Topology Student Seminar Fall 2025: Dehn Twists

Jacob Monzel

Explicated Notes from My Talk on Dehn Twists

Nir Ghadish

Fall 2025

Last updated: October 14, 2025

Contents

1 Reflection on My Talk

In hindsight, I should have covered significantly less material in significantly more detail as Nir asked many good questions which I simply did not have answers to as many of the ideas and explanations in Farb and Margalit's book have a lot more going on under the surface than I realized. I tended to view definitions and theorems in terms of representatives of classes without thinking too much about why such things are even well-defined. Below, I have tried to update my talk with a lot of these details. However, for the sake of time, I have tried to link sources where possible for explanations as some of them are quite lengthy.

2 Definition and Nontriviality

2.1 Main Idea

Idea: Dehn twists play the role for mapping class groups that elementary matrices play for linear groups. We will see in the next talk that Dehn twists generate the mapping class group!

2.2 Dehn twists and their actions on curves

Consider the annulus $A = S^1 \times [0,1]$. To orient A, we embed it in the (θ, r)-plane via the map $(\theta, t) \mapsto (\theta, t+1)$, and take the orientation induced by the standard orientation of the plane. We will let $T : A \to A$ be the "twist map" of A given by the formula

$$T(\theta, t) := (\theta + 2\pi t, t).$$

Remark 2.1. *This map is orientation preserving. Moreover, T fixes the boundary of A point wise. To see this, notice that any point on the boundary is of the form $(\theta, 0)$ or $(\theta, 1)$, so clearly when we apply our formula of T, we get the same thing back!*

Definition 2.2. *Let S be an arbitrary (oriented) surface and let α be a simple closed curve in S. Let N be a regular neighborhood of α, and choose an orientation preserving homeomorphism $\phi : A \to N$ (where A is an annulus). We obtain a homeomorphism $T_\alpha : S \to S$, called a Dehn twist about α, as follows*

$$T_\alpha(x) = \begin{cases} \phi \circ T \circ \phi^{-1}(x) & \text{if } x \in N \\ x & \text{if } x \in S - N \end{cases}$$

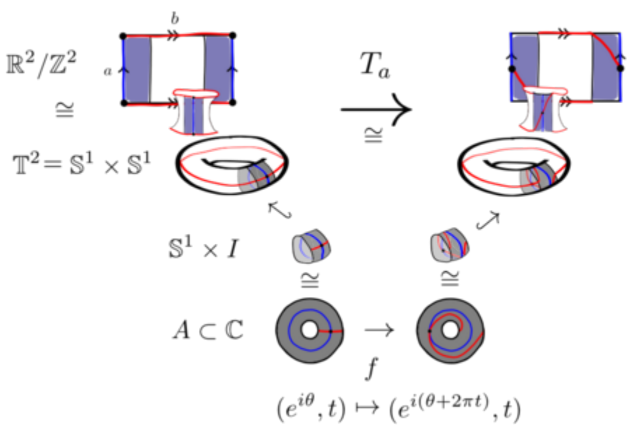
In short, perform the twist map T on the annulus N and fix every other point.

Remark 2.3. *We have a few remarks about this definition. First, what is a regular neighborhood? We had some trouble actually tracking down a definition, but we think what is described in this Math Stack Exchange Post is what we are actually looking for. Next, what is a simple closed curve? First, it is an injective map f from $[0,1]$ into our surface such that $f(a) = f(b)$. One could also just define this as some subspace that is homeomorphic to the unit circle like in Munkres. I did not explain the following point well in the talk when asked: really we only care about simple closed curves up to isotopy. The above definition works for* *any old simple closed curve, but for the purpose of this seminar, we will care about simple closed curves up to isotopy and denote them by $[a]$ since they are an equivalence class. The obvious question is then is T_a well-defined in $\mathrm{Mod}(S)$...*

Remark 2.4. *Notice that T_α obviously depends on the choice of N and the homeomorphism ϕ. However, uniqueness of regular neighborhoods (i.e. they differ by a homeomorphism that is isotopic to the identity), the isotopy class of T_α does NOT depend on either of these choices (homeomorphism since its just isotopic to the identity?) Moreover, it does not depend on choice of simple closed curve within its isotopy class either (since we can identify neighborhoods of both)! This means that if a represents this isotopy class, then T_a is a well-defined element of $\mathrm{Mod}(S)$, called the Dehn twist about a.*

The above was my explanation during the talk. As Nir pointed out, this really is not a good explanation. One should really look at the normal bundle to see this, i.e. the uniqueness of normal neighborhoods and such. Farb and Margalit seem to gloss over these facts (i.e. my explanation is what is mentioned in the text). For a full proof of this result, you can look at this post; however we note that this post only contains a (rather long) outline of the proof. As you will see later in this notes, this is a common problem with my talk: I took a lot of these explanations on faith when in fact there is a decent amount of leg work to be done. To keep these notes short, I will link proofs when I can.

Example 2.5. *Consider a torus $\mathbb{R}^2/\mathbb{Z}^2$. Take a few minutes to try to work out what our possible Dehn twists will look like.*



Let a closed curve be the line along the edge a called γ_a.

Given the choice of gluing homeomorphism in the figure, a tubular neighborhood of the curve γ_a will look like a band linked around a doughnut. This neighborhood is homeomorphic to an annulus, say

$$A(0;0,1) \;=\; \{\, z \in \mathbb{C} : 0 < |z| < 1 \,\}$$

in the complex plane.

By extending to the torus the twisting map

$$(e^{i\theta}, t) \;\mapsto\; \left(e^{i(\theta+2\pi t)}, t\right)$$

of the annulus, through the homeomorphisms of the annulus to an open cylinder to the neighborhood of γ_a, we obtain a Dehn twist of the torus along a:

$$T_a : \mathbb{T}^2 \longrightarrow \mathbb{T}^2.$$

This self-homeomorphism acts on the closed curve along b. In the tubular neighborhood it takes the curve of b once along the curve of a.

Example 2.6. *(Dehn twists on Torus) From results earlier in the text (specifically theorem 2.5), we have that the Dehn twists about the $(1,0)$ and $(0,1)$ curves in T^2 are matrices that in fact generate $\mathrm{SL}(2,\mathbb{Z}) \simeq \mathrm{Mod}(T^2)$ here. This is a first example of how Dehn twists actually generate the mapping class group!*

Remark 2.7. *We can also just think about Dehn twists as cutting along our curve α, twist a neighborhood of one boundary component by 360 degrees, then reglue the. This in fact is well-defined and is equivalent to T_α. Notice how we twist only ONE boundary component. Think about what happens if we twisted both for say our torus example (i.e. it is just the identity).*

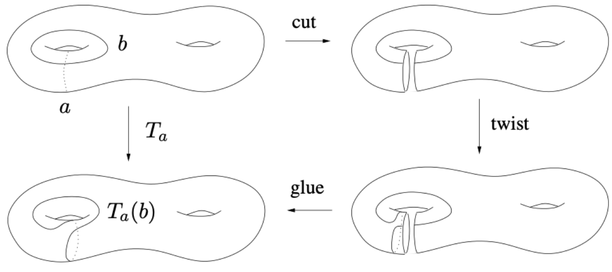


Figure 3.2 A Dehn twist via cutting and gluing.

Note: As Nir mentioned, as stated above, it is not apriori obvious this is well-defined as I claim. One must do some work to actually make this well-defined. The outline is as follows: first we need to show that for a simple closed curve, this "gluing" operation is actually a homeomorphism that preserves orientation. In fact, before that, we have to actually define this map. My idea to do that would be to just take the disjoint union of the two pieces and then identify the boundaries. Next, one needs to show this gives the same map UP TO ISOTOPY, i.e. any curve in the class $[a]$ actually gives the same map up to isotopy.

Remark 2.8. *Another issue that popped up during the talk is how does $\mathrm{Mod}(S)$ act on the isotopy classes of simple closed curves on S? For $[f] \in \mathrm{Mod}(S)$, we have $[f]([b]) :=$ the isotopy class $[f(b)]$. Again, we need to show that for any representatives f, b, we get the same thing up to isotopy.*

Note: another issue that popped up is that Farb and Margalit often denote a simple closed curve WITHOUT brackets which can be confusing. I will try to always include brackets here, so it is clear when we are talking about things up to isotopy. Note though, I often drop brackets when writing T_a etc.

Let us try to understand T_a by its action on the isotopy classes of simple closed curves in S. Recall that for isotopy classes of simple closed curves on S, $i([a],[b])$ = the minimum number of intersection points over all the representatives of a, b.

(Case 1) If $[b]$ is an isotopy class with $i([a],[b]) = 0$, then $T_{[a]}([b]) = 0$ (which makes sense because they are far away in some sense so the twisting will leave it alone if we take representatives that don't intersect since we can take a small enough tubular neighborhood).

(Case 2) When $i([a],[b]) \neq 0$, we determine $T_{[a]}([b])$ by the following rule: take representatives that cross the minimal number of times say a, b, we will modify the arcs of b that pass through a tubular region of a according to the following rule: where b intersects the tubular neighborhood, have b follow along a LEFT once around then exit RIGHT. ISSUE: I left this two things as exercises (Exercise: explain why this is well-defined. Exercise: look at the in the case of Tori). As Nir said, I really should have explained the first one why it is well-defined. See this for some explanation of this.

Remark 2.9. *The ideas in this paragraph and picture show up often in what follows. I failed to mention them during my talk when I should have:*

The action on curves via surgery. If $i(a,b)$ is large (say, more than two), it can be difficult to draw a picture of $T_a(b)$ using the "turn left–turn right" procedure given above. It is hard to plan ahead and leave enough room for all of the strands of $T_a(b)$ that run around a. A convenient way to draw $T_a(b)$ in practice is as follows. Start with one curve β in the class b and $i(a,b)$ parallel curves α_i, each in the class a, each in minimal position with β (one can also take the α_i to not have minimal position with β, but then one must take $|\alpha_i \cap \beta|$ parallel curves α_i). Of course, the result is not a simple closed curve. At each intersection point between β and some α_i, we do surgery as in Figure 3.3. The rule for the surgery is to resolve the intersection in the unique way so that if we follow an arc of β towards the intersection, the surgered arc turns left at the intersection. Again, this does not rely on any orientation of α_i or of β, but rather the orientation of the surface. After performing this surgery at each intersection, the result is a simple closed curve in the class $T_a(b)$.

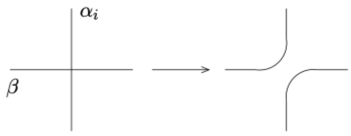


Figure 3.3 Dehn twists via surgery.

Again, one needs to check carefully that the ideas above actually are well-defined.

2.3 Nontriviality of Dehn Twists

Remark 2.10 (Nontriviality of Dehn twists)**.** *If a simple closed curve $[a]$ in a surface S is homotopic to a point or a puncture, then the Dehn twist T_a is trivial in $\mathrm{Mod}(S)$: any twisting in the annulus around $[a]$ can be undone by untwisting the disk or once–punctured disk inside.*

In contrast, if a is an essential *curve (i.e. not homotopic to a point or puncture), then T_a is nontrivial in $\mathrm{Mod}(S)$.*

The key idea is that there always exists another curve b that intersects a. Applying T_a modifies each intersection arc of b with a by dragging it once around a. This produces a new isotopy class $T_a(b)$ that cannot be isotoped back to b. Geometrically, b now "wraps around" a in a way that cannot be removed without cutting.

Intuition:

- *If a bounds a disk or puncture, T_a is trivial (the twist cancels inside).*
- *If a is essential, T_a is nontrivial (it forces other curves to change).*

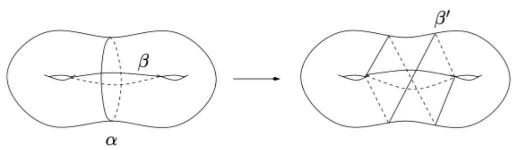


Figure 3.4 Checking that a Dehn twist about a separating simple closed curve is nontrivial.

During my talk, I did not have the full proof prepared as I aassumed it was too long to share with the amount of material I wanted to cover. However, as Nir said, this really is a result I need to prove as it uses a lot of concepts. The intution I tried to provide did not really help much. As such, here is the full proof as presented in Farb and Margalit:

Proof. If α is a nonseparating simple closed curve, then by change of coordinates, we can find a simple closed curve β with $i(\alpha, \beta) = 1$. Denote the isotopy class of β by b. As in Figure 3.2, one can draw a representative of $T_a(b)$ that intersects β once transversely. By the bigon criterion, $i(T_a(b), b)$ is actually equal to 1 (a bigon requires two intersections). Therefore $T_a(b)$ is not the same as b and so T_a is nontrivial in $\mathrm{Mod}(S)$.

Perhaps a simpler way to phrase the proof in the case that a is nonseparating is to check that T_a acts nontrivially on $H_1(S;\mathbb{Z})$; see Chapter 6 for more on this homology action. If α is a separating essential simple closed curve, then the action of T_a on $H_1(S;\mathbb{Z})$ is trivial, and so we are forced to use the more subtle machinery of the change of coordinates principle and the bigon criterion.

By the change of coordinates principle, an essential separating curve α is as depicted in Figure 3.4 (possibly with different genera and different numbers of punctures/boundary on the two sides of α). We can thus choose an isotopy class b with $i(a, b) = 2$, and we consider the isotopy class $T_a(b)$. We claim that $T_a(b) \neq b$, from which it follows that T_a is nontrivial.

We now prove the claim. In the right hand side of Figure 3.4, we show representatives β and β' of b and $T_a(b)$; the given representatives intersect four times. We will use the bigon criterion to check that all intersections are essential and so $i(T_a(b), b) = 4$, from which it follows that $T_a(b) \neq b$. To do this, note that β cuts β' into four arcs, β'_1, β'_2, β'_3, and β'_4, and similarly β' cuts β into four arcs β_1, β_2, β_3, and β_4. For each β_i there is a unique β'_j that has the same pair of endpoints on $\beta \cap \beta'$. This gives four candidates for bigons. But each of these four "candidate bigons" $\beta_i \cup \beta'_j$ is a nonseparating simple closed curve, and so none is an actual bigon. This proves the claim, and so T_a is nontrivial.

The remaining case is that α is homotopic to a boundary component of S and that α is neither homotopic to a point or a puncture. It follows that S is some surface with boundary other than the disk or the once-punctured disk. Let $\overline{S}$ denote the *double* of S, obtained by taking two copies of S and identifying corresponding boundary components. In $\overline{S}$, the curve α becomes essential. By our definition of the mapping class group for a surface with boundary, if T_a were trivial in $\mathrm{Mod}(S)$, it would be trivial in $\mathrm{Mod}(\overline{S})$, contradicting the previous cases. □

Note: this is where the concept of bigons and the change of coordinates principle really shows up. These ideas are why Farb and Margalit can just draw pictures as proofs as these pictures actually contain all of the information we need in a very packaged manner.

Note: as mentioned by Nir, you can also prove this via homology which Farb and Margalit mention as well.

3 Dehn twists and intersection numbers

The guiding principle for this section is: *twisting around a curve a forces other curves that cross a to wrap around it, creating new intersections.* Let us run through the main ideas and then I will try to give some ideas as to the proofs of them.

1. Action on curves. To reiterate: Let a and b be isotopy classes of essential (i.e. not the boundary of a disk) simple closed curves on a surface S. If $i(a, b) = 0$, then $T_a(b) = b$ (curves disjoint from a are unaffected). If $i(a, b) > 0$, then every crossing arc of b through the annulus around a is modified by looping once around a. Geometrically, each arc "turns left", follows a all the way around, then turns "right". This is why twisting visibly changes curves that intersect a.

2. Intersection growth formula. The effect of twisting can be quantified:

$$i(T_a^k(b),\, b) \;=\; |k| \cdot i(a,b)^2.$$

So each time we twist b around a, the number of intersections grows quadratically in the intersection number $i(a,b)$. This provides a precise measure of the "extra wrapping" that occurs.

Proof. We choose representative simple closed curves α and β in minima position and form a simple closed curve β' in the class of $T_a(b)$ using th surgical recipe given above. More specifically, we take $k\, i(a,b)$ paralle copies of α lying to one side of α and one copy of β lying parallel to β, an then we surger as in Figure 3.3; see the left-hand side of Figure 3.5 for picture in the case of $i(a,b) = 3$ and $k = 1$.

DEHN TWISTS 73

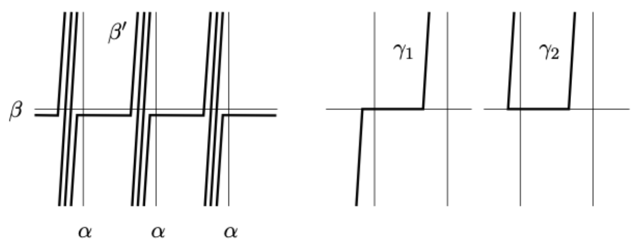


Figure 3.5 The simple closed curves in the proof of Proposition 3.2.

Simply by counting we see that

$$|\beta \cap \beta'| = |k| i(a,b)^2.$$

Thus it suffices to show that β and β' are in minimal position. By the bigon criterion we only need to check that they do not form any bigons.

We cut β and β' at the points of $\beta \cap \beta'$ and call the resulting closed arcs $\{\beta_i\}$ and $\{\beta'_i\}$. We see that there are two types of "candidate bigons," that is, simple closed curves that can be formed from one arc β_i and one arc β'_j: either the orientations of the two intersection points are the same, as for the curve γ_1 on the right hand side of Figure 3.5, or the orientations of the intersection points are different, as for γ_2 in the same picture. In a true bigon, the orientations at the two intersection points are different, and so the simple closed curve γ_1 in the first case cannot be a bigon. In the second case, if γ_2 were a bigon, then since the vertical arcs of β' are parallel to arcs of α we see that α and β form a bigon, contrary to assumption. $\square$

3. Infinite order. From the formula above, we see that the curves b and $T_a^k(b)$ become more and more complicated as k grows. They cannot be isotopic once $k \neq 0$, because their intersection numbers with b keep increasing. Hence each T_a has infinite order in $\mathrm{Mod}(S)$ (NEED TO MENTION CHANGE OF COORDINATES PRINCIPLE HERE FOR WHY WE CAN FIND A NONTRIVIAL INTERSECTING CURVE).

4 Basic facts about Dehn twists

Let a, b denote arbitrary (unorineted) isotopy classes of simple closed curves. In this section, we will collect facts about Dehn twists that will be used in future talks most likely. We will not prove any of these results and direct you to our main reference (*a Primer on Mapping Class Groups*).

Remark 4.1. *We do not give the proofs here (again) as Nir was okay with these being somewhat black-boxed for the sake of time. Moreover, they make for good exercises to try!*

Proposition 4.2. $T_a = T_a$ *iff* $a = b$.

Proposition 4.3. *For any* $f \in \mathrm{Mod}(S)$ *and any isotopy class* a *of a simple closed curves in* S*, we have*

$$T_{f(a)} = fT_a f^{-1}.$$

Remark 4.4. *I thought this was really cool that this acts by conjugation!*

Proposition 4.5. *For any* $f \in \mathrm{Mod}(S)$ *and any isotopy class* a *of simple closed curves in* S*, we have*

$$f \text{ commutes with } T_a \iff f(a) = a$$

Proof. Apply the previous two facts! □

Proposition 4.6. *If* a, b *are non-separating simple closed curves in* S*, then* T_a *and* T_b *are conjugate in* $\mathrm{Mod}(S)$.

Proposition 4.7. *For any two isotopy classes* a, b *of simple closed curves in a surface* S*, we have*

$$i(a, b) = 0 \iff T_a(b) = b \iff T_aT_b = T_bT_a$$

Remark 4.8. *All of the above propositions have analogues for when we talk about powers of Dehn twists! Exercise: try to find these based off the above facts.*

5 The center of the mapping class group

We recall that the center of a group $Z(G)$ is just the elements that commute with all other elements. This is a very natural thing to study in $\mathrm{Mod}(S)$ and from our last section, it should be clear why Dehn twists may show up!

Theorem 5.1. *For* $g \geqslant 3$*, the group* $Z(\mathrm{Mod}(S_g))$ *is trivial.*

Proof. (Sketch) Recall something is $f \in Z(\mathrm{Mod}(S_g))$ if $f \circ g = g \circ f$ for all other $g \in \mathrm{Mod}(S_g)$. The idea is to check this on Dehn twists. If f commutes with every Dehn twist, it must send a curve to something that is invariant under twisting. But we know

$$fT_af^{-1} = T_{f(a)}$$

from earlier, so if f commutes, we must have that $T_a = T_{f(a)} \iff f(a) = a$ from an earlier fact. But this means that $f(a) = a$ FOR ALL such curves, i.e. it fixes every isotopy class of curves. But the mapping class group acts faithfully for genus $\geqslant 3$. Contradiction! □

Here is the full proof which I should have presented:

THEOREM 3.10 *For $g \geq 3$ the group $Z(\mathrm{Mod}(S_g))$ is trivial.*

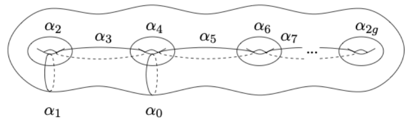


Figure 3.6 The simple closed curves used to determine the center of $\mathrm{Mod}(S)$.

Proof. By Fact 3.8, any central element f of $\mathrm{Mod}(S_g)$ must fix every isotopy class of simple closed curves in S_g. Consider the simple closed curves $\alpha_0, \ldots, \alpha_{2g}$ shown in Figure 3.6. By statement (1) of the Alexander method, f has a representative ϕ that fixes the graph $\cup\alpha_i$, and thus, ϕ induces a map ϕ_* of this graph.

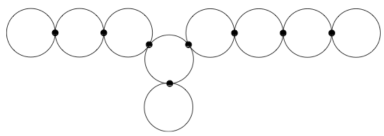


Figure 3.7 The collection of simple closed curves in Figure 3.6 form a graph in S_4 that is abstractly isomorphic to the graph Γ shown here for the case $g = 4$.

The graph $\cup\alpha_i$ is isomorphic to the abstract graph Γ shown in Figure 3.7 for the case $g = 4$. For $g \geq 3$, the only automorphisms of Γ come from flipping the three edges that form loops and swapping pairs of edges that form a loop. In particular, any automorphism of Γ must fix the three edges coming from α_4. Thus, we see that ϕ preserves the orientation of α_4, and so since ϕ is orientation preserving, it must also preserve the two sides of α_4. It

80 CHAPTER 3

follows that ϕ_* does not interchange the two edges of Γ coming from α_0, the two coming from α_3, or the two coming from α_5. Inductively, we see that ϕ_* fixes each edge of Γ with orientation. By statement (2) of the Alexander method, plus the fact that the $\{\alpha_i\}$ fill S_g, we have that ϕ is isotopic to the identity; that is, f is the identity. □

Remark 5.2. *Why this does NOT work in lower genus? In $g = 2$, we have the hyperelliptic involution: it rotates by 180 degrees of a genus 2 surface that flips every curve to itself up to isotopy. This is actually a central element. Exercise: show this!*

For $g = 1$, we know $\mathrm{Mod}(S_1) \simeq \mathrm{SL}(2, \mathbb{Z})$ which has center $\{\pm I\}$.

In short, if a mapping class is to commute with every Dehn twist, it CANNOT move curves around at all.

6 What is left?

Farb and Margalit have two more sections on Dehn twists after the material covered above. Namely, "Relations between two Dehn twists" and "Cutting, capping, and including". Realistically, there was simply not enough time to cover these topics in a 1.5 hour talk. I barely made it through the previous sections without even explicitly writing out proofs with detailed explanations (which I should have done). If and when these topics appear in later talks, I recommend consulting chapter 3 for more details.