Notes on Banach Spaces

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1 Taylor's Theorem

Theorem 1.1 (Symmetry of the second derivative)
Suppose V, W are Normed Spaces,
f is defined and differentiable near $\alpha \in V$ and maps to W,
f is twice differentiable at α.
Then $d^2 f_\alpha$ is symmetric.

Proof.
For formality, let δ^* be s.t. f is defined and differentiable in $B(\alpha, \delta^*)$.
Given $\epsilon > 0$,
Let $\delta \in (0, \delta^*)$ be s.t. $\forall v \in B(0, \delta)$, $\|df_{\alpha+v} - df_\alpha - d^2 f_\alpha v\| \leq \frac{\epsilon}{2}\|v\|$.
Now $\forall v_1, v_2 \in B(0, \frac{\delta}{2})$,
$\|df_{\alpha+v_1+v_2} - df_{\alpha+v_1} - d^2 f_\alpha v_2\| \leq \epsilon(\|v_1\| + \|v_2\|) \leq \epsilon\delta$.
Given $v_2 \in B(0, \frac{\delta}{2})$,
apply the MVT to $v_1 \mapsto f(\alpha + v_1 + v_2) - f(\alpha + v_1) - d^2 f_\alpha v_2 v_1$ on $B(0, \frac{\delta}{2})$.
We get $\forall v_1 \in B(0, \frac{\delta}{2})$,
$\|f(\alpha + v_1 + v_2) - f(\alpha + v_1) - f(\alpha + v_2) + f(\alpha) - d^2 f_\alpha v_2 v_1\| \leq \frac{1}{2}\epsilon\delta^2$.
Hence $\forall \epsilon > 0$, $\exists \delta \in (0, \delta^*)$ s.t. $\forall v_1, v_2 \in B(0, \frac{\delta}{2})$, $\|d^2 f_\alpha v_1 v_2 - d^2 f_\alpha v_2 v_1\| \leq \epsilon\delta^2$.
Given $\xi, \eta \in V \setminus \{0\}$, $\epsilon > 0$,
Let δ be as above, and $v_1 = \lambda_1\xi$,$v_2 = \lambda_2\eta$ s.t. $\|v_1\|, \|v_2\| = \frac{\delta}{4}$.
Now, $|\lambda_1\lambda_2|\|d^2 f_\alpha \xi\eta - d^2 f_\alpha \eta\xi\| \leq \epsilon\delta^2 = 16\epsilon|\lambda_1\lambda_2|\|\xi\eta\|$.
The claim follows. ∎

Corollary 1.1.1 (Symmetry of derivatives)
Suppose V, W are Normed Spaces,
f is defined and n-times differentiable near $\alpha \in V$ and maps to W,
f is $n+1$-times differentiable at α.
Then $d^{(n+1)} f_\alpha$ is symmetric.

Proof.
Given $1 \leq k \leq n$, $v_1, \ldots, v_{k-1}, v_k, v_{k+1} \in V$,
Consider $x \mapsto d^{(n-1)} f_x v_1 \ldots v_{k-1} = E_{v_{k-1}} \circ \ldots \circ E_{v_1} \circ (x \mapsto d^{(n-1)} f_x)$.
(E_v denotes the map evaluation at v)
This function is defined and differentiable near α,

with derivative $E_{v_{k-1}} \circ \ldots \circ E_{v_1} \circ (x \mapsto d^{(n)} f_x)$.
This is differentiable at α, with derivative $E_{v_{k-1}} \circ \ldots \circ E_{v_1} \circ d^{(n+1)} f_\alpha$.
Using the preceding theorem,
$d^{(n+1)} f_\alpha v_1 \ldots v_{k-1} v_k v_{k+1} = d^{(n+1)} f_\alpha v_1 \ldots v_{k-1} v_{k+1} v_k$.
So, the corollary follows. ■

Theorem 1.2 (Generalized Mean Value Theorem with domain $\mathbb{R}$)
Suppose V is a Normed Space,
$f : [0,1] \to V$ is continuous on $[0,1]$ and differentiable on $(0,1)$,
$g : [0,1] \to \mathbb{R}$ is continuous on $[0,1]$ and differentiable on $(0,1)$,
$\|df_x\| \leq dg_x(1)$ on $(0,1)$.
Then $\|f(1) - f(0)\| \leq g(1) - g(0)$.

Proof.
First, to simplify the argument suppose $f(0) = 0$ and $g(0) = 0$.
Given $\epsilon > 0$.
Let $S_\epsilon := \{ x \in [0,1] : \forall u \in [0,x], \|f(u)\| \leq g(u) + \epsilon u + \epsilon \}$.
Note $\exists x \in (0,1) \cap S_\epsilon$ by continuity.
Suppose $x \in (0,1) \cap S_\epsilon$.
Since f, g are both differentiable at x,
$\exists x' \in (x,1)$ s.t. $\forall u \in (x, x']$,
$\|f(u) - f(x) - df_x(u-x)\| \leq \frac{\epsilon}{2}(u-x)$ and $|g(u) - g(x) - dg_x(u-x)| \leq \frac{\epsilon}{2}(u-x)$.
Then, $\|f(u)\| \leq g(x) + dg_x(u-x) + \frac{\epsilon}{2}(u+x) + \epsilon \leq g(u) + \epsilon u + \epsilon$.
So $x' \in S_\epsilon$.
Now let $s := \sup S_\epsilon$.
Suppose $u \in [0,s)$. Let $x \in (u,s] \cap S_\epsilon$, then $u \in S_\epsilon$.
Also, since for each $u \in [0,s)$, $\|f(u)\| \leq g(u) + \epsilon u + \epsilon$, and $s > 0$,
$\|f(s)\| \leq g(s) + \epsilon s + \epsilon$ follows from continuity at s.
Hence $s \in S_\epsilon$.
Now, $s \notin [0,1)$, so $s = 1$.
Therefore, $\|f(1)\| \leq g(1) + 2\epsilon$.
Hence, $\|f(1)\| \leq g(1)$.
In the general case, let $f_1 := f - f(0)$ and $g_1 := g - g(0)$ and the claim is proven.
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Corollary 1.2.1 (Mean Value Theorem with domain $\mathbb{R}$)
Suppose V is a Normed Space,
$f : [0,1] \to V$ is continuous on $[0,1]$ and differentiable on $(0,1)$,
$\|df_x\| \leq M$ on $(0,1)$.
Then $\|f(1) - f(0)\| \leq M$.

We introduce some definitions.

Definition 1.1. A nonempty subset U of a Normed Space is a *region* if it is open and connected.

Definition 1.2. A *polygonal path* in a Normed Space V is a finite sequence $x_0, x_1, \ldots, x_n$ where $n \leq 1$.

Its length is $\|x_1 - x_0\| + \ldots + \|x_n - x_{n-1}\|$.
It is said to be injective if the map $l : [0, n] \to V$ s.t. $l(k) = x_k$ and is constant speed in between is injective.

Definition 1.3. Suppose U is a region in V, a Normed Space, $a, b \in U$, $d'_U(a, b)$ is defined to be the infimum of the lengths of polygonal paths from a to b.

This is well-defined per a standard result.
This is a metric on U which agrees with the induced metric if U is also convex.

Lemma 1.3 (Polygonal path skrinking)
Suppose V is a Normed Space,
$x_0, x_1, \ldots, x_m$ a polygonal path in V, where $x_0 \neq x_m$.
Then, there exists an injective polygonal path from x_0 to x_m
whose image is a subset of the original path and is of shorter or equal length.

Proof.
We induct on m,
the base case $m = 1$ is trivial.
Now, assume $m \geq 2$.
Suppose $x_0 = x_{m-1}$, then the path x_{m-1}, x_mworks, so assume otherwise.
Applying the inductive hypothesis, there is a injective polygonal path from x_0 to x_{m-1}
which is a subset of the path $x_0, \ldots, x_{m-1}$ and is of shorter or equal length.
Let the new path be $x_0 = y_0, \ldots, y_n = x_{m-1}$.
Let l be its corresponding map from $[0, n]$ to V.
($l(k) = y_k$, and constant speed in between)
If $x_{m-1} = x_m$, then this new path works, so suppose not.
Consider $l^{-1}([x_{m-1}, x_m])$
(where $[x_{m-1}, x_m]$ is the segment between x_{m-1} and x_m)
This is a closed nonempty subset of $\mathbb{R}$ which is bounded below (it contains n), so it contains its infimum, call it t^*.
Suppose $t^* = 0$, then x_0, x_m works since $x_0 \in [x_{m-1}, x_m]$.
So suppose $t^* > 0$,
Now, $y_0, \ldots, y_{\lceil t^* \rceil - 1}, l(t^*)$ is an injective path whose image is contained in the original and is of shorter or equal length to $x_0, \ldots, x_{m-1}$.
If $l(t^*) = x_m$, then $y_0, \ldots, y_{\lceil t^* \rceil - 1}, l(t^*)$ works.
Otherwise, the path $y_0, \ldots, y_{\lceil t^* \rceil - 1}, l(t^*), x_m$ works.
since $l([0, t^*))$ is disjoint from $[l(t^*), x_m] \subseteq [x_{m-1}, x_m]$.

■

The previous proof isn't very elegent, see if you can shorten it.

Theorem 1.4 (Generalized Mean Value Theorem)
Suppose V, W are Normed Spaces, U is a region in V containing a,
$f : U \to W$ is differentiable,
$g : [0, \infty) \to \mathbb{R}$ continuous on $[0, \infty)$ and differentiable on $(0, \infty)$,
g is increasing and $\|df_x\| \leq dg_{d'_U(x,a)}(1)$ on $U \setminus \{a\}$.
Then for each $x \in U$, $\|f(x) - f(a)\| \leq g(d'_U(x, a)) - g(0)$.

Proof.
The theorem is trivial if $x = a$, so assume otherwise. Suppose $x_0, x_1, \ldots, x_m$ be an injective polygonal path from a to x in U.
let its length be $l = l_1 + \ldots + l_m$, (where $l_k = \|x_k - x_{k-1}\|$)
We will let $s_k := l_1 + \ldots + l_k$.
Given $1 \leq k \leq m$, we wish to show $\|f(x_k) - f(x_{k-1})\| \leq g(s_k) - g(s_{k-1})$.
Consider $F : [0,1] \to W$ s.t. $F(t) = f(tx_k + (1-t)x_{k-1})$.
F is continuous on $[0,1]$ and differentiable on $(0,1)$,
with $dF = df_{tx_k+(1-t)x_{k-1}} \circ (\delta t \mapsto \delta t(x_k - x_{k-1}))$.
Consider also $G : [0,1] \to \mathbb{R}$ s.t. $G(t) = g(s_{k-1} + tl_k)$. G is continuous on $[0,1]$ and differentiable on $(0,1)$,
with $dG_t(1) = l_k dg_{s_{k-1}+tl_k}(1)$.
Now, $\|dF_t\| \leq l_k dg_{d'_U(tx_k+(1-t)x_{k-1},a)}(1) \leq dG_t(1)$ on $(0,1)$,
so applying the general MVT with domain $\mathbb{R}$,
$\|f(x_k) - f(x_{k-1})\| \leq g(s_k) - g(s_{k-1})$.
Hence, $\|f(x) - f(a)\| \leq g(l) - g(0)$.
Now, by the lemma above,
this inequality holds for all polygonal paths from a to x in U.
The theorem follows. ■

Corollary 1.4.1 (Mean Value Theorem (MVT))
Suppose V, W are Normed Spaces, U is a region in V,
$f : U \to W$ is differentiable, $\|df_x\| \leq M$ on U.
Then for each $a, b \in U$, $\|f(b) - f(a)\| \leq M d'_U(a,b)$.

Theorem 1.5 (Taylor's Theorem)
Suppose V, W are Normed Spaces, U is a region in V,
$f : U \to W$ is $n+1$-times differentiable,
$\|d^{n+1} f_x\| \leq M$ on U.
Then for each $a, b \in U$, $\|f(b) - \sum\limits_{i=0}^{n} \frac{1}{i!} d^{(i)} f_a (b-a)^{\otimes i}\| \leq \frac{M}{(n+1)!}(d'_U(a,b))^{n+1}$.

Proof.
In contrast to the real analytic case, here we proceed by induction.
The $n = 0$ case is simply MVT, now assume $n \geq 1$.
Given $a \in U$,
apply the inductive hypothesis to df to get
$\forall b \in U \ \|df_b - \sum\limits_{i=0}^{n-1} \frac{1}{i!} d^{(i+1)} f_a (b-a)^{\otimes i}\| \leq \frac{M}{n!}(d'_U(a,b))^n$.
Now consider the function $b \mapsto d^{(i)} f_a (b-a)^{\otimes i}$ for $i \geq 1$.
It has derivative $b \mapsto i d^{(i)} f_a (b-a)^{\otimes (i-1)}$,
this can be shown with chain rule and the fact that composition
is a bounded linear map together with the symmetry of derivatives.
Thus $b \mapsto f(b) - \sum\limits_{i=0}^{n} \frac{1}{i!} d^{(i)} f_a (b-a)^{\otimes i}$ has derivative
$b \mapsto df_b - \sum\limits_{i=0}^{n-1} \frac{1}{i!} d^{(i+1)} f_a (b-a)^{\otimes i}$.

Now apply the general MVT with the appropriate g to get the result.

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