Morse Homology

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December 2021

1 Some Contexts and Intuition

In the classification of manifolds in **Top**, it is generally hopeless to construct explicit homeomorphisms between manifolds. As an alternative, we first opt to look for topological invariants, such as homology and homotopy groups, to tell non-homeomorphic manifolds apart. However, the drawback of homology is that it is a weak invariant, as seen by the Poincare $3-$sphere. On the other hand, higher homotopy groups are difficult to compute. So, the goal is to construct tools that extracts more information and are computationally viable. Morse Theory, first considered by Marston Morse, employs the brilliant idea of using a "nice" function on a smooth manifold to reflect its topology. In particular, critical points of a nice smooth function should represent points where the topology of the manifold "changes".

As an explicit example, consider the torus embedded in $\mathbb{R}^3$ vertically on top of the $xy-$plane, as depicted in Figure 1. Then, let the smooth function f from the torus to $\mathbb{R}$ being its height. By our understanding of multivariable calculus, we know the critical points of the function are exactly the local maximum, local minimum, and saddle points. By examining the level sets, we see that the topology "changes" at all critical point, and stays the same in between two critical points, ordered by their height. A bad example would be when the

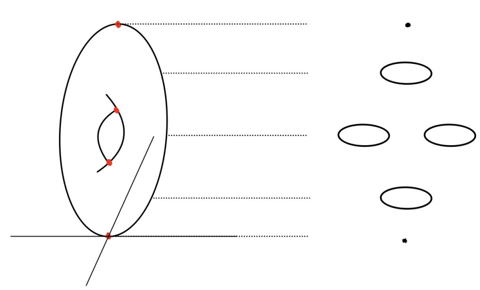

Figure 1: Torus and level sets

torus lies flat on the $xy-$plane, with the smooth function still being the height. In this case, the extremas consists two circles, which are uncountably many points and certainly do not reveal as much information as our previous case. This leads us to consider the following questions: what exactly defines a nice-enough function? And in what ways can we recover useful information about the manifold? Can we at least recover singular homology?

2 Set up for Morse Homology

To keep the section clean, we assume some basic knowledge in differential geometry such as smooth manifolds, (co)tangent bundle, etc. In the following discussion, we will assume M a Riemannian manifold with metric g.

Definition 2.1. *Given a smooth map $f : M \longrightarrow \mathbb{R}$, a point $x \in M$ is called a* ***critical point*** *if the differential*

$$df_x : T_xM \longrightarrow \mathbb{R}$$

is the 0 map.

The differential is a generalization to the derivative in euclidean analysis, and the critical point is still where the derivative vanishes.

Definition 2.2. *Let ∇ be any connection of the tangent bundle TM Given x is a critical point, we have a well-defined generalization of the second derivative $H(f,x) : T_xM \longrightarrow T^*_xM$, defined point-wise for each tangent vector $v \in T_pM$:*

$$H(f,p)(v) := \nabla_v(df)$$

The map $H(f,p)$ is called the ***Hessian****.*

Locally around a fixed critical point x, we can find a set of coordinates $\{x_1, ..., x_n\}$ on M, which corresponds to a set of basis $\{\frac{\partial}{\partial x_i}\}$ for $T_xM$and a set of basis $\{dx_i\}$ for T^*_xM. Then, locally Hessian can be expressed in matrix for as $[\frac{\partial^2 f}{\partial x_i \partial x_j}]$. Note that this matrix is symmetric by smoothness assumption and Clairaut's theorem. As a result, we can find real eigenvalues of map. We then define the **index** of a critical point p, denoted $ind(p)$, to be the number of negative eigenvalues that the hessian has. Moreover, a critical point x is called **non-degenerate** if and only if the determinant of the hessian at x is non-zero.

Definition 2.3. *A smooth map $f : M \longrightarrow \mathbb{R}$ is called* ***Morse*** *if and only if all of its critical points are non-degenerate.*

By Morse Lemma (whose proof we omit but can be found at [1]), which is a generalization of Taylor's theorem that a smooth function is approximated by its derivatives, we may deduce that the critical points of a Morse function are isolated. As a corollary, the number of critical points are finite on a compact manifold. Moreover, Morse functions are in fact dense in C^∞, so they are in this sense generic smooth functions and not exotic unicorns that makes the theory hard to execute in practice. A proof of this fact is in [1].

Definition 2.4. *Let V denote the negative gradient of f with respect to the metric g. Then given two critical points p, q, we may define a* ***negative-gradient flowline*** *between the two points as a smooth map $\gamma : \mathbb{R} \longrightarrow M$, such that*

$$\lim_{x \to -\infty} \gamma(x) = p, \lim_{x \to \infty} \gamma(x) = q$$

and $\gamma'(x) = V(\gamma(x))$. Note that $\mathbb{R}$ acts on a given flowline by changing the "speed" of the flow but does not change the geometry. Thus, we are motivated to define the moduli space $M(p, q)$, which is the space of all negative gradient flowlines from p to q modulo $\mathbb{R}$ action. Under tranversality assumptions (referred to as the Smale condition in [1]), $M(p, q)$ has the natural structure of a manifold of dimension

$$dimM(p, q) = ind(p) - ind(q) - 1$$

In particular, if $ind(p) - ind(q) = 1$, then $M(p, q)$ is a $0-$dimensional manifold, and is a finite collection of points if it is compact.

3 Morse Homology

In our following discussion, we fix our smooth manifold M and Morse function f unless indicated otherwise.

We are now tempted to define Morse homology: the ith Morse chain complex, denoted by MC_i, is the free $\mathbb{Z}_2$ module generated by critical points of index i. (We forget orientation to make our lives easier). The chain map $\partial : MC_i \longrightarrow MC_{i-1}$, is defined by the mapping of the generators:

$$\partial(p) = \sum_{q: ind(q) = ind(p) - 1} |M(p, q)| \cdot q$$

The Morse homology groups, denoted by MH_* are defined in the usual way:

$$MH_* = \frac{ker\partial}{im\partial}$$

Our big theorem here will be to show that the Morse homology group is well-defined. In fact, it is an honest homology theory that satisfies the Eilenberg-Steenrod axioms, so it is isomorphic to singular homology, but we defer that to later discussion.

Theorem 3.1. *MH_* is well-defined, i.e $\partial^2 = 0$ and $M(p, q)$ is finite when $ind(q) = ind(p) - 1$*

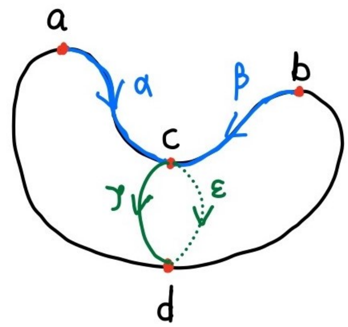

Figure 2: Hot Dog

Before proving Theorem 3.1, let us see an fun example of why this may be true. Consider M to be the hot-dog homeomorphic to the sphere embedded in $\mathbb{R}^3$, and the Morse function to be the height function, depicted in Figure 2.

In this case, finiteness of the moduli space is clearly represented by the flowlines. Moreover, since c is the only element in MC_1, and $\partial(c) = 2 \cdot d = 0 \pmod 2$, we have ∂^2 trivially true. In fact, a quick computation here would show that $MH_0 \cong MH_2 \cong \mathbb{Z}_2$, and MH_i are trivial for $i \neq 0, 2$, so we do recover singular homology of S^2 with $\mathbb{Z}_2$ coefficients.

The core idea behind proving Theorem 3.1 would be using "broken flowlines".

Definition 3.1. *Let p, q be critical points of f. Then*

$$\overline{M}(p,q) = \bigcup_{c_i \in Crit(f)} M(p, c_1) \times ... \times M(c_k, q)$$

is called the space of broken flowlines from p to q.

As the notation suggests, $\overline{M}(p,q)$ is intended to be the compactification of $M(p,q)$. In some sense, $M(p,q)$ is the interior open set, and we are adding the "boundary limit points," which are the broken flowlines, for compactification. But of course we first have to equip a topology on $\overline{M}(p,q)$ such that it should induce the product topology on the RHS. Since it is rather long and technical to define, we refer to [1].

Lemma 3.2. *$\overline{M}(p,q)$ is compact.*

Proof. A detailed proof can be found in [1]. The basic idea is showing that all sequence of flowlines in $M(p,q)$ has a convergent subsequence in $\overline{M}(p,q)$. $\square$

Corollary 3.2.1. *$M(p,q)$ is a finite collection of points when $ind(p) = ind(q) - 1$.*

Proof. Note for a flowline between p, q to have negative gradient, we must have $ind(p) > ind(q)$. Thus for $M(p, q)$ with $ind(p) = ind(q) - 1$, we have $M(p, q) = \overline{M}(p, q)$ as we do not have more "room" for broken flowlines in between. Since $M(p, q)$ is a compact $0-$dimensional manifold, it must be a finite collection of points. $\square$

Corollary 3.2.2. $\partial^2 = 0$

Proof. For simplicity of notation, let Crit_i denote the set of critical points of index i. Suppose p is a critical point of index i. By direct computation, we have

$$\begin{aligned}\partial(\partial(p)) &= \partial(\sum_{c_1 \in \mathrm{Crit}_{i-1}} |M(p, c_1)| \cdot c_1) \\ &= \sum_{c_1 \in \mathrm{Crit}_{i-1}, c_2 \in \mathrm{Crit}_{i-2}} |M(p, c_1)||M(c_1, c_2)| \cdot c_2\end{aligned}$$

Thus, it suffices to show $\sum_{c_2} |M(p, c_1)||M(c_1, c_2)| = 0$, which we identify as the cardinality of the boundary of $\overline{M}(p, c_2)$. But since $\overline{M}(p, c_2)$ is a compact one-dimensional manifold, its boundary is either 0 points or 2 points, which are both 0 mod2. $\square$

Corollary 3.2.1 and 3.2.2 together implies Theorem 3.1, and we are done.

4 Some applications

Although we are not going to prove it in this essay, it can be shown that Morse homology is isomorphic to singular homology(a potential reference is [1] and [3]). Thus, Morse homology should also enjoy results such as excision, Mayer-Vietoris, Kunneth formula, etc. On the other hand, we can also use Morse homology to show that these results are true without refering to singular homology. In particular, Poincare duality accepts a clean presentation with Morse theory.

4.1 Poincare Duality

The duality theorem in the context of Morse theory is somewhat more obvious. Note that for a Morse function f paired with negative gradient V on a manifold M, $-f$ would be a honest Morse function paired with negative gradient $-V$ on a M. Moreover, if p is a critical point of index k of f, then it would be a critical point of index $n-k$ of $-f$, where n is the dimension of the manifold. An intuitive scenario would be f being the height function of a vertical torus, and $-f$ would be the height function of the torus standing upside-down. It is clear that the critical points remain the same, but the indices are flipped. From now on, let $MC_*(f)$ denote the Morse chain complex associated to a Morse function f and with $\mathbb{Z}_2$ coefficients. We have established that

$$MC_k(f) \cong MC_{n-k}(-f)$$

With this isomorphism in mind, we take a slight detour in defining Morse cohomology

Definition 4.1. *The Morse co-chain complex associated to Morse function f with $\mathbb{Z}_2$ coefficients, denotes by $MC^*(f)$, is the dual complex to $MC_{n-*}(-f)$.*

We then have the commutative diagram

$$\begin{tikzcd}
MC^{n-k}(f) \arrow[r, "\partial_{n-k+1}"] & MC^{n-k+1}(f) \\
MC_{n-k}(-f) \arrow[u] & MC_{n-k+1}(-f) \arrow[u] \arrow[l, "\delta'_{n-k+1}"'] \\
MC_k(f) \arrow[u] \arrow[r, "\delta_k"] \arrow[uu, bend left=60, "\phi_k"] & MC_{k-1}(f) \arrow[u] \arrow[uu, bend right=60, "\phi_{k-1}"']
\end{tikzcd}$$

Here ϕ and φ are compositions of the two up arrows, which are isomorphisms as they are both dual maps of finitely generated modules. Then, ϕ and φ are both isomorphisms. If they are both chain maps, then the isomorphism between homology and cohomology groups formulated in Poincare duality should follow directly. Then chain map condition explicitly is the following:

$$\partial_{n-k+1} \circ \phi_k = \phi_{k-1} \circ \delta_k$$

Choose $a \in MC_k(f)$ and $b \in MC_{n-k+1}(-f)$, we have

$$\begin{aligned}
(\phi_{k-1} \circ \delta_k(b))(a) &= \phi_{k-1}(\sum_{a_i \in \mathrm{Crit}_{k-1}} |M_f(b, a_i)|a_i)(a) \\
&= \sum_{a_i \in \mathrm{Crit}_{k-1}} |M_f|(b, a_i) a_i^*(a) \\
&= |M_f(b, a)|
\end{aligned}$$

$$\begin{aligned}
(\partial_{n-k+1} \circ \phi_k(b))(a) &= \partial_{n-k+1}(b^*)(a) \\
&= (b^* \circ \delta_{n-k+1})(a) \\
&= b^*(\sum_{b_i \in \mathrm{Crit}_{n-k}} |M_{-f}(a, b_i)|b_i \\
&= |M_{-f}(a, b)|
\end{aligned}$$

So by noting that $|M_{-f}(a,b)| = |M_f(b,a)|$ by a reverse-flow argument(a rigorous treatment with orientation can be found in [1]), we conclude that ϕ is indeed a chain map, and deduce the following version of Poincare duality.

Theorem 4.1. *(Poincare Duality) Let M be a closed, smooth $n-$manifold. Then there is an isomorphism $MH^{n-k}(M;\mathbb{Z}_2) \cong MH_k(M;\mathbb{Z}_2)$.*

5 Classic vs Modern Morse Theory

The classical approach of Morse theory (Morse, Smale, Milnor) differs from what is described in this essay. As a crude generalization, the original Morse theory focuses on the sub-level sets in between critical points and examines their topology. Then through the process called "attaching handles" at critical points corresponding to their index, one can reconstruct the manifold up to homotopy equivalence. This process lie in the center of what is called "h-corbordism" theorem, which leads to the Smales's proof of generalized Poincare conjecture for dimension > 4 in **Top**. A introduction to this classical approach of Morse Theory can be found in [2].

The classic approach fails when dealing with infinite dimensional manifolds, in which case the handle attachment and indexing do not make sense anymore. This has led to Floer and Witten's modern formulation of Morse homology with gradient flowlines, which makes doing Morse theory possible in infinite dimensions. The infinite-dimensional analogy of Morse homology is referred to as Floer homology, and it is a very powerful tool in the context of Homological Mirror Symmetry. In broad terms, Homological mirror symmetry is an exchange between symplectic geometry and algebraic geometry, and Floer homology comes in to package up information in the sympletic side into what is called Fukaya Categories for a categorical equivalance.

However, more technical difficulties are still expected. For example, the transversality requirement and compactification become incredibly difficult to show for a generic manifold. An example of toy case where homological mirror symmetry can be explicitly constructed is on a elliptic curve, and a detailed treatment can be found here.[4]

6 References

[1] Mich'ele Audin and Mihai Damian. *Morse Theory and Floer Homology.* Universitext. Springer-Verlag London, 2014.

This book is a very detailed treatment of Morse Theory in the Floer approach. Most of the proofs and statements in this essay are credited to this source. The second half of the book also discusses Floer homology.

[2]J. Milnor, *Morse Theory*, Annals of Mathematics Studies no.51, Princeton University Press, 1963.

This set of notes presents the classical approach of Morse theory developed by Morse, Milnor and Smale. The classic approach leads to the h-corbordism theorem, which is central in solving the generalizad poincare conjecture for dimension > 4.

[3]M. Hutchings, Lecture Notes on Morse Homology(with an eye towards Floer theory and pseudoholomorphic curves). Lecture Notes, UC Berkeley, delivered Dec 15, 2002.

I referenced this set of a notes as it gives a great outline for the content covered in this essay, without piling on too much technical details.

[4]A. Port, "An Introduction to Homological Mirror Symmetry and the Case of Elliptic Curves," arxiv:1501.00730 [math sg], Jan 2015.

This is an interesting, concrete example of Floer Theory and Fukaya Category in application.