The Weil Conjecture for K3 Surfaces

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1 Statement of the theorem

Let $\mathbb{F}_q$ be a a field with q elements, $\overline{\mathbb{F}}_q$ an algebraic closure of $\mathbb{F}_q$, $\phi \in \mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ the Frobenius substitution $x \mapsto x^q$, and $F = \phi^{-1}$ the "geometric Frobenius".

Let X be a scheme (separated of finite type) over $\mathbb{F}_q$, and let $\bar{X}$ be the scheme over $\overline{\mathbb{F}}_q$ deduced by extension of scalars. For every closed point x of X, let $\deg(x) = [k(x) : \mathbb{F}_q]$ be the degree over $\mathbb{F}_q$ of the residual extension. The zeta function $Z(X,t) \in \mathbb{Z}[[t]]$ is defined by

$$Z(X,t) \coloneqq \prod_{\substack{x \in X \\ x \text{ closed}}} (1 - t^{\deg(x)})^{-1}.$$

Equivalently,

$$\log Z(X,t) \coloneqq \sum_{n>0} \#X(\mathbb{F}_{q^n}) \frac{t^n}{n}.$$

For every prime number ℓ not dividing q, the ℓ-adic cohomology with proper supports $H^i_c(\bar{X}, \mathbb{Q}_\ell)$ of $\bar{X}$ is a finite-dimensional vector space over $\mathbb{Q}_\ell$, on which $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ acts by transport of structures. According to Grothendieck, one has

$$Z(X,t) = \prod_i \det(1 - Ft, H^i_c(\bar{X}, \mathbb{Q}_\ell))^{(-1)^{i+1}}.$$

Let us take X to be a non-singular projective surface. It is known in this case that the factors

$$P_i(t) \coloneqq \det(1 - Ft, H^i(\bar{X}, \mathbb{Q}_\ell))$$

of $Z(X,t)$ are polynomials with rational integer coefficients independent of ℓ. The Weil conjecture asserts here that the roots of the polynomial $P_i(t)$ have complex absolute value $q^{-i/2}$. This conjecture is invariant under finite extension of the finite base field. For X as above, and $i \neq 2$, it follows from Weil [9]. If, for simplification, we suppose X is geometrically connected, one has indeed

$$\begin{aligned}
P_0(t) &= 1 - t \\
P_4(t) &= 1 - q^2 t \\
P_1(t) &= \det(1 - \phi t; T_\ell(\mathrm{Pic}^0(X)_{\mathrm{red}})) \\
P_3(t) &= P_1(qt).
\end{aligned}$$

Let k be a field of characteristic 0 and X a non-singular projective geometrically connected surface over k. One says that X is a *K3 surface* if X is regular and has a trivial canonical divisor, that is, if $H^1(X, \mathcal{O}_X) = 0$ and Ω^2_X is isomorphic to $\mathcal{O}_X$. For the remarkable properties of these surfaces, we refer to [8]. We will mainly use the following facts.

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Lemma 1.1. *(i) One has $h^{2,0} = h^{0,2} = 1$. (ii) One has $H^2(X, T_X) = 0$ (the infinitesimal deformations are not obstructed), the application deduced from the interior product*

$$H^1(X, T_X) \otimes H^0(X, \Omega_X^2) \to H^1(X, \Omega_X^1) \tag{1}$$

is bijective, and $H^0(X, T_X) = 0$ (X has no infinitesimal automorphisms).

Lemma 1.2. *Let Y_0 be a non-singular projective surface over a finite field $\mathbb{F}_q$. The following conditions are equivalent.*

(i) There exists a complete discrete valuation ring V of unequal characteristic with finite residue field k, a proper and smooth morphism $f_V : Y \to \operatorname{Spec}(V)$ whose generic fiber is a K3 surface, and an isomorphism $i : \mathbb{F}_q \xrightarrow{\sim} k$ such that $Y_0 \otimes_{\mathbb{F}_q} k \cong Y \otimes_V k$.

(ii) There exists an integral scheme S whose field of fractions has characteristic 0, a point $s \in S$, a proper and smooth morphism $f : Y \to S$ whose generic fiber is a K3 surface, and an isomorphism $i : \mathbb{F}_q \xrightarrow{\sim} k(s)$ such that $Y_0 \otimes_{\mathbb{F}_q} k(s) \cong Y_s$.

If the equivalent conditions of Lemma 1.2 are satisfied, one says that Y_0 lifts to a K3 surface.

Theorem 1.3. *A non-singular projective surface over a finite field $\mathbb{F}_q$ which lifts to a K3 surface verifies the Weil conjecture.*

This theorem was obtained independently by Pjateckii-Shapiro and afarevitch. It applies for example to surfaces of degree 4 in $\mathbb{P}^3$. The Betti numbers of a K3 are $(1, 0, 22, 0, 1)$; the theorem implies

$$|\#X(\mathbb{F}_q) - \#\mathbb{P}^2(\mathbb{F}_q)| \leq 21q.$$

The conjectural theory of motives of Grothendieck (in particular his theory of the "motivic Galois group") was very useful to me for constructing the proof. In this language (which will not be used in the sequel), the idea is, roughly, to construct by reduction from $\mathbb{C}$ to $\mathbb{F}_q$ an abelian variety A and an injective "morphism" of motives

$$H^2(X) \hookrightarrow H^1(A) \otimes H^1(A),$$

to deduce the Weil conjecture for X from the Weil theorem for A. However, we do not define this "morphism" by an algebraic cycle; it is therefore not a morphism in the sense of [5]. In Hodge theory, its mode of definition has the following analogue (which will not be used in the sequel).

Lemma 1.4. *Let H be a variation of Hodge structures of weight n over a scheme S. If the local system $H_{\mathbb{Z}}$ has for global sections only the section s and its multiples, then s is at every point of S of type $(n/2, n/2)$.*

Notation 1.5. 1.5.1. We will denote scalar extensions by subscripts: if M_A is a module over a (commutative with unity) ring A (or a scheme over A, for example an algebraic group over A), we will denote by M_B the module over the A-algebra B (or the scheme over B) which is deduced by extension of scalars.

1.5.2. If a group G acts on a set X, we will denote by $g * x$ the transform of x by g.

1.5.3. When an equality is the definition of one of its members, we denote it by $\coloneqq$.

2 Polarized Hodge Structures

To handle Hodge structures, we will use the formalism presented in [4] and briefly recalled below.

2.1 Basic Formalism and Definitions

Let $\mathbb{S}$ be the real algebraic group of invertible elements of the $\mathbb{R}$-algebra $\mathbb{C}$, $w : (\mathbb{G}_m)_{\mathbb{R}} \to \mathbb{S}$ the inclusion of $\mathbb{R}^*$ into $\mathbb{C}^*$ and $t : \mathbb{S} \to (\mathbb{G}_m)_{\mathbb{R}}$ the inverse of the norm; we have $t(w(x)) = x^{-2}$. Let V be a finite-dimensional real vector space. It is equivalent to give:

(a) a representation $\rho : \mathbb{S} \to \mathrm{GL}(V)$ of weight n: $\rho(w(x)) * v = x^n \cdot v$;

(b) a Hodge bigraduation of weight n of $V_{\mathbb{C}} \coloneqq V \otimes_{\mathbb{R}} \mathbb{C}$:

$$V_{\mathbb{C}} = \bigoplus_{p+q=n} V^{p,q} \quad \text{with} \quad \overline{V^{p,q}} = V^{q,p};$$

(c) a Hodge filtration of weight n of $V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$: it is a decreasing filtration with $F^p(V_{\mathbb{C}}) \oplus \overline{F^{n-p+1}(V_{\mathbb{C}})} = V_{\mathbb{C}}$.

We set $C \coloneqq \rho(i)$: this is the Weil operator. One has $Cv^{p,q} = i^{p-q}v^{p,q}$. For $\mathcal{E}$ a subset of $\mathbb{Z} \times \mathbb{Z}$, a Hodge bigraduation is said to be of type $\mathcal{E}$ if the Hodge numbers $h^{p,q} = \dim V^{p,q}$ are null for $(p,q) \notin \mathcal{E}$.

We will call a *Hodge structure of weight* n here the data of a free $\mathbb{Z}$-module of finite type $H_{\mathbb{Z}}$ and a Hodge bigraduation of weight n of $H_{\mathbb{C}} = H_{\mathbb{Z}} \otimes \mathbb{C}$. The *Tate Hodge structure* $\mathbb{Z}(n)$ is the Hodge structure of type $\{(-n,-n)\}$ given by

$$\mathbb{Z}(n)_{\mathbb{C}} = \mathbb{C} \quad \text{and} \quad \mathbb{Z}(n)_{\mathbb{Z}} = (2\pi i)^n\mathbb{Z} \subset \mathbb{C}.$$

A *polarization* of a Hodge structure of weight n, H, is a morphism of Hodge structures $\psi : H \otimes H \to \mathbb{Z}(-n)$ such that, on $H_{\mathbb{R}} = H_{\mathbb{Z}} \otimes \mathbb{R}$, the real bilinear form $(2\pi i)^n\psi(x, Cy)$ is symmetric and positive definite. The form ψ is then $(-1)^n$-symmetric. A *rational Hodge structure of weight* n consists of a $\mathbb{Q}$-vector space $H_{\mathbb{Q}}$ and a Hodge bigraduation of weight n of $H_{\mathbb{C}} = H_{\mathbb{Q}} \otimes \mathbb{C}$. We denote by $\mathbb{Q}(n)$ the rational Hodge structure underlying $\mathbb{Z}(n)$. The definition of polarizations is left to the reader.

The following statement reformulates a theorem of Riemann.

Scholium 2.1. *The functor* $A \mapsto H^1(A, \mathbb{Z})$ *identifies polarized abelian varieties with polarized Hodge structures of type* $\{(0,1),(1,0)\}$.

Remark 2.2. From the end of this section, we will only use 2.11(iii') $\implies$ (ii). To prove this implication, we can restrict ourselves to considering only connected groups, which makes 2.5 unnecessary or evident.

2.2 Compact Groups and Polarizations

Let G be a real linear algebraic group. We say that G is *compact* if $G(\mathbb{R})$ is compact and every connected component of $G(\mathbb{C})$ has a real point. The evident functor is then an equivalence from the category of algebraic linear representations of G to the category of linear representations of the compact group $G(\mathbb{R})$.

Lemma 2.3. *Every real algebraic subgroup* H *of a compact real algebraic group* G *is compact.*

Proof. We know that the map $(g, \ell) \mapsto g \cdot \exp(i\ell) : G(\mathbb{R}) \times \mathrm{Lie}(G)_{\mathbb{R}} \to G(\mathbb{C})$ is surjective. For any $h \in H(\mathbb{C})$, we can write $h = g\exp(i\ell)$ for $g \in G(\mathbb{R}), \ell \in \mathrm{Lie}(G)_{\mathbb{R}}$. Then $\bar{h}^{-1}h = \exp(-i\ell)g^{-1}g\exp(i\ell) = \exp(2i\ell) \in H(\mathbb{C})$. The Zariski closure J in $H(\mathbb{C})$ of the group generated by $\exp(i\ell)$ consists of elements j such that $\bar{j} = j^{-1}$, so $\mathrm{Lie}(J) \subset i\,\mathrm{Lie}(G)_{\mathbb{R}}$. Also, $\exp(\mathrm{Lie}(J))$ has finite index in J. So for some $n \in \mathbb{Z}$ and $\ell' \in \mathrm{Lie}(J)$, $\exp(in\ell) = \exp(\ell')$. This implies $in\ell - \ell'$ is in the kernel of exp, and that $i\ell \in \mathrm{Lie}(H)_{\mathbb{C}}$. It follows that every element of $H(\mathbb{C})$ has a real point in its connected component. $H(\mathbb{R}) = H(\mathbb{C}) \cap G(\mathbb{R})$ is a closed subgroup of the compact group $G(\mathbb{R})$, so it is compact. Thus H is compact. $\square$

Lemma 2.4. *Let G be a real algebraic group. The following conditions are equivalent:*

(i) G is compact;

(ii) For every real (resp. complex) linear representation of G, there exists a G-invariant positive definite symmetric (resp. Hermitian) bilinear (resp. sesquilinear) form;

(iii) There exists a faithful representation of G admitting a form as in (ii).

For G connected, these conditions are also equivalent to:

(iv') There exists a representation of G with finite kernel admitting a form as in (ii).

Proof. For G compact, G-invariance is equivalent to $G(\mathbb{R})$-invariance, and (ii) is classical by averaging over the compact group $G(\mathbb{R})$. (iii) $\implies$ (i): If (iii) holds, G is isomorphic to a subgroup of an orthogonal or unitary group K. Since K is compact, G is compact by Lemma 2.3. (iii') $\implies$ (i) for G connected: Condition (iii') implies that $G(\mathbb{R})$ is compact (as a finite cover of a compact group); if G is connected, $G(\mathbb{C})$ is connected and contains a real point. G is thus compact. $\square$

2.3 Cartan Involutions and C-Polarization

Let G be a real linear algebraic group. For any involutive automorphism σ of G, let $G^{(\sigma)}$ be the real form of $G_{\mathbb{C}}$ defined by the antilinear involution $g \mapsto \sigma(\bar{g})$. We say that σ is a *Cartan involution* if $G^{(\sigma)}$ is compact.

Let $C \in G(\mathbb{R})$ be such that C^2 is central. A real (resp. complex) representation of G is called *C-polarizable* if there exists on V a G-invariant bilinear (resp. sesquilinear) form ψ such that the form $(x, y) \mapsto \psi(x, Cy)$ is symmetric (resp. Hermitian) and positive definite. We then say that ψ is a *C-polarization* of V.

Lemma 2.5. *a) Whether ψ is a C-polarization of V depends only on the $G(\mathbb{R})$-conjugacy class of C. b) The following conditions are equivalent:*

(i) $\mathrm{ad}\, C$ is a Cartan involution of G;

(ii) every real (resp. complex) representation of G is C-polarizable;

(iii) G admits a faithful C-polarizable real (resp. complex) representation.

If G is connected, these conditions are also equivalent to:

(iv') G admits a C-polarizable real (resp. complex) representation with finite kernel.

Proof. a) Note that $\psi(x, gCg^{-1}y) = \psi(g^{-1}x, Cg^{-1}y)$. If ψ is G-invariant, the C-polarizability condition for C is equivalent to the gCg^{-1}-polarizability condition.

b) A sesquilinear form ψ' on a complex representation of G is $G^{(\mathrm{ad}\, C)}$-invariant if and only if the form $\psi(x, y) \coloneqq \psi'(x, C^{-1}y)$ is G-invariant. The equivalence (i) $\iff$ (ii) $\iff$ (iii) $\iff$ (iii') then follows from Lemma 2.4 applied to the group $G^{(\mathrm{ad}\, C)}$. Also note:

- The complexification (resp. realification) of a C-polarizable real (resp. complex) representation is C-polarizable.
- A C-polarization of V induces a C-polarization of any sub-representation of V.
- For V real (resp. complex), one has $V \hookrightarrow V \otimes_{\mathbb{R}} \mathbb{C}$.

$\square$

2.4 Polarized Homogeneous Representations

Convention 2.6. Let G be a reductive group over $\mathbb{Q}$, and let $w : \mathbb{G}_m \to G$ and $t : G \to \mathbb{G}_m$ be morphisms defined over $\mathbb{Q}$ such that $t \circ w(x) = x^{-2}$ and w is central. Suppose we have a homomorphism $h : \mathbb{S} \to G(\mathbb{R})$ fitting into the commutative diagram:

$$\begin{tikzcd}
(\mathbf{G}_m)_{\mathbb{R}} \arrow[r, "w"] \arrow[d, equal] & \mathbb{S} \arrow[r, "t"] \arrow[d, "h"] & (\mathbf{G}_m)_{\mathbb{R}} \arrow[d, equal] \\
(\mathbf{G}_m)_{\mathbb{R}} \arrow[r, "w"] & G_{\mathbb{R}} \arrow[r, "t"] & (\mathbf{G}_m)_{\mathbb{R}}
\end{tikzcd}$$

A representation $V_{\mathbb{Q}}$ of G defined over $\mathbb{Q}$ is called *homogeneous of weight* n if $w(x)v = x^n \cdot v$ for all $x \in \mathbb{G}_m$ and $v \in V$. Any representation is a sum of homogeneous representations. The homomorphism h endows any $V_{\mathbb{Q}}$ with a rational Hodge structure. Its weight is n if $V_{\mathbb{Q}}$ is homogeneous of weight n. We consider $\mathbb{Q}(n)$ as a representation of G via $g * \lambda := t(g)^n \cdot \lambda$.

Definition 2.7. Under the setup of Convention 2.6, a *polarization* of a homogeneous representation $V_{\mathbb{Q}}$ of weight n is a G-invariant morphism $\psi : V_{\mathbb{Q}} \otimes V_{\mathbb{Q}} \to \mathbb{Q}(n)$ such that the bilinear form $(x, y) \mapsto \psi(x, h(i)y)$ on $V_{\mathbb{R}}$ is symmetric and positive definite.

Remark 2.8. Such a polarization ψ, being G-invariant, is also $\mathbb{S}$-invariant (via h), and thus defines a polarization of the rational Hodge structure $V_{\mathbb{Q}}$.

Lemma 2.9. *For the representation $V_{\mathbb{Q}}$ to be polarizable (as a G-representation), it is necessary and sufficient that the representation $V_{\mathbb{R}}$ is $h(i)$-polarizable (in the sense of Definition 2.7).*

Proof. Let $P_{\mathbb{Q}}$ (resp. $P_{\mathbb{R}}$) be the space of G-invariant $(-1)^n$-symmetric bilinear forms on $V_{\mathbb{Q}}$ (resp. $V_{\mathbb{R}}$). We have $P_{\mathbb{R}} = \mathbb{R} \otimes P_{\mathbb{Q}}$. The condition for $\psi \in P_{\mathbb{R}}$ to define an $h(i)$-polarization defines an open cone in $P_{\mathbb{R}}$. The polarizations of $V_{\mathbb{Q}}$ are the elements of $P_{\mathbb{Q}}$ lying in this open cone. The lemma follows. ☐

Proposition 2.10. *Assume the setup of Convention 2.6.*

(a) Whether a form ψ is a polarization of V depends only on the $G(\mathbb{R})$-conjugacy class of h.

(b) The following conditions are equivalent:

- *(i)* $\operatorname{ad} h(i)$ *is a Cartan involution of* $\operatorname{Ker}(t)_{\mathbb{R}}$*;*
- *(ii) every homogeneous representation of G is polarizable;*
- *(iii) G admits a faithful family of polarizable homogeneous representations.*

If G is connected, these conditions are equivalent to:

- *(i') G admits a polarizable representation ρ such that* $\operatorname{Ker}(\rho) \cap \operatorname{Ker}(t)$ *is finite.*

Proof. Assertion (a) follows from Lemma 2.5.a). If G is connected, then $\operatorname{Ker}(t)$ is connected: otherwise t would be of the form t_0^n with $n > 1$. Since $t(w(x)) = x^{-2}$, we would have $n = 2$, and $w(-1) \in \operatorname{Ker}(t)^0$. This is absurd, since $w(-1) \in h(\operatorname{Ker}(t : \mathbb{S} \to G_{m\mathbb{R}}))$, which is connected. This said, (b) results from 2.10 and 2.8.b). ☐

3 Reminders on the Spinorial Group

For the reminders contained in this section, one may consult [3].

3.1 Clifford Algebras and Groups

If V is a free module over a ring A, equipped with a symmetric bilinear form ψ, we will denote by $C(V)$ (or by $C(V,\psi)$) the Clifford algebra of V equipped with the quadratic form $\psi(x,x)$.

Suppose that A is a field of characteristic 0, and that Q is non-degenerate. We denote by $\mathrm{CSpin}(V)$ the Clifford group. It is the algebraic group of invertible elements g of $C^+(V)$ such that $gVg^{-1}=V$. We define a morphism from $\mathrm{CSpin}(V)$ to $\mathrm{SO}(V)$ by $g\mapsto(v\mapsto gvg^{-1})$. The kernel of this morphism is reduced to scalars: we have an exact sequence of algebraic groups

$$0\to\mathbb{G}_m\xrightarrow{w}\mathrm{CSpin}(V)\to\mathrm{SO}(V)\to 0.$$

The spinorial group $\mathrm{Spin}(V)$ is the algebraic subgroup of $\mathrm{CSpin}(V)$ which is the kernel of the spinorial norm $N:\mathrm{CSpin}(V)\to\mathbb{G}_m$. Let us denote by t the inverse of N; we have a commutative diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & \mathrm{Spin}(V) \arrow[d, hook] \arrow[dr] & & \\
0 \arrow[r] & \mathbb{G}_m \arrow[r, "w"] \arrow[dr, "x\mapsto x^{-2}"'] & \mathrm{CSpin}(V) \arrow[r, "\pi"] \arrow[d, "t"] & \mathrm{SO}(V) \arrow[r] & 0 \\
 & & \mathbb{G}_m \arrow[d] & & \\
 & & 0 & &
\end{tikzcd} \tag{2}$$

By definition, we also have

$$\alpha:\mathrm{CSpin}(V)\to C^+(V)^*. \tag{3}$$

3.2 Spin Representations

We denote by $(C^+(V))_s$ the representation of $\mathrm{CSpin}(V)$ on $C^+(V)$ given by

$$g*x=\alpha(g)\cdot x. \tag{4}$$

We denote by $(C^+(V))_{\mathrm{ad}}$ the representation of $\mathrm{CSpin}(V)$ on $C^+(V)$ deduced from the action of $\mathrm{SO}(V)$ on $C^+(V)$ by transport of structures. We have

$$g*_{\mathrm{ad}}x=\alpha(g)\cdot x\cdot\alpha(g)^{-1}. \tag{5}$$

The action of $\mathrm{CSpin}(V)$ on $(C^+(V))_s$ is compatible with the right $C^+(V)$-module structure of $(C^+(V))_s$. We have an isomorphism of representations

$$\mathrm{End}_{C^+(V)}((C^+(V))_s)=(C^+(V))_{\mathrm{ad}}. \tag{6}$$

We have an isomorphism of representations

$$(C^+(V))_{\mathrm{ad}}=\bigoplus_{i\geq 0}\bigwedge^{2i}V. \tag{7}$$

3.3 Structure of Spin Representations

Suppose A is algebraically closed. We distinguish two cases.

1. $\dim(V)$ *odd*: $C^+(V)$ is a matrix algebra. Let W be a simple $C^+(V)$-module; W is a spinorial representation of $\mathrm{CSpin}(V)$: $g * w = \alpha(g) \cdot w$. We have isomorphisms of representations:

$$(C^+(V))_s = \text{sum of copies of } W,$$

$$(C^+(V))_{\mathrm{ad}} = \mathrm{End}_A(W).$$

2. $\dim(V)$ *even*: $C^+(V)$ is a product of two matrix algebras. Let W_1 and W_2 be two non-isomorphic simple $C^+(V)$-modules and let $W = W_1 + W_2$; W_1 and W_2 are the semi-spinorial representations. We have isomorphisms of representations:

$$(C^+(V))_s = \text{sum of copies of } W,$$

$$(C^+(V))_{\mathrm{ad}} = \mathrm{End}_A(W_1) \times \mathrm{End}_A(W_2).$$

In both cases, $C^+(V)$ is the bicommutant of the representation W.

Proposition 3.1. *Let Γ be a Zariski-dense subgroup of* $\mathrm{Spin}(V)$ *and let a be an A-algebra automorphism of $C^+(V)$ which commutes with the action of Γ given by $\gamma \mapsto \alpha(\gamma) \cdot x \cdot \alpha(\gamma)^{-1}$. Then, a is the identity.*

Proof. It suffices to treat the case where A is an algebraically closed field and where Γ is the entire Spin group (indeed, commuting with a is a Zariski-closed condition). With the notations of 3.4, we identify $C^+(V)$ with a subalgebra of $\mathrm{End}_A(W)$. There exists an automorphism $\tilde{a}$ of W such that $a(x) = \tilde{a} x \tilde{a}^{-1}$ $(x \in C^+(V))$.

For $\gamma \in \mathrm{Spin}(V)$, we have by hypothesis

$$\alpha(\gamma)\tilde{a}\alpha(\gamma)^{-1}\tilde{a}^{-1} \cdot x \cdot \tilde{a}\alpha(\gamma)\tilde{a}^{-1}\alpha(\gamma)^{-1} = x \quad (x \in C^+(V));$$

the commutator $(\alpha(\gamma), \tilde{a}) = \alpha(\gamma)\tilde{a}\alpha(\gamma)^{-1}\tilde{a}^{-1}$ is therefore in the center of $C^+(V)$.

1) $\dim(V)$ *odd*: here, $C^+(V) = \mathrm{End}_A(W)$, and $\det_W((\alpha(\gamma), \tilde{a})) = 1$, so that $\alpha(\gamma)\tilde{a}\alpha(\gamma)^{-1}\tilde{a}^{-1}$ is a root of unity v. Since $\mathrm{Spin}(V)$ is connected, and for $\gamma = e$, we have $v = 1$, we even have $(\alpha(\gamma), \tilde{a}) = 1$, i.e. $\alpha(\gamma)\tilde{a} = \tilde{a}\alpha(\gamma)$.

2) $\dim(V)$ *even*: here, $C^+(V) = \mathrm{End}_A(W_1) \times \mathrm{End}_A(W_2)$ and let $\tilde{a}$ permute the W_i or respect them. In both cases, we deduce from $\det_{W_i}(\alpha(\gamma)) = 1$ $(i = 1, 2)$ that $\det_{W_i}((\alpha(\gamma), \tilde{a})) = 1$ $(i = 1, 2)$, and we conclude as above that $\alpha(\gamma)\tilde{a} = \tilde{a}\alpha(\gamma)$.

In both cases, $\tilde{a}$ is in the commutant of the representation, thus commutes with the bicommutant $C^+(V)$: a is the identity. □

4 Construction of Abelian Varieties

In this section and part of the next, we present results from [7].

4.1 Setup

Let $V_{\mathbb{Z}}$ be a free $\mathbb{Z}$-module of rank $n+2$ equipped with a bilinear form of discriminant $\neq 0$

$$B : V_{\mathbb{Z}} \otimes V_{\mathbb{Z}} \to \mathbb{Z}.$$

We suppose B has signature $(n+, 2-)$. We are interested in morphisms of weight 0 $h : \mathbb{S} \to \mathrm{SO}(V_{\mathbb{R}})$ such that

1. B is a polarization of the representation $V_{\mathbb{Q}}$ of $\mathrm{SO}(V_{\mathbb{Q}})$;
2. the Hodge numbers of $V_{\mathbb{C}}$ are $h^{-1,1} = h^{1,-1} = 1$ and $h^{0,0} = n$.

4.2 Lifting to CSpin

A morphism h as above lifts uniquely to $\tilde{h} : \mathbb{S} \to \mathrm{CSpin}(V_{\mathbb{R}})$ making the diagram commutative

$$\begin{tikzcd}
(\mathbb{G}_m)_{\mathbb{R}} \arrow[r, "w"] \arrow[d, "="] & \mathbb{S} \arrow[r, "t"] \arrow[d, "\tilde{h}"] & (\mathbb{G}_m)_{\mathbb{R}} \arrow[d, "="] \\
(\mathbb{G}_m)_{\mathbb{R}} \arrow[r, "w"] & \mathrm{CSpin}(V_{\mathbb{R}}) \arrow[r, "t"] & (\mathbb{G}_m)_{\mathbb{R}}
\end{tikzcd}$$

Lemma 4.1. *Relative to $\tilde{h}$, every representation of $\mathrm{CSpin}(V_{\mathbb{Q}})$ is polarizable (2.9).*

Proof. We apply criterion 2.11.b)(iii') to the representation $V_{\mathbb{Q}}$ of $\mathrm{CSpin}(V_{\mathbb{Q}})$. $\square$

Lemma 4.2. *The Hodge structure of $C^+(V_{\mathbb{Q}})_{\mathrm{ad}}$ defined by $\tilde{h}$ is of type $\{(-1,1),(0,0),(1,-1)\}$.*

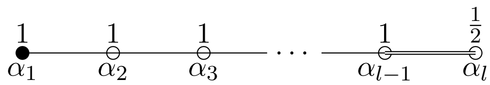
Proof. We have $C^+(V_Q)_{\mathrm{ad}} \cong \bigoplus \bigwedge^{2i} V_Q$, and we conclude using the fact that $h^{-1,1}(V_Q) = 1$. $\square$

Proposition 4.3. *The Hodge structure of $C^+(V_{\mathbb{Q}})_s$ defined by $\tilde{h}$ is of type $\{(0,1),(1,0)\}$. The Hodge structure $C^+(V_{\mathbb{Q}})_s$, equipped with the lattice $C^+(V_{\mathbb{Z}})$, thus defines an abelian variety.*

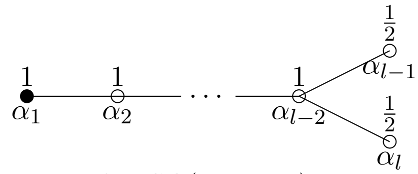
Proof. Suppose first n is odd. After extension of scalars to $\mathbb{C}$, $C^+(V_{\mathbb{C}})$ becomes a matrix algebra $C^+(V_C) \cong \mathrm{End}(W)$. On W, $\mathrm{CSpin}(V_C)_s$ acts by $g * w = \alpha(g) \cdot w$: this is the spinorial representation. The representation $C^+(V_{\mathbb{C}})_s$ is a sum of $\dim(W)$ copies of W, so it suffices to prove that the weight one representation W is purely of type $\{(0,1),(1,0)\}$. Otherwise, the representation $\mathrm{End}(W) \cong C^+(V_{\mathbb{C}})_{\mathrm{ad}}$ could not be purely of type $\{(-1,1),(0,0),(1,-1)\}$. For n even, we likewise have $C^+(V_{\mathbb{C}}) \cong \mathrm{End}(W') \oplus \mathrm{End}(W'')$, where W' and W'' are the semi-spinorial representations. As above, we prove that W' and W'' are of type $\{(0,1),(1,0)\}$, hence also $C^+(V_{\mathbb{C}})_s$. The second assertion results from 2.3 and 3.3. $\square$

4.3 Remark on Dynkin Diagrams

The construction given in this section hides the following fact (which we will not use), which can be read on the Dynkin diagram and is the real reason why everything works. Let $V_{\mathbb{Z}}$ be a polarized Hodge structure of weight 0 and type $\{(1,-1),(0,0),(-1,1)\}$, with $h^{1,-1} = 1$. Let $r : \mathbb{G}_m \to \mathrm{SO}(V_{\mathbb{C}})$ be the homomorphism such that $r(x)v^{p,q} = x^p v^{p,q}$ for $v^{p,q}$ of type (p,q). Let T be a maximal torus of $\mathrm{SO}(V_{\mathbb{C}})$, equipped with a system of simple roots Φ, $\tilde{T}$ the inverse image of T in $\mathrm{Spin}(V_{\mathbb{C}})$, and $(\omega_\alpha)_{\alpha \in \Phi}$ the fundamental weights of $\tilde{T}$. The homomorphism r admits one and only one conjugate $r_0 : \mathbb{G}_m \to \tilde{T}$ such that the rational numbers $\langle \omega_\alpha, r_0 \rangle \in \frac{1}{2}\mathbb{Z}$ are all positive. One verifies that r_0 is the homomorphism $H_{\alpha_1} : \mathbb{G}_m \to T$ relative to the root denoted α_1 below. The numbers $\langle \omega_\alpha, r_0 \rangle$ are the following:



for $SO(2, 2l-1)$,



for $SO(2, 2l-2)$.

Let ω be a dominant weight of a spinorial or semi-spinorial representation and σ the opposition involution. The point is that $\langle \omega, r_0 \rangle + \langle \sigma\omega, r_0 \rangle = 1$.

5 Construction of Families of Abelian Varieties

We will make use of the following theorem (to appear) by A. Borel.

Theorem 5.1 (A. Borel)**.** *Let X/Γ be the quotient of a Hermitian symmetric domain by a torsion-free arithmetic group. We know [2] that X/Γ is naturally a quasi-projective algebraic variety. Let S be a reduced scheme over $\mathbb{C}$ and $f : S^{an} \to X/\Gamma$ a morphism of analytic spaces. Then, f is algebraic.*

Let S be smooth over $\mathbb{C}$. For the notion of variation of Hodge structures H on S, and that of polarization of H, we refer to Griffiths [6]. We will denote by $H_{\mathbb{Z}}$ the local system of free $\mathbb{Z}$-modules on S defined by H, and by H_s the Hodge structure fiber of H at $s \in S$. For X the Siegel upper half-space, theorem 5.1 has the following corollary, which generalizes 2.3.

Scholium 5.2. *The functor $(a : A \to S) \mapsto R^1 a_* \mathbb{Z}$ identifies polarized abelian schemes A over S with polarized variations of Hodge structures of type {(0, 1), (1, 0)} on S.*

In the category of variations of Hodge structures on S, all the usual tensorial operations make sense. Thus a) if H and H' are variations of Hodge structures, we have variations of Hodge structures $H \otimes H'$, $\mathrm{Hom}(H, H')$, $\bigwedge^k H$, $\mathrm{End}(H)$, ... b) if H is a variation of Hodge structures of weight 0, and ψ is a polarization of H, we have $C(H, \psi)$ and $C^+(H, \psi)$. At every point $t \in S$, we have with the notations of 3.4, $C^+(H, \psi)_t = C^+(H_t, \psi)_{\mathrm{ad}}$.

5.1 The Period Domain and Associated Variation

Let $(V_{\mathbb{Z}}, B)$ be as in 4.1, and let X be the set of $h : \mathbb{S} \to \mathrm{SO}(V_{\mathbb{R}})$ of the type considered in 4.1. To give such an h is equivalent to giving a 2-dimensional subspace $V_{\mathbb{R}}^-$ of $V_{\mathbb{R}}$ on which B is negative definite, and an orientation of $V_{\mathbb{R}}^-$. The group $\mathrm{O}(V_{\mathbb{R}})$ acts on X by transport of structure: $(g * h)(z) = g \cdot h(z) \cdot g^{-1}$, and the description above makes it geometrically evident that X is a homogeneous space under $\mathrm{SO}(V_{\mathbb{R}})$. The stabilizer of $h \in X$ is a subgroup $\mathrm{SO}(2) \times \mathrm{SO}(n)$, the neutral component of a maximal compact subgroup.

For $h \in X$, let F_h be the corresponding Hodge filtration of $V_{\mathbb{C}}$. The function $h \mapsto F_h^1$ identifies X with an open set in the space of isotropic lines of $V_{\mathbb{C}}$. This construction equips X with a complex structure for which F_h is a holomorphic function of h. We know that for this structure, the (two) connected components of X are Hermitian symmetric domains.

Let W_Q be a weight n representation of the algebraic group $\mathrm{CSpin}(V_Q)$. Let $\underline{W}_Q$ be the constant local system on X with fiber W_Q. The group $\mathrm{CSpin}(V_Q)$ acts on $(X, \underline{W}_Q)$ by $\gamma * (w \text{ at } h) = (\gamma w \text{ at } \gamma h \gamma^{-1})$. Each $h \in X$ defines $\tilde{h} : \mathbb{S} \to \mathrm{CSpin}(V_{\mathbb{R}})$, whence a $\mathbb{Q}$-structure of Hodge on the fiber of $\underline{W}_Q$ at $h \in X$. We thus obtain a variation of $\mathbb{Q}$-structures of Hodge of weight n on S, still denoted $\underline{W}_Q$.

Let $W_{\mathbb{Z}}$ be an integer lattice in W_Q and Γ a subgroup of $\mathrm{CSpin}(V_Q)$, discrete in $\mathrm{CSpin}(V_{\mathbb{R}})$, acting freely on X and such that $\Gamma W_{\mathbb{Z}} = W_{\mathbb{Z}}$. Then, the constant local system $\underline{W_{\mathbb{Z}}}$ on X, of fiber $W_{\mathbb{Z}}$, is underlying a Γ-equivariant variation of Hodge structure W on X, as above. By passing to the quotient, it defines a variation of Hodge structures W on X/Γ.

Proposition 5.3. *Let (H, ψ) be a polarized variation of Hodge structures of type $\{(-1, 1), (0, 0), (1, -1)\}$, with $h^{1,-1} = 1$, on a smooth and connected scheme S. There exists*

(a) a finite étale surjective covering $u : S_1 \to S$;

(b) an abelian scheme A on S_1;

(c) a $\mathbb{Z}$-algebra C, and $\mu : C \to \mathrm{End}_{S_1}(A)$;

(d) an isomorphism of variations of Hodge structures

$$u^* C^+(H, \psi) \xrightarrow{\sim} \underline{\mathrm{End}}_C(R^1 a_* \mathbb{Z})$$

which induces an isomorphism of local systems of algebras

$$u^* C^+(H_{\mathbb{Z}}, \psi) \xrightarrow{\sim} \underline{\mathrm{End}}_C(R^1 a_* \mathbb{Z}).$$

Proof. Let (V_Z, B) be a $\mathbb{Z}$-module with a symmetric bilinear form, isomorphic to (H_Z, ψ) and let's resume the notations of 5.4. Let n be an integer, $\Gamma = \{\gamma \in \mathrm{SO}(V_Z) | \gamma \equiv 1 \pmod{n}\}$ and $\tilde{\Gamma} = \{\gamma \in \mathrm{CSpin}(V_Q) | \alpha(\gamma) \equiv 1 \pmod{n} \text{ in } C^+(V_Z)\}$. We assume n is large enough for Γ and $\tilde{\Gamma}$ to be torsion-free.

There exists a finite étale covering $v : S_1 \to S$ such that the local system $v^* H_Z/nZ$ is trivial. Let's choose an isomorphism σ_n between this local system and the constant local system V_Z/nZ. Let $s \in S_1$ and $\sigma : V_Z \to H_{sZ}$ be an isometry such that $\sigma \equiv \sigma_n \pmod{n}$. The Hodge structure h_s of H_s defines $\sigma^{-1}(h_s) \in X$. The image of $\sigma^{-1}(h_s)$ in X/Γ depends only on s. We thus define

$$m : S_1 \to X/\Gamma$$

and a compatible isomorphism with polarizations

$$H \cong m^* V.$$

Let $S'_1 = S_1 \times_{X/\Gamma} X/\tilde{\Gamma}$:

$$\begin{tikzcd}
 & & S_1 \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\
S & S'_1 \arrow[l, "v"'] \arrow[dr, "m"'] & & X/\tilde{\Gamma} \arrow[dl, "\pi"] \\
 & & X/\Gamma &
\end{tikzcd}$$

On S_1, we have: a) an isomorphism $(vp_1)^* H \cong (mp_1)^* V$; b) a variation of Hodge structure, the pullback by p_2 of the one on $X/\tilde{\Gamma}$ defined by the representation $(C^+(V_Q))_s$ and the lattice $C^+(V_Z)$. According to 5.2, 5.6 and 4.5, this variation of Hodge structures corresponds to an abelian scheme $a : A \to S_1$. c) Let C be the algebra $C^+(V_Z)$. The abelian scheme constructed in b) has complex multiplication by C, and the variation of Hodge structures $\mathrm{End}_C(R^1 a_* \mathbb{Z})$ is $p_2^*((C^+(V_Z))_{\mathrm{ad}})$. Combining this with a), we find an isomorphism of local systems of algebras and of variation of Hodge structures

$$C^+(vp_1^* H) \cong \mathrm{End}_C(R^1 a_* \mathbb{Z}).$$

This proves 5.7. $\square$

6 K3 Surfaces

6.1 Basic Notions

A polarized K3 surface over a field of characteristic 0 K is a K3 surface over K, say Y, equipped with $\eta \in NS(Y) = \mathrm{Pic}(Y)$ which is the class of an ample invertible sheaf. For $K = \mathbb{C}$, we will identify $NS(Y)$ with a part of $H^2(X, \mathbb{Z})$.

A family of K3 surfaces parameterized by a scheme S of characteristic 0, is a proper and flat morphism $f : Y \to S$ whose fibers are K3 surfaces. A polarization of $f : Y \to S$ is a section η of $\mathrm{Pic}_S(Y)$, which on each $s \in S$ is a polarization of $Y_s = f^{-1}(s)$. For any prime number ℓ, we identify η with a section of $R^2 f_* \mathbb{Z}_\ell$; we denote by $P^2 f_* \mathbb{Z}_\ell$ the orthogonal of η (for the cup-product); it is a sub-Z_ℓ-sheaf of $R^2 f_* \mathbb{Z}_\ell$. The cup-product and the trace morphism induce

$$\psi_\ell : P^2 f_* \mathbb{Z}_\ell(1) \otimes P^2 f_* \mathbb{Z}_\ell(1) \to \mathbb{Z}_\ell.$$

6.2 K3 Surfaces over Complex Numbers

Let's take for S a scheme of finite type over $\mathbb{C}$. Then, $R^2 f_* \mathbb{Z}$ is a variation of Hodge structures on $\mathbb{C}$; η is at every point of type $(1,1)$ and its orthogonal $P^2 f_* \mathbb{Z}$ is still a variation of Hodge structures. As above, the cup-product defines

$$\psi : P^2 f_* \mathbb{Z}(1) \otimes P^2 f_* \mathbb{Z}(1) \to \mathbb{Z}.$$

This time, the opposite of ψ is a polarization of the variation of Hodge structures $P^2 f_* \mathbb{Z}(1)$. Moreover, ψ_s is deduced from ψ via the isomorphism $(P^2 f_* \mathbb{Z}(1)_s) \otimes \mathbb{Z}_\ell = P^2 f_* \mathbb{Z}_\ell(1)_s$. For every $s \in S$, the fiber at s $(P^2 f_* \mathbb{Z}(1))_s$ is the primitive part of the cohomology of $H^2(Y_s, \mathbb{Z})$. The Hodge structure of $(P^2 f_* \mathbb{Z}(1))_s$ is of type $\{(-1,1), (0,0), (1,-1)\}$, with $h^{-1,1} = 1$ ((1.1)(i)). The action of $\pi_1(S, s)$ on $H^2(Y_s, \mathbb{Z})$ induces

$$\pi : \pi_1(S, s) \to \mathrm{O}(P^2(Y_s, \mathbb{Z})(1)_Z, \psi).$$

Proposition 6.1. *For any complex polarized K3 surface Y_0, there exists a family of polarized K3 surfaces $f : Y \to S$ as above, with $Y_s \cong Y_0$ for some $s \in S$, and such that the image of π (6.3.2) is of finite index.*

Proof. Let $\hat{f}_{\mathcal{T}} : \mathcal{Y} \to \hat{\mathcal{T}}$ be the formal versal deformation of Y_0. Since $H^0(Y_0, T^1) = 0$, $\hat{\mathcal{T}}$ is even a formal moduli scheme for Y_0, but it doesn't matter. Since $H^2(Y_0, T^1) = 0$ ((1.1)(ii)), $\hat{\mathcal{T}}$ is the spectrum of a ring of formal power series over $\mathbb{C}$. From $\hat{f}$, we deduce a variation of Hodge structures (non-polarized) on $\hat{\mathcal{T}}$. According to (1.1.1), this is the universal deformation of {the Hodge structure $H^2(Y_0, \mathbb{Z})$, equipped with the cup-product}.

Let $\hat{\eta}$ be the image of η by the composite map

$$H^2(Y_0, \mathbb{Z}) \to H^2_{DR}(Y_0/\hat{\mathcal{T}}) \to R^2 \hat{f}_* \mathcal{O}.$$

Let $\hat{S}$ be the formal subscheme of $\hat{\mathcal{T}}$ with equation $\hat{\eta} = 0$ and let $\hat{f} : \hat{Y} \to \hat{S}$ be induced by $\hat{f}_{\mathcal{T}}$. Since $h^{0,2} = 1$, $\hat{S} \subset \hat{\mathcal{T}}$ is defined by one equation; it is part of a regular system of parameters, so that $\hat{S}$ is a ring of formal series. On $\hat{S}$, η comes from an invertible sheaf $\mathcal{O}(1)$ ample on X; $\hat{f}$ is therefore algebraizable. Moreover, on $\hat{S}$, the variation of polarized Hodge structure $(P^2(\hat{Y}/\hat{S}, \mathbb{Z})(1), \psi)$ is a universal deformation of $(P^2(Y_0, \mathbb{Z})(1), \psi)$. Set $\hat{S} = \mathrm{Spec}(\hat{A})$, and express $\hat{A}$ as an inductive limit of rings of finite type over $\mathbb{C}$. Since $\hat{Y}$ is algebraizable, there exists a cartesian diagram

$$\begin{array}{ccc} \hat{Y} & \longrightarrow & Y \\ {\scriptstyle \hat{f}}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ \hat{S} & \longrightarrow & S \end{array}$$

with S of finite type over $\mathbb{C}$ and Y a family of polarized K3 surfaces over S. We can take S to be normal and connected; let $s \in S$ be the image of the closed point of $\hat{S}$. □

In fact, thanks to Artin's approximation theorem [1], one can find S such that $\hat{S}$ is its completion. S is then smooth in the neighborhood of s. The simplifications resulting from such a choice are mostly psychological. Let's show that $f : Y \to S$ verifies 6.2. Up to replacing S by a finite étale covering, we define as in 5.7 a holomorphic application $m : S \to X/\Gamma$ from S into a moduli space of polarized Hodge structures, such that the pullback of the universal family of Hodge structures on X/Γ is $(P^2(Y/S,\mathbb{Z})(1),\psi)$. By Borel (5.1), m is a morphism of schemes. The morphism m is dominant, because the composite $\hat{m} : \hat{S} \to X/\Gamma$ is. It results (see for example P. Deligne, Théorie de Hodge II. Publ. Math. IHES 40, lemma 4.4.17) that $m(\pi_1(S,s))$ is of finite index in $\pi_1(X/\Gamma, m(s)) = \Gamma$, itself of finite index in the orthogonal group.

Proposition 6.2. *Let Y_0 be a polarized K3 surface over a field K of characteristic 0. There exists:*

- *(a) a scheme S of finite type over $\mathbb{Q}$, and a family $f : Y \to S$ of polarized K3 surfaces, parameterized by S;*
- *(b) a finite extension K' of K and $v : \mathrm{Spec}(K') \to S$ such that v^*Y is isomorphic to $Y_0 \otimes_K K'$;*
- *(c) an abelian scheme $a : A \to S$ on S;*
- *(d) a $\mathbb{Z}$-algebra C and $\mu : C \to \mathrm{End}_S(A)$;*
- *(e) an isomorphism of $\mathbb{Z}_\ell$-sheaves of algebras on S*

$$u : C^+(P^2 f_*\mathbb{Z}_\ell(1), \psi_\ell) \simeq \mathrm{End}_C(R^1 a_*\mathbb{Z}_\ell).$$

Proof. We can assume K is of finite type over $\mathbb{Q}$. Let's choose a complex embedding $\sigma : K \to \mathbb{C}$ and apply 6.4 to $Y_0 \otimes_K \mathbb{C}$ to obtain a family of polarized K3 surfaces $f_1 : Y_1 \to S_1$. Apply 5.7 to f_1; we obtain a family $f_2 : Y_2 \to S_2$, still satisfying the conclusion of 6.4, and, on S_2, an abelian scheme $a_2 : A_2 \to S_2$ with complex multiplication $\mu_2 : C \to \mathrm{End}_{S_2}(A_2)$ and an isomorphism

$$u_2 : C^+(P^2 f_{2*}\mathbb{Z}_\ell(1), \psi) \simeq \mathrm{End}_C(R^1 a_{2*}\mathbb{Z}_\ell)$$

of local systems of algebras. The system of complex schemes and morphisms $\Sigma_2 = (S_2, f_2, Y_2, a_2, A_2, \mu_2)$ is defined by a finite number of equations. There thus exists an extension of finite type K_3 of K, with $K \subset K_3 \subset \mathbb{C}$, and, on K_3, a system $\Sigma_3 = (S_3, f_3, Y_3, a_3, A_3, \mu_3)$ which, by extension of scalars from K_3 to $\mathbb{C}$, provides Σ_2. □

Lemma 6.3. *There exists a unique isomorphism of $\mathbb{Z}_\ell$-sheaves of algebras*

$$u_3 : C^+(P^2 f_{3*}\mathbb{Z}_\ell(1), \psi_\ell) \simeq \mathrm{End}_C(R^1 a_{3*}\mathbb{Z}_\ell).$$

Proof. It suffices to verify that such an isomorphism exists and is unique after extension of scalars to an algebraic closure of K_3, for example $\mathbb{C}$. We have existence by construction, and uniqueness results from 3.5. Indeed, for $s \in S_2$, the image of $\pi_1(S_2, s)$ in $\mathrm{O}(P^2(Y_{2s},\mathbb{Z}),\psi)$ is of finite index, thus contains a Zariski-dense subgroup in the group SO. □

Let T be an integral scheme of finite type over $\mathbb{Q}$, with function field K_3. For a suitable T, Σ_3 is the generic fiber of a system $\Sigma = (S, f, Y, a, A, \mu)$ on T, and u_3 extends to

$$u : C^+(P^2 f_* \mathbb{Z}_\ell(1), \psi_\ell) \simeq \mathrm{End}_C(R^1 a_* \mathbb{Z}_\ell).$$

We easily verify the existence of K' as in b), and this completes the proof.

6.3 Proof of Theorem 1.3

Let's prove theorem 1.3. So let $f_V : Y \to \mathrm{Spec}(V)$ be as in 1.2(i). Let K be the field of fractions of V, and let's apply 6.5 to the surface $Y_K = Y \otimes_V K$ over K, equipped with any polarization. Up to replacing V by its normalization in a finite extension of K, we can suppose that there exists $\Sigma = (S, f, Y, a, A, \mu, u)$ as in 6.5, such that Y_K is the pullback of Y by $v : \mathrm{Spec}(K) \to S$. By pullback, we deduce an abelian variety with complex multiplication (A_K, μ) over K.

Let $\bar{K}$ be an algebraic closure of K and $\bar{k}$ the corresponding algebraic closure of k (the residue field of the normalization of V in $\bar{K}$). From u, we get an isomorphism of $\mathrm{Gal}(\bar{K}/K)$-modules

$$C^+(P^2(Y_{\bar{K}}, \mathbb{Z}_\ell)(1), \psi_\ell) \simeq \mathrm{End}_C(H^1(A_{\bar{K}}, \mathbb{Z}_\ell)). \tag{8}$$

Since Y_k is smooth over V, the inertia group I acts trivially on $H^2(Y_{\bar{K}}, \mathbb{Z}_\ell) \simeq H^2(Y_{\bar{k}}, \mathbb{Z}_\ell)$. The polarization defines an invariant η in $H^2(Y_{\bar{K}}, \mathbb{Z}_\ell)(1)$, with orthogonal $P^2(Y_{\bar{K}}, \mathbb{Z}_\ell)(1)$, and I acts trivially on $C^+(P^2(Y_{\bar{K}}, \mathbb{Z}_\ell)(1), \psi_\ell) \simeq C^+(P^2(Y_{\bar{k}}, \mathbb{Z}_\ell)(1), \psi_\ell)$.

Since I acts trivially on $\mathrm{End}_C(H^1(A_{\bar{K}}, \mathbb{Z}_\ell))$, and commutes with complex multiplication, it acts via the center of $C \otimes \mathbb{Q}_\ell$. We know moreover that a subgroup of finite index of I acts on $H^1(A_{\bar{K}}, \mathbb{Z}_\ell)$ in a unipotent way; it results that I acts via one of its finite quotients. Up to replacing K by a finite extension, we can thus suppose that A_K has good reduction over V. For A_k the special fiber of its reduction, (6.6.1) reduces to an isomorphism of $\mathrm{Gal}(\bar{k}/k)$-modules

$$C^+(P^2(Y_{\bar{k}}, \mathbb{Z}_\ell)(1), \psi_\ell) \simeq \mathrm{End}_C(H^1(A_k, \mathbb{Z}_\ell)). \tag{9}$$

According to Weil's theory for A_k, the eigenvalues of Frobenius acting on $C^+(P^2(Y_{\bar{k}}, \mathbb{Z}_\ell)(1), \psi_\ell)$ are algebraic numbers all of whose complex conjugates have absolute value 1. We have

$$\lambda^2(P^2(Y_{\bar{k}}, \mathbb{Z}_\ell)(1)) \subset C^+(P^2(Y_{\bar{k}}, \mathbb{Z}_\ell)(1), \psi_\ell),$$

and, for $\dim P^2(Y_{\bar{k}}, \mathbb{Z}_\ell) > 2$ (which is the case here), we deduce that the eigenvalues of Frobenius acting on $P^2(Y_{\bar{k}}, \mathbb{Z}_\ell)(1)$, also, have complex absolute value 1. The same statement holds for $H^2(Y_{\bar{k}}, \mathbb{Z}_\ell)(1)$, and the theorem follows.

7 Remarks

7.1 Mumford-Tate Groups

Let H be a rational Hodge structure. We will call the *Mumford-Tate group* of H the smallest algebraic subgroup G of $\mathrm{GL}(H_\mathbb{Q})$, defined over $\mathbb{Q}$, such that $G(\mathbb{R})$ contains the image of $h : \mathbb{S} \to \mathrm{GL}(H_\mathbb{R})$. For H polarizable, G is reductive. This definition differs from Mumford's: if H has weight $n \neq 0$, our G contains the homotheties. According to 2.9, every homogeneous representation of G is endowed with a natural Hodge structure.

One of the key points in the proof of 1.3 was that the Hodge structure $H^2(X, \mathbb{Q})$, for X a K3, "is expressed" via abelian varieties, in the following sense.

Definition 7.1. A rational Hodge structure H *is expressed via abelian varieties* if it belongs to the smallest category of rational Hodge structures stable by direct factor, direct sums, tensor products, and which contains the $H^1(A, \mathbb{Q})$ for A an abelian variety, as well as the $\mathbb{Q}(n)$.

In this section, we show that this property of K3s is exceptional. Consider the following property of a rational Hodge structure, with Mumford-Tate group G.

(*) The Hodge structure of the adjoint representation $\mathrm{Lie}(G)$ of G is purely of type $\{(-1,1),(0,0),(1,-1)\}$.

Proposition 7.2. *If a rational Hodge structure H is expressed via abelian varieties, then condition (*) is verified.*

Proof. Indeed, (a) the Hodge structures $H^1(A, \mathbb{Q})$ and $\mathbb{Q}(n)$ verify (*); (b) condition (*) is stable by direct factors, direct sums and tensor products. □

Remark 7.3. Analogous arguments make it possible to show that if H is expressed by means of abelian varieties, then the reductive group G_c has no "factors" of exceptional type.

7.2 Monodromy and Mumford-Tate Groups

Proposition 7.4. *Let H be a polarized variation of Hodge structures on an irreducible scheme S. For $s \in S$ outside a meager part of S, there exists a subgroup of finite index of $\pi_1(S,s)$ whose image in $\mathrm{GL}(H_{s\mathbb{Z}})$ is contained in the Mumford-Tate group of H_s.*

Proof. At every point of S, the Mumford-Tate group G_s of H_s can be described as follows. It is the algebraic subgroup of $\mathrm{GL}(H_{s\mathbb{Q}})$ which leaves fixed all tensors $t \in H_s^{\otimes n} \otimes H_s^{*\otimes m}$ $(n \geq 0)$ which are rational of type $(0,0)$. From this description, we easily deduce that, outside a meager part M of S, G_s is locally constant: for $s \notin M$, the largest subspace of type $(0,0)$ $V(n)_s \subset H_s^{\otimes n}$ is locally constant, and defines a local system $V(n)$ on S. This local system is underlying a polarized variation of Hodge structures of type $(0,0)$; there exists thus on $V(n)_s$ a positive definite quadratic form invariant by $\pi_1(S,s)$, and $\pi_1(S,s)$ acts on $V(n)_s$ through a finite group. We conclude by noting that there exists a finite sequence of integers n_i such that G_s is the algebraic subgroup of $\mathrm{GL}(H_{s\mathbb{Q}})$ which acts trivially on the $V(n_i)_s$. □

7.3 Example: Lefschetz Pencils

Consider a Lefschetz pencil of hypersurfaces of degree d and odd dimension $2r-1$.

$$\begin{tikzcd} X \arrow[r, hook] \arrow[dr, "f"'] & \mathbb{P}^{2r}(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}) \arrow[d, "pr_2"] \\ & \mathbb{P}^1(\mathbb{C}) \end{tikzcd}$$

Let $U \subset \mathbb{P}^1(\mathbb{C})$ be the complement of the finite set of $t \in \mathbb{P}^1(\mathbb{C})$ such that the hypersurface $X_t = f^{-1}(t)$ is singular, and let $s \in U$. The cup-product is an alternating form ψ on $H^{2r-1}(X_s, \mathbb{Z})$, respected by the monodromy, whence

$$m : \pi_1(U,s) \to \mathrm{Sp}(H^{2r-1}(X_s,\mathbb{Z}), \psi).$$

According to a theorem of Kajdan and Margulis, the image of m is Zariski-dense in the symplectic group. According to 7.5, for s outside a meager (in fact, countable) part of $\mathbb{P}^1(\mathbb{C})$, the Mumford-Tate group of the rational Hodge structure $H^{2r-1}(X_s, \mathbb{Q})$ is thus the entire group of symplectic similitudes. If this Hodge structure is not of Hodge level at most one (i.e. if a $h^{p,q}$ with $|q-p| > 1$ is non-zero), we deduce from 7.3 that it is not expressed via abelian varieties.

References

[1] Artin, M.: Algebraization of formal moduli I. Global analysis. Papers in honor of K. Kodaira. Princeton University Press (1969).

[2] Baily, W.L., Jr., Borel, A.: Compactification of arithmetic quotients of bounded symmetric domains. Ann. of Math. (2) **84**, 442–528 (1966).

[3] Bourbaki, N.: Algèbre – Chapitre 9. Paris: Hermann 1959. Act. Sci. et Ind. 1272.

[4] Deligne, P.: Travaux de Griffiths. Séminaire Bourbaki 376, May 1970. Lecture Notes in mathematics 180. Berlin-Heidelberg-New York: Springer 1971.

[5] Demazure, M.: Motifs des variétés algébriques. Séminaire Bourbaki 365, May 1970. Lecture Notes in mathematics 180. Berlin-Heidelberg-New York: Springer 1971.

[6] Griffiths, P. A.: Periods of integrals on algebraic manifolds. (Summary of main results and discussions of open problems and conjectures.) Bull. Am. Math. Soc. **76**, 228–296 (1970).

[7] Kuga, M., Satake, I.: Abelian varieties attached to polarized K3-surfaces. Math. Ann. **169**, 239–242 (1967).

[8] Shafarevich, I. R., et al.: Algebraic surfaces. Trudi Mat. Inst. Steklov LXXV.

[9] Weil, A.: Variétés abéliennes et courbes algébriques. Paris: Hermann 1948.

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