ECONOMETRICS 2

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Exercises: session 1

Stationary processes: introduction

Exercise # 1

For each of the following processes, check whether it is stationary or not and give its autocorrelation function when it is stationary:

1. (X_t) is an i.i.d. noise and $E(X_t^2) = \sigma^2 < \infty$
2. (X_t) is a sequence of uncorrelated random variables, each of them with zero mean and variance σ^2
3. (S_t) is a random walk with zero mean, starting at zero, which is obtained by cumulatively summing of (or "integrating") i.i.d. random variables.
   Thus $S_t, t = 0, 1, 2, ...$ is obtained by defining S_0 such that $ES_0 = 0$ and
   $$S_t = S_0 + X_1 + X_2 + ... + X_t \text{ for } t = 1, 2, \ldots$$
   where (X_t) is an i.i.d. white noise which is uncorrelated with S_0: $\forall t, \mathrm{Cov}(X_t, S_0) = 0$.
4. (X_t) is a first-order moving average or MA(1) process defined by the equation
   $$\forall t \in \mathbb{Z},\ X_t = Z_t + \theta Z_{t-1}$$
   where $(Z_t) \sim \mathrm{WN}(0, \sigma^2)$ and θ is a real-valued constant.
5. $\forall t \in \mathbb{Z},\ X_t = \varepsilon_t - \varepsilon_{t-1}$, where $(\varepsilon_t)_{t\in\mathbb{Z}}$ is a white noise with variance $\sigma^2 > 0$.
6. $(X_t)_{t\in\mathbb{Z}}$ is defined by $\forall t \in \mathbb{Z}, X_t = a + b\varepsilon_t + c\varepsilon_{t-1}$ where $(a, b, c) \in \mathbb{R}^3$ and $(\varepsilon_t)_{t\in\mathbb{Z}}$ is a white noise with variance $\sigma^2 > 0$.
7. (X_t) is such that $X_t - X_{t-1} = \varepsilon_t$, where $(\varepsilon_t)_{t\in\mathbb{Z}}$ is a white noise with variance $\sigma^2 > 0$ and $\forall t > 0, \varepsilon_t \perp X_0$.

Exercise # 2

Let (Y_t) be a stationary process with mean zero and let a and b be constants.

1. If $X_t = a + bt + s_t + Y_t$, where s_t is a seasonal component with period 12, show that $\Delta_{12} X_t = (1 - L^{12})X_t$ is stationary and express its autocovariance function in terms of the autocovariance function of (Y_t).
2. If $X_t = (a + bt)s_t + Y_t$, where s_t is a seasonal component with period 12, show that $\Delta_{12}^2 X_t = (1 - L^{12})^2 X_t$ is stationary and express its autocovariance function in terms of the autocovariance function of (Y_t).

Exercise # 3

Show that the autoregressive equation

$$\forall t \in \mathbb{Z},\ X_t = \phi X_{t-1} + Z_t$$

where(Z_t) is a WN(0,σ^2) and $\phi = \pm 1$, has *no stationary solutions*.
HINT: Suppose it does exist a stationary solution (X_t) and use the autoregressive equation to derive an expression for the variance of $X_t - \phi^{n+1} X_{t-n-1}$ which contradicts the stationarity assumption.