Notes on the $84(g-1)$ Theorem

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This note follows Chapter 7 of [FM12]. I attached the whiteboard I used for the talk, where you can find some pictures that might be helpful.

1 Nielsen Realization Theorem

We are interested in what the finite subgroups of $\mathrm{Mod}(S)$ are, and whether they can be realized as a subgroup of $\mathrm{Diff}^+(S)$. The **Nielsen realization problem** asks whether finite subgroups of $\mathrm{Mod}(S)$ can be realized as a subgroup of $\mathrm{Diff}^+(S)$. This is true for all surfaces, which is proven by Kerckhoff in [Ker83]:

Theorem 1.1. *Let $S = S_{g,n}$ and suppose $\chi(S) < 0$. Suppose $G < \mathrm{Mod}(S)$ is a finite group. Then there exists a finite group $\tilde{G} < \mathrm{Diff}^+(S)$ so that the natural projection $\mathrm{Diff}^+(S) \to \mathrm{Mod}(S)$ restricts to an isomorphism $\tilde{G} \to G$. Further, $\tilde{G}$ can be chosen to be a subgroup of isometries of some hyperbolic metric on S.*

This theorem is not true in general for any manifolds. There are counterexamples in dimension 7 and higher. The question of whether Nielse n realization theorem holds for dimension 3 remains open.

2 Orbifolds, the $84(g-1)$ Theorem, and the $4g+2$ Theorem

We will start with the following theorem by Hurwitz in 1893 that bounds the order of any finite group of hyperbolic isometries of a genus ≥ 2 surface. It in facts bounds the size of the isometry group of S.

Theorem 2.1 ($84(g-1)$ Theorem)**.** *If X is a closed hyperbolic surface of genus $g \geq 2$, then*

$$|\operatorname{Isom}^+(X)| \leq 84(g-1).$$

Remark 2.2. The number 84 appears out of nowhere, and the bound is even sharp for infinitely many genus g's. For example, when $g = 3$, the Klein quartic $x^3y + y^3z + z^3x = 0$ in $\mathbb{P}^2$ has $|\operatorname{Isom}^+(X)| = 168 = 84(3-1)$. More generally, if $g-1$ is a prime p of the form $p = 3q+1$ where q is also a prime, then there exists a closed surface X of genus g such that $|\operatorname{Isom}^+(X)| = 84(g-1)$. Such surfaces are constructed by considering the normal subgroups of the $(2,3,7)$-triangle group.

The following theorem was proved by Wiman in 1895:

Theorem 2.3 ($4g+2$ Theorem)**.** *Let X be a closed hyperbolic surface of genus $g \geq 2$. Then any element of $\operatorname{Isom}^+(X)$ has order at most $4g+2$.*

Remark 2.4. The upper bound of Theorem 2.3 is attained for every $g \geq 2$: we simply realize S_g as a regular hyperbolic $(4g+2)$-gon with angle sum 2π and with opposite sides identified, and we consider the rotation by one click.

To start with, we show that the isometry group of a hyperbolic surface is finite. We first claim that $\mathrm{Isom}(S)$ for any compact Riemannian surface S is compact. To do so, we recall the Arzelà-Ascoli theorem:

Theorem 2.5 (Arzelà-Ascoli)**.** *Let X be a compact Hausdorff space and Y a complete metric space. Then $\mathcal{F} \subset C(X,Y)$ is compact in the compact-open topology if and only if it is equicontinuous, pointwise relatively compact and closed.*

Clearly, $\mathrm{Isom}(S)$ is equicontinuous. It is pointwise relatively compact since S is compact. It is closed since it contains all its limit points. It follows that $\mathrm{Isom}(S)$ is a compact topological group, so to show that it's finite it suffices to show that it's discrete, or the connected component containing the identity is trivial. In fact, it suffices to show that $\mathrm{Isom}(X) \cap \mathrm{Diff}_0(S_g) = \{\mathrm{Id}\}$.

Suppose $\phi \in \mathrm{Isom}(X) \cap \mathrm{Diff}_0(S_g)$. Then, ϕ is isotopy to the identity. It follows that ϕ has a lift $\tilde{\phi} \in \mathrm{Diff}(\mathbb{H}^2)$ that is of a bounded distance away from $\mathrm{Id}_{\mathbb{H}^2}$. It follows that $\tilde{\phi}$ fixes $\overline{\mathbb{H}^2}$, so it must be $\mathrm{Id}_{\mathbb{H}^2}$. This shows that $\mathrm{Isom}(X)$ is finite.

The usual strategy to understand $\mathrm{Isom}^+(X)$ is to understand the quotient space $Y = X/\mathrm{Isom}^+(X)$. Notice that when $\mathrm{Isom}^+(X)$ acts freely on X, the space Y is itself a hyperbolic surface. However, elements of $\mathrm{Isom}^+(X)$ can have fixed points on X (for example, the hyperelliptic involution). It is then not even clear that whether Y is a manifold (it nevertheless IS a manifold). In particular, the space Y has an area given by

$$\mathrm{Area}(Y) = \frac{\mathrm{Area}(X)}{|\mathrm{Isom}^+(X)|} = \frac{2\pi(2g-2)}{|\mathrm{Isom}^+(X)|}$$

where the second equality comes from the Gauss-Bonnet theorem. We will show that $\mathrm{Area}(Y) \leq \pi/21$. This would prove Theorem 2.1.

To understand Y, we need to develop the theory of hyperbolic orbifolds.

Definition 2.6. A 2-dimensional (orientable) **hyperbolic orbifold** is a quotient X/G, where X is an orientable surface with a hyperbolic metric and G is a subgroup of the finite group $\mathrm{Isom}^+(X)$.

The **orbifold fundamental group** of $Y = X/G$ is the deck transformation group of the universal cover $\tilde{X} \cong \mathbb{H}^2$. Elements of the orbifold fundamental group can be represented by loops in X.

Given any point $y \in Y$, there is a neighborhood of y that is isometric to the quotient of an open ball in $\mathbb{H}^2$ by a finite group of rotations F_y of $\mathbb{H}^2$. To see this, notice that the stabilizer of y in $\mathrm{Isom}^+(X)$.

A map $\pi : X \to Y$ between 2-dimensional hyperbolic orbifolds is a **regular d-fold orbifold covering map** if it's a quotient map by an order-d subgroup of $\mathrm{Isom}^+(X)$. Suppose X has signature $(h; q_1, \ldots, q_n)$ and Y has signature $(g; p_1, \ldots, p_m)$. For every point $y \in Y$, we see that $\pi^{-1}(y)$ consists of d points in X up to multiplicity of cone points. In particular, if y is a cone point with order p where $\pi^{-1}(y)$ is a collection of cone points with order $q_1, \ldots, q_k$ (notice that there are p/q_i many cone points of degree q_i in $\pi^{-1}(y)$), then using multiplicity of orders, we have

$$\sum_{i=1}^{k} \frac{p}{q_i} = d.$$

In particular, summing up all the p_i's give us

$$\sum_{i=1}^{n} \frac{1}{q_i} = d \sum_{j=1}^{m} \frac{1}{p_j}.$$

The **Euler characteristic** of a 2-dimensional hyperbolic orbifold is an invariant of the orbifold that is multiplicative with respect to degree-d covers. The **Riemann-Hurwitz formula** gives us a way to compute the Euler characteristic of an orbifold Y:

$$\chi(Y) = (2-2g) - m + \sum_{i=1}^{m} \frac{1}{p_i} = (2-2g) - \sum_{i=1}^{m} \left(1 - \frac{1}{p_i}\right).$$

One can check that $\chi(Y)$ behaves well under degree-d covers. That is, if $\pi : X \to Y$ is a degree-d cover, then we have $\chi(X) = d\chi(Y)$.

The **orbifold Gauss-Bonnet formula** can be computed using the punctured Riemannian manifold and is given by

$$\mathrm{Area}(Y) = -2\pi\chi(Y).$$

To see this, suppose $Y = X/G$, where $|G| = d$ and $\pi : X \to Y$ is the degree-d cover. Then, we have

$$\mathrm{Area}(Y) = \frac{\mathrm{Area}(X)}{|G|} = \frac{-2\pi\chi(X)}{d} = -2\pi\chi(Y).$$

Using this, we can show that $\chi(X) \geq -1/42$, where equality is achieved by the $(2,3,7)$-triangle orbifold with signature $(0;2,3,7)$. To see this, one rules out case-by-case other orbifolds until one is left with the triangle orbifold $(0;2,3,7)$. One can follow [FM12, p209-210]. This proves Theorem 2.1.

To prove Theorem 2.3, we could use a similar method. For all hyperbolic surface X of genus $g \geq 2$, the orbifolds $X \to Y \cong X/G$ has Euler characteristics $\chi(Y) = \frac{2-2g}{|G|}$. It then suffices to show that

$$-\chi(Y) \geq \frac{2g-2}{4g+2} = \frac{1}{2} - \frac{3}{4g+2}.$$

Moreover, equality is achieved iff Y has signature $(0;2,2g+1,4g+2)$. The orbifold is achieved by letting X be a hyperbolic regular $(4g+2)$-gon and G generated by a rotation ϕ of 1-click. The center of the $(4g+2)$-gon is stabilized by $\langle\phi\rangle$, which projects to a cone point of order $4g+2$. The two vertices are stabilized by $\langle\phi^2\rangle$, which project to a cone point of order $2g+1$. The $2g+1$ many midpoints of edges are stablized by $\langle\phi^{2g+1}\rangle$, so they project to a cone point of order 2. The rest of the points have trivial stabilizers.

Remark 2.7. To obtain all $g \geq 2$ such that there exists a closed surface X of genus-g with $|\operatorname{Isom}^+(X)| = 84(g-1)$, we may consider the quotient of $\mathbb{H}^2$ under the group

3 Conjugacy Classes of Finite Subgroups

To understand finite subgroups of $\mathrm{Diff}^+(S_g)$, the book first claims the following statement, which allegedly comes from the uniformization theorem: Given any finite subgroup $G < \mathrm{Diff}^+(S_g)$, we can consider a G-invariant metric by the following costruction. Start with an arbitrary Riemannian

metric $\tilde{g}$ of S_g. Consider $\bar{g} = \sum_{\phi \in G} \phi^* \tilde{g}$. Clearly, $\bar{g}$ is G-invariant. Next, by the uniformization theorem, there exists

The book claimed that it proved the following proposition on conjugacy classes of finite subgroups. The claim is that given any finite subgroup $G < \mathrm{Diff}^+(S_g)$, the conjugacy classes of G in $\mathrm{Diff}^+(S_g)$ is finite.

Proposition 3.1. *Let $g \geq 2$. There are finitely many conjugacy classes of finite subgroups in $\mathrm{Diff}^+(S_g)$. In particular, there are finitely many conjugacy classes of finite-order elements in $\mathrm{Diff}^+(S_g)$.*

I figured out the following proof (assuming that the lifting criterion for orbifolds is the same as the lifting criterion for covering projections).

Lemma 3.2. *Let X be a manifold. Fix a discrete group G. Consider two embeddings $\rho_i : G \hookrightarrow \mathrm{Diff}^+(X)$ for $i = 1, 2$. Denote by $G_i = \rho_i(G)$. Let $Y \cong X/G_i$ be the corresponding orbifold and $p_i : X \to Y$ be the two orbifold covering maps. Consider the induced map $\tilde{p}_i : \pi_1(Y) \to \mathrm{Deck}(X \xrightarrow{p_i} Y_i)$. If $\rho_1^{-1}\tilde{p}_1 = \rho_2^{-1}\tilde{p}_2$, then ther exists an orientation-preserving homeomorphism $\phi : X \to X$ such that the following diagram commutes:*

$$\begin{tikzcd} X \arrow[rr, "\phi"] \arrow[dr, "p_1"'] & & X \arrow[dl, "p_2"] \\ & Y & \end{tikzcd}$$

Proof. This is just the lifting criterion. We know that ϕ is a homeomorphism if $p_{1*}\pi_1(X) = p_{2*}\pi_1(X)$. Notice that $p_{i*}\pi_1(X) = \ker(\tilde{p}_i)$, but then the condition $\rho_1^{-1}\tilde{p}_1 = \rho_2^{-1}\tilde{p}_2$ gives us $\ker(\tilde{p}_1) = \ker(\tilde{p}_2)$, which is precisely the desired result. □

Lemma 3.3. *Let X be a manifold. Let $G_1, G_2 < \mathrm{Diff}^+(X)$ be subgroups. Suppose $p_1 : X \to Y$ and $p_2 : X \to Y$ are orbifold covering maps with respect to the action of G_1 and G_2, and there exists an orientation-preserving homeomorphism $\phi : X \to X$ such that the following diagram commutes:*

$$\begin{tikzcd} X \arrow[rr, "\phi"] \arrow[dr, "p_1"'] & & X \arrow[dl, "p_2"] \\ & Y & \end{tikzcd}$$

Then, $G_2 = \phi G_1 \phi^{-1}$.

Proof. Notice that $G_1 = \mathrm{Deck}(X \xrightarrow{p_1} Y)$ and $G_2 = \mathrm{Deck}(X \xrightarrow{p_2} Y)$. It then suffices to show that for all $g \in G_1$ and $x \in X$, we have $p_2 \phi g \phi^{-1}(x) = p_2(x)$. Notice that

$$p_2 \phi g \phi^{-1}(x) = (p_2\phi)(g\phi^{-1})(x) = p_1(g\phi^{-1})(x) = p_1\phi^{-1}(x) = p_2(x).$$

This completes the proof. □

Proof of Proposition 3.1. Fixing a finite subgroup $G < \mathrm{Diff}^+(S_g)$, we show that the set of conjugacy classes $\{\phi G \phi^{-1} \mid \phi \in \mathrm{Diff}^+(S_g)\}$ is finite. Let $Y = S_g/G$. We claim that

$$\#\,\mathrm{Hom}(\pi_1(Y), G) \geq \#\{\phi G \phi^{-1} \mid \phi \in \mathrm{Diff}^+(S_g)\}.$$

It suffices to show that given some $G' < \mathrm{Diff}^+(S_g)$ and an isomorphism $f : G' \to G$, if the two orbifold covering maps $p, p' : X \to Y$ satify $f\tilde{p}' = \tilde{p}$, then G' is conjugate to G. First, Lemma 3.2 gives us an orientation-preserving homeomorphism $\phi : X \to X$ such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[rr, "\phi"] \arrow[dr, "p"'] & & X \arrow[dl, "p'"] \\ & Y & \end{tikzcd}$$

Then, Lemma 3.3 implies $G' = \phi G \phi^{-1}$. Now, since $\pi_1(Y)$ is finitely-generated, and G is a fixed finite group, there exist only finitely many homomorphisms from $\pi_1(Y)$ to G, so $\mathrm{Hom}(\pi_1(Y), G)$ is finite. This completes the proof. $\square$

4 Generating the MCG with Torsion Elements

Luo [Luo00] showed that for all $g \geq 3$, $\mathrm{Mod}(S_g)$ is generated by finitely many involutions using the lantern relations. Later, Korkmaz [Kor05] showed that $\mathrm{Mod}(S_g)$ can be generated by two torsion elements. Kassabov [BF04; Kas03] showed that for $g \geq 7$, $\mathrm{Mod}(S_g)$ can be generated by 4 involutions. Since $\mathrm{Mod}(S_g)$ has no finite-index cyclic subgroups, it is not generated by 2 involutions. The question of whether $\mathrm{Mod}(S_g)$ is generated by 3 involutions remains open.

References

[Ker83] Steven P. Kerckhoff. "The Nielsen Realization Problem". In: *Annals of Mathematics* 117.2 (1983). Accessed 10 Nov. 2025, pp. 235–265. DOI: 10.2307/2007076. URL: https://doi.org/10.2307/2007076.

[Luo00] Feng Luo. *Torsion Elements in the Mapping Class Group of a Surface*. 2000. arXiv: math/0004048 [math.GT]. URL: https://arxiv.org/abs/math/0004048.

[Kas03] Martin Kassabov. *Generating Mapping Class Groups by Involutions*. 2003. arXiv: math/0311455 [math.GT]. URL: https://arxiv.org/abs/math/0311455.

[BF04] Tara E Brendle and Benson Farb. "Every mapping class group is generated by 6 involutions". In: *Journal of Algebra* 278.1 (2004), pp. 187–198. ISSN: 0021-8693. DOI: https://doi.org/10.1016/j.jalgebra.2004.02.019. URL: https://www.sciencedirect.com/science/article/pii/S0021869304001632.

[Kor05] Mustafa Korkmaz. "The Title of the Article". In: *Transactions of the American Mathematical Society* 357.7 (2005). Received by editor(s): November 8, 2003; Received in revised form: January 8, 2004; Published electronically: December 28, 2004; The author was supported in part by the Turkish Academy of Sciences under the Young Scientists Award Program (MK/TÜBA-GEBİP 2003-10), pp. 3299–3310. DOI: https://doi.org/10.1090/S0002-9947-04-03605-0. URL: https://doi.org/10.1090/S0002-9947-04-03605-0.

[FM12] Benson Farb and Dan Margalit. *A Primer on Mapping Class Groups (PMS-49)*. Princeton University Press, 2012. ISBN: 9780691147949. URL: http://www.jstor.org/stable/j.ctt7rkjw (visited on 07/11/2025).

$\mathrm{SH}(S_g, \cdot)$ Theorem. [FM, Chap 9].

Thm. Let $g \geq 1$. Given $G < \mathrm{Mod}(S_g)$ finite

Then, $|G| \leq 84(g-1)$.

Rmk: this bound is sharp for infinitely many g

* the bound is realized iff S_g admits an orbifold covering map to $(2,3,7)$-triangle orbifold.

Ex: $g=3$ "Klein quartic" $x^3y + y^3z + z^3x = 0$

$\Rightarrow \mathbb{CP}^1$ $\xrightarrow{\text{ramified}}$ $\|\mathrm{Aut}(X)\| = 168 = 84(3-1)$.

To prove this.

Step 1: Within finite subgroup

Nielsen $\nearrow \mathrm{Diff}^+(S_g)$

$G \longrightarrow \mathrm{Mod}(S_g)$

Thm. [Kerckhoff, 1983]. $\forall g \geq 1$. $G < \mathrm{Mod}(S_g)$

G finite, then, $\exists Z < \mathrm{Diff}^+(S_g)$ i.e.

$\mathrm{Diff}^+(S_g) \longrightarrow \mathrm{Mod}(S_g)$ restricts to an iso.

$G \xrightarrow{\text{iso}} Z$

* Z can be chosen to be $< \mathrm{Isom}^+(X)$.

* X - closed orientable $\quad$ Rk: X is hyperbolic

$\exists \rho: \mathrm{Diff}^+(X)$ $\quad$ No loss since $g \geq 2$.

$G \longrightarrow \mathrm{Mod}(X)$

$S_g \quad \kappa(S_g) \longrightarrow \mathrm{Mod}(S_g) < \mathrm{Out}(\pi_1(S_g))$

$g \geq 2$ Dehn-Nielsen $\Rightarrow$ iso. bijection

Step 2: If X := closed hyp. surface, non-arithmetic $g \geq 1$.

then $|\mathrm{Isom}^+(X)| \leq 84$

Why? $\underline{|\mathrm{Isom}^+(X)| < C(X, g)}$

By Arzela-Ascoli $\Rightarrow$ $\mathrm{Isom}^+(X)$ is compact

X - compact metric space. $F < \mathrm{Iso}(X)$ is compact

iff F is equicontinuous, pointwise precompact, closed.

$\Rightarrow |\mathrm{Isom}^+(X)| < \infty$ iff $\mathrm{Isom}^+(X)$ is discrete

iff $\mathrm{Isom}^+(X) \cap \mathrm{Isom}^+(X) = \{1\}$

$\phi \in \mathrm{Isom}^+(X)$ ϕ maps to id.

Case 1: $\exists \phi \neq id$ $\phi: \mathbb{H}^2 \to \mathbb{H}^2$.

ϕ and id are within a bounded distance of each other.

$d(\phi(p), p) < \mathrm{Isom}(X)$

$d(\phi(p), id(p)) < \mathrm{Isom}(X)$

X

ϕ acts trivially on $\mathbb{H}^2$ $\Rightarrow$

classification of $\mathrm{Isom}(\mathbb{H}^2)$ $\Rightarrow$ $\phi = -id$

$\Rightarrow \phi = id$ ✓

$\mathrm{Isom}^+(\mathbb{H}^2)$ (Upper half plane)

$PSL(2,\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \begin{array}{l} a,b,c,d \in \mathbb{R} \\ ad-bc=1 \end{array} \right\}$

$z \longmapsto \frac{az+b}{cz+d}$

$\pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ acts identity for $z \to$ geodesic.

Along these 2 fixed pts on $\partial\mathbb{H}^2 = \mathbb{R} \cup \infty$

Then other geodesics must be $\circlearrowleft$ $\Rightarrow$ $\begin{array}{l} \text{fixed pts} \\ \text{fixed pts} \end{array}$

$\partial\mathbb{H}^2$ = {about the geodesics org} $\cup \{\infty\}$

∞
∞
∞