# Several new classes of optimal ternary cyclic codes with two or three zeros

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#### Abstract

Cyclic codes are a subclass of linear codes and have wide applications in data storage systems, communication systems and consumer electronics due to their efficient encoding and decoding algorithms. Let  $\alpha$  be a generator of  $\mathbb{F}_{3^m}^*$ , where m is a positive integer. Denote by  $\mathcal{C}_{(i_1,i_2,\cdots,i_t)}$  the cyclic code with generator polynomial  $m_{\alpha^{i_1}}(x)m_{\alpha^{i_2}}(x)\cdots m_{\alpha^{i_t}}(x)$ , where  $m_{\alpha^i}(x)$  is the minimal polynomial of  $\alpha^i$  over  $\mathbb{F}_3$ . In this paper, by analyzing the solutions of certain equations over finite fields, we present four classes of optimal ternary cyclic codes  $\mathcal{C}_{(0,1,e)}$  and  $\mathcal{C}_{(1,e,s)}$  with parameters  $[3^m-1,3^m-\frac{3m}{2}-2,4]$ , where  $s=\frac{3^m-1}{2}$ . In addition, by determining the solutions of certain equations and analyzing the irreducible factors of certain polynomials over  $\mathbb{F}_{3^m}$ , we present four classes of optimal ternary cyclic codes  $\mathcal{C}_{(2,e)}$  and  $\mathcal{C}_{(1,e)}$  with parameters  $[3^m-1,3^m-2m-1,4]$ . We show that our new optimal cyclic codes are inequivalent to the known ones.

Index Terms finite fields, linear codes, minimum distance, cyclic codes.

### 1 Introduction

Cyclic codes are a very important subclass of linear codes and have been extensively studied. Throughout this paper, let  $\mathbb{F}_{3^m}$  denote the finite field with  $3^m$  elements. An [n, k, d] linear code over  $\mathbb{F}_3$  is a k-dimensional subspace of  $\mathbb{F}_3^n$  with minimum Hamming distance d. An [n, k] cyclic code  $\mathcal{C}$  is an [n, k] linear code with the property that any cyclic shift of a codeword is another codeword of  $\mathcal{C}$ . Let  $\gcd(n, 3) = 1$ . By identifying any codeword  $(c_0, c_1, \dots, c_{n-1}) \in \mathcal{C}$  with

$$c_0 + c_1 x + c_2 x^2 + \dots + c_{n-1} x^{n-1} \in \mathbb{F}_3[x]/(x^n - 1),$$

any cyclic code of length n over  $\mathbb{F}_3$  corresponds to an ideal of the polynomial ring  $\mathbb{F}_3[x]/(x^n-1)$ . Notice that every ideal of  $\mathbb{F}_3[x]/(x^n-1)$  is principal. Thus, any cyclic code can be expressed as  $\langle g(x) \rangle$ , where g(x) is monic and has the least degree. The polynomial g(x) is called the *generator polynomial* and  $h(x) = (x^n-1)/g(x)$  is called the *parity-check polynomial* of  $\mathcal{C}$ . Let  $\alpha$  be a generator of  $\mathbb{F}_{3^m}^*$  and let  $m_{\alpha^i}(x)$  denote the minimal polynomial of  $\alpha^i$  over  $\mathbb{F}_3$ . We denote by  $\mathcal{C}_{(i_1,i_2,\cdots,i_t)}$  the cyclic code with generator polynomial  $m_{\alpha^{i_1}}(x)m_{\alpha^{i_2}}(x)\cdots m_{\alpha^{i_t}}(x)$ . In 2005, Carlet, Ding, and Yuan [1] constructed some optimal ternary cyclic

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codes  $\mathcal{C}_{(1,e)}$  with parameters  $[3^m-1,3^m-2m-1,4]$  by using perfect nonlinear monomials  $x^e$ . In 2013, Ding and Helleseth [2] constructed several classes of optimal ternary cyclic codes  $\mathcal{C}_{(1,e)}$  by using monomials including almost perfect nonlinear (APN) monomials. Moreover, they presented nine open problems on optimal ternary cyclic codes  $\mathcal{C}_{(1,e)}$ . Open problems 7.5 and 7.8 were solved in [3] and [4], respectively. In [4], Li et al. also presented several classes of optimal ternary cyclic codes with parameters  $[3^m-1, 3^m-2m-1, 4]$ or  $[3^m-1, 3^m-2m-2, 5]$ . In 2016, Wang and Wu [5] presented four classes of optimal ternary cyclic codes with parameters  $[3^m - 1, 3^m - 2m - 1, 4]$  by analyzing the solutions of certain equations over  $\mathbb{F}_{3^m}$ . It was shown that some previous results on optimal ternary cyclic codes given in [2] [4] [6] [7] are special cases of the constructions given in [5]. In 2019, another open problem 7.12 proposed in [2] was settled independently in [8] and [9]. Later, Zha et al. [10] [11] presented several classes of optimal ternary cyclic codes  $\mathcal{C}_{(1,e)}$  and  $C_{(u,v)}$ , they also proposed a link between the ternary cyclic codes  $C_{(1,e)}$  and  $C_{(\frac{3^m+1}{2},e+\frac{3^m-1}{2})}$ . In 2022, Zhao, Luo, and Sun [12] presented two families of optimal ternary cyclic codes and solved the remain problem in [5]. Recently, Ye and Liao [22] gave a counterexample of the open problem 7.13 proposed in [2] and presented three classes of optimal ternary cyclic codes  $\mathcal{C}_{(1,e)}$ . A sufficient and necessary condition for the ternary cyclic code  $\mathcal{C}_{(u,v)}$  to be optimal were given in [13]. Based on this basic result, several classes of optimal ternary cyclic codes  $C_{(u,v)}$  were also presented in [13]. Recently, Li and Liu [14] proposed some classes of optimal ternary cyclic codes  $C_{(2,e)}$  with parameters  $[3^m-1,3^m-2m-1,4]$ . There have been some other optimal ternary cyclic codes constructed in the literature, see [15] [16] [17] [18] and the references therein. We list the known optimal ternary cyclic codes  $C_{(1,e)}$  and  $C_{(u,v)}$  with parameters  $[3^m-1,3^m-2m-1,4]$  in Tables 1 and 2, respectively.

However, there are only a few constructions of optimal p-ary  $(p \ge 3)$  is a prime) cyclic codes  $\mathcal{C}_{(0,1,e)}$  and  $\mathcal{C}_{(1,e,s)}$   $(s=\frac{p^m-1}{2})$  with parameters  $[p^m-1,p^m-\frac{3m}{2}-2,4]$ . In 2019, Li, Zhu, and Liu [19] constructed a class of optimal ternary cyclic codes  $\mathcal{C}_{(0,1,e)}$  with parameters  $[3^m-1,3^m-\frac{3m}{2}-2,4]$ , where  $e=3^{\frac{m}{2}}+1$ . Recently, Wu, Liu, and Li [20] generalized the construction in [19] and presented two classes of optimal p-ary (for any odd prime p) cyclic codes  $\mathcal{C}_{(0,1,e)}$  and  $\mathcal{C}_{(1,e,s)}$  with parameters  $[p^m-1,p^m-\frac{3m}{2}-2,4]$ . We summarize some known optimal p-ary cyclic codes  $\mathcal{C}_{(0,1,e)}$  and  $\mathcal{C}_{(1,e,s)}$  with parameters  $[p^m-1,p^m-\frac{3m}{2}-2,4]$  in Table 3.

In this paper, we first present four classes of optimal ternary cyclic codes  $C_{(0,1,e)}$  and  $C_{(1,e,s)}$  with parameters  $[3^m - 1, 3^m - \frac{3m}{2} - 2, 4]$  by analyzing the solutions of certain equations over finite fields, where  $s = \frac{3^m - 1}{2}$ . Then we give four classes of optimal ternary cyclic codes with parameters  $[3^m - 1, 3^m - 2m - 1, 4]$  by analyzing the irreducible factors of certain polynomials and determining the solutions of certain equations over  $\mathbb{F}_{3^m}$ . Two of them confirmed some special cases of the open problem 7.9 proposed in [2]. We also show that the new cyclic codes constructed in this paper are inequivalent to the known ones.

The rest of this paper is organized as follows. In Section 2, we introduce some preliminaries. Four classes of optimal ternary cyclic codes  $C_{(0,1,e)}$  and  $C_{(1,e,s)}$  are given in Section 3. In Section 4, we present four classes of optimal ternary cyclic codes with two zeros. Concluding remarks are given in Section 5.

### 2 Preliminaries

Let p be a prime and m be a positive integer. The p-cyclotomic coset modulo  $p^m - 1$  containing j is defined as

$$C_i = \{j \cdot p^r \pmod{(p^m - 1)} : r = 0, 1, \dots, l_i - 1\},\$$

where  $l_j$  is the least positive integer such that  $j \cdot p^{l_j} \equiv j \mod (p^m - 1)$ . Thus the size of  $C_j$  is  $|C_j| = l_j$ . It is known that  $l_j|m$ . The smallest integer in  $C_j$  is called the coset leader of  $C_j$ .

The following lemmas will be frequently used in the sequel.

**Lemma 1.** [27] Let p be a prime and m be a positive integer. Let  $n = p^m - 1$ . For any  $1 \le e \le n - 1$ , if one of the following conditions is satisfied, then  $l_e = |C_e| = m$ :

- 1)  $1 \le \gcd(e, n) \le p 1$ ,
- 2)  $\gcd(e, n) \cdot \gcd(p^j 1, n) \not\equiv 0 \pmod{n}$  for all  $1 \le j < m$ .

**Lemma 2.** [27] Let  $e = p^k + 1$ , where  $1 \le k \le m - 1$ . Then

- 1) If m is odd, then  $|C_e| = m$ ;
- 2) If m is even, then

$$|C_e| = \begin{cases} \frac{m}{2}, & k = \frac{m}{2} \\ m, & k \neq \frac{m}{2}. \end{cases}$$

**Lemma 3.** [10] Let m be odd and  $e = \frac{3^h + 5}{2}$ , where h is an odd integer. Then  $e \notin C_1$  and  $|C_e| = m$ .

By the following bound of the Hamming distance of general linear codes, it can be shown that cyclic code  $C_{(1,e,s)}$  or  $C_{(0,1,e)}$  with parameters  $[3^m-1,3^m-\frac{3m}{2}-2,4]$  is optimal.

**Lemma 4.** [28] Let  $A_p(n,d)$  be the maximum number of codewords of a p-ary code with length n and Hamming distance at least d. If  $p \ge 3$ , t = n - d + 1 and  $r = \left\lfloor \min\left\{\frac{n-t}{2}, \frac{t-1}{p-2}\right\}\right\rfloor$ , then

$$A_p(n,d) \le \frac{p^{t+2r}}{\sum_{i=0}^r \binom{t+2r}{i} (p-1)^i}$$

Ding and Helleseth proved the following fundamental theorem about the optimality of the ternary cyclic codes  $C_{(1,e)}$ .

**Lemma 5.** [2] Let  $e \notin C_1$  and  $|C_e| = m$ . The ternary cyclic code  $C_{(1,e)}$  has parameters  $[3^m - 1, 3^m - 1 - 2m, 4]$  if and only if the following conditions are satisfied:

- 1) e is even;
- 2)  $(x+1)^e + x^e + 1 = 0$  has no solution in  $\mathbb{F}_{3^m} \setminus \mathbb{F}_3$ ; and
- 3)  $(x+1)^e x^e 1 = 0$  has no solution in  $\mathbb{F}_{3^m} \setminus \mathbb{F}_3$ .

**Lemma 6.** [13] [14] Let e be a positive integer with  $1 \le e \le 3^m - 1$  and  $|C_e| = m$ . The ternary cyclic code  $C_{(2,e)}$  has parameters  $[3^m - 1, 3^m - 2m - 1, 4]$  if and only if the following three conditions are satisfied:

- 1) e is odd;
- 2) the equation  $(1+x^2)^e (1+x^e)^2 = 0$  has no solution in  $\mathbb{F}_{3^m} \setminus \mathbb{F}_3$ ; and
- 3) the equation  $(1+x^2)^e + (1+x^e)^2 = 0$  has no solution in  $\mathbb{F}_{3^m} \setminus \mathbb{F}_3$ .

Table 1: Known optimal ternary cyclic codes  $C_{(1,v)}$  with parameters  $[3^m-1,3^m-2m-1,4]$ 

Type	v (even)	Conditions	Ref.
1	$(3^k + 1)/2$	$m \ge 2, k \text{ is odd, } \gcd(m, k) = 1$	[1]
2	$3^k + 1$	$m \ge 2$ , $m/\gcd(m,k)$ is odd	[1]
3	$(3^m - 3)/2$	$m$ is odd, $m \ge 5$	[2]
4	$(3^m+1)/4+(3^m-1)/2$	$m$ is odd, $m \ge 3$	[2]
5	$(3^{(m+1)/4} - 1)(3^{(m+1)/2} + 1)$	$m \equiv 3 \pmod{4}$	[2]
6	$(3^{(m+1)/2} - 1)/2$ or $(3^{m+1} - 1)/8$	$m \equiv 3 \pmod{4}$	[2]
7	$(3^{(m+1)/2} - 1)/2 + (3^m - 1)/2$	$m \equiv 1 \pmod{4}$	[2]
8	$(3^{m+1}-1)/8 + (3^m-1)/2$	$m \equiv 1 \pmod{4}$	[2]
9	$(3^h - 1)/2$ $3^h - 1$	m is odd, $h$ is even, $gcd(m, h) = gcd(m, h - 1) = 1$	[2]
10	$3^{h} - 1$	$\gcd(m,h) = \gcd(3^m - 1, 3^h - 2) = 1$	[2]
11	$2(3^{m-1}-1)$ or $5(3^{m-1}-1)$ or 16	$m \text{ is odd}, m \not\equiv 0 \pmod{3}$	[4]
12	$(3^m - 1)/2 - 2$ , or $(3^m - 1)/2 + 10$	$m \equiv 2 \pmod{4}$	[4]
13	$(3^m - 1)/2 - 5$ or $(3^m - 1)/2 + 7$ or 20	m is odd	[4]
14	$2(3^h + 1)$	m is odd	[3]
15	$\frac{(3^m - 3)/4}{3^h + 5}$	m is odd	[21]
16	$3^{h} + 5$	$m \equiv 0 \pmod{4}, m \ge 4, h = m/2$	[8]
		$m \equiv 2 \pmod{4}, \ m \ge 6, h = (m+2)/2$	
		$m$ is odd, $gcd(m,3) = 1$ , $2h \equiv \pm 1 \pmod{m}$	
		$m \ge 5$ is prime, $m \ne 19$ , $2h \equiv 3 \pmod{m}$	[22]
		$m \ge 5$ is prime, $m \equiv 2 \pmod{3}$ , $3h \equiv 1 \pmod{m}$	
17	$v(3^s - 1) \equiv 3^t - 1 \pmod{3^m - 1}$	$\gcd(m,t) = \gcd(m,t-s) = 1$	[5] [12]
18	$v(3^s + 1) \equiv 3^t + 1 \pmod{3^m - 1}$	$\gcd(m, t + s) = \gcd(m, t - s) = 1$	[5] [12]
19	$v \equiv (3^m - 1)/2 + 3^s + 1 \pmod{3^m - 1}$	$m$ is even, $m/\gcd(m,s)$ is odd	[5]
20	$v \equiv (3^m - 1)/2 + 3^s - 1 \pmod{3^m - 1}$	$m \text{ is even, } \gcd(m, s) = \gcd(3^m - 1, 3^s - 2) = 1$	[5] [12]
21	$(3^h + 7)/2$	$m$ is odd, $h$ is even, $1 \le h < m$	[10]
22	$(3^h + 7)/2 + (3^m - 1)/2$	$m$ is odd, $h$ is odd, $1 \le h < m$	[10]
23	$(3^{(m+1)/2} + 5)/2$	$m \equiv 1 \pmod{4}, \ m \not\equiv 0 \pmod{3}$	[10]
24	$3^h + 13$	$m \text{ is odd, } \gcd(m,3) = 1, \ 2h \equiv \pm 1 \pmod{m}$	[8]
25	$(3^{(m+1)/2} + 5)/2 + (3^m - 1)/2$	$m \equiv 3 \pmod{4}, \ m \not\equiv 0 \pmod{3}$	[10]
26	$v(3^h + 1) \equiv (3^m + 1)/2 \pmod{3^m - 1}$	m is odd	[10]
27	$5v \equiv 2 \pmod{3^m - 1}$	$m \not\equiv 0 \pmod{3}$	[10]
28	$7v \equiv 2 \pmod{3^m - 1}$	$m \not\equiv 0 \pmod{5}$ , $\gcd(m,6) = 1$ or 3	[10]
29	$5v \equiv 4 \pmod{3^m - 1}$	$m>2, m \not\equiv 0 \pmod{3}, m \not\equiv 0 \pmod{5}$	[10]
30	$5v \equiv 3^m - 3 \pmod{3^m - 1}$	$m \not\equiv 0 \pmod{3}, \ m \not\equiv 0 \pmod{4}$	[18]
31	$7v \equiv 3^m - 3 \pmod{3^m - 1}$	$m \text{ is odd}, m \not\equiv 0 \pmod{3}, m \not\equiv 0 \pmod{7}$	[18]
32	$5v \equiv 3^m - 5 \pmod{3^m - 1}$	$m \text{ is odd}, m \not\equiv 0 \pmod{5}$	[18]
33	$(3^{(m-1)/2} + 5)/2$	$m \not\equiv 0 \pmod{3}, m \equiv 3 \pmod{4}$	[22]
34	$(3^{(m-1)/2} + 5)/2 + (3^m - 1)/2$	$m \not\equiv 0 \pmod{3}, m \equiv 1 \pmod{4}$	[22]
35	$(3^m - 1)/2 - k \ (m > 1 \text{ is odd})$	$k = 7$ , or $k = 11, -19$ and $m \not\equiv 0 \pmod{9}$	[13]
36	$(3^m - 1)/2 + 3^s + 2 \ (m \text{ is odd})$	$gcd(m,s) = 1, x^{3^{s}+1} - x^{2} + 1 = 0$ has no solution in $\mathbb{F}_{3^{m}}$	[12]

Table 2: Known optimal ternary cyclic codes  $C_{(u,v)}$  with parameters  $[3^m-1,3^m-2m-1,4]$ 

Type	u	v	Conditions	Ref.	
1	$\frac{3^m+1}{2}$	$(3^k + 1)/2$	m is odd, $k$ is even, $gcd(m, k) = 1$	[23]	
		$2 \cdot 3^{(m-1)/2} + 1$	$m \ge 3, m \text{ is odd}$	[15]	
		$3^{s} + 2$	$m \text{ is odd, } m \geq 3, 4s \equiv 1 \pmod{m}, 9 \nmid m$	[17]	
		$(3^m - 1)/2 + e$	$m$ is odd, $e$ is even, $C_{(1,e)}$ has parameters $[3^m - 1, 3^m - 2m - 1, 4]$	[11]	
		$(3^{m+1}+7)/8$	$m \ge 3, m \equiv 3 \pmod{4}, 9 \nmid m, 5 \nmid m$	[24]	
		$3^h + 2 \cdot 3^i,$			
		h > 1, $m = 2h - 1$	$m \not\equiv 0 \pmod{3}, i = 0$	[25]	
		III - 2II - 1	$m \not\equiv 0 \pmod{3}, i = 1, h \not\equiv 3 \pmod{5}$	1	
			$m \not\equiv 0 \pmod{3}, \ i = 1, \ n \not\equiv 3 \pmod{5}$ $m \not\equiv 0 \pmod{3}, \ i = 2, \ h \not\equiv 27 \pmod{53}$		
			$m \neq 0 \pmod{3}, i = 2, i \neq 2i \pmod{33}$ $m \text{ is odd, } e \text{ is even, and the ternary}$		
2	$2^i$	$(3^m - 1)/2 + 2^i \cdot e$	cyclic code $\mathcal{C}_{(1,e)}$ is optimal	[11]	
3	u	$(3^k - 1)u + (3^m - 1)/2$	$m \text{ is odd, } \gcd(u, 3^m - 1) = 2,$ $k = 1, 2, 3, (m + 1)/2 \text{ and } \gcd(m, k) = 1$	[11]	
4	$\frac{3^k+1}{2}$	$(3^l + 1)/2$	$m, l, \text{ and } \frac{m}{\gcd(m,l)} \text{ are all even},$ $\gcd(m, k+l) = \gcd(m, k-l) = 1$	[11]	
5	$3^m - 6,$ $m$ is even	$(3^k + 1)/2$ , k is odd	$k = m - 1, m \not\equiv 0 \pmod{6}, m \not\equiv 0 \pmod{20}$	[13]	
			$k = 1, m \not\equiv 0 \pmod{6}, m \not\equiv 0 \pmod{20}$		
			$k = 3, 6 \nmid m, 25 \nmid m,$		
			$46 \nmid m, 78 \nmid m$		
6	2	$(3^m - 1)/2 + 2(3^k - 1)$	m, k are positive integers, $m$ is odd, $\gcd(m, k) = \gcd(3^k - 2, 3^m - 1) = 1$	[16]	
		$(3^m - 1)/2 + 2(3^k + 1)$	$m, k \in \mathbb{N}^*, m \text{ is odd, } \gcd(m, k) = 1$	[16]	
		$(3^k + 1)/2$	$k$ is even, $2 \le k \le m$	[10]	
		(3 + 1)/2	$\gcd(k+1,m) = \gcd(k-1,m) = 1$	[18]	
		$3v \equiv 5 \pmod{3^m - 1}$	$m \not\equiv 0 \pmod{3}$	[18]	
		$3v \equiv 7 \pmod{3^m - 1}$	$m \not\equiv 0 \pmod{5}, \ m \not\equiv 0 \pmod{6}$	[18]	
		$3v \equiv 11 \pmod{3^m - 1}$	$m \not\equiv 0 \pmod{4}, \ m \not\equiv 0 \pmod{9}$	[18]	
		$3v \equiv 13 \pmod{3^m - 1}$	$m \not\equiv 0 \pmod{3}, m \not\equiv 0 \pmod{4}$	[18]	
		$(3^m + 2 \cdot 3^k + 1)/2$	$m \text{ is odd, } \gcd(m,k) = 1$	[14]	
		$\frac{(3^m - 9)/2}{3^h + 2}$	m is odd	[14]	
			$h \in \mathbb{N}^*, 2h \equiv 1 \pmod{m}$	[14]	
		$\frac{3^m-1}{2} + 2(2 \cdot 3^k - 1)$	m  is odd, k = (m-1)/2	[14]	

Table 3: Known optimal cyclic codes with parameters  $[p^m - 1, p^m - \frac{3m}{2} - 2, 4]$ 

Type	e	Conditions	Ref.
$\mathcal{C}_{(1,e,s)}$	$e = \frac{p^m - 1}{2} + 1 + p^{\frac{m}{2}}$	p is an odd prime	[20]
$\mathcal{C}_{(0,1,e)}$	$e = 1 + p^{\frac{m}{2}}$	p is an odd prime	[20]

**Lemma 7.** [29] For every finite field  $\mathbb{F}_{p^m}$  and every positive integer r, the product of all monic irreducible polynomials over  $\mathbb{F}_{p^m}$  whose degrees divide r is equal to  $x^{(p^m)^r} - x$ .

**Lemma 8.** [29] Let f(x) be an irreducible polynomial over  $\mathbb{F}_{p^m}$  of degree r. Then f(x) = 0 has a root x in  $\mathbb{F}_{p^{mr}}$ . Furthermore, all the roots of f(x) = 0 are simple and are given by the r distinct elements  $x, x^{p^m}, x^{p^{2^m}}, \cdots, x^{p^{m(r-1)}}$  of  $\mathbb{F}_{p^{mr}}$ .

## 3 Four classes of optimal ternary cyclic codes with parameters

$$[3^m-1,3^m-\frac{3m}{2}-2,4]$$

In this section, we will propose four classes of optimal ternary cyclic codes  $C_{(0,1,e)}$  or  $C_{(1,e,s)}$  with parameters  $[3^m - 1, 3^m - \frac{3m}{2} - 2, 4]$  by analyzing the solutions of certain equations over  $\mathbb{F}_{3^m}$ .

# 3.1 The first two classes of optimal ternary cyclic codes with parameters $[3^m - 1, 3^m - \frac{3m}{2} - 2, 4]$

Let  $e = 2 \cdot 3^{m-1} - 3^{\frac{m}{2}-1} - 1$ , where m is an even integer. In this subsection, we will show that  $C_{(0,1,e)}$  is an optimal ternary cyclic code with parameters  $[3^m - 1, 3^m - \frac{3m}{2} - 2, 4]$ .

**Theorem 1.** Let m be an even integer and  $e = 2 \cdot 3^{m-1} - 3^{\frac{m}{2}-1} - 1$ . Then  $C_{(0,1,e)}$  is an optimal ternary cyclic code with parameters  $[3^m - 1, 3^m - \frac{3m}{2} - 2, 4]$ .

*Proof:* Note that  $3e \equiv -1 - 3^{\frac{m}{2}} \pmod{3^m - 1}$ . It is obviously that  $e \notin C_1$  and  $|C_e| = \frac{m}{2}$ . Thus the dimension of  $\mathcal{C}_{(0,1,e)}$  is  $3^m - \frac{3m}{2} - 2$ .

In the following we show that  $C_{(0,1,e)}$  does not have a nonzero codeword of Hamming weight less than 4. Clearly, the minimum distance of  $C_{(0,1,e)}$  cannot be 1. Suppose that  $C_{(0,1,e)}$  has a codeword of Hamming weight 2, then there exist two elements  $c_1, c_2 \in \mathbb{F}_3^*$  and two distinct elements  $x_1, x_2 \in \mathbb{F}_{3m}^*$  such that

$$\begin{cases}
c_1 + c_2 = 0 \\
c_1 x_1 + c_2 x_2 = 0 \\
c_1 x_1^e + c_2 x_2^e = 0.
\end{cases}$$
(1)

By the first equation of (1), we have  $c_1 = -c_2$ , thus  $x_1 = x_2$  by the second equation of (1), which is contrary to  $x_1 \neq x_2$ . Therefore,  $\mathcal{C}_{(0,1,e)}$  does not have a codeword of Hamming weight 2.

 $C_{(0,1,e)}$  has a codeword of Hamming weight 3 if and only if there exist three elements  $c_1, c_2, c_3 \in \mathbb{F}_3^*$  and three distinct elements  $x_1, x_2, x_3 \in \mathbb{F}_{3^m}^*$  such that

$$\begin{cases}
c_1 + c_2 + c_3 = 0. \\
c_1 x_1 + c_2 x_2 + c_3 x_3 = 0 \\
c_1 x_1^e + c_2 x_2^e + c_3 x_3^e = 0.
\end{cases}$$
(2)

By the first equation of (2), we have  $(c_1, c_2, c_3) = (1, 1, 1)$  or  $(c_1, c_2, c_3) = (-1, -1, -1)$ . Due to symmetry, it is sufficient to consider the case  $(c_1, c_2, c_3) = (1, 1, 1)$ . In this case, let  $x = \frac{x_1}{x_3}$  and  $y = \frac{x_2}{x_3}$ , then  $x \neq 1$ ,  $y \neq 1$ , and (2) becomes

$$\begin{cases} x + y + 1 = 0 \\ x^e + y^e + 1 = 0, \end{cases}$$
 (3)

which is equivalent to  $(x+1)^e + x^e + 1 = 0$ . Raising both sides of this equation to the power of 3 will lead to  $(x+1)^{3e} + x^{3e} + 1 = 0$ , i.e.,

$$x^{2(3^{\frac{m}{2}}+1)} + x^{2 \cdot 3^{\frac{m}{2}}+1} + x^{3^{\frac{m}{2}}+2} + x^{3^{\frac{m}{2}}} + x + 1 = 0.$$

Notice that the above equation can be factorized as

$$\left( \left( x^{3^{\frac{m}{2}}} - 1 \right) (x - 1) - \left( x^{3^{\frac{m}{2}}} - x \right) \right) \left( \left( x^{3^{\frac{m}{2}}} - 1 \right) (x - 1) + \left( x^{3^{\frac{m}{2}}} - x \right) \right) = 0.$$

As a result,  $(x^{3\frac{m}{2}} - 1)(x - 1) - (x^{3\frac{m}{2}} - x) = 0$  or  $(x^{3\frac{m}{2}} - 1)(x - 1) + (x^{3\frac{m}{2}} - x) = 0$ . Case 1),  $(x^{3\frac{m}{2}} - 1)(x - 1) - (x^{3\frac{m}{2}} - x) = 0$ . In this case,

$$x^{3^{\frac{m}{2}}+1} + x^{3^{\frac{m}{2}}} + 1 = 0. (4)$$

Taking  $3^{\frac{m}{2}}$ -th power of both sides of the above equation will lead to  $x^{3^{\frac{m}{2}}+1}+x+1=0$ , togehter with (4), we have  $x^{3^{\frac{m}{2}}}=x$ , i.e.,  $x\in\mathbb{F}_{3^{\frac{m}{2}}}$ . Thus, (4) can be simplied to  $x^2+x+1=0$ , i.e., x=1, which is contrary to  $x\neq 1$ .

Case 2),  $(x^{3^{\frac{m}{2}}} - 1)(x - 1) + (x^{3^{\frac{m}{2}}} - x) = 0$ . In this case,  $x^{3^{\frac{m}{2}} + 1} + x + 1 = 0$ . Similar to case 1), we can show that x = 1, which is contrary to  $x \notin \{0, 1\}$ .

To sum up,  $\mathcal{C}_{(0,1,e)}$  does not have a codeword of Hamming weight 3. This completes the proof.

**Example 1.** Let p=3 and m=4. Then  $e=2\cdot 3^{m-1}-3^{\frac{m}{2}-1}-1=50$ . Let  $\alpha$  be the generator of  $\mathbb{F}_{3^4}^*$  with  $\alpha^4+2\alpha^3+2=0$ . Then the code  $\mathcal{C}_{(0,1,e)}$  has parameters [80,73,4] and generator polynomial  $x^7+2x^6+x^5+x^3+2x+2$ .

Similar as the proof of Theorem 1, we can obtain another class of optimal ternary cyclic codes with parameters  $[3^m - 1, 3^m - \frac{3m}{2} - 2, 4]$ .

Corollary 1. Let m be an even integer and  $e = \frac{3^m - 1}{2} - 3^{\frac{m}{2}} - 1$ . Let  $s = \frac{3^m - 1}{2}$ . Then  $C_{(1,e,s)}$  is an optimal ternary cyclic code with parameters  $[3^m - 1, 3^m - \frac{3m}{2} - 2, 4]$ .

**Example 2.** Let p = 3 and m = 6. Then  $e = \frac{3^m - 1}{2} - 3^{\frac{m}{2}} - 1 = 336$ . Let  $\alpha$  be the generator of  $\mathbb{F}_{3^6}^*$  with  $\alpha^6 + 2\alpha^4 + \alpha^2 + 2\alpha + 2 = 0$ . Then the code  $\mathcal{C}_{(1,e,s)}$  has parameters [728, 718, 4] and generator polynomial  $x^{10} + 2x^9 + 2x^6 + 2x^5 + 2x^4 + 2x^3 + 2x^2 + 2x + 1$ .

Remark 1. It can be shown that the cyclic code  $C_{(0,1,e)}$  constructed in Theorem 1 are not covered by the known ones in Table 3. In fact, suppose that  $e = 2 \cdot 3^{m-1} - 3^{\frac{m}{2}-1} - 1$  and  $3^{\frac{m}{2}} + 1$  are in the same cyclotomic coset. Note that  $3e \equiv 2 \cdot 3^m - 3^{\frac{m}{2}} - 3 \equiv -3^{\frac{m}{2}} - 1 \pmod{3^m - 1}$ . Thus, there exists an integer  $1 \le i \le m - 1$  such that  $3e \equiv -3^{\frac{m}{2}} - 1 \equiv (3^{\frac{m}{2}} + 1) \cdot 3^i \pmod{3^m - 1}$ , i.e.,  $(3^{\frac{m}{2}} + 1)(3^i + 1) \equiv 0 \pmod{3^m - 1}$ , which implies

 $(3^{\frac{m}{2}}-1)|(3^{i}+1)$ . It is known that

$$\gcd(3^{\frac{m}{2}} - 1, 3^{i} + 1) = \begin{cases} 2, & \text{if } \frac{m/2}{\gcd(i, m/2)} \text{ is odd} \\ 3^{\gcd(i, m/2)} + 1, & \text{if } \frac{m/2}{\gcd(i, m/2)} \text{ is even.} \end{cases}$$

If  $m \equiv 2 \pmod{4}$ , then  $\gcd(3^{\frac{m}{2}} - 1, 3^i + 1) = 2$ , i.e.,  $(3^{\frac{m}{2}} - 1) \nmid (3^i + 1)$ . If  $m \equiv 0 \pmod{4}$ , then we have  $\gcd(3^{\frac{m}{2}} - 1, 3^i + 1) \leq 3^{\frac{m}{4}} + 1$ , one get  $(3^{\frac{m}{2}} - 1) \nmid (3^i + 1)$  again. As a result, the cyclic code  $\mathcal{C}_{(0,1,e)}$  given in Theorem 1 is inequivalent to the known ones. Similarly, we can show that the optimal code  $\mathcal{C}_{(1,e,s)}$  presented in Corollary 1 is not covered by the known ones in Table 3.

# 3.2 The second two classes of optimal ternary cyclic codes with parameters $[3^m - 1, 3^m - \frac{3m}{2} - 2, 4]$

**Theorem 2.** Let  $m \ge 4$  be an integer with  $m \equiv 0 \pmod{4}$ . Let  $e = \frac{3^m - 1}{2} + 3^{\frac{m}{2}} + 1$ . Then  $\mathcal{C}_{(0,1,e)}$  is an optimal ternary cyclic code with parameters  $[3^m - 1, 3^m - \frac{3m}{2} - 2, 4]$ .

Proof: By Lemma 2, we know that  $|C_{3^{\frac{m}{2}}+1}| = \frac{m}{2}$ . Since  $(3^m - 1)|e(3^l - 1)$  if and only if  $(3^m - 1)|(e + \frac{3^m - 1}{2})(3^l - 1)$ , we have  $|C_e| = |C_{3^{\frac{m}{2}}+1}| = \frac{m}{2}$ . It is clearly that  $C_{(0,1,e)}$  does not have a codeword of Hamming weight 1. In the following we show that  $C_{(0,1,e)}$  does not have a codeword of Hamming weight 2 or 3.

Suppose that  $C_{(0,1,e)}$  has a codeword of Hamming weight 2. Then there exist two elements  $c_1, c_2 \in \mathbb{F}_3^*$  and two distinct elements  $x_1, x_2 \in \mathbb{F}_{p^m}^*$  such that (1) is satisfied. By (1),  $c_1 = -c_2$  and  $x_1 = x_2$ , which is contrary to  $x_1 \neq x_2$ . Thus  $C_{(0,1,e)}$  does not have a codeword of Hamming weight 2.

 $C_{(0,1,e)}$  has a codeword of Hamming weight 3 if and only if (2) has no pairwise different nonzero solutions  $x_1, x_2, x_3 \in \mathbb{F}_{3^m}^*$ . Similar as the proof of Theorem 1, we only need to show that  $(x+1)^e + x^e + 1 = 0$  has no solution in  $\mathbb{F}_{3^m} \setminus \mathbb{F}_3$ . Let  $s = \frac{3^m - 1}{2}$  and  $k = \frac{m}{2}$ . Then  $x^s = \pm 1$  and  $(x+1)^s = \pm 1$ . We consider the following four cases according to the values of  $x^s$  and  $(x+1)^s$ .

Case 1),  $(x^s, (x+1)^s) = (1,1)$ . In this case,  $(x+1)^e + x^e + 1 = (x+1)^{3^k+1} + x^{3^k+1} + 1 = 0$ , i.e.,  $x^{3^k+1} - x^{3^k} - x + 1 = (x-1)^{3^k+1} = 0$ . Thus, x = 1.

Case 2),  $(x^s, (x+1)^s) = (1, -1)$ . In this case,  $(x+1)^e + x^e + 1 = -(x+1)^{3^k+1} + x^{3^k+1} + 1 = 0$ , i.e.,  $x^{3^k} + x = x(x^{3^k-1} + 1) = 0$ . Since  $x \neq 0$ , then  $x^{3^k-1} = -1$ . Therefore,

$$x^{s} = (x^{3^{k}-1})^{\frac{3^{k}+1}{2}} = -1$$

since  $\frac{3^k+1}{2}$  is odd due to 2|k. This is contrary to  $x^s=1$ .

Case 3),  $(x^s, (x+1)^s) = (-1, -1)$ . In this case,  $(x+1)^e + x^e + 1 = -(x+1)^{3^k+1} - x^{3^k+1} + 1 = 0$ , i.e.,  $-x^{3^k} + x^{3^k-1} + 1 = 0$ . Thus,  $(\frac{1}{x})^{3^k} + \frac{1}{x} - 1 = 0$ , i.e.,

$$\left(\frac{1}{x}+1\right)^{3^k} + \frac{1}{x}+1 = 0.$$

Since  $x \neq -1$  due to  $(x+1)^s = -1$ , we have  $(\frac{1}{x}+1)^{3^k-1} = -1$ . Therefore,

$$\left(\frac{x+1}{x}\right)^s = \left(\frac{1}{x}+1\right)^s = \left(\left(\frac{1}{x}+1\right)^{3^k-1}\right)^{\frac{3^k+1}{2}} = -1$$

due to  $\frac{3^k+1}{2}$  is odd. This is contrary to  $(\frac{x+1}{x})^s = \frac{(x+1)^s}{x^s} = \frac{-1}{-1} = 1$ .

Case 4),  $(x^s, (x+1)^s) = (-1, 1)$ . In this case,  $(x+1)^e + x^e + 1 = (x+1)^{3^k+1} - x^{3^k+1} + 1 = 0$ , i.e.,  $x^{3^k} + x - 1 = 0$ . Thus,  $(x+1)^{3^k} + x + 1 = 0$ . Since  $x \neq -1$  due to  $(x+1)^s = 1$ , we have  $(x+1)^{3^k-1} = -1$ . Therefore,

$$(x+1)^s = ((x+1)^{3^k-1})^{\frac{3^k+1}{2}} = -1$$

due to  $\frac{3^k+1}{2}$  is odd. This is contrary to  $(x+1)^s=1$ .

As a consequence,  $\mathcal{C}_{(0,1,e)}$  does not have a codeword of Hamming weight 3. This completes the proof.  $\square$ 

**Example 3.** Let p = 3 and m = 8. Then  $e = \frac{3^m - 1}{2} + 3^{\frac{m}{2}} + 1 = 3362$ . Let  $\alpha$  be the generator of  $\mathbb{F}_{38}^*$  with  $\alpha^8 - \alpha^5 + \alpha^4 - \alpha^2 - \alpha - 1 = 0$ . Then the code  $\mathcal{C}_{(0,1,e)}$  has parameters [6560, 6547, 4] and generator polynomial  $x^{13} + 2x^{11} + 2x^{10} + 2x^8 + x^7 + x^5 + 2x^4 + 2x^3 + 2$ .

Similar as the proof of Theorem 2, we can obtain another class of optimal ternary cyclic codes with parameters  $[3^m - 1, 3^m - \frac{3m}{2} - 2, 4]$ .

Corollary 2. Let  $m \equiv 0 \pmod{4}$  and  $e = 3^{\frac{m}{2}} + 1$ . Let  $s = \frac{3^m - 1}{2}$ . Then  $C_{(1,e,s)}$  is an optimal ternary cyclic code with parameters  $[3^m - 1, 3^m - \frac{3m}{2} - 2, 4]$ .

**Example 4.** Let p = 3 and m = 8. Then  $e = 3^{\frac{m}{2}} + 1 = 82$ . Let  $\alpha$  be the generator of  $\mathbb{F}_{38}^*$  with  $\alpha^8 - \alpha^5 + \alpha^4 - \alpha^2 - \alpha - 1 = 0$ . Then the code  $\mathcal{C}_{(1,e,s)}$  has parameters [6560,6547,4] and generator polynomial  $x^{13} + 2x^{11} + 2x^{10} + x^7 + 2x^3 + 2x^2 + 2x + 1$ .

**Remark 2.** It can be shown that  $e = \frac{3^m - 1}{2} + 3^{\frac{m}{2}} + 1$  and  $3^{\frac{m}{2}} + 1$  are not in the same cyclotomic coset. Actually, if there exists an integer  $1 \le i \le m - 1$  such that  $3^i(3^{\frac{m}{2}} + 1) \equiv \frac{3^m - 1}{2} + 3^{\frac{m}{2}} + 1 \pmod{3^m - 1}$ , then  $\frac{3^{m/2} - 1}{2}|(3^i - 1)$ . If  $i \ne \frac{m}{2}$ , then

$$\gcd(\frac{3^{\frac{m}{2}}-1}{2},3^i-1) \leq \gcd(3^{\frac{m}{2}}-1,3^i-1) = 3^{\gcd(m/2,i)}-1 \leq 3^{\frac{m}{4}}-1 < \frac{3^{\frac{m}{2}}-1}{2}$$

due to  $m \ge 4$ . Therefore,  $\frac{3^{m/2}-1}{2}|(3^i-1)$  implies  $i=\frac{m}{2}$ . However,  $3^{\frac{m}{2}}(3^{\frac{m}{2}}+1) \not\equiv \frac{3^m-1}{2}+3^{\frac{m}{2}}+1 \pmod{3^m-1}$ . As a result, the cyclic code  $\mathcal{C}_{(0,1,e)}$  given in Theorem 2 is inequivalent to the known ones. Similarly, we can show that the optimal code  $\mathcal{C}_{(1,e,s)}$  presented in Corollary 2 is not covered by the known ones in Table 3.

# 4 Optimal ternary cyclic codes with parameters $[3^m-1, 3^m-2m-1, 4]$

In this section, we will present a class of optimal ternary cyclic codes  $C_{(2,e)}$  and three classes of optimal ternary cyclic codes  $C_{(1,e)}$ . All of them has parameters  $[3^m - 1, 3^m - 2m - 1, 4]$ .

### 4.1 A class of optimal ternary cyclic codes $\mathcal{C}_{(2,e)}$

In this subsection, we will consider the exponents  $e = 3^{\frac{m}{2}} + 2$ . We will show that  $C_{(2,e)}$  is an optimal ternary cyclic codes with parameters  $[3^m - 1, 3^m - 2m - 1, 4]$  if  $m \equiv 2 \pmod{4}$ .

**Theorem 3.** Let  $e = 3^h + 2$ , where  $h = \frac{m}{2}$  and  $m \equiv 2 \pmod{4}$ . Then  $\mathcal{C}_{(2,e)}$  is an optimal cyclic code with parameters  $[3^m - 1, 3^m - 2m - 1, 4]$ .

Proof: Since e is odd, we have  $e \notin C_2$ . Note that  $\gcd(e(3^{\frac{m}{2}}-2), 3^m-1) = \gcd(3^m-4, 3^m-1) = 1$ , which implies  $\gcd(e, 3^m-1) = 1$ . According to Lemma 1, we have  $|C_e| = m$ . By Lemma 6, we need to show that conditions 2) and 3) are met.

We first consider the condition 2) in Lemma 6, i.e., we show that the equation  $(x^2 + 1)^e - (x^e + 1)^2 = 0$  has no solutions in  $\mathbb{F}_{3^m} \setminus \mathbb{F}_3$ . Note that

$$(x^{2}+1)^{e} - (x^{e}+1)^{2} = x^{2(3^{h}+1)} - x^{2\cdot 3^{h}} - x^{3^{h}+2} - x^{4} + x^{2} = 0.$$

The above equation can be factorized as

$$x^{2}(x^{3^{h}} - 1 - (x^{3^{h} - 1} + x))(x^{3^{h}} - 1 + x^{3^{h} - 1} + x) = 0.$$

Thus,  $x^{3^h} - 1 - (x^{3^h-1} + x) = 0$  or  $x^{3^h} - 1 + x^{3^h-1} + x = 0$ . We will show that both  $x^{3^h} - 1 - (x^{3^h-1} + x) = 0$  and  $x^{3^h} - 1 + x^{3^h-1} + x = 0$  has no solution in  $\mathbb{F}_{3^m} \setminus \mathbb{F}_3$ .

Case 1),  $x^{3^h} - 1 - (x^{3^h-1} + x) = 0$ . Note that x = 0 and x = 1 are not solutions of  $x^{3^h} - 1 - (x^{3^h-1} + x) = 0$ . Thus, by  $x^{3^h} - 1 - (x^{3^h-1} + x) = 0$ , we have  $x^{3^h} (1 - \frac{1}{x}) = x + 1$ , i.e.,

$$x^{3^h} = \frac{x^2 + x}{x - 1}. ag{5}$$

Raising both sides of (5) to the power of  $3^h$  gives

$$x^{3^{2h}} = \frac{(x^{2\cdot 3^h}) + x^{3^h}}{x^{3^h} - 1}. (6)$$

Note that  $x^{3^{2h}} = x^{3^m} = x$ . By equation (5), we have

$$x = \frac{x^4 + x^2 + 2x}{x^3 + 2x^2 + x + 2}. (7)$$

If  $x^3 + 2x^2 + x + 2 = (x+2)(x^2+1) = 0$ , then x = 1 or  $x^2 = -1$ . Since  $x \neq 1$ , we have  $x^2 = -1$ . By (5),  $x^{3^h} = 1$ , which implies x = 1. This is contrary to  $x^2 = -1$ . As a result,  $x^3 + 2x^2 + x + 2 \neq 0$ . Then by (7), one obtains  $2x^3 = 0$ , which implies x = 0.

Case 2),  $x^{3^h} - 1 + x^{3^h - 1} + x = 0$ . Note that x = 0 and x = -1 are not solutions of  $x^{3^h} - 1 + x^{3^h - 1} + x = 0$ . Thus, by  $x^{3^h} - 1 + x^{3^h - 1} + x = 0$ , we have  $x^{3^h} (1 + \frac{1}{x}) = 1 - x$ , i.e.,

$$x^{3^h} = \frac{-x^2 + x}{x+1}. (8)$$

Raising both sides of (8) to the power of  $3^h$  will lead to  $x^{3^{2h}} = x = \frac{-(x^{2 \cdot 3^h}) + x^{3^h}}{x^{3^h} + 1}$ . By (8), we have

$$x = \frac{-x^4 + x^3 - x^2 + x}{-x^3 + x^2 + 1}. (9)$$

Similar as in Case 1), one can show that  $-x^3 + x^2 + 1 \neq 0$ . Thus, according to (9), we have  $x^2 = 0$ , which implies x = 0.

Now we consider the condition 3) in Lemma 6, i.e., the equation  $(x^2+1)^e + (x^e+1)^2 = 0$  has no solution in  $\mathbb{F}_{3^m} \setminus \mathbb{F}_3$ . Note that

$$(x^2+1)^e + (x^e+1)^2 = -(x^{2\cdot(3^h+2)} + x^{2\cdot(3^h+1)} - x^{2\cdot3^h} + x^{3^h+2} - x^4 + x^2 + 1)$$

and

$$x^{2\cdot(3^{h}+2)} + x^{2\cdot(3^{h}+1)} - x^{2\cdot3^{h}} + x^{3^{h}+2} - x^{4} + x^{2} + 1$$

$$= x^{2\cdot3^{h}}(x^{4} + x^{2} - 1) + x^{3^{h}+2} - (x^{4} + x^{2} - 1) - x^{2}$$

$$= (x^{2} - 1)^{3^{h}}(x^{4} + x^{2} - 1) + x^{2}(x^{3^{h}} - 1)$$

$$= (x^{3^{h}} - 1)((x^{3^{h}} + 1)(x^{4} + x^{2} - 1) + x^{2}) = 0.$$
(10)

Therefore, we only need to show that

$$(x^{3^h} + 1)(x^4 + x^2 - 1) + x^2 = 0 (11)$$

has no solution in  $\mathbb{F}_{3^m} \setminus \mathbb{F}_3$ .

If  $x^4 + x^2 - 1 = 0$ , then from (11), we have  $x^2 = 0$ , i.e., x = 0. Thus,  $x^4 + x^2 - 1 \neq 0$ . Therefore, from (11), we have

$$x^{3^h} = \frac{-x^4 + x^2 + 1}{x^4 + x^2 - 1} \tag{12}$$

Thus,

$$x^{2\cdot 3^h} = \frac{x^8 + x^6 - x^4 - x^2 + 1}{x^8 - x^6 - x^4 + x^2 + 1} = \frac{y_1}{y_2},\tag{13}$$

where  $y_1 = x^8 + x^6 - x^4 - x^2 + 1$  and  $y_2 = x^8 - x^6 - x^4 + x^2 + 1$ . Similarly, one can obtain

$$x^{4\cdot3^h} = \frac{y_1^2}{y_2^2}. (14)$$

Raising both sides of (12) to the power of  $3^h$  gives  $x^{3^{2h}} = \frac{-x^{4 \cdot 3^h} + x^{2 \cdot 3^h} + 1}{x^{4 \cdot 3^h} + x^{2 \cdot 3^h} - 1}$ . Together with  $x^{3^{2h}} = x$ , we have

$$x = \frac{-x^{4\cdot3^h} + x^{2\cdot3^h} + 1}{x^{4\cdot3^h} + x^{2\cdot3^h} - 1}.$$

By (13) and (14), we have

$$x\left(\left(\frac{y_1}{y_2}\right)^2 + \frac{y_1}{y_2} - 1\right) = -\left(\frac{y_1}{y_2}\right)^2 + \frac{y_1}{y_2} + 1,\tag{15}$$

i.e., ,

$$x(y_1^2 + y_1y_2 - y_2^2) + y_1^2 - y_1y_2 - y_2^2 = 0. (16)$$

By  $y_1 = x^8 + x^6 - x^4 - x^2 + 1$  and  $y_2 = x^8 - x^6 - x^4 + x^2 + 1$ , (16) becomes

$$f(x) \triangleq x^{17} - x^{16} + x^{15} + x^{14} + x^{11} + x^{10} - x^9 + x^8 - x^7 - x^6 - x^3 - x^2 + x - 1 = 0.$$

Thanks to the Magma computation, the canonical factorization of f(x) over  $\mathbb{F}_3$  is given by

$$f(x) = (x-1)^5(x^4 + x - 1)(x^4 - x^3 - 1)(x^4 - x^3 + x^2 - x + 1).$$

Then by Lemma 8, f(x) = 0 has no solutions in  $\mathbb{F}_{3^m} \setminus \mathbb{F}_3$  if and only if  $m \not\equiv 0 \pmod{4}$ .

Therefore,  $C_{(2,e)}$  does not have a codeword of Hamming weight 3. This completes the proof.

**Example 5.** Let p=3 and m=6. Then  $e=3^{\frac{m}{2}}+2=29$ . Let  $\alpha$  be the generator of  $\mathbb{F}_{36}^*$  with  $\alpha^6+2\alpha^4+\alpha^2+2\alpha+2=0$ . Then the code  $\mathcal{C}_{(2,e)}$  has parameters [728,716,4] and generator polynomial  $x^{12}-x^{11}+x^{10}-x^{10}$ 

$$x^6 - x^3 - 1$$
.

Remark 3. Note that  $gcd(3^{\frac{m}{2}} + 2, 3^m - 1) = 1$ . Thus, the code  $C_{(2,e)}$  given in Theorem 3 is equivalent to  $C_{(1,2e^{-1})} = C_{(1,4\cdot3^{\frac{m}{2}}-2)}$ . Let m=6, we have  $4\cdot3^{\frac{m}{2}}-2=106$ . Magma experiments confirm that 106 and all the exponents given in Table 1 are not in the same coset. Thus, the optimal ternary cyclic codes  $C_{(1,2e^{-1})}$  given in Theorem 3 are not covered by the known ones in Table 1. Take m=6, then  $e=3^3+2=29$ . Magma experiments confirm that 29 and all the exponents given in type 6 in Table 2 are not in the same coset. Thus, the optimal ternary cyclic codes  $C_{(2,e)}$  given in Theorem 3 are not covered by the known ones in Table 2. As a result, the optimal ternary cyclic code  $C_{(2,e)}$  given in Theorem 3 is inequivalent to the known ones in Tables 1 and 2.

### 4.2 Three classes of optimal ternary cyclic codes $\mathcal{C}_{(1,e)}$

**Theorem 4.** Let m be odd and e be an even integer satisfying  $e(3^h - 1) \equiv \frac{3^m + 1}{2} \pmod{3^m - 1}$ , where  $1 \leq h \leq m - 1$ . Then  $\mathcal{C}_{(1,e)}$  has parameters  $[3^m - 1, 3^m - 2m - 1, 4]$  and is optimal if  $\gcd(3^m - 1, 3^h - 2) = 1$ .

Proof. Since m is odd, we have  $\gcd(\frac{3^m+1}{2}, 3^m-1)=2$  and consequently  $\gcd(e(3^h-1), 3^m-1)=2$ , which implies  $\gcd(e, 3^m-1)=2$  and  $\gcd(3^h-1, 3^m-1)=2$ . Hence  $\gcd(h, m)=1$ ,  $e \notin C_1$ , and  $|C_e|=m$ . According to Lemma 5,  $C_{(1,e)}$  has parameters  $[3^m-1, 3^m-2m-1, 4]$  if the equation  $(1+x)^e=\pm(x^e+1)$  has no solution in  $\mathbb{F}_{3^m}\setminus\mathbb{F}_3$ . Raising both sides of the above equation to the  $(3^h-1)$ -th power will lead to  $(1+x)^{e(3^h-1)}=(x^e+1)^{3^h-1}$ , i.e.,

$$(1+x)^{s}(1+x)(x^{e}+1) = x^{e\cdot 3^{h}} + 1 = x^{e(3^{h}-1)} \cdot x^{e} + 1 = x^{s} \cdot x^{1+e} + 1$$
(17)

due to the fact that  $e(3^h - 1) \equiv 1 + s \pmod{3^m - 1}$ , where  $s = \frac{3^m - 1}{2}$ . We will discuss the solutions of equation (17) in the following four cases.

Case 1),  $(1+x)^s = x^s = 1$ . In this case, (17) becomes  $(x^e+1)(x+1) = x^{e+1}+1$ , i.e.,  $x^e+x = x(x^{e-1}+1) = 0$  and consequently  $x^{e-1} = -1$  due to  $x \neq 0$ . Then we have  $x^{2(e-1)} = 1$ . Note that

$$\gcd(\frac{3^{m}-1}{2}, (e-1)(3^{h}-1)) = \gcd(\frac{3^{m}-1}{2}, e(3^{h}-1) - (3^{h}-1))$$

$$= \gcd(\frac{3^{m}-1}{2}, \frac{3^{m}+1}{2} - 3^{h} + 1) = \gcd(\frac{3^{m}-1}{2}, 3^{h} - 2) = 1.$$
(18)

Hence  $gcd(3^m - 1, e - 1) = 1$  and  $gcd(3^m - 1, 2(e - 1)) = 2$ , which implies  $x^2 = 1$ , i.e.,  $x = \pm 1$ .

Case 2),  $(x+1)^s = -1$ ,  $x^s = 1$ . In this case,  $-(x+1)(x^e+1) = x^{e+1} + 1$ , i.e.,  $x^{e+1} - x^e - x + 1 = (x^e-1)(x-1) = 0$ , which implies x = 1 or  $x^e = 1$ . Since  $\gcd(e, 3^m - 1) = 2$ , we have  $x^2 = 1$ , i.e.,  $x = \pm 1$ .

Case 3),  $(x+1)^s = -1$ ,  $x^s = -1$ . In this case, (17) becomes  $-(x+1)(x^e+1) = -x^{e+1}+1$ , i.e.,  $x^{e+1}+x^e+x+1-x^{e+1}+1=0$  and

$$x^e + x + 2 = 0 (19)$$

Raising the both sides of (19) to the  $3^h$ -th power will lead to

$$x^{e \cdot 3^h} + x^{3^h} + 2 = 0 (20)$$

By (19) and (20), we have  $x^{e \cdot 3^h} - x^e + x^{3^h} - x = 0$ . Note that

$$x^{e \cdot 3^h} = x^e \cdot x^{e(3^h - 1)} = x^{e + 1 + s} = -x^{e + 1}$$

Thus, we have

$$x^{e \cdot 3^h} - x^e + x^{3^h} - x = -x^{e+1} - x^e + x^{3^h} - x = -x^e(x+1) + x^{3^h} - x = (x-1)(x+1) + x^{3^h} - x = 0$$

due to  $x^e = 1 - x$ . Therefore,  $x^{3^h} = x - x^2 + 1 = -(x^2 - x - 1) = -(x^2 + 2x + 1) - 1$ , i.e.,  $(x + 1)^{3^h - 2} = -1$ . Then we have  $(x+1)^{2(3^h-2)}=1$ . Together with  $(x+1)^{3^m-1}=1$ , we can deduce that  $(x+1)^2=1$  due to  $gcd(3^h - 2, 3^m - 1) = 1$ . Hence x = 0 or 1.

Case 4),  $(x+1)^s = 1$ ,  $x^s = -1$ . In this case, (17) becomes  $(x+1)(x^e+1) = -x^{e+1}+1$ , i.e.,  $2x^{e+1}+x^e+x=1$ 0. Let  $y = \frac{1}{x}$ , we have

$$\frac{2}{y^{e+1}} + \frac{1}{y^e} + \frac{1}{y} = 0,$$

which is  $y^e + y + 2 = 0$ . According to Case 3),  $y \in \mathbb{F}_3$ . Hence,  $x \in \mathbb{F}_3$ .

As a result, (17) has no solution in  $\mathbb{F}_{3^m} \setminus \mathbb{F}_3$ . This completes the proof.

**Remark 4.** For any  $1 \le h \le m-1$ , there exists exactly one e satisfying the conditions of Theorem 4. We list some examples as follows:

- 1) For h = 1,  $e = \frac{3^m 1}{2} + \frac{3^m + 1}{4}$ ;
- 2) For h=2,  $e=\frac{3^m-1}{2}+\frac{3^{m+1}-1}{8}\cdot\frac{3^m+1}{4}$  when  $m\equiv 1\pmod 4$ , and  $e=\frac{3^{m+1}-1}{8}\cdot\frac{3^m+1}{4}$  with  $m\equiv 3\pmod 4$ ; 3) For h=3,  $e=\frac{3^{m+1}-1}{26}\cdot\frac{3^m+1}{4}$  when  $m\equiv 5\pmod 6$ ,  $e=\frac{3^m-1}{2}+\frac{3^{m+2}-1}{26}\cdot\frac{3^m+1}{4}\cdot\frac{3^{m+1}-1}{8}$  when  $m\equiv 1\pmod 12$ , and  $e=\frac{3^{m+2}-1}{26}\cdot\frac{3^m+1}{4}\cdot\frac{3^{m+1}-1}{8}$  when  $m\equiv 1\pmod 12$ .

**Theorem 5.** Let m be a positive integer with gcd(m,6) = 1. Let  $e = \frac{3^{\frac{m+3}{2}} + 5}{2}$ , where  $m \equiv 3 \pmod{4}$ . Then the ternary code  $C_{(1,e)}$  has parameters  $[3^m-1,3^m-1-2m,4]$  and is optimal if  $m \not\equiv 0 \pmod{13}$ .

*Proof.* Note that  $h = \frac{m+3}{2}$  is odd due to  $m \equiv 3 \pmod{4}$ , then by Lemma 3,  $e \notin C_1$  and  $|C_e| = m$ . Since h is odd, we have  $3^h \equiv -1 \pmod{4}$ . Thus  $3^h + 5 \equiv 0 \pmod{4}$ . This shows that  $e = \frac{3^h + 5}{2}$  is even. Thus, Condition 1) in Lemma 5 is satisfied. Conditions 2) and 3) in Lemma 5 are met if and only if the following equation has no solution in  $\mathbb{F}_{3^m} \setminus \mathbb{F}_3$ :

$$(x+1)^{2e} - x^{2e} + x^e - 1 = 0. (21)$$

Note that  $(x+1)^{2e} = (x+1)^{3^h+5} = (x^{3^h}+1)(x^3+1)(x+1)^2$ . Thus, (21) can be rewritten as

$$-x^{3^{h}+3} + x^{3^{h}+2} + x^{3^{h}+1} - x^{3^{h}} + x^{3^{h}-1} + x^{4} - x^{3} + x^{2} + x - 1 + x^{\frac{3^{h}+3}{2}} = 0,$$

i.e., ,

$$(x^{4} - x^{3} + x^{2} + x - 1)(1 - x^{3^{h} - 1}) - x^{\frac{3^{h} + 3}{2}}(x^{\frac{3^{h} - 1}{2}} - 1) = 0.$$
(22)

Suppose  $x^{\frac{3^h-1}{2}} = 1$ . From  $x^{3^m-1} = 1$ , we have x = 1 since  $\gcd(3^m - 1, \frac{3^h-1}{2}) = 1$  due to  $\gcd(h, m) = 1$  and h is odd. Therefore,  $x^{\frac{3^h-1}{2}} \neq 1$ . Thus, (22) turns to

$$(x^4 - x^3 + x^2 + x - 1)(1 + x^{\frac{3^h - 1}{2}}) + x^{\frac{3^h + 3}{2}} = 0,$$

i.e., ,

$$x^{\frac{3^{h}-1}{2}}(x^{4}-x^{3}-x^{2}+x-1) = -(x^{4}-x^{3}+x^{2}+x-1).$$

If  $x^4 - x^3 - x^2 + x - 1 = 0$ , then  $x^4 - x^3 + x^2 + x - 1 = 0$ , which implies  $x^2 = 0$ . Therefore,  $x^4 - x^3 - x^2 + x - 1 \neq 0$ . As a result,

$$x^{\frac{3^{h}-1}{2}} = -\frac{x^4 - x^3 + x^2 + x - 1}{x^4 - x^3 - x^2 + x - 1}$$

Hence we have

$$x^{3^h} = \left(\frac{x^4 - x^3 + x^2 + x - 1}{x^4 - x^3 - x^2 + x - 1}\right)^2 \cdot x = \frac{x^9 + x^8 + x^4 - x^3 + x^2 + x}{x^8 + x^7 - x^6 + x^5 + x + 1} \stackrel{\triangle}{=} \frac{f(x)}{g(x)},\tag{23}$$

where  $f(x) = x^9 + x^8 + x^4 - x^3 + x^2 + x$  and  $g(x) = x^8 + x^7 - x^6 + x^5 + x + 1$ .

Raising both sides of (23) to the  $3^h$ -th power, we have

$$x^{3^{2h}} = \frac{f(x^{3^{h}})}{g(x^{3^{h}})} = \frac{x^{9 \cdot 3^{h}} + x^{8 \cdot 3^{h}} + x^{4 \cdot 3^{h}} - x^{3 \cdot 3^{h}} + x^{2 \cdot 3^{h}} + x^{3^{h}}}{x^{8 \cdot 3^{h}} + x^{7 \cdot 3^{h}} - x^{6 \cdot 3^{h}} + x^{5 \cdot 3^{h}} + x^{3^{h}} + 1}$$

$$= \frac{f^{9}(x) + f^{8}(x)g(x) + f^{4}(x)g^{5}(x) - f^{3}(x)g^{6}(x) + f^{2}(x)g^{7}(x) + f(x)g^{8}(x)}{f^{8}(x)g(x) + f^{7}(x)g^{2}(x) - f^{6}(x)g^{3}(x) + f^{5}(x)g^{4}(x) + f(x)g^{8}(x) + g^{9}(x)}$$

$$\triangleq \frac{h_{1}(x)}{h_{2}(x)}.$$
(24)

Note that  $x^{3^{2h}} = x^{3^{m+3}} = x^{27}$ . By (24),  $x^{27} = \frac{h_1(x)}{h_2(x)}$ , i.e.,  $h_2(x) \cdot x^{27} - h_1(x) = 0$ . Let  $h(x) = h_2(x) \cdot x^{27} - h_1(x)$ . With Magma program, h(x) can be factorized into the product of irreducible factors over  $\mathbb{F}_3$  as

$$h(x) = x(x+1)(x-1)^9(x^9 - x^7 - x^5 + x^4 + x^3 + x^2 - 1)(x^9 - x^7 - x^6 - x^5 + x^4 + x^2 - 1) \cdot k(x),$$

where k(x) is the product of six irreducible polynomials of degree 13 over  $\mathbb{F}_3$ . Therefore, (21) has no solution in  $\mathbb{F}_{3^m} \setminus \mathbb{F}_3$  if  $m \not\equiv 0 \pmod{13}$ . This completes the proof.

Similar as the proof of Theorem 5, we have the following theorem.

**Theorem 6.** Let m be a positive integer with  $m \equiv 1 \pmod{6}$ . Let  $e = \frac{3^{\frac{m+2}{3}} + 5}{2}$ . Then the ternary code  $C_{(1,e)}$  has parameters  $[3^m - 1, 3^m - 1 - 2m, 4]$  and is optimal.

Remark 5. Theorems 5 and 6 give positive support of the open problem 7.9 in [2].

### 5 Conclusions

In this paper, four classes of optimal ternary cyclic codes  $C_{(0,1,e)}$  and  $C_{(1,e,s)}$  were constructed by analyzing the solutions of certain equations over  $\mathbb{F}_{3^m}$ . Moreover, by analyzing the irreducible factors of certain polynomials and the solutions of certain equations over  $\mathbb{F}_{3^m}$ , we presented four classes of optimal ternary cyclic codes with parameters  $[3^m - 1, 3^m - 2m - 1, 4]$ . It is shown that our new optimal cyclic codes are inequivalent to the known ones.

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