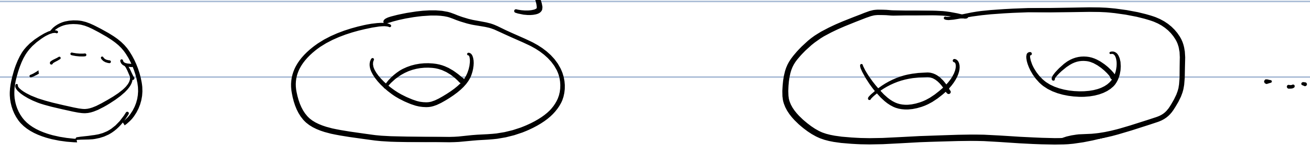
Talk 1. Basic surface topology

A surface is a 2-dim manifold possibly with boundary
Let $S_g = S^2 \# \underbrace{T^2 \# T^2 \cdots \# T^2}_{g}$



S^2 = genus 0 $\qquad T^2$ = genus 1 $\qquad S_2$ = genus 2.

<u>Classification of surfaces</u>:
Any compact connected oriented surface (possibly with boundary) $\cong S_g$ with $b \geq 0$ open disks removed.

From now on, a "surface" will mean a <u>compact</u>, <u>connected</u> <u>oriented</u> surface possibly with <u>punctures</u> (marked pts) which is determined by a triple (g, b, n)

<u>Rmk</u>: Boundaries have compact collars but punctures have noncompact collars.

Let $S_{g,n}$ be a surface with genus g, n punctures and without boundary.
The Euler characteristic $\chi(S) = 2 - 2g - (b+n)$.

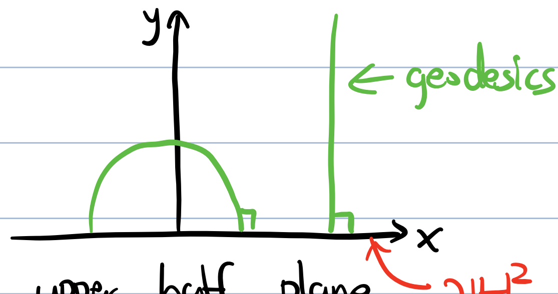
I. The hyperbolic structure

Let $\mathbb{H}^2$ be the hyperbolic plane.
several models:

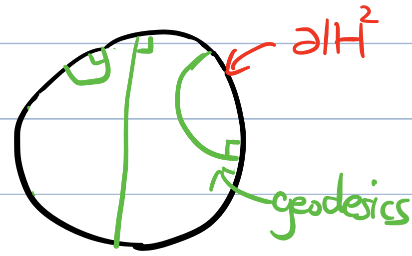
- <u>The upper half-plane model</u>:
$\mathbb{H}^2 = \{(x,y) \in \mathbb{R}^2 : y > 0\}$ equipped with metric $ds^2 = \dfrac{dx^2+dy^2}{y^2}$

- **The Poincare disk model:**
  $\mathbb{H}^2 = \{z \in \mathbb{C} : |z| < 1\}$ equipped with metric $ds^2 = 4\frac{dx^2+dy^2}{(1-r^2)^2}$

Prop: Any simply-connected Riemannian 2-manifold with constant sectional curvature -1 is isometric to $\mathbb{H}^2$.



upper half plane

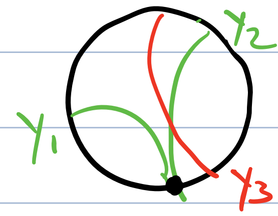


Poincare disk

Boundary of $\mathbb{H}^2$: the boundary of $\mathbb{H}^2$ can be viewed as <u>points at infinity</u> of $\mathbb{H}^2$.

We can define $\partial\mathbb{H}^2$ as equivalence classes $[\gamma]$ of unit speed geodesics where $\gamma_1 \sim \gamma_2$ if

$$\sup_{t \geq 0} d(\gamma_1(t), \gamma_2(t)) < \infty$$



$\gamma_1 \sim \gamma_2$ but $\gamma_1 \nsim \gamma_3$

Geodesics of $\mathbb{H}^2$: In the models geodesics in $\mathbb{H}^2$ are precices round circle arcs that are perpendicular to $\partial\mathbb{H}^2$

Isometries of $\mathbb{H}^2$: $\mathrm{Isom}(\mathbb{H}^2) \cong PSL_2(\mathbb{R})$

acting on the upper half plane by Möbius transformations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \left(z \longmapsto \frac{az+b}{cz+d}\right)$$

- **Rigidity:** For any given $p \in \mathbb{H}^2$, $f \in \mathrm{Isom}(\mathbb{H}^2)$ is determined by $f(p)$ and $f_* : T_p\mathbb{H}^2 \to T_{f(p)}\mathbb{H}^2$
- **Transitivity:** For any $p, q \in \mathbb{H}^2$ and linear isometry $g : \mathbb{R}^2 \to \mathbb{R}^2$, $\exists\, f \in \mathrm{Isom}(\mathbb{H}^2)$ s.t. $q = f(p)$, $g = f_* : T_p\mathbb{H}^2 \to T_q\mathbb{H}^2$.

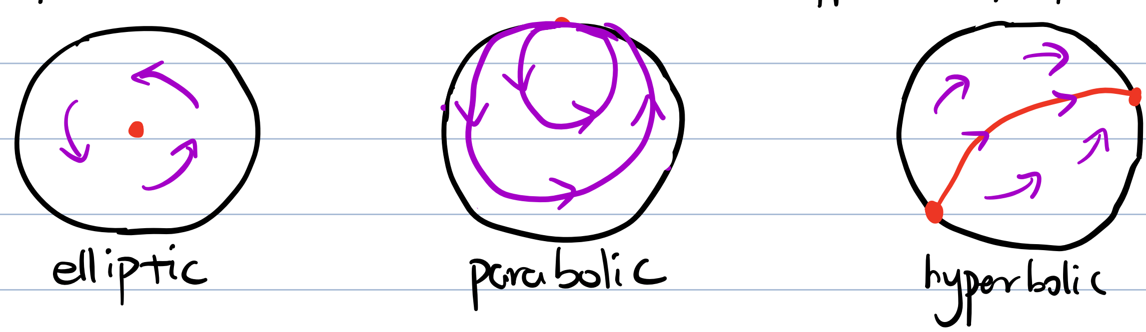
- Classification:

Every $f \in \mathrm{Isom}^+(\mathbb{H}^2)$ induces $\overline{\mathbb{H}}^2 \to \overline{\mathbb{H}}^2$, so f has at least one fixed point in $\overline{\mathbb{H}}^2$.

i) f is called <u>elliptic</u> if f has a fixed point in $\mathbb{H}^2$.
Here $f \sim$ a rotation in the Poincare disk.
and f doesn't have fixed points on $\partial \mathbb{H}^2$.

ii) f is called <u>parabolic</u> if f has exactly one fixed point on $\partial \mathbb{H}^2$
Here $f \sim$ a translation in the upper half plane.

iii) f is called <u>hyperbolic</u> if f has two fixed points on $\partial \mathbb{H}^2$
Here $f \sim (z \mapsto \lambda z,\ \lambda > 0)$ in the upper half plane.



(If f has more than 2 fixed points, $f = id$)

As in $PSL_2(\mathbb{R})$,
elliptic $\Leftrightarrow |tr| < 2$
parabolic $\Leftrightarrow tr = \pm 2$
hyperbolic $\Leftrightarrow |tr| > 2$

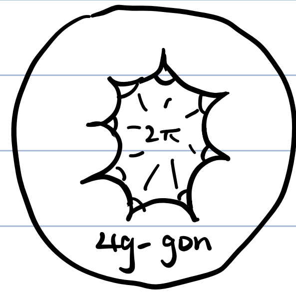
<u>Def</u>: We say that a manifold S admits a <u>hyperbolic metric</u>, if there is a complete Riemannian metric on S, of constant sectional curvature -1 (i.e. locally isometric to open subsets of $\mathbb{H}^n$), and boundaries are totally geodesic (geodesics in ∂S are geodesics in S)

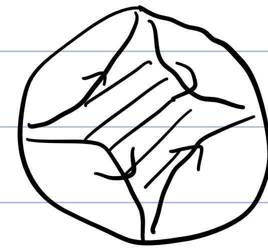
<u>Theorem</u>: Any surface S (possibly w/ boundaries, punctures) with $\chi(S) < 0$ admits a hyperbolic metric.

Rmk: Guass-Bonet tells us $\int_S K = 2\pi\chi(S)$

So hyperbolic $\Rightarrow$ $\chi(S) < 0$

Sket of proof for S_g:



()

The universal cover of S is $\mathbb{H}^2$

so $\pi_1 S \hookrightarrow \mathrm{Isom}\,\mathbb{H}^2$ as a Deck transformation group

Def: A <u>free homotopy</u> of loops in S is an unbased homotopy

Prop: loops in $\pi_1 S$ correspond to either parabolic or hyperbolic isometry

$\{\text{parabolic}\} \longleftrightarrow \left\{\begin{array}{l}\gamma \text{ can be freely homotoped into} \\ \text{a neighborhood of a puncture}\end{array}\right\}$

Thm: Suppose S is hyperbolic. Then the centralizer of any element in $\pi_1 S$ is $\mathbb{Z}$. In particular, if $\pi_1 S$ has nontrivial center, $\pi_1 S \cong \mathbb{Z}$ but not possible, so $\pi_1 S$ has trivial center.

II. Simple closed curves

Def: A <u>closed curve</u> is a map $S^1 \to S$

A closed curve is called <u>essential</u>

if it is not homotopic to a point, a puncture or a boundary component.

A closed curve is called <u>primitive</u>

if $\alpha \neq \beta^k$ for some $k>0$, $\beta \in \pi_1 S$

A closed curve is <u>a multiple</u> if it factors through

$S^1 \xrightarrow{\times n} S^1$ for $n > 1$.

A **lift** of α to the universal cover $\tilde{S}$ is a map $\mathbb{R} \to \tilde{S}$ that projects to α.

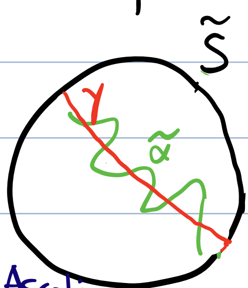
Prop: $\left\{ \begin{array}{c} \text{closed curves in } S \\ \text{up to free homotopy} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{conjugacy classes} \\ \text{of } \pi_1 S \end{array} \right\}$

• Closed geodesics represents free homotopy classes.

Thm: S hyperbolic. If α is a closed curve that is not homotopic into a neighborhood of a puncture, then α is homotopic to a unique closed geodesic γ.

Pf: Exsitence:

Nir: let $\gamma \in \langle \alpha \rangle$ has minimal length
existentence by Arzela-Ascoli



Uniqueness: $d(\gamma(t), \tilde{\alpha}(t))$ bounded because of compactness of S. ☑

Therefore $\left\{ \begin{array}{c} \text{conjugcy classes} \\ \text{of } \pi_1(S) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{oriented closed} \\ \text{geodesics in } S \end{array} \right\}$

Def (Simple closed curves): A closed curve α is simple if $\alpha : S^1 \to S$ is injective.

Thm: Simple closed curves are primitive.

Thm: α simple closed curve in S, γ be the closed geodesic freely homotopic to α then γ is simple.

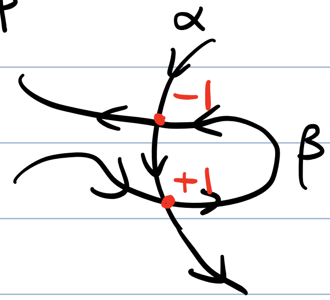
Hence $\left\{ \begin{array}{c} \text{free homotpy classes} \\ \text{of } \alpha \text{ where } \alpha \text{ simple} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{simple closed} \\ \text{geodesics} \end{array} \right\}$

- Intersection numbers

Let α, β be transverse oriented simple closed curves.
a, b be their free homotopy classes

<u>Def</u>: the <u>algebraic intersection number</u>

$\hat{i}(a,b) = \sum_{\alpha\cap\beta} \pm 1$, where ± 1 determined by orientation



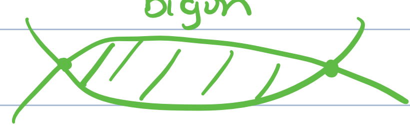
(can be computed using homology so doesn't depend on free homotopy classes)

the <u>geometric intersection number</u>

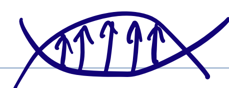
$$i(a,b) = \min_{\substack{\alpha\in a\\ \beta\in b}} \{ |\alpha\cap\beta| \}$$

<u>Def</u> (minimal position): Two simple closed curves are in <u>minimal position</u> if $|\alpha\cap\beta| = i(\alpha,\beta)$

<u>Thm</u> (Bigon criterion): Two transverse simple closed curves are in minimal position
$\Leftrightarrow$ they do not form a <u>bigon</u>.



$\Rightarrow$ is trivial because we can always homotope



through a bigon to reduce intersecti

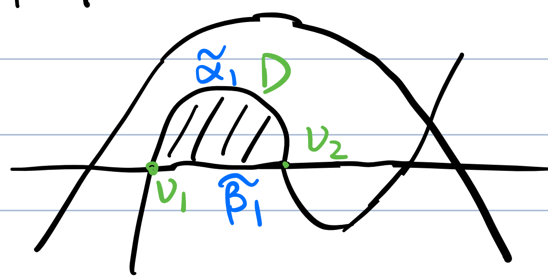
<u>Cor</u>: $|\alpha\cap\beta| = 1$ are in minimal position.
<u>Cor</u>: simple closed geodesics are in minimal position.

<u>Sketch of proof of Bigon Criterion</u>:

<u>Lemma</u>: If α, β does not form a bigon, then

$p: \tilde{S} = \mathbb{R}^2 \to S$, $|\tilde{p}^{-1}(\alpha) \cap \tilde{p}^{-1}(\beta)| \le 1$.

Pf: Suppose $\tilde{p}^{-1}(\alpha)$ and $\tilde{p}^{-1}(\beta)$ intersects at ≥ 2 pts
we can find an innermost disk in the picture
of $\tilde{p}^{-1}(\alpha) \cup \tilde{p}^{-1}(\beta)$



and we claim that D embeds in S, form a bigon
(i) $p(v_1) \neq p(v_2)$ because of orientation
(ii) $p(x) \neq p(y)$ for $x \in \tilde{\alpha}_1$, $y \in \tilde{\beta}_1$ because otherwise
$\tilde{p}^{-1}(\beta)$ passes through x
(iii) $p(x) \neq p(y)$ for $x, y \in \tilde{\alpha}_1$ because otherwise
a transform of v_1 lies between x, y
and $\tilde{p}^{-1}(\beta)$ passes through all transforms of v_1
— (i) (ii) (iii) implies $p|_{\partial D}$ is an embedding
(iv) $p(x) \neq p(y)$ for $x, y \in D \setminus \partial D$ because otherwise
suppose $y = \phi(x)$ for some deck transformation ϕ
we know $\phi(\partial D) \cap \partial D = \phi$ now
so $\phi(\partial D)$ either contains D or contained in D

$\Downarrow$ $\qquad\qquad\qquad\qquad$ $\Downarrow$

$\phi^{-1}(D) \subseteq D$ $\qquad\qquad$ $\phi(D) \subseteq D$

Hence ϕ has a fixed point and $\phi = id$.
$\Rightarrow$ p embeds D into S. $\square$.

Now for the bigon criterion theorem, suppose α, β are not
in minimal position.
Then a homotopy $H: S^1 \times I \to S$ homotopes α to
reduce intersection number with β.

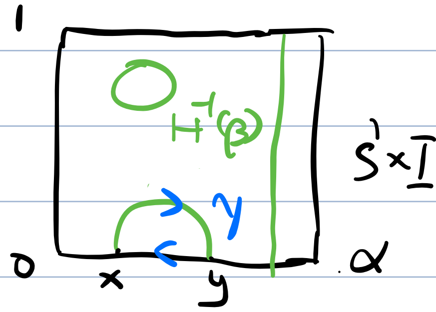
Consider the 1-dim submfd $H^{-1}(\beta) \subset S^1 \times I$

$H^{-1}(\beta)$ could be

(i) an arc connecting one boundary

(ii) an arc connecting two boundaries

(iii) a circle inside.



Since H reduces intersections, we must have type (i)

form a loop γ from type (i) + an arc in $S^1 \times \{0\}$

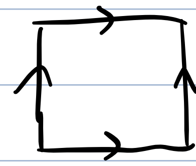
then $H(\gamma) \subset S$ is null-homotopic, hence lifts to $\tilde{S}$

and $x, y \in \tilde{p}^{-1}(\alpha) \cap \tilde{p}^{-1}(\beta)$, contradiction!

Hence α, β are in minimal position. $\boxed{//}$

• Torus

On T^2 stories are simpler:



- T^2 can be given a flat metric
- $\{\text{closed curves}\}/\text{free homotopy} \longleftrightarrow \{(p,q) : p, q \in \mathbb{Z}\}$

  $\{\text{simple closed curves}\}/\text{free homotopy} \longleftrightarrow \{(p,q) : \gcd(p,q) = 1\}$
- $\hat{i}((p,q),(p',q')) = pq' - p'q$

  $i((p,q),(p',q')) = |pq' - p'q|$

• Arcs

All analysis above works for $\underline{\text{arcs}}$: $I \to S$ with endpoints lying in punctures or boundaries.

III. The change of coordinates principle

A prototype theorem:

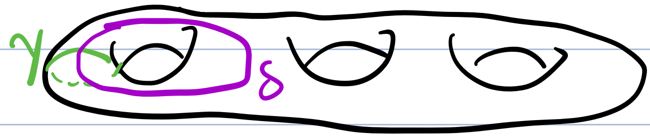
$\underline{\text{Thm}}$: α, β non-separating simple closed curves

then $\exists$ homeomorphism $h: S \to S$ s.t. $h(\alpha) = \beta$

Pf: Denote by S_α the surface obtained by cutting along α.
Then $S_\alpha \cong S_\beta$
because they have the same genus + # of boundary component.
A homeomorphism $S_\alpha \to S_\beta$ will give
$h: S \to S$ s.t. $h(\alpha) = \beta$.

Example: Given nonseperating simple closed curve γ.
How to find a simple closed curve δ
with $i(\gamma, \delta) = 1$?

Solution:

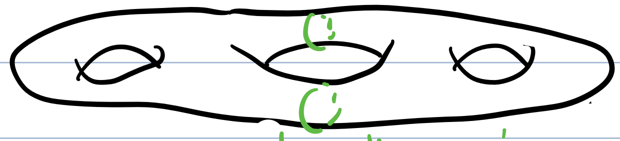


A priori γ could be very complicated.
But by change of coordinates principle we can assume γ is as above. Then δ can be easily constructed.

Other examples of the change of coordinate principle:

1. Separating curves:
Suppose a seperating s.c.c. γ has type $(k, g-k)$
if S_γ has components of genus $k, g-k$
Then any s.c.c. α, β with the same types
$\exists$ homeo $h: S \to S$ s.t. $h(\alpha) = \beta$.

2. Bounding pairs:
A bounding pair is a pair of disjoint, homologous, nonseperating s.c.c. such that their union separates.
The type $\{k, g-k\}$ is similarly defined.
Then any two bounding pairs are topological equivalent.



3. k disjoint s.c.c. whose union doesn't seperate
4. A pair of s.c.c. (α, β) with $|\alpha \cap \beta| = 1$
5. A pair of s.c.c. (α, β) with $i(\alpha \cap \beta) = |\alpha \cap \beta| = 2$
   and $\hat{i}(\alpha, \beta) = 0$. and $\alpha \cup \beta$ doesn't seperate
6. Chains of s.c.c. $(\alpha_1, \cdots, \alpha_n)$ with
   $|\alpha_i \cap \alpha_{i+1}| = 1$ and $|\alpha_i \cap \alpha_j| = 0$ otherwise
   and $\cup \alpha_i$ doesn't seperate.

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