M1 APE, Econometrics 2
Exercises: session 2
MA processes

2018-2019

Exercise 1

1) As $Z_t \sim WN$, (X_t) is an MA(2) process, by definition. Like any other finite order MA process, it is stationary. Thus, it has an autocovariance (resp. autocorrelation) function, which can be denoted as $\gamma_X(h)$ (resp. $\rho_X(h)$) and which is an even function (by property).

$\forall t \in \mathbb{Z}, \forall h \in \mathbb{Z}, \gamma_X(h) \equiv \mathrm{Cov}(X_t, X_{t-h}) = \mathrm{Cov}(Z_t + \theta Z_{t-2}, Z_{t-h} + \theta Z_{t-h-2})$

- <u>h=0:</u> $\gamma_X(0) = \mathrm{Cov}(X_t, X_{t-0}) = \mathrm{Cov}(Z_t, Z_t) + \theta^2 \mathrm{Cov}(Z_{t-2}, Z_{t-2}) = \left(1+\theta^2\right)V(Z_t) = 1+\theta^2$
- <u>|h|=1:</u> $\gamma_X(1) = \gamma_X(-1) = \mathrm{Cov}(X_t, X_{t-1}) = \mathrm{Cov}(Z_t + \theta Z_{t-2}, Z_{t-1} + \theta Z_{t-3}) = 0$
- <u>|h|=2:</u> $\gamma_X(2) = \gamma_X(-2) = \mathrm{Cov}(X_t, X_{t-2}) = \mathrm{Cov}(Z_t + \theta Z_{t-2}, Z_{t-2} + \theta Z_{t-4}) = \theta V(Z_{t-2}) = \theta$
- <u>|h|>2:</u> $\gamma_X(h) = 0$

$$\gamma_X(h) = \begin{cases} 1+\theta^2 = 1.64 & \text{if } h=0 \\ \theta = 0.8 & \text{if } |h|=2 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \rho_X(h) \equiv \frac{\gamma_X(h)}{\gamma_X(0)} = \begin{cases} 1 & \text{if } h=0 \\ \frac{\theta}{1+\theta^2} = \frac{0.8}{1.64} & \text{if } |h|=2 \\ 0 & \text{otherwise} \end{cases}$$

We recognize the classic forms of the autocovariance and autocorrelation functions of an MA(2), like in lesson 3, slides 13-17 (here: $\theta_2 = \theta$, $\theta_1 = 0$ and $\sigma^2 = 1$).

<u>Reminders:</u>
- A finite order MA(q) process ($q < \infty$) is always stationary.
- The autocovariance $\gamma(h)$ and the autocorrelation $\rho(h)$ of an MA(q) process are null for $|h| > q$.

2) <u>Reminder:</u> $V\left(\sum_{i=1}^n X_i\right) = \sum_{i=1}^n V(X_i) + 2\sum_{1\le i<j\le n} \mathrm{Cov}(X_i, X_j)$

$$\begin{aligned}
V\left(\frac{1}{4}\sum_{i=1}^4 X_i\right) &= \left(\frac{1}{4}\right)^2 V\left(\sum_{i=1}^4 X_i\right) \\
&= \frac{1}{16}\big(V(X_1)+V(X_2)+V(X_3)+V(X_4)\big) \\
&\quad + \frac{2}{16}\big(\mathrm{Cov}(X_1,X_2)+\mathrm{Cov}(X_1,X_3)+\mathrm{Cov}(X_1,X_4)+\mathrm{Cov}(X_2,X_3)+\mathrm{Cov}(X_2,X_4)+\mathrm{Cov}(X_3,X_4)\big) \\
&= \frac{1}{16}4\gamma_X(0) + \frac{1}{8}\big(\gamma_X(1)+\gamma_X(2)+\gamma_X(3)+\gamma_X(1)+\gamma_X(2)+\gamma_X(1)\big) \qquad \textit{as } (X_t) \textit{ is stationary} \\
&= \frac{1}{4}\gamma_X(0) + \frac{3}{8}\underbrace{\gamma_X(1)}_{\substack{=0 \\ \text{according to} \\ \text{question 1}}} + \frac{1}{4}\gamma_X(2) + \frac{1}{8}\underbrace{\gamma_X(3)}_{\substack{=0 \\ (X_t)\text{ is} \\ \text{an MA(2)}}} \\
&= \frac{1}{4}\gamma_X(0) + \frac{1}{4}\gamma_X(2) \\
V\left(\frac{1}{4}\sum_{i=1}^4 X_i\right) &= \frac{1}{4}\left(1+\theta+\theta^2\right)
\end{aligned}$$

Hence, for $\theta = 0.8$, $V\left(\frac{1}{4}\sum_{i=1}^{4} X_i\right) = \frac{1}{4}\left(1 + 0.8 + 0.8^2\right) = \frac{1}{4} \times 2.44 = 0.61$

3) For $\theta = -0.8$, $V\left(\frac{1}{4}\sum_{i=1}^{4} X_i\right) = \frac{1}{4}\left(1 + (-0.8) + (-0.8)^2\right) = \frac{1}{4} \times 0.84 = 0.21 < 0.61$
From the two previous questions, we know that $\gamma_X(h)$ is positive for $h = 0$, equal to θ if $|h| = 2$ (and hence has the same sign), and is null otherwise. Thus, when $\theta = -0.8 < 0$, some covariances of the variables within the sample are negative (namely $\text{Cov}(X_1, X_3)$ and $\text{Cov}(X_2, X_4)$), which reduces the variance of the sample mean.

Exercise 2

1) As $u_t \sim WN$, (y_t) is an MA(2) process, by definition.

2)a) $\forall t \in \mathbb{Z}, \forall h \in \mathbb{Z}, \text{Cov}(x_t, x_{t-h}) = \text{Cov}(u_{3t} - \theta_1 u_{3t-1} - \theta_2 u_{3t-2}, u_{3(t-h)} - \theta_1 u_{3(t-h)-1} - \theta_2 u_{3(t-h)-2})$
It is obvious that $\gamma_x(h) = \begin{cases} (1+\theta_1^2+\theta_2^2)\sigma_u^2 & \text{if } h = 0 \\ 0 & \text{if } h \neq 0 \end{cases}$ which yields $\rho_x(h) = \begin{cases} 1 & \text{if } h = 0 \\ 0 & \text{if } h \neq 0 \end{cases}$

b) (x_t) is a white noise by definition since it has:

- zero mean $\left(E(x_t) = E(y_{3t}) = E(u_{3t}) - \theta_1 E(u_{3t-1}) - \theta_2 E(u_{3t-2}) = 0\right)$
- finite and constant variance
- no autocorrelation

3) $\forall t \in \mathbb{Z}, \forall h \in \mathbb{Z}$,

$$\begin{aligned}\text{Cov}(z_t, z_{t-h}) &= \text{Cov}\left(\frac{1}{3}\left(y_{3t-2} + y_{3t-1} + y_{3t}\right), \frac{1}{3}\left(y_{3(t-h)-2} + y_{3(t-h)-1} + y_{3(t-h)}\right)\right) \\ &= \frac{1}{9}\text{Cov}\left(y_{3t-2} + y_{3t-1} + y_{3t}, y_{3t-3h-2} + y_{3t-3h-1} + y_{3t-3h}\right)\end{aligned}$$

Developing this expression yields:

$$\text{Cov}\left(z_t, z_{t-h}\right) = \frac{1}{9}\left[\begin{array}{c} \text{Cov}\left(y_{3t-2}, y_{3t-3h-2}\right) + \text{Cov}\left(y_{3t-2}, y_{3t-3h-1}\right) + \text{Cov}\left(y_{3t-2}, y_{3t-3h}\right) \\ +\text{Cov}\left(y_{3t-1}, y_{3t-3h-2}\right) + \text{Cov}\left(y_{3t-1}, y_{3t-3h-1}\right) + \text{Cov}\left(y_{3t-1}, y_{3t-3h}\right) \\ \text{Cov}\left(y_{3t}, y_{3t-3h-2}\right) + \text{Cov}\left(y_{3t}, y_{3t-3h-1}\right) + \text{Cov}\left(y_{3t}, y_{3t-3h}\right) \end{array}\right]$$

Since (y_t) is stationary as a finite order MA process: $\forall t, h$, $\text{Cov}(y_t, y_{t-h}) = \gamma_y(h) = \gamma_y(-h)$. Thus:

$$\text{Cov}(z_t, z_{t-h}) = \frac{1}{9}\left[\begin{array}{c} \gamma_y(3h) + \gamma_y(3h-1) + \gamma_y(3h-2) \\ +\gamma_y(3h+1) + \gamma_y(3h) + \gamma_y(3h-1) \\ +\gamma_y(3h+2) + \gamma_y(3h+1) + \gamma_y(3h) \end{array}\right]$$

Finally:

$$\forall t, h\ \text{Cov}(z_t, z_{t-h}) = \frac{1}{9}\left[3\gamma_y(3h) + 2\gamma_y(3h+1) + \gamma_y(3h+2) + 2\gamma_y(3h-1) + \gamma_y(3h-2)\right]$$

All the γ_y terms are finite and do not depend on t. Hence, so does $\text{Cov}\left(z_t, z_{t-h}\right)$.
$\Rightarrow (z_t)$ is stationary.

♣

We can now easily compute the autocovariances of (z_t) depending on the $\gamma_y(h)$ terms.

- <u>h=0:</u> $\gamma_z(0) \equiv \text{Cov}(z_t, z_{t-0}) = \frac{1}{3}\gamma_y(0) + \frac{2}{9}\gamma_y(1) + \frac{1}{9}\gamma_y(2) + \frac{2}{9}\gamma_y(-1) + \frac{1}{9}\gamma_y(-2) = \frac{1}{3}\gamma_y(0) + \frac{4}{9}\gamma_y(1) + \frac{2}{9}\gamma_y(2)$
- <u>h=1:</u> $\gamma_z(1) \equiv \text{Cov}(z_t, z_{t-1}) = \frac{1}{3}\gamma_y(3) + \frac{2}{9}\gamma_y(4) + \frac{1}{9}\gamma_y(5) + \frac{2}{9}\gamma_y(2) + \frac{1}{9}\gamma_y(1) = \frac{1}{9}\gamma_y(1) + \frac{2}{9}\gamma_y(2)$,
since (y_t) is an MA(2) process and $\gamma_y(h) = 0$ for $|h| > 2$.

- <u>h=2:</u> $\gamma_z(2) \equiv \text{Cov}(z_t, z_{t-2}) = \frac{1}{3}\gamma_y(6) + \frac{2}{9}\gamma_y(7) + \frac{1}{9}\gamma_y(8) + \frac{2}{9}\gamma_y(5) + \frac{1}{9}\gamma_y(4) = 0$

- <u>|h|>2:</u> $\gamma_z(h) = 0$

$$\gamma_z(h) = \begin{cases} \frac{1}{3}\gamma_y(0) + \frac{4}{9}\gamma_y(1) + \frac{2}{9}\gamma_y(2) & \text{if } h = 0 \\ \frac{1}{9}\gamma_y(1) + \frac{2}{9}\gamma_y(2) & \text{if } |h| = 1 \\ 0 & \text{if } |h| > 1 \end{cases}$$

This yields an easy computation of the autocorrelations of (z_t) in terms of $\gamma_y(h)$:

$$\rho_z(h) \equiv \frac{\gamma_z(h)}{\gamma_z(0)} \begin{cases} 1 & \text{if } h = 0 \\ \frac{\gamma_y(1)+2\gamma_y(2)}{3\gamma_y(0)+4\gamma_y(1)+2\gamma_y(2)} & \text{if } |h| = 1 \\ 0 & \text{if } |h| > 1 \end{cases}$$

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Finally, we can compute the $\gamma_y(h)$ terms in order to compute $\rho_z(1)$ explicitly.

$\forall t \in \mathbb{Z}, \forall h \in \mathbb{Z}, \gamma_y(h) = \text{Cov}(y_t, y_{t-h}) = \text{Cov}(u_t - \theta_1 u_{t-1} - \theta_2 u_{t-2}, u_{t-h} - \theta_1 u_{t-h-1} - \theta_2 u_{t-h-2})$

- <u>h=0:</u> $\gamma_y(0) = \text{Cov}(u_t - \theta_1 u_{t-1} - \theta_2 u_{t-2}, u_t - \theta_1 u_{t-1} - \theta_2 u_{t-2}) = \left(1 + \theta_1^2 + \theta_2^2\right)\sigma_u^2$

- <u>|h|=1:</u> $\gamma_y(1) = \gamma_y(-1) = \text{Cov}(u_t - \theta_1 u_{t-1} - \theta_2 u_{t-2}, u_{t-1} - \theta_1 u_{t-2} - \theta_2 u_{t-3})$
$= -\theta_1 V(u_{t-1}) + \theta_1\theta_2 V(u_{t-2}) = (\theta_2 - 1)\theta_1\sigma_u^2$

- <u>|h|=2:</u> $\text{Cov}(u_t - \theta_1 u_{t-1} - \theta_2 u_{t-2}, u_{t+2} - \theta_1 u_{t+1} - \theta_2 u_t) = -\theta_2\sigma_u^2$

- <u>|h|>2:</u> $\gamma_y(h) = 0$ since (y_t) is an MA(2)

$$\Rightarrow \gamma_y(h) = \begin{cases} \left(1 + \theta_1^2 + \theta_2^2\right)\sigma_u^2 & \text{if } h = 0 \\ (\theta_2 - 1)\theta_1\sigma_u^2 & \text{if } |h| = 1 \\ -\theta_2\sigma_u^2 & \text{if } |h| = 2 \\ 0 & \text{if } |h| > 2 \end{cases}$$

So that $\rho_z(h)$ can be rewritten: $\begin{cases} 1 & \text{if } h = 0 \\ \frac{\theta_1\theta_2 - \theta_1 - 2\theta_2}{3 - 4\theta_1 - 2\theta_2 + 4\theta_1\theta_2 + 3\theta_1^2 + 3\theta_2^2} & \text{if } |h| = 1 \\ 0 & \text{if } |h| > 1 \end{cases}$

Exercise 3

1) $y_t = x_t + e_t = u_t + \theta u_{t-1} + e_t$
(y_t) is the sum of two independent stationary processes, so it is stationary too.

Indeed:

- $\forall t \in \mathbb{Z},\ Ey_t = Ex_t + Ee_t = 0$: does not depend on t

- as (e_t) is independent of (u_t), it is also independent of (x_t) so that:

  $\forall t \in \mathbb{Z},\ \forall h \in \mathbb{Z},\ \text{Cov}(y_t, y_{t-h}) = \text{Cov}(x_t + e_t, x_{t-h} + e_{t-h}) = \text{Cov}(x_t, x_{t-h}) + \text{Cov}(e_t, e_{t-h}) = \gamma_x(h) + \gamma_e(h)$

  which does not depend on t.

Thus, the autocovariance function of (y_t) exists and is an even function.

$\forall t \in \mathbb{Z}, \forall h \in \mathbb{Z}, \gamma_y(h) = \text{Cov}(y_t, y_{t-h}) = \text{Cov}(u_t + \theta u_{t-1} + e_t, u_{t-h} + \theta u_{t-h} + e_{t-h})$

- <u>h=0:</u> $\gamma_y(0) = \text{Cov}(u_t + \theta u_{t-1} + e_t, u_t + \theta u_{t-1} + e_t)$
$$\begin{aligned} &= \left(1+\theta^2\right)V(u_t) + V(e_t) \quad \text{because } \text{Cov}(e_t, u_s) = 0\ \forall (t,s) \\ &= \left(1 + \left(\tfrac{1}{\sqrt{2}}\right)^2\right) \times 1 + \tfrac{1}{2} \\ &= 2 \end{aligned}$$
- <u>|h|=1:</u> $\gamma_y(1) = \gamma_y(-1) = \text{Cov}(u_t + \theta u_{t-1} + e_t, u_{t-1} + \theta u_{t-2} + e_{t-1}) = -\theta V(u_{t-1}) = -\theta = -\frac{1}{\sqrt{2}}$
- <u>|h|>1:</u> $\gamma_y(h) = 0$

2) (z_t) is an MA(1) so it is stationary. Its autocovariance function can be denoted as $\gamma_z(h)$ and is an even function.

$\forall t \in \mathbb{Z}, \forall h \in \mathbb{Z}, \gamma_z(h) = \text{Cov}(z_t, z_{t-h}) = \text{Cov}(\varepsilon_t - \psi\varepsilon_{t-1}, \varepsilon_{t-h} - \psi\varepsilon_{t-h-1})$

- <u>h=0:</u> $\gamma_z(0) = V(\varepsilon_t - \psi\varepsilon_{t-1}) = \left(1+\psi^2\right)\sigma_\varepsilon^2$
- <u>|h|=1:</u> $\gamma_z(1) = \gamma_z(-1) = \text{Cov}(\varepsilon_t - \psi\varepsilon_{t-1}, \varepsilon_{t-1} - \psi\varepsilon_{t-2}) = -\psi\sigma_\varepsilon^2$
- <u>|h|>1:</u> $\gamma_z(h) = 0$

As $\gamma_y(h) = \gamma_z(h) = 0$ for $|h| > 1$, the two autocovariance functions will be equal iff the two following statements hold:

$$\left(1+\psi^2\right)\sigma_\varepsilon^2 = 2 \quad (1)$$
$$-\psi\sigma_\varepsilon^2 = -\frac{1}{\sqrt{2}} \quad (2)$$

Computing the ratio $\frac{(2)}{(1)}$, we obtain:
$$\frac{\psi}{1+\psi^2} = \frac{1}{2\sqrt{2}}$$
or, equivalently:
$$\psi^2 - 2\sqrt{2}\psi + 1 = 0$$
As $\Delta = \left(-2\sqrt{2}\right)^2 - 4 \times 1 \times 1 = 8 - 4 = 4 > 0$, this last equations has 2 roots given by:

$$\psi_1 = \frac{2\sqrt{2}-2}{2} = \sqrt{2}-1 \text{ and } \psi_2 = \frac{2\sqrt{2}+2}{2} = \sqrt{2}+1$$

As we suppose $|\psi| < 1$, the only acceptable solution is $\psi_1 = \sqrt{2}-1$. Equation (2) then yields:

$$\sigma_\varepsilon^2 = \frac{1}{\sqrt{2}\psi} = \frac{1}{\sqrt{2}(\sqrt{2}-1)}$$

We can check that $\sqrt{2}-1 > 0$ so $\sigma_\varepsilon^2 > 0$ as expected for a variance.

A weakly stationary process is entirely defined by its mean and its autocovariance function.
Here, for $\psi = \sqrt{2}-1$ and $\sigma_\varepsilon^2 = \frac{1}{\sqrt{2}(\sqrt{2}-1)}$, we have:

- $\forall t \in \mathbb{Z}, E(y_t) = E(z_t) = 0$
- $\forall h \in \mathbb{Z}, \gamma_y(h) = \gamma_z(h)$

Thus the two processes (y_t) and (z_t) can be considered as identical: the MA equations satisfied by (z_t) give an MA representation of (y_t).

Exercise 4

1) Using equation ($\mathscr{E}$) from t to $t-h$ (with $h \geq 0$) and multiplying each line by a growing power of θ reveals a telescoping sum:

$$\begin{aligned} x_t &= \epsilon_t & -\theta\epsilon_{t-1} \\ \theta x_{t-1} &= \theta\epsilon_{t-1} & -\theta^2\epsilon_{t-2} \\ &\vdots \\ \theta^h x_{t-h} &= \theta^h\epsilon_{t-h} & -\theta^{h+1}\epsilon_{t-h-1} \end{aligned}$$

Summing each side then yields the desired outcome:

$$\forall h \geq 0, \sum_{k=0}^{h} \theta^k x_{t-k} = \epsilon_t - \theta^{h+1}\epsilon_{t-h-1}$$

2) $V\big(\theta^{h+1}\epsilon_{t-h-1}\big) = \big(\theta^{h+1}\big)^2 V(\epsilon_{t-h-1}) = \theta^{2(h+1)}\sigma^2$.

Since $|\theta| < 1$, $\lim_{h\to\infty} \theta^{2(h+1)}\sigma^2 = 0$ so that $\lim_{h\to\infty} V\big(\theta^{h+1}\epsilon_{t-h-1}\big) = 0$.
As $E\big(\theta^{h+1}\epsilon_{t-h-1}\big) = 0$, we obtain that:

$$\theta^{h+1}\epsilon_{t-h-1} \text{ converges to 0 in quadratic mean.}$$

We'll denote this as: $\lim_{h\to\infty} \theta^{h+1}\epsilon_{t-h-1} = 0$.

Remark: in the whole course, limits for random variables are limits in quadratic mean.

3)i) We will start by computing $E\big((\theta^h x_{t-h})^2\big)$.

$$E\big((\theta^h x_{t-h})^2\big) = E\big(\theta^{2h} x_{t-h}^2\big) = \theta^{2h} E\big(x_{t-h}^2\big) = \theta^{2h}\big(V(x_{t-h}) + E(x_{t-h})^2\big)$$

Considering that

- $\forall t \in \mathbb{Z},\ E(x_t) = E(\epsilon_t) - \theta E(\epsilon_{t-1}) = 0$
- $\forall t \in \mathbb{Z},\ V(x_t) = V(\epsilon_t - \theta\epsilon_{t-1}) = V(\epsilon_t) + (-\theta)^2 V(\epsilon_{t-1}) - 2\theta\,\mathrm{Cov}(\epsilon_t, \epsilon_{t-1}) = (1+\theta^2)\sigma^2$

we get

$$E\big((\theta^h x_{t-h})^2\big) = \theta^{2h}(1+\theta^2)\sigma^2.$$

This immediately yields

$$\sum_{h=0}^{+\infty} E\big((\theta^h x_{t-h})^2\big) = (1+\theta^2)\sigma^2 \sum_{h=0}^{+\infty} \theta^{2h} = \frac{(1+\theta^2)\sigma^2}{1-\theta^2} \qquad \text{(geometric sequence)}$$

which is obviously finite as σ^2 is finite (part of the definition of a white noise), and $|\theta| < 1$.

ii) We know that:

- $\forall h \geq 0,\ \sum_{k=0}^{h} \theta^k x_{t-k} = \epsilon_t - \theta^{h+1}\epsilon_{t-h-1}$ (from question 1)
- $\lim_{h\to\infty} \theta^{h+1}\epsilon_{t-h-1} = 0$ (from question 2)

Thus, $\sum_{k=0}^{+\infty} \theta^k x_{t-k} = \epsilon_t$ (admitting that 3)i) is sufficient to prove that $\sum_{k=0}^{+\infty} \theta^k x_{t-k}$ is well defined).

Remark: This is the AR(∞) representation of the MA(1) process (x_t).
See the course, and the exercises sessions 4 and 5.