Introduction to Mapping Class Groups

David Zhu

September 19, 2025

First, let us recall some basic definitions and theorems from the previous lecture.

Theorem 0.1 (Classification of Surfaces)**.** A **surface** is a connect sum of a 2-sphere with g tori, possibly iwth $b \geq 0$ disjoint open disks removed, and $n \geq 0$ points removed from the interior.

Definition 0.1.1. A **simple closed curve** γ on a surface S is an embedding

$$\gamma : S^1 \to S$$

The theme of this talk is that we can study homeomorphisms of surfaces via their action on simple closed curves.

Definition 0.1.2. Let α, β be two simple closed curves on a surface S. An **isotopy** from α to β is a homotopy

$$H : S^1 \times [0,1] \to S$$

from α to β, with the additional property that $H(S^1 \times \{t\})$ is simple for each $t \in [0,1]$.

Maybe draw a picture here.

There are a few important facts on isotopy of curves that we will need.

Proposition 0.1.1. Two essential (not homotopic to a point, a puncture, or a boundary component) simple closed curves are isotopic iff they are homotopic.

The non-trivial direction uses hyperbolic geometry. This propositions allows us to conflate isotopies with homotopies of surfaces.

Definition 0.1.3. An **isotopy** of a surface S is a homotopy $H : S \times I \to S$ such that H_t is a homeomorphism for all t.

Given a isotopy of simple closed curves on S, we may promote it to an **ambient isotopy** of the surface.

Proposition 0.1.2 (Extension of Isotopies)**.** Let S be a surface, and $F : S^1 \times I \to S$ an smooth isotopy of simple closed curves. Then, there is an isotopy $H : S \times I \to S$ so that $H_{S \times \{0\}}$ is the identity and $H_{F(S^1 \times 0) \times I} = F$.

One can prove this fact by constructing a smooth vector field supported in a neighborhood of the two curves that takes one curve to the other. Then, one extends the vector field to the entire S and integrate.

1 Mapping Class Groups

Definition 1.0.1 (Mapping Class Group). Let $\mathrm{Homeo}^+(S, \partial S)$ be the group of orientation preserving homeomorphisms of S that fixes the boundary pointwise. The **mapping class group** of S, denoted by $\mathrm{Mod}(S)$, is the group

$$\mathrm{Mod}(S) := \pi_0(\mathrm{Homeo}^+(S, \partial S))$$

A homotopy is path in the mapping space; similarly an isotopy is a path in the space of homeomorphisms. Thus, we may intepret $\pi_0(\mathrm{Homeo}^+(S, \partial S))$ as the group of isotopy classes of elements of $\mathrm{Homeo}^+(S, \partial S)$, where isotopies are required to fix the boundary.

Theorem 1.1 (Baer, 1920). Homotopic (rel boundary) orientation-preserving homeomorphisms on a compact surface are isotopic (rel boundary).

By the theorem, we see

$$\mathrm{Mod}(S) \cong \mathrm{Homeo}^+(S, \partial S)/\text{homotopy}$$

Remark 1.1.1. Throught the book and this talk, when we say some curve/boundary/collection is fixed, we mean fixed as a set, as apposed to being fixed pointwise.

Remark 1.1.2 (Punctures versus Boundary). A mapping class must fix the boundary components pointwise by definition; however, viewing punctures as marked points, we see that a mapping class is allowed to permute the marked points.

There are a couple of surfaces whose mapping class groups we can immediately determine to be trivial: an orientation-preserving homeomorphism of S^2 must have degree 1; homotopy classes of maps between spheres are completely determined by their degree, and therefore any orientation preserving homeomorphism is homotopic to identity. Thus,

$$\mathrm{Mod}(S^2) \cong e$$

For $S_{0,1}$, the once marked sphere, we can identify $S_{0,1}$ with $\mathbb{R}^2$, and any orientation preserving homeomorphism of $\mathbb{R}^2$ is homotopic to identity via straightline homotopy. Thus,

$$\mathrm{Mod}(S_{0,1}) \cong e$$

We can also compute $\mathrm{Mod}(\mathbb{D}^2)$ with relative ease.

Lemma 1.2 (Alexander's Trick). The MCG of a disk and MCG of a once punctured disk are both trivial.

Proof. Given an homeomorphism ϕ of $\mathbb{D}^2$ that fixes the boundary pointwise, we will construct an explicity isotopy from ϕ to the identity.

Let

$$F(x,t) = \begin{cases} (1-t)\phi(\frac{x}{1-t}) & 0 \le |x| < 1-t \\ x & 1-t \le |x| \le 1 \end{cases}$$

At $t = 0$ we are starting with ϕ on the entire disk, but as time move on, we are performing a normalized ϕ on the smaller disk of radius $1-t$, and identity on the complement. □

Note that the argument also works verbatim for a disk with one puncture at the origin.

The triviality of the MCG of the (once puncture) disk will be an important tool for our understanding of MCG for general surfaces. We will prematurely outline the general framework of the "Alexander method": for a surface S, we will find a collection of simple closed curves/arcs, along which S will be cut into disjoint disks, possibly with 1 puncture. If a orientation preserving homeomorphism (up to isotopy) fixes the curves pointwise, then applying the Alexander trick tells us the homeomorphism is isotopic to the identity.

2 More Computations

The first non-trivial example of mapping class groups we can compute is that of the thrice punctured sphere.

Proposition 2.0.1. We have

$$\mathrm{Mod}(S_{0,3}) \cong \Sigma_3$$

Proof. We show the homomorphism

$$f : \mathrm{Mod}(S_{0,3}) \to \Sigma_3$$

that sends a mapping class to its action on the three marked points. The map is obviously surjective, since we can realize transpositions as some rotation that swaps the two antipodal points, with the third point on the pole being fixed. Thus, it suffices to show injectivity: if an orientation preserving homeomorphism fixes the three marked points, then it must be homotopic to the identity.

Let ϕ be such an homeomorphism. Consider any arc α connecting two of the marked points, and $\phi(\alpha)$ will be an arc that with the same endpoints as α. Viewing the third point as the point at infinity, we can treat α and $\phi(\alpha)$ as two curves in the plane. If ϕ does not fix α pointwise on the nose, then up to isotopy, we can assume they intersect transversally. Then, we may reduce intersection of α and $\phi(\alpha)$ one by one by isotoping one curve past any bounded disk. In the end, they will bound an open disk, which immediately gives us they are isotopic. Using extension of isotopies, we may isotope ϕ to a orientation preserving homeomorphism $\tilde{\phi}$ that fixes α pointwise.

Cutting $S_{0,3}$ along α, we see that ϕ induces a orientation preserving homeomorphism of the once-marked disc that is identity on the boundary. By Alexander trick, we know $\tilde{\phi}$ is isotopic to the identity.

□

The same argument gives us

Corollary 2.0.1. We have

$$\mathrm{Mod}(S_{0,2}) \cong \Sigma_2$$

The annulus $A = S^1 \times I$ gives us an infinite mapping class group.

Proposition 2.0.2. We have

$$\mathrm{Mod}(A) = \mathbb{Z}$$

Proof. We define a homomorphism

$$\rho : \mathrm{Mod}(A) \to \mathbb{Z}$$

as follows: let ϕ be a homeomorphism representating a mapping class in $\mathrm{Mod}(A)$. Such a homeomorphism liftS to a map of the universal cover $\tilde{A} = \mathbb{R} \times I$ that fixes $(0, 0)$

$$\begin{tikzcd} \tilde{A} \arrow[r, dashed, "\tilde{\phi}"] \arrow[d, "\pi"'] \arrow[dr] & \tilde{A} \arrow[d, "\pi"'] \\ A \arrow[r, "\phi"] & A \end{tikzcd}$$

Since ϕ is required to fix the boundary of A pointwise, we see that $\tilde{\phi}$ must restrict to an integer translation on $\mathbb{R} \times \{1\}$. We may then define $\rho(\phi)$ as $\tilde{\phi}|_{\mathbb{R}\times\{1\}}(0)$. To see this is well-defined, consider the arc β on the annulus that lifts to $\{0\} \times I$. We see that $\rho(\phi)$ corresponds precisely to the class $[\phi(\beta) \cdot \beta] \in \pi_1(A) \cong \mathbb{Z}$. Pictorially, this corresponds to how many times the arc winds around the annulus. A generator is winding this once, which is called a "Dehn twist". Insert pic.

To see ρ is surjective, consider the linear transformation

$$M_n := \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$$

on $\mathbb{R}^2$ restricted to the strip $\mathbb{R} \times I$. The linear transformation is equivariant with respect to the deck transformation, and in particular is an integer translation on the boundary of the strip. Therefore, M_n descends to a homeomorphism of the annulus that fixes the boundary circles pointwise. It follows from construction that $\rho(M_n) = n$.

To show injectivity, suppose f is in the kernel of ρ, which means the lift $\tilde{f}$ fixes $\mathbb{R} \times \{0, 1\}$ pointwise. We therefore knows that f must induce the identity map on $\pi_1(A)$. Thus,

$$\tilde{f}(\tau \cdot x) = f_*(\tau) \cdot \tilde{\phi}(x) = \tau \cdot \tilde{\phi}(x)$$

so $\tilde{f}$ is compatible with the deck transform. It is then straightforward to check the straightline homotopy of $\tilde{f}$ to the identity map on $\tilde{A}$ is equivariant with the deck transformation as well, thus descending to a homotopy between f and the identity map of A, which fixes the boundary of A pointwise along the homotopy. $\square$

Proposition 2.0.3. We have

$$\mathrm{Mod}(S_{1,0}) \cong \mathrm{SL}_2(\mathbb{Z})$$

Proof. The homomorphism

$$\rho : \mathrm{Mod}(T^2) \to \mathrm{SL}_2(\mathbb{Z})$$

is given by the action of a mapping class on $H_1(T^2, \mathbb{Z}; \mathbb{Z}) \cong \mathbb{Z}^2$. Since the homeomorphism is orientation preserving, the image of ρ lands in $\mathrm{SL}_2(\mathbb{Z})$. (the algebraic intersection number of (p, q) and (p', q') is the determinant of the 2×2 matrix formed by the two vectors, and orientation preserving homeomorphism preserves the algebraic intersection number).

We may identify the class of a meridian circle α with the generator $(1, 0)$ and a equatorial circle β with the generator $(0, 1)$. For surjectivity, it suffices to note that each $M \in \mathrm{SL}_2(\mathbb{Z})$ induces an orientation preserving linear automorphism of $\mathbb{R}^2$ compatible with the deck transformation, and descends to a homeomorphism of the torus. By construction, the image of such homeomorphism under ρ is just M again.

For injectivity, we may appeal to the Alexander method: suppose $\rho(\phi)$ is the identity matrix in $\mathrm{SL}_2(\mathbb{Z})$. This means that $f(\alpha)$ is homotopic to α and $f(\beta)$ is isotopic to beta. Our first goal is to find a single isotopy of the torus that takes α, β to $\phi(\alpha), \phi(\beta)$ simultaneously, so we may assume ϕ fixes α, β pointwise, up to isotopy: by isotopy extension, we know ϕ is isotopic to some ϕ' that fixes α pointwise; if we then cut along α, we see β and $\phi'(\beta)$ are now two homotopic arcs on the annulus, sharing the same endpoints. By our characterization of $\mathrm{Mod}(A)$, it is immediate that ϕ' is the trivial mapping class on the annulus. Combining the two isotopies together does the job.

The final step fit the recurrent theme: since we may assume f fixes α, β pointwise, we may cut along the two simple closed curves, and get a disk. Apply the Alexander trick finishes.

□

Note that this exact proof also works for the once-marked torus $S_{1,1}$.

3 Alexander Method

We now give a precise statement of the Alexander method: Let S be a compact surface, possibly with marked points, and let $\phi \in \mathrm{Homeo}^+(S, \partial S)$. Let $\{\gamma_i\}$ be a finite collection of essential simple closed curves/arcs in S that satisfies

- The γ_i are pairwise in minimal position.
- The γ_i are pairwise non-isotopic.
- For distinct i, j, k, at least one of $\gamma_i \cap \gamma_j$, $\gamma_i \cap \gamma_k$, $\gamma_j \cap \gamma_k$ is empty.

These three conditions allows us to regard $\cup\gamma_i$ as a graph in S. We say $\{\gamma_i\}$ **fills** S if cutting along all γ_i produces disjoint union of disks and once-marked disks.

Remark 3.0.1. The third bullet is required in order to view $\cup\gamma_i$ as a graph, since triple intersections do not work well with isotopy, and we do not have a good notion of "minimal intersection".

Proposition 3.0.1. Suppose $\phi \in \mathrm{Homeo}^+(S, \partial S)$ permutes $\{\gamma_i\}$ up to individual isotopy, then $\phi(\cup\gamma_i)$ is isotopic to $\cup\gamma_i$ rel ∂S. As a result, ϕ induces a automorphism of the graph Γ formed by $\cup\gamma_i$.

Before proving the proposition, we first proof a direct corollary, which gives us a procedure to identify a mapping class up to some finite order.

Corollary 3.0.1 (Alexander Method)**.** Let ϕ be a representative for a mapping class of S. Suppose there is a collection $\{\gamma_i\}$ that fills S, and ϕ permutes $\{\gamma_i\}$ up to individual isotopy, then ϕ represents a finite order element in the mapping class group.

Proof. Note ϕ induces an automorphism of a finite graph Γ formed by $\cup\gamma_i$, which is automatically of finite order. Up to isotopy, ϕ^k induces the identity map on Γ, and applying Alexander trick shows that ϕ^k is the trivial mapping class group. □

Proposition 3.0.1 is a direct corollary of the stronger lemma:

Lemma 3.1. Let S be a compact surface, possibly with marked points, and let $\gamma_1, ..., \gamma_n$ be a collections of essential simple closed curve/arcs that satisfy the three conditions. If γ'_1,..., γ'_n is another collection such that γ_i is isotopic to γ'_i rel boundary, then there is an isotopy of S rel boundary that takes γ_i to γ'_i simultaneously.

4 Hyperelliptic Involution

An important mapping class of genus g surfaces is the Hyperelliptic involution.

Definition 4.0.1. A **hyperelliptic involution** of S_g is a mapping class such that is of order 2, and acts on $H_1(S_g;\mathbb{Z})$ by $-\mathrm{Id}$.

We may realize a hyperelliptic involution as a rigid roation by embedding S_g in $\mathbb{R}^3$. Imagine a torus laying flat on the $x-y$ plane, then a rotation along x-axis by 180 degrees is a hyperelliptic involution.

An example of a hyperelliptic involution of the torus is realized in the way above, and the quotient of the torus by the action is homeomorphic to a 2-sphere with 4 marked points, corresponding to the fixed points of the action. We will use this quotient to determine the mapping class group of $S_{0,4}$.

Proposition 4.0.1. We have

$$\mathrm{Mod}(S_{0,4}) \cong \mathrm{PSL}_2(\mathbb{Z}) \rtimes (\mathbb{Z}/2 \times \mathbb{Z}/2)$$

Proof. We will realize $\mathrm{Mod}(S_{0,4})$ as a split extension; in other words, we will find a split surjective homomorphism

$$\sigma : \mathrm{Mod}(S_{0,4}) \to \mathrm{PSL}_2(\mathbb{Z})$$

with kernel $\mathbb{Z}/2 \times \mathbb{Z}/2$. The split surjection is constructe das follows: an orientation preserving homeomorphism ϕ of $S_{0,4}$ has two lifts to T^2, call them $\tilde{\phi}$ and $i\tilde{\phi}$. We can then define $\sigma(\phi) := \rho(\tilde{\phi})$, viewed as a class in $\mathrm{PSL}_2(\mathbb{Z})$ since the lifts differ by an involution. The right inverse of σ is induced by the action of $\mathrm{PSL}_2(\mathbb{Z})$ on T^2, up to some involution, which descends to a well-defined orientation preserving homeomorphism of $S_{0,4}$. It is the right inverse of σ by construction.

The hyperelliptic involution of $S_{0,4}$ lives in the kernel of σ; conversely, given any f in the kernel, $\tilde{f}$ acts trivially on H_1, and thus trivially on the collection of simple closed curves. In fact, the hyperelliptic involution induces a bijection between simple closed curve on T^2 and $S^{0,4}$, so f fixes the meridian and equatorial circle of $S_{0,4}$. Since f permutes the 4 marked points, we may precompose f with some hyperelliptic involution k of $S_{0,4}$ so fk fixes both curves and the 4 marked points by isotopy. We may then apply Alexander trick to show that fk is the identity element in the mapping class group, so f is in the subgroup generated by the hyperelliptic involution of $S_{0,4}$.

□