Quantitative macroeconomics - Problem Set 3

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1 Problem 1 The Bellman equation and its properties.

We simplify the problem as a choice of $\{k_{t+1}\}$:

$$i_t = k_{t+1} - (1-\delta)k_t$$

And then:

$$c_t \leq (1-\delta)k_t + z_t k_t^\theta - k_{t+1}$$

Now the problem simplifies to solving:

$$\max_{\{c_t, k_{t+1}\}} \sum_{t=0}^{T} \beta^t \frac{c_t^{1-\sigma} - 1}{1-\sigma}$$

$$s.t. \quad c_t \leq (1-\delta)k_t + z_t k_t^\theta - k_{t+1}$$

At the optimum the resource constraints will be binding so we can rewrite the program as :

$$\max_{\{k_{t+1}\}} \sum_{t=0}^{T} \beta^t \frac{\left((1-\delta)k_t + z_t k_t^\theta - k_{t+1}\right)^{1-\sigma} - 1}{1-\sigma}$$

$$\iff \max_{\{k_{t+1}\}} \sum_{t=0}^{T} \beta^t F(k_{t+1}, k_t) - 1 \quad \text{(SP)}$$

with $F(k_{t+1}, k_t) = \dfrac{\left((1-\delta)k_t + z_t k_t^\theta - k_{t+1}\right)^{1-\sigma} - 1}{1-\sigma}$

Let's define Γ, X, β and F as following:

$X = \left\{k_t \in X, k_t \in [0, max(k_0, k*)]\right\}$ set of possible state variable k_t

$\Gamma : X -> X$ feasible values of the next period capital for a given level of capital stock k.

The possible values of k_{t+1} have to be such that $k_t \geq 0$ and $c_t \geq 0$.

$$\begin{aligned}
&c_t \geq 0 \\
\iff &(1-\delta)k_t + z_t k_t^\theta - k_{t+1} \geq 0 \\
\iff &k_{t+1} \leq (1-\delta)k_t + z_t k_t^\theta
\end{aligned}$$

So $\Gamma = \left\{k_{t+1} \in \Gamma, k_{t+1} \in [0, (1-\delta)k_t + z_t k_t^\theta]\right\}$

$\beta > 0$ the constant discount factor

$F(k_{t+1}, k_t) = \dfrac{\left((1-\delta)k_t + z_t k_t^\theta - k_{t+1}\right)^{1-\sigma} - 1}{1-\sigma}$ with $F : X \times X- > \mathbb{R}$ the one-period return function which maps all feasible combination of today's and tomorrow's capital stock. Is F bounded ?

Than we can determine the functional equation:
$v(k_{t+1}) = sup_{\{k_{t+1} \in \Gamma(k_t)\}}[F(k_t, k_{t+1}) + \beta k_{t+1}], \quad \text{all} \quad k_t \in X$

1.1 The return function F is not bounded, but (FE) nevertheless maps C(X), the space of continuous bounded functions on R, into itself. (Make sure you understand why). (FE) defines a contraction mapping using Blackwell's sufficient conditions. (Again, make sure you understand why). Can you further characterise v given the functional forms of f and Γ?