# Counting $l$-adic sheaves

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to Gérard Laumon, on the occasion of his sixtieth birthday

## 1 Introduction

### 1.1

Let $X_0$ be a projective, smooth, absolutely connected curve of genus $g$, over a finite field $\mathbb{F}_q$ of characteristic $p$, and let $X$ be the one deduced from it by extension of scalars to an algebraic closure $F$ of $\mathbb{F}_q$:

$$\begin{array}{ccc} X & \longrightarrow & X_0 \\ \uparrow & & \uparrow \\ \mathrm{Spec}(F) & \longrightarrow & \mathrm{Spec}(\mathbb{F}_q) \end{array} \tag{1}$$

We will denote Frob: $X \to X$ the Frobenius endomorphism "$x \mapsto x^q$" of the $F$-scheme $X$. Fix a prime number $l \neq p$, an algebraic closure $\overline{\mathbb{Q}}_l$ of $\mathbb{Q}_l$, and let $E$ be the set of isomorphism classes of rank 2 irreducible smooth $\overline{\mathbb{Q}}_l$-sheaves on $X$. The inverse image by Frob induces a permutation of $E$. We will denote it by $\phi$.

### 1.2

In the article [Dr], which remains for me as mysterious as it was 31 years ago, Drinfeld calculates the number of fixed points of $\phi : E \to E$.

A $\overline{\mathbb{Q}}_l$-sheaf $\mathcal{L}_0$ of rank one on $\mathrm{Spec}(\mathbb{F}_q)$ is determined up to isomorphism by the unit $\lambda$ of $\overline{\mathbb{Q}}_l$ such that the geometric Frobenius $\mathrm{Fr} \in \mathrm{Gal}(F/\mathbb{F}_q)$ acts by multiplication by $\lambda$ on the fiber of $\mathcal{L}_0$ at the geometric point $\mathrm{Spec}(F)$. The $\mathbb{F}_q$-twist of $\mathcal{F}_0$ on $X_0$ by $\mathcal{L}_0$ is the tensor product with the inverse image of $\mathcal{L}_0$ on $X_0$. By abuse of language, we will also say "$\mathbb{F}_q$-twist by $\lambda$".

Drinfeld uses that the isomorphism class of a smooth $\overline{\mathbb{Q}}_l$-sheaf $\mathcal{F}$ on $X$ is fixed by $\mathrm{Frob}^*$ if and only if $\mathcal{F}$ is the inverse image of a $\overline{\mathbb{Q}}_l$-sheaf $\mathcal{F}_0$ on $X_0$, and that, if $\mathcal{F}$ is irreducible, $\mathcal{F}_0$ is unique up to $\mathbb{F}_q$-twist. The problem solved by [Dr] becomes that of counting the rank 2 smooth $\overline{\mathbb{Q}}_l$-sheaves on $X_0$, taken up to $\mathbb{F}_q$-twist, and considering only those which are irreducible and remain so after inverse image on $X$.

In 1981, we almost had the correspondence between irreducible rank 2 smooth $\overline{\mathbb{Q}}_l$-sheaves on $X_0$ and everywhere unramified cuspidal automorphic representations for $\mathrm{GL}(2, k(X_0))$. Thanks to this, Drinfeld reduced the problem to an application of the trace formula for $\mathrm{GL}(2)$. This reduction shows that the sought number is independent of $l$. The formula obtained by Drinfeld has the miraculous properties (A) and (B) below.

For $n \geq 1$, let $N_n$ be the number of fixed points of the $n$-th iterate $\phi^n : E \to E$ of $\phi$. Calculating $N_n$ is the problem above, with $X_0/\mathbb{F}_q$ replaced by $(X_0 \otimes_{\mathbb{F}_q} \mathbb{F}_{q^n})/\mathbb{F}_{q^n}$.

(A) The function $n \mapsto N_n$ has the form

$$N_n = \sum a_i \beta_i^n, \tag{2}$$

for integers $a_i$ and suitable Weil numbers $\beta_i$. The $\beta_i$ are monomials in $q$ and in the eigenvalues of the $\mathrm{Frob}^*$ endomorphism of $H^1(X, \mathbb{Q}_l)$. The genus $g$ being fixed, which monomials $\beta_i$ appear and with what multiplicities $a_i$ does not depend on the curve considered.

The trace formula expresses $N_n$ as a sum of several terms. Taken separately, these terms do not all have the form (2): to obtain (2), they must be combined astutely.

Let $\Sigma$ be a compact Riemann surface of genus $g$. Consider the local systems of complex vector spaces on $\Sigma$. Let $M$ be the moduli space of those which are irreducible of rank 2. The space $M$ and the set

$E$ are empty if $g = 0$ or 1. Otherwise, $M$ is a connected complex symplectic manifold. Its complex dimension is therefore an even integer $2N$.

(B) In (2), the dominant term is $q^N$: one of the $\beta_i$ is $q^N$, its multiplicity $a_i$ is 1, and the other $\beta_i$ satisfy $|\beta_i| < q^N$.

### 1.3

The formula (2) is reminiscent of a Lefschetz trace formula where, in good cases, the number of fixed points of the iterates $T^n$ ($n \geq 1$) of an endomorphism $T$ of a space $S$ is

$$\mathrm{Tr}(T^{*n}, H_c^*(S)) := \sum (-1)^i \,\mathrm{Tr}(T^{*n}, H_c^i(S)). \tag{3}$$

The "?" is there to remind that one must consider a cohomology with support conditions. The support conditions to be imposed depend on the behavior at infinity of $T$. For example, if for an exhaustion function $f$ on $S$ we have $f(T(x)) > f(x)$ (resp. $f(T(x)) < f(x)$) for $f(x)$ large enough, the cohomology to consider is the cohomology with compact support (resp. ordinary).

Suppose that $H_c^i$ is of finite dimension, so that (3) makes sense. The right-hand side of (3) is then of the form (2): it is $\sum a(\beta)\beta^n$, where $\beta$ runs through the eigenvalues of $T^*$ and where the integer $a(\beta)$ is the alternating sum of the multiplicities of $\beta$ as an eigenvalue of the endomorphisms $T^*$ of the $H_c^i(S)$.

Unfortunately, I have no idea how to consider the set $E$ of isomorphism classes of irreducible rank 2 smooth $\mathbb{Q}_l$-sheaves on $X$ as a "space" having a cohomology $H_c^*(E)$ such that one can hope that the number of fixed points of $\phi^n$ is given by a formula of type (3).

### 1.4

Even if one does not know how to think geometrically about $E$, if $f \in E$ is the isomorphism class of $\mathcal{F}$, we know what the formal completion $E_f$ of $E$ at $f$ should be. A finite-dimensional local $\mathbb{Q}_l$-algebra $A$ is augmented towards $\mathbb{Q}_l$. A deformation of $\mathcal{F}$ over $\mathrm{Spec}(A)$ is a smooth $\mathbb{Q}_l$-sheaf $\mathcal{F}_A$ on $X$, equipped with an $A$-module structure and an isomorphism $\mathcal{F}_A \otimes_A \mathbb{Q}_l \to \mathcal{F}$, such that at a point $x$ (and therefore at any point $x$ of $X$), $(\mathcal{F}_A)_x$ is a free $A$-module. The functor on $\mathrm{Spec}(A)$ of isomorphism classes of deformations of $\mathcal{F}$ over $\mathrm{Spec}(A)$ is pro-representable. Let us denote by $R_\mathcal{F}$ the complete local $\mathbb{Q}_l$-algebra with residue field $\mathbb{Q}_l$ such that $\mathrm{Spec}(R_\mathcal{F})$ pro-represents it. There is no obstruction to deformations, and $R_\mathcal{F}$ is thus isomorphic to a formal power series algebra $\mathbb{Q}_l[[t_1, \ldots, t_m]]$. Because $\mathcal{F}$ is irreducible, its only automorphisms are multiplications by a $\lambda \in \mathbb{Q}_l^*$. They extend to any deformation. If $\mathcal{F}$ and $\mathcal{F}'$ are two representatives of the isomorphism class $f$, two isomorphisms of $\mathcal{F}$ with $\mathcal{F}'$ thus induce the same isomorphism between $R_\mathcal{F}$ and $R_{\mathcal{F}'}$: $E_f := \mathrm{Specf}(R_\mathcal{F})$ does not depend, up to unique isomorphism, on $f$.

The Zariski tangent space of $E$ at $f$ is $H^1(X, \mathrm{End}(\mathcal{F}))$. The symmetric bilinear form $\mathrm{Tr}(fg)$ on $\mathrm{End}(\mathcal{F})$ is an auto-duality. It induces on $H^1(X, \mathrm{End}(\mathcal{F}))$ a symplectic form with values in $H^2(X, \mathbb{Q}_l) = \mathbb{Q}_l(-1)$. The same construction gives on $\mathbb{C}$ the symplectic structure of $M$. The formal completion $E_f$ is thus smooth of even dimension, the same as that of $M$.

The inverse image functor by Frob: $X \to X$ transforms a deformation of $\mathcal{F}$ into a deformation of $\mathrm{Frob}^*\mathcal{F}$. It thus induces a morphism

$$\hat{\phi}_f : E_f \to E_{\phi(f)}.$$

The argument that shows that, because Frob induces an equivalence of étale sites, $\phi$ is bijective, also shows that the $\hat{\phi}_f$ are isomorphisms. Another proof: let $\omega$ be the symplectic form defined above on the tangent space at $f$. If $[t] \in H^1(X, \mathrm{End}(\mathcal{F}))$ is the class of the tangent vector $t$ to $\mathcal{F}$, the class of $d\hat{\phi}_f(t)$ in $H^1(X, \mathrm{End}(\mathrm{Frob}^*\mathcal{F}))$ is $\mathrm{Frob}^*([t])$. For two tangent vectors $t_1$ and $t_2$, $\omega(d\hat{\phi}_f(t_1), d\hat{\phi}_f(t_2))$ is therefore the inverse image, by $\mathrm{Frob}^*$, of $\omega(t_1, t_2) \in \mathbb{Q}_l(-1)$, identified with $H^2(X, \mathbb{Q}_l)$. Because Frob: $X \to X$ is of degree $q$, the endomorphism $\mathrm{Frob}^*$ of $H^2(X, \mathbb{Q}_l)$ is multiplication by $q$, and

$$\omega(d\hat{\phi}_f(t_1), d\hat{\phi}_f(t_2)) = q\omega(t_1, t_2),$$

which is also written $(d\hat{\phi}_f)^*\omega = q\omega$. The differential $d\hat{\phi}_f$ is therefore an isomorphism. Question: can we hope that $\Lambda^N\omega$ provides a cohomology class responsible for the dominant term $q^N$ of (2).

The application $d\hat{\phi}_f$ between Zariski tangent spaces identifies with

$$\mathrm{Frob}^* : H^1(X, \mathrm{End}(\mathcal{F})) \to H^1(X, \mathrm{End}(\mathrm{Frob}^*\mathcal{F})).$$

If $f$ is a fixed point of $\phi$, $\mathcal{F}$ is the inverse image of $\mathcal{F}_0$ on $X_0$. We know that we can choose $\mathcal{F}_0$ to be pure, and $\mathrm{End}(\mathcal{F}_0)$ is therefore pure of weight 0. The eigenvalues of $d\hat{\phi}_f$ on the Zariski tangent space of $E_f$ at $f$ are therefore Weil numbers of weight 1. In particular, no eigenvalue of $d\hat{\phi}_f$ at $f$ is equal to 1. This is what makes it reasonable that a Lefschetz formula (3) could count the fixed points of $\phi^n$, all taken with multiplicity one.

### 1.5

If $Y_0$ is a finite type scheme over $\mathbb{F}_q$, the Frobenius endomorphism Frob of $Y := Y_0 \otimes_{\mathbb{F}_q} F$ induces a permutation Frob of $Y(F)$. The preceding shows that the nature of $(E, \phi)$ is different from that of $(Y, \mathrm{Frob})$:

(i) $E$ is "over $\mathbb{Q}_l$", thus an object of characteristic 0.

(ii) The $\hat{\phi}_f$ are isomorphisms. We would like to say that $\phi$ is of "degree one".

(iii) The fixed points of $\phi$ are isolated because the eigenvalues of $d\hat{\phi}$ at a fixed point are Weil numbers of weight 1. Those of Frob are isolated because $d\mathrm{Frob} = 0$.

(iv) If $E$ is of dimension $2N$, in the sense that the $E_f$ are, the order of magnitude of the number of fixed points of $\phi^n$ is $(q^n)^N$, rather than $(q^n)^{2N}$.

### 1.6

The moduli space $M$ is a complex algebraic variety. Contrary to what a hasty analogy might suggest, $E$ is not naturally the space of $\mathbb{Q}_l$-points of an algebraic variety over $\mathbb{Q}_l$. Let $x \in X(F)$. The set $E$ identifies with the set of isomorphism classes of irreducible continuous representations $V$ of $\pi_1(X, x)$, of dimension 2 over $\mathbb{Q}_l$. As coordinates on $E$, we would like to take the $\mathrm{Tr}(\gamma, V)$ for $\gamma \in \pi_1(X, x)$. In the complex case, this construction provides on $M$ its "Betti" structure of a complex algebraic variety. Here, $\pi_1(X, x)$ being a profinite group, any representation $V$ admits an invariant lattice, and the $\mathrm{Tr}(\gamma, V)$ are values in the valuation ring $\mathbb{Z}_l$ of the valued field $\mathbb{Q}_l$.

The point of view "$E$ as a space of representations of $\pi_1(X, x)$" does not help to understand that the number of fixed points of $\phi$ is independent of $l$. Another mystery: why, the genus $g$ being fixed, does the number of fixed points of $\phi$ admit a uniform description in terms of $H^1(X, \mathbb{Q}_l)$, of its symplectic structure with values in $\mathbb{Q}_l(-1)$ and of its Frob* endomorphism, whereas $\pi_1(X, x)$ is a quotient of the profinite completion of the topological $\pi_1$ of $\Sigma$, which quotient depends on $X$?

### 1.7

Let $E_r$ be the set of isomorphism classes of irreducible smooth $\mathbb{Q}_l$-sheaves of rank $r$ on $X$. Thanks to Lafforgue [L], the number of fixed points of the bijection Frob*: $E_r \to E_r$ has an automorphic interpretation. For $S_0 \subset X_0$ a finite set of closed points of $X$, and $S = S_0 \otimes_{\mathbb{F}_q} F$, we can also consider smooth irreducible $\mathbb{Q}_l$-sheaves on $X - S$, with prescribed ramification at each $s \in S$. In all cases that have been calculated, the miracles (A) and (B) discovered by Drinfeld persist, mutatis mutandis. The purpose of this paper is to give an overview of what is known.

## 2 What to count?

### 2.1

Fix an integer $r \geq 1$, and let $X_0, S_0/\mathbb{F}_q$ and $X, S/F$ be as in 1.1 and 1.7. For $s \in S$, let $X_{(s)}$ be the henselization of $X$ at $s$ and $X^*_{(s)} := X_{(s)} - \{s\}$. Suppose given, for each $s \in S$, a smooth $\mathbb{Q}_l$-sheaf of rank $r$, $\mathcal{R}(s)$, on $X^*_{(s)}$. Only its isomorphism class matters. The morphism Frob: $X \to X$ induces a morphism still denoted Frob from $X^*_{(s)}$ to $X^*_{(\mathrm{Frob}(s))}$. We assume that $\mathcal{R}(s)$ is isomorphic to the inverse image of $\mathcal{R}(\mathrm{Frob}(s))$ by Frob:

$$\mathcal{R}(s) \simeq \mathrm{Frob}^*(\mathcal{R}(\mathrm{Frob}(s))). \tag{4}$$

Let $\mathcal{R} := (\mathcal{R}(s))_{s \in S}$ be the family of $\mathcal{R}(s)$ and let us denote by $E(\mathcal{R})$ the set of isomorphism classes of irreducible smooth $\mathbb{Q}_l$-sheaves of rank $r$, $\mathcal{F}$, on $X - S$, such that for all $s \in S$, the inverse image $\mathcal{F}|_{X^*_{(s)}}$

of $\mathcal{F}$ on $X^*_{(s)}$ is isomorphic to $\mathcal{R}(s)$. The inverse image by Frob: $X \to X$ induces a permutation of $E(\mathcal{R})$. We will denote it by $\phi$.

The questions we ask ourselves are the following.

(i) Calculate, for $n \geq 1$, the number $N_n(\mathcal{R})$ of fixed points of the $n$-th iterate $\phi^n$ of $\phi$. We note that replacing $(X_0, S_0)$ by $(X_0, S_0) \otimes_{\mathbb{F}_q} \mathbb{F}_{q^n}$ replaces $\phi$ by $\phi^n$.

(ii) Determine if the miracles 1.2 (A) (B) subsist.

(iii) If so, why?

## 2.2

Let $\mathcal{F}$ be a smooth $\mathbb{Q}_l$-sheaf on $X - S$. As in 1.2, we have:

**Lemma 2.1.** *(i) For the isomorphism class of $\mathcal{F}$ to be fixed under Frob*, it is necessary and sufficient that $\mathcal{F}$ be isomorphic to the inverse image on $X$ of a $\mathbb{Q}_l$-sheaf $\mathcal{F}_0$ on $X_0$. Moreover, if $\mathcal{F}$ is irreducible, (ii) such an $\mathcal{F}_0$ is unique up to $\mathbb{F}_q$-twist, and (iii) the $\mathbb{F}_q$-twists of $\mathcal{F}_0$ by $\lambda \in \mathbb{Z}_l^*$ (1.2) are pairwise non-isomorphic.*

According to 2.2.1, the number of fixed points of $\phi$ is the number of classes, modulo $\mathbb{F}_q$-twist, of smooth $\mathbb{Q}_l$-sheaves of rank $r$, $\mathcal{F}_0$, on $X_0 - S_0$, such that

(a) the inverse image $\mathcal{F}$ of $\mathcal{F}_0$ on $X - S$ is irreducible;

(b) for each $s \in S$, $\mathcal{F}|_{X^*_{(s)}}$ is isomorphic to $\mathcal{R}(s)$.

## 2.3

Let $K := k(X_0)$. For each closed point $x$ of $X_0$, let $k(x)$ be the residue field of $x$, $\deg(x) := [k(x) : \mathbb{F}_q]$ its degree, $K_x$ the completion of $K$ at $x$, and $\mathcal{O}_x$ the valuation ring of $v_x$ of $K_x$. Let $\mathbb{A}$ be the adele ring of $K$. The sum $v := \sum \deg(x) v_x : \mathbb{A}^* \to \mathbb{Z}$ is trivial on $K^*$. We have $||a|| = q^{-v(a)}$.

By Lafforgue, $\mathcal{F}_0$ corresponds to a cuspidal automorphic representation $\pi$ of $\mathrm{GL}(r, \mathbb{A})$. If $\mathcal{F}'_0$ is the $\mathbb{F}_q$-twist of $\mathcal{F}_0$ by $\lambda$, the corresponding representation $\pi'$ is the $\mathbb{F}_q$-twist of $\pi$ by $\lambda$, defined as follows. As a representation, it is the tensor product of $\pi$ by the one-dimensional representation $g \mapsto \lambda^{v \det g}$. As a space of functions on $\mathrm{GL}(r, \mathbb{A})$, it is the space of products of $f$ in $\pi$ by the character $g \mapsto \lambda^{v \det g}$.

Translated into the automorphic language, problem 2.1 (i) is that of counting, up to $\mathbb{F}_q$-twist, the cuspidal automorphic representations $\pi$ of $\mathrm{GL}(r, \mathbb{A})$, unramified outside of $S_0$, and such that

(a') For all $n \geq 1$, $\pi$ remains cuspidal after base change from $K$ to $K \otimes_{\mathbb{F}_q} \mathbb{F}_{q^n}$. This condition expresses the irreducibility 2.2 (a) of $\mathcal{F}$. It is sufficient to check it for $n$ dividing $r$.

(b') For each $s_0 \in S_0$ and $s \in S$ above $s_0$, the local component $\pi_{s_0}$ of $\pi$ satisfies a condition dictated by $\mathcal{R}(s)$ and by the local Langlands correspondence.

**Lemma 2.2.** *If an automorphic cuspidal $\pi$ satisfies (a'), the $\mathbb{F}_q$-twists of $\pi$ are pairwise distinct.*

This is the translation of 2.2.1 (iii). If $d$ is an idèle satisfying $v(d) = 1$, it is therefore equivalent to count, up to $\mathbb{F}_q$-twist, the $\pi$ satisfying (a') (b'), or to count only those whose central character $\omega_\pi$ satisfies $\omega_\pi(d) = 1$, and to divide by $r$.

In finite characteristic, the notion of automorphic representation is purely algebraic. This allows one to take the automorphic representations $\pi$ as having values in $\mathbb{Q}_l$, rather than $\mathbb{C}$. If one does so, $\pi_{s_0}$ is an irreducible admissible $\mathbb{Q}_l$-representation of $\mathrm{GL}(r, K_{s_0})$, and the local Langlands correspondence associates to it an isomorphism class of $F$-semi-simple continuous $\mathbb{Q}_l$-representations of dimension $r$ of the local Weil group $W(\bar{K}_{s_0}/K_{s_0})$. Condition (b') is that its restriction to the inertia subgroup is given by $\mathcal{R}(s)$ (cf 2.4 below).

Almost all known results on question 2.1 (i) are obtained via the dictionary 2.3.

## 2.4

In 2.1, we used the language of $\mathbb{Q}_l$-sheaves. This spared us from having to choose base points. An equivalent language is that of Galois representations. Here is the translation.

Let $\overline{k(X)}$ be an algebraic closure of the rational function field $k(X)$ of $X$, and $k(X)_{nr}$ the largest sub-extension unramified outside of $S$. The Galois group $\mathrm{Gal}(\overline{k(X)}_{nr}/k(X))$ is the fundamental group of $X$ at the geometric point $\bar\eta := \mathrm{Spec}(\overline{k(X)})$ of $X$: the functor "fiber at $\bar\eta$" is an equivalence of categories, from smooth (resp. and irreducible of rank $r$) $\mathbb{Q}_l$-sheaves on $X - S$, to linear (resp. and irreducible of rank $r$) $\mathbb{Q}_l$-representations of $\mathrm{Gal}(\overline{k(X)}_{nr}/k(X))$.

The scheme $X^*_{(s)}$ is the spectrum of an extension $k(X^*_{(s)})$ of $k(X)$. If we choose an embedding of this extension into $\overline{k(X)}$, the corresponding inertia group $I_s := \mathrm{Gal}(\overline{k(X)}/k(X^*_{(s)}))$ is also the fundamental group of $X^*_{(s)}$ at $\bar\eta$. It maps into $\pi_1(X,\bar\eta)$:

$$\pi_1(X^*_{(s)},\bar\eta) = I_s = \mathrm{Gal}(\overline{k(X)}/k(X^*_{(s)})) \to \mathrm{Gal}(\overline{k(X)}_{nr}/k(X)) = \pi_1(X,\bar\eta). \tag{5}$$

The fiber at $\bar\eta$ of $\mathcal{R}(s)$ is a representation $\rho_s$ of $I_s$. Let $\mathcal{F}$ be a smooth $\mathbb{Q}_l$-sheaf on $X - S$, with inverse image $\mathcal{F}|_{X^*_{(s)}}$ on $X^*_{(s)}$. The fibers at the geometric point $\bar\eta$ are respectively a representation of $\pi_1(X-S,\bar\eta) = \mathrm{Gal}(\overline{k(X)}_{nr}/k(X))$ and its restriction to $\pi_1(X^*_{(s)},\bar\eta) = I_s$, by (5). That $\mathcal{F}|_{X^*_{(s)}}$ is isomorphic to $\mathcal{R}(s)$ is equivalent to this restriction to $I_s$ being isomorphic to $\rho_s$.

Condition (4) is equivalent to the $\rho_s$, for $s$ above $s_0$, coming from a representation of a decomposition group $D_{s_0} \subset \mathrm{Gal}(\overline{k(X)}/k(X_0))$ at $s_0$. It implies that the representations $\rho_s$ are quasi-unipotent.

## 2.5

I hope that problem 2.1 (i) is more tractable when a smooth $\mathbb{Q}_l$-sheaf $\mathcal{F}$ such that $\mathcal{F}|_{X^*_{(s)}} \simeq \mathcal{R}(s)$ for all $s \in S$ is automatically irreducible. This is the case if the following condition is satisfied. Consider an integer $0 < r' < r$ and the data, for each $s \in S$, of a smooth sub-$\mathbb{Q}_l$-sheaf $\mathcal{R}'(s)$ of rank $r'$ of $\mathcal{R}(s)$. Let $\mathcal{R}''(s) := \mathcal{R}(s)/\mathcal{R}'(s)$. The condition is that

(2.5.1) Whatever $r'$ and the $\mathcal{R}'(s)$, either there is no smooth $\mathbb{Q}_l$-sheaf $\mathcal{F}'$ of rank $r'$ on $X - S$ such that $\mathcal{F}'|_{X^*_{(s)}} \simeq \mathcal{R}'(s)$ for all $s \in S$, or there is no smooth $\mathbb{Q}_l$-sheaf $\mathcal{F}''$ of rank $r - r'$ on $X - S$ such that $\mathcal{F}''|_{X^*_{(s)}} \simeq \mathcal{R}''(s)$ for all $s \in S$.

Condition (2.5.1) is satisfied if one of the $\mathcal{R}(s)$ is irreducible. For other cases, see 2.7.4, 2.8 and 2.9.

## 2.6

Let us set

$$\hat{\mathbb{Z}}(1) := \lim_{\leftarrow} \mu_n(F). \tag{6}$$

The projective limit is taken over the set of subgroups $\mu_n(F)$ of $F^*$ ($n$ prime to $p$), ordered by inclusion. The transition morphism from $\mu_{mn}(F)$ to $\mu_n(F)$ is $x \mapsto x^m$. We denote $\mathbb{F}_{q^k}$ the subfield with $q^k$ elements of $F$. The subgroups $\mathbb{F}^*_{q^k-1}$ of $F^*$ are cofinal among the $\mu_n(F)$. Since $(q^{kd}-1)/(q^k-1) = 1+q^k+\cdots+q^{k(d-1)}$, the transition morphism from $\mu_{q^{kd}-1} = \mathbb{F}^*_{q^{kd}}$ to $\mu_{q^k-1} = \mathbb{F}^*_{q^k}$ is the norm morphism, and

$$\hat{\mathbb{Z}}(1) \simeq \lim_{\leftarrow} \mathbb{F}^*_{q^k} \tag{7}$$

(projective limit along the norm morphisms between the $\mathbb{F}^*_{q^k}$). For a product $\prod k_\alpha$ of finite extensions of $\mathbb{F}_q$ and $\iota : k \to k_\alpha \to \mathbb{F}_{q^k} \to F$, we denote $\mathrm{pr}_\iota$ the composite

$$\mathrm{pr}_\iota : \hat{\mathbb{Z}}(1) \to \mathbb{F}^*_{q^k} \to k^*_\alpha \to k^*. \tag{8}$$

For $k = \mathbb{F}_{q^k}$ and $\iota$ the identical inclusion, $\mathrm{Gal}(F/\mathbb{F}_{q^k})$ acts on $\hat{\mathbb{Z}}(1)$ and (8) induces an isomorphism of the coinvariants with $\mathbb{F}^*_{q^k}$.

The local fundamental group $\pi_1(X^*_{(s)},\bar\eta) = I_s$ is an extension of $\hat{\mathbb{Z}}(1)$ by a pro-$p$-group $P_s$. The quotient $I_s/P_s \simeq \hat{\mathbb{Z}}(1)$ is the tame fundamental group of $X^*_{(s)}$. The morphism Frob: $X^*_{(s)} \to X^*_{(\mathrm{Frob}(s))}$ induces a morphism between fundamental groups, which are identified with $\hat{\mathbb{Z}}(1)$, this morphism is multiplication by $q$.

Suppose that the $\mathcal{R}(s)$ are semi-simple and tame, that is to say that the corresponding representations $\rho_s$ of the inertia groups $I_s$ (2.4) are semi-simple and trivial on $P_s$. Let $\mathcal{R}(s)$ be the multiset of $r$ characters of finite order of $\hat{\mathbb{Z}}(1)$, such that $\rho_s$ is isomorphic to the sum of the one-dimensional representations of $I_s$ they define. Condition (4) is that $\mathcal{R}(s)$ is deduced from $\mathcal{R}(\mathrm{Frob}(s))$ by multiplication by $q$ in the group of characters.

## 2.7

If $\mathcal{L}$ is a rank one $\mathbb{Q}_l$-sheaf on $X-S$, the restriction of $\mathcal{L}$ to $X^*_{(s)}$ defines a character $l_s$ of the fundamental group $I_s$ of $X^*_{(s)}$. Because $l \neq p$, the restriction of $l_s$ to $P_s$ is of finite order: there exists a power $P$ of $p$ such that the $l_s^P$ factor through characters, still denoted $l_s^P$, of $\hat{\mathbb{Z}}(1) = I_s/P_s$. We have

$$\prod_s l_s^P = 1. \tag{9}$$

The analogue on $\mathbb{C}$ of (9) is that, for $\Sigma$ as in 1.2 and $S$ a finite part of $\Sigma$, the sum over $S$ of a small positive circle around each $s \in S$ is a boundary in $\Sigma - S$. In rank $r$, (9) applied to $\Lambda^r\mathcal{F}$ provides a necessary condition for a sheaf $\mathcal{F}$ on $X-S$ with $\mathcal{F}|_{X^*_{(s)}} \simeq \mathcal{R}(s)$ to exist, and interesting cases where the hypothesis of the irreducibility criterion (2.5.1) is satisfied. Let $\rho_s$ be the representation of $I_s$ corresponding to $\mathcal{R}(s)$. Suppose as in 2.6 that the representations $\rho_s$ are semi-simple and tame. Let $\mathcal{R}(s)$ be the multiset of characters of $\hat{\mathbb{Z}}(1)$ of sum $\rho_s$. Suppose that

$$\prod_{s\in S}\prod_{\varepsilon\in\mathcal{R}(s)} \varepsilon = 1. \tag{10}$$

According to (9), this is a necessary condition for a sheaf $\mathcal{F}$ on $X-S$ to exist such that $\mathcal{F}|_{X^*_{(s)}} \simeq \mathcal{R}(s)$ for all $s \in S$. Consider the condition

(2.7.3) Whatever $0 < r' < r$ and the family $(\mathcal{R}'(s))_{s\in S}$ of sub-multisets with $r'$ elements of the multisets $\mathcal{R}(s)$, $\prod_{s\in S}\prod_{\varepsilon\in\mathcal{R}'(s)}\varepsilon \neq 1$.

**Lemma 2.3.** *Condition (2.7.3) implies (2.5.1), and thus the irreducibility of any smooth sheaf of rank $r$, $\mathcal{F}$, on $X-S$ satisfying $\mathcal{F}|_{X^*_{(s)}} \simeq \mathcal{R}(s)$ for all $s \in S$.*

*Remark* 2.4. There exist $\mathcal{R}(s)$ as above, giving rise to multisets $\mathcal{R}(s)$ satisfying (10), for which (2.7.3) is false, but where condition (2.5.1) is nevertheless satisfied. Here is an example. We take $X = \mathbb{P}^1$, $r = 4$, $|S| = 3$ and multisets of characters $\mathcal{R}(s)$ of the following type

$$\{a_1, a_1, a_3, a_4\}, \{b_1, b_1, b_3, b_4\}, \{c_1, c_2, c_3, c_4\}.$$

We suppose that the product of all these characters is 1 (10), and that $a_4b_4c_4 = 1$, so that (2.7.3) is false. If the $a_i, b_i, c_i$ are "general", condition (2.5.1) is satisfied, because there is no $\mathcal{F}$ of rank 3 with local monodromy given by

$$\{a_1, a_1, a_3\}, \{b_1, b_1, b_3\}, \{c_1, c_2, c_3\}.$$

The verification is by reduction to the complex case, by lifting $(X, S)$ to characteristic 0. This reduction leads to the following lemma.

**Lemma 2.5.** *Let $u_1, u_3, v_1, v_3, w_1, w_2, w_3$ be in $\mathbb{Q}_l^*$. We suppose that $u_1v_1w_i \neq 1$ for $i = 1, 2, 3$. Under this hypothesis, there are no elements $U, V, W$ in $\mathrm{GL}(3, \mathbb{Q}_l)$, conjugate respectively to the diagonal matrices $(u_1, u_1, u_3)$, $(v_1, v_1, v_3)$, $(w_1, w_2, w_3)$, and such that $UVW = 1$.*

*Proof.* Let $L$ be the intersection of the kernels of $U - u_1$ and $V - v_1$. These kernels being of dimension $\geq 2$, $L$ is non-zero. On $L$, $U = u_1$, $V = v_1$ and $W = (UV)^{-1} = 1/u_1v_1$. Since $1/u_1v_1$ is not an eigenvalue of $W$, this is impossible. □

*Remark* 2.6. If the $\mathcal{R}(s)$ are tame, that the semi-simplifications of the corresponding representations $\rho_s$ of $I_s$ are sums of multisets of characters $\mathcal{R}(s)$ of $\hat{\mathbb{Z}}(1) = I_s/P_s$, and that the $\mathcal{R}(s)$ satisfy (2.7.3), it still follows from (9) that the hypothesis of the irreducibility criterion (2.5.1) is satisfied.

## 2.8

Let $\Sigma$ be a compact Riemann surface of genus $g$. Suppose an injection of $S$ into $\Sigma$ is chosen, by which we identify $S$ with a finite set of points in $\Sigma$. If $\Sigma_{(s)}$ is a small disk around $s$, the fundamental group of $\Sigma^*_{(s)} := \Sigma_{(s)} - \{0\}$ is $\mathbb{Z}$, generated by a positive turn around $s$. Suppose given for each $s \in S$ an element $U_s$ of $\mathrm{GL}(r, \mathbb{C})$. Let $U := (U_s)_{s\in S}$ be the family of $U_s$ and let us denote $M(U)$ the moduli space of irreducible complex local systems of rank $r$ on $\Sigma - S$, with local monodromy at $s$ conjugate to $U_s$. Let us choose

(2.10.1) A generator "1" of $\hat{\mathbb{Z}}(1)$, given by an isomorphism between the projective system of $\mu_n(F)$ and that of $\mathbb{Z}/n$ ($n$ prime to $p$).

(2.10.2) An embedding into $\mathbb{C}$ of the cyclotomic subfield of $\mathbb{Q}_l$.

These choices being made, a character of finite order $\chi : \hat{\mathbb{Z}}(1) \to \mathbb{Q}_l^*$ provides the root of unity $\chi^* \in \mathbb{C}$ image of $\chi(\text{"1"})$. If the $\mathcal{R}(s)$ are semi-simple and tame, and define as in 2.6 multisets $\mathcal{R}(s) = \{\chi_1(s), \ldots, \chi_r(s)\}$ of characters of $\hat{\mathbb{Z}}(1)$, we will denote $\mathcal{R}^*$ the family of diagonal matrices $(\chi_1^*(s), \ldots, \chi_r^*(s))$ for $s \in S$, and we will say that $M(\mathcal{R}^*)$ corresponds to $E(\mathcal{R})$.

**Conjecture 2.7.** *Under the hypotheses and with the notations of 2.10,*

*(i) As a function of n, the number of fixed points $N_n(\mathcal{R})$ of $\phi^n$ is of the form*

$$N_n(\mathcal{R}) = \sum a_i \beta_i^n \tag{11}$$

*for integers $a_i$ and suitable numbers $\beta_i$;*

*(ii) The sum of the $a_i$ is equal to the Euler-Poincaré characteristic of the moduli space $M(\mathcal{R}^*)$ corresponding to $E(\mathcal{R})$;*

*(iii) As in 1.2 (B), the dominant term in* (11) *is $q$ to the power* $\dim_{\mathbb{C}}(M(\mathcal{R}^*))/2$.

As will be explained in 6.6, if $g > 0$, we have $\chi(M(\mathcal{R}^*)) = 0$. For a refinement of 2.11 (i) (ii), see 6.3 and 6.7.

## 2.9

If the representation $\rho_s$ is tame, not necessarily semi-simple, the choices (2.10.1) (2.10.2) still allow to associate to $\rho_s$ a conjugacy class $U_s$ in $\mathrm{GL}(r, \mathbb{C})$. The construction $\rho_s \to U_s$ is characterized by the properties that it extends that of 2.10, is compatible with $\oplus$ and $\otimes$, and that for $\rho_s$ unipotent, given by $I_s \to \hat{\mathbb{Z}}(1) \to \mathrm{GL}(r, \mathbb{Q}_l)$, $U_s$ is unipotent, with the same Jordan decomposition as the unipotent element of $\mathrm{GL}(r, \mathbb{Q}_l)$ image of "1".

I conjecture that 2.11 remains valid in this more general framework. Using the analogy between the exponential and Artin-Schreier sheaves, it is possible to define complex analogues for certain wild representations $\rho_s$. Rather than moduli spaces of local systems on $\Sigma - S$, the complex analogues are moduli spaces of vector bundles with algebraic connection on $\Sigma - S$, with given formal completions at each $s \in S$.

## 2.10

The complex analogues $M(\mathcal{R}^*)$ of the $E(\mathcal{R})$ are in general non-compact. One obtains partial compactifications as follows:

(2.13.1) In definition 2.10, rather than requiring that the local monodromy at $s$ is conjugate to $U_s$, demand that it be in the closure of the conjugacy class of $U_s$.

(2.13.2) Omit the requirement that the considered local systems be irreducible, and define the moduli spaces as classifying them up to semi-simplification.

One can complete $E(\mathcal{R})$ in the same way:

(2.13.1)' Let $m$ be the projection $I_s \to \hat{\mathbb{Z}}(1) \to \hat{\mathbb{Z}}_l(1)$, composed with any isomorphism between $\hat{\mathbb{Z}}_l(1)$ and $\mathbb{Z}_l$. The representation $\rho_s$ of $I_s$ corresponding to $\mathcal{R}$ can be written uniquely in the form $\rho_s(g) = \rho'_s(g)\rho''_s(m(g))$, where $\rho'_s$ is a representation of a finite quotient of $I_s$, and where $\rho''_s$ is of the form $x \mapsto \exp(xN)$ for a nilpotent endomorphism $N$ that commutes with the image of $\rho'_s$. In definition 2.1, rather than requiring that the local monodromy be conjugate to $\rho_s$, demand that it be conjugate to a representation $g \mapsto \rho'_s(g)\exp(m(g)N')$, where $N'$ is in the closure of the orbit of $N$ under the centralizer, in the linear group, of the image of $\rho'_s$.

(2.13.2)' As in (2.13.2).

In some cases, the formulas giving the number of fixed points of $\phi$ become simpler if $E(\mathcal{R})$ is replaced by such a completion. See 4.5.

# 3 Arinkin's Method

## 3.1

Let $s_0$ be a closed point of $X_0$. One can identify the closed points $s$ of $X$ both with the $F$-points of the $F$-scheme $X$, and with the $F$-points of the $\mathbb{F}_q$-scheme $X_0$. In doing so, we identify the $s$ above $s_0$ with the $F$-points of $\operatorname{Spec}(k(s_0))$, that is to say with the embeddings of $k(s_0)$ into $F$ that extend the embedding of $\mathbb{F}_q$ into $F$. If we choose such an embedding $s$, a character $\varepsilon_0 : k(s_0)^* \to \mathbb{Q}_l^*$ provides by 2.6 a character $\varepsilon := \varepsilon_0 \circ \mathrm{pr}_s : \hat{\mathbb{Z}}(1) \to k(s_0)^* \to \mathbb{Q}_l^*$. If $s = \mathrm{Frob}(s')$, the character $\varepsilon'$ of $\hat{\mathbb{Z}}(1)$ defined by $\varepsilon_0$ and $s'$ is $\varepsilon^q$.

## 3.2

For each $s_0 \in S_0$, let us give ourselves a multiset $\mathcal{R}(s_0)$ of $r$ characters $k(s_0)^* \to \mathbb{Q}_l^*$. By 3.1, we deduce for every $s \in S$ a multiset of $r$ characters $\mathcal{R}(s)$ of $\hat{\mathbb{Z}}(1)$. The corresponding $\mathbb{Q}_l$-sheaves $\mathcal{R}(s)$ on $X^*_{(s)}$ (see 2.6) satisfy (2.1.1). Let us denote $\mathcal{R}$ the family $(\mathcal{R}(s))_{s\in S}$. We will assume that

$$\prod_{s\in S}\prod_{\varepsilon\in\mathcal{R}(s)} \varepsilon = 1. \tag{12}$$

According to (2.7.1), this is a necessary condition for a smooth $\mathbb{Q}_l$-sheaf $\mathcal{F}$ on $X - S$ to exist such that $\mathcal{F}|_{X^*_{(s)}} \simeq \mathcal{R}(s)$. If $s_0$ is of degree $d$, that $\varepsilon_0$ is a character of $k(s_0)^*$ and that for each $s$ above $s_0$, $\varepsilon[s] := \varepsilon_0 \circ \mathrm{pr}_s$ is the corresponding character of $\hat{\mathbb{Z}}(1)$, the product of these $\varepsilon[s]$ is the character that 3.1 attaches to $\varepsilon_0^{N_{k(s_0)/\mathbb{F}_q}(q^{d-1})/(q-1)}$ and to any $s$ above $s_0$. It is also the character that 3.1 attaches to $\varepsilon_0|^d_{\mathbb{F}_q^*}$ and to $\mathbb{F}_q \to F$. The hypothesis (12) can thus be rewritten

$$\prod_{s_0\in S_0}\prod_{\varepsilon_0\in\mathcal{R}(s_0)} \varepsilon_0|_{\mathbb{F}_q^*} = 1. \tag{13}$$

## 3.3

Suppose moreover that the $\mathcal{R}(s)$ satisfy the general position condition (2.7.3). I learned from Arinkin how, under this hypothesis, to construct an algebraic variety $Z$ over $\mathbb{F}_q$ such that, for all $n \geq 1$, the number $N_n(\mathcal{R})$ of fixed points of the permutation $\phi^n$ of $E(\mathcal{R})$ is equal to the number of $\mathbb{F}_{q^n}$-points of $Z$:

$$N_n(\mathcal{R}) = |Z(\mathbb{F}_{q^n})|. \tag{14}$$

Under the hypotheses of 3.3, conjecture 2.11 (i) is therefore true. Be aware that $Z$ will only be defined up to a decomposition into locally closed parts.

## 3.4

For each $s_0 \in S_0$, let $\varepsilon[s_0](i)$ $(0 \leq i \leq a(s_0))$ be the characters of $k(s_0)^*$ that appear in $\mathcal{R}(s_0)$, arranged in some order. Let $n[s_0](i)$ be the multiplicity of $\varepsilon[s_0](i)$ in the multiset $\mathcal{R}(s_0)$, and $n[s_0]$ the family of integers $(n[s_0](i))_{0\leq i\leq a(s_0)}$. For $s \in S$ above $s_0$, we denote $\varepsilon[s](i)$ the character $\varepsilon[s_0](i)\circ \mathrm{pr}_s$ of $\hat{\mathbb{Z}}(1)$ (3.1) and we set $n[s] := n[s_0]$. The multiset $\mathcal{R}(s)$ is

$$\mathcal{R}(s) := \{\text{the } \varepsilon[s](i), \text{with multiplicities } n[s](i)\}. \tag{15}$$

Let $\mathcal{E}_0$ be a vector bundle of rank $r$ on $X_0$. A parabolic structure $a_0$ of type $(n(s_0))_{s_0\in S_0}$ on $\mathcal{E}_0$ is the data, for each $s_0 \in S_0$, of a finite filtration $F(s_0)$ of the $k(s_0)$-vector space $(\mathcal{E}_0)_{s_0}$, such that $\mathrm{Gr}^i_{F(s_0)}((\mathcal{E}_0)_{s_0})$ is of dimension $n[s_0](i)$. We define similarly the parabolic structures for $\mathcal{E}$ on $X$. If $\mathcal{E}_0$ on $X_0$ is equipped with a parabolic structure $a_0$ of type $(n(s_0))_{s_0\in S_0}$, its inverse image $\mathcal{E}$ on $X$ inherits a parabolic structure $a$ of type $(n(s))_{s\in S}$. We say that $(\mathcal{E}_0, a_0)$ is geometrically indecomposable if its inverse image $\mathcal{E}$ on $X$ does not admit a non-trivial decomposition $\mathcal{E} = \mathcal{E}' \oplus \mathcal{E}''$ compatible with the parabolic structure $a$, i.e. inducing decompositions of the $F \otimes_{\mathbb{F}_q} (\mathcal{E}_0)_{s_0}$. Let $\mathrm{End}(\mathcal{E}_0, a_0)$ (resp. $\mathrm{End}(\mathcal{E}, a)$) be the algebra of endomorphisms of $\mathcal{E}_0$ (resp. $\mathcal{E}$) respecting the filtrations $F(s_0)$ of the $(\mathcal{E}_0)_{s_0}$ (resp. $F(s)$ of the $\mathcal{E}_s$). Each of the following conditions is equivalent to geometric indecomposability:

(3.4.2) The algebra $\mathrm{End}(\mathcal{E}, a) = \mathrm{End}(\mathcal{E}_0, a_0) \otimes_{\mathbb{F}_q} F$ has no idempotent other than 0 and 1.

(3.4.3) The quotient of the algebra $\mathrm{End}(\mathcal{E}_0, a_0)$ by its radical is reduced to $\mathbb{F}_q$.

Fix an integer $d$. Consider the vector bundles $\mathcal{E}_0$ of rank $r$ and degree $d$ on $X_0$, equipped with a parabolic structure $a$ of type $(n[s_0])_{s_0\in S_0}$. Let $Z(\mathbb{F}_q)$ be the set of isomorphism classes of $(\mathcal{E}_0, a)$ as above and geometrically indecomposable. We define $Z(\mathbb{F}_{q^n})$ similarly by replacing $X_0/\mathbb{F}_q$ by $X_0 \otimes_{\mathbb{F}_q} \mathbb{F}_{q^n}/\mathbb{F}_{q^n}$. I learned from Arinkin the following theorem.

**Theorem 3.1.** *Under the hypotheses of 3.2 and 3.3 and with the notations of 3.4, the number $N_n(\mathcal{R})$ of fixed points of $\phi^n$ (2.1) is*

$$N_n(\mathcal{R}) = |Z(\mathbb{F}_{q^n})|. \tag{16}$$

By replacing $X_0$ by $X_0 \otimes_{\mathbb{F}_q} \mathbb{F}_{q^n}$, we can reduce to the case $n = 1$. We should be careful that the proof does not suggest a natural bijection between $E(\mathcal{R})^\phi$ and $Z(\mathbb{F}_q)$. It gives a family of one-dimensional vector spaces indexed by $E(\mathcal{R})^\phi$, with direct sum $V$, another family, with direct sum $V'$, indexed by $Z(\mathbb{F}_q)$ and an isomorphism of vector spaces from $V$ to $V'$.

## 3.5 Sketch of proof of 3.5 ⇒ (3.3.1)

For any scheme $T$ over $\mathbb{F}_q$, let $X_T := X_0 \times_{\mathbb{F}_q} T$ and $S_T := S_0 \times_{\mathbb{F}_q} T$. Let us define $Z(T)$ as being the set of isomorphism classes of vector bundles of rank $r$ on $X_T$, equipped with a parabolic structure along $S_T$, and such that the conditions of 3.2 and 3.3 are satisfied on each geometric fiber of $X_T \to T$: degree $d$, type of the parabolic structure, indecomposability. Naively, we would like to take for the scheme $Z$ of (3.3.1) a scheme that represents the functor $Z$. For several reasons, this functor is not representable:

(a) The considered objects having non-trivial automorphisms, for example homotheties, this functor is not a sheaf, even for the Zariski topology. For the Pic functor, one bypasses the same problem by passing to the associated sheaf. Here, the situation is worse, because, for $(\mathcal{E}_T, \alpha_T)$ a vector bundle with parabolic structure on $X_T$, and for $(\mathcal{E}_t, \alpha_t)$ the induced bundle on the fiber $X_t$ of $X_T \to T$ at $t \in T$

(b) the dimension of the group of automorphisms of $(\mathcal{E}_t, \alpha_t)$ (equal to $\dim \mathrm{End}(\mathcal{E}_t, \alpha_t)$) is not always locally constant in $t$.

(c) The set of $t$ for which $(\mathcal{E}_t, \alpha_t)$ is geometrically indecomposable is not always locally closed.

Because of (c), one cannot even hope that $Z$ comes from an algebraic stack $\mathcal{C}$ (that is to say that $Z(T)$ is the set of isomorphism classes in $\mathcal{C}(T)$). However

(d) The automorphism groups are connected algebraic groups. If $k' \subset k''$ are finite extensions of $\mathbb{F}_q$, this ensures that $Z(k') \simeq Z(k'')^{\mathrm{Gal}(k''/k')}$.

(e) In (b) above, the function $t \mapsto \dim \mathrm{End}(\mathcal{E}_t, \alpha_t)$ is constructible; in (c), the set of $t$ such that $(\mathcal{E}_t, \alpha_t)$ is geometrically indecomposable is constructible.

The properties (d) and (e) allow, by "cutting $Z$ into pieces" and taking the associated (representable) sheaves, to obtain $Z$ as promised by (3.3.1).

## 3.6 Sketch of proof of 3.5 (for $n = 1$)

We apply the strategy 2.2. We must show that $|Z(\mathbb{F}_q)|$ is the number of cuspidal automorphic representations $\pi$ of $\mathrm{GL}(r, \mathbb{A})$, counted up to $\mathbb{F}_q$-twist, which are unramified outside of $S_0$ and whose local component $\pi_{s_0}$ at $s_0 \in S_0$ is of the following type. By the local Langlands correspondence, the representation $\pi_{s_0}$ of $\mathrm{GL}(r, K_{s_0})$ corresponds to an $F$-semi-simple $\mathbb{Q}_l$-representation $\rho$ of the Weil group $W(\bar{K}_{s_0}/K_{s_0})$. We want first of all that $\rho$ be semi-simple (in the language of the Weil-Deligne group: $N = 0$; in the language of Arthur: trivial action of the local SL(2) or SU(2)). The group $W(\bar{K}_{s_0}/K_{s_0})^{ab}$ is $K^*_{s_0}$. Its inertia group $\mathcal{O}^*_{s_0}$ has as quotient $k(s_0)^*$. We want the restriction of $\rho$ to the inertia group to factor through the representation of $k(s_0)^*$ sum of the $\varepsilon_0$ in $\mathcal{R}(s_0)$. Let $P[s_0] < \mathrm{GL}(r, \mathcal{O}_{s_0})$ be the parahoric subgroup inverse image of the standard parabolic subgroup $P[s_0]$ of $\mathrm{GL}(r, k(s_0))$ of type $n(s_0)$. The reductive quotient $L[s_0]$ of $P[s_0]$ is $\prod \mathrm{GL}(n[s_0](i), k(s_0))$. Let us denote $U[s_0]$ the kernel of $P[s_0] \to L[s_0]$ and $\varepsilon[s_0]$ the character

$$\varepsilon[s_0] = \prod \varepsilon[s_0](i)^{\det g_i} : P[s_0] \to P[s_0] \to L[s_0] \to \mathbb{Q}_l^*$$

of the parahoric $P[s_0]$. By construction, the $\varepsilon[s_0](i)$ are pairwise distinct. We will admit the following assertion:

**Assertion 3.1.** *The irreducible admissible representations of* $\mathrm{GL}(r, K_{s_0})$ *of the type defined in 3.7 are those which admit a stable line by $P$ on which $P$ acts by the character $\varepsilon[s_0]$. This line is unique.*

## 3.7

Let $A'$ be the set of cuspidal automorphic representations $\pi$ of $\mathrm{GL}(r, \mathbb{A})$, admitting a line $L(\pi)$ fixed by the $\mathrm{GL}(r, \mathcal{O}_x)$ for $x \neq S_0$, and stable by $P[s_0]$ for $s_0 \in S$, the action of $P[s_0]$ being given by $\varepsilon[s_0]$. The line $L(\pi)$ is unique if it exists. By Lafforgue [L] and 3.8, $N_1(\mathcal{R})$ is the number of classes modulo $\mathbb{F}_q$-twist in $A'$. Fix an idèle $d$ of valuation 1 and let $A$ be the set of $\pi$ in $A'$ whose central character $\omega_\pi$ is 1 at $d$. As observed after 2.3.1, we have

$$N_1(\mathcal{R}) = |A' \text{ mod } \mathbb{F}_q\text{-twist}| = |A \text{ mod } \mathbb{F}_q\text{-twist by } \zeta \in \mu_r(\mathbb{Q}_l)| = \frac{1}{r}|A|. \tag{17}$$

## 3.8

Let $A$ be the vector space of functions on $\mathrm{GL}(r, \mathbb{A})$ which are

(a) left invariant by $\mathrm{GL}(r, K) \cdot d^{\mathbb{Z}}$;

(b) right invariant under $\mathrm{GL}(r, \mathcal{O}_x)$, for $x \notin S_0$;

(c) transform by $\varepsilon[s_0]$ for the right action of $P[s_0]$, for $s_0 \in S_0$.

A key point of the proof of 3.5 is the following lemma.

**Lemma 3.2.** *The vector space $A$ is the direct sum of the lines $L(\pi)$, for $\pi$ in $A$. It is therefore of dimension $rN_1(\mathcal{R})$.*

This lemma is the automorphic counterpart, and a consequence, of the fact that the irreducibility of the considered $\mathbb{Q}_l$-sheaves $\mathcal{F}_0$ results from the local hypotheses made. It remains to verify that the functions in $A$ are automatically cuspidal. This done, if we write the space of cuspidal functions on $\mathrm{GL}(r, K) \cdot d^{\mathbb{Z}} \setminus \mathrm{GL}(r, \mathbb{A})$ as a direct sum of irreducible representations $\pi$ of $\mathrm{GL}(r, \mathbb{A})$, the $\pi$ in $A$ contribute $L(\pi)$ to $A$, the others do not contribute. That the functions in $A$ are cuspidal results from the fact that in the spectral decomposition of functions on $\mathrm{GL}(r, K) \cdot d^{\mathbb{Z}} \setminus \mathrm{GL}(r, \mathbb{A})$, the induced representations giving rise to Eisenstein series do not admit a non-zero vector satisfying 3.10 (b) (c).

## 3.9

The space $A$, sum of the $L(\pi)$ for $\pi \in A$, identifies with the space of functions $f$ on

$$\mathrm{GL}(r, K) d^{\mathbb{Z}} \setminus \mathrm{GL}(r, \mathbb{A}) / \prod_{x \notin S_0} \mathrm{GL}(r, \mathcal{O}_x) \prod_{s_0 \in S_0} U[s_0] \tag{18}$$

such that, for each $s_0 \in S_0$, the right action of $P[s_0]$ satisfies $f(gp) = \varepsilon[s_0](p)f(g)$. Let $\nu$ be the application from (3.12.1) to $\mathbb{Z}/r$ induced by $v \circ \det : \mathrm{GL}(r, \mathbb{A}) \to \mathbb{Z}$. Let $\bar{A}$ be the quotient of $A$ by the action by $\mathbb{F}_q$-twist of $\mu_r(\mathbb{Q}_l)$, and for each orbit $B$, let $V_B$ be the subspace of $A$ sum of the $L(\pi)$ for $\pi$ in $B$. According to (2.3.1), we have $\dim(V_B) = r$. For $m \in \mathbb{Z}/r$, let $(3.12.1)_m$ be the part of (3.12.1) where $\nu = m$, $\varphi_m$ its characteristic function and $L(\pi, m) = \{\varphi_m f | f \in L(\pi)\}$. The dimension of $L(\pi, m)$ is $\leq 1$. For each character $\eta$ of $\mathbb{Z}/r$, if $\pi'$ is the $\mathbb{F}_q$-twist of $\pi$ by $\eta(1)$, the multiplication by $\eta\nu$ transforms $L(\pi)$ into $L(\pi')$. For $\pi \in B$, $L(\pi, m)$ is thus independent of $\pi$; we set $L(B, m) := L(\pi, m)$. The space $V_B$ is stable by multiplications by the $\eta\nu$, thus by multiplication by the $\varphi_m$: for $\pi \in B$,

$$V_B = \bigoplus_{m \in \mathbb{Z}/r} L(B, m).$$

Each $L(B, m)$ is therefore of dimension one. The sum $A_m$ of the $L(B, m)$ is of dimension $|A|/r = N_1(\mathcal{R})$. It identifies with the space of functions $f$ on $(3.12.1)_m$ such that, for each $s_0 \in S_0$, the right action of $P[s_0]$ satisfies $f(gp) = \varepsilon[s_0](p)f(g)$. Let $\mathrm{GL}(r, \mathbb{A})^{(d)}$ be the set of $g \in \mathrm{GL}(r, \mathbb{A})$ such that $v \det(g) = d$. If $\bar{d}$ is the class of $d$ modulo $r$, the set

$$\mathrm{GL}(r, K) \setminus \mathrm{GL}(r, \mathbb{A})^{(d)} / \prod_{x \notin S_0} \mathrm{GL}(r, \mathcal{O}_x) \prod_{s_0 \in S_0} U[s_0] \tag{19}$$

maps bijectively to $(3.12.1)_{\bar{d}}$. From now on,

**Lemma 3.3.** *$N_1(\mathcal{R})$ is the dimension of the space, still denoted $A_d$, of functions $f$ on (19) which, for the right action of the $P[s_0]$ ($s_0 \in S_0$) satisfy $f(gp) = \varepsilon[s_0](p)f(g)$.*

## 3.10

The set (19) is interpreted as the set of isomorphism classes of triples $(\mathcal{E}_0, \alpha, \beta)$ where $\mathcal{E}_0$ is a vector bundle of rank $r$ and degree $d$ on $X_0$, and where $\alpha$ and $\beta$ are structures of the following types on $\mathcal{E}_0$:

(a) $\alpha$ is a parabolic structure of type $(n[s_0])_{s_0 \in S_0}$;

(b) $\beta$ is the data of bases of the vector spaces $\mathrm{Gr}^i_{F(s_0)}((\mathcal{E}_0)_{s_0})$.

We will identify a basis as in (b) with an isomorphism

$$\beta_{s_0,i} : k(s_0)^{n[s_0](i)} \xrightarrow{\sim} \mathrm{Gr}^i_{F(s_0)}((\mathcal{E}_0)_{s_0}).$$

With this interpretation, the space $A_d$ of 3.13 becomes the space of functions $f(\mathcal{E}_0, \alpha, \beta)$, for $\mathcal{E}_0, \alpha, \beta$ as above, such that

(i) $f(\mathcal{E}_0, \alpha, \beta)$ depends only on the isomorphism class of $(\mathcal{E}_0, \alpha, \beta)$;

(ii) for $s_0 \in S_0$ and $g \in L[s_0] = \prod \mathrm{GL}(n_i, k(s_0))$, we have

$$f(\mathcal{E}_0, \alpha, \beta g) = \varepsilon[s_0](g) f(\mathcal{E}_0, \alpha, \beta).$$

For $T$ an automorphism of $(\mathcal{E}_0, \alpha)$, let us denote $\mathrm{Gr}^i_{F(s_0)}(T_{s_0})$ the automorphism of $\mathrm{Gr}^i_{F(s_0)}((\mathcal{E}_0)_{s_0})$ induced by $T$, and let us set

$$\chi(T) := \prod_{s_0} \prod_{i} \varepsilon[s_0](i)(\det \mathrm{Gr}^i_{F(s_0)}(T_{s_0})). \tag{20}$$

Whatever $\beta$, hypothesis (i) gives that $f(\mathcal{E}_0, \alpha, T(\beta)) = f(\mathcal{E}_0, \alpha, \beta)$. Hypothesis (ii) gives that $f(\mathcal{E}_0, \alpha, T(\beta)) = \chi(T) f(\mathcal{E}_0, \alpha, \beta)$. If there exists an automorphism $T$ such that $\chi(T) \neq 1$, we thus have $f(\mathcal{E}_0, \alpha, \beta) = 0$. On the other hand, if for any automorphism $T$ of $(\mathcal{E}_0, \alpha)$ we have $\chi(T) = 1$, $f$ can be freely prescribed on one $(\mathcal{E}_0, \alpha, \beta)$, and the value at $(\mathcal{E}_0, \alpha, \beta)$ determines the value at the $(\mathcal{E}_0, \alpha, \beta')$. The dimension of $A_d$ is therefore the number of isomorphism classes of $(\mathcal{E}_0, \alpha)$ as above, such that for any automorphism $T$ one has $\chi(T) = 1$. Theorem 3.5 results then from the characterization (3.4.3) of geometric indecomposability and the following proposition.

**Proposition 3.4.** *Let $\mathcal{E}_0$ be a vector bundle of rank $r$ on $X_0$, equipped with a parabolic structure of type $(n(s_0))_{s_0 \in S_0}$. The following conditions are equivalent:*

*(i) The quotient of the $\mathbb{F}_q$-algebra* $\mathrm{End}(\mathcal{E}_0, \alpha)$ *by its radical is reduced to* $\mathbb{F}_q$*.*

*(ii) Whatever the automorphism $T$ of* $(\mathcal{E}_0, \alpha)$*,* $\chi(T)$ *defined by* (20) *is 1.*

We will deduce that (i) implies (ii) from (3.2.2), and that the negation of (i) implies the negation of (ii) from the general position hypothesis (2.7.3).

*Proof of (i) ⇒ (ii).* If $\lambda$ is the image in $\mathbb{F}_q^*$ of the automorphism $T \in \mathrm{End}(\mathcal{E}_0, \alpha)^*$, we have

$$\varepsilon[s_0](i)(\det \mathrm{Gr}^i_{F(s_0)}(T_{s_0})) = \varepsilon[s_0](i)^{n[s_0](i)}(\lambda),$$

so that (3.2.2) implies that $\chi(T) = 1$.

*Proof of (ii) ⇒ (i).* The quotient of the $\mathbb{F}_q$-algebra $\mathrm{End}(\mathcal{E}_0, \alpha)$ by its radical is a product of matrix algebras over finite extensions of $\mathbb{F}_q$. If it is not reduced to $\mathbb{F}_q$, it contains a commutative separable $\mathbb{F}_q$-algebra of dimension $> 1$. This algebra admits a lifting in $\mathrm{End}(\mathcal{E}_0, \alpha)$. Suppose then that $\mathrm{End}(\mathcal{E}_0, \alpha)$ contains a commutative separable subalgebra $k$ of dimension $> 1$. We must show that the character (3.14.1) of $\mathrm{End}(\mathcal{E}_0, \alpha)^*$ is non-trivial. We will prove that its restriction $\chi$ to $k^*$ is non-trivial.

**Split case.** Suppose first that $k$ is a product of copies of $\mathbb{F}_q$: $k \simeq \mathbb{F}_q^I$, and that the closed points $s_0 \in S_0$ are of degree 1: $\mathbb{F}_q \xrightarrow{\sim} k(s_0)$. The structure of $\mathbb{F}_q^I$-module of $\mathcal{E}_0$ provides a decomposition

$$\mathcal{E}_0 = \bigoplus_{\iota \in I} \mathcal{E}_0^\iota \tag{21}$$

such that $(\lambda_\iota)_{\iota \in I} \in \mathbb{F}_q^I$ acts on $\mathcal{E}_0^\iota$ by multiplication by $\lambda_\iota$. The decomposition (21) is compatible with the parabolic structure. For $\iota \in I$, let us set

$$n^\iota[s_0](i) = \dim \mathrm{Gr}^i_{F(s_0)}((\mathcal{E}_0^\iota)_{s_0}).$$

The sum over $i$ of the $n^\iota[s_0](i)$ is the rank of the bundle $\mathcal{E}_0^\iota$. The restriction of $\chi$ to the factor of index $\iota$ of $(F_q^I)^* = \prod_{\iota \in I} \mathbb{F}_q^*$ is

$$\lambda \mapsto \prod_{s_0 \in S_0} \prod_i \varepsilon[s_0](i)^{n^\iota[s_0](i)}(\lambda). \tag{22}$$

Its composite with the morphism $\hat{\mathbb{Z}}(1) \to \mathbb{F}_q^*$ of 3.1 is the character

$$\prod_{s \in S} \prod_i \varepsilon[s](i)^{n^\iota[s_0](i)}.$$

of $\hat{\mathbb{Z}}(1)$. Hypothesis (2.7.3) ensures that this character is non-trivial.

**Reduction of the general case to the split case.** It suffices to show that the non-triviality of $\chi$ is invariant by an extension of scalars from $\mathbb{F}_q$ to a finite extension $\mathbb{F}_{q^n}$. By such an extension of scalars, $X_0$ becomes a curve $X'$ over $\mathbb{F}_{q^n}$, $S_0$ becomes a divisor $S'$ of $X'$, and $k$ becomes $k' := k \otimes_{\mathbb{F}_q} \mathbb{F}_{q^n}$. For $s' \in S'$ above $s_0 \in S_0$, we set $n[s'] := n[s_0]$ and we denote $\varepsilon[s'](i)$ the composite of $\varepsilon[s_0](i)$ with the norm $N_{k(s')/k(s_0)}$. The bundle $\mathcal{E}'$ on $X'$ inverse image of $\mathcal{E}_0$ inherits a parabolic structure $\alpha'$ of type $(n(s'))_{s' \in S'}$. If we repeat for $\mathcal{E}'$ on $X'/\mathbb{F}_{q^n}$ the construction (3.14.1), we obtain a character $\chi'$ of $k'^*$. The curve $X/F$, the integers $n[s](i)$ and the characters $\varepsilon[s](i)$ of $\hat{\mathbb{Z}}(1)$ come from both $X_0/\mathbb{F}_q$ and $X'/\mathbb{F}_{q^n}$. To prove that the triviality of $\chi$ is equivalent to that of $\chi'$, it suffices to prove that

$$\chi' = \chi \circ N_{k'/k}. \tag{23}$$

The norm $N_{k'/k}$ is indeed surjective. The identity (23) results from the identity (3.16.1) that follows, applied to each $\mathrm{Gr}^i_{F(s_0)}((\mathcal{E}_0)_{s_0})$, endowed with its structure of $(k, k(s_0))$-bimodule. Let $k_0$ be a finite extension of $\mathbb{F}_q$, $N_0$ a $k_0$-vector space of finite dimension, $k$ a product of finite extensions of $\mathbb{F}_q$, and $\rho : k \to \mathrm{End}_{k_0}(N_0)$. The $(k, k_0)$ bimodule $N$ defines a homomorphism

$$[N] : k^* \to k_0^* : \lambda \mapsto \det(\rho(\lambda)).$$

Let us extend the scalars from $\mathbb{F}_q$ to $\mathbb{F}_{q^n}$. We obtain $k_0'$, $k'$ and a $(k', k_0')$-bimodule $N'$. This bimodule defines

$$[N'] : k'^* \to k_0'^* : \lambda \mapsto \det_{k_0'}(\rho(\lambda)).$$

**Lemma 3.5.** *The diagram*

$$\begin{array}{ccc} k'^* & \xrightarrow{[N']} & k_0'^* \\ {\scriptstyle N_{k'/k}}\uparrow & & {\scriptstyle N_{k_0'/k_0}}\uparrow \\ k^* & \xrightarrow{[N]} & k_0^* \end{array} \tag{24}$$

*is commutative.*

*Proof.* Extend the scalars from $\mathbb{F}_q$ to $F$. The diagram (3.16.1) embeds in an analogous diagram (3.16.2), where $k$ and $k_0$ are replaced by $k_F := k \otimes_{\mathbb{F}_q} F$, $k_{0F} := k_0 \otimes_{\mathbb{F}_q} F$, where $N_0$ is replaced by a $(k_F, k_{0F})$-bimodule $N$, and where the extension $\mathbb{F}_{q^n}$ of $\mathbb{F}_q$ is replaced by a product $F^J$ of $n$ copies of $F$.

$$\begin{array}{ccc} (k_F \otimes_F F^J)^* & \xrightarrow{[N\otimes_F F^J]} & (k_{0F} \otimes_F F^J)^* \\ \uparrow & & \uparrow \\ k_F^* & \xrightarrow{[N]} & k_{0F}^* \end{array} \tag{25}$$

We have $(k_F \otimes_F F^J)^* = (k_F^*)^J$, $(k_{0F} \otimes_F F^J)^* \simeq (k_{0F}^*)^J$ and the vertical "norm" applications are the product applications. The first horizontal line is the product of copies indexed by $J$ of the second horizontal line. The commutativity of 3.16.2 results from this. □

# 4 Trace formula.

## 4.1

In [DF], the counting problem 2.1 (i) is solved when $|S_0| \geq 2$ and the imposed local monodromies are unipotent, with a single Jordan block. In automorphic language, this means asking the local components $\pi_{s_0}$ ($s_0 \in S_0$) to be of the form

$$\text{special representation } \otimes \chi \det g, \tag{26}$$

for $\chi$ an unramified character of $K_{s_0}^*$. The hypothesis $|S_0| \geq 2$ allows one to pass from automorphic representations for $\mathrm{GL}(r, K)$ to automorphic representations for the multiplicative group of a division algebra $D$ of dimension $r^2$ over $K$. It is reassuring to use the trace formula only in a case with a compact quotient (modulo the center), but it should be possible to use instead the trace formula for $\mathrm{GL}(n)$, for a suitable test function, with the simplification brought by the fact that one wants to detect automorphic representations which at two places are of the discrete series. After all, this is how one relates $\mathrm{GL}(r, K)$ and $D^*$. We will simplify the explanations that follow by staying with $\mathrm{GL}(r, K)$, but admitting that the only non-zero terms in the required trace formula are those associated with the conjugacy classes of elliptic elements of finite order of $\mathrm{GL}(r, K)$. Let us choose an embedding $\mathbb{F}_{q^r} \to M_r(\mathbb{F}_q)$. It defines a morphism $\mathbb{F}_{q^r}^* \to \mathrm{GL}(r, \mathbb{F}_q) \to \mathrm{GL}(r, K)$, by which the set of orbits of $\mathrm{Gal}(\mathbb{F}_{q^r}/\mathbb{F}_q)$ in $\mathbb{F}_{q^r}^*$ maps bijectively to the set of conjugacy classes to consider. Let $T(\gamma)$ be the term of the trace formula (giving the number of fixed points of $\phi$) thus associated to $\gamma$ in $\mathbb{F}_{q^r}^*$. It depends only on the sub-extension $\mathbb{F}_{q^m}$ of $\mathbb{F}_{q^r}$ generated by $\gamma$. Let us set $T(m) := T(\gamma)$. The number $c_m$ of elements of $\mathbb{F}_{q^m}^*$ that generate the extension $\mathbb{F}_{q^m}$ of $\mathbb{F}_q$ is

$$c_m = \sum_{a|m} \mu(a)(q^{m/a} - 1).$$

The number of fixed points of $\phi$ is

$$N_1 = \sum_{m|r} \frac{c_m}{m} T(m) = \sum_{m|r} \frac{1}{m} \sum_{ab=m} \mu(a)(q^b - 1)T(ab) = \sum_b \frac{1}{b} \sum_{a|r/b} \frac{\mu(a)}{a}(q^b - 1)T(ab). \tag{27}$$

The number of fixed points of $\phi^n$ ($n \geq 1$) is obtained similarly, after extending the scalars from $\mathbb{F}_q$ to $\mathbb{F}_{q^n}$.

**Surprise 4.2.** When one lets $n$ vary, each $T(m)$ is not in general, as a function of $n$, of the form (1.2.1). On the other hand, in (4.1.2), each term of the sum over $b$ is of this form. I have no explanation, neither geometric nor automorphic, for 4.2, only a proof. I also have no explanation for the

**Surprise 4.3.** Under the hypotheses of 4.1, the number $N_1$ of fixed points of $\phi$ is divisible by $q$.

## 4.2

If $r = 2$, the trace formula is a little less frightening. Drinfeld [Dr] used it to treat the case where $S_0$ is empty. Flicker (unpublished) treated the case where $|S_0| = 1$ and where the local monodromy at each $s \in S$ is unipotent with a single Jordan block (that is, since $r = 2$, non-trivial). Assembling the information thus obtained, one obtains the

**Surprise 4.5.** Let $E$ be the set of isomorphism classes of irreducible smooth $\mathbb{Q}_l$-sheaves of rank 2 on $X - S$, with unipotent local monodromy (both non-trivial and trivial). The number of fixed points of $\phi := \mathrm{Frob}^*$ on $E$ depends only on $X_0/\mathbb{F}_q$ and on $|S|$. It is to this surprise that 2.13 alludes. If, as in [DF], one requires that the local monodromy at each $s \in S$ be unipotent with a single Jordan block, the number of fixed points depends on $X_0$, on $|S|$, and also on the action of Frobenius on $S$ (seen as a conjugacy class in the symmetric group $S_{|S|}$).

# 5 Examples

## 5.1

Let us take $X_0 = \mathbb{P}^1$, $|S| = 4$, $r = 2$ and let $E$ be the set of isomorphism classes of irreducible smooth $\mathbb{Q}_l$-sheaves of rank 2 on $X - S$, with non-trivial unipotent local monodromy at each $s \in S$. In this case, the number of fixed points of the iterates of the permutation $\phi = \mathrm{Frob}^*$ of $E$ (notations of 2.1) is given by

$$N_n = q^n. \tag{28}$$

If $|S_0| \geq 2$, (5.1.1) is an application of the main result of [DF]. The curious result below ([DF] §7) allows to deduce the case where $S_0$ is reduced to a closed point of degree 4.

**Proposition 5.1.** *Every permutation $\sigma$ of type (2,2) of $S$ extends to a projectivity of $\mathbb{P}^1$. The action of this projectivity on $E$ is trivial.*

The proof is by reduction to an analogous theorem on $\mathbb{C}$.

## 5.2

It is natural to complete the set $E$ of 5.1 into $\bar{E}$, the set of isomorphism classes up to semi-simplification of smooth $\mathbb{Q}_l$-sheaves of rank 2 on $X - S$, with unipotent local monodromy at each $s \in S$. Because $X_0 = \mathbb{P}^1$ and $|S| = 4$, this completion $\bar{E}$ differs from $E$ only by the addition of a point corresponding to the constant $\mathbb{Q}_l$-sheaf $\mathbb{Q}_l^2$. We still denote $\phi$ the permutation of $\bar{E}$ induced by Frob*. The number $\bar{N}_n$ of fixed points of $\phi^n$ acting on $\bar{E}$ is

$$\bar{N}_n = q^n + 1 \quad \text{(fixed points on } \bar{E}). \tag{29}$$

## 5.3

The complex analogue of $\bar{E}$ is the moduli space $M$ of complex local systems of rank 2 on $\mathbb{P}^1 - S$, for $|S| = 4$, with unipotent local monodromy. These local systems are taken up to semi-simplification. If one chooses a base point $o \in \mathbb{P}^1 - S$, the functor $\mathcal{F} \mapsto \mathcal{F}_o$ is an equivalence from local systems to representations of $\pi_1(\mathbb{P}^1 - S, o)$, and one can take as coordinates on $M$ the traces of the elements of $\pi_1$. The space $M$ is affine, purely of dimension 2, with a unique singular point corresponding to the constant local system $\mathbb{C}^2$. The singularity is of type $D_4$. The non-zero Betti numbers of $M$ are

$$b_0 = 1, b_2 = 1. \tag{30}$$

## 5.4

I have not done a complete verification, but the situation seems quite similar for $X_0 = \mathbb{P}^1$, $|S| = 3$, $r = 3$, unipotent local monodromy. Again, if one of the local monodromies is not a single Jordan block, the $\mathbb{Q}_l$-sheaf is an iterated extension of constant local systems $\mathbb{Q}_l$. The analogue of 5.3 holds for $\sigma$ a cyclic permutation of the three points of $S$. The number of fixed points of $\phi^n$ is again $q^n + 1$ (with "1" given by the local systems with reducible unipotent monodromy, all semi-simplified to a constant local system). The unique singular point of the complex analogue is of type $E_6$.

### 5.5

Let us take $X_0 = \mathbb{P}^1$, $|S| = 4$, $r = 2$ and suppose that as in 3.4 the imposed local monodromy at each $s_0 \in S_0$ is given by two distinct characters $\alpha'[s_0], \alpha''[s_0]$ of $k(s_0)^*$. Suppose (3.2.2) and the general position hypothesis (2.7.3) are satisfied. With the help of (3.5.1), one obtains

**Proposition 5.2.** *Under the hypotheses of 5.3, we have*

$$N_1 = q + 1 + |S_0(\mathbb{F}_q)|. \tag{31}$$

Arinkin has verified that for $X_0 = \mathbb{P}^1$ and $r = 2$, if a bundle $\mathcal{E}_0$ of rank 2 on $X_0$ with parabolic structure $a_0$ at $S_0$ is indecomposable, then $\mathrm{End}(\mathcal{E}_0, a_0)$ is reduced to the scalars. If one defines $Z$ as in 3.6, and one passes to the associated sheaf for the Zariski topology, one obtains a functor representable by an algebraic space. In the case $|S| = 4$, if one takes the degree $d = 1$, it is represented by the sum of two copies of $X_0$, glued along $X_0 - S_0$. The number of points of this scheme over $\mathbb{F}_q$ is given by (5.7.1).

### 5.6

Let us pass to the complex analogue. On $\mathbb{P}^1 - S$, with $|S| = 4$, we consider the moduli space of complex local systems of rank 2 with imposed local monodromy at each $s \in S$. At $s \in S$, we ask that the local monodromy be conjugate to $A_s = \mathrm{diag}(a_s, b_s)$. We suppose $\prod a_s b_s = 1$ and that if for each $s$, $c_s$ is one of $a_s$ or $b_s$, we never have $\prod c_s = 1$. Let $A := (A_s)_{s \in S}$ and let $M(A)$ be the moduli space, 2.10. It is smooth purely of dimension 2. We can look at $M(A)$, for $A_s$ close to 1, as a deformation of $M$ considered in 5.4. In this deformation, the space of vanishing cycles is of dimension 4 (since the singularity of $M$ is of type $D_4$). At infinity, the geometry does not change and so we have as non-zero Betti numbers

$$b_0 = 1, b_2 = 5. \tag{32}$$

### 5.7

Let us take $X_0 = \mathbb{P}^1$, $|S| = 3$, $r = 3$ and suppose that as in 3.4 the imposed local monodromy at each $s_0 \in S_0$ is given by three distinct characters of $k(s_0)^*$. Suppose (3.2.2) and the general position hypothesis 2.7.3 are satisfied. With the help of (3.5.1) one obtains here that

$$N_1 = q + 1 + 2|S_0(\mathbb{F}_q)|. \tag{33}$$

The associated sheaf to $Z$ as in 3.6 is indeed, for $d = 1$, represented by the sum of three copies of $X_0$, glued along $X_0 - S_0$. If $S_0$ consists of three rational points over $\mathbb{F}_q$, (5.9.1) says that $N_1 = (q + 1) + 6$, as suggested by the $E_6$ singularity of the complex analogue, for a unipotent local monodromy (cf 5.5).

## 6 Rank 1

### 6.1

Let us denote $E(1, \emptyset)$ the set $E(\mathcal{R})$ of 2.1 for $r = 1$ and $S_0 = \emptyset$. The tensor product endows it with an abelian group structure. Class field theory identifies $E(1, \emptyset)^\phi$ with the group of characters with values in $\mathbb{Q}_l^*$ of the finite group $\mathrm{Pic}^0(X_0)(\mathbb{F}_q)$. This theory indeed identifies the abelianized fundamental group of $X_0$ with the profinite completion of $\mathrm{Pic}(X_0)$. The group $\mathrm{Pic}(X_0)$ maps to $\mathbb{Z}$ by the degree application, with kernel the finite group $\mathrm{Pic}^0(X_0)(\mathbb{F}_q)$ of the $\mathbb{F}_q$-points of the Jacobian $\mathrm{Pic}^0(X_0)$. The isomorphism classes of smooth $\mathbb{Q}_l$-sheaves $\mathcal{F}_0$ of rank one on $X_0$ are thus identified with the characters $\chi$ of $\mathrm{Pic}(X_0)$ with values in $\mathbb{Z}_l^*$, the $\mathbb{F}_q$-twist by $\lambda$ corresponds to the product by the character $\lambda^{\deg}$ of $\mathbb{Z}$, and, attaching to $\mathcal{F}_0$ the restriction of $\chi$ to $\mathrm{Pic}^0(X_0)(\mathbb{F}_q)$, one obtains a bijection between classes of $\mathbb{F}_q$-twist of smooth $\mathcal{F}_0$ of rank 1 on $X_0$ and characters of $\mathrm{Pic}^0(X_0)(\mathbb{F}_q)$.

### 6.2

Whatever $r \geq 1$ and $\mathcal{R}$ as in 2.1, the tensor product induces an action of the abelian group $E(1, \emptyset)$ on $E(\mathcal{R})$, and of $E(1, \emptyset)^\phi$ on $E(\mathcal{R})^\phi$. If $r = 1$, and if $\mathcal{R}$ satisfies the compatibility (2.7.1), class field theory shows that $E(\mathcal{R})^\phi$ is a principal homogeneous space under $E(1, \emptyset)^\phi$. We thus have

$$|E(\mathcal{R})^\phi| = |E(1, \emptyset)^\phi| = |\,\mathrm{Pic}^0(X_0)(\mathbb{F}_q)| \quad (\text{if } r = 1 \text{ and } (2.7.1)). \tag{34}$$

If $r > 1$, taking the exterior power $\Lambda^r$ provides an application

$$\det : E(\mathcal{R}) \to E(\Lambda^r \mathcal{R})$$

such that, for the action, denoted $\otimes$, of $E(1, \emptyset)$, we have

$$\det(l \otimes \mathcal{F}) = l^{\otimes r} \otimes \det(\mathcal{F}).$$

Similarly, after passing to the fixed points of $\phi = \mathrm{Frob}^*$, for $\det : E(\mathcal{R})^\phi \to E(\Lambda^r \mathcal{R})^\phi$ and the action of $E(1, \emptyset)^\phi$. The action of $E(1, \emptyset)^\phi$ on $E(\mathcal{R})^\phi$ is not necessarily free, and the fibers of

$$\det : E(\mathcal{R})^\phi \to E(\Lambda^r \mathcal{R})^\phi$$

do not necessarily all have the same number of elements. Nevertheless, in all the cases that could be calculated $|E(1, \emptyset)^\phi|$ divides $|E(\mathcal{R})^\phi|$. As a function of $n$, $|\operatorname{Pic}^0(X_0)(\mathbb{F}_{q^n})|$ has the form (2.11.1):

$$\begin{aligned} |\operatorname{Pic}^0(X_0)(\mathbb{F}_{q^n})| &= \sum (-1)^i \operatorname{Tr}(\mathrm{Frob}^{*n}, H^i(\operatorname{Pic}^0(X), \mathbb{Q}_l)) \\ &= \sum (-1)^i \operatorname{Tr}(\mathrm{Frob}^{*n}, \Lambda^i H^1(X, \mathbb{Q}_l)). \end{aligned}$$

I conjecture the following precise form of conjecture 2.11 (i).

**Conjecture 6.1.** *As a function of $n$, the number of fixed points $N_n(\mathcal{R})$ of $\phi^n$ is of the form*

$$N_n(\mathcal{R}) = |\operatorname{Pic}^0(X_0)(\mathbb{F}_{q^n})| \cdot \sum c_k \gamma_k^n \tag{35}$$

*for integers $c_k$ and suitable numbers $\gamma_k$.*

### 6.3

Here is an analogue of 6.3 in algebraic geometry. Let $A_0$ be an abelian variety over $\mathbb{F}_q$ and $f_0 : X_0 \to A_0$ a proper and smooth morphism. Although the fibers of $f : X_0(\mathbb{F}_{q^n}) \to A_0(\mathbb{F}_{q^n})$ do not necessarily have the same number of elements (example: $X_0 = A_0$ and $f_0 =$ multiplication by an integer prime to $p$), $|A_0(\mathbb{F}_{q^n})|$ divides $|X_0(\mathbb{F}_{q^n})|$. Indeed, the $R^i f_! \mathbb{Q}_l$ are smooth $\mathbb{Q}_l$-sheaves on $A_0$, and the (abelian) fundamental group of $A_0$ acts on these sheaves. We know that this action is semi-simple. Let $(R^i f_! \mathbb{Q}_l)^0$ be the subsheaf of invariants. It comes from a subsheaf $(R^i f_{0!} \mathbb{Q}_l)^0$ of $R^i f_{0!} \mathbb{Q}_l$ on $A_0$, and the latter is the inverse image on $A_0$ of a sheaf on $\operatorname{Spec}(\mathbb{F}_q)$, which we will denote $H^i(X_0/A_0)$. Because the fundamental group acts trivially on the cohomology, we have

$$H^i(A, (R^j f_! \mathbb{Q}_l)^0) \xrightarrow{\sim} H^i(A, R^j f_! \mathbb{Q}_l). \tag{36}$$

The Grothendieck trace formula for the number of $\mathbb{F}_q$-points of $X_0$ can thus be rewritten

$$|X_0(\mathbb{F}_q)| = \operatorname{Tr}(\mathrm{Frob}^*, (\sum (-1)^i H^i(A, \mathbb{Q}_l)) \otimes (\sum (-1)^j H^j(X/A))). \tag{37}$$

If the $\gamma_k$ are the eigenvalues of Frobenius on the $H^i(X/A)$, and the $c_k$ the alternating sum of their multiplicities, we deduce from (6.4.2) that

$$|X_0(\mathbb{F}_{q^n})| = |A_0(\mathbb{F}_{q^n})| \cdot \sum c_k \gamma_k^n, \tag{38}$$

a formula analogous to (6.3.1). The promised divisibility of $|X_0(\mathbb{F}_{q^n})|$ by $|A_0(\mathbb{F}_{q^n})|$ results from the fact that the eigenvalues of Frobenius on $H^i(X/A)$, contained in the cohomology of a fiber of $X_0 \to A_0$, are algebraic integers.

### 6.4

In 6.4, the crucial point is not that $f_0$ is proper and smooth, but that the $R^i f_! \mathbb{Q}_l$ are smooth. One can avoid having to suppose that the $R^i f_! \mathbb{Q}_l$ are semi-simple by considering, rather than the subsheaf of invariants for the action of $\pi_1(A)$, the largest subsheaf on which the action is unipotent. Suppose that there exists an action of $A_0$ on $X_0$, and an integer $r$, such that $f_0$ is equivariant for the action of $A_0$ on $A_0$ by $x : y \mapsto rx + y$. Under this hypothesis, the inverse images of the $R^i f_! \mathbb{Q}_l$ on $A$ by the multiplication by $r : A \to A$, are constant sheaves. This implies both their smoothness and their semi-simplicity.

### 6.5

If $g \geq 1$, conjecture 6.3 implies that in the conjectural expression (2.11.1) for $N_n(\mathcal{R})$, we have $\sum a_i = 0$. Let us pass to the complex analogues as in 2.10. Let $M(1, \emptyset)$ be the group of isomorphism classes of complex local systems of rank 1 on $\Sigma$. It is isomorphic to $(\mathbb{C}^*)^{2g}$ and, if the compatibility $\prod \det R^*(s) = 1$ is verified, $M(\det R^*)$ is a principal homogeneous space under $M(1, \emptyset)$. The group $M(1, \emptyset)$ acts on $M(R^*)$ and on $M(\det R^*)$. Let us denote $\otimes$ the action. The application induced by $\Lambda^r$:

$$\det : M(R^*) \to M(\det R^*)$$

verifies $\det(l \otimes \mathcal{F}) = l^{\otimes r} \otimes \det(\mathcal{F})$. This implies that det is a fibration, that the $R^i \det_! \mathbb{Z}$ are locally constant, and that the action of $\pi_1 M(\det R^*)$ on the $R^i \det_! \mathbb{Z}$ factors through its quotient $\pi_1/r\pi_1 \simeq (\mathbb{Z}/r\mathbb{Z})^{2g}$. If $g \geq 1$, we deduce, by arguments analogous to those of 6.4, that $\chi(M(R^*)) = 0$, in agreement with 2.11 (ii). Let $(R^i \det_! \mathbb{Q})^0$ be the local sub-system of $R^i \det_! \mathbb{Q}$ of invariants under the action of $\pi_1$, and $\chi(M(R^*)/M(\det R^*))$ the alternating sum of the ranks of the $(R^i \det_! \mathbb{Q})^0$.

**Conjecture 6.2.** *With the notations of 6.3 and 6.6, we have*

$$\sum c_k = \chi(M(R^*)/M(\det R^*)).$$

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