M1 APE, Econometrics 2
Exercises: session 1
Stationary processes: introduction

2018-2019

Exercise 1

1) $\forall t \in \mathbb{Z},\ E(X_t) = 0$
$\forall t \in \mathbb{Z},\ V(X_t) = E(X_t^2) - E(X_t)^2 = \sigma^2 - 0 = \sigma^2$
$\forall t \in \mathbb{Z},\ \forall h \in \mathbb{Z},\ \text{Cov}(X_t, X_{t-h}) = \begin{cases} \sigma^2 & \text{if } h = 0 \\ 0 & \text{if } h \neq 0 \end{cases}$
$\Rightarrow (X_t)$ is stationary and $\gamma_X(h) = \begin{cases} \sigma^2 & \text{if } h = 0 \\ 0 & \text{if } h \neq 0 \end{cases}$

As (X_t) is stationary, $\rho_X(h)$ also exists and is defined as:
$\rho_X(h) \equiv \dfrac{\gamma_X(h)}{\gamma_X(0)} \equiv \dfrac{\gamma_X(h)}{V(X_t)} = \begin{cases} 1 & \text{if } h = 0 \\ 0 & \text{if } h \neq 0 \end{cases}$
Remarks:

- An i.i.d. noise is strictly stationary (admitted).
- Strict stationarity implies weak stationarity (if the process has a finite second moment).

2) $\forall t \in \mathbb{Z},\ E(X_t) = 0$
$\forall t \in \mathbb{Z},\ V(X_t) = \sigma^2$
$\forall t \in \mathbb{Z},\ \forall h \in \mathbb{Z},\ \text{Cov}(X_t, X_{t-h}) = \begin{cases} \sigma^2 & \text{if } h = 0 \\ 0 & \text{if } h \neq 0 \text{ (the random variables } X_t \text{ are uncorrelated)} \end{cases}$
$\Rightarrow (X_t)$ is stationary and $\gamma_X(h) = \begin{cases} \sigma^2 & \text{if } h = 0 \\ 0 & \text{if } h \neq 0 \end{cases}$

As (X_t) is stationary, $\rho_X(h)$ also exists and is defined as:
$\rho_X(h) \equiv \dfrac{\gamma_X(h)}{\gamma_X(0)} \equiv \dfrac{\gamma_X(h)}{V(X_t)} = \begin{cases} 1 & \text{if } h = 0 \\ 0 & \text{if } h \neq 0 \end{cases}$

3) $\forall t \in \mathbb{Z},\ E(S_t) = E\left(S_0 + \sum_{i=1}^t X_i\right) = E(S_0) + E\left(\sum_{i=1}^t X_i\right) = 0 + \sum_{i=1}^t E(X_i) = 0$
$\forall t \in \mathbb{Z},\ V(S_t) = V\left(S_0 + \sum_{i=1}^t X_i\right) = V(S_0) + V\left(\sum_{i=1}^t X_i\right) + 2\,\text{Cov}\left(S_0, \sum_{i=1}^t X_i\right)$

$$= V(S_0) + \underbrace{\sum_{i=1}^{t} V(X_i)}_{\substack{\text{uncorrelation} \\ \text{of the } X_i}} + 2\sum_{i=1}^t \underbrace{\text{Cov}(S_0, X_i)}_{=0\ \forall i}$$

$$= V(S_0) + t\sigma^2 \qquad \textit{(explosive variance)}$$

As $V(X_t)$ depends on t, (X_t) is not a stationary process.

Remark: It can further be noticed that:
$\forall(t,h),\ \mathrm{Cov}(S_t,S_{t-h}) = \mathrm{Cov}(S_0+\sum_{i=1}^{t}X_i, S_0+\sum_{i=1}^{t-h}X_i) = V(S_0)+\sum_{i=1}^{t-h}\mathrm{Cov}(X_i,X_i) = V(S_0)+(t-h)\sigma^2$

Thus: $\forall(t,h),\ \mathrm{Cor}(S_t,S_{t-h}) \equiv \frac{\mathrm{Cov}(S_t,S_{t-h})}{\sqrt{V(S_t)}.\sqrt{V(S_{t-h})}} = \frac{V(S_0)+(t-h)\sigma^2}{\sqrt{V(S_0)+t\sigma^2}.\sqrt{V(S_0)+(t-h)\sigma^2}}$

When $t\to+\infty$, we can neglect the $V(S_0)$ terms so that $\mathrm{Cor}(S_t,S_{t-h}) \approx \frac{(t-h)\sigma^2}{\sqrt{t\sigma^2}\sqrt{(t-h)\sigma^2}}$, which can be simplified as $\sqrt{\frac{t-h}{t}} = \sqrt{1-\frac{h}{t}}$.

Reminding that for a fixed h and a large t $\left(\text{so that } \frac{h}{t} \text{ is near } 0\right)$, the Taylor expansion at order 1 leads to $\sqrt{1-\frac{h}{t}} \approx 1-\frac{1}{2}\frac{h}{t}$, we can see that for a fixed and large t, the autocorrelation decreases linearly in h which will be later mentioned as a signal for non-stationarity (see the course).

4) From the course, you know that a finite order MA process is always stationary.
In this question we will check it for the specific case of an MA(1). But in the next exercises, we will simply use this stationarity result.

$\forall t\in\mathbb{Z},\ E(X_t) = E(Z_t)+\theta E(Z_{t-1}) = 0$
$\forall t\in\mathbb{Z},\ V(X_t) = V(Z_t+\theta Z_{t-1}) \overset{uncorrelation}{=} V(Z_t)+\theta^2 V(Z_{t-1}) = \sigma^2+\theta^2\sigma^2 = (1+\theta^2)\sigma^2$
$$\begin{aligned}\forall(t,h),\ \mathrm{Cov}(X_t,X_{t-h}) &= \mathrm{Cov}(Z_t+\theta Z_{t-1}, Z_{t-h}+\theta Z_{t-h-1})\\ &= \mathrm{Cov}(Z_t,Z_{t-h})+\theta\,\mathrm{Cov}(Z_{t-1},Z_{t-h})+\theta\,\mathrm{Cov}(Z_t,Z_{t-h-1})+\theta^2\,\mathrm{Cov}(Z_{t-1},Z_{t-h-1})\end{aligned}$$

- $\underline{h=0}$: $\mathrm{Cov}(X_t,X_{t-0}) = V(X_t) = (1+\theta^2)\sigma^2$
- $\underline{h=-1}$: $\begin{aligned}[t]\mathrm{Cov}(X_t,X_{t+1}) &= \mathrm{Cov}(Z_t+\theta Z_{t-1}, Z_{t+1}+\theta Z_t)\\ &= \mathrm{Cov}(Z_t,Z_{t+1})+\theta\,\mathrm{Cov}(Z_{t-1},Z_{t+1})+\theta\,\mathrm{Cov}(Z_t,Z_t)+\theta^2\,\mathrm{Cov}(Z_{t-1},Z_t)\\ &= 0+0+\theta\sigma^2+0 = \theta\sigma^2\end{aligned}$
- $\underline{h=1}$: $\mathrm{Cov}(X_t,X_{t-1}) = \mathrm{Cov}(Z_t+\theta Z_{t-1}, Z_{t-1}+\theta Z_{t-2}) = \theta V(Z_{t-1}) = \theta\sigma^2$
- Otherwise: $\mathrm{Cov}(X_t,X_{t-h}) = 0$

$$\Rightarrow (X_t) \text{ is stationary and } \gamma_X(h) = \begin{cases}(1+\theta^2)\sigma^2 & \text{if } h=0\\ \theta\sigma^2 & \text{if } |h|=1\\ 0 & \text{if } |h|>1\end{cases}$$

As (X_t) is stationary, $\rho_X(h)$ also exists and is defined as:
$$\rho_X(h) \equiv \frac{\gamma_X(h)}{\gamma_X(0)} = \begin{cases}1 & \text{if } h=0\\ \dfrac{\theta}{1+\theta^2} & \text{if } |h|=1\\ 0 & \text{if } |h|>1\end{cases}$$
Reminder: The autocorrelation $\rho(h)$ of an MA(q) process is null for $|h|>q$.

5) We recognize the MA(1) process of question 4 with $\theta=-1$. (X_t) is stationary, and the results for expectation, variance, autocovariances and autocorrelations are the same, with $\theta=-1$.

6) We recognize an MA(1) with a constant term. It is stationary, like any other finite order MA process.

$\forall t\in\mathbb{Z},\ E(X_t) = a$
$\forall(t,h),\ \mathrm{Cov}(X_t,X_{t-h}) = \mathrm{Cov}(a+b\varepsilon_t+c\varepsilon_{t-1}, a+b\varepsilon_{t-h}+c\varepsilon_{t-h-1}) = \mathrm{Cov}(b\varepsilon_t+c\varepsilon_{t-1}, b\varepsilon_{t-h}+c\varepsilon_{t-h-1})$
- $\underline{h=0}$: $\mathrm{Cov}(X_t,X_{t-0}) = V(X_t) = (b^2+c^2)\sigma^2$
- $\underline{|h|=1}$: $\mathrm{Cov}(X_t,X_{t-1}) = \mathrm{Cov}(b\varepsilon_t+c\varepsilon_{t-1}, b\varepsilon_{t-1}+c\varepsilon_{t-2}) = bc\,\mathrm{Cov}(\varepsilon_{t-1},\varepsilon_{t-1}) = bc\sigma^2$
- $\underline{|h|>1}$: $\mathrm{Cov}(X_t,X_{t-h}) = 0$

$$\text{so } \gamma_X(h) = \begin{cases}(b^2+c^2)\sigma^2 & \text{if } h=0\\ bc\sigma^2 & \text{if } |h|=1\\ 0 & \text{if } |h|>1\end{cases} \quad\text{and}\quad \rho_X(h) = \begin{cases}1 & \text{if } h=0\\ \dfrac{bc}{b^2+c^2} & \text{if } |h|=1\\ 0 & \text{if } |h|>1\end{cases}$$

7) This is a simple random walk process like in question 3. For all $t \in \mathbb{Z}$, X_t can be rewritten as:
$X_t = X_{t-1} + \varepsilon_t = (X_{t-2} + \varepsilon_{t-1}) + \varepsilon_t = ... = X_0 + \sum_{i=1}^{t} \varepsilon_i$
We recognize the same process as in question 3, with the role of S_t played here by X_t, and the role of X_t played here by ε_t, for all t. The results are the same.

Reminders on the non-stationarity

The next two exercises deal with non-stationarity. There are two types of non-stationarity:

- *Deterministic non-stationarity*: deterministic trend and seasonality.
  The series are non-stationary in expectation.
  → Exercise 2

- *Stochastic non-stationarity*: unit root.
  The series are non-stationary in variance.
  → Exercise 3

Exercise 2

Reminders on the lag operator L:

- Sometimes called the backshift operator B
- $L^h X_t = X_{t-h}$
- $\Delta_h X_t = (1-L^h)X_t = X_t - L^h X_t = X_t - X_{t-h}$
- $(1-L^i)(1-L^j) = (1-L^j)(1-L^i)$ *commutative property*
- $(1-L^h)^2 = 1 - 2L^h + L^{2h}$ *distributive property*
- $(1-L)(X_t + Y_t) = 1(X_t+Y_t) - L(X_t+Y_t)$
  $= (1-L)X_t + (1-L)Y_t$

1) This process exhibits *additive* seasonality.

First, explicit $\Delta_{12} X_t$:

$$\begin{aligned}
\Delta_{12} X_t &= X_t - X_{12} \\
&= (a + bt + s_t + Y_t) - (a + b(t-12) + s_{t-12} + Y_{t-12}) \\
&= 12b + \underbrace{(s_t - s_{t-12})}_{=0} + (Y_t - Y_{t-12}) \\
\Delta_{12} X_t &= 12b + \Delta_{12} Y_t
\end{aligned}$$

⇒ Both the seasonality and the deterministic trend are eliminated with this transformation.

Secondly, we will check if $(\Delta_{12} X_t)$ is stationary (and thus if $\gamma_{\Delta_{12} X}$ exists), and express $\mathrm{Cov}(\Delta_{12} X_t, \Delta_{12} X_{t-h})$.

$E(\Delta_{12} X_t) = 12b + E(Y_t) - E(Y_{t-12}) = 12b \quad \forall t$

$\mathrm{Cov}(\Delta_{12} X_t, \Delta_{12} X_{t-h}) = \mathrm{Cov}(\Delta_{12} Y_t, \Delta_{12} Y_{t-h}) = \mathrm{Cov}(Y_t - Y_{t-12}, Y_{t-h} - Y_{t-h-12}) \quad \forall t, h$
Since (Y_t) is assumed to be stationary, $\mathrm{Cov}(Y_t, Y_{t-h}) = \gamma_Y(h) = \gamma_Y(-h) \quad \forall t, h$
$\mathrm{Cov}(\Delta_{12} X_t, \Delta_{12} X_{t-h}) = 2\gamma_Y(h) - \gamma_Y(h-12) - \gamma_Y(h+12) \quad \forall t, h$

⇒ $(\Delta_{12} X_t)$ is stationary and $\gamma_{\Delta_{12} X}(h) = 2\gamma_Y(h) - \gamma_Y(h-12) - \gamma_Y(h+12) \quad \forall t, h$

2) This process exhibits *multiplicative* seasonality.

First, explicit $\Delta_{12}^2 X_t$:

$$
\begin{aligned}
\Delta_{12} X_t &= X_t - X_{t-12} \\
&= \Big((a+bt)s_t + Y_t\Big) - \Big(\big(a+b(t-12)\big)s_{t-12} + Y_{t-12}\Big) \\
&= (a+bt)\underbrace{(s_t - s_{t-12})}_{=0} + 12bs_{t-12} + \Delta_{12} Y_t \\
\Delta_{12} X_t &= 12bs_{t-12} + \Delta_{12} Y_t
\end{aligned}
$$

$$
\begin{aligned}
\Delta_{12}^2 X_t &= \Delta_{12}\Delta_{12} X_t \\
&= \Delta_{12}(12bs_{t-12} + \Delta_{12} Y_t) \\
&= \Delta_{12}(12bs_{t-12}) + \Delta_{12}^2 Y_t \\
&= 12b\underbrace{(s_{t-12} - s_{t-24})}_{=0} + \Delta_{12}^2 Y_t \\
\Delta_{12}^2 X_t &= \Delta_{12}^2 Y_t \\
\Delta_{12}^2 X_t &= \Delta_{12}(Y_t - Y_{t-12}) = \Delta_{12}(Y_t) - \Delta_{12}(Y_{t-12}) = Y_t - Y_{t-12} - (Y_{t-12} - Y_{t-24}) = Y_t - 2Y_{t-12} + Y_{t-24}
\end{aligned}
$$

$\Rightarrow$ Both the seasonality and the deterministic trend are eliminated with these transformations.

Secondly, we will check if $(\Delta_{12}^2 X_t)$ is stationary (and thus if $\gamma_{\Delta_{12}^2 X}$ exists), and express $\mathrm{Cov}(\Delta_{12}^2 X_t, \Delta_{12}^2 X_{t-h})$.

$E(\Delta_{12} X_t) = E(Y_t) - 2E(Y_{t-12}) + E(Y_{t-24}) = 0 \quad \forall t$

$\mathrm{Cov}(\Delta_{12}^2 X_t, \Delta_{12}^2 X_{t-h}) = \mathrm{Cov}(Y_t - 2Y_{t-12} + Y_{t-24}, Y_{t-h} - 2Y_{t-h-12} + Y_{t-h-24}) \quad \forall t, h$

Developing this expression yields:

$$
\begin{aligned}
\mathrm{Cov}(\Delta_{12}^2 X_t, \Delta_{12}^2 X_{t-h}) = &+\mathrm{Cov}(Y_t, Y_{t-h}) - 2\,\mathrm{Cov}(Y_t, Y_{t-h-12}) + \mathrm{Cov}(Y_t, Y_{t-h-24}) \\
&- 2\,\mathrm{Cov}(Y_{t-12}, Y_{t-h}) + 4\,\mathrm{Cov}(Y_{t-12}, Y_{t-h-12}) - 2\,\mathrm{Cov}(Y_{t-12}, Y_{t-h-24}) \\
&+ \mathrm{Cov}(Y_{t-24}, Y_{t-h}) - 2\,\mathrm{Cov}(Y_{t-24}, Y_{t-h-12}) + \mathrm{Cov}(Y_{t-24}, Y_{t-h-24})
\end{aligned}
$$

Since (Y_t) is assumed to be stationary, $\mathrm{Cov}(Y_t, Y_{t-h}) = \gamma_Y(h) = \gamma_Y(-h) \quad \forall t, h$

$$
\begin{aligned}
\mathrm{Cov}(\Delta_{12}^2 X_t, \Delta_{12}^2 X_{t+h}) = &+\gamma_Y(h) - 2\gamma_Y(h+12) + \gamma_Y(h+24) \\
&- 2\gamma_Y(h-12) + 4\gamma_Y(h) - 2\gamma_Y(h+12) \\
&+ \gamma_Y(h-24) - 2\gamma_Y(h-12) + \gamma_Y(h)
\end{aligned}
$$

Finally,

$$\mathrm{Cov}(\Delta_{12}^2 X_t, \Delta_{12}^2 X_{t-h}) = 6\gamma_Y(h) - 4\gamma_Y(h-12) - 4\gamma_Y(h+12) + \gamma_Y(h-24) + \gamma_Y(h+24) \quad \forall t, h$$

$\Rightarrow$ $(\Delta_{12}^2 X_t)$ is stationary and

$$\gamma_{\Delta_{12}^2 X}(h) = 6\gamma_Y(h) - 4\gamma_Y(h-12) - 4\gamma_Y(h+12) + \gamma_Y(h-24) + \gamma_Y(h+24) \quad \forall t, h$$

Exercise 3

First, rewrite X_t as a function of X_{t-n-1}.

$$\begin{aligned} X_t &= \phi X_{t-1} + Z_t \\ &= \phi(\phi X_{t-2} + Z_{t-1}) + Z_t \\ &= \ldots \\ X_t &= \phi^{n+1} X_{t-(n+1)} + \sum_{i=0}^{n} \phi^i Z_{t-i} \end{aligned} \tag{1}$$

We will write $V(X_t - \phi^{n+1} X_{t-n-1})$ under two different forms, and show that they cannot be equal under the assumption that (X_t) is stationary for $|\phi| = 1$.

$$\begin{aligned} V(X_t - \phi^{n+1} X_{t-n-1}) &= V(X_t) + V(-\phi^{n+1} X_{t-n-1}) + 2\,\mathrm{Cov}(X_t, -\phi^{n+1} X_{t-n-1}) \\ &= V(X_t) + (-\phi^{n+1})^2 V(X_{t-n-1}) + 2(-\phi^{n+1})\,\mathrm{Cov}(X_t, X_{t-n-1}) \\ &= V(X_t) + |\phi|^{2(n+1)} V(X_{t-n-1}) - 2\phi^{n+1}\,\mathrm{Cov}(X_t, X_{t-n-1}) \\ &= V(X_t) + V(X_{t-n-1}) - 2\phi^{n+1}\,\mathrm{Cov}(X_t, X_{t-n-1}) && \text{for } |\phi| = 1 \\ &= 2\sigma_X^2 - 2\phi^{n+1}\gamma_X(n+1) && \textbf{if } (X_t) \textbf{ is stationary} \\ &= 2\left[\sigma_X^2 - \phi^{n+1}\gamma_X(n+1)\right] \end{aligned}$$

As (X_t) is assumed to be stationary, σ_X^2 and $\gamma_X(n+1)$ are supposed to be finite. ϕ^{n+1} is also finite: it can be equal to -1 or 1 depending on whether ϕ =1 or -1, and whether n+1 is odd or even.
⇒ **If (X_t) is stationary, then $V(X_t - \phi^{n+1} X_{t-n-1})$ is bounded for $|\phi| = 1$, and converges to $2\sigma_X^2$ with n.**

Using (1), we can also write $V(X_t - \phi^{n+1} X_{t-n-1})$ as:

$$\begin{aligned} V(X_t - \phi^{n+1} X_{t-n-1}) &= V\Big(\sum_{i=0}^{n} \phi^i Z_{t-i}\Big) \overset{uncorrelation}{=} \sum_{i=0}^{n} V(\phi^i Z_{t-i}) \\ &= \sum_{i=0}^{n} \phi^{2i} V(Z_{t-i}) \overset{even\ power}{=} \sum_{i=0}^{n} |\phi|^{2i} V(Z_{t-i}) \\ &= \sum_{i=0}^{n} V(Z_{t-i}) = \sum_{i=0}^{n} \sigma^2 \\ V(X_t - \phi^{n+1} X_{t-n-1}) &= (n+1)\sigma^2 \end{aligned}$$

This variance is explosive and tends to infinity with n. We saw previously that if (X_t) is stationary, then $V(X_t - \phi^{n+1} X_{t-n-1})$ is necessarily bounded for $|\phi| = 1$. Thus (X_t) can not be stationary for $|\phi| = 1$.