Line Bundles on Elliptic Curve

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A Mirror Symmetry Story

-Originated as physics phenomenon observed by string theorists.

-Kontsevich(1994): Homological Mirror Symmetry

$$D^b(Coh(M)) \approx Fk^0(M')$$

$$Algebraic\ Geometry \Leftrightarrow Symplectic\ Geomtry$$

-For elliptic curves, holomorphic line bundles generate $D^b(Coh(M))$.

What Are Holomorphic Line Bundles?

Definition: Given a topological space B, which we call the base space, a holomorphic line bundle over B is a topological space E, together with a **holomorphic** map $\pi: E \to B$ that satisfies the following

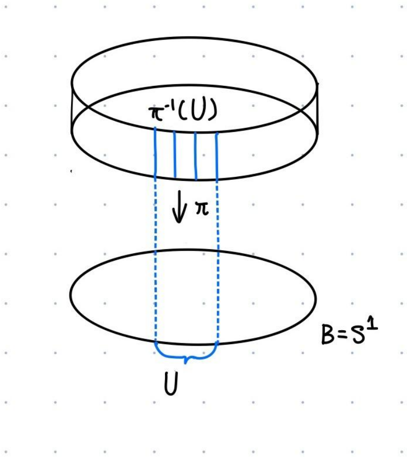
1. For each $x \in B$, the fiber $\pi^{-1}(x)$ has the structure of a 1-dimensional **complex vector space.**
2. For each $x \in B$, there is an open neighborhood U of x and a fiber-preserving **biholomorphism** $\phi: \pi^{-1}(U) \to U \times \mathbb{C}$ such that

$$\phi|_{\pi^{-1}(p)}: \pi^{-1}(p) \to \{p\} \text{ x } \mathbb{C}$$

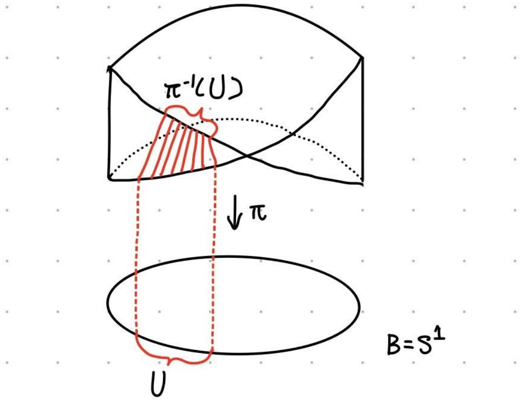
is a vector space isomorphism for every $P \in U$.

We may extend the definition in multiple ways: fiber bundles; smooth vector bundles; holomorphic vector bundles.

Real Line Bundles over S^1



The trivial bundle



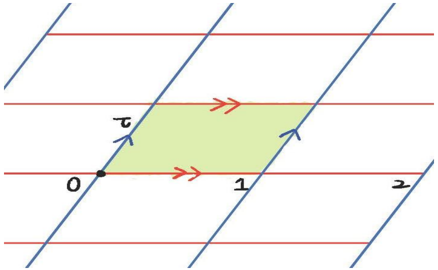
The mobius strip

What Is an Elliptic Curve?

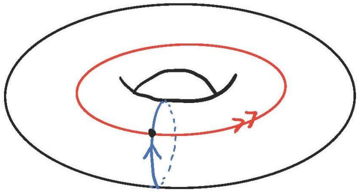
- We define an elliptic curve E_τ to be the Riemann surface given by

$$E_\tau := \mathbb{C}/\Lambda \cong \mathbb{C}/1\mathbb{Z} \oplus \tau\mathbb{Z}$$

where Λ is a lattice generated by $\langle 1, \tau \rangle$.



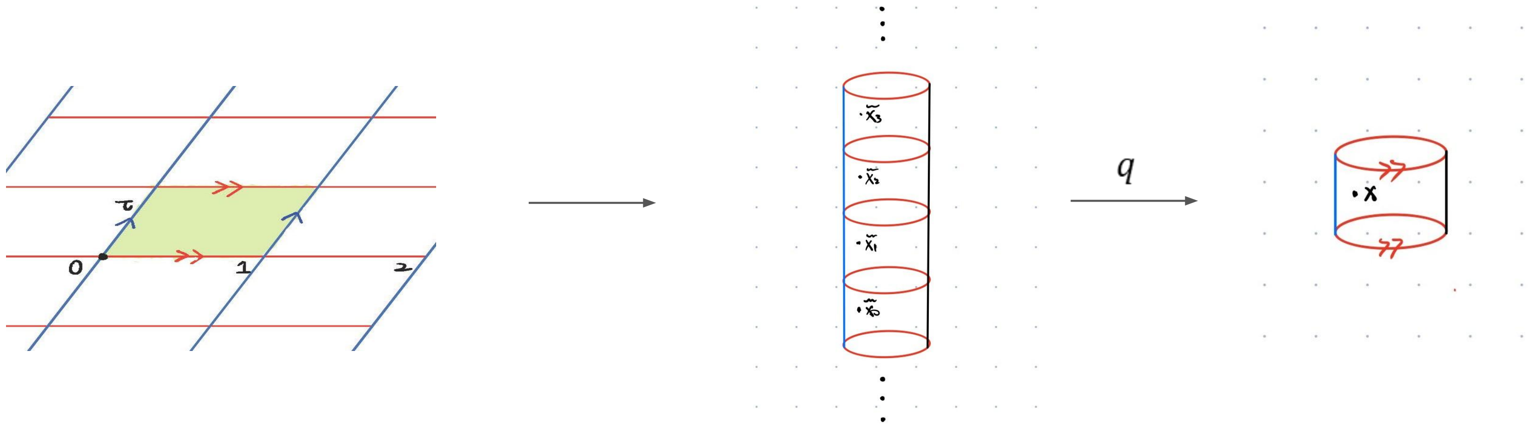
Fundamental Domain



It's a torus!

Line Bundles on E_τ

- The trick is to consider $E_\tau \cong \mathbb{C}^\times/\tau\mathbb{Z}$, where we quotient out the one $\mathbb{Z}$-action first and identify $\mathbb{C}/\ 1\mathbb{Z} \cong \mathbb{C}^\times$. Then, then quotient map $q : \mathbb{C}^\times \to \mathbb{C}^\times/\tau\mathbb{Z} \cong E_\tau$ is a natural fiber bundle, where the fibers over a point are the orbit of the $\tau\mathbb{Z}$-action on any lift of that point. The action is free and transitive.



Principal G-Bundle

- **Definition:** A principal G-bundle, where G denotes any topological group, is a fiber bundle $\pi: P \to B$ together with a continuous right action $P \times G \to P$, such that G preserves the fibers of P and acts freely and transitively. In particular. Each fiber is homeomorphic to G itself.

In our case, the principal $\mathbb{Z}$ -bundle over the base space $E_\tau \cong \mathbb{C}^\times/\mathbb{Z}$, is the fiber bundle $q: \mathbb{C}^\times \to \mathbb{C}^\times/\mathbb{Z}$ with the prescribed $\mathbb{Z}$-action.

What Can We Do About it?

- **Goal: Build line bundles over $\boldsymbol{E_\tau}$ given a principal $\mathbb{Z}$-bundle over $\boldsymbol{E_\tau}$.**

Here is a general idea: $\mathbb{C}^\times$ is a fiber bundle that has fiber $\mathbb{Z}$ over E_τ. We want to replace the fibers with vector space $\mathbb{C}$ (with structures).

A somewhat natural thing to do is starting with $\mathbb{C}^\times \times \mathbb{C}$. We know that $\mathbb{Z}$ acts principally on $\mathbb{C}^\times$. Thus, we can hope that an appropriate $\mathbb{Z}$-action on the product will lead to $\mathbb{C}^\times \times \mathbb{C}$ descending to line bundle after quotienting the $\mathbb{Z}$-action.

Associated Bundle to a Principal Bundle

- Choose any $\mathbb{Z}$-action on $\mathbb{C}$: $\rho : \mathbb{Z} \to Aut(\mathbb{C})$. Then on the product $\mathbb{C}^\times \times \mathbb{C}$, we have a "$\mathbb{Z}$-equivariant" action $\mathbb{Z} \to Aut(\mathbb{C}^\times \times \mathbb{C})$: given $k \in \mathbb{Z}$, the action is given by

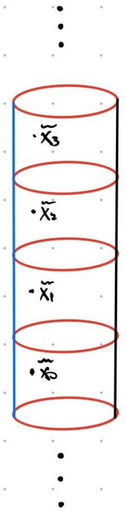
$$(c_1, c_2) \cdot k = (c_1 \cdot k, k^{-1} \cdot c_2)$$

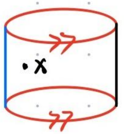
Then we can define the associated line bundle $\mathbb{C}^\times \times_{\mathbb{Z}} \mathbb{C}$ by the quotient map:

$$\begin{aligned} \pi_{\mathbb{Z}}: \mathbb{C}^\times \times \mathbb{C}/\mathbb{Z} &\to \mathbb{C}^\times/\mathbb{Z} \cong E_\tau \\ [c_1, c_2] &\mapsto q(c_1) \end{aligned}$$

Some checks

- Is the map well-defined?
- What do the fibers look line?
- Is it a vector bundle? (is local-trivalisation satisfied?)
- Is the complex structures we want on $\mathbb{C}$ preserved?





$$\pi_{\mathbb{Z}}: \mathbb{C}^{\times} \times \mathbb{C}/\mathbb{Z} \to \mathbb{C}^{\times}/\mathbb{Z} \cong E_{\tau}$$

$$[c_1, c_2] \quad \mapsto \quad q(c_1)$$

Some Food For Thought...

- How do we construct others?
- How do we know when we have found them all?
- How do we define morphisms in our category?

Thank you!