ECONOMETRICS 2

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Exercises: session 4

Stationary processes and forecasts

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Preliminary note: This tutorial session is the last one before the midterm exam. A part of it may be devoted to the questions you have about the course or the previous tutorial sessions.

Reminder:

1. If $Y, Z_1, \dots, Z_q$ are real random variables, the best linear forecast of Y given $Z_1, \dots, Z_q$ is defined by: $BLF(Y|Z_1, \dots, Z_q) = b_0 + b_1 Z_1 + \dots + b_q Z_q$ where:

   i) $\mathrm{E}\left(Y - (b_0 + b_1 Z_1 + \dots + b_q Z_q)\right) = 0$ (the forecast error has expectation 0)

   ii) $\mathrm{Cov}\left(Y - (b_0 + b_1 Z_1 + \dots + b_q Z_q), Z_j\right) = 0$ for $j = 1, \dots, q$
   (all the information in $Z_1, \dots, Z_q$ has been used to compute the forecast).

2. In the same way, and in the framework of this course, if Y is a real random variable, and if $(Z_k)_{k\in\mathbb{N}}$ is a sequence of real random variables, the best linear forecast of Y given $Z_k, k \in \mathbb{N}$ is defined by: $BLF(Y|Z_k, k \in \mathbb{N}) = b_0 + \sum_{k=1}^{+\infty} b_k Z_k$ where:

   i) $\mathrm{E}\Big(Y - (b_0 + \sum_{k=1}^{+\infty} b_k Z_k)\Big) = 0$ (the forecast error has expectation 0)

   ii) $\mathrm{Cov}\Big(Y - (b_0 + \sum_{k=1}^{+\infty} b_k Z_k), Z_j\Big) = 0$ for $j \in \mathbb{N}$
   (all the information in the Z_k's has been used to compute the forecast).

Exercise # 1

Let us consider the AR process (y_t) which has been studied in Exercises session 3, exercise #2, and which is defined by

$$y_t = 0.5 + \frac{5}{6} y_{t-1} - \frac{1}{6} y_{t-2} + v_t$$

where (v_t) is a WN with variance σ_v^2 and has been proved to be the innovation process of (y_t).

1. Compute $y^*_{t+1|t} = BLF(y_{t+1}|\underline{y_t})$ and the variance of the associated forecast error.
2. Compute $y^*_{t+2|t} = BLF(y_{t+2}|\underline{y_t})$ and the variance of the associated forecast error.
3. Compute $y^*_{t+3|t} = BLF(y_{t+3}|\underline{y_t})$ and the variance of the associated forecast error.
4. For $h \geq 1$ (and $t \geq 2$) does $BLF(y_{t+h}|\underline{y_t})$ coincide with $BLF(y_{t+h}|y_t, \dots, y_1)$?

Exercise # 2

Suppose that (X_t) is a stationary process with $\mathrm{E}X_t = m$, and that its autocorrelation function is denoted as $\rho(.)$.

1. i) Show that among the predictors of X_{t+h} which can be written as $aX_t + b$, the best one (*i.e.* the predictor for which the forecast error has expectation 0 and minimum variance) is obtained by choosing $a = \rho(h)$ and $b = m(1 - \rho(h))$.

   ii) Check that this predictor coincides with $BLF(X_{t+h}|X_t)$.

2. i) Using the same method as in 1 (i), compute the best predictor of X_{t+h} among the predictors which can be written as $a_0 X_t + a_1 X_{t-1} + b$.

   ii) Check that this predictor coincides with $BLF(X_{t+h}|X_t, X_{t-1})$.

Remark: It could be proved exactly in the same way that, for any $p > 0$, the best predictor of X_{t+h} among the predictors which can be written as $a_0 X_t + a_1 X_{t-1} + \cdots + a_p X_{t-p} + b$, coincides with $BLF(X_{t+h}|X_t, X_{t-1}, \ldots, X_{t-p})$.

Exercise # 3 : AR(∞) representation of a MA(1), innovation and forecasts.

This exercise proves, using the simple MA(1) example, the same results as those which have been obtained during the course for the general case of MA(q) processes.

Let (x_t) the MA(1) process defined by the following equation:

$$x_t = m + \varepsilon_t - \theta \varepsilon_{t-1}$$

where ε_t is a WN(0,σ^2) and θ a real number such that $|\theta| < 1$.

1. Compute Ex_t.

2. Explain why $(1 - \theta L)$ is invertible. Using this inverse, show that (x_t) has an AR(∞) representation and give this representation.

3. i) Using the definition of x_t, prove that $\forall k > 0,\ Cov(\varepsilon_t, x_{t-k}) = 0$.

   ii) Using (i) and the AR(∞) representation of (x_t), compute $BLF(x_t|\underline{x_{t-1}})$

   iii) Prove that (ε_t) the innovation process of (x_t)

4. i) Give the best linear forecast $x^*_{t+1|t}$ of x_{t+1} at time t as a function of the x_{t-k}'s. What is the forecast error variance ?

   ii) Give the best linear forecast $x^*_{t+2|t}$ of x_{t+2} at time t as a function of the x_{t-k}'s. What is the forecast error variance ?

   iii) In practise, the x_t's are observed only for $t \geq 1$ and the corresponding truncation of $x^*_{t+1|t}$ is taken to get an approximate forecast of x_{t+1} at time t. If this approximation is denoted as $\hat{x}_{t+1|t}$ what is the variance of the associated forecast error $x_{t+1} - \hat{x}_{t+1|t}$ (give its formula, but don't compute it) ? What can you say of this variance when $t \to \infty$ and why ?