M1 APE, Econometrics 2
Solution tutorial 3
Stationary processes and AR processes

2017-2018

Exercise 1

1) In order to prove that (u_t) is the innovation process of (x_t) it is sufficient to prove that (x_t) is written under its canonical representation.

$x_t = 0.2 + 0.8x_{t-1} + u_t \Leftrightarrow \Phi(L)x_t = 0.2 + u_t$ with $\Phi(L) = 1 - 0.8L$

This is indeed the canonical representation of (x_t) since the root of $\Phi(z)$ is $\frac{1}{0.8} = 1.25$, with modulus greater than one. Thus (u_t) is the innovation process of (x_t).

If (u_t) is the innovation process of (x_t) then we know in particular that: $\forall h > 0, \text{Cov}(u_t, x_{t-h}) = 0$

2) (x_t) is stationary so $\forall t,\ E(x_t) = m_x$. We then have:

$x_t - 0.8x_{t-1} = 0.2 + u_t$

$\Rightarrow E(x_t) - 0.8E(x_{t-1}) = 0.2 + E(u_t)$

$\Leftrightarrow (1 - 0.8)m_x = 0.2 + 0$

$\Leftrightarrow m_x = 1$

Remark: we have here $\Phi(L)x_t = \mu + u_t$, with $\Phi(L) = 1 - \phi L = 1 - 0.8L$ and $\mu = 0.2$.

We can also directly get the result from the course: $E(x_t) = m_x = \dfrac{\mu}{1-\phi}$.

3) We have:

$$
\begin{array}{rrcl}
 & \gamma_x(0) & \equiv & V(x_t) \\
\Leftrightarrow & \gamma_x(0) & = & V(0.2 + 0.8x_{t-1} + u_t) \\
\Leftrightarrow & \gamma_x(0) & = & 0.8^2 V(x_{t-1}) + V(u_t) + 2 \times 0.8 \times \text{Cov}(u_t, x_{t-1}) \\
\Leftrightarrow & \gamma_x(0) & = & 0.64\gamma_x(0) + \sigma_u^2 + 0 \qquad (u_t) \text{ is the innovation so it cannot be correlated with the past of } (x_t) \\
\Leftrightarrow & \left[\gamma_x(0)\right. & = & \left.\dfrac{\sigma_u^2}{0.36}\right]
\end{array}
$$

$$
\begin{array}{rcl}
\gamma_x(1) & \equiv & \text{Cov}(x_t, x_{t-1}) \\
 & = & \text{Cov}(0.2 + 0.8x_{t-1} + u_t, x_{t-1}) \\
 & = & 0.8V(x_{t-1}) + \text{Cov}(u_t, x_{t-1}) \\
 & = & 0.8\gamma_x(0) + 0 \qquad (u_t) \text{ is the innovation so it cannot be correlated with the past of } (x_t) \\
\left[\gamma_x(1)\right. & = & \left.0.8\gamma_x(0) = \dfrac{0.8}{0.36}\sigma_u^2\right]
\end{array}
$$

4) Similarly:

$$\begin{aligned}
\gamma_x(h) &\equiv \text{Cov}(x_t, x_{t-h}) \\
&= \text{Cov}(0.2 + 0.8x_{t-1} + u_t, x_{t-h}) \\
&= 0.8\,\text{Cov}(x_{t-1}, x_{t-h}) + \text{Cov}(u_t, x_{t-h}) \\
&= 0.8\gamma_x(h-1) + 0 \qquad \text{Cov}(u_t, x_{t-h}) = 0 \ \forall h \geq 1 \text{ because } (u_t) \text{ is the innovation process of } (x_t) \\
\big[\gamma_x(h) &= 0.8\gamma_x(h-1) \ \forall h \geq 1\big]
\end{aligned}$$

As expected, we get the Yule-Walker equations.
Iterating the previous equality until $h = 1$ we can compute the value of any $\gamma_x(h)$:

$$\begin{aligned}
\gamma_x(h) &= 0.8\big(0.8\gamma_x(h-2)\big) \quad \forall h \geq 1 \\
&= \ldots \\
&= 0.8^h \gamma_x(0) \quad \forall h \geq 0 \\
\left[\gamma_x(h) \right. &= \left. \frac{0.8^h}{0.36}\sigma_u^2 \quad \forall h \geq 0\right]
\end{aligned}$$

From this, we can simply compute the autocorrelations with:

$$\begin{aligned}
\rho_x(h) &\equiv \frac{\gamma_x(h)}{\gamma_x(0)} \quad \forall h \\
&= \frac{0.8^h \gamma_x(0)}{\gamma_x(0)} \quad \forall h \geq 0 \\
\left[\rho_x(h) \right. &= \left. 0.8^h \quad \forall h \geq 0\right]
\end{aligned}$$

5) (x_t) can be expressed as $\Phi(L)x_t = 0.2 + u_t$, with $\Phi(L) = 1 - 0.8L$. The root of $\Phi(L)$ has a modulus greater than one so this polynomial is invertible.

$$\Phi^{-1}(L) = \sum_{i=0}^{\infty} 0.8^i L^i$$

Note that this result generalizes the classic formula which can be written for $|a| < 1$:

$$\frac{1}{1-a} = \sum_{i=0}^{\infty} a^i$$

Due to the stationarity of (x_t), this formula can indeed be extended to the present framework, involving the lag operator L, as it has been seen in the course:

$$(1-\phi L)^{-1} = \frac{1}{1-\phi L} = \sum_{i=0}^{\infty} (\phi L)^i = \sum_{i=0}^{\infty} \phi^i L^i \quad \text{since } |\phi| = 0.8 < 1$$

This leads to the infinite MA representation of (x_t)

$$\begin{aligned}
\Phi(L)x_t &= 0.2 + u_t \\
\Leftrightarrow x_t &= \Phi^{-1}(L)(0.2 + u_t) \\
\Leftrightarrow x_t &= \Phi^{-1}(L)(0.2) + \Phi^{-1}(L)(u_t) \\
\Leftrightarrow x_t &= \left(\sum_{i=0}^{\infty} 0.8^i L^i\right)(0.2) + \left(\sum_{i=0}^{\infty} 0.8^i L^i\right)(u_t) \\
\Leftrightarrow x_t &= \left(\sum_{i=0}^{\infty} 0.8^i\right)0.2 + \sum_{i=0}^{\infty} 0.8^i u_{t-i} \\
\Leftrightarrow x_t &= 0.2 \times \frac{1}{1-0.8} + \sum_{i=0}^{\infty} 0.8^i u_{t-i} \\
\Leftrightarrow \left[x_t \right. &= \left. 1 + \sum_{i=0}^{\infty} 0.8^i u_{t-i}\right]
\end{aligned}$$

Following a unit shock u_t at period t, the effect on x will be 0.8^0 at period t, 0.8^1 at period $t+1$, 0.8^2 at period $t+2$, etc...

Remark: it can be directly written that:

$$\Phi(L)^{-1}\mu = \Phi(1)^{-1}\mu = \frac{\mu}{\Phi(1)} = \frac{\mu}{1-\phi} = \frac{0.2}{1-0.8} = 1$$

Exercise 2

1) $y_t = 0.5 + \frac{5}{6}y_{t-1} - \frac{1}{6}y_{t-2} + v_t \;\Leftrightarrow\; \Phi(L)y_t = 0.5 + v_t$ with $\Phi(L) = 1 - \frac{5}{6}L + \frac{1}{6}L^2 = \left(1-\frac{1}{2}L\right)\left(1-\frac{1}{3}L\right)$.

- The two roots of $\Phi(z)$ are 2 and 3, and both have a modulus greater than one, so that the previous equation does admit a stationary solution (y_t).
- Since the roots of $\Phi(z)$ have modulus greater than one, this is the canonical representation of (y_t) and (v_t) necessarily is the innovation process of (y_t)

2) (y_t) is stationary so $\forall t,\ E(y_t) = m_y$. We then get:
$y_t - \frac{5}{6}y_{t-1} + \frac{1}{6}y_{t-2} = 0.5 + v_t$
$\Rightarrow E(y_t) - \frac{5}{6}E(y_{t-1}) + \frac{1}{6}E(y_{t-2}) = 0.5 + E(v_t)$
$\Leftrightarrow \frac{1}{3}m_y = 0.5 + 0$
$\Leftrightarrow \left[E(y_t) = m_y = 1.5\right]\ \forall t$
Remark: we have here $\Phi(L)x_t = \mu + v_t$, with $\Phi(L) = 1 - \frac{5}{6}L + \frac{1}{6}L^2$ and $\mu = 0.5$.
We simply get the result from the course: $E(y_t) = m_y = \frac{\mu}{\Phi(1)}$.

3) There are two ways to compute the autocovariances of a process: either we can center the process, either we can directly compute the covariances. In this question, we will center y_t, pre-multiply it by its most convenient lag, and take the expectation to make appear the different $\gamma_y(h)$ terms. In the next one, we'll directly compute the covariances, in order to use both methods.

Denoting $\widetilde{y}_t = y_t - m_y$, we have:

$$\gamma_y(h) = \gamma_{\widetilde{y}}(h) = E(\widetilde{y}_t\widetilde{y}_{t-h}) \quad \forall h$$

The equation defining (y_t) can then be rewritten as:

$$\begin{aligned}
&\Phi(L)y_t = \mu + v_t\\
\Leftrightarrow\ &\Phi(L)y_t = \Phi(1)m_y + v_t\\
\Leftrightarrow\ &\Phi(L)(y_t - m_y) = v_t\\
\Leftrightarrow\ &\Phi(L)\widetilde{y}_t = v_t\\
\Leftrightarrow\ &\left[\widetilde{y}_t = \frac{5}{6}\widetilde{y}_{t-1} - \frac{1}{6}\widetilde{y}_{t-2} + v_t\right]
\end{aligned} \tag{1}$$

Pre-multiplying (1) by $\widetilde{y}_t$ yields:

$$\begin{aligned}
&\widetilde{y}_t^2 = \frac{5}{6}\widetilde{y}_t\widetilde{y}_{t-1} - \frac{1}{6}\widetilde{y}_t\widetilde{y}_{t-2} + \widetilde{y}_t v_t\\
\Leftrightarrow\ &E(\widetilde{y}_t^2) = \frac{5}{6}E(\widetilde{y}_t\widetilde{y}_{t-1}) - \frac{1}{6}E(\widetilde{y}_t\widetilde{y}_{t-2}) + E(\widetilde{y}_t v_t)\\
\Leftrightarrow\ &\gamma_y(0) = \frac{5}{6}\gamma_y(1) - \frac{1}{6}\gamma_y(2) + E\left[\left(\frac{5}{6}\widetilde{y}_{t-1} - \frac{1}{6}\widetilde{y}_{-2} + v_t\right)v_t\right]\\
\Leftrightarrow\ &\gamma_y(0) = \frac{5}{6}\gamma_y(1) - \frac{1}{6}\gamma_y(2) + \frac{5}{6}E(\widetilde{y}_{t-1}v_t) - \frac{1}{6}E(\widetilde{y}_{t-2}v_t) + E(v_t^2)\\
\Leftrightarrow\ &\left[\gamma_y(0) = \frac{5}{6}\gamma_y(1) - \frac{1}{6}\gamma_y(2) + \sigma_v^2\right] \qquad \text{because } (v_t) \text{ is the innovation of } (y_t) \text{ and of } (\widetilde{y}_t)
\end{aligned} \tag{2}$$

Pre-multiplying (1) by $\widetilde{y}_{t-1}$ yields:

$$\widetilde{y}_{t-1}\widetilde{y}_t = \frac{5}{6}\widetilde{y}_{t-1}^2 - \frac{1}{6}\widetilde{y}_{t-1}\widetilde{y}_{t-2} + \widetilde{y}_{t-1}v_t$$

$$\Leftrightarrow E(\widetilde{y}_{t-1}\widetilde{y}_t) = \frac{5}{6}E(\widetilde{y}_{t-1}^2) - \frac{1}{6}E(\widetilde{y}_{t-1}\widetilde{y}_{t-2}) + E(\widetilde{y}_{t-1}v_t)$$

$$\Leftrightarrow \gamma_y(1) = \frac{5}{6}\gamma_y(0) - \frac{1}{6}\gamma_y(1) \qquad \text{because } (v_t) \text{ is the innovation of } (y_t) \text{ and of } (\widetilde{y}_t) \tag{3}$$

$$\Leftrightarrow \frac{7}{6}\gamma_y(1) = \frac{5}{6}\gamma_y(0)$$

$$\Leftrightarrow \left[\gamma_y(1) = \frac{5}{7}\gamma_y(0)\right] \tag{4}$$

Pre-multiplying (1) by $\widetilde{y}_{t-2}$ yields:

$$\widetilde{y}_{t-2}\widetilde{y}_t = \frac{5}{6}\widetilde{y}_{t-2}\widetilde{y}_{t-1} - \frac{1}{6}\widetilde{y}_{t-2}^2 + \widetilde{y}_{t-2}v_t$$

$$\Leftrightarrow E(\widetilde{y}_{t-2}\widetilde{y}_t) = \frac{5}{6}E(\widetilde{y}_{t-2}\widetilde{y}_{t-1}) - \frac{1}{6}E(\widetilde{y}_{t-2}^2) + E(\widetilde{y}_{t-2}v_t)$$

$$\Leftrightarrow \left[\gamma_y(2) = \frac{5}{6}\gamma_y(1) - \frac{1}{6}\gamma_y(0)\right] \qquad \text{because } (v_t) \text{ is the innovation of } (y_t) \text{ and of } (\widetilde{y}_t) \tag{5}$$

Solving the system given by the equations (2), (4) and (5) yields:

$$\left[\gamma_y(0) = 2.1\sigma_v^2\right] \qquad \left[\gamma_y(1) = 1.5\sigma_v^2\right] \qquad \left[\gamma_y(2) = 0.9\sigma_v^2\right]$$

4)i) In this question, we will rather compute the autocovariances using $\text{Cov}(y_t, y_{t-h})$.

$$\gamma_y(h) \equiv \text{Cov}(y_t, y_{t-h})$$

$$\Leftrightarrow \gamma_y(h) = \text{Cov}\left(0.5 + \frac{5}{6}y_{t-1} - \frac{1}{6}y_{t-2} + v_t, y_{t-h}\right)$$

$$\Leftrightarrow \gamma_y(h) = \frac{5}{6}\text{Cov}(y_{t-1}, y_{t-h}) - \frac{1}{6}\text{Cov}(y_{t-2}, y_{t-h}) + \text{Cov}(v_t, y_{t-h})$$

$$\Leftrightarrow \gamma_y(h) = \frac{5}{6}\gamma_y(h-1) - \frac{1}{6}\gamma_y(h-2) + 0 \qquad \text{Cov}(v_t, y_{t-h}) = 0\ \forall h \geq 1 \text{ because } (v_t) \text{ is the innovation process of } (y_t)$$

$$\left[\Leftrightarrow \gamma_y(h) = \frac{5}{6}\gamma_y(h-1) - \frac{1}{6}\gamma_y(h-2) \quad \forall h \geq 1\right] \tag{6}$$

For $h = 2$, this is consistent with equation (5) for example.

<u>Reminder:</u>
The previous results concerning the relations between the $\gamma_y(h)$ terms (equations (2), (3), (5), and (6)) correspond to the Yule-Walker equations.
For an AR(p) process expressed as $X_t = \sum_{k=1}^{p}\phi_k X_{t-k} + \epsilon_t$, the Yule-Walker equations are:

$$\begin{cases} \gamma_X(0) = \sum_{k=1}^{p}\phi_k\gamma_X(k) + \sigma_\epsilon^2 & \text{for } h = 0 \\ \gamma_X(h) = \sum_{k=1}^{p}\phi_k\gamma_X(h-k) & \text{for } h > 0 \end{cases}$$

We can now compute the value of $\gamma_y(h)$ for every h:
- iteratively, using (6) and the values of $\gamma_y(0), \gamma_y(1)$ and $\gamma_y(2)$ from the previous question;
- or with the general form of the solutions for a recurrence relation (next question).

ii) We know that:
- Equation (6), the general form of the Yule-Walker equations for $h \geq 1$, yields $\Phi(L)\gamma_y(h) = 0$
- $\Phi(z)$ has two roots, equal to 2 and 3.

The general form of the solutions for such a recurrence relation is:

$$\gamma_y(h) = \alpha\left(\frac{1}{2}\right)^h + \beta\left(\frac{1}{3}\right)^h \quad \forall h \geq 1$$

This is a general result, and you can check it works in this particular case (left to the reader).

iii) We can compute α and β with the initial conditions.

$$\begin{cases} \gamma_y(1) = \frac{1}{2}\alpha + \frac{1}{3}\beta \\ \gamma_y(2) = \frac{1}{4}\alpha + \frac{1}{9}\beta \end{cases} \Leftrightarrow \begin{pmatrix} \frac{3}{2} \\ \frac{9}{10} \end{pmatrix} \sigma_v^2 = \begin{pmatrix} \frac{1}{2} & \frac{1}{3} \\ \frac{1}{4} & \frac{1}{9} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

We get:

$$\alpha = \frac{\begin{vmatrix} \frac{3}{2} & \frac{1}{3} \\ \frac{9}{10} & \frac{1}{9} \end{vmatrix}}{\begin{vmatrix} \frac{1}{2} & \frac{1}{3} \\ \frac{1}{4} & \frac{1}{9} \end{vmatrix}} \sigma_v^2 = \frac{\frac{1}{6} - \frac{3}{10}}{\frac{1}{18} - \frac{1}{12}} \sigma_v^2 = \frac{-\frac{4}{30}}{-\frac{1}{36}} \sigma_v^2 = \frac{4}{30} \times 36\sigma_v^2 = \frac{24}{5}\sigma_v^2$$

$$\beta = \frac{\begin{vmatrix} \frac{1}{2} & \frac{3}{2} \\ \frac{1}{4} & \frac{9}{10} \end{vmatrix}}{\begin{vmatrix} \frac{1}{2} & \frac{1}{3} \\ \frac{1}{4} & \frac{1}{9} \end{vmatrix}} \sigma_v^2 = \frac{\frac{9}{20} - \frac{3}{8}}{\frac{1}{18} - \frac{1}{12}} \sigma_v^2 = \frac{\frac{3}{40}}{-\frac{1}{36}} \sigma_v^2 = -\frac{3}{40} \times 36\sigma_v^2 = -\frac{27}{10}\sigma_v^2$$

iv) $\gamma_y(h)$ decreases exponentially to 0, like for any other stationary process.

5) In order to compute the MA(∞) representation of (y_t) we need to inverse its AR lag polynomial $\Phi(L) = \left(1 - \frac{1}{2}L\right)\left(1 - \frac{1}{3}L\right)$. This lag polynomial is invertible because all of its roots (2 and 3) are bigger than one in modulus.

$$\begin{aligned} \left(1 - \frac{1}{2}L\right)\left(1 - \frac{1}{3}L\right) y_t &= 0.5 + v_t \\ \Leftrightarrow y_t &= \left(1 - \frac{1}{2}L\right)^{-1} \left(1 - \frac{1}{3}L\right)^{-1} (0.5 + v_t) \\ \Leftrightarrow y_t &= \Psi(L)(0.5 + v_t) \\ \Leftrightarrow y_t &= \Psi(1)0.5 + \Psi(L) v_t \end{aligned}$$

For the first term, we have:

$$\Psi(1)0.5 = \Phi(1)^{-1} 0.5 = \frac{0.5}{\Phi(1)} = m_y = 1.5$$

For the second term, we have:

$$\Psi(L) v_t = \sum_{k=0}^{\infty} \left(\psi_k L^k\right) v_t = \sum_{k=0}^{\infty} \psi_k v_{t-k}$$

with $\psi_0 = 1$ and $\sum_{k=0}^{\infty} |\psi_k| < \infty$.

The MA(∞) representation of (y_t) is then:

$$\left[y_t = m_y + \sum_{k=0}^{\infty} \psi_k v_{t-k} = 1.5 + \sum_{k=0}^{\infty} \psi_k v_{t-k} \right]$$

The ψ_k terms can be explicited:

$$\Psi(L) = \left(1 - \frac{1}{2}L\right)^{-1} \left(1 - \frac{1}{3}L\right)^{-1} = \left(\sum_{i=0}^{\infty} \left(\frac{1}{2}\right)^i L^i\right) \left(\sum_{j=0}^{\infty} \left(\frac{1}{3}\right)^j L^j\right) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{1}{2^i} \frac{1}{3^j} L^{i+j} = \sum_{i=0}^{\infty} \sum_{k=i}^{\infty} \frac{1}{2^i} \frac{1}{3^{k-i}} L^k = \sum_{k=0}^{\infty} \underbrace{\left(\sum_{i=0}^{k} \frac{1}{2^i} \frac{1}{3^{k-i}}\right)}_{\psi_k} L^k$$

The first three ψ_k terms are:

$$\psi_0 = \frac{1}{2^0} \frac{1}{3^0} = 1$$

$$\psi_1 = \frac{1}{2^0} \frac{1}{3^1} + \frac{1}{2^1} \frac{1}{3^0} = \frac{1}{3} + \frac{1}{2} = \frac{5}{6}$$

$$\psi_2 = \frac{1}{2^0} \frac{1}{3^2} + \frac{1}{2^1} \frac{1}{3^1} + \frac{1}{2^2} \frac{1}{3^0} = \frac{1}{9} + \frac{1}{6} + \frac{1}{4} = \frac{19}{36}$$

Following a unit shock v_t at period t, the effect on y will be 1 in t, $\frac{5}{6}$ in $t+1$, $\frac{19}{36}$ in $t+2$, etc...

Exercise 3

1) The polynomial $\left(1-\phi z^4\right)$ is a polynomial of degree 4 with real coefficients.
It thus has four complex roots (and when these roots are not real, they are complex conjugate).
If i is the complex number such that $i^2=-1$, these 4 roots are:

$$z_0=\phi^{-1/4},\ z_1=i\phi^{-1/4},\ z_2=-\phi^{-1/4},\ z_3=-i\phi^{-1/4}$$

which can also be written as

$$z_k=\phi^{-1/4}e^{2ik\pi/4}, k=0,\dots,3$$

We are in a case where the autoregressive polynomial has complex non real roots (this is linked to seasonal behaviour), but all these roots have modulus $\phi^{-1/4}$, with

$$0<\phi<1\Rightarrow 0<\phi^{1/4}<1\Rightarrow \phi^{-1/4}>1$$

Thus, this equation has indeed a stationary solution (y_t). Further, since the roots of $\Phi(z)$ have modulus bigger than 1, $\left(1-\phi L^4\right)y_t=u_t$ is the canonical representation of (y_t), and (u_t) is its innovation.

2) The variance can be simply computed as:

$$\begin{aligned}
& V(y_t)=V(\phi y_{t-4}+u_t)\\
\Leftrightarrow\ & V(y_t)=\phi^2V(y_{t-4})+V(u_t)+2\phi\mathrm{Cov}(y_{t-4},u_t)\\
\Leftrightarrow\ & \gamma_y(0)=\phi^2\gamma_y(0)+\sigma_u^2+0 \qquad \text{because } (u_t) \text{ is the innovation of } (y_t)\\
\Leftrightarrow\ & (1-\phi^2)\gamma_y(0)=\sigma_u^2\\
\Leftrightarrow\ & \left[\gamma_y(0)=\frac{\sigma_u^2}{1-\phi^2}\right]
\end{aligned}$$

3) Remembering that $\mathrm{Cov}(u_t,y_{t-h})=0\ \ \forall h>0$ because (u_t) is the innovation process of (y_t),

- $\gamma_y(1)=\mathrm{Cov}(y_t,y_{t-1})=\mathrm{Cov}(\phi y_{t-4}+u_t,y_{t-1})=\phi\gamma_y(3)$
  and $\gamma_y(3)=\mathrm{Cov}(y_t,y_{t-3})=\mathrm{Cov}(\phi y_{t-4}+u_t,y_{t-3})=\phi\gamma_y(1)$
  so that $\gamma_y(1)=\phi\gamma_y(3)\Rightarrow\gamma_y(1)=\phi^2\gamma_y(1)\Leftrightarrow(1-\phi^2)\gamma_y(1)=0\Leftrightarrow\gamma_y(1)=0$
- $\gamma_y(4)=\mathrm{Cov}(y_t,y_{t-4})=\mathrm{Cov}(\phi y_{t-4}+u_t,y_{t-4})=\phi V(y_{t-4})=\phi\gamma_y(0)=\frac{\phi}{1-\phi^2}\sigma_u^2$
- $\gamma_y(5)=\mathrm{Cov}(y_t,y_{t-5})=\mathrm{Cov}(\phi y_{t-4}+u_t,y_{t-5})=\phi\gamma_y(1)=0$
- $\gamma_y(8)=\mathrm{Cov}(y_t,y_{t-8})=\mathrm{Cov}(\phi y_{t-4}+u_t,y_{t-8})=\phi\gamma_y(4)=\frac{\phi^2}{1-\phi^2}\sigma_u^2$

4) The hth partial autocorrelation, usually denoted $r(h)$ is the coefficient b_{hh} of y_{t-h} in the computation of $BLF(y_t|y_{t-1},...,y_{t-h})$.

(u_t) is the innovation process of (y_t), so $\mathrm{Cov}(u_t,y_{t-h})=0\ \ \forall h>0$, and $\mathrm{BLF}(u_t|y_{t-1},...,y_{t-h})=0\ \ \forall h>0$.
Thus,

$$\begin{aligned}
\mathrm{BLF}(y_t|y_{t-1},...,y_{t-h}) &= \mathrm{BLF}(\phi y_{t-4}+u_t|y_{t-1},...,y_{t-h})\\
&= \mathrm{BLF}(\phi y_{t-4}|y_{t-1},...,y_{t-h})+\mathrm{BLF}(u_t|y_{t-1},...,y_{t-h})\\
&= \mathrm{BLF}(\phi y_{t-4}|y_{t-1},...,y_{t-h})
\end{aligned}$$

For $h\geq 4$, $\mathrm{BLF}(y_t|y_{t-1},...,y_{t-h})=\phi y_{t-4}$ so we can deduce that $r(4)=\phi$, and $r(h)=0\ \ \forall h>4$.

$r(1)$ is the coefficient b_{11} of y_{t-1} in the computation of $BLF(y_t|y_{t-1})$.
We have $BLF(y_t|y_{t-1})=b_{01}+b_{11}y_{t-1}$, with b_{01} and b_{11} defined by:

$$\begin{cases}E(y_t-(b_{01}+b_{11}y_{t-1}))=0\\ \mathrm{Cov}(y_t-(b_{01}+b_{11}y_{t-1}),y_{t-1})=0\end{cases}\Leftrightarrow\begin{cases}b_{01}=0\\ \gamma_y(1)=b_{11}\gamma_y(0)\end{cases}\Leftrightarrow\begin{cases}b_{01}=0\\ b_{11}=\frac{\gamma_y(1)}{\gamma_y(0)}=0\end{cases}$$

(there is no constant in the definition of (y_t) so $\forall t,\ E(y_t)=0$)

5) Using the results from question 3 for the autocovariances,

$$\left[\rho_y(1)=\frac{\gamma_y(1)}{\gamma_y(0)}=0\right] \qquad \left[\rho_y(4)=\frac{\gamma_y(4)}{\gamma_y(0)}=\phi\right] \qquad \left[\rho_y(5)=\frac{\gamma_y(5)}{\gamma_y(0)}=0\right] \qquad \left[\rho_y(8)=\frac{\gamma_y(8)}{\gamma_y(0)}=\phi^2\right]$$

6) For an AR(p) process defined as $x_t = c+\phi_1 x_{t-1}+...+\phi_p x_{t-p}+\epsilon_t$, the Yule-Walker equations with the autocorrelations are:

$$\rho_x(h) \quad = \quad \phi_1\rho_x(h-1)+...+\phi_p\rho_x(h-p) \qquad \text{for } h\geq 1$$

In the specific case of the AR(4) process (y_t), we have $p=4$, $\phi_1=\phi_2=\phi_3=0$ and $\phi_4=\phi$. So the Yule-Walker equations are:

$$\rho_y(h)=\phi\rho_y(h-4) \qquad \text{for } h\geq 1$$

For $h=4m$, where m is an integer, this yields:

$$\begin{aligned}
\rho_y(4m) &= \phi\rho_y(4m-4) \quad \text{for } m\geq 1\\
&= \phi\rho_y\big(4(m-1)\big)\\
&= \phi\Big[\phi\rho_y\Big(4\big((m-1)-1\big)\Big)\Big]\\
&= ...\\
&= \phi^m\rho_y(0)\\
\Big[\rho_y(4m) &= \phi^m \quad \text{for } m\geq 0\Big]
\end{aligned}$$

As we would expect from a stationary process, the autocorrelation decreases exponentially to 0 (ignoring the null terms in the autocorrelation function).

Exercise 4

1) By definition of the best linear forecast:

$$\forall k\in\{1,...,h\},\ \mathrm{Cov}(x_t-\mathrm{BLF}(x_t|x_{t-1},...,x_{t-h}),x_{t-k})=0$$

Using $\mathrm{BLF}(x_t|x_{t-1},...,x_{t-h})=b_{1h}x_{t-1}+\cdots+b_{hh}x_{t-h}$, this can be written as:

$$\forall k\in\{1,...,h\},\ \gamma(k)=b_{1h}\gamma(k-1)+...+b_{hh}\gamma(k-h)$$

We then get:

$$\begin{cases}\gamma(1)=b_{1h}\gamma(0)+...+b_{hh}\gamma(h-1)\\ \gamma(2)=b_{1h}\gamma(1)+b_{2h}\gamma(0)+...+b_{hh}\gamma(h-2)\\ \quad\vdots\\ \gamma(h)=b_{1h}\gamma(h-1)+...+b_{hh}\gamma(0)\end{cases}$$

$$\Leftrightarrow \begin{cases}\rho(1)=b_{1h}+b_{2h}\rho(1)+...+b_{hh}\rho(h-1)\\ \rho(2)=b_{1h}\rho(1)+b_{2h}+b_{3h}\rho(1)+...+b_{hh}\rho(h-2)\\ \quad\vdots\\ \rho(h)=b_{1h}\rho(h-1)+...+b_{hh}\end{cases}$$

$$\Leftrightarrow \underset{(h\times 1)}{\begin{pmatrix}\rho(1)\\ \rho(2)\\ \vdots\\ \rho(h)\end{pmatrix}} = \underset{(h\times h)}{\begin{pmatrix}1 & \rho(1) & \dots & \rho(h-1)\\ \rho(1) & 1 & \ddots & \vdots\\ \vdots & \ddots & \ddots & \rho(1)\\ \rho(h-1) & \dots & \rho(1) & 1\end{pmatrix}} \underset{(h\times 1)}{\begin{pmatrix}b_{1h}\\ b_{2h}\\ \vdots\\ b_{hh}\end{pmatrix}}$$

2) In this application, (x_t) is an MA(1) process with $x_t = \epsilon_t - \theta\epsilon_{t-1}$, where $|\theta| < 1$ and $\epsilon_t \sim WN(0,\sigma^2)$.

$$\gamma_x(h) = \begin{cases} \left(1+\theta^2\right)\sigma^2 & \text{if } h = 0 \\ -\theta\sigma^2 & \text{if } |h| = 1 \\ 0 & \text{if } |h| > 1 \end{cases} \quad \text{and} \quad \rho_x(h) \equiv \frac{\gamma_X(h)}{\gamma_X(0)} = \begin{cases} 1 & \text{if } h = 0 \\ -\frac{\theta}{1+\theta^2} & \text{if } |h| = 1 \\ 0 & \text{if } |h| > 1 \end{cases}$$

Denoting $\rho = \rho_x(1) = -\frac{\theta}{1+\theta^2}$, the general form of the previous equation is:

$$\underset{(h\times 1)}{\begin{pmatrix} \rho \\ 0 \\ \vdots \\ 0 \end{pmatrix}} = \underset{(h\times h)}{\begin{pmatrix} 1 & \rho & 0 & \dots & 0 \\ \rho & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \rho \\ 0 & \dots & 0 & \rho & 1 \end{pmatrix}} \underset{(h\times 1)}{\begin{pmatrix} b_{1h} \\ b_{2h} \\ \vdots \\ b_{hh} \end{pmatrix}}$$

<u>For $h = 1$:</u>

$$\rho = 1 \times b_{11} \Leftrightarrow \left[r(1) = b_{11} = \rho \right]$$

<u>For $h = 2$:</u>

$$\begin{pmatrix} \rho \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \begin{pmatrix} b_{12} \\ b_{22} \end{pmatrix}$$

We get:

$$\begin{aligned} b_{12} &= \frac{\rho}{1-\rho^2} \\ \left[r(2) \right. &= b_{22} = \left. \frac{-\rho^2}{1-\rho^2} \right] \end{aligned}$$

<u>For $h = 3$:</u>

$$\begin{pmatrix} \rho \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & \rho & 0 \\ \rho & 1 & \rho \\ 0 & \rho & 1 \end{pmatrix} \begin{pmatrix} b_{13} \\ b_{23} \\ b_{33} \end{pmatrix} \Leftrightarrow \begin{cases} \rho = b_{13} + \rho b_{23} \\ 0 = \rho b_{13} + b_{23} + \rho b_{33} \\ 0 = \rho b_{23} + b_{33} \end{cases} \Leftrightarrow \begin{cases} \rho = b_{13} - \frac{\rho^2}{1-\rho^2} b_{13} \\ -\rho b_{13} = b_{23} - \rho^2 b_{23} = (1-\rho^2) b_{23} \\ b_{33} = -\rho b_{23} \end{cases}$$

This yields:

$$\begin{aligned} b_{13} &= \frac{\rho}{1 - \frac{\rho^2}{1-\rho^2}} = \frac{\rho(1-\rho^2)}{1-2\rho^2} \\ b_{23} &= -\frac{\rho^2}{1-2\rho^2} \\ \left[r(3) \right. &= b_{33} = \left. \frac{\rho^3}{1-2\rho^2} \right] \end{aligned}$$

<u>For $h = 4$:</u>

$$\begin{pmatrix} \rho \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & \rho & 0 & 0 \\ \rho & 1 & \rho & 0 \\ 0 & \rho & 1 & \rho \\ 0 & 0 & \rho & 1 \end{pmatrix} \begin{pmatrix} b_{14} \\ b_{24} \\ b_{34} \\ b_{44} \end{pmatrix} \Leftrightarrow \begin{cases} \rho = b_{14} + \rho b_{24} \\ 0 = \rho b_{14} + b_{24} + \rho b_{34} \\ 0 = \rho b_{24} + b_{34} + \rho b_{44} \\ 0 = \rho b_{34} + b_{44} \end{cases} \Leftrightarrow \begin{cases} \rho = b_{14} + \rho \left(\frac{-\rho b_{14}}{1 - \frac{\rho^2}{1-\rho^2}} \right) \\ -\rho b_{14} = b_{24} - \frac{\rho^2}{1-\rho^2} b_{24} \\ -\rho b_{24} = b_{34}(1-\rho^2) \\ b_{44} = -\rho b_{34} \end{cases}$$

$$\rho = b_{14} - b_{14}\frac{\rho^2(1-\rho^2)}{(1-\rho^2)-\rho^2} \Leftrightarrow \rho = b_{14}\frac{1-2\rho^2-\rho^2(1-\rho^2)}{1-2\rho^2} \Leftrightarrow \rho = b_{14}\frac{1-3\rho^2+\rho^4}{1-2\rho^2} \Leftrightarrow b_{14} = \frac{\rho(1-2\rho^2)}{1-3\rho^2+\rho^4}$$

$$-\rho b_{14} = \left(1-\frac{\rho^2}{1-\rho^2}\right)b_{24} \Leftrightarrow -\rho b_{14} = \frac{1-2\rho^2}{1-\rho^2}b_{24} \Leftrightarrow b_{24} = \frac{-\rho(1-\rho^2)}{1-2\rho^2}b_{14}$$

$$b_{34} = \frac{-\rho}{1-\rho^2}b_{24} = \frac{-\rho}{1-\rho^2} \times \frac{-\rho^2(1-\rho^2)}{1-3\rho^2+\rho^4} = \frac{\rho^3}{1-3\rho^2+\rho^4}$$

$$\left[r(4) = b_{44} = \frac{-\rho^4}{1-3\rho^2+\rho^4}\right]$$

We can see that:

- $r(1)$ tends to 0 like ρ
- $r(2)$ tends to 0 like ρ^2
- $r(3)$ tends to 0 like ρ^3
- $r(4)$ tends to 0 like ρ^4

From this, we can conjecture that $r(h)$ decreases exponentially to 0 with h for an MA(1).

This is actually true for any stationary process (exactly like $\gamma(h)$ and $\rho(h)$) but the general demonstration is beyond the program of this course.