M1 APE, Econometrics 2
Solution tutorial 4
Stationary processes and forecasts

2017-2018

Exercise 1

1)i) We can compute a and b using the two conditions defining this linear predictor (forecast error of expectation 0 and of minimum variance).

The first condition yields:

$$\begin{aligned} E\big(X_{t+h}-(aX_t+b)\big)=0 \quad &\Leftrightarrow m-am-b=0 \quad \text{as } (X_t) \text{ is stationary} \\ &\Leftrightarrow \big[b=m(1-a)\big] \end{aligned}$$

Secondly, the variance of the forecast error can be rewritten as:

$$\begin{aligned} V\big(X_{t+h}-(aX_t+b)\big) &= V(X_{t+h}-aX_t) \\ &= V(X_{t+h})+(-a)^2V(X_t)+2\times(-a)\times \text{Cov}(X_{t+h},X_t) \\ &= \big(1+a^2\big)\gamma(0)-2a\gamma(h) \qquad \text{as } (X_t) \text{ is stationary} \end{aligned}$$

Let $f(a)=\big(1+a^2\big)\gamma(0)-2a\gamma(h)$. We have:

$$f'(a)=0 \Leftrightarrow 2a\gamma(0)-2\gamma(h)=0 \Leftrightarrow a=\frac{\gamma(h)}{\gamma(0)} \Leftrightarrow \big[a=\rho(h)\big]$$
$$f''(a)=2\gamma(0)>0 \ \forall a \quad \text{so } a=\rho(h) \text{ is indeed a minimum}$$

This predictor is thus obtained by choosing $a=\rho(h)$ and $b=m(1-a)=m(1-\rho(h))$.

ii) $BLF(X_{t+h}|X_t)=\alpha X_t+\beta$ has been define d in the course by the two following properties:

$$\begin{cases} E\big(X_{t+h}-(\alpha X_t+\beta)\big)=0 \\ \text{Cov}\big(X_{t+h}-(\alpha X_t+\beta),X_t\big)=0 \end{cases}$$

$$\Leftrightarrow \begin{cases} m-\alpha m-\beta=0 \\ \gamma(h)-\alpha\gamma_y(0)=0 \end{cases} \quad \Leftrightarrow \begin{cases} \beta=m(1-\alpha)=m(1-\rho(h))=b \\ \alpha=\frac{\gamma(h)}{\gamma(0)}=\rho(h)=a \end{cases}$$

The previous linear predictor does coincide with BLF($X_{t+h}|X_t$).
This explains why BLF($X_{t+h}|X_t$) is named as *best* linear forecast: it is the best one since it is associated to a forecast error which has minimum variance.

Remark: The coefficients a, b, α and β obviously depend on the forecast horizon h and we should rather denote them a_h, b_h, α_h and β_h (but we don't, in order to have simpler notations).

2)i) We can compute a_0, a_1 and b using the two conditions defining this linear predictor (forecast error of expectation 0 and of minimum variance).

The first condition yields:

$$\begin{aligned} E\big(X_{t+h} - (a_0 X_t + a_1 X_{t-1} + b)\big) = 0 \quad & \Leftrightarrow m - a_0 m - a_1 m - b = 0 \quad \text{as } (X_t) \text{ is stationary} \\ & \Leftrightarrow \big[b = m(1 - a_0 - a_1)\big] \end{aligned} \tag{1}$$

Secondly, the variance of the forecast error can be rewritten as:

$$\begin{aligned} V\big(X_{t+h} - (a_0 X_t + a_1 X_{t-1} + b)\big) &= V(X_{t+h} - a_0 X_t - a_1 X_{t-1}) \\ &= V(X_{t+h}) + a_0^2 V(X_t) + a_1^2 V(X_{t-1}) \\ &\quad -2a_0 \operatorname{Cov}(X_{t+h}, X_t) - 2a_1 \operatorname{Cov}(X_{t+h}, X_{t-1}) + 2a_0 a_1 \operatorname{Cov}(X_t, X_{t-1}) \\ &= \big(1 + a_0^2 + a_1^2\big)\gamma(0) - 2a_0\gamma(h) - 2a_1\gamma(h+1) + 2a_0 a_1 \gamma(1) \qquad \text{as } (X_t) \text{ is stationary} \end{aligned}$$

Let $g(a_0, a_1) = \big(1 + a_0^2 + a_1^2\big)\gamma(0) - 2a_0\gamma(h) - 2a_1\gamma(h+1) + 2a_0 a_1 \gamma(1)$.

The first derivatives are:

$$\begin{aligned} \frac{\partial g}{\partial a_0}(a_0, a_1) &= 2a_0\gamma(0) - 2\gamma(h) + 2a_1\gamma(1) \\ \frac{\partial g}{\partial a_1}(a_0, a_1) &= 2a_1\gamma(0) - 2\gamma(h+1) + 2a_0\gamma(1) \end{aligned}$$

The hessian matrix (second derivatives matrix) is:

$$H_g(a_0, a_1) = \begin{pmatrix} 2\gamma(0) & 2\gamma(1) \\ 2\gamma(1) & 2\gamma(0) \end{pmatrix} = 2\begin{pmatrix} \gamma(0) & \gamma(1) \\ \gamma(1) & \gamma(0) \end{pmatrix}$$

We can see that $H_g(a_0, a_1)$ is positive definite, since the leading principal minors (determinants of the upper-left successive submatrices) are strictly positive:

- the upper left term is $2\gamma(0) > 0$
- the determinant is $\det\big(H_g(a_0, a_1)\big) = 4\big(\gamma(0)^2 - \gamma(1)^2\big)$
  with $\big(\gamma(0)^2 - \gamma(1)^2\big) > 0$ since, by Schwarz inequality: $(\operatorname{Cov}(X_t, X_{t-1}))^2 \le V(X_t)V(X_{t-1})$.

We thus have a global minimum in (a_0, a_1) such that $\frac{\partial g}{\partial a_0}(a_0, a_1) = \frac{\partial g}{\partial a_1}(a_0, a_1) = 0$.

The first order conditions yield:

$$\begin{cases} a_0\gamma(0) - \gamma(h) + a_1\gamma(1) = 0 \\ a_1\gamma(0) - \gamma(h+1) + a_0\gamma(1) = 0 \end{cases} \tag{2}$$

$$\Leftrightarrow \begin{cases} a_0 - \rho(h) + a_1\rho(1) = 0 \\ a_1 - \rho(h+1) + a_0\rho(1) = 0 \end{cases} \quad \text{dividing by } \gamma(0)\ (\neq 0)$$

$$\Leftrightarrow \begin{cases} a_0 = \rho(h) - a_1\rho(1) \\ a_1\big(1 - \rho(1)^2\big) = \rho(h+1) - \rho(h)\rho(1) \end{cases}$$

$$\Leftrightarrow \begin{cases} a_0 = \dfrac{\rho(h) - \rho(1)\rho(h+1)}{1 - \rho(1)^2} \\ a_1 = \dfrac{\rho(h+1) - \rho(1)\rho(h)}{1 - \rho(1)^2} \end{cases}$$

ii) $BLF(X_{t+h}|X_t) = \alpha_0 X_t + \alpha_1 X_{t-1} + \beta$ has been defined by:

$$\begin{cases} E\big(X_{t+h} - (\alpha_0 X_t + \alpha_1 X_{t-1} + \beta)\big) = 0 \\ \text{Cov}\big(X_{t+h} - (\alpha_0 X_t + \alpha_1 X_{t-1} + \beta), X_t\big) = 0 \\ \text{Cov}\big(X_{t+h} - (\alpha_0 X_t + \alpha_1 X_{t-1} + \beta), X_{t-1}\big) = 0 \end{cases} \Leftrightarrow \begin{cases} \beta = (1 - \alpha_0 - \alpha_1) m \\ \gamma(h) = \alpha_0 \gamma(0) + \alpha_1 \gamma(1) \\ \gamma(h+1) = \alpha_0 \gamma(1) + \alpha_1 \gamma(0) \end{cases}$$

This system corresponds to the equation (1) and the system of two equations (2), so that $\alpha_0 = a_0$, $\alpha_1 = a_1$, $\beta = b$. We have thus proved that $\text{BLF}(X_{t+h}|X_t, X_{t-1})$ is the best predictor of X_{t+h} as a linear function of X_t and X_{t-1} since it minimizes the variance of the forecast error.

Remarks:

- *Again, all the coefficients obviously depend on the forecast horizon h and should actually be indexed by h, but we don't do in order to have simpler notations.*
- *As it is written in the text, the proof would be identical for* $\text{BLF}(X_{t+h}|X_t, ..., X_{t-p})$ *which has been characterized in the course by*
  $$\begin{cases} \text{BLF}(X_{t+h}|X_t, ..., X_{t-p}) = \alpha_0 X_t + ... + \alpha_p X_{t-p} + \beta \\ E\big(X_{t+h} - (\alpha_0 X_t + ... + \alpha_p X_{t-p} + \beta)\big) = 0 \\ \text{Cov}\big(X_{t+h} - (\alpha_0 X_t + ... + \alpha_p X_{t-p} + \beta), X_{t-j}\big) = 0 \quad \forall j = \{0, ..., p\} \end{cases}$$
  and which can be proved to be the linear predictor which minimizes the forecast error variance. This property remains true for $\text{BLF}(X_{t+h}|\underline{X_t})$ *but the proof is more complicated.*

Exercise 2

Remember that in problem set 3 exercise 2, we saw that the equation defining (y_t) can be rewritten as $\big(1 - \frac{1}{3}L\big)\big(1 - \frac{1}{2}L\big) y_t = 0.5 + v_t$ and that (v_t) is the innovation process of (y_t).

1) For a forecast horizon $h = 1$, we have:

$$\begin{aligned} y^*_{t+1|t} &= \text{BLF}(y_{t+1}|\underline{y_t}) \\ &= \text{BLF}\left(0.5 + \frac{5}{6} y_t - \frac{1}{6} y_{t-1} + v_{t+1}|\underline{y_t}\right) \\ &= 0.5 + \frac{5}{6} y_t - \frac{1}{6} y_{t-1} + \text{BLF}(v_{t+1}|\underline{y_t}) \\ y^*_{t+1|t} &= 0.5 + \frac{5}{6} y_t - \frac{1}{6} y_{t-1} \qquad \text{since } (v_t) \text{ is the innovation process of } (y_t) \end{aligned}$$

$$e^*_{t+1|t} = y_{t+1} - y^*_{t+1|t} = v_{t+1}$$

$$V\big(e^*_{t+1|t}\big) = V(v_{t+1}) = \sigma^2_v$$

2) For a forecast horizon $h = 2$, we have:

$$\begin{aligned}
y^*_{t+2|t} &= \mathrm{BLF}(y_{t+2}|\underline{y_t}) \\
&= \mathrm{BLF}\left(0.5 + \frac{5}{6}y_{t+1} - \frac{1}{6}y_t + v_{t+2}|\underline{y_t}\right) \\
&= 0.5 + \frac{5}{6}\mathrm{BLF}(y_{t+1}|\underline{y_t}) - \frac{1}{6}y_t + \mathrm{BLF}(v_{t+2}|\underline{y_t}) \\
y^*_{t+2|t} &= 0.5 + \frac{5}{6}y^*_{t+1|t} - \frac{1}{6}y_t \qquad \text{since } (v_t) \text{ is the innovation process of } (y_t)
\end{aligned}$$

$$e^*_{t+2|t} = y_{t+2} - y^*_{t+2|t} = \frac{5}{6}(y_{t+1} - y^*_{t+1|t}) + v_{t+2} = \frac{5}{6}v_{t+1} + v_{t+2}$$

$$\begin{aligned}
V\left(e^*_{t+2|t}\right) &= \left(\frac{25}{36} + 1\right)\sigma^2_v \qquad \text{since } (v_t) \sim \mathrm{WN} \\
V\left(e^*_{t+2|t}\right) &= \frac{61}{36}\sigma^2_v
\end{aligned}$$

3) For a forecast horizon $h = 3$, we have:

$$\begin{aligned}
y^*_{t+3|t} &= \mathrm{BLF}(y_{t+3}|\underline{y_t}) \\
&= \mathrm{BLF}\left(0.5 + \frac{5}{6}y_{t+2} - \frac{1}{6}y_{t+1} + v_{t+3}|\underline{y_t}\right) \\
&= 0.5 + \frac{5}{6}\mathrm{BLF}(y_{t+2}|\underline{y_t}) - \frac{1}{6}\mathrm{BLF}(y_{t+1}|\underline{y_t}) + \mathrm{BLF}(v_{t+3}|\underline{y_t}) \\
y^*_{t+3|t} &= 0.5 + \frac{5}{6}y^*_{t+2|t} - \frac{1}{6}y^*_{t+1|t} \qquad \text{since } (v_t) \text{ is the innovation process of } (y_t)
\end{aligned}$$

$$\begin{aligned}
e^*_{t+3|t} &= y_{t+3} - y^*_{t+3|t} \\
&= \frac{5}{6}\left(y_{t+2} - y^*_{t+2|t}\right) - \frac{1}{6}\left(y_{t+1} - y^*_{t+1|t}\right) + v_{t+3} \\
&= \frac{5}{6}\left(\frac{5}{6}v_{t+1} + v_{t+2}\right) - \frac{1}{6}v_{t+1} + v_{t+3} \\
&= v_{t+3} + \frac{5}{6}v_{t+2} + \left(\frac{25}{36} - \frac{1}{6}\right)v_{t+1} \\
e^*_{t+3|t} &= v_{t+3} + \frac{5}{6}v_{t+2} + \frac{19}{36}v_{t+1}
\end{aligned}$$

$$V\left(e^*_{t+3|t}\right) = \left(1 + \frac{25}{36} + \left(\frac{19}{36}\right)^2\right)\sigma^2_v$$

4) *There is a typo in the text: read* $h \geq 1$ *(and* $t \geq 2$*)*

<u>For $h = 1$:</u>

$$\begin{aligned}
\mathrm{BLF}(y_{t+1}|y_t, ..., y_1) &= \mathrm{BLF}\left(0.5 + \frac{5}{6}y_t - \frac{1}{6}y_{t-1} + v_{t+1}|y_t, ..., y_1\right) \\
&= 0.5 + \frac{5}{6}y_t - \frac{1}{6}y_{t-1} + 0 \qquad \text{since } \mathrm{Cov}(v_{t+1}, y_{t-k}) = 0 \quad \forall k \geq 0 \\
&= \mathrm{BLF}(y_{t+1}|\underline{y_t})
\end{aligned}$$

For $h = 2$:

$$\begin{aligned}
\text{BLF}(y_{t+2}|y_t,...,y_1) &= \text{BLF}\left(0.5 + \frac{5}{6}y_{t+1} - \frac{1}{6}y_t + v_{t+2}|y_t,...,y_1\right)\\
&= 0.5 + \frac{5}{6}\text{BLF}(y_{t+1}|y_t,...,y_1) - \frac{1}{6}y_t + 0 \qquad \text{since } \text{Cov}(v_{t+2}, y_{t-k}) = 0 \quad \forall k \geq 0\\
&= 0.5 + \frac{5}{6}y^*_{t+1|t} - \frac{1}{6}y_t\\
&= \text{BLF}(y_{t+2}|\underline{y_t})
\end{aligned}$$

For $h = 3$:

$$\begin{aligned}
\text{BLF}(y_{t+3}|y_t,...,y_1) &= \text{BLF}\left(0.5 + \frac{5}{6}y_{t+2} - \frac{1}{6}y_{t+1} + v_{t+3}|y_t,...,y_1\right)\\
&= 0.5 + \frac{5}{6}\text{BLF}(y_{t+2}|y_t,...,y_1) - \frac{1}{6}\text{BLF}(y_{t+1}|y_t,...,y_1) + 0 \qquad \text{since } \text{Cov}(v_{t+3}, y_{t-k}) = 0 \quad \forall k \geq 0\\
&= 0.5 + \frac{5}{6}y^*_{t+2|t} - \frac{1}{6}y^*_{t+1|t}\\
&= \text{BLF}(y_{t+3}|\underline{y_t})
\end{aligned}$$

It is easy to iterate and to see that $\text{BLF}(y_{t+h}|y_t,...,y_1) = \text{BLF}(y_{t+h}|\underline{y_t})$ for any $h \geq 1$.

Remark: **This property is true for any AR process, but <u>not</u> for an MA(q) or ARMA(p,q) process!**

Exercise 3

1) In order to prove that (u_t) is the innovation process of (x_t) it is sufficient to prove that (x_t) is written under its canonical representation.
$(1 - 0.8L)x_t = (1 - 0.4L)u_t \Leftrightarrow \phi(L)x_t = \theta(L)u_t$ with $\phi(L) = 1 - 0.8L$ and $\theta(L) = 1 - 0.4L$.
This is indeed the canonical representation of (x_t) since the respective roots of the two lag polynomials $\left(\frac{1}{0.8} = 1.25 \text{ and } \frac{1}{0.4} = 2.5\right)$ are all greater than one in modulus.
Thus (u_t) is the innovation process of (x_t).

2) The root of $\theta(L)$ is greater than one in modulus so $\theta(L)$ is invertible and we can compute the AR(∞) representation of (x_t) as:

$$\theta^{-1}(L)\phi(L)x_t = u_t$$

$\theta^{-1}(L)$ can be expressed like:

$$\theta^{-1}(L) = (1 - 0.4L)^{-1} = \sum_{i=0}^{\infty} 0.4^i L^i$$

and the infinite lag polynomial $\theta^{-1}(L)\phi(L)$ can then be simplified as:

$$\begin{aligned}
\theta^{-1}(L)\phi(L) &= \left(\sum_{i=0}^{\infty} 0.4^i L^i\right)(1 - 0.8L)\\
&= \sum_{i=0}^{\infty} 0.4^i L^i - 0.8\sum_{i=0}^{\infty} 0.4^i L^{i+1}\\
&= 0.4^0 L^0 + \sum_{i=1}^{\infty} 0.4^i L^i - 0.8\sum_{i=1}^{\infty} 0.4^{i-1} L^i\\
&= 1 + \sum_{i=1}^{\infty}\left(0.4^i - 0.8 \times 0.4^{i-1}\right)L^i\\
&= 1 + \sum_{i=1}^{\infty} 0.4^{i-1}(0.4 - 0.8)L^i
\end{aligned}$$

$$\boxed{\theta^{-1}(L)\phi(L) = 1 - \sum_{i=1}^{\infty} 0.4^i L^i}$$

Finally, the AR(∞) representation of (x_t) can be expressed as:

$$\begin{aligned}\theta^{-1}(L)\phi(L)x_t &= u_t\\ \left[\left(1-\sum_{i=1}^{\infty}0.4^iL^i\right)x_t \right. &= \left. u_t\right]\end{aligned}$$

3) The root of $\phi(L)$ is greater than one in modulus so $\phi(L)$ is invertible and we can compute the MA(∞) representation of (x_t) as:

$$x_t = \phi^{-1}(L)\theta(L)u_t$$

$\phi^{-1}(L)$ can be expressed like:

$$\phi^{-1}(L) = (1-0.8L)^{-1} = \sum_{i=0}^{\infty}0.8^iL^i$$

and the infinite lag polynomial $\phi^{-1}(L)\theta(L)$ can then be simplified as:

$$\begin{aligned}\phi^{-1}(L)\theta(L) &= \left(\sum_{i=0}^{\infty}0.8^iL^i\right)(1-0.4L)\\ &= \sum_{i=0}^{\infty}0.8^iL^i - 0.4\sum_{i=0}^{\infty}0.8^iL^{i+1}\\ &= 0.8^0L^0 + \sum_{i=1}^{\infty}0.8^iL^i - 0.4\sum_{i=1}^{\infty}0.8^{i-1}L^i\\ &= 1+\sum_{i=1}^{\infty}\left(0.8^i - 0.4\times 0.8^{i-1}\right)L^i\\ &= 1+\sum_{i=1}^{\infty}0.8^{i-1}(0.8-0.4)L^i\\ &= 1+0.4\sum_{i=1}^{\infty}0.8^{i-1}L^i\\ &= 1+\frac{0.8}{2}\sum_{i=1}^{\infty}0.8^{i-1}L^i\\ \left[\phi^{-1}(L)\theta(L)\right. &= \left. 1+\frac{1}{2}\sum_{i=1}^{\infty}0.8^iL^i\right]\end{aligned}$$

Finally, the MA(∞) representation of (x_t) can be expressed as:

$$\begin{aligned}x_t &= \phi^{-1}(L)\theta(L)u_t\\ \left[x_t\right. &= \left. \left(1+\frac{1}{2}\sum_{i=1}^{\infty}0.8^iL^i\right)u_t\right]\end{aligned}$$

4) First, remember that if (u_t) is the innovation of (x_t), then:

$$E(u_tx_{t-1}) = \underbrace{\text{Cov}(u_t,x_{t-1})}_{\substack{=0 \text{ as } (u_t) \text{ is the}\\ \text{innovation of } (x_t)}} + \underbrace{E(u_t)}_{=0}E(x_{t-1}) = 0$$

Using this result, we can compute $E(u_tx_t)$ directly from the ARMA representation of (x_t):

$$\begin{aligned}&x_t - 0.8x_{t-1} = u_t - 0.4u_{t-1}\\ &u_tx_t - 0.8u_tx_{t-1} = u_t^2 - 0.4u_tu_{t-1}\\ &E(u_tx_t) - 0.8\underbrace{E(u_tx_{t-1})}_{\substack{=0\\ \text{(innovation)}}} = \underbrace{E(u_t^2)}_{=\sigma_u^2} - 0.4\underbrace{E(u_tu_{t-1})}_{\substack{=0\\ (u_t)\text{ WN}}}\\ &\left[E(u_tx_t) = \sigma_u^2\right]\end{aligned}$$

$E(u_{t-1}x_t)$ can then be computed like:

$$\begin{aligned}
E(u_{t-1}x_t) &= E\big[u_{t-1}(0.8x_{t-1}+u_t-0.4u_{t-1})\big] \\
&= 0.8\underbrace{E(u_{t-1}x_{t-1})}_{=\sigma_u^2}+\underbrace{E(u_{t-1}u_t)}_{\substack{=0\\(u_t)\text{ WN}}}-0.4\underbrace{E(u_{t-1}^2)}_{=\sigma_u^2} \\
\Big[E(u_{t-1}x_t) &= 0.4\sigma_u^2\Big]
\end{aligned}$$

5) Remember that for a stationary and centered process (x_t),

$$\begin{aligned}
E(x_kx_j) &= \mathrm{Cov}(x_k,x_j)+E(x_k)E(x_j) && \text{by property of the covariance} \\
&= \mathrm{Cov}(x_k,x_j) && \text{as } (x_t) \text{ is centered} \\
&= \gamma_x(k-j) && \text{as } (x_t) \text{ is stationary}
\end{aligned}$$

Here, (x_t) is stationary (ARMA and no unit root), and also already centered (no constant). We can directly compute $\gamma_x(0)$ without centering the process first:

$$\begin{aligned}
&x_t-0.8x_{t-1}=u_t-0.4u_{t-1} \\
&x_t^2-0.8x_tx_{t-1}=x_tu_t-0.4x_tu_{t-1} \\
&E(x_t^2)-0.8E(x_tx_{t-1})=E(x_tu_t)-0.4E(x_tu_{t-1}) \\
&\gamma_x(0)-0.8\gamma_x(1)=\sigma_u^2-0.4\times 0.4\sigma_u^2 \\
&\Big[\gamma_x(0)=0.8\gamma_x(1)+0.84\sigma_u^2\Big]
\end{aligned} \tag{3}$$

We can compute $\gamma_x(1)$ the same way:

$$\begin{aligned}
&x_t-0.8x_{t-1}=u_t-0.4u_{t-1} \\
&x_{t-1}x_t-0.8x_{t-1}^2=x_{t-1}u_t-0.4x_{t-1}u_{t-1} \\
&E(x_{t-1}x_t)-0.8E(x_{t-1}^2)=E(x_{t-1}u_t)-0.4E(x_{t-1}u_{t-1}) \\
&\gamma_x(1)-0.8\gamma_x(0)=0-0.4\sigma_u^2 \\
&\Big[\gamma_x(1)=0.8\gamma_x(0)-0.4\sigma_u^2\Big]
\end{aligned} \tag{4}$$

(3) and (4) are the Yule-Walker equations of this ARMA process for $h=0$ and $h=1$. Solving them yields:

$$\left[\gamma_x(0)=\frac{0.84-0.8\times 0.4}{1-0.8^2}\sigma_u^2=\frac{13}{9}\sigma_u^2\right]$$

$$\left[\gamma_x(1)=0.8\times\frac{13}{9}\sigma_u^2-0.4\sigma_u^2=\frac{34}{45}\sigma_u^2\right]$$

6) As before:

$$\begin{aligned}
&x_t-0.8x_{t-1}=u_t-0.4u_{t-1} \\
&x_{t-h}x_t-0.8x_{t-h}x_{t-1}=x_{t-h}u_t-0.4x_{t-h}u_{t-1} \\
&E(x_{t-h}x_t)-0.8E(x_{t-h}x_{t-1})=E(x_{t-h}u_t)-0.4E(x_{t-h}u_{t-1})
\end{aligned}$$

As (u_t) is the innovation process of (x_t), $E(x_{t-h}u_t)=E(x_{t-h}u_{t-1})=0 \quad \forall h\geq 2$. This yields:

$$\begin{aligned}
\Big[\gamma_x(h) &= 0.8\gamma_x(h-1) \quad \forall h\geq 2\Big] \\
\gamma_x(h) &= 0.8\big(0.8\gamma_x(h-2)\big) \\
&\vdots \\
\left[\gamma_x(h) \right.&= \left. 0.8^{h-1}\gamma_x(1)=\frac{34}{45}0.8^{h-1}\sigma_u^2 \quad \forall h\geq 1\right]
\end{aligned}$$

$\gamma_x(h)$ exponentially decreases to 0 in absolute value (and even without taking it in absolute value here), as we would expect from a stationary process.