The monodromy determination and McMullen's theorem

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Outline

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Classical Problem in Number Theory

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- This is known as the **Fermat's last theorem**, which is proved by Wiles in 1995.
- We have a nice way to deal with the weakened version of this problem.

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- Consider the Riemann surface $C \subset \mathbb{P}^2(\mathbb{C})$ given by $x^n + y^n = z^n$.
- Treat this as a family C/B spread out over the prime $p \in \mathbb{Z}$, with fibers $C_p \simeq C \mod p$. Also $g(C) \geq 2$. Actually $B = \mathrm{Spec}(\mathbb{Z}) - \{\text{primes}\}$.

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- An integral solution gives a coherent family of points on C_p, and hence a section of C/B.
- Finite Fermat follows from an arithmetic version of the finiteness of sections for families C/B.

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Goal

Make sense of this theorem!

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Let $\mathcal{S}$ be the set of isotopy classes of simple loops (i.e. cannot be shrunk to a point) in S, and

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There is a natural group action $\mathrm{Mod}(S)$ on $(\mathcal{S}, i)$.

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Any $f \in \mathrm{Mod}(S)$ *is either*

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3. ***pseudo-Anosov***: *there exists an expansion factor* $K > 1$ *such that* $i(f^n(\alpha), \beta)$ *grows like* K^n *for all* $\alpha, \beta \in \mathcal{S}$*.*

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Theorem (Finiteness Theorem, a.k.a. Geometric Shafaravich Conjecture)

For a given base B with genus $g \geq 2$, there are only finitely many truly varying families C/B with fibers of genus g.

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- How to understand this?

Monodromy Representation

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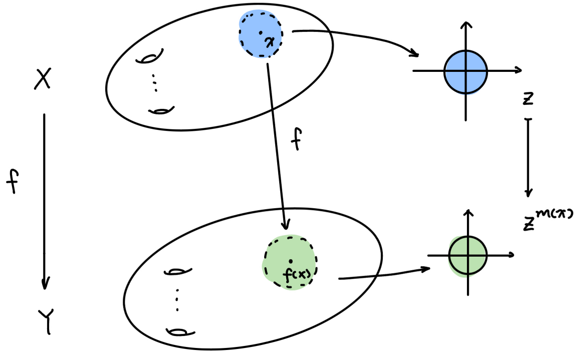
Let $f : X \to Y$ be a non-constant holomorphic map of Riemann surfaces.



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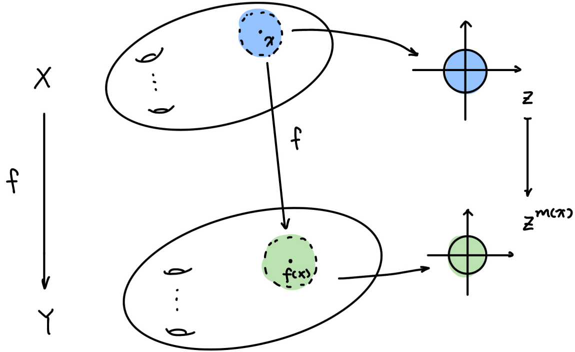

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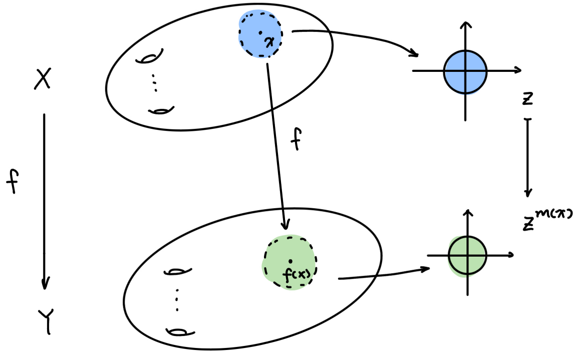

For any $x \in X$, there is a local chart such that the local expression of f is $f(z) = z^{m(x)}$ for some integer $m(x) \geq 1$. Call $f(x)$ the **branched points** if $m(x) > 1$, and denote $\mathcal{B}$ the set of branched points (**branched locus**).

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Let $f : X \to Y$ be a non-constant holomorphic map of compact Riemann surfaces of degree d, and $\mathcal{B} = \{b_1, \cdots, b_n\} \subset Y$ be the branched locus (finiteness follows from Riemann-Hurwitz theorem).

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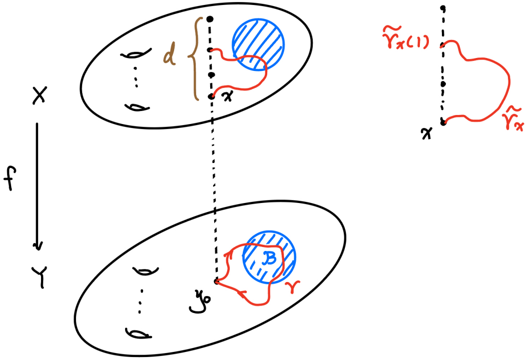

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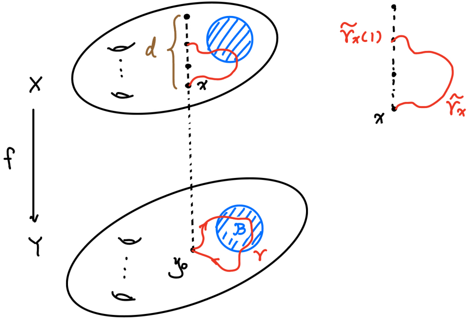

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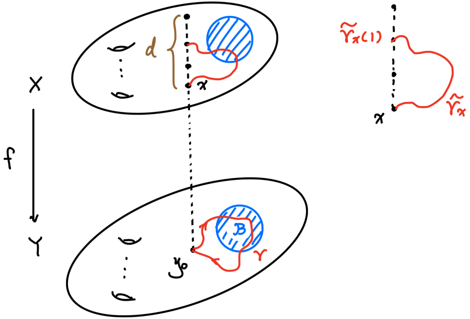

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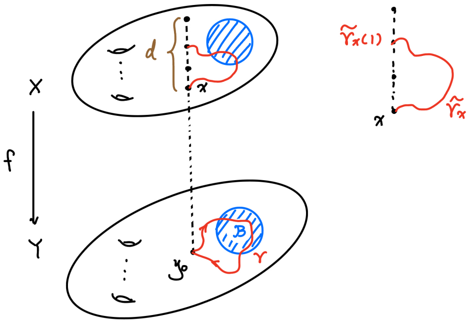

Every $x \in f^{-1}(y_0)$, γ lifts to a path $\widetilde{\gamma}_x$ with the other endpoint $\widetilde{\gamma}_x(1) \in f^{-1}(y_0)$. This gives rise to a function

$$\sigma_\gamma : f^{-1}(y_0) \to f^{-1}(y_0)$$

sending x to $\widetilde{\gamma}_x(1)$. This σ_γ is actually a permutation of elements in $f^{-1}(y_0)$.

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For any $y_0 \in Y - \mathcal{B}$, one obtains a group homomorphism

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The group homomorphism Φ above is called the **monodromy representation** of $f : X \to Y$.

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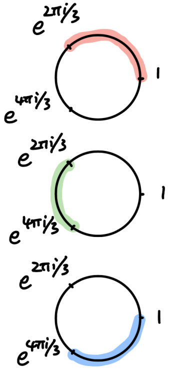

Lifts | Permutation

$\gamma_1(t) = e^{2\pi i/3 \cdot t} \rightsquigarrow \sigma_\gamma(1) = e^{2\pi i/3}$

$\gamma_2(t) = e^{2\pi i/3 \cdot (t+1)} \rightsquigarrow \sigma_\gamma(e^{2\pi i/3}) = e^{4\pi i/3}$

$\gamma_3(t) = e^{4\pi i/3 \cdot (t + \frac{2\pi i}{3})} \rightsquigarrow \sigma_\gamma(e^{4\pi i/3}) = 1$

$\Rightarrow \quad \sigma_\gamma = (1 \ 2 \ 3) \in S_3.$

Teichmüller Spaces

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is isotopic to a biholomorphic map $h : X_1 \to X_2$. In other word, there is an isotopy between f_1 and f_2. The space

$$T(S) \coloneqq \{(X, f) \mid f : S \to X\} / \sim$$

is called a **Teichmüller space** of S.

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Something of personal interest: Take $S = S^n$, then

$$\pi_0(\mathcal{M}_0(S^n)) = \Theta_n,$$

where the latter is the group of smooth structures on S^n. This is deeply intertwined with the celebrated Kervaire invariant one problem – one of my all-time mathematical obsessions!

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To put them in the same picture: there is a natural complex structure on $\mathcal{M}_g$ (inherited from T_g, the **universal covering of** $\mathcal{M}_g$, see Tom's talk), such that for all family C/B, it determines a holomorphic classifying map

$$\begin{aligned} F : B &\to \mathcal{M}_g \\ t &\mapsto [C_t = \pi^{-1}(t)] \end{aligned}$$

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This is the generalized monodromy representation in the new setting C/B, instead of a simple covering $C \to B$ of fixed degree.

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Recall in Tom' talk:

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Let $\widetilde{B}$ be the universal cover of B in the family C/B, we have a commutative diagram

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So in the case of $g(B) < 2$, $\widetilde{B} \cong \mathbb{C}$ ($g(B) = 1$) or $\hat{\mathbb{C}}$ ($g(B) = 0$), and thus $\widetilde{F}$ is trivial. Hence, all C/B is trivial.

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$$ds^2 = \frac{4|dz|^2}{(1-|z|^2)^2}.$$

2. The Teichmüller metric on $\mathcal{M}_g$ is
$$d(X,Y) = \frac{1}{2}\log\inf\{K \geq 1 : \text{there exists } K\text{-quasiconformal map } f : X \to Y \text{ respecting markings}\}.$$

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implying

$$\int_\epsilon^r \frac{2dt}{1 - t^2} = \log \frac{1 + r}{1 - r} - \log \frac{1 + \epsilon}{1 - \epsilon}.$$

When $\epsilon \to 0$, the length tends to $\log \frac{1+r}{1-r}$, which tends to 0 as $r \to 0$. So Δ^* has a cusp at 0.

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Similarly, the circle $S^1(r) = \{z : |z| = r\}$ has the hyperbolic metric (by setting $z = re^{i\theta}$ for $\theta \in [0, 2\pi]$)

$$ds = \frac{d\theta}{\log(1/r)},$$

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implying the length of $S^1(r) \to 0$ as $r \to 0$. By the modular Schwarz lemma, $F : \Delta^* \to \mathcal{M}_g$ shrinks the distance. It follows that

$$\begin{tikzcd} F_*(\pi_1(B)) \arrow[r, equal] & {[f] \in \mathcal{M}_g} \\ \widetilde{\gamma} \in \widetilde{B} \arrow[u] \arrow[r] & \widetilde{\gamma}_f \in T_g \arrow[u] \end{tikzcd}$$

and $\tau(f) = \inf \text{length}(\widetilde{\gamma}_f) = 0$ because the hyperbolic metric for γ tends to 0.

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and $\tau(f) = \inf \text{length}(\widetilde{\gamma}_f) = 0$ because the hyperbolic metric for γ tends to 0. By Nielsen-Thurston, f is reducible or of finite order. (If $\tau(f) > 0$, then pseudo-Anosov since f moves every surface some distance.)

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Theorem (Rigidity)

A truly varying family C/B is determined up to finitely many choices by its monodromy

$$F_* : \pi_1(B) \to \mathrm{Mod}_g = \pi_1(\mathcal{M}_g).$$

Outline

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Theorem

Given a genus $g \geq 1$ and a base B, there exists a genus $h \geq 2$ and a finite-to-one map

$$\{C/B \text{ with fibers of genus } g + \text{sections } s : B \to C\} \to \{D/B \text{ with fibers of genus } h\}.$$

For each $t \in B$, the surface D_t is a covering of C_t branched over the single point $s(t) \in C(t)$.

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References

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Thank you!