H}^i_{\text{Bredon}}(S^{n\rho_G};\underline{\mathbb{Z}}) \cong \tilde{H}^{i-n}_{\text{Bredon}}(S^{n(\rho_G-1)};\underline{\mathbb{Z}}) \cong 0$$

for $i \leq n+1$. Thus to prove the corollary, the last remaining case is $n = 1$ and $i = 3$. Since the groups are finitely generated, and $S^{\rho_G - 1}/G$ is simply connected, the universal coefficient theorem says

$$\tilde{H}^3_{\text{Bredon}}(S^{\rho_G}; \underline{\mathbb{Z}}) \cong H^2(S^{\rho_G - 1}/G; \mathbb{Z})$$

is torsion free, so it suffices to show $H^3(S^{\rho_G}/G; \mathbb{Q}) = 0$. A classic result in the theory of transfer homomorphisms states that if a finite group G acts simplicially on a simplicial complex X, then $H^n(X/G; \mathbb{Q})$ is isomorphic to the G-invariant part of $H^n(X; \mathbb{Q})$. In our case, $H^3(S^{\rho_G}/G; \mathbb{Q}) = 0$ is the G-invariant part of $H^3(S^{|G|}; \mathbb{Q})$, which is 0 when $G \neq C_3$

□

1.5.2 A general example

We will also consider an example with non-constant coefficients. Let M denote the C_2 coefficient system [1] depicted by

$$\begin{tikzcd}
C_2/C_2 : & \mathbb{Z}x \oplus \mathbb{Z}y \arrow[d, "x+2y"] \\
C_2/e : & \mathbb{Z} \arrow[loop below, "Id"]
\end{tikzcd}$$

and we will compute $H^*_{\text{Bredon}}(S^{2\sigma}; M)$. Recall the cell structure of $S^{2\sigma}$ given in the first example. For $n = 0$, we look for

$$\begin{tikzcd}
C_2/C_2 : & \mathbb{Z} \oplus \mathbb{Z} \arrow[r, dashed] \arrow[d, "Id"'] & \mathbb{Z}x \oplus \mathbb{Z}y \arrow[d, "x+2y"] \\
C_2/e : & \mathbb{Z} \oplus \mathbb{Z} \arrow[r, dashed] & \mathbb{Z}
\end{tikzcd}$$

which is completely determined by what happens at the C_2/C_2 level, so $\text{Hom}_{\mathcal{CS}}(C_0(X), M) \cong \mathbb{Z}^4$. For $n = 1$, we look for

$$\begin{tikzcd}
C_2/C_2 : & 0 \arrow[r, dashed] \arrow[d] & \mathbb{Z}x \oplus \mathbb{Z}y \arrow[d, "x+2y"] \\
C_2/e : & \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \arrow[r, dashed] \arrow[loop below, "Swap"] & \mathbb{Z} \arrow[loop below, "Id"]
\end{tikzcd}$$

which is determined by the C_2/e level and $\text{Hom}_{\mathcal{CS}}(C_1(X), M) \cong \mathbb{Z}$. The case where $n = 2$ is analogous to $n = 1$, and our Bredon cochain complex is

$$0 \longrightarrow \mathbb{Z}^4 \xrightarrow{d^0} \mathbb{Z}f \xrightarrow{d^1} \mathbb{Z}g \longrightarrow 0$$

Note d^1 is still the zero map as in the case of the constant coefficient. Let $(a, b, c, d) \in \mathbb{Z}^4$ denote the morphism that sends $(1, 0)$ to (a, b) and $(0, 1)$ to (c, d) on C_2/C_2. This corresponds to the morphism that sends $(1, 0)$ to $a + 2b$ and $(0, 1)$ to $c + 2d$ on C_2/e. Then, $d^0(a, b, c, d) = (a + c + 2b + 2d)f$, which is clearly surjective. Thus, we may conclude

$$H^n_{\text{Bredon}}(S^{2\sigma}; M) = \begin{cases} \mathbb{Z}^3 & n = 0 \\ \mathbb{Z} & n = 2 \\ 0 & \text{otherwise} \end{cases}$$

[1] This is in fact the contravariant part of the C_2 Burnside Mackey functor, which is introduced in Section 2.3.

1.5.3 Free Action

Now let fix some arbitrary coefficients system ($\mathcal{O}_G$-module) $\underline{M}$. It turns out that if the G-action on a G-CW complex is free, then the Bredon (co)homology also reduces to cellular homology. We outline the proof for cohomology: suppose the G-action on X is free, then the group $C_n(X)(G/H)$ is trivial for all subgroups H except for $H = e$. Again, we reduce computing $\mathrm{Hom}_{\mathcal{CS}}(C_n(X), \underline{M})$ reduces to morphisms at G/e level.

$$\begin{tikzcd} C_n(X)(G/e) \arrow[loop above, "G"] \arrow[r, dashed] & \underline{M}(G/e) \arrow[loop above, "G"] \end{tikzcd}$$

Note that each $C_n(X)(G/e)$ is freely generated by the n-cells of X, and it has the free G-action. Moreover, a morphism of the diagram is equiavalent to specifying a morphism $\mathbb{Z} \to M(G/e)$ for each orbit of the G-action, and we have

$$\mathrm{Hom}_{CS}(C_n(X), \underline{M}) \cong \mathrm{Hom}(C_n(X)/G, M(G/e))$$

It follows that

Theorem 1.39. If G acts freely on X, we have

$$H^*_{\text{Bredon}}(X; \underline{M}) \cong H^*_{CW}(X/G; M(G/e))$$

and the similar statement for homology holds as well.

Example 1.40. Consider the unit sphere $S(n\sigma)$ in the n-dimensional C_2 representation given by the n-times direct sum of the sign representation. The induced C_2-action on $S(n\sigma)$ is the antipodal action. Thus,

$$H^*_{\text{Bredon}}(S(n\sigma); \underline{M}) \cong H^*(S(n\sigma)/C_2; M(C_2/e)) \cong H^*(\mathbb{RP}^{n-1}; M(C_2/e))$$

and we know how to compute the cohomology of real projective space.

2 The Equivariant Stable Homotopy Category

2.1 Orthogonal G-spectra

One way to motivate the correct definition of equivariant stable homotopy category is to represent the equivariant cohomology theories. In particular, we want a form of the suspension isomorphism

$$H^*(X) \cong H^{*+V}(\Sigma^V X)$$

In the non-equivariant case, we only take V to be trivial representations, which means we are smashing with a sphere with trivial G-action. However, it turns out that the cohomology would carry more useful data if we want to keep track of all possible representations V. This means that once we define the homotopy category of equivariant spectra, we then have to invert functors Σ^V for all G representation V to get the correct homotopy category.

We eventually want to take colimits with respect to G representations. If we only consider isomorphism classes of representations, we run into sign issues. Thus, we need a device to keep track of actual representations and control the size.

Definition 2.1. A **complete G-universe** U is an infinite dimensional real inner product space with a G-action, such that it is a direct sum of countably many copies of each irreducible representations of G.

Definition 2.2. A **G-prespectrum** X is a collection of pointed G-spaces $X(V)$, one for each finite dimensional subspace $V \subset U$ of a given G universe. The collection is equipped with structure maps

$$\Sigma^W X(V) \to X(V \oplus W)$$

The structure maps must be associative. A G-prespectrum is a **G-spectrum** (or Ω-G-spectrum) if the adjoint structure maps

$$X(V) \to \Omega^W X(V \oplus W)$$

are homeomorphisms [1].

[MM02] define a (non-equivariant) morphism $X \to Y$ between G-spectra to be a collection of continuous maps between G-spaces $X(V) \to Y(V)$ compatible with structure maps. A **G-map** $X \to Y$ is a morphism where the maps $X(V) \to Y(V)$ are also equivariant. Sometimes in literature this representation graded spectrum is referred to as the **genuine G-spectrum**. The category of genuine G-prespectra with G-maps will be denoted $\mathbf{Sp}^G$, and the category of genuine G-prespectra with non-equivariant morphisms will be denoted $\mathbf{Sp}_G$

Definition 2.3. The **homotopy groups** of a G-prespectrum X are

$$\pi_q^H(X) := \mathrm{colim}_{V \subset U} \pi_q^H \Omega^V X(V)$$

$$\pi_{-q}^H(X) := \mathrm{colim}_{\mathbb{R}^m \subset V} \pi_q^H \Omega^{V-\mathbb{R}^m} X(V)$$

where $V - \mathbb{R}^m$ denotes the orthogonal complement. A **weak-equivalence** between G-spectra is a π_*^H equivalence for all $H \leq G$.

[1] One can replace homeomorphism with weak equivalence, and [LMM86] has a procedure to convert a spectrum whose structure maps are weak equivalences to an equivalent one whose structure maps are homeomorphisms.

Note that since each representation V is contained in $n\rho_G$ for large enough n, we can take a cofinal colimit

$$\pi_q^H(X) \cong \mathrm{colim}_n \pi_q^H(\Omega^{n\rho_G} X(n\rho_G))$$

As hinted before, the correct homotopy category of genuine G-spectra should be obtained by inverting Σ^V for all finite representation V, which amounts to inverting the weak equivalences in $\mathbf{Sp}^G$. We introduce two possible ways to define the category of orthogonal G-spectra, which will serve as a point-set model for $\mathbf{Sp}^G$.

Definition 2.4 (Schwede)**.** An **orthogonal G-spectrum** is an orthogonal spectrum equipped with a G-action through automorphisms of orthogonal spectra. Explicitly, this amounts to the data of

1. A sequence of $O(n) \times G$ spaces X_n
2. structure maps $\sigma_n : X_n \wedge S^1 \to X_{n+1}$ that is G-equivariant with respect to the trivial G-action on S^1, and $O(n) \times O(m)$ equivariant

A morphism of orthogonal G-spectra is a morphism of orthogonal spectra that commutes with the G-action.

It is not immediate that this is a genuine G-prespectrum as in Definition 2.2. It turns out that the action of the orthogonal groups encode enough information so that we can evaluate an orthogonal G-spectrum on a G-representation. To see this, let V be a n-dimensional inner product space, and let $L(\mathbb{R}^n, V)$ be the space of linear isometries from $\mathbb{R}^n$ to V. We have a canonical $O(n)$-action on $L(\mathbb{R}^n, V)$ by precomposition, so we may define

$$X(V) := L(\mathbb{R}^n, V)_+ \wedge_{O(n)} X_n$$

Suppose V is a G-representation, then $X(V)$ is equipped with the G-action defined by $g \cdot [\varphi, x] = [g\varphi, gx]$. To define the structure maps

$$\sigma_{V,W} : X(V) \wedge S^W \to X(V \oplus W)$$

we set $m = \dim(W)$, and choose an isometry $\gamma \colon \mathbb{R}^m \to W$. Then, define

$$\sigma_{V,W}([\varphi, x] \wedge w) := [\varphi \oplus \gamma, \sigma^m(x \wedge \gamma^{-1}(w))] \in X(V \oplus W)$$

where $\sigma^m \colon X_n \wedge S^m \to X_{n+m}$ is the structure map built in the orthogonal spectra. It is straightforward to verify that the definition does not depend on the choice of γ and has the correct equivariance. The procedure alludes to the [MM02] definition of orthogonal G-spectra using diagram spectra, which makes the closed symmetric monoidal structure more apparent.

Definition 2.5. Fix a G-universe U, and let V, V' be finite dimension subspaces of U. Further, let $I_G(V, V')$ [2]be the space of linear G-equivariant isometries from V to V'. The **complement bundle** is defined to be the subbundle of the product bundle $I_g(V, V') \times V'$ consisting of pairs (f, x) such that $x \in V' \setminus f(V)$.

We now define the diagram for orthogonal G-spectra.

Theorem 2.6. Let $\tilde{I}_G$ be the diagram (associated to a fixed G-universe U) where the objects are finite dimensional subspaces of U. The enriched morphisms $\tilde{I}_G(V, V')$ is defined to be the Thom space of the complement bundle $E(V, V')$, where composition

$$\tilde{I}_G(V', V'') \wedge \tilde{I}_G(V, V') \to \tilde{I}_G(V, V'')$$

is given by $(g, y) \circ (f, x) = (g \circ f, g(x) + y)$

The symmetric monoidal structure is given by the operation $\otimes$ being the direct sum of representations on objects and $(f, x) \otimes (f', x') = (f \oplus f', x + x')$ on enriched morphisms.

Let X be an $\tilde{I}_G$-space given by the functor $F : \tilde{I}_G \to G\mathbf{Top}_*$, where $X(V) := F(V)$. When $dim(V) = dim(V')$, the data of $\tilde{I}_G(V, V')$ corresponds to the action $L(V, V')_+ \wedge X(V) \to X(V')$; when $V \subset V'$, the data of $\tilde{I}_G(V, V') \cong O(V') \wedge_{O(V' \ominus V)} S^{V' \ominus V}$ gives us the structure morphisms

$$X(V) \wedge S^{V' \ominus V} \to X(V')$$

and the compatibility conditions are encoded in the axioms of an enriched functor.

Definition 2.7 (Mandell-May)**.** The category of orthogonal G-spectra is the functor category

$$\mathrm{Fun}(\tilde{I}_G, G\mathbf{Top}_*)$$

of $\mathbf{Top}_*$-enriched functors.

Theorem 2.8. The category of orthogonal G-spectra in Defintition 2.4 is equivalent to the category of orthogonal G-spectra in Definition 2.7.

The proof of the theorem can be found in Chapter 5 of [MM02]. Using the theorem, we can use the two point-set models interchangably. [HHR16] used the equivalence notably in their construction of the norm and symmetric powers.

Schwede's definition helps us write down a few examples of orthogonal G-spectra with explicit formulas:

Example 2.9 (Sphere Spectrum)**.** The equivaraint sphere spectrum $\mathbb{S}$ is given by

$$\mathbb{S}_n := S^n$$

and the $O(n)$-action is the natural action induced from $\mathbb{R}^n$, and a group G-acts trivially on $\mathbb{S}_n$ for all n. Note that we have a G-equivariant homeomorphism

$$\mathbb{S}(V) := L(\mathbb{R}^n, V)_+ \wedge S^n \to S^V$$

defined by $[\varphi, x] \mapsto \varphi(x)$.

Example 2.10 (Suspension Spectra)**.** Given a pointed G-space X, we may define a suspension spectrum $\Sigma^\infty_G X$ by

$$(\Sigma^\infty_G X)_n := X \wedge S^n$$

The $O(n)$-action is on the S^n-coordinate, and the G-action is on the X-coordinate. Note that we have

$$\Sigma^\infty_G X(V) = X \wedge S^n \wedge L(\mathbb{R}^n, V) \cong X \wedge S^V$$

[2]We can drop the G-equivariance assumption on $I(V, V')$, and the resulting category of orthogonal G-spectra is $\mathbf{Sp}_G$.

Example 2.11 (Loop Spectrum)**.** Let V be a G-representation and X a G-spectrum. The **loop spectrum** $\Omega^V X$ is defined by

$$(\Omega^V X)_n := \Omega^V(X_n) = \mathrm{Maps}(S^V, X_n)$$

The $O(n)$-action is given by the induced action from X_n, and G-acts by conjugation: for $\varphi \in \mathrm{Maps}(S^V, X_n)$,

$$(g \cdot \varphi)(v) = g \cdot \varphi(g^{-1} \cdot v)$$

The structure maps are given by the composition

$$\mathrm{Maps}(S^V, X_n) \wedge S^1 \xrightarrow{f} \mathrm{Maps}(S^V, X_n \wedge S^1) \xrightarrow{\sigma_n \circ -} \mathrm{Maps}(S^V, X_{n+1})$$

where $f(\varphi \wedge s)(v) := \varphi(v) \wedge s$.

The value of a the loop spectrum at a representation is checked to be $\Omega^V X(W) = \mathrm{Maps}(S^V, X(W))$.

Example 2.12 (Suspension by a representation)**.** Let V be a G-representation and X a G-spectrum. The **suspension** $\Sigma^V X$ is defined by

$$(\Sigma^V X)_n := S^V \wedge X_n$$

We have $O(n)$-acts on X_n, and G-acts diagonally, and the structure maps are given by the obvious composite. The value of $\Sigma^V X$ at an inner product space W is checked to be

$$\Sigma^V X(W) = S^V \wedge X(W)$$

Example 2.13 (Free G-spectrum)**.** This will give the equivariant analog of the shifted suspension spectrum. For a G-representation V, the **free G-spectrum** F_V is defined levelwise by

$$(F_V)_n := \tilde{I}_G(V, \mathbb{R}^n)$$

The structure maps are given by the composite

$$\tilde{I}_G(V, \mathbb{R}^n) \wedge S^1 \to \tilde{I}_G(V, \mathbb{R}^n) \wedge \tilde{I}_G(\mathbb{R}^n, \mathbb{R}^{n+1}) \to \tilde{I}_G(V, \mathbb{R}^{n+1})$$

where the first map is induced by the inclusion of a fiber $S^1 \to \tilde{I}_G(\mathbb{R}^n, \mathbb{R}^{n+1})$.

[HHR16] uses the notation S^{-V} and the name Yoneda spectrum for F_V. We will keep with their notation in Chapter 5 when discussing the slice spectral sequence.

Given a finite group G and a G-set J, let $B_J G$ denote the category with objects elements in J and morphisms $j \to j'$ corresponding to an element $g \in G$ such that $g \cdot j = j'$. This is the **translation category** of J. In particular, if J is a singleton, then $B_J G$ is the usual groupoid BG.

Definition 2.14. Given a functor $X : B_J G \to \mathbf{Sp}$, the indexed wedge, product, smash product are defined to be

$$\bigvee_{i\in J} X_i \qquad \prod_{i\in J} X_i \qquad \bigwedge_{i\in J} X_i$$

The data of the functor X determines a G-action on the three constructions, therefore extending the functor to $\mathbf{Sp}^G$.

These constructions depend on the fact that orthogonal spectra with G action can be prolonged to genuine G-spectra, and their homotopical properties are discussed in the next section.

Definition 2.15. The **norm** functor $N_H^G : \mathbf{Sp}^H \to \mathbf{Sp}^G$ is defined by sending an H-spectrum X to the indexed smash product

$$\bigwedge_{i\in G/H} X_i$$

There are many model structures one can put on $\mathbf{Sp}^G$, with weak equivalences being the equivariant π_*-equivalences, as in Definition 2.3. [MM02] introduced a positive stable model structure on orthogonal G-spectra, where the generating cofibrations are

$$\{F_V S^n \to F_V D^{n+1} : V^G \neq 0, n \geq 0\}$$

The requirement that V contains a nonzero fixed vector is the "positivity," and having this helps defining a model structure on commutative ring spectra. However, [HHR16] needed a model structure with more generating cofibrations such that the induction and restriction will form a Quillen adjunction, and indexed smash products will be homotopical on cofibrant objects.

Theorem 2.16 (Positive Complete Model Structure)**.** There is a model structure on $\mathbf{Sp}^G$, with generating cofibrations the set

$$\{G_+ \wedge_H F_V S^n_+ \to G_+ \wedge_H F_V D^{n+1}_+ : H \leq G, V^H \neq 0\}$$

and weak equivalences defined in 2.3.

2.2 Homotopical Properties of $\mathbf{Sp}^G$

In this section, we list some expected properties of $\mathrm{Ho}(\mathbf{Sp}^G)$. The best reference for these is the appendices of [HHR16].

The category $\mathrm{Ho}(\mathbf{Sp}^G)$ is bicomplete, and the (co)limits are defined levelwise. The equivariant homotopy groups takes fiber sequences to long exact seqeces. In particular, we can use this to show that the homotopy groups take wedges to direct sums and finite products to products. This implies

Theorem 2.17. The catgeory $\mathrm{Ho}(\mathbf{Sp}^G)$ is additive, with coproducts given by wedge product of spectra. The hom groups will be denoted by

$$[-,-]_G$$

The diagram spectra perspective gives us:

Theorem 2.18. The category of orthogonal G-spectra is closed symmetric monoidal, with smash product as the tensor product and the sphere spectrum as the tensor unit. Moreover, the positive complete model structure satisfies the pushout product axiom, so $\mathrm{Ho}(\mathbf{Sp}^G)$ inherits the closed symmetric monoidal structure.

Theorem 2.19. For every representation V, there is a π_*-isomorphism

$$F_V \wedge X \to \Omega^V X$$

A corollary is that

$$S^{-V} := F_V \cong \Omega^V \mathbb{S} \cong \mathrm{Maps}(S^V, \mathbb{S})$$

which implies S^{-V} is the Spanier-Whitehead dual of S^V.

Theorem 2.20. Smashing with S^V and taking Ω^V are homotopical functors. Their derived functors Σ^V and Ω^V are inverse equivalences on $\mathrm{Ho}(\mathbf{Sp}^G)$.

$$\begin{tikzcd} \mathrm{Ho}(\mathbf{Sp}^G) \arrow[r, bend right, "\Omega^V"'] & \mathrm{Ho}(\mathbf{Sp}^G) \arrow[l, bend right, "\Sigma^V"'] \end{tikzcd}$$

Using the positive complete model structure, the suspension and evaluation functor[3] are not a pair of Quillen adjunction between $\mathbf{Sp}^G$ and $G\mathbf{Top}_*$, since $\Sigma^\infty_G S^0$ is not cofibrant by positivity requirement. However, the two functors can still be derived without using the model structure and descend to an adjunction on the homotopy categories.

Theorem 2.21. The suspension functor Σ^∞ can be left derived, and the evaluation functor $\mathbf{Ev}_0$ can be right derived. The respective derived functors form an adjunction

$$\begin{tikzcd} \mathrm{Ho}(\mathbf{Sp}^G) \arrow[r, bend right, "\Omega^\infty_G"'] \arrow[r, phantom, "\perp"] & \mathrm{Ho}(G\mathbf{Top}_*) \arrow[l, bend right, "\Sigma^\infty_G"'] \end{tikzcd}$$

The derived functor Ω^∞ is still computed as

$$\Omega^\infty X \cong \mathrm{Hocolim}_{V \subset U} \Omega^V X(V)$$

A corollary of this is the formula

$$\pi_k^H X \cong [\Sigma^\infty_G G/H_+ \wedge S^k, X]_G$$

Given a subgroup $H \leq G$, there is a homotopical functor $\mathrm{Res}^G_H \colon \mathbf{Sp}^G \to \mathbf{Sp}^H$ by forgetting the G-action to the H-action. As on the space level, there is a left adjoint

Definition 2.22. Given an orthogonal H-spectrum X, the **induced G-spectrum**, denoted by $G \wedge_H X$, is given level-wise by

$$(G \wedge_H X)_n := G \wedge_H X_n$$

[3] named "zero-space functor" by [HHR16]

The coinduced G-spectrum can be identified as the indexed wedge

$$G \wedge_H X \cong \bigvee_{i \in G/H} iH \wedge_H X \cong \bigvee_{G/H} X$$

The restriction also has a right adjoint

Definition 2.23. Given an orthogonal H-spectrum X, the **coinduced G-spectrum**, denoted by $\mathrm{Maps}_H(G, X)$, is given level-wise by

$$(\mathrm{Maps}_H(G, X))_n := \mathrm{Maps}_H(G, X_n)$$

The induced G-spectrum can be identified as the indexed product

$$\mathrm{Maps}_H(G, X) \cong \prod_{i \in H \backslash G} \mathrm{Maps}_H(Hi, X) \cong \prod_{G/H} X$$

From the theory of finite group representation, we know that induction are coinduction are isomorphic. The analog is also true for orthogonal G-spectra

Theorem 2.24. The formation of arbitrary indexed wedges and finite indexed products are homotopical. Moreover, suppose J is a finite G-set, then the canonical map

$$\bigvee_{i \in J} X_i \to \prod_{i \in J} X_i$$

is a weak equivalence.

By taking $J = G/H$, we obtain the Wirthmuller isomorphism

Theorem 2.25 (Wirthmuller Isomorphism)**.** Let H be a subgroup of a finite group G, and X an orthogonal H-spectrum. Then, there is a morphism of orthogonal spectra

$$\Psi_X : G_+ \wedge_H X \to \mathrm{Maps}_H(G, X)$$

which is an π_*-isomorphism.

Corollary 2.26. Orbit are self-dual in $\mathbf{Sp}^G$.

Proof. Take $X = \mathbb{S}$. Then, Theorem 2.25 gives us the isomorphisms

$$\Sigma^\infty_G G/H_+ \cong G_+ \wedge_H \mathbb{S} \cong \mathrm{Maps}_H(G, \mathbb{S}) \cong \mathrm{Maps}(\Sigma^\infty_G G/H_+, \mathbb{S}) \cong D\Sigma^\infty_G G/H_+$$

. where the first and third isomorphism follows from the fact that the sphere spectrum has trivial G-action. $\square$

2.3 Transfer and Mackey functor

Unstably, there is a morphism between G sets $G/H \to G/K$ iff H is subconjugate to K, by Proposition 1.11. But passing to the stable category, we also have transfer maps

$$\Sigma^\infty_G G/K_+ \to \Sigma^\infty_G G/H_+$$

which goes the "wrong way." The geometric construction goes as follows: for every subgroup $H \le G$, we may choose a finite G representation V such that it admits a G-equivariant embedding

$$i\colon G/H_+ \to V$$

One could do this by choosing a basis of V which correspond to elements of G/H_+, and let G act on the basis. Such an embedding induces a map

$$G/H_+ \wedge D(V) \to V$$

which is the disk bundle of the embedding i. By taking one-point compactification, the Pontryagin-Thom collapse map

$$S^V \to \frac{G/H_+ \wedge D(V)}{G/H_+ \wedge S(V)} \cong G/H_+ \wedge S^V$$

is the desired transfer map. For subgroups $H \le K \le G$, we may take the transfer map

$$S^V \to K/H_+ \wedge S^V$$

where V is WLOG a G representation that restricts to K. Inducing up to G gives us

$$G/K_+ \wedge S^V \to G/H_+ \wedge S^V$$

This provides us the stable transfer map. The existence of transfers supply internal structure to the collection of homotopy groups $\{\pi_*^H(X) : H \le G\}$ of a G-spectrum X:

1. There is a **restriction map**

$$\mathrm{Res}^G_H\colon \pi_*^G(X) \to \pi_*^H(i^*X) \cong \pi_*^H(X)$$

   induced by the inclusion $i\colon H \to G$. It is clear that $\mathrm{Res}^K_K = Id$ for any subgroup $K \le G$, and map is transitive in the sense that

$$\mathrm{Res}^H_K \circ \mathrm{Res}^G_H = \mathrm{Res}^G_K$$

   for subgroup $K \le H \le G$.

2. for $g \in G$ and subgroup $H \le G$, there is the conjugation map

$$c_g\colon H \to H^g \colon = g^{-1}Hg$$

   which induces a map

$$\pi_*^{H^g}(X) \to \pi_*^H(c_g^*X)$$

   Moreover, since left multiplication by g is an isomorphism between $G/H \to G/H^g$, it also induces an isomorphism $c_g^*X \to X$, so we have the composite

$$c_g\colon \pi_*^{H^g}(X) \to \pi_*^H(c_g^*X) \cong \pi_*^H(X)$$

   which we call the **conjugation map**. It is also easy to verify that $c_g \circ c_h = c_{gh}$ for all $g, h \in G$.

3. The final piece is the **transfer map**

$$\mathrm{Tr}^G_H\colon \pi_*^H(X) \to \pi_*^G(X)$$

   which we will construct in the remainder of the section.

Given an H-spectrum Y, there is a projection map of H-spectra

$$pr\colon G \wedge_H Y \to H \wedge_H Y \cong Y$$

given by

$$pr(g \wedge_H y) = \begin{cases} gy & g \in H \\ * & \text{otherwise} \end{cases}$$

The projection map then factors as

$$G \wedge_H Y \xrightarrow{\Psi_Y} \mathrm{Maps}_H(G, Y) \xrightarrow{\mathrm{ev}_0} Y$$

Then, the composite

$$\pi_*^G(G \wedge_H Y) \xrightarrow{\mathrm{Res}_H^G} \pi_*^H(G \wedge_H Y) \xrightarrow{pr_*} \pi_*^H(Y)$$

is an isomorphism. We can now define the transfer map Tr_H^G for a G-spectrum X as the composite

$$\pi_*^H(X) \to \pi_*^G(G \wedge_H X) \to \pi_*^G(X)$$

where the first map is the inverse of the composite above, and the second map is the action map $g \wedge_H x \mapsto gx$. One can also check that the transfer maps are also transitive.

The transfer map and the restriction map also satisfy the double-coset formula

Proposition 2.27. For subgroups $H, K \leq G$

$$\mathrm{Res}_K^G \circ \mathrm{Tr}_H^G = \bigoplus_{x \in H \backslash G / K} \mathrm{Tr}_{x^{-1}Hx \cap K}^K \circ c_x \circ \mathrm{Res}_{H \cap xKx^{-1}}^H$$

as maps $\pi_*^H(X) \to \pi_*^K(X)$.

Proof. The important observation is the splitting

$$\mathrm{Res}_K^G(G \wedge_H X) \cong \bigvee_{x \in H \backslash G / K} K \wedge_{K \cap xHx^{-1}} \mathrm{Res}_{K \cap xHx^{-1}}^{xHx^{-1}}(c_g^* X)$$

so we only have to check the formula on each wedge summand since homotopy groups turns wedges to direct sum. The checks are straightforward, and details can be found in [Sch23].

□

It turns out that the data above precisely constitute a Mackey functor

Definition 2.28. A **Mackey functor** $\underline{M} : \mathrm{GSet}^{\mathrm{Fin}} \to \mathrm{Ab}$ consists of the following data:

- An abelian group $\underline{M}(G/H)$ for every subgroup $H \leq G$;
- For every subgroup $H \leq K \leq G$, a restriction map

$$\mathrm{Res}_H^K : \underline{M}(G/K) \to \underline{M}(G/H)$$

  and a transfer map

$$\mathrm{Tr}_H^K : \underline{M}(G/H) \to \underline{M}(G/K)$$

  .
- For every subgroup $H \leq G$ and $g \in G$, a conjugation homomorphism

$$c_g : \underline{M}(G/H) \to \underline{M}(G/H)$$

The three types of morphisms must also satisfy the following relations:

1. Tr_H^H and Res_H^H are both identity maps for all subgroups $H \leq G$;
2. The restriction is transitive, i.e

$$\mathrm{Res}_H^K \circ \mathrm{Res}_K^G = \mathrm{Res}_H^G$$

   and similarly for transfer and conjugation.
3. The restriction and transfer maps are compatible with conjugation, and satisfy the double coset formula.

Mackey functors will serve the role of coefficient systems in the cohomology theories represented by G-specrtra. In particular, the take into account of the transfer maps, which is not present in the orbit category $\mathcal{O}_G$. From now on, we will denote the Mackey functor valued homotopy groups of a G-spectrum X as $\underline{\pi_*}X$.

There is also a formula-free definition of Mackey functors in terms of the Burnside category.

Definition 2.29. Let C be a category with finite limits and colimits. Then, $\mathbf{Span}(C)$ is the category with the same objects as C, and morphisms isomorphism classes of spans. Composition of morphisms is given by pullback of spans.

Given two morphisms $X \leftarrow Z \rightarrow Y$ and $X \leftarrow Z' \rightarrow Y$, the disjoint union operation $X \leftarrow Z \coprod Z' \rightarrow Y$ endows the homsets a monoid structure. Let $\mathbf{Span}^+(C)$ denote the preadditive completion of $\mathbf{Span}(C)$, i.e universally group completing the homsets.

Definition 2.30. Let G be a finite group. Then, the **Burnside Category** is defined to be $\mathcal{A}(G) := \mathbf{Span}^+(\mathrm{GSet}^{\mathrm{Fin}})$.

The Burnside category can be viewed as formally adding the transfers to each morphism in the orbit category. We have seen that transfers exist in genuine G-spectra, and the following is not surprising:

Proposition 2.31. The Burnside category is equivalent to the full subcatgory of $\mathrm{Ho}(\mathbf{Sp}^G)$ spanned by the suspension spectra of finite G-sets.

Proof. We give a sketch of the proof, and the details can be found in Section 9 of [LMM86]. The equivalence of categories is constructed by sending each morphism

$$G/H \xleftarrow{f} G/K \xrightarrow{g} G/L$$

to the composite

$$\Sigma^\infty_G G/H \xrightarrow{\tau(f)} \Sigma^\infty_G G/K \xrightarrow{\Sigma^\infty_G g} \Sigma^\infty_G G/L$$

where $\tau(f)$ is the stable transfer map associated to f. To see that this is an isomorphism on morphisms, we note

$$[\Sigma^\infty_G G/H, \Sigma^\infty_G G/L]_G \cong [\Sigma^\infty_G S^0, \Sigma^\infty_G G/L]_H = \pi_0^H(\Sigma^\infty_G G/L)$$

and the tom Dieck splitting [4] tells us that $\pi_0^H(\Sigma^\infty_G G/L)$ corresponds to free abelian group generated by diagrams of the form

$$\Sigma^\infty_G S^0 \xrightarrow{\tau} \Sigma^\infty_G H/K \to \Sigma^\infty_G G/L$$

where K runs through each conjugacy class of subgroups of H. $\square$

Definition 2.32. A **Mackey functor** is an additive functor $F : \mathcal{A}(G)^{op} \to \mathbf{Ab}$.

We immediately see that the homotopy groups of G-spectra is a Mackey functor

$$\Sigma^\infty G/H_+ \mapsto [\Sigma^\infty G/H_+ \wedge S^n, -] \cong \pi_n^H(-)$$

using Proposition 2.33. Using this perspective, we also see the category of Mackey functors is abelian, and we can do homological algebra.

We now try to explicitly reconcile the two definitions of Mackey functor:

Proposition 2.33. The two definitions of Mackey functors given above are equivalent.

Proof. Considering the relation between the Burnside category and the orbit category, we can define a functor

$$i_* \colon \mathcal{O}_G \to \mathcal{A}(G)$$

by sending a morphism $f \colon G/H \to G/K$ to the span $G/H \xleftarrow{Id} G/H \xrightarrow{f} G/K$; similarly, we have a functor

$$i^* \colon \mathcal{O}_G^{op} \to \mathcal{A}(G)$$

by sending a morphism $f \colon G/H \to G/K$ to the span $G/K \xleftarrow{f} G/H \xrightarrow{Id} G/H$. Given a Mackey functor $\underline{M} \colon \mathcal{A}(G)^{op} \to \mathbf{Ab}$, precomposition with i_* gives a coefficients system, which we will denote M^*, and precomposition with i^* is a covariant funcor $\mathcal{O}_G \to \mathrm{Ab}$, which we will denote as M_*. By construction, M^* and M_* agree on objects, and they determine the value of the Mackey functor on objects. By additivity, the data is determined by the disjoint orbits. We would also like to know how the morphisms commute: if we have the pullback square of orbits

$$\begin{tikzcd}
W \arrow[r, "\beta"] \arrow[d, "\alpha"'] & Y \arrow[d, "\delta"] \\
X \arrow[r, "\gamma"'] & Z
\end{tikzcd}$$

then $M_*(\beta)M^*(\alpha) = M^*(\delta)M_*(\gamma)$, as illustrated by the following composition diagram.

$$\begin{tikzcd}
 & & W \arrow[dl, "\alpha"] \arrow[dr, "\beta"] \arrow[dd, phantom, "\vee", very near start] & & \\
 & X \arrow[dl, "Id"'] \arrow[dr, "\gamma"] & & Y \arrow[dl, "\delta"'] \arrow[dr, "Id"] & \\
X & & Z & & Y
\end{tikzcd}$$

[4]to be introduced in the next subsection

From the above discussion, we have another equivalent characterization: A Mackey functor is a pair of functors $M^*\colon \mathrm{GSet}^{\mathrm{Fin}} \to \mathbf{Ab}$ and $M_*\colon \mathrm{GSet}^{\mathrm{Fin}} \to \mathbf{Ab}$ such that the following axioms hold:

1. M^* and M_* agree on objects;
2. For every pullback square of orbits

$$\begin{tikzcd} W \arrow[r, "\beta"] \arrow[d, "\alpha"'] & Y \arrow[d, "\delta"] \\ X \arrow[r, "\gamma"'] & Z \end{tikzcd}$$

   then $M_*(\beta)M^*(\alpha) = M^*(\delta)M_*(\gamma)$;
3. Both functors maps disjoint unions to direct sums.

Given a G-map between orbits $G/H \to G/K$, there is a decomposition by Proposition 1.11

$$G/H \xrightarrow{c_g} G/gHg^{-1} \xrightarrow{\pi} G/K$$

It thus suffices to understand how the image of conjugation and projection commute.

Let $c_g^* := M^*(c_g)$ and $c_*^g := M_*(c_g)$. We have the following pullback square

$$\begin{tikzcd} G/H \arrow[r, "Id"] \arrow[d, "Id"'] \arrow[dr, phantom, "\lrcorner", very near start] & G/H \arrow[d, "c_g"] \\ G/H \arrow[r, "c_g"'] & G/gHg^{-1} \end{tikzcd}$$

So in fact we have $c_g^* = (c_*^g)^{-1}$.

Given a canonical projection $\pi\colon G/H \to G/K$, denote $\mathrm{Res}_H^K := M^*(\pi)$ and $\mathrm{Tr}_H^K := M_*(\pi)$. Now consider the following commutative diagram:

$$\begin{tikzcd} G/H \arrow[r, "\pi"] \arrow[d, "c_g"'] & G/G \arrow[d, "Id"] \\ G/gHg^{-1} \arrow[r, "\pi"'] & G/G \end{tikzcd}$$

which implies $\mathrm{Tr}_H^G \circ c_*^g = \mathrm{Tr}_{H^g}^G$ and $\mathrm{Res}_{H^g}^G(x) = c_g^* \circ \mathrm{Res}_H^G(x)$ for all g. Finally, consider the pullback diagram

$$\begin{tikzcd} G/H \times G/K \arrow[r, "\beta"] \arrow[d, "\alpha"'] & G/K \arrow[d, "\delta"] \\ G/H \arrow[r, "\gamma"'] & G/G \cong * \end{tikzcd}$$

By the G-isomorphism $G/H \times G/K \cong G \times_H G/K$, it is easy to see the the class $(1, xK)_H$ has stabilizer $H \cap xKx^{-1}$, and the orbits are indexed by a class in the double coset $H\backslash G/K$. Thus,

$$G/H \times G/K \cong \coprod_{H\backslash G/K} G/H \cap xKx^{-1}$$

We may use this decomposition to compute the composition of transfers and restrictions: the composition $\mathrm{Res}^G_K \circ \mathrm{Tr}^g_H \colon M(G/H) \to M(G/K)$ is given by the value of the Mackey functor on the diagram

$$\begin{tikzcd}
 & & \coprod_{H\backslash G/K} G/H \cap xKx^{-1} \arrow[dl] \arrow[dr] & & \\
 & G/H \arrow[dl] \arrow[dr] & & G/K \arrow[dl] \arrow[dr] & \\
G/H & & G/G & & G/K
\end{tikzcd}$$

By additivity, it is determined by

$$\begin{tikzcd}
 & G/H \cap xKx^{-1} \arrow[dl] \arrow[dr] & \\
G/H & & G/K
\end{tikzcd}$$

which is the composition $\mathrm{Tr}^K_{x^{-1}Hx\cap K} \circ c_x \circ \mathrm{Res}^H_{H\cap xKx^{-1}}$. This gives us the double coset formula, as in Proposition 2.27. $\square$

We now give a few more examples of Mackey functors:

Example 2.34 (Constant Mackey functor). For every abelian group A, we have the constant Mackey functor $\underline{A}$, which assigns every orbit the abelian group A, and restriction maps the identity morphism. One may check that this forces the transfer maps $\mathrm{Tr}^K_H : \mathbb{Z} \to \mathbb{Z}$ to be multiplication by $|K/H|$, and the conjugation maps to be the identity.

In the case where $G = C_2$ and $A = \mathbb{Z}$, the Mackey functor is described by the following diagram:

$$\begin{tikzcd}
\mathbb{Z} \arrow[d, bend right, "Id"'] \\
\mathbb{Z} \arrow[u, bend right, "\times 2"'] \arrow[loop below, "Id"]
\end{tikzcd}$$

In the case where $G = C_2$ and $A = \mathbb{Z}/2$, the Mackey functor is described by the following diagram:

$$\begin{tikzcd}
\mathbb{Z}/2 \arrow[d, bend right, "Id"'] \\
\mathbb{Z}/2 \arrow[u, bend right, "\times 0"'] \arrow[loop below, "Id"]
\end{tikzcd}$$

Remark 2.35. Note that the "constant" Mackey functor $\underline{\mathbb{Z}}$ breaks up to a constant coefficient system $\mathbb{Z}$, but the covariant part is **NOT** the constant functor. We will run into this problem later.

The double coset formula in 2.28 seems complicated, but for C_p-Mackey functors, the axioms can be reduced to the following:

Proposition 2.36. A C_p-Mackey functor $\underline{M}$ consists of the following data:

$$\begin{tikzcd} \underline{M}(C_p/C_p) \arrow[d, bend right, "\mathrm{Res}_e^{C_p}"'] \\ \underline{M}(C_p/e) \arrow[u, bend right, "\mathrm{Tr}_e^{C_p}"'] \arrow[loop below, "c_g"] \end{tikzcd}$$

where we ignore the some of all of the obvious identity morphisms. The morphisms satifies the following:

1. $c_g \circ \mathrm{Res}_e^{C_p} = \mathrm{Res}_e^{C_p}$.
2. $\mathrm{Tr}_e^{C_p} \circ c_g = \mathrm{Tr}_e^{C_p}$
3. $\mathrm{Res}_e^{C_p} \circ \mathrm{Tr}_e^{C_p} = \sum_{\gamma \in C_p} c_\gamma$

In particular, we can completely determine the data of a C_p Mackey functor with just the restriction or the transfer map.

Example 2.37 (Burnside Mackey Functor). The Burnside Mackey Functor A_G assigns every orbit G/H the Grothendieck group of the symmetric monoidal category of finite H-sets under coproduct. The transfer and restriction maps come from induction and the forgetful functor, respectively, between H-Sets and K-Sets.

When $G = C_p$, the Burnside Mackey Functor A_{C_p} is given by the following: there are two subgroups e and C_p; the corresponding Grothendieck groups are $A_{C_p}(e) = \mathbb{Z}$, generated by the singleton with trivial action, and $A_{C_p}(C_p) = \mathbb{Z} \oplus \mathbb{Z}$, generated by the singleton with trivial action and a C_p-orbit. The induced G set $C_p \times_e *$ is the C_p-orbit, and forgetting the G action on the C_p-orbit gives us a disjoint union of p singletons with trivial action. Diagrammatically, we have the following

$$\begin{tikzcd} \mathbb{Z} \oplus \mathbb{Z} \arrow[d, bend right, "(1,p)"'] \\ \mathbb{Z} \arrow[u, bend right, "(0,Id)"'] \end{tikzcd}$$

2.4 Fixed Points and tom Dieck Splitting

For a G-spectrum X, its **naive fixed points** are simply the fixed points being taken level-wise. The construction is not homotopical. For example, consider the $\underline{\pi}_*$-isomorphism of genuine C_2-spectra

$$S^{-\sigma} \wedge S^{\sigma} \to S^0$$

But $(S^{-\sigma} \wedge S^{\sigma})_n^G$ is contractible, while $(S^0)^G = S^0$. The solution is to do fibrant replacement of X first:

Definition 2.38. The **categorical fixed points functor** $(-)^H : \mathbf{Sp}^G \to \mathbf{Sp}$ is defined to be

$$X \mapsto \mathrm{Maps}_G(G/H_+, RX)$$

It is easy to verify that the fibrant replacement does not change the homotopy groups. One notes that the categorical fixed points detect weak equivalences. Since we are taking the fixed points of the fibrant replacement, it is immediate that the categorical fixed points functor does not commute with the suspension functor Σ_G^∞. For example, take X to have trivial G-action. Then, the fibrant replacement of $\Sigma_G^\infty X$ has zero space

$$\mathrm{Hocolim}_{V\subset U}\Omega^V\Sigma^V X$$

which has non-trivial G-action. The tom Dieck splitting theorem gives a description of the fixed points of the fibrant replacement.

Theorem 2.39 (tom Dieck Splitting)**.** There is a weak equivalence

$$(\Sigma_G^\infty X)^G \cong \bigvee_{(H)\leq G} \Sigma_G^\infty EWH_+ \wedge_{WH} X^H$$

and an isomorphism

$$\pi_*^G(\Sigma_G^\infty X) \cong \bigoplus_{(H)\leq G} \pi_*^{WH}(\Sigma^\infty EWH_+ \wedge X^H)$$

A proof of the tom Dieck splitting can be found in [Sch23] and [LMM86]. A corollary is that we can compute the 0th homotopy group of the sphere spectrum:

$$\pi_0^G(\Sigma^\infty S^0) \cong \bigoplus_{(H)\leq G} \pi_*^{WH}(\Sigma_G^\infty EWH_+) \cong \bigoplus_{(H)\leq G} \mathbb{Z}$$

and $\underline{\pi_0}(\Sigma_G^\infty S^0)$ is the Burnside Mackey functor.

Definition 2.40. The **homotopy fixed points functor** $()^{hG}$ is defined to be

$$X \mapsto \mathrm{Maps}_G(EG_+, X)$$

One notes that the homotopy fixed points functor also does not commute with the suspension functor.

Example 2.41. Take $X = \mathbb{S}$, and let $\mathbb{Z}/p$ act trivially on the sphere spectrum, so that

$$\pi_0(\Sigma^\infty (S^0)^{h\mathbb{Z}/p}) \cong \pi_0(\mathbb{S}) \cong \mathbb{Z}$$

By the resolved Segal conjecture for cyclic groups [Rav81], we have

$$\pi_0(\mathbb{S}^{h\mathbb{Z}/p}) \cong \mathbb{Z} \oplus \mathbb{Z}_p$$

We also have a type of fixed points that commutes with the suspension functor and smash products:

Definition 2.42. The **geometric fixed points functor** $\Phi^H : \mathbf{Sp}^G \to \mathbf{Sp}^G$ is defined to be

$$\Phi^G X(V) := X(V \otimes \rho_G)^G$$

The geometric fixed points functor is the unique functor that is derived symmetric monoidal, preserves homotopy colimits, and commutes with the suspension functor. The construction of the geometric fixed point functor can be found in [Sch23].

3 $RO(G)$-graded cohomology Theories

3.1 Representability

Recall that a reduced cohomology theory $\tilde{h}^n(-)$ on CW complexes corresponds to some Ω-spectra E, and we have

$$\tilde{h}^n(X) = [\Sigma^\infty X, \Sigma^n E]$$

by Brown representability. But with genuine G-spectra, we can suspend with non-trivial representations, so the equivariant cohomology theories should be graded by our universe of representations.

Definition 3.1. Fix a group G and a universe U. Let $RO(G;U)$ be the category with objects G-representations embeddable in U, with morphisms are G-equivariant isometric isomorphisms. Two morphisms $f, g\colon V \to W$ are homotopic if their induced morphisms $f', g'\colon S^V \to S^W$ are stably homotopic.

Definition 3.2. An $RO(G)$**-graded cohomology theory** is a functor

$$E : \mathrm{Ho}(RO(G;U)) \times \mathrm{Ho}(G\mathbf{Top})^{op} \to \mathbf{Ab}$$

written $(V, X) \mapsto E^V(X)$ on objects, together with natural isomorphisms

$$\sigma_W : E^V(X) \to E^{V\oplus W}(\Sigma^W X)$$

such that the following axioms are satisfied:

1. For each representation V, the functor E^V sends cofiber sequences to exact sequences, and wedges to direct products.
2. For each map $\alpha : W \to W'$ in $RO(G;U)$, the map

$$\begin{tikzcd} E^V(X) \arrow[r, "\sigma_W"] \arrow[d, "\sigma_{W'}"] & E^{V\oplus W}(\Sigma^W X) \arrow[d, "(Id\oplus\alpha, Id)"] \\ E^{V\oplus W'}(\Sigma^{W'} X) \arrow[r, "(Id\oplus Id, \alpha)"'] & E^{V\oplus W'}(\Sigma^W X) \end{tikzcd}$$

3. We have $\sigma_0 = Id$ and the following transitivity diagram

$$\begin{tikzcd} E^V(X) \arrow[rr, "\sigma_W"] \arrow[dr, "\sigma_{W\oplus Z}"'] & & E^{V\oplus W}(\Sigma^W X) \arrow[dl, "\sigma_Z"] \\ & E^{V\oplus W\oplus Z}(\Sigma^{W\oplus Z} X) & \end{tikzcd}$$

Definition 3.3. The **formal difference** of a pair of representations $(V, W) \in \mathrm{Ho}(RO(G;U)) \times \mathrm{Ho}(RO(G;U))^{op}$ is written as $V \ominus W$. We define

$$E^{V\ominus W}(X) := E^V(\Sigma^W X)$$

We can naturally extend the **Ab**-valued cohomology theories to a Mackey-functor valued cohomology theory:

Definition 3.4. Given an $RO(G)$-graded cohomology theory E, the associated **Mackey functor-valued cohomology**, denoted $\underline{E}$, is defined to be

$$E^V(X)(G/H) := E^V(G/H \times X)$$

the structure is described by the diagram

$$\begin{tikzcd} \underline{E}^V(X) \arrow[d, bend right, "\pi^*"'] \\ E^V(G \times X) \arrow[u, bend right, "\mathrm{tr}"'] \arrow[loop below, "G"] \end{tikzcd}$$

where the transfer map is induced by the covering space structure $\pi : G \times X \to X$.

From representability and Definition 2.33, the above definition is the natural one.

Proposition 3.5. Every $RO(G)$-graded cohomology theory is represented by an G-spectrum.

Proof. It is straightforward to show that for each $V \in RO(G)$, E^V is a non-equivariant cohomology theory, which is represented by some G-CW complex $X(V)$ by Brown representability. The isomorphisms $\sigma_W : E^V(X) \to E^{V \oplus W}(\Sigma^W X)$ induces a structure map

$$\overline{\sigma_W} : X(V) \to \Omega^W X(V \oplus W)$$

up to homotopy. However, we want structure maps on the nose. For this, we may choose a cofinal sequence of representations

$$V_0 \subseteq V_1 \subseteq \ldots$$

in our universe U. Then, we choose a sequence of homotopy equivalences $X_i \to \Omega X_{i+1}$ like the usual Brown representability, and define the other structure maps by composition using cofinality. $\square$

Corollary 3.6. Every $\mathbb{Z}$-graded cohomology theory on $\mathbf{Sp}^G$ can be extended to an $RO(G)$-graded theory.

Proof. First, note that every $\mathbb{Z}$-graded cohomology theory of $\mathbf{Sp}^G$ is represented by Theorem A.37. Since the loop and suspension functor are inverse equivalence on $\mathrm{Ho}(\mathbf{Sp}^G)$, the entire theory is determined on the $\mathbb{Z}$-graded component, and we have the extension. $\square$

For an equivariant cohomology theory H^* on $G\mathbf{Top}_*$ with values in a coefficient system M, we have the following:

Theorem 3.7. A $\mathbb{Z}$-graded cohomology theory on $G\mathbf{Top}_*$ with coefficient system M extends to an $RO(G)$-graded theory iff M extends to a Mackey functor.

Proof. The forward direction really follows from representability and the definition of Mackey functor using the Burnside category, but we can see this more explicitly: if $H^*(-, M)$ extends to an $RO(G)$-graded theory, then the transfer map

$$\tau \colon S^V \to G/H_+ \wedge S^V$$

induces a transfer homomorphism

$$H^{V+n}(\Sigma^V(G/H_+ \wedge X); M) \to H^{V+n}(\Sigma^V X; M)$$

and by the suspension isomorphism, this corresponds to a transfer map

$$H^n(X; M|H) \cong H^n(G/H_+ \wedge X; M) \to H^n(X; M)$$

By taking $n = 0$ and $X = S^0$, we get a transfer homomorphism $M(G/H) \to M(G/G)$, so one sees that the coefficient system must extend to a Mackey functor.

For the converse, the idea is to find a $\mathbb{Z}$-graded cohomology theory on $\mathbf{Sp}^G$ corresponding to M. Corollary 3.6 then extends the $\mathbb{Z}$-graded theory to an $RO(G)$-graded one. Starting with a G-CW spectra X, one has a skeletal filtration

$$X_0 \subseteq X_1 \subseteq ...$$

Let $\underline{C_n}(X) := \underline{\pi_n}(X_n/X_{n-1})$ be a chain complex of Mackey functors, and consider the cochain complex complex of abelian groups

$$C^n(X) \colon = \mathrm{Hom}_{\mathrm{Mac}}(\underline{C_n}(X); M)$$

The homology of the cochain complex, which we denote by $H^*(-, M)$, give us a well-defined cohomology theory on G-CW spectra. By G-CW approximation, we may extend the theory to the entire $\mathbf{Sp}^G$. One then computes that

$$H^0(G/H_+ \wedge S^0; M) = \mathrm{Hom}_{\mathrm{Mac}}(\underline{\pi_0}(G/H_+ \wedge S^0); M) \cong \mathrm{Hom}_G(*; M(G/H)) \cong M(G/H)$$

where the isomorphism comes from the universal property of the Burnside Mackey functor. Moreover, all higher cohomology groups vanish and we get our desired $RO(G)$-graded cohomology theory.

□

An immediate consequence of the cohomology theory constructed above is that we explicitly constructed the Eilenberg-Maclane spectrum in the equivariant setting.

Definition 3.8. For a Mackey functor M, the **Eilenberg-Maclane spectrum** associated to M is a spectrum $H\underline{M}$ that satisfies

$$\underline{\pi_n}(H\underline{M}) = \begin{cases} M & n = 0 \\ 0 & \text{otherwise} \end{cases}$$

3.2 The $RO(C_2)$ cohomology of a point

For this section, we are going to let $H^*_G(X; M)$ denote the $RO(G)$-graded cohomology theory represented by the Eilenberg-Maclane spectrum $H\underline{M}$. Before we proceed in the computation, we first list some C_2 Mackey functors that will come up:

1. The constant Mackey functor $\underline{\mathbb{Z}}$

$$\begin{tikzcd} \mathbb{Z} \arrow[d, bend right, "Id"'] \\ \mathbb{Z} \arrow[u, bend right, "\times 2"'] \arrow[loop below, "Id"] \end{tikzcd}$$

2. The opposite constant Mackey functor $\underline{\mathbb{Z}}^{op}$, where we reverse all arrows in $\underline{\mathbb{Z}}$

$$\begin{tikzcd} \mathbb{Z} \arrow[d, bend right, "\times 2"'] \\ \mathbb{Z} \arrow[u, bend right, "Id"'] \arrow[loop below, "Id"] \end{tikzcd}$$

   The opposite of other Mackey functors is defined similarly.

3. The Mackey functor $\underline{\mathbb{Z}}_{-}$

$$\begin{tikzcd} 0 \arrow[d, bend right, "0"'] \\ \mathbb{Z} \arrow[u, bend right, "0"'] \arrow[loop below, "\times -1"] \end{tikzcd}$$

   where the fixed point set has the nontrivial C_2-action.

4. The Mackey functor $\underline{\mathbb{Z}}_{-}^{op}$

$$\begin{tikzcd} \mathbb{Z}/2 \arrow[d, bend right, "0"'] \\ \mathbb{Z} \arrow[u, bend right, "pr"'] \arrow[loop below, "\times -1"] \end{tikzcd}$$

5. For any group G, we define the Mackey functor $\langle G \rangle$ to be the following

$$\begin{tikzcd} G \arrow[d, bend right, "0"'] \\ 0 \arrow[u, bend right, "0"'] \end{tikzcd}$$

3.2.1 Computing $H^{p,q}_{C_2}(*;\underline{\mathbb{Z}})$

Proposition 3.9. The representation ring of C_2 is given by

$$RO(C_2) \cong \mathbb{Z}[\sigma]/(\sigma^2 - 1)$$

As an abelian group, we have $RO(C_2) \cong \mathbb{Z}^2$, generated by the trivial representation and the sign representation. We now calculate the ordinary $RO(C_2)$-cohomology of a point with respect to the constant Mackey functor $\underline{\mathbb{Z}}$. Our indexing convention[5] is

$$H^{p,q}_{C_2}(*;\underline{\mathbb{Z}})$$

[5]Some other sources use the motivic indexing convention, where p is the total dimension of the representation and q is the dimension of the sign representation

where p is the dimension of the trivial representation, and q is the dimension of the sign representation.

Our plan for the calculation is to use the suspension isomorphism to reduce the $RO(G)$-graded cohomology to Bredon cohomology/homology of representation spheres, which are easier to compute.

If $q \leq 0$, then by the suspension isomorphism we have

$$H^{p,q}_{C_2}(*;\underline{\mathbb{Z}}) := \tilde{H}^{p,q}_{C_2}(S^0;\underline{\mathbb{Z}}) \cong \tilde{H}^{p,0}_{C_2}(S^{-q\sigma};\underline{\mathbb{Z}}) \cong \tilde{H}^p_{\text{Bredon}}(S^{-q\sigma};\underline{\mathbb{Z}})$$

Recall that $S^{-q\sigma}$ is the one-point compactification of the sign-representation on $\mathbb{R}^{-q}$. By Theorem 1.35, we have the reduction

$$\tilde{H}^p_{\text{Bredon}}(S^{-q\sigma};\underline{\mathbb{Z}}) \cong \tilde{H}^p(S^{-q\sigma}/C_2;\mathbb{Z})$$

In example 1.37, we saw $S^{-q\sigma}/C_2$ is homotopy equivalent to $\Sigma\mathbb{RP}^{-q-1}$. Using suspension isomorphism, we have

$$H^{p,q}_{C_2}(*;\underline{\mathbb{Z}}) \cong \tilde{H}^{p-1}(\mathbb{RP}^{-q-1})$$

which we know how to compute.

When $q > 0$, we apply the Spanier-Whitehead duality to turn the $RO(G)$-graded cohomology to Bredon homology:

Proposition 3.10 (Spanier-Whitehead Duality)**.** The Spanier-Whitehead dual of S^V is S^{-V}, and we have

$$\tilde{H}^{p,q}_{C_2}(S^0;\underline{\mathbb{Z}}) := [S^0, S^{p+q\sigma} \wedge H\mathbb{Z}] \cong [S^{-p-q\sigma}, S^0 \wedge H\mathbb{Z}] =: \tilde{H}^{C_2}_{-p,-q}(S^0;\underline{\mathbb{Z}})$$

and we reduce the problem again to computing $\tilde{H}^{C_2}_{-p,-q}(S^0;\underline{\mathbb{Z}}) \cong \tilde{H}^{\text{Bredon}}_{-p}(S^{q\sigma};\underline{\mathbb{Z}})$. However, we should note that the constant Mackey functor does not restrict to the constant covariant coefficient system, so Theorem 1.36 does not apply here. We thus turn to the cofiber sequence (which could also be used in computing the Bredon cohomology if we did not have the explicit homotopy equivalence)

$$S(V)_+ \to D(V)_+ \to S^V$$

where $S(V)$ and $D(V)$ are the unit sphere and unit disk in the representation V, respectively. Moreover, $D(n)_+$ is C_2-homotopy equivalent to S^0, and the C_2-action on $S(V)_+$ is free. By the cofiber long exact sequence, we have

$$0 \longrightarrow \tilde{H}^{C_2}_{p+1,0}(S^{q\sigma};\underline{\mathbb{Z}}) \longrightarrow \tilde{H}^{C_2}_{p,0}(S(q\sigma)_+;\underline{\mathbb{Z}}) \longrightarrow 0$$

for $p \geq 1$, and we have

$$\tilde{H}^{C_2}_{p+1,0}(S^{q\sigma};\underline{\mathbb{Z}}) \cong \tilde{H}^{\text{Bredon}}_p(S(q\sigma)_+;\mathbb{Z})$$

Applying Theorem 1.39, we get

$$\tilde{H}^{C_2}_{p+1,0}(S^{q\sigma};\underline{\mathbb{Z}}) \cong H_p(S(q\sigma)/C_2;\mathbb{Z}) \cong H_p(\mathbb{RP}^{q-1};\mathbb{Z})$$

When $p = 0, 1$, we open up the LES

$$0 \longrightarrow \tilde{H}^{C_2}_{1,0}(S^{q\sigma};\underline{\mathbb{Z}}) \longrightarrow \tilde{H}^{C_2}_{0,0}(S(q\sigma)_+;\underline{\mathbb{Z}}) \cong \mathbb{Z} \xrightarrow{\gamma} \tilde{H}^{C_2}_{0,0}(S^0;\underline{\mathbb{Z}}) \cong \mathbb{Z} \longrightarrow \tilde{H}^{C_2}_{0,0}(S^{q\sigma};\underline{\mathbb{Z}}) \longrightarrow 0$$

Proposition 3.11. The map γ is multiplication by 2. Thus, $\tilde{H}_{1,0}(S^{q\sigma};\underline{\mathbb{Z}}) = 0$ and $\tilde{H}_{0,0}(S^{q\sigma};\underline{\mathbb{Z}}) = \mathbb{Z}/2$

Proof. There are a couple of ways to identify the middle map γ: the first possibility is to note when $q = 1$, the middle map is

$$H_{0,0}^{C_2}(C_2; \underline{\mathbb{Z}}) \to H_{0,0}^{C_2}(*; \underline{\mathbb{Z}})$$

induced by the map of orbits $C_2/e \to C_2/C_2$. Thus, we can identify the map as the transfer map of the Mackey functor $\underline{\mathbb{Z}}$, which is $\times 2$. The exactness of the sequence then gives us $\tilde{H}_{0,0}^{C_2}(S^\sigma, \underline{\mathbb{Z}}) \cong \mathbb{Z}/2$. For $q \geq 2$, we see that $\tilde{H}_{0,0}^{C_2}(S^{q\sigma}, \underline{\mathbb{Z}}) \cong \mathbb{Z}/2$ as well, since the 0th Bredon homology is determined by the 1-skeleton. Thus, the middle map is multiplication by 2 for all q. Another possibility is noting that the middle map is induced by the inclusion

$$H_0(S(q\sigma)/C_2; \mathbb{Z}) \cong \mathbb{Z} \to H_0(D(q\sigma)/C_2; \mathbb{Z}) \cong \mathbb{Z}$$

which is multiplication by 2.

□

Thus, the $RO(G)$-graded cohomology of a point at fixed point level is

$$H_{C_2}^{p,q}(*, \underline{\mathbb{Z}}) = \begin{cases} \mathbb{Z} & p = -q \text{ and } p \text{ even} \\ \mathbb{Z}/2 & 1 < p \leq -q \text{ and } p \text{ odd} \\ \mathbb{Z}/2 & 1 < -p < q \text{ and } p \text{ even} \\ \mathbb{Z}/2 & p = 0 \text{ and } q > 0 \\ 0 & \text{otherwise} \end{cases}$$

At the C_2/e level, we have the adjuntion

$$\begin{aligned} \tilde{H}_{C_2}^{p,q}((C_2)_+ \wedge S^0, \underline{\mathbb{Z}}) &:= [(C_2)_+ \wedge S^0, \Sigma^{p+q\sigma} H\underline{\mathbb{Z}}]_G \\ &\cong [S^0, \Sigma^{p+q\sigma} H\underline{\mathbb{Z}}]_e \\ &\cong [S^0, \Sigma^{p+q} H\mathbb{Z}] \\ &\cong \tilde{H}_{C_2}^{p+q}(S^0; \mathbb{Z}) \end{aligned}$$

Thus, it is only non-trivial when $p = -q$, and our last step is to determine the structure of the cohomology as a Mackey functor: by Proposition 2.36, almost all of the Mackey functors here are completely determined by the two groups alone:

1. When $1 < p < -q$ and p is odd, or $1 < -p < q$ and q is even, or $p = 0$ and $q > 0$, the C_2/C_2 level is $\mathbb{Z}/2$, and C_2/e level is 0. Thus, the corresponding Mackey functor is $\langle \mathbb{Z}/2 \rangle$;
2. When $1 < p = -q$, with p odd, the C_2/C_2 level is $\mathbb{Z}/2$, and C_2/e level is $\mathbb{Z}$. Thus, the Mackey functor is $\underline{\mathbb{Z}}_-^{op}$;
3. When $p = -q \leq 1$, with p odd, the C_2/C_2 level is 0, and C_2/e level is $\mathbb{Z}$. Thus, the Mackey functor is $\underline{\mathbb{Z}}_-$;

The remaining case is when $p = -q$ is even, and we need to determine a transfer/restriction map between $\mathbb{Z}$. We use the cofiber sequence

$$(C_2)_+ \to S^0 \to S^\sigma$$

and the LES gives us

$$\tilde{H}_{C_2}^{2n,-2n-1}(S^0; \underline{\mathbb{Z}}) \longrightarrow \tilde{H}_{C_2}^{2n,-2n}(S^0; \underline{\mathbb{Z}}) \xrightarrow{\text{Res}} \tilde{H}_{C_2}^{2n,-2n}((C_2)_+; \underline{\mathbb{Z}}) \longrightarrow \tilde{H}_{C_2}^{2n+1,-2n-1}(S^0; \underline{\mathbb{Z}}) \longrightarrow \ldots$$

We have calculated that the middle two groups are both $\mathbb{Z}$; when $n > 0$, we have computed

$$\tilde{H}_{C_2}^{2n+1,-2n-1}(S^0; \underline{\mathbb{Z}}) \cong \mathbb{Z}/2$$

and

$$\tilde{H}^{2n,-2n-1}_{C_2}(S^0;\underline{\mathbb{Z}}) \cong 0$$

so the restriction map is multiplication by 2; when $n < 0$, we see that

$$\tilde{H}^{2n+1,-2n-1}_{C_2}(S^0;\underline{\mathbb{Z}}) \cong 0$$

and

$$\tilde{H}^{2n,-2n-1}_{C_2}(S^0;\underline{\mathbb{Z}}) \cong 0$$

so the restriction map is an isomorphism. By Proposition 2.36, knowing the restriction determines the entire structure of the Mackey functor: when $n > 0$, we have $H^{2n,-2n}_{C_2}(*;\underline{\mathbb{Z}}) \cong \underline{\mathbb{Z}}^{op}$; when $n \leq 0$, we have $H^{2n,-2n}_{C_2}(*;\underline{\mathbb{Z}}) \cong \underline{\mathbb{Z}}$. We now have the following:

Theorem 3.12. The $RO(C_2)$-graded $H\underline{\mathbb{Z}}$ cohomology of a point is given by

$$H^{p,q}(*;\underline{\mathbb{Z}}) = \begin{cases} \underline{\mathbb{Z}}^{op} & 0 < p = -q \text{ and } p \text{ even} \\ \underline{\mathbb{Z}} & p = -q \leq 0 \text{ and } p \text{ even} \\ \underline{\mathbb{Z}}_{-}^{op} & 1 < p = -q \text{ and } p \text{ odd} \\ \underline{\mathbb{Z}}_{-} & p = -q \leq 1 \text{ and } p \text{ odd} \\ \langle \mathbb{Z}/2 \rangle & 1 < p < -q \text{ and } p \text{ odd} \\ \langle \mathbb{Z}/2 \rangle & 0 \leq -p < q \text{ and } p \text{ even} \\ 0 & \text{otherwise} \end{cases}$$

3.2.2 Computing $H^{p,q}_{C_2}(*;A_{C_2})$

The results are largely the same when will replace the constant Mackey functor $\underline{\mathbb{Z}}$ with the Burnside Mackey functor A_{C_2}. However, we will streamline the computation a bit by considering LES of Mackey functors instead of abelian groups. Again, we will reduce to computing the reduced cohomology of S^0

$$H^{p,q}_{C_2}(*;A_{C_2}) \cong \tilde{H}^{p,q}_{C_2}(S^0;A_{C_2})$$

From now on, we will suppress the coefficient A_{C_2} unless otherwise specified. Our primary tool is still the cofiber sequence

$$S(\sigma)_+ \cong C_{2+} \to S^0 \to S^\sigma$$

So the first step is understanding $\tilde{H}^{p,q}_{C_2}(C_{2+})$. At C_2/C_2 level, we have

$$\tilde{H}^{p,q}_{C_2}(C_{2+}) := [C_{2+}, \Sigma^{p+q\sigma}_{C_2} HA_{C_2}]_{C_2} = [S^0, \Sigma^{p+q\sigma}_{C_2} HA_{C_2}]_e = \tilde{H}^{|p+q|}(S^0;\mathbb{Z})$$

Using the same adjunction, we see the C_2/e level will be

$$\tilde{H}^{|p+q|}(C_{2+};\mathbb{Z}) \cong \tilde{H}^{|p+q|}(S^0;\mathbb{Z}) \oplus \tilde{H}^{|p+q|}(S^0;\mathbb{Z})$$

with the C_2-Weyl action. The restriction and transfer are the obvious diagonal and fold maps. As a result, we get

Proposition 3.13. When $p+q=0$, the Mackey functor $\tilde{H}^{p,q}_{C_2}(C_{2_+})$ is represented by the following diagram

$$\begin{tikzcd} \mathbb{Z} \arrow[d, bend right, "\Delta"'] \\ \mathbb{Z}\oplus\mathbb{Z} \arrow[u, bend right, "\nabla"'] \arrow[loop below, "swap"] \end{tikzcd}$$

where Δ is the diagonal map and ∇ is the fold map.

As a result, we know $\tilde{H}^{p,q}(C_{2_+})$ is the zero Mackey functor if $p+q\neq 0$. Then, we can apply the long exact sequence

$$\ldots \longrightarrow \tilde{H}^{p-1,q}_{C_2}(C_{2_+}) \longrightarrow \tilde{H}^{p,q}_{C_2}(S^\sigma) \longrightarrow \tilde{H}^{p,q}_{C_2}(S^0) \longrightarrow \tilde{H}^{p,q}_{C_2}(C_{2_+}) \longrightarrow \ldots$$

Thus, if $p+q\neq 0,1$, we have the isomorphism

$$\tilde{H}^{p,q-1}_{C_2}(S^0) \cong \tilde{H}^{p,q}_{C_2}(S^\sigma) \cong \tilde{H}^{p,q}_{C_2}(S^0)$$

If we plot the Mackey functors onto the plane, where X-axis represents index p, and Y-axis represents index $p+q$, then every lattice point not on $y=0,1$ is isomorphic to the lattice point below it. Moreover, by dimension axiom, we have $\tilde{H}^{p,0}_{C_2}(S^0)=0$ if $p\neq 0$. Thus, all lattice points on the diagonal will be 0, and by the previous observation, all of the 1st and 3rd quadrant will be zero Mackey functor, as depicted in Figure **??**.

For the rest of the missing pieces, it suffices to compute the values at $p+q=0,1,-1$. These computations can be done again using the cofiber sequence

$$S(q\sigma)_+ \to D(q\sigma)_+ \to S^{q\sigma}$$

As a result, we know $\tilde{H}^{p,q}(C_{2_+})$ is the zero Mackey functor if $p+q\neq 0$. Then, we can apply the long exact sequence

$$\ldots \longrightarrow \tilde{H}^{p-1,q}_{C_2}(C_{2_+}) \longrightarrow \tilde{H}^{p,q}_{C_2}(S^\sigma) \longrightarrow \tilde{H}^{p,q}_{C_2}(S^0) \longrightarrow \tilde{H}^{p,q}_{C_2}(C_{2_+}) \longrightarrow \ldots$$

Thus, if $p+q\neq 0,1$, we have the isomorphism

$$\tilde{H}^{p,q-1}_{C_2}(S^0) \cong \tilde{H}^{p,q}_{C_2}(S^\sigma) \cong \tilde{H}^{p,q}_{C_2}(S^0)$$

If we plot the Mackey functors onto the plane, where X-axis represents index p, and Y-axis represents index $p+q$, then every lattice point not on $y=0,1$ is isomorphic to the lattice point below it. Moreover, by dimension axiom, we have $\tilde{H}^{p,0}_{C_2}(S^0)=0$ if $p\neq 0$. Thus, all lattice points on the diagonal will be 0, and by the previous observation, all of the 1st and 3rd quadrant will be zero Mackey functor, as depicted in Figure 3.

For the rest of the missing pieces, it suffices to compute the values at $p+q=0,1,-1$. These computations can be done again using the cofiber sequence

$$S(q\sigma)_+ \to D(q\sigma)_+ \to S^{q\sigma}$$

Proposition 3.14. If $p \geq 0$ and $p+q=-1$, then

$$\tilde{H}^{p,q}_{C_2}(S^0) \cong \begin{cases} \mathbb{Z} & p=0 \\ \mathbb{Z}/2 & p \text{ odd and greater or equal to } 3 \\ 0 & \text{otherwise} \end{cases}$$

as abelian groups.

Proof. Since $p \geq 0$ and $p+q=-1$, then

$$\tilde{H}^{p,q}_{C_2}(S^0) \cong \tilde{H}^{p,-p-1}_{C_2}(S^0) \cong \tilde{H}^{p,0}_{C_2}(S^{(p+1)\sigma})$$

When $p \geq 2$, the LES gives the isomorphism

$$\tilde{H}^{p,0}_{C_2}(S^{(p+1)\sigma}) \cong \tilde{H}^{p-1,0}_{C_2}(S((p+1)\sigma))$$

Again by Proposition 1.39, $\tilde{H}^{p-1,0}_{C_2}(S(p+1)\sigma) \cong \tilde{H}^{p-1}(\mathbb{RP}^p;\mathbb{Z})$, which is $\mathbb{Z}/2$ iff p is odd. When $p=0$, we open up the LES

$$\tilde{H}^{0,0}_{C_2}(S^\sigma) \to \tilde{H}^{0,0}_{C_2}(D(\sigma)_+) \to \tilde{H}^{0,0}_{C_2}(S(\sigma)_+)$$

where the second arrow is identified as the restriction map in the Burnside Mackey functor, so $\tilde{H}^{0,0}_{C_2}(S^\sigma) \cong \mathbb{Z}$ [6]. For $p=1$, we have done the explicit Bredon cohomology computation in section 1.5.2 to conclude that $\tilde{H}^{1,0}_{C_2}(S^{2\sigma})=0$. $\square$

Proposition 3.15. If $p \leq 0$ and $p+q=1$, then

$$\tilde{H}^{p,q}_{C_2}(S^0) \cong \begin{cases} \mathbb{Z} & p=0 \\ \mathbb{Z}/2 & p \text{ even and less or equal to } -2 \\ 0 & \text{otherwise} \end{cases}$$

as abelian groups.

Proof. The argument is entirely symmetric to that of the previous proposition by first using duality to turn cohomology into homology, and then apply the LES to the same cofiber sequence. $\square$

To determine the Mackey functor structure, we note

Proposition 3.16. When $p+q=-1$, we have

$$\tilde{H}^{p,q}_{C_2}(S^0) \cong \tilde{H}^{p,q+1}_{C_2}(S^\sigma) \cong ker(\tilde{H}^{p,q+1}_{C_2}(S^0) \to \tilde{H}^{p,q+1}_{C_2}(C_{2+}))$$

as Mackey functors. When $p+q=1$, we have

$$\tilde{H}^{p,q}_{C_2}(S^0) \cong \tilde{H}^{C_2}_{-p,-q}(S^0) \cong coker(H^{C_2}_{-p,1-q}(C_{2+}) \to H^{C_2}_{-p,1-q}(S^0))$$

as Mackey functors

[6] This is essentially the only difference from the constant Mackey functor case.

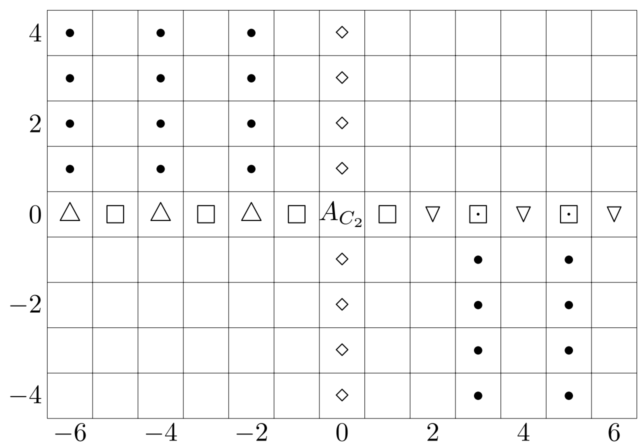


Figure 3: $\bullet = \langle \mathbb{Z}/2 \rangle$, $\diamond = \langle \mathbb{Z} \rangle$, $\triangle = \underline{\mathbb{Z}}$, $\triangledown = \underline{\mathbb{Z}}^{op}$, $\square = \underline{\mathbb{Z}}_-$, $\boxdot = \underline{\mathbb{Z}}_-^{op}$

Proof. The first part follows directly from the cohomology LES of the cofiber sequence; the second part follows from the homology LES after applying Spanier-Whitehead duality. $\square$

By adjunction,

$$\tilde{H}^{p,q+1}_{C_2}(S^0) \to \tilde{H}^{p,q+1}_{C_2}(C_{2+})$$

has trivial kernel at C_2/e level since the cohomology reduces to ordinary nonequivariant cohomology; similarly,

$$H^{C_2}_{-p,1-q}(C_{2+}) \to H^{C_2}_{-p,1-q}(S^0)$$

has trivial cokernel at C_2/e. Thus, the Mackey functors are all of the form $\langle \mathbb{Z} \rangle$ or $\langle \mathbb{Z}/2 \rangle$.

We are left with what happens when $p + q = 0$. The usual strategy as in the constant Mackey functor case works: for $p > 0$, we compute $\tilde{H}^{p,-p}_{C_2}(S^0) = \tilde{H}^{p,0}_{C_2}(S^{p\sigma})$ via the cofiber sequence

$$S(p\sigma)_+ \to S^0 \to S^{p\sigma}$$

The LES is

$$\tilde{H}^{p-1,0}_{C_2}(S^0) \to \tilde{H}^{p-1,0}_{C_2}(S(p\sigma)_+) \to \tilde{H}^{p,0}_{C_2}(S^{p\sigma}) \to \tilde{H}^{p,0}_{C_2}(S^0)$$

when $p > 1$, we have

$$\tilde{H}^{p-1,0}_{C_2}(S(p\sigma)_+) \cong \tilde{H}^{p,0}_{C_2}(S^{p\sigma})$$

Proposition 3.17. For $p > 1$, the Mackey functor $\tilde{H}^{p,-p}_{C_2}(S^0)$ is $\underline{\mathbb{Z}}_-^{op}$ when p is odd, and $\underline{\mathbb{Z}}^{op}$ when p is even.

Proof. The C_2/e level are going to be $\mathbb{Z}$ by usual adjunction; At C_2/C_2, the abelian group $\tilde{H}^{p-1,0}_{C_2}(S(p\sigma)_+)$ is isomorphic to $\tilde{H}^{p-1}(\mathbb{RP}^{p-1};\mathbb{Z})$ since the action is free. When p is odd, the group is $\mathbb{Z}/2$ and the Mackey functor structure $\underline{\mathbb{Z}}_-^{op}$ is forced; when p is even, the restriction map is determined by

$$\tilde{H}^{2p,-2p}_{C_2}(S^\sigma) \to \tilde{H}^{2p,-2p}_{C_2}(S^0) \cong \mathbb{Z} \xrightarrow{\text{Res}} \tilde{H}^{2p,-2p}_{C_2}(C_{2+}) \cong \mathbb{Z} \to \tilde{H}^{2p+1,-2p}_{C_2}(S^\sigma)$$

The group on the left is zero, and the group on the right is $\mathbb{Z}/2$. Thus, middle map is multiplication by 2 and the Mackey functor structure of $\tilde{H}^{2p,-2p}_{C_2}(S^0)$ is $\underline{\mathbb{Z}}^{op}$. $\square$

When $p = 1$, we see that $\tilde{H}^{1,0}_{C_2}(S^\sigma)$ is the cokernel of the canonical morphism $\tilde{H}^{0,0}_{C_2}(S^0) \to \tilde{H}^{0,0}_{C_2}(C_{2+})$

$$\begin{tikzcd}
\mathbb{Z}x \oplus \mathbb{Z}y \arrow[r, "x+2y"] \arrow[d, bend right] & \mathbb{Z} \arrow[r, dashed] \arrow[d, bend right] & 0 \arrow[d, bend right] \\
\mathbb{Z} \arrow[r, "(0,Id)"] \arrow[u, bend right] & \mathbb{Z} \oplus \mathbb{Z} \arrow[r, dashed] \arrow[u, bend right] & \mathbb{Z} \arrow[u, bend right]
\end{tikzcd}$$

which is $\underline{\mathbb{Z}}_-$. For $p < 0$ we use Spanier-Whitehead duality. Let $q = -p > 1$, the same procedure yields

$$\tilde{H}^{C_2}_{q-1,0}(S(q\sigma)_+) \cong \tilde{H}^{C_2}_{q,0}(S^{q\sigma})$$

and

Proposition 3.18. For $q > 1$, the Mackey functor $\tilde{H}^{C_2}_{q-1,0}(S(q\sigma)_+)$ is $\underline{\mathbb{Z}}_-$ when q is odd, and $\underline{\mathbb{Z}}$ when q is even.

For $q = 1$, $\tilde{H}^{1,0}_{C_2}(S^\sigma)$ is the kernel of the canonical morphism $\tilde{H}^{C_2}_{0,0}(C_{2+}) \to \tilde{H}^{C_2}_{0,0}(S^0)$

$$\begin{tikzcd}
0 \arrow[r, dashed] \arrow[d, bend right] & \mathbb{Z} \arrow[r, "(0,Id)"] \arrow[d, bend right] & \mathbb{Z} \oplus \mathbb{Z} \arrow[d, bend right] \\
\mathbb{Z} \arrow[r, dashed] \arrow[u, bend right] & \mathbb{Z}x \oplus \mathbb{Z}y \arrow[r, "x+2y"] \arrow[u, bend right] & \mathbb{Z} \arrow[u, bend right]
\end{tikzcd}$$

which is also $\underline{\mathbb{Z}}_-$. We can conclude:

Theorem 3.19. The $RO(C_2)$-graded HA_{C_2} cohomology of a point is given by

$$H^{p,q}(*; A_{C_2}) = \begin{cases} \underline{\mathbb{Z}}^{op} & 0 < p = -q \text{ and } p \text{ even} \\ \underline{\mathbb{Z}} & p = -q < 0 \text{ and } p \text{ even} \\ \underline{\mathbb{Z}}_-^{op} & 1 < p = -q \text{ and } p \text{ odd} \\ \underline{\mathbb{Z}}_- & p = -q \leq 1 \text{ and } p \text{ odd} \\ \langle \mathbb{Z}/2 \rangle & 1 < p < -q \text{ and } p \text{ odd} \\ \langle \mathbb{Z}/2 \rangle & 0 \leq -p < q \text{ and } p \text{ even} \\ \langle \mathbb{Z} \rangle & p = 0 \text{ and } q \neq 0 \\ A_{C_2} & p = q = 0 \\ 0 & \text{otherwise} \end{cases}$$

4 The Slice Spectral Sequence

The goal is to produce a spectral sequence to compute the equivariant homotopy groups. The motivation comes from the usual Postnikov tower for spaces/spectra: suppose we have a Postnikov filtration of a spectrum E

$$\begin{tikzcd}
 & P^{n+1}_{n+1}E \arrow[d, hook] & P^n_n E \arrow[d, hook] & P^{n-1}_{n-1}E \arrow[d, hook] & \\
... \arrow[r] & P^{n+1}E \arrow[r] & P^n E \arrow[r] & P^{n-1}E \arrow[r] & ...
\end{tikzcd}$$

where

1. $X \cong \text{holim}_n P^n E$
2. The **n-th Postnikov section** $P^n E$ satisfies
$$\pi_k(P^n E) = \begin{cases} \pi_k(E) & k \leq n \\ 0 & k > n \end{cases}$$
3. Each morphism $P^n X \to P^{n-1}E$ is a fibration, and the fiber $P^n_n E$ satisfies
$$\pi_k(P^n_n E) = \begin{cases} \pi_n(E) & k = n \\ 0 & \text{otherwise} \end{cases}$$

   In other words, the fibers are shifted Eilenberg-Maclane spectra.

It is easy to verify that the fiber of the morphism $X \to P^n E$, denoted as $P_{n+1}E$, is the n-connected cover of E. Recall that the n-th Postnikov section of a space/spectrum can be realized as the left Bousfield localization with respect to the single morphism

$$f\colon S^{n+1} \to *$$

X is f-local iff $\Omega^{n+1}X$ is contractible iff $\pi_k(X) = 0$ for $k > n$. Effectively, the procedure kills all maps coming from S^k for $k > n$, giving us a functor $P^n\colon \mathbf{Top}_* \to \mathbf{Top}_*^{\leq n}$, where $\mathbf{Top}_*^{\leq n}$ denote the subcategory of spaces with vanishing homotopy group above dimension n.

The Postnikov filtration induces the Atiyah-Hirzebruch spectral sequence in the following way: for a space or spectrum X, we have the fiber sequence

$$X \wedge P^n_n E \to X \wedge P^n E \to X \wedge P^{n-1}E$$

and taking the homotopy groups gives us the exact couple

$$\begin{tikzcd}
\oplus\pi_k(X \wedge P^n E) \arrow[rr] & & \oplus\pi_k(X \wedge P^n E) \arrow[dl] \\
 & \oplus\pi_k(X \wedge P^n_n E) \arrow[ul] &
\end{tikzcd}$$

and we identify $\oplus\pi_k(X \wedge P^n_n E) \cong \oplus H_{k-n}(X, \pi_n(E))$. Take $q = n$ and $p = k - n$, the exact couple then gives us the convergent spectral sequence

$$E^2_{p,q} := H_p(X, \pi_q(E)) \Rightarrow E_{p+q}(X)$$

which is the usual Atiyah-Hirzebruch spectral sequence. Motivated by Voevodsky's work on motivic cohomology, Dugger in [Dug05] first developed the equivariant version of the Postnikov tower and the associated slice spectral sequence, which was later further developed and used by HHR in [HHR16].

For the following discussion, we will use the regular slice cells as developed by Ullman [Ull13] and in HHR's book [HHR21]. We suppress the Σ_G^∞ notation when the context is clear.

4.1 Localizing Subcategory and Slice Connectivity

The naive filtration by cells of the form $G/H_+ \wedge S^n$ does not work, since it will give us a spectral sequence with non-computable input. Motivated by [Dug05], [HHR16] devised a new filtration, together with a new notion of connectivity that made the spectral sequence computable in their case of need. The idea of a localizing subcategory imitates the subcategory of n-connected spaces, and its complement will imitate the subcategory of n-coconnected spaces.

Definition 4.1. A subcategory τ of a topological model category M is a **localizing subcategory** if it satisfies the following:

1. closed under arbitrary coproducts;
2. closed under weak equivalence;
3. closed under cofibers and extensions: given a cofiber sequence

$$X \to Y \to Z$$

with $X \in \tau$, then Y is in τ iff Z is in τ.

We say a localizing subcategory is **generated by** a set of objects T if it is the smallest localizing subcategory containing T.

Proposition 4.2. Given a localizing subcategory τ, and suppose it is generated by a set of objects $T = \{T_\alpha\}$. Then, the generating set can be replaced by the singleton consisting the object

$$\bigvee_\alpha T_\alpha$$

Proof. Clearly τ contains the localizing subcategory generated by $\bigvee_\alpha T_\alpha$ by the first axiom; conversely, each generating object T_β is the cofiber of the morphism

$$\bigvee_\alpha T_\alpha \to \bigvee_\alpha T_\alpha$$

by collapsing T_β to the basepoint and the identity map on other wedge summands.

□

Definition 4.3. Given a localizing subcategory τ of M, define its **complement**, denoted $\tau^\perp$, as the full subcategory of M with objects Y such that $M(X, Y)$ is contractible for all $X \in \tau$.

Example 4.4. If $M = \mathbf{Top}_*$, the the subcategory of n-connected spaces $\mathbf{Top}_*^{>n}$ is generated by S^{n+1} as a localizing subcategory: axioms 1 and 3 allows us to build all CW complexes with cells of dimension greater or equal to $n+1$, and axiom 2 gives us the CW approximation to get the entire subcategory. The complement of $\mathbf{Top}_*^{>n}$ is the subcategory $\mathbf{Top}_*^{\leq n}$.

The slice cells will generate the localizing subcategory from which we will define a new version of connectivity and Postnikov tower in the equivariant setting.

Definition 4.5. For a subgroup $H \leq G$, let ρ_H to denote its regular representation. The (regular) **slice cells** are spectra of the form

$$\widehat{S}(m, H) = G_+ \wedge_H S^{m\rho_H}$$

for $m \in \mathbb{Z}$ and $H \leq G$. The **dimension** of a cell $\widehat{S}(m, H)$ is defined the be $m|H|$, the dimension of the underlying sphere.

Definition 4.6. A G-spectrum X is called **slice n-coconnected**, written as $X < n$, if for every slice sphere $\widehat{S}$ with $dim(\widehat{S}) \geq n$, the G-space $\mathrm{Maps}(\widehat{S}, X)$ is contractible. A G-spectrum Y is called **slice n-connected**, written as $Y > n$, if is belongs to the localizing subcategory generated by the set

$$\{\widehat{S}(m, H) : dim(\widehat{S}(m, H) > n)\}$$

We denote the full subcategory of slice n-coconnected G-spectra as $\mathbf{Sp}^G_{<n}$, and the full subcategory of slice n-connected G-spectra as $\mathbf{Sp}^G_{>n}$.

Remark 4.7. The definition is different from the original in [HHR16]: the dimension is shifted by 1.

It is readily verifiable that Y being slice n-connected is the same as $\mathbf{Sp}_G(Y, X)$ being contractible for all X in $\mathbf{Sp}^G_{<n}$.

Directly from definition, we see that the restriction and induction functors are compatible with slice connectivity, which aids in inductive arguments.

Proposition 4.8. Suppose $H \leq G$, X is a G-spectrum and Y is an H-spectrum. Then,

$$X < n \Leftrightarrow \mathrm{Res}^G_H < n$$

$$X > n \Leftrightarrow \mathrm{Res}^G_H > n$$

$$Y > n \Leftrightarrow G_+ \wedge_H Y > n$$

$$Y < n \Leftrightarrow G_+ \wedge_H Y < n$$

We observe some relationships between slice connectivity and regular connectivity of spectra:

Proposition 4.9. If $n \geq 0$, then $G_+ \wedge_H S^n$ is slice $(n-1)$-connected; if $n < 0$, then $G_+ \wedge_H S^n$ is slice $(n|G| - 1)$-connected.

Proof. For the first part, we induct on the order of $|G|$. The case is trivial when $|G| = 1$; Using Proposition 4.8, the inductive hypothesis implies that $G/H_+ \wedge S^n \geq n$ when H is a proper subgroup. From the fiber sequence

$$S(n\rho - n)_+ \to S^0 \to S^{n\rho - n}$$

we may suspend by S^n and get

$$S(n\rho - 1)_+ \wedge S^n \to S^n \to S^{n\rho}$$

Since we know $S(n\rho-1)_+ \wedge S^n$ can be decomposed into cells of the form $G/H_+ \wedge S^k$, where H is proper and $k \geq n$, it is also slice $(n-1)$-connected. By extension property, the middle term S^n is slice $n-1$-connected.

For the second part, we may write

$$G/H_+ \wedge S^{-n} = (G/H_+ \wedge S^{n\rho-n}) \wedge S^{-n\rho}$$

and we just showed that $G/H_+ \wedge S^{n\rho-n} \geq 0$, so that $G/H_+ \wedge S^{-n} \geq -|n||G|$ by [HHR21] Proposition 11.1.23 ☐

Corollary 4.10. For a G-spectrum X, it is slice (-1)-connected iff it is (-1)-connected. Equivalently, X is slice 0-coconnected iff it is 0-coconnected.

Proof. First, we observe every generator $\widehat{S}(m,H)$ in $\mathbf{Sp}^G_{>(-1)}$ is 0-connected if $m>0$: since homotopy group Mackey functor commutes with induction, it suffices to show that $S^{m\rho_H}$ is 0-connected for every subgroup H. Since the regular representation contains a copy of the trivial representation, we are able to write

$$S^{m\rho_H} \cong \Sigma^m S^{m\rho_H - m}$$

and $S^{m\rho_H-m}$ is an honest suspension spectrum of a G-CW complex. Thus,

$$\underline{\pi_k}(S^{m\rho_H}) \cong \underline{\pi_{k-m}}(\Sigma^m S^{m\rho_H-m})$$

which is trivial if $k \leq 0$. Thus, the only generator that is potentially not 0-connected is $\widehat{S}(0,H)$, which is (-1)-connected. Thus, all spectra in the localizing subcategory are (-1)-connected.

Conversely, $G \wedge_H S^0$ is a generator, so the localizating subcategory also contains all (-1)-connected G-CW spectra and thus all (-1)-connected spectra.

☐

Corollary 4.11. For a G-spectrum X, it is slice 0-connected iff it is 0-connected. Equivalently, X is slice 1-coconnected iff it is 1-coconnected;

Proof. The proof is similar to that of previous corollary, except now we have to show that $G \wedge_H S^1$ is a generator in $\mathbf{Sp}^G_{>0}$. This follows from Proposition 4.9. ☐

Proposition 4.12. For $n \geq 0$, Y being slice n-connected implies $\lceil \frac{n}{|G|} \rceil$-connected; for $n<0$, Y being slice n-connected implies n-connected.

Proof. A cell decomposition of S^{ρ_G} is given by G-cells of dimension 1 to $|G|$; thus, for $n \geq 0$, a slice sphere $G/H_+ \wedge S^{n\rho_G}$ can be decomposed into cells of the form $G/H_+ \wedge S^k$ with $n \leq k \leq n|G|$. The class of spectra that can be decomposed in $G/H_+ \wedge S^m$ for $m \geq \lceil \frac{n}{|G|} \rceil$ is a localizing subcategory that contains slice spheres of dimension $\geq n$. Thus, every slice n-connected spectrum can be decomposed into cells that are $\lceil \frac{n}{|G|} \rceil$-connected, which implies $\underline{\pi_k}(Y) = 0$ for $k \leq \lceil \frac{n}{|G|} \rceil$. When $n<0$, Spanier-Whitehead duality provides a cell decomposition of $S^{n\rho}$ whose dimension ranges from $n|G|$ to n. The rest of the argument follows similarly. ☐

We have an example of slice connectivity being different from regular connectivity:

Example 4.13. Consider the fiber sequence

$$F_f \to S^1 \xrightarrow{f} \Sigma H\underline{\mathbb{Z}}$$

The the homotopy group Mackey functor $\underline{\pi_1}(S^1)$ is the Burnside Mackey functor, which is given by the tom Dieck splitting. Thus, $\underline{\pi_1}(F_f)$ is the augmentation ideal of the Burnside Mackey functor and thus non-trivial. Therefore, F_f is not 1-connected. However, we will later show in Example 4.28 that F_f is slice 1-connected.

4.2 The Slice Tower and the Slice Spectral Sequence

The construction of the slice tower is as follows: let P^n be the Bousfield localization of $\mathbf{Sp}^G$ with respect to $\mathbf{Sp}^G_{>n}$. Then, let P_{n+1} be the homotopy fiber of canonical map $X \to P^n X$, and we have the fiber sequence

$$P_{n+1}X \to X \to P^n X$$

Since $\mathbf{Sp}^G_{>n} \subset \mathbf{Sp}^G_{>n-1}$, there is natural transformation between functors $P^n \to P^{n-1}$.

The following definitions are analogous to the Postnikov tower construction:

Definition 4.14. Given a G-spectrum X, the spectrum $P^n X$ is the **nth slice section** of X, and $P_{n+1}X$ is the **slice n-connected cover** of X. The **n-th slice** of X, denoted $P^n_n X$, is the fiber of the map $P^n X \to P^{n-1}x$. The **slice tower** of X is the tower of fibrations

$$\begin{tikzcd}
 & P^{n+1}_{n+1}X \arrow[d, hook] & P^n_n X \arrow[d, hook] & P^{n-1}_{n-1}X \arrow[d, hook] & \\
\ldots \arrow[r] & P^{n+1}X \arrow[r] & P^n X \arrow[r] & P^{n-1}X \arrow[r] & \ldots
\end{tikzcd}$$

There are a few things we want the tower to satisfy:

1. $P^n X$ is slice $n+1$-coconnected;
2. $P_{n+1}X$ is slice n-connected;
3. $\mathrm{hocolim}_n P^n X$ is weakly contractible;
4. $\mathrm{holim}_n P^n X \cong X$.

Property $3, 4$ and the general properties of the spectral sequence associated to a tower of fibrations will give us the strongly convergent spectral sequence

$$E_2^{t-s,t} : \pi^G_{t-s} P^t_t X \Rightarrow \pi^G_{t-s} X$$

which is the **slice spectral sequence**. The spectral sequence can be intepreted as a spectral sequence of abelian groups or Mackey functors; the grading t can also be promoted into a $RO(G)$-grading. For the remainder of the subsection, we will prove the four aforementioned properties of the slice tower.

The construction of $P^n X$ is similar to the classical Postnikov tower: let $W_0 X = X$, and define $W_k X$ to be the pushout

$$\begin{tikzcd}
\bigvee_{L_k} \widehat{S} \arrow[r] \arrow[d] & W_{k-1}X \arrow[d] \\
\bigvee_{L_k} C\widehat{S} \arrow[r] & W_k X
\end{tikzcd}$$

where L_k is the set of all maps from a slice sphere $\widehat{S}$ of dimension $> n$ to $\mathrm{colim}_k W_{k-1}X$. It follows that

$$[\widehat{S}, \mathrm{colim}_k W_k X]_G = 0$$

Define $P^n X = \mathrm{colim}_k W_k X$. To see this implies $\mathbf{Sp}_G(\widehat{S}, P^n X)$ being weakly contractible, see [HHR21] Lemma 11.1.15. From the construction, it follows that

Proposition 4.15. There is a functorial fiber sequence

$$P_{n+1}X \to X \to P^n X$$

with $P_{n+1}X$ slice n-connected and $P^n X$ slice $(n+1)$-coconnected.

We can determine the connectivity of the slice section:

Theorem 4.16. Let X be a G-spectrum. Then, $X \to P^n X$ induces an isomorphism on $\underline{\pi_k}$ for

$$\begin{cases} k < \lceil \frac{n+1}{|G|} \rceil & n \geq 0 \\ k < n+1 & n < 0 \end{cases}$$

Moreover, $\underline{\pi_k} P^n(X) = 0$ for

$$\begin{cases} k \geq n+1 & n \geq 0 \\ k \geq \lceil \frac{n+1}{|G|} \rceil & n < 0 \end{cases}$$

Proof. We look at the LES induced by the fiber sequence in Proposition 4.15

$$\ldots \longrightarrow \underline{\pi_k} P_{n+1}X \longrightarrow \underline{\pi_k} X \longrightarrow \underline{\pi_k} P^n X \longrightarrow \underline{\pi_{k-1}} P_{n+1}X \longrightarrow \ldots$$

Then, the first part of the theorem follows directly from the vanishing $\pi_k P_{n+1}X$ by combining the fact that $P_{n+1}X$ is slice n-connected and Proposition 4.12. The second part of the theorem follows the slice connectivity of $G/H_+ \wedge S^k$ from Proposition 4.9. ☐

Corollary 4.17. The homotopy colimit of the slice tower $\{P^n X\}$ is weakly contractible, and the homotopy limit is weakly equivalent to X.

We have a useful lemma that utilizes ordinary connectivity to show slice connectivity. The proof of the lemma is in [HHR21] Proposition 11.3.3.

Lemma 4.18. Let k be an integer not divisible by any prime factor of $|G|$. Suppose X is slice k connected, and the non-equivariant spectrum $\mathrm{Res}^G_e X$ is $(k+1)$-connected, then X is slice $(k+1)$-connected.

4.3 Identifying the Slices

In general, it is very difficult to determine the slices of a given spectrum. However, we may determine some lower slices and relations between slices, which will already let us compute the slice spectral sequence for $K\mathbb{R}$.

Definition 4.19. A spectrum X is an **n-slice** if

$$X \cong P_n^n X$$

Definition 4.20. A map of spectrum $f : X \to Y$ is a **P^n-equivalence** if $P^n X \to P^n Y$ is a weak equivalence.

It is immediate that a P^n-equivalence induces a weak equivalence between n-slices; moreover, the LES to the fiber sequence

$$\mathbf{Sp}_G(Y, Z) \to \mathbf{Sp}_G(X, Z) \to \mathbf{Sp}_G(\mathrm{Fib}(f), Z)$$

implies the following:

Lemma 4.21. Suppose the fiber of $X \to Y$ is slice n-connected; then $X \to Y$ is a slice P^n-equivalence.

Proposition 4.22. If X is an n-slice, then $\underline{\pi_k}(Y) = 0$ unless

$$\begin{cases} \lceil \frac{n}{|G|} \rceil \leq k \leq n & n \geq 0 \\ n \leq k \leq \lceil \frac{n+1}{|G|} \rceil & n < 0 \end{cases}$$

Proof. The claim follows from a direct computation using Theorem 4.16. $\square$

Corollary 4.23. For any spectrum X,

$$P_0^0 X = H\underline{\pi_0} X$$

and

$$P_1^1 X = \Sigma H\underline{M}$$

for some Mackey functor $\underline{M}$.

Proof. Proposition 4.22 implies that the homotopy group Mackey functors of $P_0^0 X$ and $P_1^1 X$ are concentrated in degree 0 and 1, respectively. Moreover, the LES gives us that $\underline{\pi_0} P_0^0 X \cong \underline{\pi_0} P^0(X) \cong \underline{\pi_0} X$. $\square$

We will improve the identification of $P_1^1 X$ later on.

Proposition 4.24. A spectrum X is an n-slice iff $\Sigma^{\rho_G} X$ is an $(n + |G|)$-slice. As a result, we have

$$P_{n+|G|}^{n+|G|} \Sigma^{\rho_G} X \cong \Sigma^{\rho_G} P_n^n X$$

Proof. The result follows from the fact that smashing with S^{ρ_G} is an equivalence of categories

$$(-) \wedge S^{\rho_G} : \mathbf{Sp}^G_{>n} \to \mathbf{Sp}^G_{>n+|G|}$$

The proof of the fact is given in [HHR21] Proposition 11.1.23. □

We will use the remainder of the subsection to identify the 1-slices:

Theorem 4.25. A spectrum Y is a 1-slice iff it is of the form $\Sigma H\underline{M}$ for some Mackey functor $\underline{M}$ whose restriction maps are injective.

Proof. For the forward direction, suppose we have subgroups $K \leq H \leq G$. Then, for a map $f : G/K \to G/H$, the cofiber Cf is a wedge of circles whose single suspension is slice 1-connected. Thus, $[\Sigma Cf, \Sigma H\underline{M}] = [X, H\underline{M}] = 0$ by the assumption that $\Sigma H\underline{M}$ is a 1-slice and thus slice 2-coconnected. Then, the LES of the cofiber sequence implies

$$\underline{M}(G/H) \cong [G/H_+, H\underline{M}] \xrightarrow{\text{Res}} [G/K_+, H\underline{M}] \cong \underline{M}(G/K)$$

is injective.

For the converse, suppose $\underline{M}$ is a Mackey functor whose restriction maps are injective. Then, $\Sigma H\underline{M}$ fits into the cofiber sequence

$$P_2\Sigma H\underline{M} \longrightarrow \Sigma H\underline{M} \longrightarrow P^1\Sigma H\underline{M}$$

It is clear that $P_2\Sigma H\underline{M}$ is a shifted Eilenberg-Maclane spectrum, with homotopy groups concentrated in degree 1. It suffices to show that $\underline{\pi_1}(P_2\Sigma H\underline{M})(S) = 0$ for every finite G-set S. Let M_1, M_2, M_3 be the $\underline{\pi_1}$ of the three spectrum, such that there is a short exact sequence of Mackey functors

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$$

One observes that every restriction map of a Mackey functor $\underline{N}$ being a monomorphism is equivalent to the criterion that $N(S) \to N(G \times S)$, induced by the action $G \times S \to S$ for every finite G-set S, is injective. In particular, in the diagram

$$\begin{tikzcd}
0 \arrow[r] & M_1(S) \arrow[r] \arrow[d] & M_2(S) \arrow[r] \arrow[d] & M_3(S) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M_1(G \times S) \arrow[r] & M_2(G \times S) \arrow[r] & M_3(G \times S) \arrow[r] & 0
\end{tikzcd}$$

the middle map is a monomorphism by assumption; the bottom right map is can be identified as the map on π_1 induced by $\text{Res}^G_e X \to \text{Res}^G_e P^1 X$, which is an isomorphism. Thus by diagram chasing, we see that the leftmost map is a monomorphism, with $M_1(G \times S) = 0$. Thus, $M_1(S) = 0$ for every finite G-set S, as desired.

□

Corollary 4.26. Given a Mackey functor $\underline{M}$, define $\underline{M'}$ to be

$$\underline{M'}(G/H) := ker(\text{Res} : \underline{M}(G/H) \to \underline{M}(G/e))$$

and let $M'' := M/M''$. Then, $P^1\Sigma H\underline{M} \cong \Sigma H\underline{M}''$

Proof. By construction, $\Sigma H\underline{M}''$ is a 1-slice by the previous proposition; moreover, we have the fiber sequence

$$\Sigma H\underline{M}' \longrightarrow \Sigma H\underline{M} \longrightarrow \Sigma H\underline{M}''$$

where $\Sigma H\underline{M}'$ clearly 0-connected by the LES and slice 0-connected by Proposition 4.16. Moreover, $M'(G/e) = 0$ by construction, so that it satisfies the hypothesis of Lemma 4.18. Thus, the slice 1-connectedness of the fiber implies $\Sigma H\underline{M} \to \Sigma H\underline{M}''$ is a P^1-equivalence, and $P^1_1\Sigma\underline{M} \cong \Sigma H\underline{M}''$. ☐

Corollary 4.27. For any spectrum X, its 1-slice is given by

$$P^1_1 X \cong \Sigma H\underline{\pi''_1} X$$

where $\underline{\pi''_1}X$ is constructed in the same way as $\underline{M}''$ the corollary above.

Proof. First, we note that X and P_1X have the same 1-slice: P^0P_1X is clearly weakly contractible, so the 1-slice of P_1X is just $P^1P_1X = P^1_1X$. Then, it suffices to show that the map

$$f : P_1X \to \Sigma H\underline{\pi_1}$$

is a P^1-equivalence. By Lemma 4.21, it suffices to show that the fiber F_f of f is slice 1-connected. Using the LES associated to the fiber sequence, it is immediate that F_f is 1-connected and thus slice 0-connected by 4.16. In particular, $\pi_1(F_f)$ vanishes, so 4.18 implies F_f is slice 1-connected as well. ☐

Example 4.28. Consider the fiber sequence in Example 4.13. From the homotopy groups, we know that F_f is 0-connected and thus slice 0-connected. Moreover, if we restrict to the underlying non-equivariant spectrum, the map

$$\pi_1 S^1 \to \pi_1 \Sigma H\mathbb{Z}$$

is an isomorphism, so it is immediate that F_f is also nonequivariantly 1-connected. By Lemma 4.18, F_f is slice 1-connected.

Alternatively, it also suffices to show that $P^1_1 F_f$ is weakly contractible. By the previous corollary, we know that $\underline{M}'' := P^1_1 F_f$ is determined by $\underline{\pi_1}F_f$, which is the kernel of the canonical map

$$\underline{\pi_0}(S^0) = A_G \to \underline{\mathbb{Z}}$$

where a finite H-set in $A_G(H)$ is sent to its cardinality. Let $\underline{M}'$ denote the kernel, and we have the commutative diagram of restriction maps.

$$\begin{tikzcd}
\underline{M}'(G/H) \arrow[r] \arrow[d] & \underline{A_G}(G/H) \arrow[r] \arrow[d] & \mathbb{Z} \arrow[d, "\times 1"] \\
\underline{M}'(G/e) \arrow[r] & \underline{A_G}(G/e) = \mathbb{Z} \arrow[r, "\times 1"] & \mathbb{Z}
\end{tikzcd}$$

In particular, $\underline{M}'(G/e)$ is the trivial group, and by constrution $\underline{M}'' = 0$, as desired.

The success of [HHR16] is their identification

$$P^n_n MU^{C_8} \cong \begin{cases} * & n \text{ odd} \\ H\underline{\mathbb{Z}} \wedge W & n \text{ even} \end{cases}$$

where W is a wedge of slice spheres. Thus, the input to the slice spectral sequence is determined to be Bredon homology of regular representation spheres.

4.4 The Case of $K\mathbb{R}$

Introduced by Atiyah in [Ati66], the spectrum $K\mathbb{R}$ is a genuine C_2 spectrum whose homotopy groups are easy to compute using the slice spectral sequence.

Definition 4.29. A **real space** is a space equipped with a C_2 action. A **real vector bundle** over a real space X is a complex vector bundle $\pi : E \to X$ such that π is C_2-equivariant (the C_2 action on E being the complex conjugation), and the C_2 action on X induces an antilinear map on the fibers.

Definition 4.30. The group completion of the monoid of real vector bundles over a real space X is denoted $K\mathbb{R}(X)$.

With complex conjugation $X = \mathbb{CP}^1$ is a real space equivalent to $S^{1+\sigma}$, and its tautological bundle is a real vector bundle.

The representing spectrum $K\mathbb{R}$ admits the structure of a genuine C_2-spectrum: let

$$\beta : S^{1+\sigma} \cong \mathbb{CP}^1 \to \mathbb{Z} \times BU$$

be the equivariant classifying map for the tautological bundle over $\mathbb{CP}^1$. Using the multiplication on BU, there is an induced map

$$\beta' : \Sigma^{1+\sigma}\mathbb{Z} \times BU \to \mathbb{Z} \times BU$$

Let $\rho = 1 + \sigma$ be the regular C_2 representation. We define

$$K\mathbb{R}(n\rho) := \mathbb{Z} \times BU$$

for all n, and the structure maps be β'. To fill in the gaps, since every C_2-representation V is contained in some ρ^n for some minimal n, and we may define

$$K\mathbb{R}(V) := \Omega^{n\rho - V} K\mathbb{R}(n\rho)$$

so that the adjoint structure maps are the obvious identity maps and their compositions with β'. Moreover, Atiyah proved the following periodicity theorem

Theorem 4.31. There are isomorphisms

$$K\mathbb{R}(\Sigma^{1+\sigma} X) \cong K\mathbb{R}(X)$$

and

$$K\mathbb{R}(\Sigma^8 X) \cong K\mathbb{R}(X)$$

The first isomorphism implies the adjoint maps to β' are also equivalences, and $K\mathbb{R}$ is thus an genuine C_2-spectrum, as defined in Definition 2.2.

Proposition 4.32. The restriction of $K\mathbb{R}$ to the trivial group is

$$\mathrm{Res}_e^{C_2} K\mathbb{R} \cong KU$$

and the C_2 fixed point spectrum of $K\mathbb{R}$ is

$$(K\mathbb{R})^{C_2} \cong KO$$

The first part is obvious by construction, and a reference for the second part is [Kar01].

By Bott periodicity, we know that $\underline{\pi_0} K\mathbb{R} \cong \underline{\mathbb{Z}}$[1] and $\underline{\pi_1} K\mathbb{R} \cong \langle \mathbb{Z}/2 \rangle$. So the identification of 0-slice and 1-slice gives us that

$$P_0^0 K\mathbb{R} \cong H\underline{\pi_0} K\mathbb{R} \cong H\underline{\mathbb{Z}}$$

and

$$P_1^1 K\mathbb{R} \cong H\underline{\pi_1}'' K\mathbb{R} = *$$

Moreover, using the fact that slices commute with suspension with regular representation and the periodicity of $K\mathbb{R}$, we get

Proposition 4.33. The slices for $K\mathbb{R}$ is given by

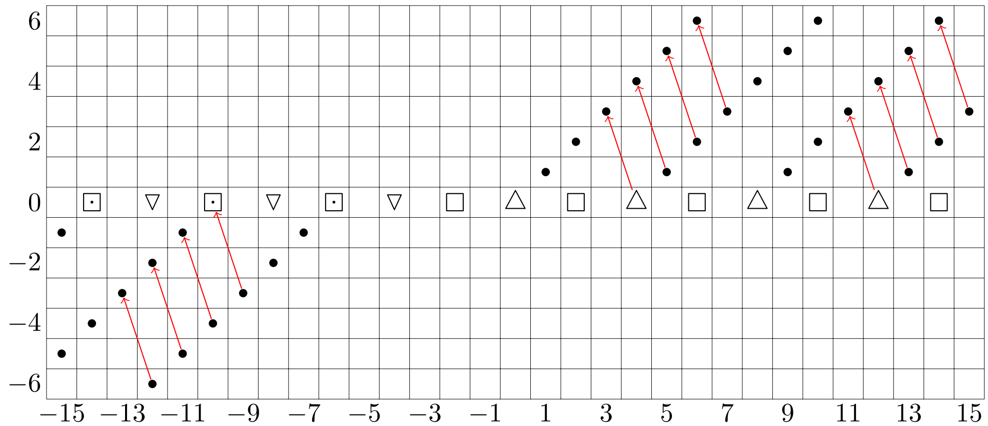
$$P_t^t K\mathbb{R} = \begin{cases} \Sigma^{\frac{t}{2}\rho} H\underline{\mathbb{Z}} & t \text{ even} \\ 0 & \text{otherwise} \end{cases}$$

By Theorem 3.12, we already know all the Mackey functors on the E_2-page of the slice spectral sequence for $K\mathbb{R}$: when t is even, we have

$$\begin{aligned} \pi_{t-s}^G P_t^t K\mathbb{R} &\cong \pi_{t-s}^G \Sigma^{\frac{t}{2}\rho} H\underline{\mathbb{Z}} \\ &\cong [S^{t-s}, S^{\frac{t}{2}(1+\sigma)} \wedge H\underline{\mathbb{Z}}] \\ &\cong [S^0, S^{(-\frac{t}{2}+s)+\frac{t}{2}\sigma} \wedge H\underline{\mathbb{Z}}] \\ &\cong H^{-\frac{t}{2}+s, \frac{t}{2}}(*, \underline{\mathbb{Z}}) \end{aligned}$$

It is clear that the nontrivial differentials start on E_3 page, which is expressed in the following $(t-s, s)$-grid:

[1] We know the map $KU(*) \to KO(*)$ induces the multiplication by 2 map as transfer.



$\bullet = \langle \mathbb{Z}/2 \rangle$, $\triangle = \underline{\mathbb{Z}}$, $\triangledown = \underline{\mathbb{Z}}^{op}$, $\square = \underline{\mathbb{Z}}_-$, $\boxdot = \underline{\mathbb{Z}}_-^{op}$

Since $\pi_* K\mathbb{R}$ is 8-periodic, we may deduce the following:

1. The spectral sequence immediately implies $\underline{\pi_{-5}} K\mathbb{R} = 0$, and $\underline{\pi_3} K\mathbb{R} = 0$ by periodicity. Thus, the first differential in the first quadrant as depicted must kill $E^{3,6}$; by multiplicativity of the spectral sequence, all other $\bullet$ connected by a red arrow in the first qudrant are killed.
2. As a consequence, $\underline{\pi_7} K\mathbb{R} = 0$, and $\underline{\pi_{-9}} K\mathbb{R} = 0$ by periodicity. All $\bullet$ connected by a red arrow in the third quadrant are then killed.

Lemma 4.34. There is only one possible surjection $\underline{\mathbb{Z}} \to \langle \mathbb{Z}/2 \rangle$, and its kernel is $\underline{\mathbb{Z}}^{op}$; similarly, there is only one possible injection $\langle \mathbb{Z}/2 \rangle \to \underline{\mathbb{Z}}_-^{op}$, and its cokernel is $\underline{\mathbb{Z}}_-$.

Proof. The surjection is given by the following

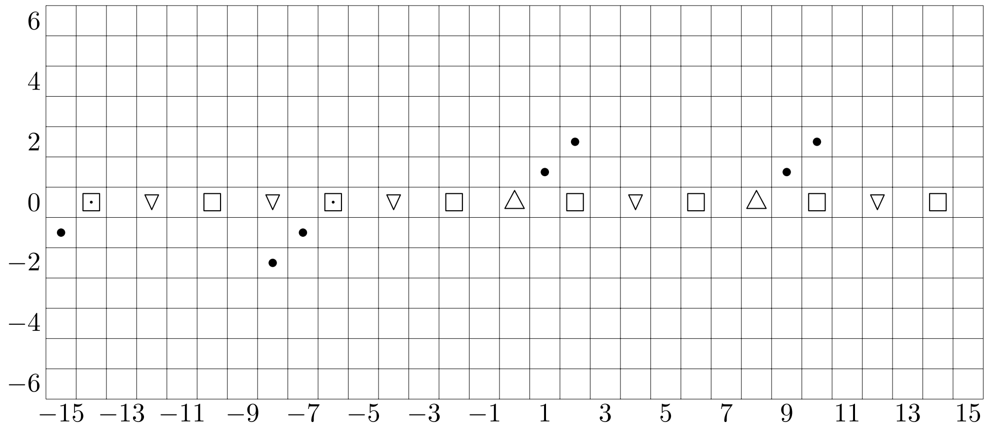
$$\begin{tikzcd}
\mathbb{Z} \arrow[r, "\times 2"] \arrow[d, bend right, "\times 2"'] & \mathbb{Z} \arrow[r, "pr"] \arrow[d, bend right, "Id"'] & \mathbb{Z}/2 \arrow[d] \\
\mathbb{Z} \arrow[r, "Id"] \arrow[u, bend right, "Id"'] \arrow[loop below, "Id"] & \mathbb{Z} \arrow[r] \arrow[u, bend right, "\times 2"'] \arrow[loop below, "Id"] & 0
\end{tikzcd}$$

and the injection is given by

$$\begin{tikzcd}
\mathbb{Z}/2 \arrow[r, "Id"] \arrow[d] & \mathbb{Z}/2 \arrow[r] \arrow[d, bend right, "0"'] & 0 \arrow[d] \\
0 \arrow[r] & \mathbb{Z} \arrow[r] \arrow[u, bend right, "pr"'] \arrow[loop below, "\times -1"] & \mathbb{Z} \arrow[loop below, "\times -1"]
\end{tikzcd}$$

$\square$

Thus, the spectral sequence degenerates on E_4:



The two extension problems at $\underline{\pi_{8n+2}}$ when $n \geq 1$ and $\underline{\pi_{8n}}$ when $n \leq -1$ is given by periodicity: we must have $\underline{\pi_2} K\mathbb{R} \cong \underline{\pi_{-6}} K\mathbb{R} \cong \underline{\mathbb{Z}}_-^{op}$ and $\underline{\pi_{-8}} K\mathbb{R} \cong \underline{\pi_0} K\mathbb{R} \cong \underline{\mathbb{Z}}$.

Theorem 4.35. The homotopy groups of $K\mathbb{R}$ is given by

$$\underline{\pi_n} K\mathbb{R} \cong \begin{cases} \underline{\mathbb{Z}} & n = 0 \text{ mod } 8 \\ \langle \mathbb{Z}/2 \rangle & n = 1 \text{ mod } 8 \\ \underline{\mathbb{Z}}_-^{op} & n = 2 \text{ mod } 8 \\ \underline{\mathbb{Z}}^{op} & n = 4 \text{ mod } 8 \\ \underline{\mathbb{Z}}_- & n = 6 \text{ mod } 8 \\ 0 & \text{otherwise} \end{cases}$$

A Nonequivariant Stable Homotopy Theory

The appendix will provide a quick overview of nonequivariant stable homotopy theory, including the theory of orthogonal spectra needed for the equivariant counterpart.

A.1 Motivations

The first motivation is Brown reresentability theorem, which is about represeting reduced cohomology theories

Theorem A.1 (Brown Representability-CW)**.** Let $\tilde{h}^*$ be a reduced cohomology theory on pointed CW-complexes. Then, for each $n \in \mathbb{Z}$, there exists a connected pointed CW complex K_n such that

$$\tilde{h}^n(X) \cong [X, K_n]$$

for all n. Moreover, the K_n are determined up to homotopy equivalence.

In fact, there are more structure to the set $\{K_n\}$: using the suspension axiom and loop-suspension adjunction, we see that there must be an isomorphism

$$[X, K_n] \cong \tilde{h}^n(X) \cong \tilde{h}^{n+1}(\Sigma X) \cong [\Sigma X, K_n] \cong [X, \Omega K_n]$$

Taking X to be K_n, we note that the identity map from the LHS corresponds uniquely to a map $\alpha_n \colon K_n \to \Omega K_n$, which we will call the **structure map.** By naturality and taking $X = S^k$, the structure map induces a weak equivalence

$$K_n \cong \Omega K_n$$

This motivates the definition for Ω-spectra.

Definition A.2. A **(sequential) spectrum** is a sequence of pointed topological spaces $\{X_n\}$ with **structure maps**.

$$\Sigma X_n \to X_{n+1}$$

An **Ω-spectrum** is a (sequential) spectrum whose adjoint structure maps

$$X_n \to \Omega X_{n+1}$$

are weak equivalences. A morphism of spectra $X \to Y$ is a sequence of maps $X_n \to Y_n$ compatible with the structure maps.

We use the adjective sequential when necessary to distinguish from structured types of spectra such as symmetric spectra or orthogonal spectra. So we see that a reduced cohomology theory corresponds to an Ω-spectrum. One can also show that an Ω-spectrum defines a reduced cohomology theory as well, and present Brown representability theorem in the following way:

Theorem A.3 (Brown Representability Theorem)**.** Every reduced cohomology theory on the category of basepointed CW complexes has the form $\tilde{h}^n(X) = [X, K_n]$ for some Ω-spectrum K.

Example A.4 (Eilenberg-Maclane Spectrum)**.** The Ω-spectrum that represents reduced ordinary cohomology with coefficients in G is given by the Eilenberg-Maclane spaces $HG := \{K(G,n)\}$, with adjoint structure maps homotopy equivalences

$$K(G,n-1) \cong \Omega K(G,n)$$

Example A.5. The Ω-spectrum that represents reduced complex topological K-theory is given by $\{KU_n\}$, where

$$KU_n = \begin{cases} BU \times \mathbb{Z} & n - \text{even} \\ \Omega BU & n - \text{odd} \end{cases}$$

In particular, Bott-periodicity shows that the adjoint structure maps are weak-equivalences.

The stable homotopy category will be a category of spectra, such that an "equivalence" of spectra will give equivalent reduced cohomology theories.

The second motivation is stable homotopy groups and stable maps. The heuristics is: in most cases the homotopy classes of maps $[X,Y]$ is very difficult to compute. However, the stable maps $\varinjlim_n [\Sigma^n X, \Sigma^n Y]$ give us nice approximation and are sometimes easier to understand. Spanier-Whitehead duality is a nice example of this, stated in Theorem A.19.

Definition A.6. Let X and Y be pointed CW-complexes. The set of **stable homotopy classes of maps** from X to Y is defined to be

$$[X,Y]^s := \varinjlim_k [\Sigma^k X, \Sigma^k Y]$$

One form of the Freudenthal suspension theorem says that if Y is n-connected and X has dimension less than $2n+1$, then the suspension map $[X,Y] \to [\Sigma X, \Sigma Y]$ is bijective. In this case, we see that the colimit actual stabilizes after at a finite stage.

Definition A.7. For a pointed CW-complex, the n-th **stable homotopy group** is defined to be

$$\pi_n^s(X) := \varinjlim_k \pi_{n+k}(\Sigma^k X) = \varinjlim_k [\Sigma^k S^n, \Sigma^k X] = [S^n, X]^s$$

These stable homotopy groups will be the homotopy groups in the stable homtopy category. Moreover, stable homotopy groups actually defines a reduced homology theory: the two axioms that are not trivial to check are the LES and the wedge axiom. The key point is that Blakers-Massey theorem plus the LES of homotopy groups of a pair will give us the LES of stable homotopy groups of a pair; the wedge axiom follows from that $\Sigma^i X \bigvee \Sigma^i Y$ is the $2i-1$ skeleton of $\Sigma^i X \times \Sigma^i Y$. Generalizing this, we have

Theorem A.8. Let K be a CW-complex. The sequence $\tilde{h}_i(X) = \pi_i^s(X \wedge K)$ forms a reduced homology theory on the category of based CW-complexes.

We can also define the homotopy groups of a spectrum as a generalization:

Definition A.9. Suppose $K = \{K_i\}$ is a spectrum. Then the **homotopy groups** of K is defined to be:

$$\pi_n(K) := \varinjlim \pi_{n+i}(K_i)$$

where the inductive limit is induced by the suspension structure maps.

Definition A.10. Given a spectrum $K = \{K_n\}$ and a space X, there is an associated **smash spectrum** $K \wedge X$, where $(K \wedge X)_n = K_n \wedge X$. The structure maps are given by

$$\Sigma(K \wedge X)_n = \Sigma K_n \wedge X \to K_{n+1} \wedge X = (K \wedge X)_{n+1}$$

Similarly, there is a smash spectrum $X \wedge K$, where $(X \wedge K)_n := X \wedge K_n$, and the structure maps are given by the composition

$$S^1 \wedge X \wedge K_n \xrightarrow{\text{twist} \wedge Id} X \wedge S^1 \wedge K_n \xrightarrow{Id \wedge \sigma_n} X \wedge K_{n+1}$$

Note that smashing on the left would require a natural twist map to make things work; moreover, the twist isomorphism $K_n \wedge X \cong X \wedge K_n$ provides an isomorphism between $X \wedge K$ and $K \wedge X$.

Definition A.11. Given a spectrum $K = \{K_n\}$ and a space A, there is an associated function spectrum $F(A, K)$, where $(F(A, K))_n = \mathrm{Top}_*(A, K_n)$. The structure maps are given by

$$\mathrm{Top}_*(A, K_n) \xrightarrow{\circ\sigma_n} \mathrm{Top}_*(A, \Omega K_{n+1}) \cong \Omega\mathrm{Top}_*(A, K_n)$$

Example A.12. We define the suspension of a spectrum K to be

$$\Sigma K := S^1 \wedge K$$

as in Definition A.10. Similarly, we define the loop of a spectrum K to be

$$\Omega K := F(S^1, K)$$

as in Definition A.11.

Theorem A.13. Let K be a spectrum. Given a CW complex X, the sequence

$$h_n(X) := \pi_n(X \wedge K)$$

is a reduced homology theory.

The proof mostly follows from the same tactics in proving the stable homotopy groups being a reduced homology theory. Here are a couple of examples:

Example A.14 (Singular homology)**.** If $K = HG$ is the Eilenberg-Maclane spectrum, then $h_i(X) := \pi_i(X \wedge K)$ is isomorphic to singular homology with coefficients in G. We only have to check the dimension axiom

$$h_i(S^0) := \varinjlim_n \pi_{n+i}(K(G,n))$$

which is trivial when $i \neq 0$ and $\mathbb{Z}$ when $i = 0$.

Example A.15. Given a topological space X, we have the associated **suspension spectrum** $\Sigma^\infty X = \{\Sigma^k X\}$. Then, the stable homotopy groups of X is given by the homotopy groups of its associated suspension spectrum. The **sphere spectrum** $\mathbb{S}$ defined by

$$\mathbb{S}_n := S^n$$

is the suspension spectrum of S^0. By definition, we have

$$\pi_n^s(S^0) \cong \pi_n(\mathbb{S})$$

so the stable homotopy groups of spheres is a homology theory.

We now need a "correct" weak equivalence of spectra. Let $\mathbb{S}^{(n)}$ be the spectrum defined by

$$\mathbb{S}_d^{(n)} := \begin{cases} S^d & d > n \\ * & d \leq n \end{cases}$$

It is easy to see that $\mathbb{S}$ and $\mathbb{S}^{(n)}$ have the same homotopy groups, which gives us the stable homotopy groups of spheres. However, it is easy to see there is no naive level-wise homotopy equivalence between $\mathbb{S}$ and $\mathbb{S}^{(n)}$. The correct notion is then a form of weak equivalence of spectra:

Definition A.16. A morphism of spectra $f : X \to Y$ is called a **π_*-isomorphism** if the induced map

$$f_* : \pi_n(X) \to \pi_n(Y)$$

is an isomorphism for all $n \in \mathbb{Z}$.

It is easy to see that the canonical morphism $\mathbb{S}^{(n)} \to \mathbb{S}$ now induces an π_*-isomorphism.

We also have a generalization of the suspension spectrum, defined in Example A.15.

Example A.17. The **shifted suspension spectrum** $F_d K$ of a pointed topological space K with $d \in \mathbb{N}$ is defined by

$$(F_d K)_n := \begin{cases} S^{n-d} \wedge K & n \geq d \\ * & n < d \end{cases}$$

In particular, we see that the suspension spectrum of a space X is simply $F_0 X$. It also follows from definition that we have

$$\pi_n^s(X) \cong \pi_{n-m}(F_m X)$$

Using shifts, we see that spectra can have non-trivial negative homotopy groups. For example

$$\pi_{-1}(F_2 S^0) = \pi_1^s(S^0) \cong \mathbb{Z}/2$$

The shifted suspension spectrum is the stable "desuspension" of a topological space, which shifts the stable homotopy groups.

A.2 Duality

We now have spectra and stable morphisms that represent cohomology and homology. Moreover, we want to recover some of the classical duality theorems in the case of singular (co)homology.

Theorem A.18 (Alexander Duality)**.** If K is a compact, locally contractible, nonempty, proper subspace of S^n. Then we have the isomorphism

$$\tilde{H}^{n-i-1}(K;\mathbb{Z}) \cong \tilde{H}_i(S^n - K;\mathbb{Z})$$

The theorem implies that different embeddings of some nice enough space in S^n have isomorphic homology, and we say that the embedding is the Spanier-Whitehead dual to its complement. However, different embeddings do not necessarily have the same homotopy type, as there are tons of examples in knot theory. Nevertheless, Spanier and Whitehead prove the following theorem, which states that the homotopy types are the same after sufficiently many suspensions.

Theorem A.19 (Spanier-Whitehead Duality)**.** Let X be a compact simplicial complex. Let $f, g : X \to S^n$ be two simplicial embeddings. Then for some sufficiently large M, the M-fold suspensions $\Sigma^M(S^n \setminus f(X))$ and $\Sigma^M(S^n \setminus g(X))$ are homotopy equivalent.

To study this phenomenon more formally, Spanier-Whitehead proposed the now-called Spanier-Whitehead category:

Definition A.20 (Spanier-Whitehead Category)**.** The **Spanier-Whitehead category**, or S-**category** for short, has objects pairs (X, n), where X is a pointed finite CW complex, and n is an integer. The morphisms are defined by

$$\mathrm{Hom}_S((X,n),(Y,m)) := \varinjlim_{q \geq \max(|m|,|n|)} [\Sigma^{q+n}X, \Sigma^{q+m}Y]$$

We can think of (X, m) as the m-fold suspension of X. If $m = n = 0$, we see that $\mathrm{Hom}_S((X,0),(Y,0)) = \varinjlim_k [\Sigma^k X, \Sigma^k Y]$, which are simply the stable morphisms and that the different embeddings $X \to S^n$ become actually isomorphic in this category by Theorem A.19. If $m > 0$, we may denote $(X, -m)$ as $\Sigma^{-m}X$ to represent formal desuspensions. We can also think of this as "inverting smashing with spheres".

Definition A.21. Let $\mathcal{C}$ be a symmetric monoidal category with the tensor product denoted by $\wedge$, and unit object S^0. Then, X, Y are **dual** if there are morphisms $X \wedge Y \to S^0$ and $S^0 \to Y \wedge X$ such that the following compositions

$$X \cong X \wedge S^0 \to X \wedge Y \wedge X \to S^0 \wedge X \cong X$$

$$Y \cong S^0 \wedge Y \to Y \wedge X \wedge Y \to Y \wedge S^0 \cong Y$$

are the identities in the symmetric monoidal category. Given an object A in $\mathcal{C}$, we denote its dual as DA, if it exists.

Note that the definition implies the adjunction

$$\mathrm{Hom}_{\mathcal{C}}(X \wedge A, Y) \cong \mathrm{Hom}_{\mathcal{C}}(X, DA \wedge Y)$$

By the identity $\Sigma(X \wedge Y) \cong X \wedge \Sigma Y$, we see that the smash product defined by

$$(X, m) \wedge (Y, n) := (X \wedge Y, m + n)$$

endows the S-category with a symmetric monoidal structure, with S^0 being the unit. Let us see some explicit dualities

Example A.22. S^n is dual to $S^{-n} := (S^0, -n)$, the formal $n-$fold desuspension of S^0. This follows directly from definition.

Example A.23 (Alexander duality in S-category)**.** Let X be a compact finite CW complex with an embedding $X \to S^n$. Let us denote the complement $S^n \setminus X$ as $D_n X$. Then, one may construct a map $X \wedge D_n X \to S^n$, and a desuspension of such map exhibits X and $\Sigma^{-n} D_n X$ as duals.

Let Y be a finite discrete set of points. Then, Y embeds in S^1, and the complement is homotopy equivalent to Y again. This shows that Y is self-dual. This example will come up later when we deal with finite G-sets and equivariant embeddings.

Example A.24 (Atiyah Duality)**.** Suppose M is a closed smooth manifold with a smooth embedding $M \to \mathbb{R}^n$. Let Tv be the Thom space of the normal bundle of the embedding. The Pontrjagin-Thom construction yields a map

$$Tv \wedge M_+ \to \Sigma^n M_+ \xrightarrow{\text{collapse } M_+} S^n$$

which exhibits Tv and $\Sigma^{-n} M_+$ as duals.

Example A.25 (Poincaré Duality)**.** Let M be a closed orientable manifold of dimension m, with an embedding in $M \to \mathbb{R}^n$. Let $H\mathbb{Z}$ be the Eilenberg-Maclane spectrum that represents singular cohomology. From Atiyah duality, we have $[DM, H\mathbb{Z}] \cong [S^0, M_+ \wedge H\mathbb{Z}] =: \tilde{H}_*(M_+)$. On the other hand, $[DM, H\mathbb{Z}] = [\Sigma^{-n} Tv, H\mathbb{Z}] \cong \tilde{H}^*(\Sigma^{-n} Tv)$. By the Thom isomorphism and the suspension isomorphism

$$\tilde{H}^*(M_+) \cong \tilde{H}^{*+(n-m)}(Tv) \cong \tilde{H}^{*-m}(\Sigma^{-n} Tv)$$

and we establish the classical Poincaré duality $\tilde{H}^*(M_+) \cong \tilde{H}_{*-m}(M_+)$.

The Spanier-Whitehead category lacks arbitrary products and coproducts since it only contains the finite CW complexes. However, it embeds in the stable homotopy category, which we will construct later.

A.3 The Desired Stable Homotopy Category

Building upon the motivations in the previous section, we want to build a category $\mathcal{SHC}$ that captures the stable phenomena and cohomology theories/homology theories. In particular, we want it to satisfy the following properties (not necessarily axioms):

1. There is a functor $\Sigma^\infty : \mathrm{HoTop}_* \to \mathcal{SHC}$, together with an adjoint $\Omega^\infty : \mathcal{SHC} \to \mathrm{HoTop}_*$.
2. Let A, B be CW complexes, where A has finite dimension. Then, there is a natural isomorphism

$$[\Sigma^\infty A, \Sigma^\infty B] \cong [A, B]^s$$

3. The morphisms of $\mathcal{SHC}$ has the structure of graded abelian groups, with bilinear composition. We denote that graded components as $[-,-]_*$. Moreover, given a reduced cohomology theory E^* there exists an object K in Sp such that
$$E^*(A) \cong [\Sigma^\infty A, K]_{-*}$$
4. For every object K in $\mathcal{SHC}$, one defines a reduced homology theory via
$$E_n(A) := \pi(K \wedge A)$$
5. $\mathcal{SHC}$ has a closed symmetric monoidal structure, with $\wedge$ as the tensor product and $F(-,-)$ as internal hom. The object $\Sigma^\infty S^0$ is the identity object.

It was clear historically that this stable homotopy category should be the "homotopy category" of spectra, but naively things do not work out easily. The Spanier-Whitehead category contains only finite complexes, which means it does not have arbitrary products and coproduts, and the wedge axiom does not hold.

Adams considered the category of spectra. A natural candidate for homotopies in this category is a map $H: X \wedge I_+ \to Y$ However, it turns out there are not enough maps in this category, such that non-homotopy equivalent spectra might represent isomorphic cohomology theories. In [Ada74], Adams first resolved the issue by consider equivalences classes of "cofinal spectra." Later, Bousfield and Friedlander's put a model structure on the category of spectra in [BF78], and considered its homotopy category. This homotopy category turns out satisfies all the properties that people wanted, and is referred to as the stable homotopy category. A detailed account of the construction can be found in [BR20].

However, people still searched for a point-set model of spectra that satisfies the desired properties without going through the homotopy category. G. Lewis in [Lew91] showed that it is impossible to put a symmetric monoidal structure on the category of spectra that also satisfies a list of desirable axioms.

Nevertheless, people later constructed model categories of structured spectra, such as orthogonal spectra and symmetric spectra, whose homotopy category is equivalent to the stable homotopy category. Point-set wise, their symmetric monoidal structures turn out to be incredibly useful in defining ring spectra, along with A_∞/E_∞-rings. Another modern construction is to consider the $(\infty,1)$-category of spectra. The next section will discuss the case of orthogonal spectra more in detail.

A.4 The Stable Model Structure of Sequential Spectra

We would like a model structure on the category of sequential spectra. Naively, we could use the level-wise serre fibrations and level-wise weak equivalence, which indeed gives us a model structure.

Theorem A.26 (Level-wise model)**.** There is a cofibrantly generated model structure on sequential spectra, where the generating sets are given by

$$I_{level} := \{F_d S^{n-1}_+ \to F_d D^n_+ | n, d \in \mathbb{N}\}$$

$$J_{level} := \{F_d D^n_+ \to F_d (D^n \times [0,1])^n_+ | n, d \in \mathbb{N}\}$$

The weak equivalences, fibrations, and cofibrations are levelwise weak equivalences, Serre fibrations, and Serre cofibrations, respectively.

This is reminiscent of the Serre model structure on $\mathbf{Top}_*$. However, as observed in the remarks above Defintion A.16, the correct weak equivalence should be the π_*-isomorphisms. Because of representability, we want Ω-spectra to be the fibrant objects. Thus, the solution is to add more maps to the generating set of acyclic cofbrations. The details can be found in [BR20].

Theorem A.27 (Stable Model Structure)**.** There is a cofibrantly generated model structure on sequential spectra, where

- The weak equivalences are the π_*-isomorphisms.
- The cofibrations are the levelwise Serre cofibrations.
- The fibrations are called the stable fibrations, which are levelwise fibrations that also satisfy the addition property that the induced map

$$X_n \to Y_n \times_{\Omega Y_{n+1}} \Omega X_{n+1}$$

is a weak equivalence for all n.

In particular, the fibrant objects are precisely the class of Ω-spectra.

Definition A.28. The **stable homotopy category**, or $\mathcal{SHC}$ for short, is the homotopy category of the category of sequential spectra equipped with the stable model structure. The homsets in the stable homotopy category is denoted by $[-,-]$.

Consider the suspension and loop endofunctors on sequential spectra by the constrution given in Example A.12. The most important fact about the stable homotopy category is that the suspension functor and loop functor form a Quillen equivalence and thus is an equivalence of the stable homotopy category.

Lemma A.29. Given a spectrum X, we have isomorphisms of homotopy groups

$$\pi_{n+k}(\Sigma^k X) \cong \pi_n(X)$$

and

$$\pi_n(\Omega^k X) \cong \pi_{n+k}(X)$$

Proof. We will prove the first statement in the case $k = 1$, while the other follows similarly. By definition, there is a factorization

$$\begin{tikzcd}
\dots \arrow[r] & \pi_{n+i}(X_i) \arrow[r] \arrow[dr, "\Sigma"'] & \pi_{n+i+1}(X_{i+1}) \arrow[r] \arrow[dr, "\Sigma"'] & \pi_{n+i+2}(X_{i+2}) \arrow[r] & \dots \\
& & \pi_{n+i+1}(\Sigma X_i) \arrow[u, "\sigma"'] & \pi_{n+i+2}(\Sigma X_{i+1}) \arrow[u, "\sigma"'] &
\end{tikzcd}$$

where Σ is suspension and σ is induced by the structure map of X. Moreover, the following diagram commutes up to a sign

$$\begin{tikzcd}
\pi_{n+i+1}(X_{i+1}) \arrow[r, "\Sigma"] & \pi_{n+i+2}(\Sigma X_{i+1}) \\
\pi_{n+i+1}(\Sigma X_i) \arrow[u, "\sigma"'] \arrow[r, "\Sigma"] & \pi_{n+i+2}(\Sigma\Sigma X_i) \arrow[u, "\sigma"]
\end{tikzcd}$$

which means the top row and the bottom of the factorization compute the same colimit, i.e

$$\pi_n(X) = \mathrm{colim}_i \pi_{n+i}(X_i) \cong \mathrm{colim}_i \pi_{n+1+i}(\Sigma X_i) = \pi_{n+1}(\Sigma X)$$

$\square$

Corollary A.30. A map of spectra $f : X \to Y$ is a π_*-isomorphism if and only if

$$\Sigma^k f : \Sigma^k X \to \Sigma^k Y$$

is a π_*-isomorphism for all $k \in \mathbb{N}$. Similarly, f is an π_*-isomorphism if and only if

$$\Omega^k f : \Omega^k X \to \Omega^k Y$$

is a π_*-isomorphism for all $k \in \mathbb{N}$.

Theorem A.31. There is a Quillen auto-equivalence on the category of sequential spectra equipped with the stable model structure:

$$\begin{tikzcd} \mathcal{S}^{\mathbb{N}} \arrow[r, bend right, "\Omega"'] \arrow[r, phantom, "\perp"] & \mathcal{S}^{\mathbb{N}} \arrow[l, bend right, "\Sigma"'] \end{tikzcd}$$

given by the suspension-loop adjunction.

Proof. Space level suspension preserves Serre cofibrations, so the functor Σ preserves the levelwise Serre cofibrations; it also preserves π_*-isomorphisms by Corollary A.30. By standard model category theory, the adjoint pair Σ and Ω is a pair of Quillen adjunction. To show it is a Quillen equivalence, it is equivalent to showing that the derived adjunction counit is a weak equivalence. In particular, it suffices to show that the composition $X \to \Omega\Sigma X$ is a π_*-isomorphism for all X. To see this is true, note that we have

$$\pi_{n+i}(X_i) \to \pi_{n+i}(\Omega\Sigma X_i) \cong \pi_{n+i+1}(\Sigma X_i)$$

and the argument is the same as Lemma A.29.

□

By directly examining the generating cofibrations, we have a Quillen adjunction between the evaluation functor $\mathrm{Ev}_d : \mathcal{S}^{\mathbb{N}} \to \mathbf{Top}_*$, which sends a spectrum to its dth level, and the shifted suspension functor.

Theorem A.32. For $d \in \mathbb{N}$, the shifted suspension functor F_d and the evaluatin functor Ev_d form a Quillen adjunction

$$\begin{tikzcd} \mathcal{S}^{\mathbb{N}} \arrow[r, bend right, "\mathrm{Ev}_d"'] \arrow[r, phantom, "\perp"] & \mathbf{Top}_* \arrow[l, bend right, "F_d"'] \end{tikzcd}$$

In particular, the case where $d = 0$ gives us the Quillen adjunction between $F_0 = \Sigma^\infty$ and Ev_0. This is what we were looking for in the first bullet point of A.3.

We can explicitly spell out the derived functor of Ev_0, which we will call Ω^∞. Let X be a sequential spectra, our fibrant replacement will be a Ω-spectrum $R_\infty X$ and a π_*-isomorphism $X \to R_\infty X$. We construct the fibrant replacement as follows: let $R_0 X = X$, and for $k \geq 1$, let

$$(R_k X)_n := \Omega^k X_{n+k}$$

with adjoint structure maps given by the composition

$$\Omega^k X_{n+k} \xrightarrow{\Omega^k \sigma_{n+k}} \Omega^k \Omega X_{n+k+1} \xrightarrow{\cong} \Omega\Omega^k X_{n+k+1}$$

where the second isomorphism is from the associativity of composition. There are natural maps $R_k \to R_{k+1}$ induced by the adjoint structure maps, where on each level is given by

$$(R_kX)_n := \Omega^k X_{n+k} \xrightarrow{\Omega^k \sigma_{n+k}} \Omega^{k+1} X_{n+k+1} =: (R_{k+1}X)_n$$

Each map $R_kX \to R_{k+1}X$ is then a π_*-isomorphism, by the same argument as in Lemma A.29. We may then define

$$R_\infty X := \text{hocolim}_k R_k X$$

To verify that R_∞ is an Ω-spectrum, we see that that its level n is given by the pointwise colimit

$$(R_\infty X)_n = \text{hocolim}_k \Omega^k X_{n+k}$$

with adjoint structure maps

$$\text{hocolim}_k \Omega^k X_{n+k} \xrightarrow{\cong} \text{hocolim}_k \Omega^k \Omega X_{n+k+1}$$

which is a homotopy equivalence. By associativity and the fact that homotopy colimits commute with the loop space functor, we have the homotopy equivalence

$$\text{hocolim}_k \Omega^k X_{n+k} \to \Omega \text{hocolim}_k \Omega^k X_{n+k+1}$$

as desired.

Having explicitly defined the fibrant replacements, we can now define the functor Ω^∞:

Definition A.33. There is a functor $\Omega^\infty : \mathcal{SHC} \to \text{HoTop}_*$ defined by

$$\Omega^\infty X := R\text{Ev}_0 X = \text{Ev}_0 R_\infty X = \text{hocolim}_n \Omega^n X_n$$

As the notation suggests, the space $\Omega^\infty X$ is an infinite loop space for all spectrum X by construction. The adjunction between Σ^∞ and Ω^∞ gives us the following:

Proposition A.34. Let A be a compact space, and X a spectrum. Then,

$$[\Sigma^\infty A, X] \cong \text{colim}[\Sigma^n A, X_n]$$

Proof. By adjunction, we have

$$[\Sigma^\infty A, X] \cong [A, \Omega^\infty X] = [A, \text{hocolim}\Omega^n X_n]$$

By compactness of A, the colimit can be pull out and we have

$$[A, \text{hocolim}\Omega^n X_n] \cong \text{colim}[A, \Omega^n X_n] \cong \text{colim}[\Sigma^n A, X_n]$$

□

If we take $X = \Sigma^\infty B$ for some space B, then the proposition says that morphisms in the stable category are the stable maps. Moreover, we have the direct corollary

Corollary A.35. For $k \in \mathbb{N}$, the homotopy groups of a spectrum X can be identified by

$$\pi_k(X) \cong [\Sigma^\infty S^k, X]$$

$$\pi_{-k}(X) \cong [F_k\mathbb{S}, X]$$

A.5 Triangulated Structure and Brown Representability

For any model category, one can define the analogous notions of the suspension and loop functor. A model category is called **stable** if the suspension functor is invertible. The stable homotopy category is such an example. Some nice consequences of stability is that homotopy fiber sequences and homotopy cofiber sequences agree, and the fiber F_f of a morphism f is weakly equivalent to $\Omega C(f)$, the loop of the cofiber of f.

Moreover, the homotopy category of a stable model category is triangulated. In short, a triangulated category is an additive category $\mathcal{C}$, equipped with auto-equivalence Σ, and a distinguished class of exact triangles of the form

$$X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X$$

Additionally, the distinguished triangles have to satisfy a list of axioms. The details can be found in Chapter 10 of [Wei94].

Theorem A.36. The stable homotopy category is triangulated.

The proof of the triangulated structure of $\mathcal{SHC}$ can be found in Chapter 4 of [BR20]. We now introduce a general form of Brown Representability, which is due to [Nee96].

Theorem A.37 (Brown Representability-Triangulated Categories)**.** Let $\mathcal{C}$ be a compactly generated triangulated category. A functor $E : \mathcal{C}^{op} \to \mathbf{Ab}$ that satisfies:

1. E takes corpoducts to products, i.e

$$E(\coprod_i X_i) \cong \prod_i E(X_i)$$

2. E is a cohomological functor, i.e given a distinguished triangle

$$X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X$$

then E sends the exact triangle to a long exact sequence of abelian groups

$$E(Z) \longrightarrow E(Y) \longrightarrow E(X) \longrightarrow E(\Sigma^{-1}Z) \longrightarrow E(\Sigma^{-1}Y) \longrightarrow ...$$

is representable. Such functors are called **Brown Functors**.

The stable homotopy category is triangulated, and in fact has compact generator the sphere spectrum. We define a **cohomology theory on spectra** to be a Brown functor on $\mathcal{SHC}$. An application of Yoneda lemma gives us the following:

Corollary A.38. The stable homotopy category is equivalent to the category of cohomology theories on spectra.

Given a cohomology theory E on spectra, represented by the spectrum Y, there is a natural extension to a graded structure, defined by

$$E^n(X) := \begin{cases} [\Sigma^{-n}X, Y] & n \leq 0 \\ [X, \Sigma^n Y] & n > 0 \end{cases}$$

so the long exact sequence can be rewritten as

$$... \longrightarrow E^n(Z) \longrightarrow E^n(Y) \longrightarrow E^n(X) \longrightarrow E^{n+1}(Z) \longrightarrow ...$$

Given a CW complex A, we have

$$E^n(\Sigma^\infty A) \cong [\Sigma^\infty A, \Sigma^n Y] \cong [\Sigma^n F_n A, \Sigma^n Y] \cong [F_n A, Y] \cong [A, Y_n]$$

where the first isomorphism is by representability, the second arrow is by the fact that $\Sigma^n F_n A$ is π_*-isomorphic to $\Sigma^\infty A$, the third isomorphism is by Σ being an auto-equivalence of $\mathcal{SHC}$, and the final isomorphism is by the adjunction between shifted suspension and evaluation. This recovers the our original Brown representability for pointed CW complexes, stated in Theorem A.1.

Remark A.39 (Hyperphantom Maps)**.** Even though Brown representability gives a bijection between the isomorphism classes of the objects of the stable homotopy category and cohomology theories on pointed CW complexes, it is not an equivalence of categories since it is not faithful. We may construct a non-nullhomotopic map of spectra that induces the zero map on cohomology theories on spaces(more general than CW complexes), and such maps are called **hyperphantom maps**. An example is given in this MO post.

Similarly, phantoms maps also show that the analogous result of Corollary A.38 for homology does not hold, since we can write every spectra as a filtered colimit of finite spectra, and homology (homotopy groups) commutes with homotopy colimits.

A.6 Orthogonal Spectra

The final missing piece of the stable homotopy category is the closed symmetric monoidal structure, with the commutative smash product and the function spectra. The original construction is due to Adams, which is referred to as the "handicrafted" smash product, is not practical in use. The modern treatment use of structured categories spectra, with orthogonal spectra being the more favorable one since not only do we get the closed symmetric monoidal structure before passing to the homotopy category, its model structure has π_*-isomorphism as the correct form of weak equivalence.

Definition A.40. An **orthogonal spectrum** consists of the following data:

1. A sequence of pointed topological spaces X_n
2. A basepoint-preserving left $O(n)$-action on X_n.
3. Structure maps $\sigma_n : S^1 \wedge X_n \to X_{n+1}$
4. The interated structure maps
$$S^m \wedge X_n \to X_{n+m}$$
define by the composition
$$S^m \wedge X_n = S^{m-1} \wedge S^1 \wedge X_n \xrightarrow{Id_{S^{m-1}\wedge\sigma_n}} S^{m-1} \wedge X_{n+1} \to \dots \to X_{n+m}$$
is $O(m)\times O(n)$-equivariant. Specifically, $O(m)$-acts on S^m canonically since S^m is the one-point compactification of $\mathbb{R}^m$; $O(m) \times O(n)$, through orthogonal sum, is identified as a subgroup of $O(m+n)$ and acts by restriction on X_{n+m}.

Definition A.41. A **morphism of orthogonal spectra** $f : X \to Y$ is a sequences of maps $f_n : X_n \to Y_n$ that are $O(n)$-equivariant, and compatible with the structure maps such that

$$\begin{tikzcd} S^1 \wedge X_n \arrow[r, "Id\wedge f_n"] \arrow[d, "\sigma_n"] & S^1 \wedge Y_n \arrow[d, "\sigma_n"] \\ X_{n+1} \arrow[r, "f_{n+1}"] & Y_{n+1} \end{tikzcd}$$

commutes for all n.

Example A.42. The sequential sphere spectrum $\mathbb{S}$ is an orthogonal spectrum, with $O(n)$ acting canonically on S^n. The structure map

$$S^k \wedge S^n \to S^{k+n}$$

is $O(k) \times O(n)$ equivariant.

Example A.43. For a space A, the orthogonal shifted suspension spectrum $F_d A$ is defined by

$$(F_d A)_n = \begin{cases} O(n)_+ \wedge_{O(n-d)} S^{n-d} \wedge A & n \geq d \\ * & n < d \end{cases}$$

the group $O(n-d)$ acts on the first $n-d$ coordinates on $O(n)$. The nontrivial structure map is given by the composite of

$$\Sigma(O(n)_+ \wedge_{O(n-d)} S^{n-d} \wedge A) \to O(n)_+ \wedge_{O(n-d)} S^{1+n-d} \wedge A$$

with

$$O(n)_+ \wedge_{O(n-d)} S^{1+n-d} \wedge A \to O(1+n)_+ \wedge_{O(1+n-d)} S^{1+n-d} \wedge A$$

Our goal now is to show that there exists a commutative tensor product on orthogonal spectra. First, we may examine why the naive symmtric product structure on sequential spectra does not work: it is natural to define the smash product of sequential spectra as

$$(X \wedge Y)_n := \bigvee_{i+j=n} X_i \wedge Y_j$$

and we want the sphere spectrum to be the identity element respect to this smash product. The axioms of the symmtric monoidal category forces the diagram

$$\begin{tikzcd} S^n \wedge S^m \arrow[r] \arrow[d, "\text{twist}"'] & S^{n+m} \\ S^m \wedge S^n \arrow[ur] & \end{tikzcd}$$

to commute. However, the degree of the maps differ by $(-1)^k$ because of the twist, so it does not commute even up to homotopy. The extra data of the orthogonal action is to record and cancel out the twist, as we will see below.

The perspective of diagram spectra gives us a philosophically correct construction of the symmetric monoidal product of spectra, as well as the internal hom and their adjunction.

Definition A.44. A **diagram** $\mathfrak{D}$ is a small category that is enriched in $\mathbf{Top}_*$ and symmetric monoidal. The category of $\mathfrak{D}$-spaces is the functor category $\mathrm{Fun}(\mathfrak{D}, \mathbf{Top}_*)$ of enriched functors.

Note that the data of an enriched functor $F : \mathfrak{D} \to \mathbf{Top}_*$ is a collection of maps of pointed spaces

$$F(a, b) : \mathfrak{D}(a, b) \to \mathbf{Top}_*(Fa, Fb)$$

By adjunction, we may rewrite each map $F(a, b)$ as a map $\mathfrak{D}(a, b) \wedge Fa \to Fb$.

Example A.45. Consider the diagram category $\mathcal{O}$, whose objects are the non-negative integers, and morphisms given by

$$\mathcal{O}(a, b) = \begin{cases} O(a)_+ & a = b \\ * & \text{otherwise} \end{cases}$$

The symmetric monidal product on $\mathcal{O}$ is given on objects by $(a, b) \mapsto a + b$, and on mapping spaces

$$\mathcal{O}(a, b) \wedge \mathcal{O}(c, d) \to \mathcal{O}(a + c, b + d)$$

which is non-trivial iff $a = b$ and $c = d$. In this case, the morphism is given by the block sum of matrices

$$O(a) \times O(c) \to O(a + c)$$

$$(A, B) \mapsto \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$$

The category of $\mathcal{O}$-spaces is denoted $\mathcal{O}\mathbf{Top}_*$.

By the remark above, the data of an $\mathcal{O}$-space is a sequence of spaces X_n, together with maps

$$O(n)_+ \wedge X_n \to X_n$$

which are associative and unital. This corresponds precisely to data of an $O(n)$-action on X_n.

The strategy is that through Day convolution, we may transport the monoidal product structure on our diagram D to the diagram space.

Definition A.46. Let $\mathfrak{D}$ be a diagram with symmtric product $+$, and $F, G : \mathfrak{D} \to \mathbf{Top}_*$ be enriched functors. The **convolution product** $F \otimes G$ is defined to be the left Kan extension $\wedge \circ (F, G)$ along $+$.

And here is a lemma on computing the left Kan extension

Lemma A.47. Let A, B, C be V-enriched categories. Moreover, assume that A, B are small. Given enriched functors

$$F : A \to B \text{ and } G : A \to C$$

the left Kan extension $\mathrm{Lan}_G F$ of F along G is given by the enriched coend

$$\mathrm{Lan}_G F(b) = \int^{a \in C} C(Ga, b) \otimes Fa$$

Using the lemma, we see that the convolution product has the following form

$$(F \otimes G)c = \int^{a,b \in \mathfrak{D}} \mathfrak{D}(a+b, c) \wedge Fa \wedge Gb$$

Corollary A.48. The diagram space $\mathcal{O}\mathbf{Top}_*$ is symmetric monoidal, with unit the functor that send 0 to S^0 and every other number to a point.

Proof. The convolution product given in Definition A.46 gives us the desired tensor product in $\mathcal{O}\mathbf{Top}_*$. For objects F, G in $\mathcal{O}\mathbf{Top}_*$, unravelling the definition of the enriched coend as a coequalizer gives us precisely

$$(F \otimes G)_c = \bigvee_{a+b=c} O(a)_+ \wedge_{O(a)\times O(b)} Fa \wedge Gb$$

by Lemma A.47. It is straightforward to check that the unit is the functor that defined by $0 \mapsto S^0$ and $n \mapsto *$ for $n \neq 0$. $\square$

The sphere spectrum $\mathbb{S}$ in $\mathcal{O}\mathbf{Top}_*$ is the functor that sends n to S^n, with the standard induced action from $O(n)_+$.

Lemma A.49. The sphere spectrum $\mathbb{S}$ is a commutative monoid in $\mathcal{O}\mathbf{Top}_*$.

Proof. The structure map

$$\mathbb{S} \otimes \mathbb{S} \to \mathbb{S}$$

is given by

$$\begin{aligned}(\mathbb{S} \otimes \mathbb{S})_c &= \bigvee_{a+b=c} O(c)_+ \wedge_{O(a)\times O(b)} S^a \wedge S^b \\ &\cong \bigvee_{a+b=c} O(c)_+ \wedge_{O(a)\times O(b)} S^{a+b} \\ &\to S^a\end{aligned}$$

where the homotopy equivalence on the second line is given by the canonical homomotopy equivalence $S^a \wedge S^b \cong S^{a+b}$, and the map on the third line is given by evaluating each $O(c)$-action on S^{a+b} on each wedge summand.

One checks that the unit map $1 \to \mathbb{S}$ is given by sending $S^0 \to S^0$ and $* \to S^n$ for $n \neq 0$, and satisfies the unitary condition. Associativity is straightforward but messy to check. Commutativity is given by the commutativity of the diagram

$$\begin{tikzcd} O(c)_+ \wedge_{O(a)\times O(b)} S^a \wedge S^b \arrow[r] \arrow[d, "\chi_{a,b}\wedge \mathrm{twist}"'] & S^{a+b} \\ O(c)_+ \wedge_{O(b)\times O(a)} S^b \wedge S^a \arrow[ur] & \end{tikzcd}$$

where $\chi_{a,b}$ is the conjugation by block permutation matrices. $\square$

Recall that the tensor product of two R-modules A, B is the coequalizer of the diagram

$$\alpha_1, \alpha_2 : A \otimes R \otimes B \to A \otimes B$$

where the tensor is in $\mathbf{Ab}$ and the two maps are given by the action on A and B, respectively. Philosophically, the suspension structure maps should realize the category of spectra as modules over the sphere spectrum. We will make this precise in $\mathcal{O}\mathbf{Top}_*$:

Definition A.50. An **$\mathbb{S}$-module** is a object F in $\mathcal{O}\mathbf{Top}_*$ if there is an associative and unital map

$$\mathbb{S} \otimes F \to F$$

Theorem A.51. The category of orthogonal spectra is equivalent to the category of $\mathbb{S}$-modules in $\mathcal{O}\mathbf{Top}_*$.

Proof. Given an associative and unital map

$$\mathbb{S} \otimes F \to F$$

the formula in Lemma A.47 contains the data of a map

$$\bigvee_{a+b=c} O(c)_+ \wedge_{O(a)\times O(b)} S^a \wedge F_b \to F_{a+b}$$

Associativity implies that each map $S^a \wedge F_b \to F_{a+b}$ is a composition of structure maps $S^1 \wedge F_b \to F_{b+1}$. Moreover, the balanced product over $O(a) \times O(b)$ dictates that the maps $S^a \wedge F_b \to F_{a+b}$ are $O(a) \times O(b)$ equivariant. This recovers Definition A.40. □

We now can realize the smash product of spectra as tensor product of $\mathbb{S}$-modules, namely

$$X \wedge Y = X \otimes_{\mathbb{S}} Y = \mathrm{coeq}(X \otimes \mathbb{S} \otimes Y \rightrightarrows X \otimes Y)$$

where the two maps are the canonical $\mathbb{S}$-action on Y, and the composition of the twist map $X \otimes \mathbb{S} \to \mathbb{S} \otimes X$ and the $\mathbb{S}$-action on X. We may write out the smash product in explicit formulas: given orthogonal spectra X, Y, define $(X \wedge Y)_n$ to be the coequalizer of the maps

$$\alpha_X, \alpha_Y : \bigvee_{p+1+q=n} O(n)_+ \wedge_{O(p)\times Id \times O(q)} X_p \wedge S^1 \wedge Y_q \to \bigvee_{p+q=n} O(n)_+ \wedge_{O(p)\times O(q)} X_p \wedge Y_q$$

in the category of $O(n)$-spaces, where α_X is induced by the maps

$$X_p \wedge S^1 \wedge Y_q \xrightarrow{Id \wedge \sigma_Y} X_p \wedge Y_{q+1} \xrightarrow{Id \wedge \chi_{q,1}} X_p \wedge Y_{1+q}$$

on each summand, and α_Y is induced by the maps

$$X_p \wedge S^1 \wedge Y_q \xrightarrow{\mathrm{twist} \wedge Id} S^1 \wedge X_p \wedge Y_q \xrightarrow{\sigma_X \wedge Id} X_{p+1} \wedge Y_q$$

Theorem A.52. The category of $\mathfrak{D}$-spaces is closed symmetric monoidal with respect to the Day convolution product. The internal hom object $F(-,-)$ is given by the end

$$F(G, H)a = \int_{b \in \mathfrak{D}} \mathbf{Top}_*(Gb, H(a \otimes b))$$

Proof. The adjunction follows from manipulation of ends and coends. □

And we may define the internal hom spectra as $\mathbb{S}$-modules similarly to the algebra case

$$F_{\mathbb{S}}(X,Y) := \mathrm{eq}(F_{\mathcal{O}\mathbf{Top}_*}(X,Y) \rightrightarrows F_{\mathcal{O}\mathbf{Top}_*}(X \otimes \mathbb{S}, Y))$$

We can also package the data of $\mathbb{S}$-module into a new diagram $\mathcal{O}_{\mathbb{S}}$, so that the category of orthogonal spectra will be isomorphic to the category of $\mathcal{O}_{\mathbb{S}}$-spaces.

Definition A.53. Let $\mathcal{O}_{\mathbb{S}}$ be the $\mathbf{Top}_*$-enriched category, whose objects are non-negative integers. The enriched morphisms are are

$$\mathcal{O}_{\mathbb{S}}(a,b) := \begin{cases} * & a > b \\ \mathcal{O}\mathbf{Top}_*(\mathbb{S} \otimes \mathcal{O}(b,-), \mathbb{S} \otimes \mathcal{O}(a,-)) & a \leq b \end{cases}$$

The composition is given by composition in $\mathcal{O}\mathbf{Top}_*$.

We can also write out the formula for the enriched morphisms

$$\mathcal{O}\mathbf{Top}_*(\mathbb{S} \otimes \mathcal{O}(b,-), \mathbb{S} \otimes \mathcal{O}(a,-)) \cong \mathbb{S} \otimes \mathcal{O}(a,-)b \cong O(b)_+ \wedge_{O(b-a)} S^{b-a}$$

where the first isomorphism is by the enriched Yoneda lemma.

Lemma A.54. Let A be a pointed topological space. For $d \in \mathcal{O}$, there is a natural isomorphism

$$(\mathbb{S} \otimes \mathcal{O}(d,-)) \wedge A \cong F_a A$$

where the smash is applied termwise.

Proof. We simply check by hand

$$(\mathbb{S} \otimes \mathcal{O}(d,-))c \wedge A \cong \bigvee_{a+b=c} O(c)_+ \wedge_{O(a) \times O(b)} S^a \wedge \mathcal{O}(d,b) \wedge A$$

The wedge summand is trivial when $b \neq d$, and when $b = d$, we get a summand of the form

$$O(c)_+ \wedge_{O(c-d) \times O(d)} S^{c-d} \wedge O(d) \wedge A \cong O(c)_+ \wedge_{O(c-d)} S^{c-d} \wedge A$$

which is by definition $F_d A$. □

Theorem A.55. The category of orthogonal spectra $\mathbf{Sp}$ is equivalent to the category of $\mathcal{O}_{\mathbb{S}}$-spaces.

Proof. For one direction, we show that each object in $\mathcal{O}_{\mathbb{S}}$ defines an orthogonal spectrum: given a $\mathbf{Top}_*$-enriched functor $X : \mathcal{O}_{\mathbb{S}} \to \mathbf{Top}_*$, we have the natural isomorphism

$$\mathcal{O}_{\mathbb{S}}(a,a) \cong \mathcal{O}(a,a)$$

so each X naturally restricts to a functor in $\mathcal{O}\mathbf{Top}_*$; for the $\mathbb{S}$-module structure, we see that there is a map

$$S^b \to O(a+b)_+ \wedge_{O(b)} S^b \cong \mathcal{O}_{\mathbb{S}}(a, a+b)$$

by sending $x \in S^b$ to (e, x). This induces maps

$$S^b \wedge X(a) \to \mathcal{O}_{\mathbb{S}}(a, a+b) \wedge X(a) \to X(a+b)$$

which gives us the desired structure maps.

For the other direction, let X be an orthogonal spectrum, and let $a \leq b$. Then, the structure maps and the orthogonal group action gives us the map

$$O(b)_+ \wedge_{O(b-a)} S^{b-a} \wedge X(a) \to O(b)_+ \wedge_{O(b-a)} X(b) \to X(b)$$

which is unital and associative. This defines an enriched functor $\mathcal{O}_{\mathbb{S}} \to \mathbf{Top}_*$. □

Using the Day convolution, we see that the smash product of spectra is given by the formula

$$(X \wedge Y)a := \int^{b,c \in \mathcal{O}_{\mathbb{S}}} \mathcal{O}_{\mathbb{S}}(b+c, a) \wedge X_b \wedge X_c$$

and the internal hom is given by

$$F_{\mathbb{S}}(G, H)b := \int_{c \in \mathcal{O}_{\mathbb{S}}} \mathbf{Top}_*(G_c, H_{c+b})$$

We may finally prove the adjunction that give us the closed symmetric monoidal structure of $\mathbf{Sp}$.

Corollary A.56. We have the natural isomorphism

$$\mathbf{Sp}(X \wedge Y, Z) \cong \mathbf{Sp}(X, F_{\mathbb{S}}(Y, Z))$$

and $\mathbf{Sp}$ is closed symmetric monoidal.

Proof. We have the sequence of isomorphisms

$$\begin{aligned}
\mathbf{Sp}(X \wedge Y, Z) &= \int_{b,c \in \mathcal{O}_{\mathbb{S}}} \mathbf{Top}_*(X_b \wedge Y_c, Z_{b+c}) \\
&\cong \int_{b,c \in \mathcal{O}_{\mathbb{S}}} \mathbf{Top}_*(X_b, \mathbf{Top}_*(Y_c, Z_{b+c})) \\
&\cong \int_{b \in \mathcal{O}_{\mathbb{S}}} \mathbf{Top}_*(X_b, \int_{c \in \mathcal{O}_{\mathbb{S}}} \mathbf{Top}_*(Y_c, Z_{b+c})) \\
&\cong \int_{b \in \mathcal{O}_{\mathbb{S}}} \mathbf{Top}_*(X_b, F_{\mathbb{S}}(Y, Z)b) \\
&\cong \mathbf{Sp}(X, F_{\mathbb{S}}(Y, Z))
\end{aligned}$$

□

Corollary A.57. There are natual isomorphisms of orthogonal spectra

$$F_a A \wedge F_b B \cong F_{a+b}(A \wedge B)$$

for pointed topological spaces A, B.

Proof. We have the sequence of isomorphisms

$$\begin{aligned} F_nA \wedge F_mB &= \int^{b,c\in\mathcal{O}_{\mathbb{S}}} \mathcal{O}_{\mathbb{S}}(b+c,-)\wedge\mathcal{O}_{\mathbb{S}}(n,b)\wedge A\wedge\mathcal{O}_{\mathbb{S}}(m,c)\wedge B \\ &\cong \mathcal{O}_{\mathbb{S}}(n+m,-)\wedge A\wedge B \\ &\cong F_{n+m}(A\wedge B) \end{aligned}$$

where the middle isomorphisms is deduced from the enriched Yoneda Lemma. $\square$

Theorem A.58. The category of orthogonal spectra admits a cofibrantly generated stable model structure, with generating cofibration

$$I^{\mathcal{O}} := \{F_d(S^{n-1}_+ \to D^n_+) : n > 0\}$$

and weak equivalences π_*-isomorphisms.

By general theory, a closed symmetric monoidal structure on a model category descends to one on the homotopy category when the model category satisfies the pushout product axiom. The proof of **Sp** satisfying the assumption can be found in [BR20]. Moreover, we have

Theorem A.59. The stable model category of orthogonal spectra is Quillen equivalent to the stable model category of sequential spectra. In particular, $\mathcal{SHC}$ is closed symmetric monoidal.

References

[Ati66] M. F. Atiyah. "K-theory and reality". In: *Quart. J. Math. Oxford Ser. (2)* 17 (1966), pp. 367–386. ISSN: 0033-5606,1464-3847. DOI: 10.1093/qmath/17.1.367. URL: https://doi.org/10.1093/qmath/17.1.367.

[Bre67] Glen E. Bredon. *Equivariant cohomology theories.* Vol. No. 34. Lecture Notes in Mathematics. Springer-Verlag, Berlin-New York, 1967, vi+64 pp. (not consecutively paged).

[Ada74] J. F. Adams. *Stable homotopy and generalised homology.* Chicago Lectures in Mathematics. University of Chicago Press, Chicago, Ill.-London, 1974, pp. x+373.

[BF78] A. K. Bousfield and E. M. Friedlander. "Homotopy theory of Γ-spaces, spectra, and bisimplicial sets". In: *Geometric applications of homotopy theory (Proc. Conf., Evanston, Ill., 1977), II.* Vol. 658. Lecture Notes in Math. Springer, Berlin-New York, 1978, pp. 80–130. ISBN: 3-540-08859-8.

[Rav81] Douglas C. Ravenel. "The Segal conjecture for cyclic groups". In: *Bull. London Math. Soc.* 13.1 (1981), pp. 42–44. ISSN: 0024-6093,1469-2120. DOI: 10.1112/blms/13.1.42. URL: https://doi.org/10.1112/blms/13.1.42.

[LMM86] Jr. L. Gaunce Lewis, J. Peter May, and with contributions by J. E. McClure Mark Steinberger. *Equivariant Stable Homotopy Theory.* Vol. 1213. Lecture Notes in Mathematics. Springer-Verlag, 1986.

[Lew91] L. Gaunce Lewis Jr. "Is there a convenient category of spectra?" In: *J. Pure Appl. Algebra* 73.3 (1991), pp. 233–246. ISSN: 0022-4049,1873-1376. DOI: 10.1016/0022-4049(91)90030-6. URL: https://doi.org/10.1016/0022-4049(91)90030-6.

[Wei94] Charles A. Weibel. *An introduction to homological algebra.* Vol. 38. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 1994, pp. xiv+450. ISBN: 0-521-43500-5; 0-521-55987-1. DOI: 10.1017/CBO9781139644136. URL: https://doi.org/10.1017/CBO9781139644136.

[Nee96] Amnon Neeman. "The Grothendieck duality theorem via Bousfield's techniques and Brown representability". In: *J. Amer. Math. Soc.* 9.1 (1996), pp. 205–236. ISSN: 0894-0347,1088-6834. DOI: 10.1090/S0894-0347-96-00174-9. URL: https://doi.org/10.1090/S0894-0347-96-00174-9.

[Kar01] Max Karoubi. "A descent theorem in topological K-theory". In: *K-Theory* 24.2 (2001), pp. 109–114. ISSN: 0920-3036,1573-0514. DOI: 10.1023/A:1012785711074. URL: https://doi.org/10.1023/A:1012785711074.

[MM02] M. A. Mandell and J. P. May. "Equivariant orthogonal spectra and S-modules". In: *Mem. Amer. Math. Soc.* 159.755 (2002), pp. x+108. ISSN: 0065-9266,1947-6221. DOI: 10.1090/memo/0755. URL: https://doi.org/10.1090/memo/0755.

[Dug05] Daniel Dugger. "An Atiyah-Hirzebruch spectral sequence for KR-theory". In: *K-Theory* 35.3-4 (2005), pp. 213–256. ISSN: 0920-3036,1573-0514. DOI: 10.1007/s10977-005-1552-9. URL: https://doi.org/10.1007/s10977-005-1552-9.

[Ull13] John Ullman. "On the slice spectral sequence". In: *Algebr. Geom. Topol.* 13.3 (2013), pp. 1743–1755. ISSN: 1472-2747,1472-2739. DOI: 10.2140/agt.2013.13.1743. URL: https://doi.org/10.2140/agt.2013.13.1743.

[HHR16] M. A. Hill, M. J. Hopkins, and D. C. Ravenel. "On the nonexistence of elements of Kervaire invariant one". In: *Ann. of Math. (2)* 184.1 (2016), pp. 1–262. ISSN: 0003-486X,1939-8980. DOI: 10.4007/annals.2016.184.1.1. URL: https://doi.org/10.4007/annals.2016.184.1.1.

[Blu17] A. Blumberg. "The Burnside Category". Lecture Notes. 2017. URL: https://people.math.rochester.edu/faculty/doug/otherpapers/blumberg-burnside.pdf.

[BR20] David Barnes and Constanze Roitzheim. *Foundations of stable homotopy theory*. Vol. 185. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2020, pp. vi+423. ISBN: 978-1-108-48278-3. DOI: 10.1017/9781108636575. URL: https://doi.org/10.1017/9781108636575.

[HHR21] Michael A. Hill, Michael J. Hopkins, and Douglas C. Ravenel. *Equivariant stable homotopy theory and the Kervaire invariant problem*. Vol. 40. New Mathematical Monographs. Cambridge University Press, Cambridge, 2021, pp. ix+870. ISBN: 978-1-108-83144-4. DOI: 10.1017/9781108917278. URL: https://doi.org/10.1017/9781108917278.

[Sch23] S. Schwede. "Lectures on Equivariant Stable Homotopy Theory". Lecture Notes. 2023. URL: https://www.math.uni-bonn.de/~schwede/equivariant.pdf.