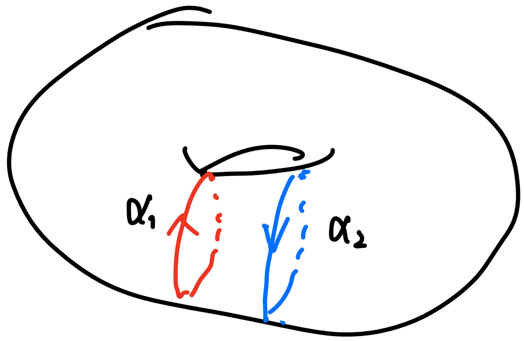
Orientation

A simple closed curve on S

$$\alpha : S^1 \xrightarrow{\text{inj}} S$$



An isotopy

$$H : S^1 \times I \longrightarrow S, \quad H_t \text{ embedding}$$

Problem: α_1, α_2 NOT isotopic as maps / oriented curves.

Def

Two s.c.c α, β are isotopic as unoriented curves

if $\alpha \sim \beta$ or $\alpha \sim \bar{\beta}$ as maps

Equivalently, if there is an isotopy

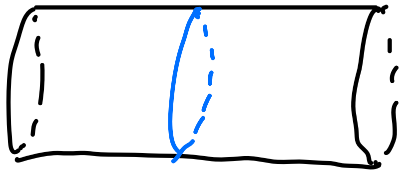
$$H : S^1 \times I \longrightarrow S$$

where $H(S^1 \times \{0\}) = \alpha(S^1)$ and
$H(S^1 \times \{1\}) = \beta(S^1)$ as embedded image

Rmk: Farb-Margalit says: "... fix the isotopy class of α" w/o saying "as map" or "oriented", they mean isotopy class of α as unoriented s.c.c.

Where we only need unoriented curves

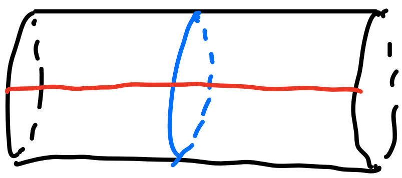
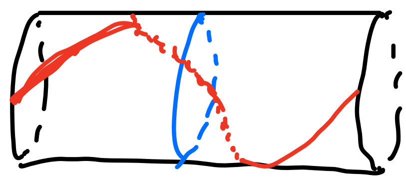
- geometric intersection $i(\alpha, \beta)$
- Dehn Twists T_a : orientation of a does not matter



a

surface has an orientation $\Rightarrow$ inside v.s outside

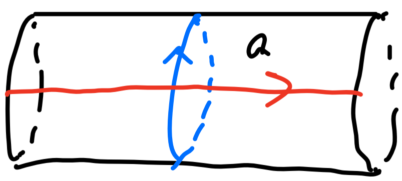
$\Rightarrow$ place hand on "top" & twist

 $\mapsto$ 

b

On homology:

$\hat{i}(a,b) = 1$



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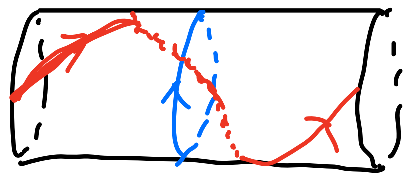
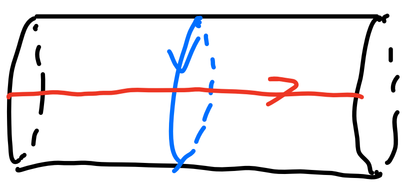
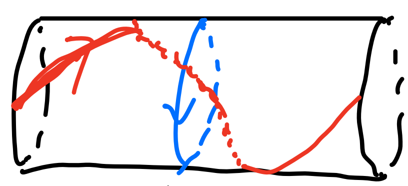


b $\qquad$ b

$$[a] \xmapsto{T_b} [a] + [b]$$

$\hat{i}(a,b) = -1$

a

$\mapsto$ a


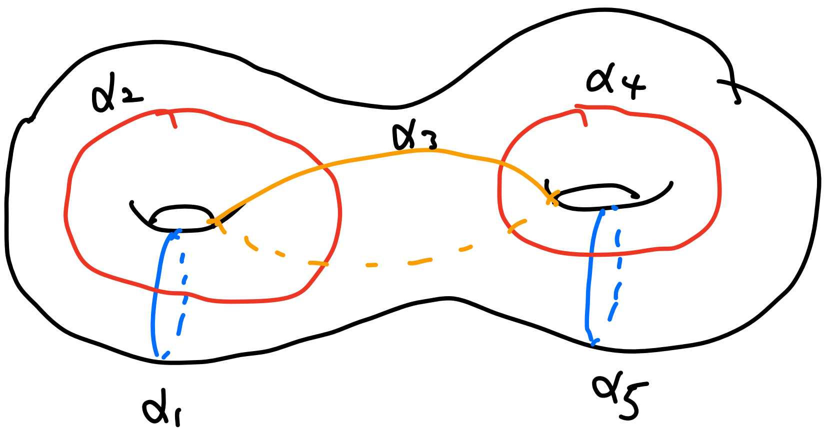
b $\qquad$ b

$$[a] \xmapsto{T_b} [a] - [b]$$

Fact T_a commutes w/ f iff $f(a) \sim a$ as unoriented curves

Cor hyperelliptic involution ι generate $Z(S_g)$ for $g = 1, 2$

" ι fixes every isotopy class of unoriented s.c.c on S_1, S_2 "



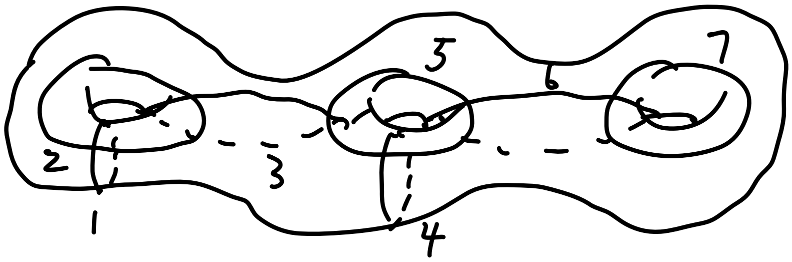
- $\{T_{\alpha_i}\}$ generate $\mathrm{Mod}(S_2)$

- Alexander method

If $\{\gamma_i\}$ is a collection of s.c.c on S, satisfying

- pairwise non-isotopic (as unoriented curves)
- pairwise min. pos.
- For triples i, j, k, at least one $\gamma_i \cap \gamma_j, \gamma_i \cap \gamma_k, \gamma_j \cap \gamma_k = \phi$

e.g



$\gamma_1 - \gamma_7$

And $\{\gamma_i'\}$ is another collection s.t

$\gamma_i \sim \gamma_i'$ as unoriented s.c.c

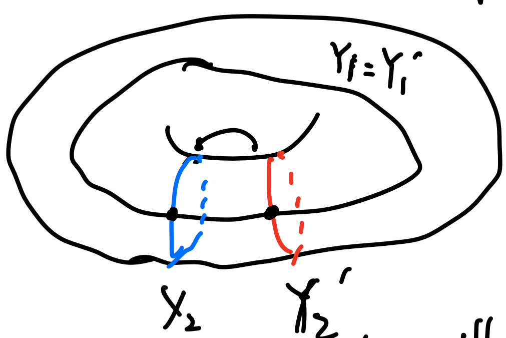
Then there is an ambient isotopy of S

$$H: S \times I \longrightarrow S$$

" taking γ_i to γ_i' ", which means

$H_1(\gamma_i) = \gamma_i'$ as embedded subsets.

<u>Rmk</u> We cannot take γ_i as maps !



since the intersection pt will be moved.

<u>Upshot</u> If $\gamma_i' = \phi(\gamma_i)$ for some $\phi \in \mathrm{Homeo}^+(S)$, the

H takes - intersections to intersections

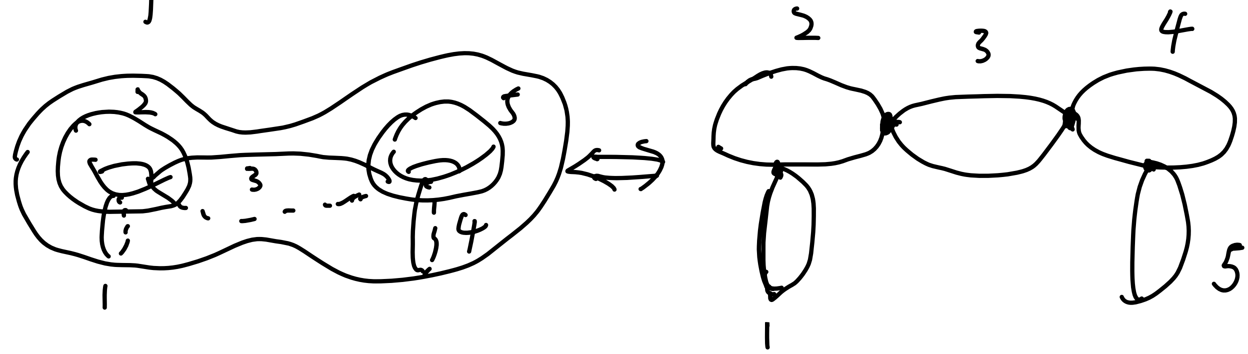
- edges to edges (of same vertices)

so we can view $\cup \gamma_i$ as a graph w/ vertices intersections

and ϕ induces an automorphism of the graph.

- If ϕ induces Id on the graph $\Rightarrow$ ϕ is isotopic to Id

Eg Hyperelliptic inv i on S_2

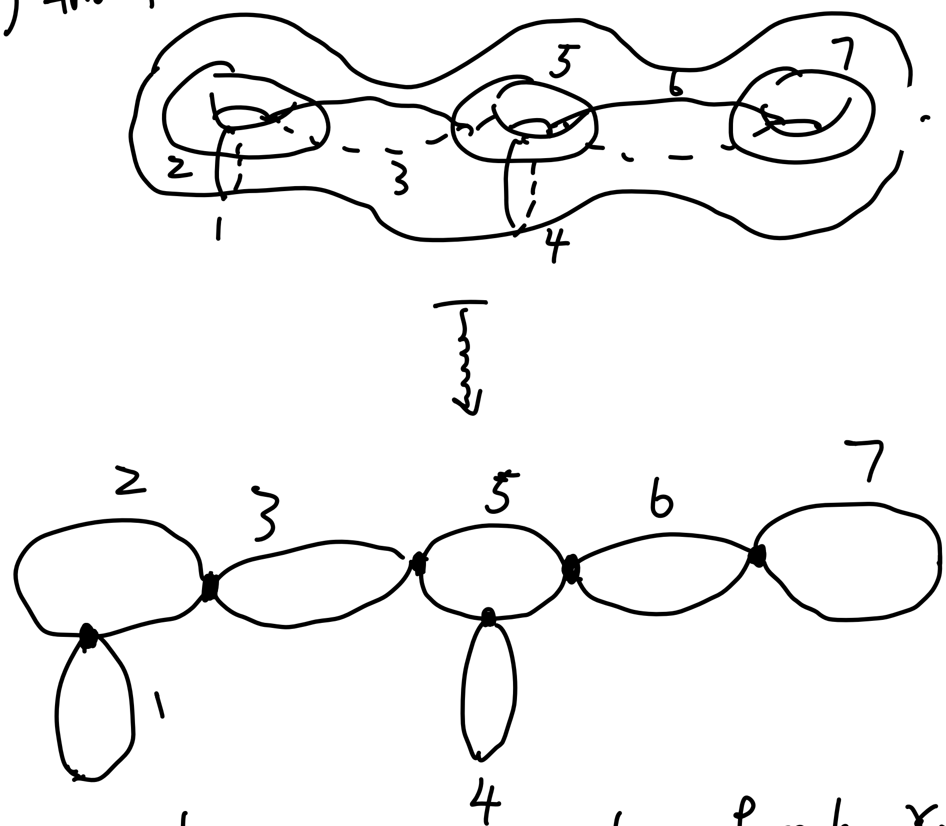


"edges flipped"

order 2 element

Eg $Z(\mathrm{Mod}(S_g))$ trivial, $g \geq 3$

• hyperelliptic no longer fixes [4]



If $\phi \in Z(S_g)$, ϕ preserves isotopy class of each γ_i above

- vertices of 5 are fixed, so are the edges
- orientation of 5 is fixed
- ϕ being orientation preserving $\Rightarrow$ orientation of 3, 4, 6 are fixed as well
- keep going out

When we need oriented curves

- isotopy / homotopy / homology
- algebraic intersection $\hat{i}([\alpha],[\beta])$ "signed count of intersection"

Symplectic Representation

Let V be a real v.s, $\dim V$ even.

Def A symplectic form on V is

$$\omega : V \times V \longrightarrow \mathbb{R}$$

- bilinear
- Alternating $\omega(v,v)=0$
- Non-degen $\omega(v,u)=0$ for all $u \Rightarrow v=0$

The pair (V,ω) is called a symplectic v.s.

L.A Fact
Every symplectic vector space (V,ω) has a Darboux basis

$$\{x_1, y_1, \ldots, x_n, y_n\}$$ "Symplectic Gram-Schmidt"

s.t $\omega(x_i, y_j) = \delta_{ij}$

$\omega(x_i, x_j) = \omega(y_i, y_j) = 0$

Consequence

$$\omega(u,v) = u^T J v \qquad \text{where } J = \begin{array}{c} \begin{array}{cccc} x_1 & y_1 & x_2 & y_2 \end{array} \\ \begin{bmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & 0 & 1 & \\ & & -1 & 0 & \\ & & & & \ddots \end{bmatrix} \end{array}$$

Def $Sp(2g, \mathbb{R}) := \{ A \in GL(2g, \mathbb{R}) : A^T J A = J \}$ $\quad \omega(Au, Av) = \omega(u,v)$

= linear automorphisms that preserves ω

$$Sp(2g, \mathbb{Z}) = Sp(2g, \mathbb{R}) \cap GL(2g, \mathbb{Z})$$

- $\det(A) = 1$ if $A \in Sp(2g, \mathbb{Z})$ "orientation preserving"

Check directly • $Sp(2, \mathbb{Z}) = SL(2, \mathbb{Z})$

The alg. intersection

$$i : H_1(S_g; \mathbb{R}) \times H_1(S_g; \mathbb{R}) \longrightarrow \mathbb{R}$$

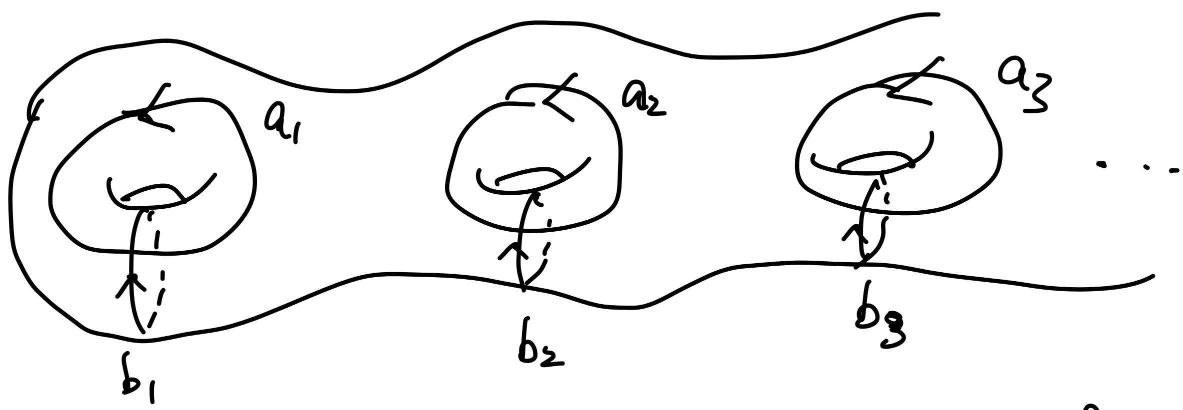
$$([\alpha]^* \cup [\beta]^*) \cap [S_g] \quad \text{oriented}$$

is :
- bilinear
- Alternating
- Non-degen

$*$ If $[\alpha], [\beta]$ are represented by α, β that intersect transversally, then

$\langle [\alpha]^* \cup [\beta]^*, [S_g] \rangle =$ signed count of the intersection

Back to topology :



H

- $\{[a_1], [b_1], [a_2], [b_2], \ldots \}$ is an ordered basis for $H_1(S_g)$ and we can view $H_1(S_g; \mathbb{Z}) \subset H_1(S_g; \mathbb{R})$ as a lattice
- $\hat{i} = \sum_{i=1}^{g} [a_i]^* \wedge [b_i]^*$ (Checking basis pairing)
- $i(a_i, b_j) = \hat{i}([a_i], [b_j])$

This is the "standard geometric symplectic basis".

Any other basis H' that satisfies the above is called "geometric symplectic basis".

<u>Fact 1</u> Given any non-separating s.c.c γ, we can complete it to a geometric symplectic basis.

<u>Proof</u> : change of coordinates

$$f(b_1) = \gamma$$

Later we will show that

$$(\bar{\Psi} T_b^k)[a] = [a] + k\hat{i}([a],[b])[b]$$

Pf Extend b to a geometric basis $\{b, v, x_2, y_2, \ldots\}$

so $[\bar{T}_b^k(v)] = [v] + k[b]$ and T_b^k fixes other basis curves.

Now let $[a]$ be some arbitrary class. Its $[v]$-th coordinate is $\hat{i}([a],[b])$, so the general formula follows from linearity.

$$(\bar{\Psi} T_b^k)(\hat{i}([a],[b])[v])$$

$$= \hat{i}[a],[b][v] + k\hat{i}([a],[b])[b]$$

The representation

We have $\phi \in \mathrm{Homeo}^+(S) \curvearrowright H_1(S_g;\mathbb{R})$. Homotopy invariance gives us

$$\Psi_{\mathbb{R}}: \mathrm{Mod}(S) \longrightarrow \mathrm{Aut}(H_1(S_g;\mathbb{R}))$$

preserving alg int $\Rightarrow$ $\mathrm{Im}(\Psi) \subset Sp(2g,\mathbb{R})$

preserving the lattice $H_1(S_g;\mathbb{Z})$

$$\Downarrow$$

$$\Psi: \mathrm{Mod}(S) \longrightarrow Sp(2g,\mathbb{Z})$$ "The symplectic representation"

Thm Ψ is a surjective homomorphism.

"$Sp(2g,\mathbb{Z})$ is the part of $\mathrm{Mod}(S)$ that acts non-trivially on non-separating s.c.c"

<u>Fact</u> $Sp(2g, \mathbb{Z})$ can be generated by

4 types "Burkhardt's generators"

Realize these geometrically
as the image of some mapping class under Ψ

1. Transvection: $(x_1, y_1, x_2, y_2, \ldots) \mapsto (x_1 + y_1, y_1, x_2, y_2, \ldots)$

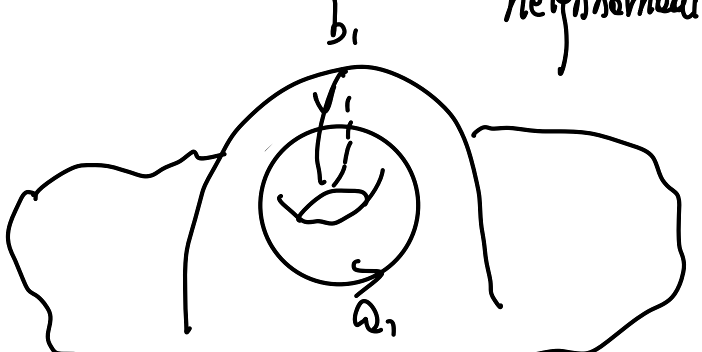
we saw $T_{b_1}([a_1]) = [a_1] + [b_1]$

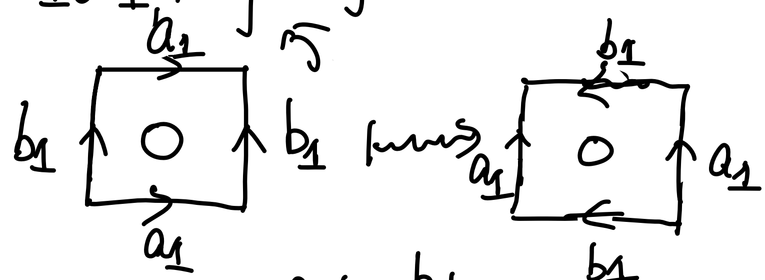
$[a_1] = (1, 0, 0, \ldots)$ in the standard basis

so transvection $= \Psi(T_{b_1})$

2. Factor Rotation: $(x_1, y_1, x_2, y_2, \ldots) \mapsto (-y_1, x_1, x_2, y_2, \ldots)$

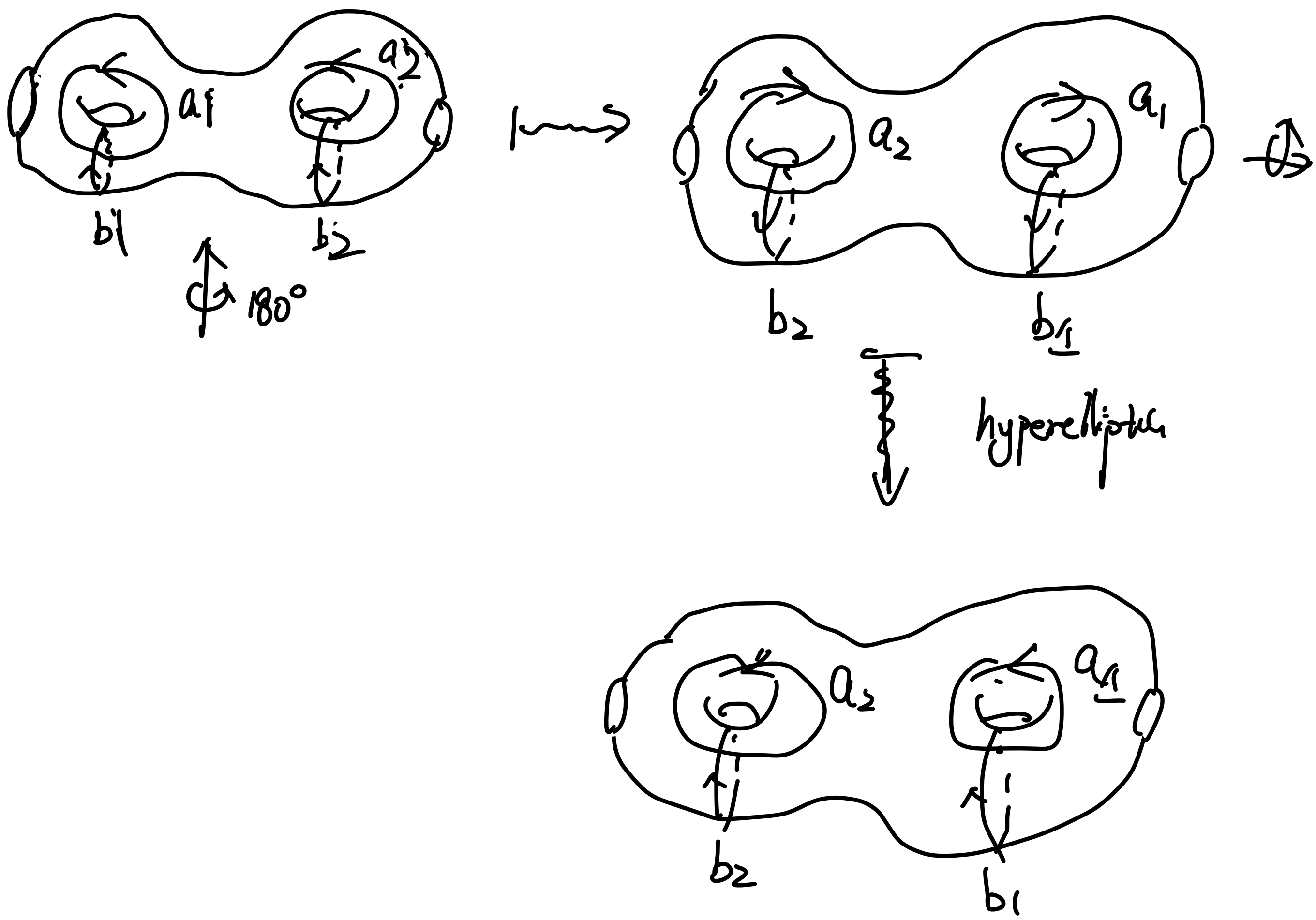
neighborhood $a_1 \cup b_1$, disjoint from a_i, b_i, $i \geq 2$



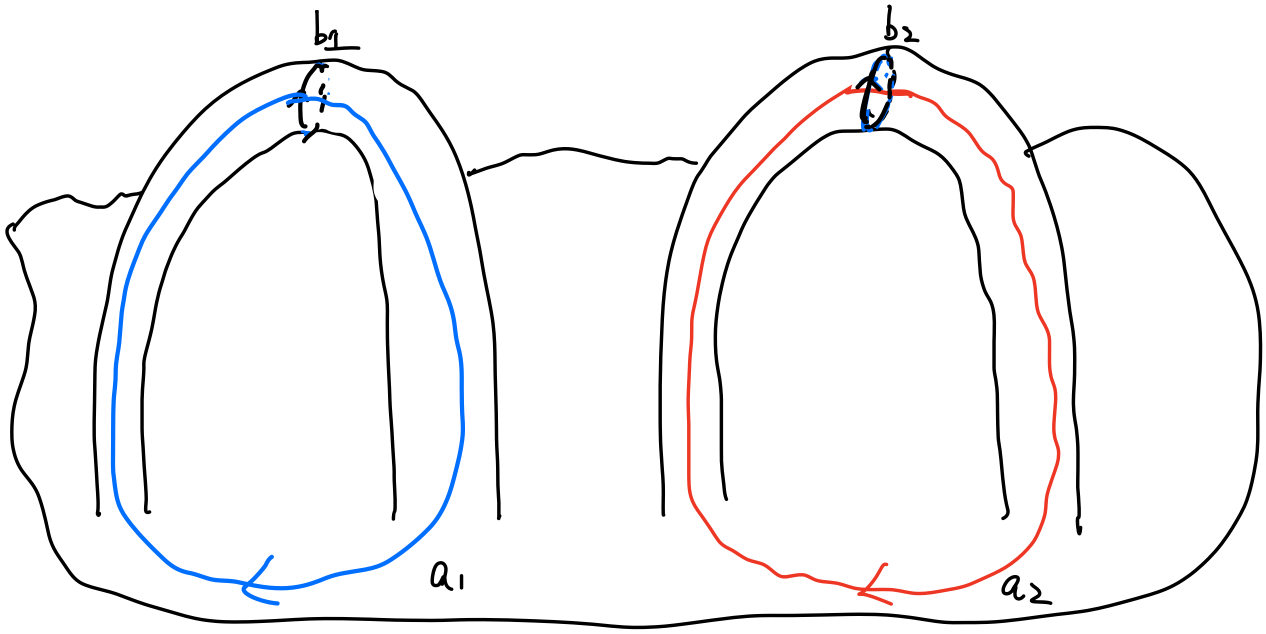


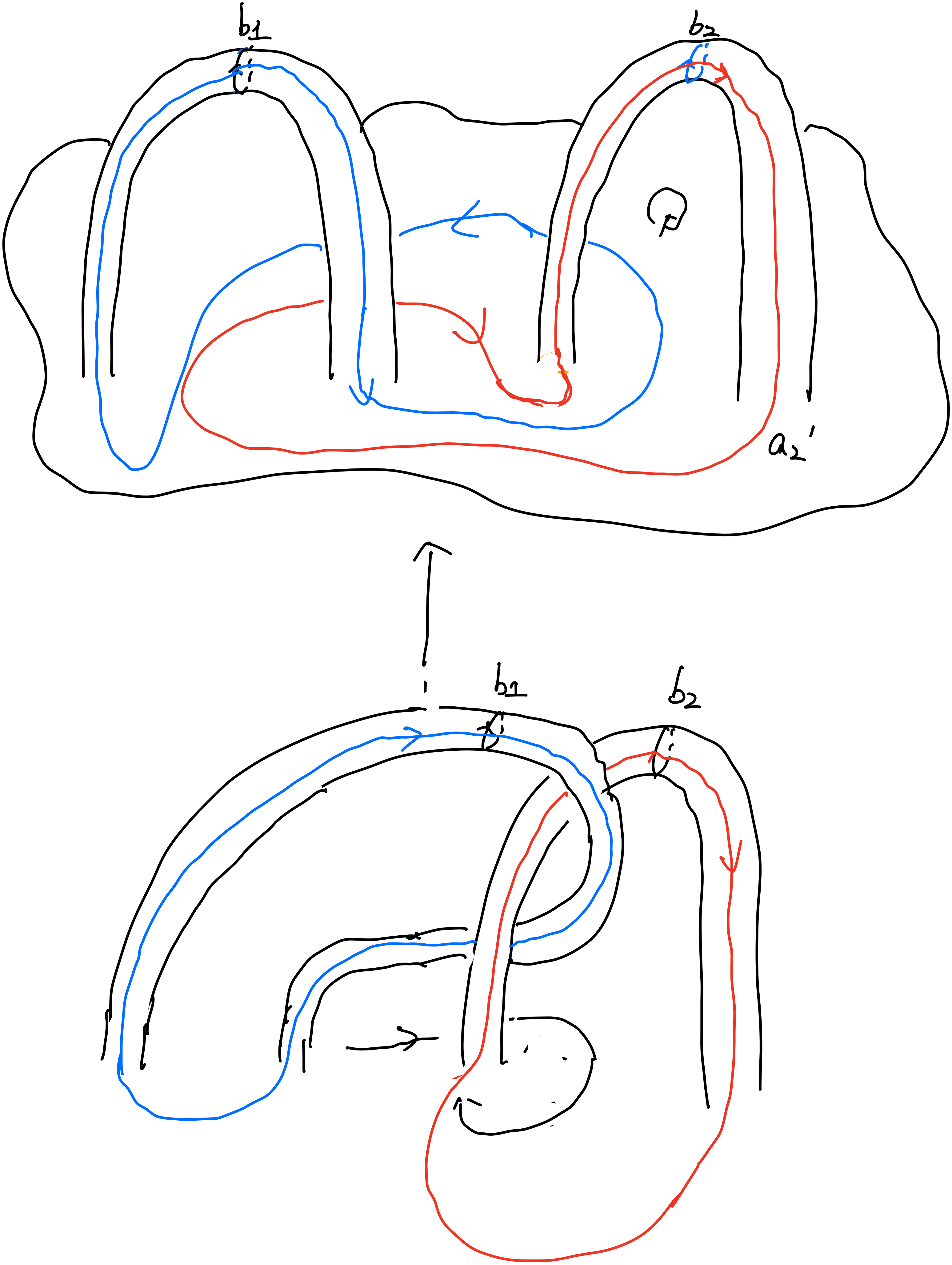
$a_1 \mapsto b_1$
$b_1 \mapsto -a_1$

3. Factor swap: $(\dots x_i, y_i, x_{i+1}, y_{i+1} \dots) \mapsto (\dots x_{i+1}, y_{i+1}, x_i, y_i, \dots)$



4. Factor mix: $(x_1, y_1, x_2, y_2, \dots) \mapsto (x_1 - y_2, y_1, x_2 - y_1, y_1)$





b1
b2
a2'
b1
b2

Minimality of Humphries Generating Set

Thm $\mathrm{Mod}(S_g)$ is generated by $2g+1$ Dehn twists.

Rmk If we relax by allowing product of Dehn twists (Wajnryb) $\mathrm{Mod}(S_g)$ can be generated by 2 elements.

Prop for $\mathrm{Mod}(S_g)$, $g \geq 2$
$2g+1$ is the min # of generators if we only use Dehn twists.

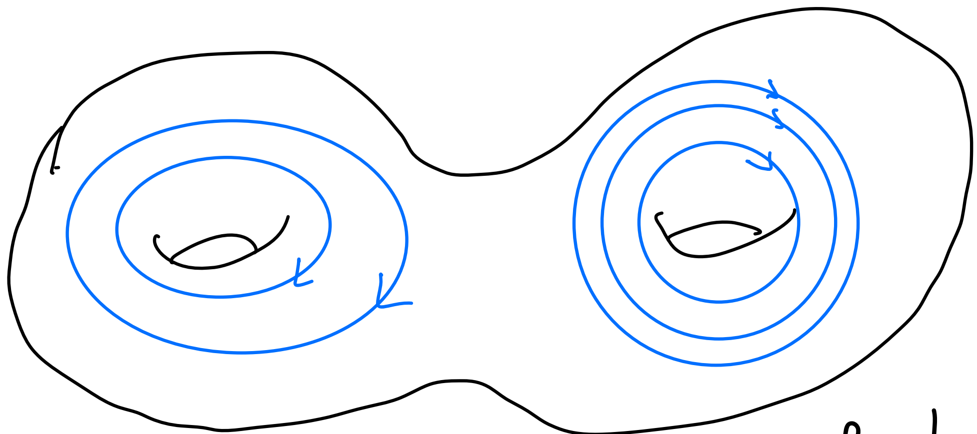
Fact 2 A non-zero element in $H_1(S_g; \mathbb{Z})$ is represented by a s.c.c iff it is primitive.

Idea of Pf $\supseteq$ Fact 1 implies any representable $\alpha \in H_1(S_g; \mathbb{Z})$ can be extended to a basis in $\mathbb{Z}^{2g}$

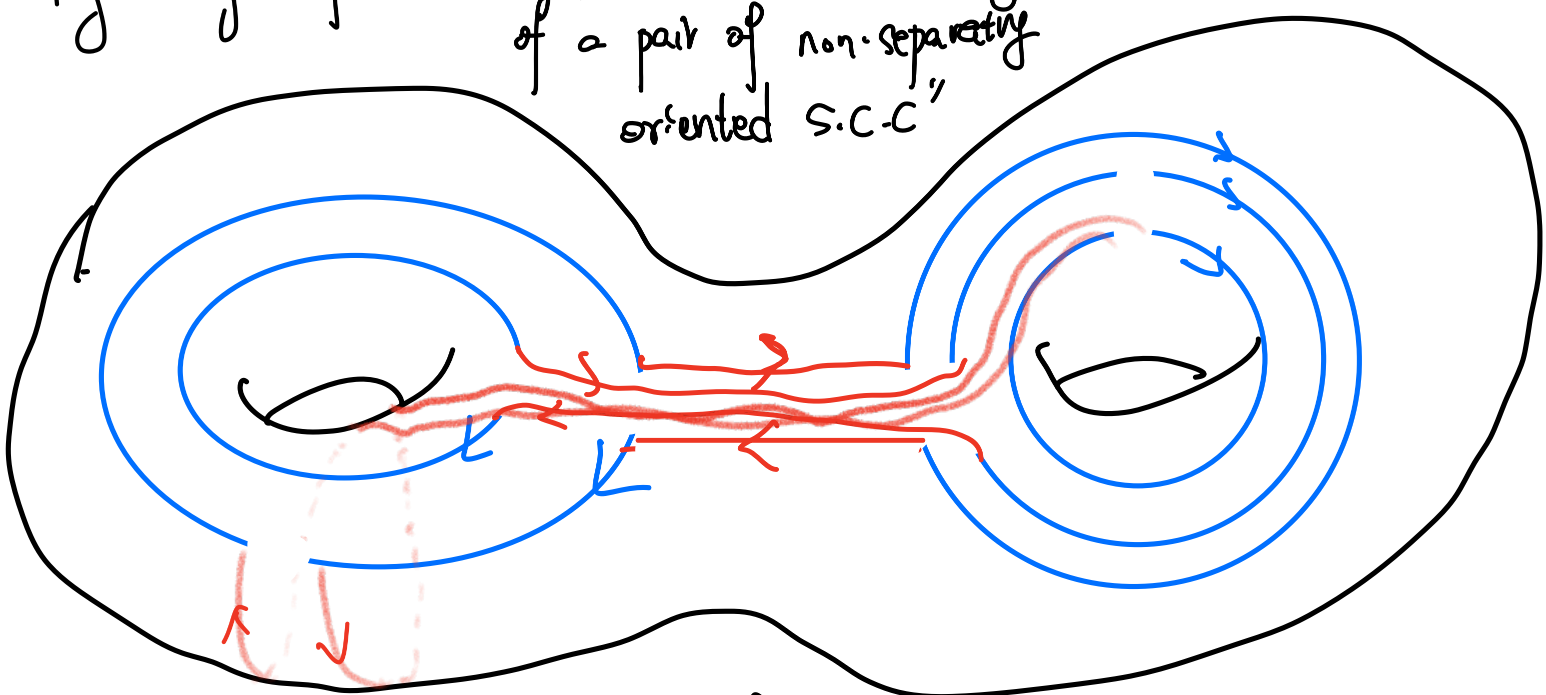
$\Leftarrow$ We know the case $g=1$; let

$$(x_1, y_1, x_2, y_2, \ldots, x_g, y_g)$$

be primitive. Each (x_i, y_i) represents $\gcd(x_i, y_i)$ parallel copies of a s.c.c in $N(a_i \cup b_i)$. The idea is connect the disjoint parallel copies of s.c.c into a single s.c.c without changing homology.



" By change of coordinates, $\exists$ an arc connecting left side to left side of a pair of non-separating oriented s.c.c "



Starting from leftmost.

"Topological Euclidean algo": connecting $\gcd(x_1, y_1)$ parallel curves to $\gcd(x_2, y_2)$ parallel curves gives you

$$\gcd(\gcd(x_1, y_1), \gcd(x_2, y_2)) = \gcd(x_1, y_1, x_2, y_2)$$

curves. Continue inductively gives 1 curve, by primitive hypothesis.

<u>Fact</u> Image of Dehn twists under $\Psi: \mathrm{Mod}(S_g) \to Sp(2g, \mathbb{Z})$ correspond to elements whose fixed set in $\mathbb{R}^{2g}$ has codimension 1; which we call "transvection";

A transvection is the image of some power of a Dehn twist,

<u>Pf</u>

part 1: Change of coordinates: T_b, extend b to a geometric symplectic basis.

part 2: Suppose $A \in Sp(2g, \mathbb{Z})$ does not fix v (primitive)

Extend v to a symplectic $(v, w, x_2, y_2, \ldots x_g, y_g)$

$$A(v) = v + kw \quad (A \text{ preserves Alg. int } \#)$$

$$A = \Psi(T_b^k), \quad [b] = w$$

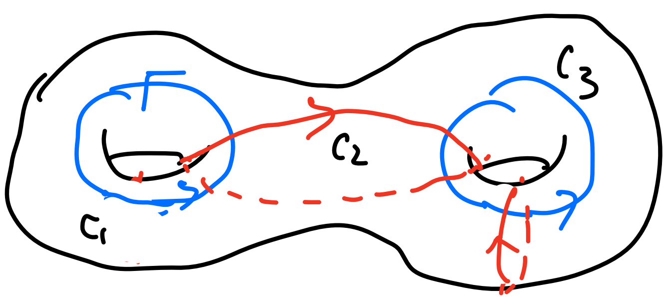
By Φ surjective & Fact, suffices to show

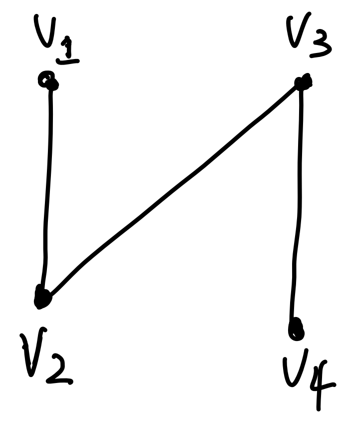
$$Sp(2g, \mathbb{Z}/2)$$

cannot be generated by $\leq 2g$ transvections.

Pf $< 2g$: The subgroup generated by $\leq 2g-1$ elements has fix set codimension $\geq 2g-1$; no non-trivial element is fixed by the whole group

$\neq 2g$: Let $v_1, \ldots, v_{2g}$ be any basis of $(\mathbb{Z}/2\mathbb{Z})^{2g}$, and define $\tau_{v_i} = \Phi(T_c) \bmod 2$, where $[c] \underset{\text{mod } 2}{=} \pm v_i$. We WTS $\{\tau_{v_i}\}$ does not generate $Sp(2g, \mathbb{Z}/2)$





Define a graph Γ: Vertex - $\{v_i\}$

edge - $\omega(v_i, v_j) \neq 0$

Given $w \in (\mathbb{Z}/2)^{2g}$, $w = \sum_i c_i v_i$. Define a subgraph

$G(w)$ as full subgraph spanned by v_i where $c_i \neq 0$

Claim: $\chi(G(w)) = \chi(G(\tau_{v_i}(w)))$ mod 2

If $\omega(v_i, w) = 0$, nothing to show;

If $\omega(v_i, w) = 1$:

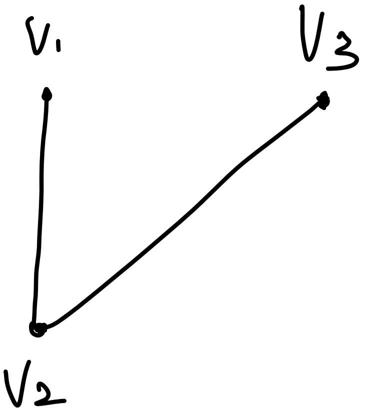
Dehn twists adds a class v_i

- add mod 2 the v_i-vertex $+1$
- add mod 2 the edges connecting v_i-vertex to other vertices

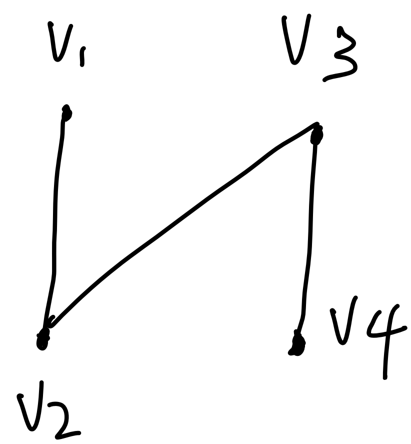
$\omega(v_i, w) = 1 \Leftrightarrow$ odd # of edges

τ_{v_2} $\quad$ $\tau_{v_2}(w) = w$

$w = (1,1,1,0)$



τ_{v_4}



__Fact__ $Sp(2g, \mathbb{Z}/2)$ acts transitively on $(\mathbb{Z}/2\mathbb{Z})^{2g}$

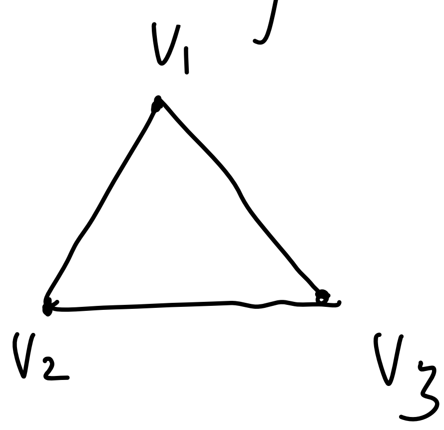
suffice to show $G(w)$ and $G(w')$ w/ different mod 2 χ.

- single vertex $\chi = 1$

- If G is not complete

$\chi = 0$

If G is complete, and $g \geq 2$



$\chi = 0$

$\square$

The Torelli Group

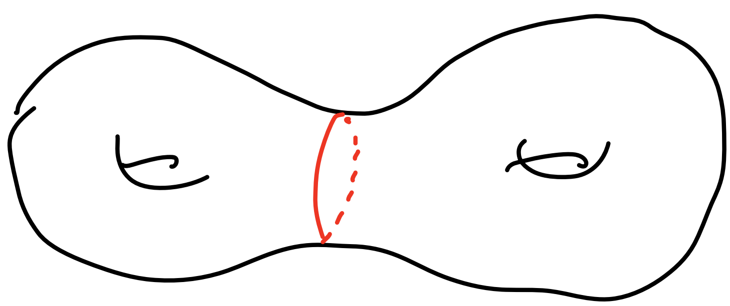
<u>Def</u> The Torelli Group $\mathcal{I}(S_g) \triangleleft \mathrm{Mod}(S_g)$ is the kernel of $\Psi : \mathrm{Mod}(S_g) \to Sp(2, \mathbb{Z})$

" subgroups of $\mathrm{Mod}(S_g)$ that acts trivially on nonseparating s.c.c "

<u>Ex:</u> $\mathrm{Mod}(S_2) \cong SL(2, \mathbb{Z}) = Sp(2, \mathbb{Z})$

so $\mathcal{I}(S_2)$ is trivial

<u>Ex</u>: Dehn twist along separating curves γ

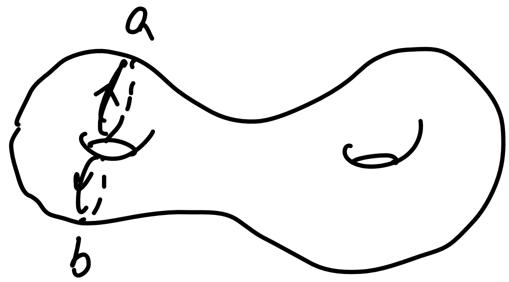


separating curves $\Leftrightarrow$ 0 in H_1

we can find a basis for H_1 where each element is disjoint from γ (cut along γ and find basis on each)

Ex: Bounding pair maps: bounding pair is a pair of homologous, disjoint, non separating s.c.c. (a,b)

A bounding pair map $:= T_b T_a^{-1}$



Thm $I(S_g)$ is torsion free

Thm (Birman, Powell) $I(S_g)$ is generated by an infinite collection of all Dehn twists about separating s.c.c and bounding pair maps.

Thm (Johnson) For $g \geq 3$, $I(S_g)$ is generated by a finite # of bounding pair maps.