ECONOMETRICS 2

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Exercises: session 6

ARIMA processes.

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Exercise 1: ARIMA process

Let us consider the ARIMA(1,1,1) process (x_t) defined by:

$$\forall t > 0,\ (1-L)(1-\phi L)x_t = \mu + (1-\theta L)\varepsilon_t$$

where:

. $(\varepsilon_t) \sim \mathrm{WN}(0, \sigma^2)$

. $|\phi| < 1$, $|\theta| < 1$ and $\phi \neq \theta$

. the initial condition is $Z_0 = (x_0, x_{-1}, \varepsilon_0)'$

. $\forall s > 0,\ \mathrm{Cov}(\varepsilon_s, Z_0) = 0$.

1. Define $m_t = EX_t$.
   - i) Give the recursive equation which is satisfied by m_t.
   - ii) When $\mu = 0$, show that $\forall a \in \mathbb{R}, \forall b \in \mathbb{R}, m_t = a + b\phi^t$ is a solution of this equation (it can be shown that it is indeed the general solution of this equation).
   - iii) When $\mu \neq 0$, show that $\forall a \in \mathbb{R}, \forall b \in \mathbb{R}, m_t = a + b\phi^t + \frac{\mu}{1-\phi}t$ is a solution of this equation (it can be shown that it is indeed the general solution of this equation).
2. Give the Wold representation of (X_t) and compute its coefficients explicitly.
   In what follows, this representation is denoted as
   $$(1-L)X_t = m + C(L)\varepsilon_t$$
   with: $C(L) = \sum_{k=0}^{+\infty} c_k L^k$, $c_0 = 1$, $\sum_{k=0}^{+\infty} |c_k| < +\infty$.
3.
   - i) Using the Wold representation, express X_{t+h} as a function of X_{t-1}, m, h and the ε_s's for any $h > 0$.
   - ii) Using (i), compute the impact of a shock ε_t on X_{t+h}.
   - iii) What is the limit of this impact when $h \to +\infty$? Compare with the results obtained from the Beveridge-Nelson decomposition.

Best linear forecast and innovation of an ARIMA process

The aim of this exercise is to see, on a simple example, how a best linear forecast can be computed for an ARIMA process, taking into account the fact that the process does not start at $t = -\infty$. Let us consider the ARIMA(1,1,1) process (x_t) defined by:

$$\forall t > 0, \ (1 - L)(1 - \phi L)x_t = (1 - \theta L)\varepsilon_t \qquad \text{(E)}$$

where:

. $(\varepsilon_t) \sim \mathrm{WN}(0, \sigma^2)$

. $|\phi| < 1$ and $|\theta| < 1$

. the initial condition is $Z_0 = (x_0, x_{-1}, \varepsilon_0)'$

. $\forall s > 0, \ \mathrm{Cov}(\varepsilon_s, Z_0) = 0$.

The information set which is used to compute best linear forecasts is

$$I_t = \{x_t, x_{t-1}, \dots, x_1, Z_0\} = \{x_t, x_{t-1}, \dots, x_1, x_0, x_{-1}, \varepsilon_0\}.$$

1. Denote $z_t = (1 - L)(1 - \phi L)x_t = x_t - (\phi + 1)x_{t-1} + \phi x_{t-2}$.
   i) Using equation (E) for $s = 1, \dots, t$, prove that:
   $$\varepsilon_t = z_t + \theta z_{t-1} + \cdots + \theta^{t-1} z_1 + \theta^t \varepsilon_0.$$
   ii) Deduce from (i) that: $\forall t > 0, \ BLF(\varepsilon_t | I_t) = \varepsilon_t$.

2. i) Denote $y_t = (1 - L)x_t = x_t - x_{t-1}$. Write x_t as a function of $x_0, y_1, \dots y_t$.
   ii) Using equation (E) for $s = 1, \dots, t$, prove that:
   $$y_t = \phi^t y_0 + \sum_{s=0}^{t-1} \phi^s(\varepsilon_{t-s} - \theta\varepsilon_{t-s-1}) = \phi^t y_0 + \sum_{s=0}^{t} b_s \varepsilon_{t-s}$$
   with $b_0 = 1$, $b_s = \phi^s - \theta\phi^{s-1}$ for $s = 1, \dots, t-1$, and $b_t = -\theta\phi^{t-1}$.
   iii) Using (i) and (ii), prove that x_t can be written as a linear function of $\varepsilon_t, \dots, \varepsilon_1, Z_0$ (you don't need to compute explicitly this function).
   iv) Using (iii), prove that: $\forall k > 0, \ \mathrm{Cov}(\varepsilon_t, x_{t-k}) = 0$.
   v) Prove that $BLF(\varepsilon_t | I_{t-1}) = 0$.

3. Deduce from (1), (2) and equation (E) that $\varepsilon_t = x_t - BLF(x_t | I_{t-1})$ so that ε_t can be considered as the innovation of x_t in the framework which is considered here.

4. Show that $BLF(x_{t+1} | I_t)$ can be written as a function of $x_t, \dots, x_1, Z_0$ (don't compute it entirely). How would you compute $BLF(x_{t+h} | I_t)$ for $h \geq 2$?