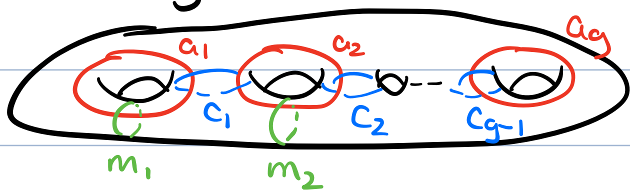
Talk 3. Generating the mapping class group

In this talk we aim to prove the following theorem:

Thm (Dehn–Lickorish) For $g \geq 0$, $\mathrm{Mod}(S_g)$ is generated by finitely many Dehn twist on nonseperating simple closed curves.

- $g=1$: $\mathrm{Mod}(T^2) = SL_2(\mathbb{Z})$ is generated by finitely many elementary matrices.

- # of generators: $2g+1$ (minimal)



- A general statement: For surfaces with punctures $S_{g,n}$.

$$1 \to \mathrm{PMod}(S_{g,n}) \to \mathrm{Mod}(S_{g,n}) \to \Sigma_n \to 1$$

Pure mapping class group: fixing n marked points.

We will actually prove that

Thm: $\mathrm{PMod}(S_{g,n})$ is generated by finitely many Dehn twist on nonseperating s.c.c

Two main ingredients:

I. The complex of curves

II. The Birman exact sequence

I. The complex of curves

Def: For a surface S, we associate a simplicial complex denoted by $C(S)$ to it, whose 1-skeleton is given by

- Vertices: Isotopy classes of essential s.c.c.

• Edges: There is an edge between a, b iff $i(a,b)=0$
And there is a k-simplex for $(k+1)$ vertices iff
those $(k+1)$-vertices are pairwise connected by an edge.

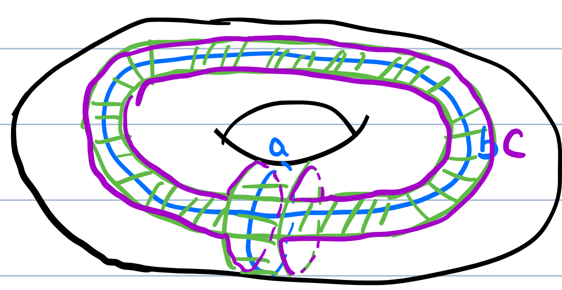
The heart of the proof of finite generation is the following fact.

Thm: $C(S_{g,n})$ is connected
except for $g=0,\ n\leq 4$ or $g=1,\ n\leq 1$.

• Exceptions:
- $C(S^2/S_{0,1}/S_{0,2}/S_{0,3})=\phi,$
- $C(T^2/S_{1,1}/S_{0,4})=$ discrete countable

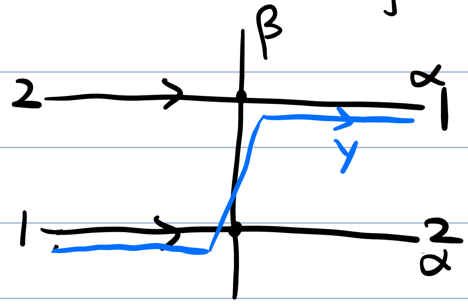
PF of connectivity:
$C(S)$ is connected $\Leftrightarrow$
$\forall$ classes of s.c.c. $a, b,$ $\exists$ a sequence of s.c.c.
$a=c_1, c_2, \cdots, c_{k-1}, c_k=b$
such that $i(c_i, c_{i+1})=0.$

Now fix $a, b.$
• If $i(a,b)=0$ nothing to prove
• If $i(a,b)=1,$
Let c be the boundary of a tubular neighborhood of $\alpha\cup\beta$.

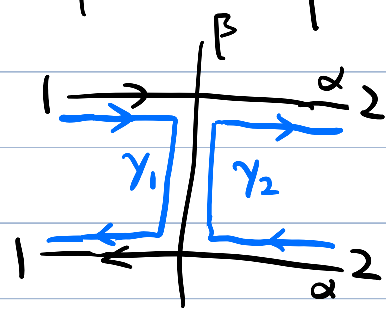


c is essential because $n\geq 2$
Then $i(a,c)=i(c,b)=0$

- If $i(a,b) \geq 2$, we use induction.
  Suppose statement is true for two curves with intersection number smaller than $i(a,b)$.
  We need to find a scc c s.t. $i(a,c), i(c,b) < i(a,b)$
  Let $\alpha \in a$, $\beta \in b$ in minimal position.
  Look at two adjacent intersection pts on β:



Case 1



Case 2.

$i(\alpha, \gamma) = 1$, $i(\gamma, \beta) < i(\alpha, \beta)$

- γ_1, γ_2 not null-homotopic because of minimal position
- If both γ_1, γ_2 bounds a punctured disc, then α bounds a twice punctured disc. On the other side of α, construct similar γ_3, γ_4. If γ_3, γ_4 both bounds a punctured disc, then $S = S_{0,4}$
  Therefore one of $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ is essential. let that one be c.
- $i(a,c) = 0$, $i(c,b) < i(a,b)$ $\boxed{\checkmark}$

Def: The complex of nonseparating curves $\mathcal{N}(S)$ is the subcomplex of $\mathcal{C}(S)$ spanned by nonseparting curves.

Thm: For $g \geq 2$. $\mathcal{N}(S_{g,n})$ is connected.

- **Exceptions**: $\mathcal{N}(S_{1,n})$ is <u>not</u> connected for any n because of a surjection $\mathcal{N}(S_{1,n}) \twoheadrightarrow \mathcal{C}(T^2)$

Pf:

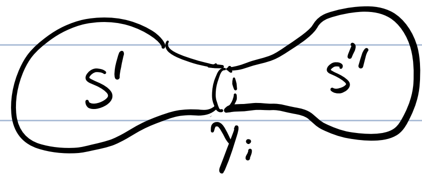
- First assume $n = 0$ or 1. Fix nonseperating a, b.

$\exists \ a = c_1, c_2, \cdots, c_n = b$ with $i(c_i, c_{i+1}) = 0$

Suppose c_i is seperating, $\gamma_i \in c_i$

Let S', S'' be the 2 components of $S_{g,n} \setminus \gamma_i$

both S', S'' have positive genus.



c_{i-1}, c_{i+1} have representatives in S'. S''

- If c_{i-1}. c_{i+1} both have representatives in S' (or S'') replace c_i by any non-seperating s.c.c in S'' (or S')
- If c_{i-1} c_{i+1} have representatives lying in different components. remove c_i from the sequence.

$\Rightarrow$ We will get $a = c_1, \cdots, c_n = b$ with c_i nonseperating and $i(c_i, c_{i+1}) = 0$.

- Next consider $n \geq 2$. Use induction on n.

The only problem in the procedure above is when c_{i-1} and c_{i+1} lie in S', and S'' has genus 0.

In this case S' has fewer punctures.

by induction $\exists \ c_{i-1} = d_1, \cdots, d_k = c_{i+1}$ nonseperating.

$$i(d_j, d_{j+1}) = 0$$

replace c_i by the sequence $(d_2, \cdots, d_{k-1})$. ☑

Def: Let $\hat{N}(S)$ be the 1-dim simplicial complex with

- vertices: classes of nonseparating s.c.c.
- edges: there is an edge between $a.b$ if $i(a, b) = 1$

Thm: For $g \geq 2$ $\hat{N}(S_{g,n})$ is connected.

Pf: $\forall$ nonseparating a, b,

$\exists$ a sequence of nonsep s.c.c. $a = c_1, \cdots, c_k = b$
s.t. $i(c_i, c_{i+1}) = 0$
By the change of coordinates principle,
$\exists$ nonseparating d_i s.t. $i(c_i, d_i) = i(d_i, c_{i+1}) = 1$.

Rmk. For $g=1$, also true
Use induction on n similar to the proof of
$\mathcal{N}(S_{g,n})$ connected when $g \geq 2$.

II. The Birman exact sequence.

Let S be a surface,
S^* be the surface S with a point x marked (in the interior).
There is a natural forgetful map

$$\mathcal{F} : \operatorname{Mod}(S^*) \to \operatorname{Mod}(S)$$

It is surjective by extension of isotopy.

Given $\alpha \in \pi_1(S, x)$, view α as a self-isotopy of $x \hookrightarrow S$
extend this isotopy to the whole S, i.e.
an isotopy $F_t : S \to S$ s.t. $F_t(x) = \alpha(t)$, $F_0 = id$.
Let $\phi_\alpha = F_1$
Now define a map

$$\mathcal{P} : \pi_1(S, x) \longrightarrow \operatorname{Mod}(S^*)$$

by sending $\alpha \longmapsto [\phi_\alpha]$.
Suppose $\mathcal{P}$ is well-defined, then

Thm (Birman exact sequence) Suppose $\chi(S) < 0$
then $\exists$ an exact sequence

$$1 \to \pi_1(S, x) \xrightarrow{\mathcal{P}} \operatorname{Mod}(S^*) \xrightarrow{\mathcal{F}} \operatorname{Mod}(S) \to 1$$

Pf: It is immediate that $\mathrm{Im}\,\mathcal{P} = \mathrm{Ker}\,\mathcal{F}$.

To show that $\mathcal{P}$ is injective,

if $\alpha \in \pi_1(S,x)$ is nontrivial

then the induced map of ϕ_α on $\pi_1(S,x)$

is $I_\alpha : \pi_1(S,x) \to \pi_1(S,x)$,

$$\beta \longmapsto \alpha\beta\alpha^{-1}$$

Since $\pi_1(S,x)$ is centerless, $I_\alpha \neq id$ hence $\phi_\alpha \neq id$.

Hence the entire content of the theorem above is to say

$\mathcal{P}$ is well defined.

In our case, we have

$$1 \to \pi_1(S_{g,n}) \to \mathrm{PMod}(S_{g,n+1}) \to \mathrm{PMod}(S_{g,n}) \to 1$$

☆ Defining $\mathcal{P}$

We claim that

$$\underset{\text{homeos fixing } x}{\mathrm{Homeo}^+(S,x)} \to \mathrm{Homeo}^+(S) \xrightarrow[\text{evaluating on } x]{\varepsilon} S$$

is a fiber bundle. Therefore, there is an exact sequence

$$\pi_1(\mathrm{Homeo}^+(S)) \to \pi_1(S,x) \to \pi_0(\mathrm{Homeo}^+(S,x)) \to$$
$$\pi_0(\mathrm{Homeo}^+(S)) \to \pi_0(S) = 0$$

$\pi_1(\mathrm{Homeo}^+(S)) = 0$ by contractibility of components of $\mathrm{Homeo}^+(S)$

the map $\pi_1(S,x) \to \pi_0(\mathrm{Homeo}^+(S,x))$ is lifting a loop in the total space, which is exactly the procedure $\mathcal{P}$ does. Hence this map is $\mathcal{P}$.

The proof of the claim is basically finding a local section of ε.

Exceptions: For $S = T^2$, we have

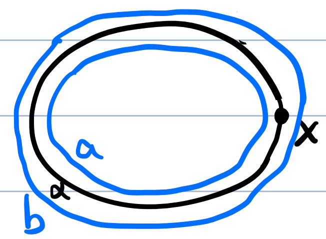
$\pi_1(\mathrm{Homeo}^+(T^2)) \cong \pi_1(T^2) \cong \mathbb{Z}^2$, so

$\to \mathbb{Z}^2 \xrightarrow{\cong} \mathbb{Z}^2 \xrightarrow{0} \mathrm{Mod}(S_{1,1}) \to \mathrm{Mod}(T^2) \to 1$

so $\mathrm{Mod}(T^2) \cong \mathrm{Mod}(S_{1,1})$.

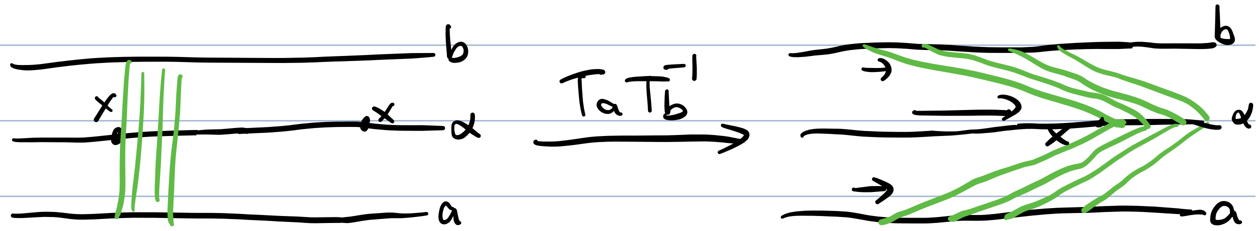
- $\mathcal{P}$ in terms of Dehn twists

Fact: Suppose $\alpha \in \pi_1(S, x)$ is a simple loop



Then $\mathcal{P}(\alpha) = T_a T_b^{-1}$

Why?



(on the infinite strip $I \times \mathbb{R}$)

The isotopy between id and $T_a T_b^{-1}$
by pushing the middle line backwards
moves x along α

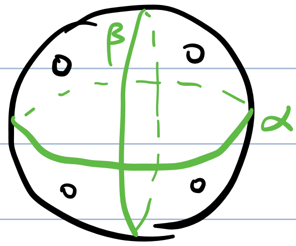
II. Finite generation.

* The genus 0 case

- $\mathrm{PMod}(S_{0,n}) = 1$ for $n \le 3$.
- For $n = 4$, by the Birman exact seq,

$$1 \to \pi_1(S_{0,3}) \xrightarrow{\mathcal{P}} \mathrm{PMod}(S_{0,4}) \to 1$$

Hence $\mathrm{PMod}(S_{0,4}) \cong \pi_1(S_{0,3}) \cong F_2$ generated by



$T_\alpha \cdot T_\beta$

- In general

$$1 \to \pi_1(S_{0,n-1}) \to PMod(S_{0,n}) \to PMod(S_{0,n-1}) \to 1$$

free

iterated extensions of free groups

And

<u>Thm</u>: $PMod(S_{0,n})$ is generated by finitely many Dehn twists.

* Genus ≥ 1.

Recall the main theorem.

<u>Thm</u>: $PMod(S_{g,n})$ is generated by finitely many Dehn twists on nonseperating s.c.c. for $g \geq 1$.

<u>Proof</u>: $Mod(S_{g,n})$ acts on $\hat{N}(S_{g,n})$

We need a lemma.

<u>Lemma</u>: Suppose G acts on a connected 1-dim simplical complex X by simplicial maps. Suppose G acts transitively on X, and also transitively on pairs of vertices connected by an edge. Let v, w two vertices that are connected by an edge, $h \in G$ s.t. $v = h(w)$. Then G is generated by h and the stabilizer of v in G.

Now we do double induction on g and n:

Base case: Statement true for $Mod(T^2) = Mod(S_{1,1})$.

- Induction on n:

Suppose $PMod(S_{g,n})$ is generated by $\{T_{\alpha_i}\}$
for nonseparating $\{\alpha_i\}$

By the Birman exact sequence

$$1 \to \pi_1(S_{g,n}) \xrightarrow{P} PMod(S_{g,n+1}) \to PMod(S_{g,n}) \to 1$$

Im P is generated by Dehn twists because
$\pi_1(S_{g,n})$ is generated by simple nonseparating

Each $T_{\alpha_i} \in PMod(S_{g,n})$ can be lifted to $T_{\tilde{\alpha}_i} \in PMod(S_{g,n+1})$
by choosing a lift of α_i in $S_{g,n+1}$ (avoiding the extra point)

Hence $PMod(S_{g,n+1})$ is generated by finite Dehn twists
on nonsep s.c.c.

Now we have that statement is true for $PMod(S_{1,n})$, $n \geq 0$.

- Induction on g.

Let $g \geq 2$ and suppose that statement is true for
$PMod(S_{g-1,n})$, $\forall n$.

$Mod(S_g)$ acts on $\hat{N}(S_g)$
transitively on vertices and (change of coordinates)
transitively on pairs of vertices connected by an edge
so we can apply the lemma.

Let a be an isotopy class of a nonsep s.c.c.
$b, \ldots,$ $i(a,b)=1$.

We have that $T_a T_b T_a(b) = a$
so $Mod(S_g)$ is generated by $T_a, T_b,$
and $Mod(S_g, a)$, the subgroup of $Mod(S_g)$
with $f(a) = a$

$\alpha \in a$, there is an exact sequence

$$1 \to \langle T_\alpha \rangle \to \mathrm{Mod}(S_g, a) \to \mathrm{PMod}(S_g \setminus \alpha) \to 1$$

$S_g \setminus \alpha$ is homeomorphic to $S_{g-1,2}$

hence by induction hypothesis

$\mathrm{PMod}(S_g \setminus \alpha)$ is generated by T_{γ_i}, $\{\gamma_i\}$ nonsep scc

Lift γ_i to $\tilde{\gamma}_i$ in S_g, then

$\mathrm{Mod}(S_g, a)$ is generated by $\{T_\alpha, T_{\tilde{\gamma}_i}\}$. $\boxtimes$.

IV. A glance at relations

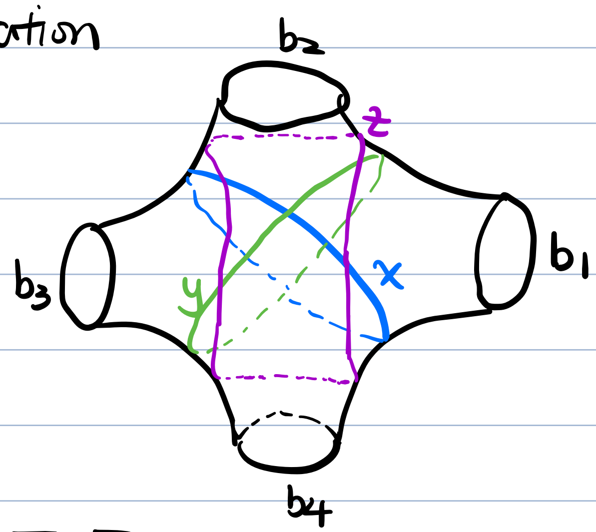
- **The k-chain relation**

Let $c_1, c_2, \cdots, c_k$ curves in minimal position, choose a regular neighborhood of $c_1 \cup c_2 \cup \cdots \cup c_n$ the boundary of this neighborhood consists of one s.c.c. d if k even and two s.c.c. d_1, d_2 if k odd. Then

$$(T_{c_1} T_{c_2} \cdots T_{c_k})^{2k+2} = T_d, \quad k \text{ even}$$

$$(T_{c_1} T_{c_2} \cdots T_{c_k})^{k+1} = T_{d_1} T_{d_2} \quad k \text{ odd}.$$

- **The lantern relation**

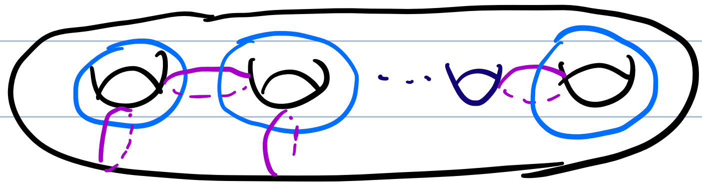


$$T_x T_y T_z = T_{b_1} T_{b_2} T_{b_3} T_{b_4}$$

- The hyperelliptic relations

Let $c_1, c_2, \cdots, c_{2g+1}$ be a chain of s.c.c. in S_g with $i(c_i, c_{i+1})=1$ and $i(c_i,c_j)=0$ otherwise.

Any such chain is homeomorphic to



Then $(T_{c_{2g+1}} \cdots T_{c_1} T_{c_1} \cdots T_{c_{2g+1}})^2 = 1$

$[T_{c_{2g+1}} \cdots T_{c_1} T_{c_1} \cdots T_{c_{2g+1}}, T_{c_{2g+1}}] = 1$

<u>Thm</u>: $\mathrm{Mod}(S_g)$ is finitely presented.

<u>Thm</u> (Wajnryb): $\mathrm{Mod}(S_g)$ is the group generated by $2g+1$ Dehn twist T_{c_i} with the following relations

1) Disjointness relations

$$T_{c_i} T_{c_j} = T_{c_j} T_{c_i} \quad \text{if} \quad i(c_i, c_j) = 0$$

2) Braid relations

$$T_{c_i} T_{c_j} T_{c_i} = T_{c_j} T_{c_i} T_{c_j} \quad \text{if} \quad i(c_i, c_j) = 1$$

3) 3-chain relations

4) Lantern relations

5) Hyperelliptic relations.