**ECONOMETRICS 2**

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**Exercises: session 2**

**MA processes**

## Exercise # 1

Let $(X_t)$ be the moving-average process of order 2 defined by

$$X_t = Z_t + \theta Z_{t-2}$$

where $(Z_t)$ is WN(0,1)

1. Compute the autocovariance and autocorrelation functions for this process when $\theta = 0.8$.

2. ~~Compute the variance~~ of the sample mean $(X_1 + X_2 + X_3 + X_4)/4$ when $\theta = 0.8$

3. Repeat the previous question when $\theta = -0.8$ and compare your answer with the result obtained in 2).

## Exercise # 2

Let $(y_t)$ be a time-process defined by:

$$y_t = (1 - \theta_1 L - \theta_2 L^2) u_t$$

where $(u_t)$ is a WN(0,$\sigma_u^2$).

1. What kind of process is $(y_t)$ ?

2. We build the quarterly time series $x_t$ in the following way:
$x_1 = y_3,\ x_2 = y_6,\ x_3 = y_9,\ x_4 = y_{12}$ and, more generally,

$$x_t = y_{3t}$$

   (a) Compute the autocorrelations of $(x_t)$.

   (b) Which family does the $(x_t)$ process belong to ?

3. We build another quarterly time series $(z_t)$ by averaging $y_t$ observations belonging to the same quarter:

$$z_1 = \frac{1}{3}(y_1 + y_2 + y_3),\ z_2 = \frac{1}{3}(y_4 + y_5 + y_6),\ z_3 = \frac{1}{3}(y_7 + y_8 + y_9),\ z_4 = \frac{1}{3}(y_{10} + y_{11} + y_{12})$$

and, more generally:

$$z_t = \frac{1}{3}(y_{3t-2} + y_{3t-1} + y_{3t})$$

Compute the autocorrelations of $(z_t)$.

## Exercise # 3

Let $(x_t)$ be the MA(1) process defined by

$$x_t = u_t - \theta u_{t-1}$$

where $(u_t)$ is WN(0,1) and $\theta = \frac{1}{\sqrt{2}}$.
Suppose that $x_t$ is not observable and that the process which is observed is $(y_t)$ defined by:

$$y_t = x_t + e_t$$

where $(e_t)$ is a measurement error and is WN$(0, \frac{1}{2})$. We suppose that $(e_t)$ is a process which is independent of $(u_t)$, so that $\forall (t,s)$, $u_t$ is independent of $e_s$.

1. Compute the autocovariances of $(y_t)$.
2. Prove that it is possible to find a white noise $(\varepsilon_t)$ with variance $\sigma_\varepsilon^2$ and a real number $\psi$, $|\psi| < 1$, so that the autocovariance function of $(y_t)$ coincides with the autocovariance function of the MA(1) process $z_t = \varepsilon_t - \psi \varepsilon_{t-1}$.
   *Hint* Compute the autovariances of order 0 and 1 for both process and find conditions on $\psi$ and $\sigma_\varepsilon^2$ ensuring equality of the autocovariance functions. Solve the corresponding equations.

## Exercise # 4

Let $(x_t)$ be the MA(1) process defined by the following equation $(\mathcal{E})$:

$$x_t = \varepsilon_t - \theta \varepsilon_{t-1} \qquad (\mathcal{E})$$

where $(\varepsilon_t)$ is WN$(0,\sigma^2)$ and $|\theta| < 1$.

1. Using equation $(\mathcal{E})$ for different values of $t$, prove that, for any $h > 0$:

$$x_t + \theta x_{t-1} + \cdots + \theta^h x_{t-h} = \varepsilon_t - \theta^{h+1} \varepsilon_{t-h-1}$$

2. Compute the variance of $\theta^{h+1}\varepsilon_{t-h-1}$. What can you say of $\lim_{h \to +\infty} \theta^{h+1}\varepsilon_{t-h-1}$?
3. i) Prove that $\sum_{h=0}^{+\infty} \mathbb{E}\left((\theta^h x_{t-h})^2\right) < +\infty$
   ii) Using (2), prove that $\varepsilon_t = \sum_{h=0}^{+\infty} \theta^h x_{t-h}$ (you will admit that (i) is sufficient to prove that $\sum_{h=0}^{+\infty} \theta^h x_{t-h}$ is well defined).