ECONOMETRICS 2

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Exercises : session 7

Stationary VAR processes and forecasts.

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Exercise 1 : stationary VAR process and innovation process

The aim of this exercise is to check, on a very simple example, properties which have been stated and used during the course.

Let us consider the following VAR model which is denoted by (M) and defined by :

$$\begin{pmatrix} x_t \\ y_t \end{pmatrix} = \begin{pmatrix} 2/3 & -1/6 \\ 1/3 & 1/6 \end{pmatrix} \begin{pmatrix} x_{t-1} \\ y_{t-1} \end{pmatrix} + \begin{pmatrix} u_t \\ v_t \end{pmatrix}$$

This equation can be also denoted as : $X_t = \Phi X_{t-1} + \varepsilon_t$ or $\Phi(L)X_t = \varepsilon_t$ with $\Phi(L) = I_2 - \Phi L$.

(ε_t) is supposed to be a white noise, with covariance matrix $\Sigma = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$.

1. Write $\Phi(L)$ explicitly .
2. Denote by $\tilde{\Phi}(L)$ the transposed of $\Phi(L)$'s comatrix.
   Check that $\tilde{\Phi}(L) = \begin{pmatrix} 1 - \frac{1}{6}L & -\frac{1}{6}L \\ \frac{1}{3}L & 1 - \frac{2}{3}L \end{pmatrix}$ and that $\tilde{\Phi}(L)\Phi(L) = \det \Phi(L) I_2$.
3. Check that $\det \Phi(L) = (1 - \frac{1}{2}L)(1 - \frac{1}{3}L)$ so that $(\det \Phi(L))^{-1} = \sum_{k=0}^{\infty} a_k L^k$, with $a_0 = 1$ and $\sum_{k=0}^{\infty} |a_k| < \infty$ (don't compute the a_k's, $k > 0$).
4. i) Using (3), prove that X_t can be written as $X_t = \sum_{k=0}^{\infty} A_k \varepsilon_{t-k}$ where the matrices A_k's can be written as functions of the coefficients a_k's.

   ii) If $a_{ij}^{(k)}$, $i, j \in \{1, 2\}$ is the general term of A_k, compute $a_{11}^{(k)}$ as a function of the a_k's and prove that $\sum_{k=0}^{\infty} |a_{11}^{(k)}| < \infty$.

   iii) More generally, using the same kind of computations, prove that :
   $\exists M \ / \ \forall i, j \in \{1, 2\}, \ \sum_{k=0}^{\infty} |a_{ij}^{(k)}| < M \sum_{k=0}^{\infty} |a_k|$.

   iv) If $\|A\|$ is defined by $\|A\| = \max_{i,j} |a_{ij}|$, prove that $\sum_{k=0}^{\infty} \|A_k\| < \infty$.

   v) Explain why $X_t = \sum_{k=0}^{\infty} A_k \varepsilon_{t-k}$ indeed defines a stationary process which is a solution of (M).
5. i) Using the previous results, prove that, for any $j > 0$, the following results hold :

   $$\text{Cov}(u_t, x_{t-j}) = 0, \ \text{Cov}(u_t, y_{t-j}) = 0, \ \text{Cov}(v_t, x_{t-j}) = 0, \ \text{Cov}(v_t, y_{t-j}) = 0$$

   ii) Using (i), compute $BLF(x_t|\underline{X_{t-1}})$, and $BLF(y_t|\underline{X_{t-1}})$.

   iii) Prove that (ε_t) is the innovation process of (X_t).

   iv) Do you think that $BLF(x_t|\underline{X_{t-1}}) = BLF(x_t|\underline{X_{t-1}}, y_t)$?

Exercise 2 : forecasting with a stationary VAR

In this exercise, we use the same VAR process (X_t) as in #1 :

$$\begin{pmatrix} x_t \\ y_t \end{pmatrix} = \begin{pmatrix} 2/3 & -1/6 \\ 1/3 & 1/6 \end{pmatrix} \begin{pmatrix} x_{t-1} \\ y_{t-1} \end{pmatrix} + \begin{pmatrix} u_t \\ v_t \end{pmatrix} \qquad (M)$$

1. In this question, we suppose that $\varepsilon_t \sim \mathcal{N}(0, \Sigma)$ and that $\Sigma = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$.

   i) Give a confidence region for X_{t+1} at confidence level $(1-\alpha)$, based on the optimal forecast at time t, $X^*_{t+1|t} = BLF(X_{t+1}|\underline{X_t})$.

   ii) Give a confidence interval for x_{t+1} at confidence level $(1-\alpha)$, based on the optimal forecast at time t, $x^*_{t+1|t} = BLF(x_{t+1}|\underline{X_t})$.

   iii) Compute the best linear forecast of X_{t+2} at time t : $X^*_{t+2|t} = BLF(X_{t+2}|\underline{X_t})$.
   Compute the associated forecast error, and its covariance matrix : just give the formula of this covariance matrix but don't compute it entirely.

   iv) Using (iii), give a confidence region for X_{t+2} at confidence level $(1-\alpha)$, based on $X^*_{t+2|t}$.

   v) Compute explicitly the forecast error $x_{t+2} - x^*_{t+2|t}$ associated to x_{t+2} where $x^*_{t+2|t} = BLF(x_{t+2}|\underline{X_t})$, and compute its variance entirely.
   Give a confidence interval for x_{t+2} at confidence level $(1-\alpha)$, based on $x^*_{t+2|t}$.

2. i) Prove that $(1-\frac{1}{6}L)(1-\frac{2}{3}L)x_t + \frac{1}{18}L^2 x_t = (1-\frac{1}{6}L)u_t - \frac{1}{6}Lv_t$.

   Hint : you can either multiply the first equation by $(1-\frac{1}{6}L)$ and then use the second equation, or multiply the entire system by $\tilde{\Phi}(L)$: use both methods in order to fix them !

   ii) Let $w_t = (1-\frac{1}{6}L)u_t - \frac{1}{6}v_t$. Prove that (w_t) is an MA(1) process. If $w_t = (1-\theta L)\eta_t$ is its canonical representation, how would you compute θ and σ^2_η (don't compute them) ?
   What kind of process is (x_t) and what is its canonical representation ?

   iii) How would you compute $BLF(x_t|\underline{x_{t-1}})$ as a function of the x_{t-k}'s (you are not asked to compute it entirely) ? Is it equal to $BLF(x_t|\underline{X_{t-1}})$?
   Compare the forecast error variances associated to these two forecasts.

3. i) Find $a \in \mathbb{R}$ such that $\mathrm{Cov}(u_t, v_t - au_t) = 0$.

   ii) If $A = \begin{pmatrix} 1 & 0 \\ -a & 1 \end{pmatrix}$, explain why the initial model (M) can be equivalently written as

   $$AX_t = A\Phi X_{t-1} + \eta_t \qquad (\tilde{M})$$

   where $\eta_t = A\varepsilon_t$ is denoted as $\eta_t = (u_t \;\; w_t)'$.

   iii) Use model $(\tilde{M})$ to compute $BLF(y_t|\underline{X_{t-1}}, x_t)$. Does it coincide with $BLF(y_t|\underline{X_{t-1}})$?