Quant Macro I

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Please email any mistakes or typos to `axelle.ferriere@psemail.eu`.

Textbooks. The key references for this course is Ljungqvist and Sargent (2012) (LS).

Acknowledgment. The class is borrowing from class material of Professors Á. Ábrahám, D. Krueger, T. Sargent, and G. Violante.

Contents

Part I

Competitive Equilibria with Heterogeneity

Chapter 1

Exchange Economy with Complete Markets

We study an infinite-horizon pure exchange economy with complete markets. After having described the economic environment, we discuss the planner's problem and various competitive equilibrium concepts. This chapter follows quite closely chapter 8 of Ljungqvist and Sargent (2012). A discussion of similar topics can also be found in chapters 19 and 20 of Mas-Colell et al. (1995).

1.1 Environment

For all $t \geq 0$ let s_t be a random variable taking values in a finite set S, which contains all possible realisations and is taken to be the same each period. An event history at time t is denoted by $s^t = (s_0, \ldots, s_t)$, a $t+1$-dimensional vector which describes the realisations of all events up to period t. Letting (with abuse of notation on the index) S^t denote the $t+1$-fold product of S we have that s^t lives in S^t. For each t, let π_t denote the distribution of histories over S^t so that $\pi_t(s^t) \geq 0$ is the unconditional probability of observing s^t. For $\pi_t(s^t)$ to be a valid probability measure over S^t we require

$$\sum_{s^t \in S^t} \pi_t(s^t) = 1.$$

We assume that the realization of s^t is observed by all agents in the economy at the beginning of period t.[1] As decisions and equilibrium objects now depend on the history, we will index commodities both by time and by event histories s^t. We assume that all agents share the same belief over the histories, which is also the objective probability distribution.

We consider an economy with I agents, indexed by $i \in I \equiv \{1, \ldots, I\}$. There is a single good, which is non-storable. The endowment of agent i of the good in period t,

[1] We typically assume that trading occurs *after* s^0 is realized. We bake this in by assuming the distribution π_0 is degenerate at the desired realization.

history s^t is $y_t^i(s^t)$, therefore y_t^i is a function which maps all possible realizations of the random variable s^t into a nonnegative real number. Each agent chooses a history-dependent consumption stream, $\{c_t^i(s^t)\}_{t=0}^{\infty}$ to maximise expected lifetime utility which is assumed to be time-separable across dates and histories and given by

$$\sum_{t=0}^{\infty} \sum_{s^t \in S^t} \beta^t \pi_t(s^t) u(c_t^i(s^t)).$$

We make the usual assumptions on u (increasing, strictly concave and twice continuously differentiable) as well as the Inada condition $\lim_{c \to 0} u'(c) = +\infty$.

An *allocation* specifies consumption for each agent in each date and history. An allocation is feasible if

$$\sum_{i \in I} c_t^i(s^t) \leq \sum_{i \in I} y_t^i(s^t)$$

for all $t \geq 0$ and all $s^t \in S^t$.

1.2 Pareto Efficient Allocations

We now reformulate the concept of Pareto efficiency in the context of a model with uncertainty. We say an allocation $\{c_t^i(s^t)\}_{i,t,s^t}$ is *Pareto efficient* if it is feasible and there is no other feasible allocation $\{\hat{c}_t^i(s^t)\}_{i,t,s^t}$ such that

$$\sum_{t=0}^{\infty} \sum_{s^t \in S^t} \beta^t \pi_t(s^t) u(\hat{c}_t^i(s^t)) \geq \sum_{t=0}^{\infty} \sum_{s^t \in S^t} \beta^t \pi_t(s^t) u(c_t^i(s^t))$$

for all $i \in I$ and the inequality is strict for some $i \in I$. In words, if an allocation if Pareto efficient no reallocation can make all agents weakly better off and some agent strictly so. Characterizing the set of Pareto efficient allocations from the definition is not easy. We therefore introduce the concept of "planner problem" or, more formally, a problem of maximisation of a social welfare function. Specifically, we focus on *linear* social welfare functions so that the planner problem amounts to maximising a weighted sum of each agent's utility. The planner chooses a *feasible* consumption sequence for each agent to maximise

$$\sum_{i=1}^{I} \lambda_i \sum_{t=0}^{\infty} \sum_{s^t \in S^t} \beta^t \pi_t(s^t) u(c_t^i(s^t))$$

where $\lambda = (\lambda_1, \ldots, \lambda_I)$ with $\lambda_i \geq 0$ for all i is called the vector of *Pareto weights* and captures the weight put by the planner on each agent's utility.

What we need now is a result connecting the Pareto efficient allocations with the set of solutions to the planner problem, which is easy to find. Luckily, it turns out that if we solve the planner problem for $\lambda \gg 0$ the allocation found is Pareto efficient. Conversely, provided that the utility possibility set is convex (which is the case here), for every Pareto efficient allocation there exists a vector λ such that it is the solution to a planner problem with Pareto weights λ.[2] What this result tells us is that we can characterise all Pareto efficient allocations by solving the social planner problem for all λ. We therefore proceed to solve the planner problem in sequential formulation.

As usual, given the assumptions on preferences, the first-order conditions characterise the global maximum. It is convenient to rewrite the multiplier on the feasibility constraint $\mu_t(s^t)$ as $\theta_t(s^t) \equiv \mu_t(s^t)/(\beta^t \pi_t(s^t))$. All we are doing is a rescaling so the problem is completely unchanged – it's as if we were multiplying both sides of the feasibility constraint in (t, s^t) by $\pi_t(s^t)\beta^t$. This will however lighten notation considerably, and is a "trick" that is very often convenient. The Lagrangian of the problem is

$$\mathcal{L} = \sum_{t=0}^{\infty} \sum_{s^t \in S^t} \beta^t \pi_t(s^t) \sum_{i \in I} \Big[\lambda_i u(c_t^i(s^t)) + \theta_t(s^t)\big(y_t^i(s^t) - c_t^i(s^t)\big) \Big]$$

The first-order conditions for consumption, for all $t \geq 0$ and $s^t \in S^t$, are

$$\lambda_i u'(c_t^i(s^t)) = \theta_t(s^t) \quad \text{for all} \quad i \in I,$$

from which we obtain, taking the ratio between two generic agents i and j,

$$\frac{u'(c_t^i(s^t))}{u'(c_t^j(s^t))} = \frac{\lambda_j}{\lambda_i} \quad \text{for all} \quad i, j \in I. \tag{1.1}$$

This shows that, in any Pareto efficient allocation, the ratio of marginal utilities across households is constant across time t and history s^t. In other words, regardless of the incomes of agents i and j in period t and history s^t, the ratio of their marginal utilities is entirely determined by the ratio of the weight the planner puts on their utility. This means that they are not *directly* affected by the realisation of their own endowments in each period. This is why we say that at a Pareto efficient allocation in this economy there is *perfect insurance*. Rewriting consumption of a generic agent i as a function of agent 1 we obtain

$$c_t^i(s^t) = u'^{-1}\Big(u'(c_t^1(s^t)) \frac{\lambda_1}{\lambda_i} \Big) \quad \text{for all} \quad i \in I,$$

which we can substitute in the feasibility constraint to obtain

$$\sum_{i \in I} u'^{-1}\Big(u'(c_t^1(s^t)) \frac{\lambda_1}{\lambda_i} \Big) = \sum_{i \in I} y_t^i(s^t) \tag{1.2}$$

for all $t \geq 0$ and all $s^t \in S^t$. This equation tells us that consumption of agent 1 in period t and history s^t only depends on the aggregate endowment in period t and history s^t. Since through equation (1.1) all ratios marginal utilities are pinned down, this equation tells

[2] A proof of this result is beyond the scope of this course. Chapter 16 (specifically section 16.E) of Mas-Colell et al. (1995) has a discussion and proof.

us that the whole Pareto efficient allocation only depends on time and history through the total endowment in that period/history. Because of this we say that in the Pareto efficient allocations of this economy there is a form of *history independence*. To reiterate, all that affects individual utilities in each period/history is the *aggregate endowment* so, for example, without aggregate risk (aggregate endowment is constant in time/histories), individual households face no risk at all.

We now move on to study which allocations arise in this economy when they are determined by a competitive equilibrium. As usual in these settings, we consider two trading schemes and define the appropriate competitive equilibrium concepts in each case. Eventually we will compare the allocation arising in a competitive equilibrium with the Pareto efficient allocations.

1.3 Arrow-Debreu Equilibrium

The first market structure we consider is as follows. At date 0, before any uncertainty is realized, it is possible to trade claims to time t consumption, contingent on history s^t having realized. What is therefore traded is a *commitment* to deliver, or receive, a certain amount of consumption at a given that if a certain event history realizes. At time 0 the price of a claim for one unit of the good at time t and history s^t is $q_t^0(s^t)$. For each $t \geq 0$ we therefore have a function $q_t^0 : S^t \to \mathbb{R}_+$ which maps each possible history at time t into a nonnegative real number, a price. We call a *price system* a sequence of functions, $\{q_t^0(\cdot)\}_{t=0}^{\infty}$.

Household. Since trading occurs at a single date, each agent $i \in I$ is faced with a single budget constraint given by

$$\sum_{t=0}^{\infty} \sum_{s^t \in S^t} q_t^0(s^t) c_t^i(s^t) \leq \sum_{t=0}^{\infty} \sum_{s^t \in S^t} q_t^0(s^t) y_t^i(s^t). \tag{1.3}$$

At time 0, the agent chooses a sequence of contingent consumption decisions maximising expected lifetime utility,

$$\sum_{t=0}^{\infty} \sum_{s^t \in S^t} \beta^t \pi_t(s^t) u(c_t^i(s^t)),$$

taking prices as given, subject to the budget constraint (1.3).

Equilibrium. We define as an *Arrow-Debreu competitive equilibrium* an allocation and a price system such that, i) given the price system, the allocation solves each household's problem and ii) markets clear date by date, history by history, that is,

$$\sum_{i \in I} c_t^i(s^t) = \sum_{i \in I} y_t^i(s^t) \tag{1.4}$$

for all $t \geq 0$ and all $s^t \in S^t$.

We proceed to finding competitive Arrow-Debreu equilibria for this economy. As usual, summing up the agents' budget constraint we obtain one of the market clearing conditions, so that we have one more unknown than equations. It is therefore customary to normalize the time-0 price to unity. Letting μ_i denote the multiplier on household i's budget constraint we can write the Lagrangian as

$$\mathcal{L}^i = \sum_{t=0}^{\infty} \sum_{s^t \in S^t} \beta^t \pi_t(s^t) u(c_t^i(s^t)) + \mu_i \sum_{t=0}^{\infty} \sum_{s^t \in S^t} q_t^0(s^t)[y_t^i(s^t) - c_t^i(s^t)],$$

from which we obtain first-order conditions

$$\beta^t \pi_t(s^t) u'(c_t^i(s^t)) = \mu_i q_t^0(s^t) \tag{1.5}$$

for all $i \in I$, $t \geq 0$ and $s^t \in S^t$. Combining the conditions for two agents i and j we obtain

$$\frac{u'(c_t^i(s^t))}{u'(c_t^j(s^t))} = \frac{\mu_i}{\mu_j} \quad \text{for all} \quad i, j \in I, \tag{1.6}$$

which looks very much like what we obtained for the planner's problem in Equation (1.1). We can do the same manipulation as in Section 1.2, rewriting consumption of i as a function of agent 1 to obtain

$$c_t^i(s^t) = u'^{-1}\Big(u'(c_t^1(s^t))\frac{\mu_i}{\mu_1}\Big) \quad \text{for all} \quad i \in I,$$

which we can substitute in the market clearing condition to obtain

$$\sum_{i \in I} u'^{-1}\Big(u'(c_t^1(s^t))\frac{\mu_i}{\mu_1}\Big) = \sum_{i \in I} y_t^i(s^t) \tag{1.7}$$

for all $t \geq 0$ and all $s^t \in S^t$.

Hence the Arrow-Debreu competitive equilibrium allocation has the same properties as the Pareto efficient allocations. Moreover, the Arrow-Debreu equilibrium allocation is a particular Pareto efficient allocation which can be obtained by solving the planner problem with welfare weights $\lambda_i = \mu_i^{-1}$ for all i. Of course, as it is a Pareto efficient allocation, it also exhibits the properties of perfect insurance and history independence. Effectively, we have taken a direct route to showing that for this particular economy the first welfare theorem holds.

1.3.1 Negishi's Algorithm for Computing Equilibria

In most cases, computing the competitive equilibrium by hand is quite hard. The following algorithm may help speeding up the process of finding a solution with a computer:

1. Choose a value for μ_1 (it's a normalization) and guess $\mu_2, \dots, \mu_I$.
2. Compute the allocation using equations (1.6) and (1.7) above, that is, the optimality conditions without prices and market clearing.
3. Substitute the allocation just found into any household's first-order condition, to obtain prices $q_t^0(s^t) = \beta^t \pi_t(s^t) u'(c_t^i(s^t))/\mu_i$.
4. For each agent $i \in I$, check that the budget constraint holds with equality. That is, check that $\sum_{t=0}^{\infty} \sum_{s^t \in S^t} q_t^0(s^t)(y_t^i(s^t) - c_t^i(s^t)) = 0$. If it is greater than zero for household i, lower μ_i, otherwise raise it.
5. Iterate to convergence.

Example. Let $I = 1, 2$, $S = \{0, 1\}$, and the process $\{s_t\}_{t=0}^{\infty}$ be independent and identically distributed over time with $P(s_t = 0) = P(s_t = 1) = 1/2$ for each $t \geq 1$. Hence the distribution for a history $s^t \in S^t$ is simply $\pi_t(s^t) = (1/2)^t$. We take $s_0 \in [0, 1]$ as given and let Arrow-Debreu trading occur after s_0 is realized. Utility takes the CRRA form $u(c) = c^{1-\rho}/(1-\rho)$ with $\rho > 0$. Endowments are $y_t^1(s^t) = s_t$, $y_t^2(s^t) = 1 - s_t$, so that the total endowment is constant each period: there is *no aggregate uncertainty* in the economy. Characterize the set of Pareto efficient allocations and find the competitive equilibrium and prices.

1.4 Sequential Trading of Arrow Securities

So far we considered a rather weird market structure where agents exchanged in a single market claims to future history-contingent consumption. We saw how the Arrow-Debreu competitive equilibrium allocation was Pareto efficient. We now consider a different market structure. In each period-history agents are allowed to trade financial contracts that pay one unit of consumption in the next period, contingent on a given history realization. Furthermore, we assume that there are available for trade as many of these contracts as there are possible next-period histories conditional on the current one. Such contracts are called *Arrow securities*. In particular, we let $a_{t+1}^i(s^t, s_{t+1})$ denote agent i's demand in period t, history s^t, of the Arrow security that delivers one unit of consumption in period $t+1$ if and only if the event s_{t+1} occurs.

Letting $q_{t+1}^t(s^t, s_{t+1})$ denote the price of one unit of the Arrow security traded in period t, history s^t and delivering one unit of consumption in period $t+1$, history (s^t, s_{t+1}), we can write agent i's budget constraint in period t, history s^t as

$$c_t^i(s^t) + \sum_{s_{t+1} \in S} a_{t+1}^i(s^t, s_{t+1}) q_{t+1}^t(s^t, s_{t+1}) \leq y_t^i(s^t) + a_t^i(s^t),$$

where $a_t^i(s^t)$ is the asset position at the beginning of s^t, that is, the number of Arrow securities i bought in $t-1$, history s^{t-1} that offered a payment contingent on (s^{t-1}, s_t) occurring.

As the problem is written now, it does not have a solution. For a given candidate optimum, it is possible to improve upon it by running a "Ponzi scheme". This is done by repaying some extra consumption in the present by debt that rolls over to infinity. To see this, suppose otherwise that there is a consumption plan c that reaches the optimum. Consider an alternative plan $\tilde{c}$ identical to c except for $\tilde{c}_0(s_0) = c_0(s_0) + k$, $k > 0$. We wish to show that if c is feasible, so is $\tilde{c}$, which is an improvement on the plan. To do this, notice that the agent can borrow k more in period 0 by selling a riskless portfolio. She can iterate this every period, rolling the debt forward each time, which is growing to infinity over time. To prevent such deviations, we impose a limit on how much the agent can borrow each period. We do this with a state-contingent borrowing constraint of the form

$$a^i_{t+1}(s^t, s_{t+1}) \geq -A^i_{t+1}(s^t, s_{t+1})$$

for all $i \in I$, in all histories $s^t \in S^t$ and for all $s_{t+1} \in S$, for some $A^i_{t+1}(\cdot) \geq 0$.

To start with, we wish to impose a borrowing limit that never binds at the optimum. That is, a limit such that the agent, knowing that she cannot run Ponzi schemes, will always find it optimal to not reach it. We call this kind of limit a *natural borrowing limit.* Let $Q(s^\tau, s^t)$ denote the product of Arrow security prices along the path from s^τ to s^t, with $Q(s^t, s^t) \equiv 1$. Let the natural borrowing limit be given by

$$\sum_{\tau=t+1}^{\infty} \sum_{s^\tau \in S^\tau | s^t} Q(s^{t+1}, s^\tau) y^i_\tau(s^\tau), \tag{1.8}$$

where the second sum runs only over histories that are consistent with being in history s^t in period t. It is easy to verify that the quantity above is obtained by iterating forward the budget constraint by substituting out Arrow securities and then setting consumption to zero in all histories. Such a limit will never bind if $\lim_{c \to 0} u'(c) = +\infty$ as, if it is hit, there is a positive probability on a consumption path with zero consumption in some period, which can be improved by an infinitesimal decrease in the debt level today.

Note that any kind of borrowing limit will prevent Ponzi schemes. For example, we could set $A^i_{t+1}(s^t, s_{t+1}) = 0$ all the time. In this case the households are prevented from holding *any* level of debt. This kind of limit is, of course, binding in equilibrium. We will study this case and other forms of market incompleteness in more detail in chapter **??**.

Equilibrium. Given this market structure, we can define an appropriate equilibrium notion. A *competitive equilibrium with sequential trading of one-period Arrow securities* is an initial distribution of wealth, $\{a^i_0(s_0)\}^I_{i=1}$, a collection of borrowing limits $\{A^i_t(s^t)\}_{i,t,s^t}$, an allocation $\left\{c^i_t(s^t)\right\}_{i,t,s^t}$, asset positions $\left\{\{a^i_{t+1}(s^t, s_{t+1})\}_{s_{t+1}}\right\}_{i,t,s^t}$, and prices $\{q^t_{t+1}(s^t, s_{t+1})\}_{s^t, s_{t+1}}$ such that:

1. Given the initial asset position, borrowing limits, and prices, the consumption and asset

positions solve the agent's maximisation problem;

2. For all t and all s^t the allocation and asset positions are such that markets clear, that is,

$$\sum_{i\in I} c_t^i(s^t) = \sum_{i\in I} y_t^i(s^t) \quad \text{and} \quad \sum_{i\in I} a_{t+1}^i(s^t, s_{t+1}) = 0 \quad \forall s_{t+1} \in S.$$

Having defined an equilibrium concept, we proceed to solving the agent's problem. We attach multiplier $\beta^t \pi_t(s^t)\eta_t^i(s^t)$ to the budget constraint in history s^t and multiplier $\beta^t \pi_t(s^t)\nu_t^i(s^t, s_{t+1})$ to the borrowing constraint in history s^t for holdings of the Arrow security with payoff in history (s^t, s_{t+1}). The Lagrangian for household i, for a given $a_0^i(s_0)$, is

$$\begin{aligned}\sum_{t=0}^{\infty} \sum_{s^t\in S^t} & \beta^t \pi_t(s^t)\Big\{u(c_t^i(s^t)) \\ & + \eta_t^i(s^t)\Big[y_t^i(s^t) + a_t^i(s^t) - c_t^i(s^t) - \sum_{s_{t+1}\in S} a_{t+1}^i(s^t, s_{t+1}) q_{t+1}^t(s^t, s_{t+1})\Big] \\ & + \sum_{s_{t+1}\in S} \nu_t^i(s^t, s_{t+1})\Big[A_{t+1}^i(s^t, s_{t+1}) - a_{t+1}^i(s^t, s_{t+1})\Big]\Big\}\end{aligned}$$

The first-order conditions with respect to consumption are

$$u'(c_t^i(s^t)) - \eta_t^i(s^t) = 0$$

while the ones with respect to the Arrow security paying in (s^t, s_{t+1}) are

$$\beta^t \pi_t(s^t)\Big[-\eta_t^i(s^t) q_{t+1}^t(s^t, s_{t+1}) - \nu_t^i(s^t, s_{t+1})\Big] + \beta^{t+1}\pi_{t+1}(s^t, s_{t+1})\eta_{t+1}^i(s^t, s_{t+1}) = 0$$

Notice that the borrowing constraint never binds, since, by assumption, marginal utility tends to infinity as consumption tends to zero. This means it can never be optimal for the household to consume nothing in any given period. This implies the multiplier on the borrowing constraint, $\nu_t^i(s^t)$, is zero in all periods and histories. We can therefore combine the two first-order conditions to obtain the familiar Euler equation

$$q_{t+1}^t(s^t, s_{t+1}) u'(c_t^i(s^t)) = \beta \pi_{t+1}(s^t, s_{t+1}|s^t) u'(c_{t+1}^i(s^t, s_{t+1})),$$

which, condensing notation, can be written as

$$q_{t+1}^t(s^{t+1}) u'(c_t^i(s^t)) = \beta \pi_{t+1}(s^{t+1}|s^t) u'(c_{t+1}^i(s^{t+1})).$$

Taking ratios for two generic agents i and j,

$$\frac{u'(c_t^i(s^t))}{u'(c_t^j(s^t))} = \cdots = \frac{u'(c_0^i(s_0))}{u'(c_0^j(s_0))} \quad \text{for all} \quad i, j \in I,$$

Which shows the usual insurance property: the ratio of marginal utilities between two agents in each history does not depend on the history and is therefore constant over time. Defining $\alpha_0^i \equiv u'(c_0^i(s^0))^{-1}$ we can write

$$\frac{u'(c_t^i(s^t))}{u'(c_t^1(s^t))} = \frac{\alpha_0^1}{\alpha_0^i}$$

for all $i \in I$ and all histories. Summing up across agents and using market clearing, we obtain the usual result that individual consumption only depends on the history through the *current* aggregate endowment. It follows that, also with this trading arrangement, the first welfare theorem holds.

1.4.1 Equivalence of Allocations

Having shown that both trading arrangement lead to Pareto efficient allocations we would like to find conditions under which the two allocations coincide. To make progress on the former, notice that if the prices in the two equilibria are such that

$$q_{t+1}^t(s^t, s_{t+1}) = \frac{q_{t+1}^0(s^{t+1})}{q_t^0(s^t)}$$

in all periods and histories the optimality conditions in the two models coincide. All that is left to do is to find the initial distribution of wealth in the sequential trading arrangement that decentralises the Arrow-Debreu allocation at *these* prices. It can be shown (see Section 8.8.7 in Ljungqvist and Sargent (2012)) that the initial wealth vector must be chosen to be zero. This is intuitive, as it forces agents to finance consumption with their endowment stream, just like in the Arrow-Debreu arrangement.

1.5 Recursive Competitive Equilibrium

So far we have derived some properties of the equilibrium allocation, such as perfect insurance and history-independence, without any assumptions about the process of endowments. To move further it is however useful to simplify the problem in a very specific way, which will lead to large improvement in tractability. We assume that the process $\{s_t\}_{t=0}^{\infty}$ is Markov with state space S. We let $\pi_0 : S \to [0, 1]$ denote the initial distribution for the random variable s_0. Transition probabilities between states s and s' in S are written as $\mathbf{P}(s_{t+1} = s'|s_t = s) = \pi(s'|s)$. The chain induces a sequence of probability measures $\pi_t(s^t)$ on histories in S^t via the recursions

$$\pi_t(s^t) = \pi(s_t|s_{t-1})\pi(s_{t-1}|s_{t-2}) \dots \pi(s_1|s_0)\pi_0(s_0)$$

We also assume that the households' endowments in period t are time-invariant measurable functions of s_t, that is, what determines i's income in period t, history s^t is *only* the event realization in period t, s_t. We can therefore write i's income as $y^i(s_t)$. It easy to check that, both for the Arrow-Debreu and sequential markets equilibria, all equilibrium objects inherit this property.

With these assumptions, we can formulate the agent's decision problem recursively. We work in a setting where agents can trade a full set of Arrow securities each period. Notice that the state for agent i in period t is given by the realisation of the event, s_t, and her holding of Arrow securities that paid in that state, $a^i(s_t)$. We generally refer to the latter quantity as the agent's "assets" or "savings". The recursive problem is to find a solution to the functional equation

$$v^i(a, s) = \max_{c, a'(\cdot)} \left\{ u(c) + \beta \sum_{s' \in S} \pi(s'|s) v^i(a'(s'), s') \right\}$$

subject to the budget constraint,

$$c + \sum_{s' \in S} Q(s'|s) a'(s') = y^i(s) + a$$

and borrowing constraint

$$a'(s') \geq -A'(s') \quad \text{for all} \quad s' \in S.$$

Notice that the choices of the households are consumption, c, and holdings of Arrow securities, as captured by $a' : S \to \mathbb{R}$. Optimal policy functions are given by

$$h(a, s) = c \quad \text{and} \quad g(a, s; s') = a'(s').$$

A *recursive competitive equilibrium* with sequential trading of one-period Arrow securities is an initial distribution $\{a_0^i(s_0)\}_{i=1}^I$, a collection of borrowing limits $\{A^i(s)\}_{i=1}^I$, a pricing kernel $Q(s'|s)$, a set of value functions $\{v^i(a, s)\}_{i=1}^I$ and decision rules $\{h^i(a, s), g^i(a, s; s')\}_{i=1}^I$ such that

1. For all i, given a_0^i and given the pricing kernel, the value functions and decisions rules solve the household problem.
2. For all $\{s^t\}_{t=0}^\infty$, the consumption and asset portfolios $\{c_t^i(s^t), a_{t+1}^i(s^{t+1})\}$ implied by the decision rules and value functions are such that $\sum_{i \in I} c_t^i(s^t) = \sum_{i \in I} y^i(s_t)$ and $\sum_i a_{t+1}^i(s^{t+1}) = 0$.

Part II

Problems

Problem Set

Deadline: **Tuesday, Dec 12, at 12pm.** By email to Eustache.

Complete Markets. **1. An exchange economy.** An economy consists of two infinitely lived consumers named $i = 1, 2$. There is one nonstorable consumption good. Consumer i consumes c_t^i at time t. Consumer i ranks consumption streams by

$$\sum_{t=0}^{\infty} \beta^t u(c_t^i)$$

where $\beta \in (0, 1)$ and $u(c)$ is increasing, strictly concave, and twice continuously differentiable. Consumer 1 is endowed with a stream of the consumption good $y_t^1 = 1, 0, 0, 1, 0, 0, 1, ...$ Consumer 2 is endowed with a stream of the consumption good $0, 1, 1, 0, 1, 1, 0, ...$

1.1. Planner allocation. Write the Planner problem and characterize the optimal allocation as a function of the Pareto weights.

1.2. Competitive Equilibrium. Assume that there are complete markets with time-0 trading. *1.2.1.* Define and compute a competitive equilibrium. *1.2.2.* Suppose that one of the consumers markets a derivative asset that promises to pay .05 units of consumption each period. What would the price of that asset be?

2. An exchange economy again. An economy consists of two consumers, named $i = 1, 2$. The economy exists in discrete time for periods $t \geq 0$. There is one good in the economy, which is not storable and arrives in the form of an endowment stream owned by each consumer.

The endowments to consumers $i = 1, 2$ are $y_t^1 = s_t$ and $y_t^2 = 1$, where s_t is a random variable governed by a two-state Markov chain with values $s_t = \bar{s}_1 = 0$ or $s_t = \bar{s}_2 = 1$. The Markov chain has time invariant transition probabilities denoted by $\pi(s_{t+1} = s'|s_t = s) = \pi(s'|s)$, and the probability distribution over the initial state is $\pi_0(s)$. The aggregate endowment at t is $Y(s_t) = y_t^1 + y_t^2$.

Let c^i denote the stochastic process of consumption for agent i. Household i orders consumption streams according to

$$U(c^i) = \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \ln[c_t^i(s^t)] \pi_t(s^t),$$

where $\pi_t(s^t)$ is the probability of the history $s^t = (s_0, s_1, \ldots, s_t)$.

2.1. Give a formula for $\pi_t(s^t)$. *2.2.* Let $\theta \in (0, 1)$ be a Pareto weight on household

1, and $1-\theta$ the Pareto weight on household 2. Write down the Planner problem, and solve it, taking θ as a parameter. *2.3.* Define a competitive equilibrium with history-dependent Arrow-Debreu securities traded once and for all at time 0. Be careful to define all of the objects that compose a competitive equilibrium. *2.4.* Compute the competitive equilibrium price system (i.e., find the prices of all of the Arrow-Debreu securities). *2.5.* Tell the relationship between the solutions (indexed by θ) of the Pareto problem and the competitive equilibrium allocation.

Incomplete Markets. **3. An income fluctuation problem.** A single consumer has preferences over sequences of a single consumption good that are ordered by $\sum_{t=0}^{\infty} \beta^t \log(c_t)$, where $\beta \in (0,1)$. The one good is not storable. The consumer has an endowment sequence of the one good $y_t = \lambda^t$ where $\lambda > 0$ and $\lambda\beta < 1$. The consumer can borrow or lend at a constant and exogenous risk-free net interest rate r that satisfiers $(1+r)\beta = 1$.

The consumer's budget constraint at time t is given by

$$c_t + \frac{1}{1+r} a_{t+1} = y_t + a_t$$

for all $t \geq 0$, where a_t is assets due at t. The consumer starts with zero assets: $a_0 = 0$.

3.1. Tight borrowing constraints. We first assume a tight borrowing constraint $a_t \geq 0$ $\forall t \geq 0$. Thus, the consumer can lend but not borrow. *3.1.1* Assume that $\lambda < 1$. Compute the household's optimal plan for $\{c_t, a_{t+1}\}_{t=0}^{\infty}$. *3.1.2* Assume that $\lambda > 1$. Compute the household's optimal plan for $\{c_t, a_{t+1}\}_{t=0}^{\infty}$.

3.2. Natural borrowing constraints. We now assume that the consumer is subject to the natural borrowing constraint associated with the given endowment sequence. *3.2.1* Consume the natural borrowing limits for all $t \geq 0$. *3.2.2* Assume that $\lambda > 1$. Compute the household's optimal plan for $\{c_t, a_{t+1}\}_{t=0}^{\infty}$.

Bibliography

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