Teichmüller Space and the Moduli of Complex Curves

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1 Teichmüller Space

1.1 Definition

Let S be a surface with $\chi(S) < 0$. (The reason for this assumption is that now S admits a hyperbolic metric) By a **hyperbolic structure** on S we mean a diffeomorphism $S \to X$, where X is a hyperbolic surface with finite volume and a totally geodesic boundary. The pair (X, ϕ) will be called a marked hyperbolic surface, with ϕ the marking. Two hyperbolic structures are homotopic if there is an isometry on the $X's$, mapping one marking to the same homotopy class of the other. Then define $Teich(S)$ to be the set of hyperbolic structures modulo homotopy.

The purpose of this definition is that we can pullback the hyperbolic metrics on the $X's$ along the markings to give S hyperbolic metric. Thus equivalently, we can define $Teich(S)$ to be the set of hyperbolic metrics on S modulo pullback via diffeomorphisms of S to itself.

There is a change of marking: let (X, ϕ) and (Y, ψ) be two hyperbolic structures, then they give a bijection from $Homeo(S)$ to $Homeo(X, Y)$, given by $f \mapsto \psi f \phi^{-1}$.

1.2 Length Functions

Let $\mathcal{X} \in Teich(S)$ be a representative of an isotopy class of hyperbolic structures. Let $\mathcal{S}$ denote the isotopy classes of simple closed curves in S. Then define the length function $\ell_{\mathcal{X}}$ on $\mathcal{S}$, sending a curve $c \in \mathcal{S}$ to the length of the unique geodesic in the isotopy class of $\phi(c)$. (uniqueness is shown in Proposition 1.3. The rough idea is that the universal cover of S, since it is a simply connected hyperbolic surface as well, is necessarily isometric to $\mathbb{H}^2$ with the metric $\frac{dx^2+dy^2}{y^2}$, and any geodesic in $\mathbb{H}^2$ is determined by where it meets $\partial\mathbb{H}^2$.

1.3 $Teich(\mathbb{T}^2)$

Recall that our definition only works for surfaces of negative Euler characteristic. Now $\chi(\mathbb{T}^2) = 0$, and we define $Teich(\mathbb{T}^2)$ to be the set of unit-area flat metrics on $\mathbb{T}^2$. Equivalently, in the definition we can replace this by homotopy classes of (X, ϕ), where instead of a hyperbolic surface, X is now a surface with a flat metric. We will prove that $Teich(\mathbb{T}^2) \simeq_{\mathsf{Set}} \mathbb{H}^2$. Then $Teich(\mathbb{T}^2)$ can be given a topology by declaring the bijection to be a homeomorphism.

A quick explanation is as follows (slightly different from Farb-Margalit): Firstly, we notice that $Teich(\mathbb{T}^2)$ is in bijection with orientation-preserving isomorphisms of $\mathbb{R}^2$ modulo dilation or rotation. This is because maps between flat metrics on $\mathbb{T}^2$ correspond to lattices in $\mathbb{R}^2$ modulo homothety and rotation, as the two basis vectors of the lattices (forming its fundamental domain) can be taken to be the orthonormal basis under a metric. Equivalently these correspond to orientation-preserving automorphisms of $\mathbb{R}^2$ modulo dilation and rotation. The flat metrics are the same if A is a rotation or dilation. Alternatively, we can say that the flat metrics on $\mathbb{T}^2$ correspond to lattices in $\mathbb{R}^2$ modulo homothety and rotation. Now consider the Iwasawa decomposition (in the theory of Shimura varieties, this is called the horospherical decomposition) $GL_2^+(\mathbb{R}) = \mathbb{R}^+ \cdot SL_2(\mathbb{R}) = \mathbb{R}^+ \cdot (KAN)$, with $K = SO_2(\mathbb{R}), A = \{diag(r, r^{-1}) : r > 0\}$ and N the unipotent subgroup $\left\{ \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} : x \in \mathbb{R} \right\}$. It is then clear that AN is in bijection with $\mathbb{H}^2$.

Another explanation is as follows: giving a metric structure and a symplectic structure (the area form is a natural symplectic form on $\mathbb{T}^2$) is equivalent to giving $\mathbb{T}^2$ a complex structure. Then $\mathbb{C}/\Gamma$, with Γ a lattice, can always be uniformized into $\mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$. with $\tau \in \mathbb{H}^2$.

1.4 Teichmüller Spaces of Curves of Higher Genus

1.4.1 $Teich(S_g)$ as a set

We first recall that $Isom(\mathbb{H}^2) \simeq PGL_2(\mathbb{R})$ and $Isom^+(\mathbb{H}^2) \simeq PSL_2(\mathbb{R})$. Denote $DF(\pi_1(S_g), PSL_2(\mathbb{R}))$ the set of faithful representations of $\pi_1(S_g)$ on $PSL_2(\mathbb{R})$ with discrete image. $Isom(\mathbb{H}^2)$ acts on it via conjugation. Then we have the bijection

$$Teich(S_g) \leftrightarrow DF(\pi_1(S_g), PSL_2(\mathbb{R})) \Big/ PGL_2(\mathbb{R})$$

Here is the rough explanation: let (X, ϕ) be a representative in $Teich(S_g)$. Then $\tilde{X}$ is isometric to $\mathbb{H}^2$. Now ϕ gives an isomorphism from $\pi_1(S_g)$ to $\pi_1(X)$, and they act by deck transformations. This gives a faithful $\rho : \pi_1(S_g) \to PSL_2(\mathbb{R})$. Note that changing the isometry $\tilde{X} \to \mathbb{H}^2$ amounts to conjugating ρ. This also does not depend on the representative chosen for $[(X, \phi)]$, since changing ϕ by isotopy lifts to an isometry of $\mathbb{H}^2$ to itself, where points move a finite distance.

Thus points of $\partial\mathbb{H}^2$ will stay fixed, and then we note that isometries of $\mathbb{H}^2$ are determined by their actions on $\partial\mathbb{H}^2$.

On the other hand, for any $\rho \in DF(\pi_1(S_g), PSL_2(\mathbb{R}))$, it induces a covering space action on $\mathbb{H}^2$. Then let $X = \mathbb{H}^2/\rho(\pi_1(S_g))$, and we know that X is diffeomorphic to S_g. Note that ρ induces an isomorphism from $\pi_1(S_g)$ to $\pi_1(X)$. To get a diffeomorphism from S_g to X, we use the following arguments: notice that S_g and X are $K(\pi_1(S_g), 1)-$spaces, so homotopy equivalences between them is unique (modulo homotopy of homotopy equivalences.) Now every homotopy equivalence on a closed surface is homotopic to a homeomorphism, and for closed surfaces every homeomorphism is homotopic to a diffeomorphism. This gives the marking.

1.4.2 Algebraic Topology on $Teich(S_g)$

The so-called algebraic topology of $Teich(S_g)$ is by identifying $DF(\pi_1(S_g), PSL_2(\mathbb{R}))$ as a subset of $Hom(\pi_1(S_g), PSL_2(\mathbb{R})$. Since closed surfaces have a unique smooth structure, that homeomorphism is homotopic to a diffeomorphism), which has the compact open topology. Then give $Teich(S_g)$ the quotient topology.

Now we have the following proposition:

Proposition 1.1. *Let $\gamma \in \pi_1(S_g)$. Then the evaluation-of-trace-on-γ function on $DF(\pi_1(S_g), PSL_2(\mathbb{R}))/PGL_2(\mathbb{R})$ is a continuous function. If for $\mathcal{X} \in Teich(S_g)$, $\rho_{\mathcal{X}}$ is the corresponding representation, then $\ell_{\mathcal{X}}(\gamma) = 2\cosh^{-1}(tr(\rho_{\mathcal{X}}(\gamma)/2)$. Specifically we know that $\ell_{\mathcal{X}}$ is continuous on $Teich(S_g)$.*

Here is a rough sketch of where the formula comes from. Let $A \in Isom^+(\mathbb{H}^2) = PSL_2(\mathbb{R})$. Then let $d(A) = \inf_{x\in\mathbb{H}^2} d(x, Ax)$. If A is elliptic/parabolic, then $d(A) = 0$. If A is hyperbolic, then $d(A)$ is given by the expression in the proposition, with trace of A being the absolute value. Now every $\rho(\gamma)$ is hyperbolic, since if it has an elliptic point, then that point is of finite order. By faithfulness of ρ, there is an element of $\pi_1(S_g)$ that has finite order, impossible. On the other hand, if it were parabolic, then $\mathbb{H}^2/Im(\rho)$, with $Im(\rho)$ the image of $\rho \leq PSL_2(\mathbb{R})$ would have a cusp, so it would not be compact. In $\mathbb{H}^2$, the set of points satisfying $d(\rho(\gamma))$ is the unique geodesic in $\mathbb{H}^2$ that is fixed by translation by $\rho(\gamma)$; $\rho(\gamma)$ acts on it via translation by $d(\rho(\gamma))$, and this geodesic projects down on S_g as the unique geodesic in the free homotopy class of γ.

1.4.3 Metric Topology on $Teich(S_g)$

By the Riemann uniformization theorem, any Riemann surface of $g \geq 2$ is biholomorphic to Δ/Γ, with Δ the unit disc and Γ a group of biholomorphic automorphisms of Δ. Any biholomorphic automorphism of Δ takes the form $z \mapsto e^{i\theta}\frac{z-a}{1-\bar{a}z}$, for $a \in \Delta$, so by computation it preserves the hyperbolic metric on the Poincare disc. Thus Δ/Γ has an induced hyperbolic structure. Thus we have the bijection between isomorphism (biholomorphic homeomorphism) classes of Riemann surfaces homeomorphic to S_g and the isometry classes of hyperbolic surfaces homeomorphic to S_g. Thus $Teich(S_g)$ can be identified with the set of conformal classes of Riemannian metrics on S_g.

Let $f : \mathbb{C} \to \mathbb{C}$, with $df = f_z dz + f_{\overline{z}} d\overline{z}$. If f is an orientation-preserving homeomorphism that has only finitely many non-smooth points, then define $K_f(p) = \frac{1+|\mu_f(p)|}{1-|\mu_f(p)|}$, where $\mu_f = f_z/f_{\overline{z}}$. Note that $\log(K_f(p))/2$ is the distance between $\mu_f(p)$ and 0 in the Poincaré disc model for $\mathbb{H}^2$. It can be interpreted as follows: df maps a circle in $T_p\mathbb{R}^2$ to an ellipse in $T_{f(p)}\mathbb{R}^2$, and $K_f(p)$ is the ratio between the lengths of the two axes. Say f is quasiconformal if $K_f = sup_p K_f(p) < \infty$. Now define the Teichmüller distance on $Teich(S_g)$ as follows: $d_{Teich}(\mathcal{X}, \mathcal{Y}) = \frac{1}{2} \log K_h$, where h is the unique diffeomorphism from X to Y that is quasiconformal, and such that K_h is minimal (existence proved by Teichmüller). It can actually be shown that d_{Teich} gives a metric topology on $Teich(S_g)$.

1.4.4 Dimension of $Teich(S_g)$

Here is a heuristic argument that $dim_{\mathbb{R}} Teich(S_g) = 3g - 3$. Firstly, note that $dim_{\mathbb{R}} PGL_2(\mathbb{R}) = 3$, so the dimension of the Teichmüller space should be $dim_{\mathbb{R}} DF(\pi_1(S_g), PSL_2(\mathbb{R}))$. Now $\pi_1(S_g) = \langle r_1, \cdots, r_{2g} | [r_1, r_2] \cdots [r_{2g-1}, r_{2g}] = I \rangle$. So choosing elements $\rho(r_i) \in PSL_2(\mathbb{R})$ each adds dimension 3, and there is one relation, so it reduces the dimension by 3.

The formal argument is called the Fenchel-Nielsen Homeomorphism, which states that $DF(\pi_1(S_g), PSL_2(\mathbb{R}))$ is homeomorphic to $\mathbb{R}_+^{3g-3} \times \mathbb{R}^{3g-3}$, with the first $3g - 3$ coordinates coming from the following fact: S_g can be seen as $g - 1$ pairs of pants ($S_{0,3}$ with 3 boundary components) glued along simple closed curves. The Teichmüller Space of a pair of pants is homeomorphic to $\mathbb{R}_+^3, \mathcal{X} \mapsto (\ell_{\mathcal{X}}(a_1), \ell_{\mathcal{X}}(a_2), \ell_{\mathcal{X}}(a_3))$, with a_1, a_2, a_3 being the boundary components. Then the other $3g - 3$ coordinates provide information about the simple closed curves where the pairs of pants glue.

2 Topological construction of $\mathcal{M}_g$

We will construct $\mathcal{M}_g$, the coarse moduli space of genus-g Riemann surfaces.

2.1 The construction

Note that since Riemann surfaces are symplectic (with the area form being symplectic), $Teich(S_g)$ can be understood equivalently as the set of complex structures of S_g. Now $Mod(S_g)$ acts on $Teich(S_g)$ by $f \cdot [(\phi, X)] = [(\phi \circ f^{-1}, X)]$. Then define $\mathcal{M}_g = Teich(\mathbb{T}_g)/Mod(\mathbb{T}_g)$. A result of Mumford says that $\forall r > 0, \{X; \inf_{\gamma \text{ s.c.c}} \ell_X(\gamma) \geq r\}$ is a compact subset of $\mathcal{M}_g$.