ALGEBRAIC K-THEORY OF CERTAIN OPERATOR ALGEBRAS

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In this article, we study the algebraic K-theory of the algebra of compact operators in a separable complex Hilbert space and certain C*-algebras naturally associated with it, such as $\mathcal{K} \otimes A$ where A is any C*-algebra. We can state the following conjecture:

Conjecture: The groups $K_n(\mathcal{K} \otimes A)$, $n \in \mathbb{Z}$, are periodic with period 2 with respect to n and are isomorphic to the topological K-groups of the Banach algebra A.

Since $\mathcal{K} \otimes A$ does not have a unit element, the groups $K_n(\mathcal{K} \otimes A)$ must be interpreted as those appearing in the exact sequence

$$0 = K_{n+1}(\mathcal{B}\otimes A/\mathcal{K}\otimes A) \to K_{n+1}(\mathcal{B}\otimes A) \to K_n(\mathcal{K}\otimes A) \to K_n(\mathcal{B}\otimes A/\mathcal{K}\otimes A) = 0.$$

Therefore $K_n(\mathcal{K} \otimes A) \cong K_{n+1}(\mathcal{B} \otimes A/\mathcal{K} \otimes A)$. In this formula, $\mathcal{B}$ is the algebra of all continuous operators in the Hilbert space and $\mathcal{B}/\mathcal{K}$ is the quotient unitary algebra (often called the "Calkin algebra").

In support of this conjecture, we can cite the theorem of Brown and Schochet [4]: $K_1(\mathcal{B}/\mathcal{K}) = 0$, as well as the following theorem that we prove in this article:

Theorem 0.1. *The conjecture is true for $n \leq 0$. Furthermore, for any n, the natural homomorphism*

$$K_n^{top}(A) \to K_n(\mathcal{K} \otimes A)$$

is surjective, the kernel being a direct summand in $K_n(\mathcal{K} \otimes A)$.

By considering inductive limits of suitable rings, as an application of the preceding theorem, we can construct examples of unitary rings such that $K_n(A) \cong K_n^{top}(A)$. In particular, $K_n(A) \cong K_{n+2}(A)$ for all $n \in \mathbb{Z}$.

Here is the detailed organization of this article. In the first three sections, we recall well-known definitions and results in algebraic or topological K-theory. The only new result worth mentioning is perhaps Theorem 3.6, which also appears implicitly in [10].

The fourth section outlines the main features of a theory of multiplicative structures in K-theory. The only new thing added to the traditional presentation (cf. [1] [15]) is the slightly more detailed treatment than usual of multiplicative structures in the case of rings without a unit element. In particular, if

$$\phi : A \times B \to C$$

is a k-bilinear map such that $\phi(aa', bb') = \phi(a,b)\phi(a',b')$, we can define a "cup-product"

$$K_i(A) \times K_j(B) \to K_{i+j}(C)$$

for i and $j \geq 0$ provided that $i + j \neq 0$.

In the fifth section, we prove the theorem mentioned above. In fact, we prove a slightly stronger result with a slightly different definition of $K_n(\mathcal{K} \otimes A)$. This group is defined here as the kernel of the homomorphism

$$K_n(\mathcal{K}^+ \otimes A) \to K_n(A)$$

where $\mathcal{K}^+$ denotes the algebra to which a unit element has been added. If we denote by $\tilde{K}_n(\mathcal{K} \otimes A)$ the group denoted $K_n(\mathcal{K} \otimes A)$ above, which is in fact $K_{n+1}(\mathcal{B} \otimes A/\mathcal{K} \otimes A)$, we can define successive homomorphisms

$$K_n(\mathcal{K} \otimes A) \to \tilde{K}_n(\mathcal{K} \otimes A) \to K_n^{top}(\mathcal{K} \otimes A) \to K_n^{top}(A).$$

Then $K_n(\mathcal{K} \otimes A) \cong \tilde{K}_n(\mathcal{K} \otimes A)$ for $n < 0$, but we have not been able to determine if $K_n(\mathcal{K} \otimes A) \cong \tilde{K}_n(\mathcal{K} \otimes A)$ for $n > 0$ (even for $n = 1$). It is with this definition of $K_n(\mathcal{K} \otimes A)$ that we prove the theorem mentioned above. It is clear that the same theorem for the group $\tilde{K}_n(\mathcal{K} \otimes A)$ follows.

Finally, in section 6, we give the explicit description of the ring A such that $K_n(A) \cong K_n^{top}(A)$ for all $n \in \mathbb{Z}$.

1 REMINDERS ON THE GROUP K_0

1.1. For any unitary ring A, the group $K_0(A)$ [2][16] is the Grothendieck group of the category $\mathcal{P}(A)$ of finitely generated projective A-modules (1). An equivalent definition is the following. Denote by $\mathrm{Proj}(A^n)$ the set of $n \times n$ matrices p with coefficients in A such that $p^2 = p$. Let $\widetilde{\mathrm{Proj}}(A^n)$ be the quotient set of $\mathrm{Proj}(A^n)$ by the equivalence relation

$$p \sim p' \iff \exists \alpha \in GL_n(A) \text{ such that } \alpha p \alpha^{-1} = p'.$$

Then $K_0(A)$ can be identified with the inductive limit

$$\varinjlim \left(\widetilde{\mathrm{Proj}}(A^2) \to \widetilde{\mathrm{Proj}}(A^4) \to \cdots \to \widetilde{\mathrm{Proj}}(A^{2n}) \to \widetilde{\mathrm{Proj}}(A^{2n+2}) \to \dots \right).$$

The map i_n is induced by $p \mapsto p \oplus p_0 \oplus \cdots \oplus p_0$ where p_0 is the projector of A^2 defined by the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. The isomorphism between $K_0(A)$ and this inductive limit is induced by the map from $\mathrm{Proj}(A^{2n})$ to $K_0(A)$ defined by

$$p \mapsto [\mathrm{Im}\ p] - [\mathrm{Im}\ p_0 \oplus \cdots \oplus p_0].$$

This definition of $K_0(A)$ is obviously functorial in A. Precisely, if $f : A \to B$ is a homomorphism, f defines a functor $\mathcal{P}(A) \to \mathcal{P}(B)$, hence a homomorphism

$K_0(A) \to K_0(B)$ by the correspondence $M \mapsto M \otimes_A B$. In terms of projectors, f defines a map $\mathrm{Proj}(A^n) \to \mathrm{Proj}(B^n)$ by the formula

$$p = (p_{ji}) \mapsto p' = (f(p_{ji})).$$

(1) To fix ideas, we will consider, for example, the category of right A-modules. However, it is easily shown that the category of left A-modules has an isomorphic K_0 group.

1.2. Let A be a "pseudo-ring" (that is, a ring not necessarily having a unit element) equipped with a k-algebra structure, k being a commutative ring. We define the ring A_k^+ as the group $A \oplus k$ equipped with the multiplication defined by the formula

$$(a, \lambda)(a', \lambda') = (aa' + \lambda a' + \lambda' a, \lambda\lambda').$$

This ring has the couple $(0, 1)$ as unit element, and we define $K_0(A)_k$ as the kernel of the natural homomorphism

$$K_0(A_k^+) \to K_0(k).$$

It is not difficult to see that the inclusion $A \to A_k^+$ induces an isomorphism $K_0(A) \cong K_0(A)_k$ [14]. We will therefore simply write $K_0(A)$ instead of $K_0(A) \cong K_0(A)_k$. This definition is functorial with respect to ring homomorphisms not necessarily having a unit element.

The construction of A_k^+ can be generalized to the case where k is not necessarily commutative, for example where A is a k-bimodule. The multiplication is then defined by the formula

$$(a, \lambda)(a', \lambda') = (aa' + \lambda a' + a\lambda', \lambda\lambda').$$

We will see important examples of this in sections 5 and 6.

1.3. Examples. Because of the title of this article, we will limit ourselves to examples from functional analysis.

a) $A = C_F(X)$, the ring of continuous functions on a compact space X with values in $F = \mathbb{R}$ or $\mathbb{C}$. In this case, it is well known that $K_0(A)$ is identified with the topological K-theory of X (our $K_F(X)$; cf. [13]). In particular, $K_{\mathbb{C}}(X) \otimes \mathbb{Q} \cong \bigoplus_i H^{2i}(X; \mathbb{Q})$ and $K_{\mathbb{R}}(X) \otimes \mathbb{Q} \cong \bigoplus_i H^{4i-1}(X; \mathbb{Q})$, H^* denoting Čech cohomology. If $X = S^n$, $K_{\mathbb{R}}(S^n) \cong \mathbb{Z}$ if n is even and $K_{\mathbb{R}}(S^n) = 0$ if n is odd, $K_F(X)$ generally denoting the group $\mathrm{Coker}[K_F(\mathrm{Point}) \to K_F(X)]$ (cf. [13]).

b) $A = C_0(X)$, the ring of continuous functions on the locally compact space X that tend to 0 at infinity. Then $K_0(A) \cong \mathrm{Ker}[K_F(X^+) \to K_F(\{\infty\})]$, X^+ denoting the Alexandroff compactification of X.

c) Let X be any paracompact space and let $A = C_F^b(X)$ be the ring of bounded continuous functions on X. Any finitely generated projective A-module E can be interpreted as $\mathrm{Im}(p(x))$ where $p : X \to \mathrm{Proj}(F^n)$ is a family of bounded projectors. In particular, E can be regarded as a direct summand of a trivial vector bundle [13]. Conversely, any vector bundle E that is a direct summand of a trivial bundle can be written in this way. Indeed, if we set $E \oplus E'$ = trivial bundle of rank n, we can always write that E is the image of a family of self-adjoint projectors p. If J denotes the family of involutions $2p-1$, the polar decomposition of J allows us to show that J is homotopic to a family of unitary involutions J', hence bounded. Thus the fibers $\mathrm{Im}(\frac{1-J}{2})$ and $\mathrm{Im}(\frac{1-J'}{2})$ are homotopic, hence isomorphic [19].

On the other hand, if E and E' are the images of two families of bounded self-adjoint projectors, an isomorphism between E and E' is homotopic to a unitary isomorphism, hence bounded. This shows that $K_0(A)$ is identified with the Grothendieck group of the category of vector bundles on X that are direct summands of trivial bundles. For example, if X is contractible, $K_0(A) \cong \mathbb{Z}$.

Before choosing other examples, let us state a theorem that will be very useful to us:

1.4. Theorem (Density Theorem). Let A and B be two unitary Banach algebras and let $i : A \to B$ be a continuous injection satisfying the following two properties:

1) $i(A)$ is dense in B.

2) If we identify $M_n(A)$ with a subalgebra of $M_n(B)$ by means of i, we have $GL_n(A) = GL_n(B) \cap M_n(A)$ for all n.

Then i induces an isomorphism $K_0(A) \cong K_0(B)$.

Sketch of the proof (cf. [12]). Let E be a finitely generated projective B-module, the image of a projector $p \in \mathrm{Proj}(B^n)$. Since A is dense in B, there exists $q' \in M_n(A)$ such that $\|i(q') - p\| < \frac{1}{2}$. Moreover, $\mathrm{Spec}(q') = \mathrm{Spec}(i(q'))$ is concentrated around 0 and 1 because $\mathrm{Spec}(p) \subset \{0, 1\}$. The holomorphic functional calculus then allows us to construct a projector $q \in \mathrm{Proj}(A^n)$ such that $i(q)$ is close to p. It follows that p and $i(q)$ are conjugate. Therefore, the homomorphism $K_0(A) \to K_0(B)$ is surjective. Injectivity follows from the fact that if $i(q)$ and $i(q')$ are conjugate by an element of $GL_n(B)$, they are conjugate by an element of $GL_n(A)$ by hypothesis 2. Thus, $K_0(A)$ and $K_0(B)$ are isomorphic.

2 Examples and Generalizations

1.5. **Remark.** If B is commutative, condition 2) of the density theorem is obviously equivalent to the condition 2') $A^* = B^* \cap A$

1.6. **Generalizations**

a) The preceding theorem also applies to C^*-Banach algebras without a unit element, provided that we interpret $GL_n(\mathbb{C})$ as $\mathrm{Ker}[GL_n(\mathbb{C}^+) \to GL_n(\mathbb{C})]$.

b) Let (A_r) be an increasing sequence of Banach algebras (the injection $A_r \longrightarrow A_{r+1}$ being continuous). Let $i_r : A_r \to B$ be a continuous injection of A_r into a Banach algebra B such that the diagram

$$\begin{tikzcd} A_r \arrow[r] \arrow[d] & B \arrow[d, equal] \\ A_{r+1} \arrow[r] & B \end{tikzcd}$$

commutes. We further assume that the following two properties are verified:

i) $GL_n(B) \cap M_n(A_r) = GL_n(A_r)$

ii) $\cup A_r$ is dense in B.

Then $K_0(B) \cong \lim_{\longrightarrow} K_0(A_r)$. The proof of this generalization is a simple transcription of the original theorem.

1.7. **Example.** Let H be a separable Hilbert space over the base field $F = \mathbb{R}$ or $\mathbb{C}$. We can therefore write $H = F \oplus \cdots \oplus F \oplus \ldots$ (Hilbert sum of countably many copies of F). In this form, we immediately see that $M_r(F)$ is a subalgebra of the algebra $\mathcal{K}$ of compact operators of H. By setting $A_r = M_r(F)$ and $B = \mathcal{K}$, we are in the conditions for applying the preceding generalization. Therefore $K_0(\mathcal{K}) \cong \lim_{\longrightarrow} K_0(M_r(F)) \cong \mathbb{Z}$, according to Morita's theorem or by a direct application of the definition of K_0 given in I.1 in terms of projectors.

1.8. **Example.** Let X be a compact differentiable manifold and let A (resp. B) be the algebra of C^s differentiable functions (resp. the algebra of continuous functions $C(X)$). Then the canonical injection $A \to B$ satisfies the hypotheses of the density theorem. Therefore $K_0(A) \cong K_0(B)$.

1.9. **Example.** Let A be any complex Banach algebra and let $A\langle t, t^{-1}\rangle$ be the Banach algebra of Laurent series $\sum_{n\in\mathbb{Z}} a_n t^n$ such that $\sum \|a_n\| < +\infty$. By setting $t = \exp(i\theta)$, we see that $A\langle t, t^{-1}\rangle$ can be viewed as a subalgebra of the algebra $A(S^1)$ of continuous functions on S^1 with values in A. On the other hand, if we consider the algebra $A^2(S^1)$ of C^2 differentiable functions on S^1 with values in A, this can be considered as a subalgebra of $A\langle t, t^{-1}\rangle$ according to the expression of an element of $A^2(S^1)$ as the sum of its Fourier series. If we consider the commutative diagram

$$\begin{tikzcd} A^2(S^1) \arrow[r] & A\langle t, t^{-1}\rangle \arrow[r] \arrow[d] & A(S^1) \arrow[d, equal] \\ & A(S^1) \arrow[r, equal] & A(S^1) \end{tikzcd}$$

we see that $A^2(S^1)$ and $A\langle t, t^{-1}\rangle$ satisfy the hypotheses of the density theorem. Therefore $K_0(A\langle t, t^{-1}\rangle) \cong K_0(A(S^1)) \cong K_0(A^2(S^1))$.

1.10. **Example.** Let A be the convolution algebra $L^1(\mathbb{R}^n)$. The Fourier transform allows us to define a continuous homomorphism $\phi : A \to B$ where B is the algebra of continuous functions on $\mathbb{R}^n$ that tend to 0 at infinity. According to Wiener's theorem, the hypotheses of the density theorem are satisfied. Therefore $K_0(L^1(\mathbb{R}^n)) \cong K_0(C_0(\mathbb{R}^n)) \cong \mathbb{Z}$ for n even and $= 0$ for n odd.

1.11. **Example.** A similar argument applies to the convolution algebra $L^1(\mathbb{T}^n)$ or $L^1(\mathbb{T}^n \times \mathbb{R}^p)$. We then find the topological K-theory of the locally compact space $\mathbb{T}^n \times \mathbb{R}^p$ [13].

1.12. **Example.** Let A be a "flasque" algebra in the sense of [14]. Then $K_0(A) = 0$. A typical example of a flasque algebra is the algebra of endomorphisms of an infinite-dimensional vector space.

1.13. **Example.** Let $\mathcal{B}$ be the algebra of continuous operators in an infinite-dimensional Hilbert space H and let $\mathcal{K}$ be the ideal of compact operators of H.

Then the quotient algebra $\mathcal{B}/\mathcal{K}$ (which is the Calkin algebra) has a K_0 group reduced to 0 (cf. [6]). Note, however, that the Calkin algebra is not flasque.

3 The groups K_i and K_i^{top} for $i > 0$

Let us begin by recalling some well-known definitions in topological K-theory.

2.1. **Definition.** Let A be a Banach algebra. Then, for $i > 0$, we set $K_i^{top}(A) = \pi_{i-1}(GL(A)) = \lim_{\longrightarrow} \pi_{i-1}(GL_n(A))$.

2.2. In this definition, the group $GL_n(A)$ is equipped with its natural topology and the group $GL(A) = \lim_{\longrightarrow} GL_n(A)$ is equipped with the inductive limit topology [note that every compact set of $GL(A)$ is included in $GL_n(A)$ for n large enough; which allows us to demonstrate the isomorphism $\pi_{i-1}(GL(A)) \cong \lim_{\longrightarrow} \pi_{i-1}(GL_n(A))$].

The following two theorems are proven in [14] and [13].

Theorem 3.1 (2.3)**.** *Let $0 \to A' \to A \to A'' \to 0$ be a "short exact sequence" of Banach algebras (A' being equipped with the induced topology and A'' with the quotient topology). We then have the exact sequence $K_i^{top}(A') \to K_i^{top}(A) \to K_i^{top}(A'') \to K_{i+1}^{top}(A') \to K_{i+1}^{top}(A) \to K_{i+1}^{top}(A'')$ for $i \geq 0$ (by convention we set $K_0 = K_0^{top}$).*

Theorem 3.2 (2.4)**.** *Let A be a complex (resp. real) Banach algebra. Then $K_i^{top}(A) \cong K_{i+8}^{top}(A)$ [resp. $K_i^{top}(A) \cong K_{i+2}^{top}(A \otimes \mathbb{C})$] for $i > 0$.*

2.5. **Examples.** Let $A = C_0(X)$. Then $K_i^{top}(A) \otimes \mathbb{Q} \cong \bigoplus H^{2i+1}(X;\mathbb{Q})$. If $A = C_{\mathbb{R}}(X)$, $K_i^{top}(A) \otimes \mathbb{Q} \cong \bigoplus H^{4i+1}(X;\mathbb{Q})$. If X is any topological space and if $A = C_F(X)$, we have $K_i^{top}(A) \cong \lim_{\longrightarrow}[X, GL_n(F)] \cong \lim_{\longrightarrow}[X, O(n)]$ if $F = \mathbb{R}$ (resp. $\lim_{\longrightarrow}[X, U(n)]$ if $F = \mathbb{C}$).

The density theorem is also valid for the groups K_i^{top}. Precisely, we have the following theorem:

Theorem 3.3 (2.6)**.** *Let A and B be two Banach algebras and let $i : A \to B$ be a continuous injection satisfying the hypotheses of the density theorem 1.4. Then i induces an isomorphism $K_i^{top}(A) \cong K_i^{top}(B)$ for all $i \geq 0$.*

2.7. **Examples.** We can reproduce examples 1.7-13 adapted to the groups K_i^{top}. Thus:

a) $K_i^{top}(\mathcal{K}) = \mathbb{Z}$ for i even and $K_i^{top}(\mathcal{K}) = 0$ for i odd if $\mathcal{K}$ is the algebra of compact operators in a complex Hilbert space. In the case of a real Hilbert space, the groups are respectively $\mathbb{Z}, \mathbb{Z}/2, \mathbb{Z}/2, 0, \mathbb{Z}, 0, 0, 0$ for $i = 0, 1, 2, 3, 4, 5, 6$ and 7 mod 8 ([13]).

b) The groups K_i^{top} of the algebra of C^s differentiable functions are isomorphic to the groups K_i^{top} of the algebra of continuous functions (on a compact manifold).

c) We have $K_i^{top}(L^1(\mathbb{T}^n \times \mathbb{R}^p))$ isomorphic to the group K_i^{-1} of the locally compact space $\mathbb{T}^n \times \mathbb{R}^p$ [13].

d) If A is "topologically" flasque (cf. [14]), $K_i^{top}(A) = 0$. This is the case, for example, of the algebra $\mathcal{B}$ of endomorphisms of an infinite-dimensional Hilbert space.

e) If A is the Calkin algebra $\mathcal{B}/\mathcal{K}$, Theorem 2.3 applied to the exact sequence $0 \to \mathcal{K} \to \mathcal{B} \to \mathcal{B}/\mathcal{K} \to 0$ shows that $K_i^{top}(\mathcal{B}/\mathcal{K}) \cong K_{i-1}^{top}(\mathcal{K})$ for $i > 0$.

2.8. We will now recall some classical definitions of algebraic K-theory (valid for any unital ring A). These definitions are due to Bass (for K_1), Milnor (for K_2), and Quillen (for K_i, $i > 2$). Thus: $K_1(A) = GL(A)/GL'(A)$, $GL'(A)$ being the commutator subgroup of $GL(A) = \lim_{\longrightarrow} GL_n(A)$. $K_2(A) = H_2(GL'(A); \mathbb{Z})$ (second homology group of the discrete group $GL'(A)$ with coefficients in $\mathbb{Z}$). $K_i(A) = \pi_i(BGL(A)^+)$ for $i \geq 1$, $BGL(A)^+$ being a certain space obtained from the classifying space of the discrete group $GL(A)$ (cf. [7][15][17][18]).

If A is again a Banach algebra, we can define a homomorphism $K_i(A) \to K_i^{top}(A)$ in the following way. If we denote by $GL(A)$ (resp. $GL(A)^{top}$) the group $GL(A)$ equipped with the discrete topology (resp. the usual topology), the map $BGL(A) \to BGL(A)^{top}$ at the level of classifying spaces induces a homomorphism $K_i(A) \to K_i^{top}(A)$ which plays a fundamental role in this article. This homomorphism is by definition the identity for $i = 0$. For $i = 1$, we have the following proposition:

Theorem 3.1 (2.9)**.** *We have an exact sequence $0 \to A^{*0} \to K_1(A) \to K_1^{top}(A) \to 0$ where A^{*0} denotes the connected component of the group A^* of invertible elements of A. If A is commutative, the map $A^{*0} \to K_1(A)$ is injective.*

Proof. It is well known that $GL'(A)$ is the group $E(A)$ generated by elementary matrices. The only non-obvious point in the proposition is the surjectivity of $A^{*0} \to \ker(K_1(A) \to K_1^{top}(A))$. It suffices to prove by induction on n that $GL_n(A)^0$ (the connected component of the identity in $GL_n(A)$) is generated by $E_n(A)$ and $GL_{n-1}(A)^0$. Since $GL_n(A)^0$ is generated by any neighborhood of the identity, we need to show that a matrix $M \in GL_n(A)^0$ close to the identity is a product of elements of $GL_{n-1}(A)^0$ and $E_n(A)$. If M is such a matrix, its top-left coefficient is invertible. A succession of elementary operations then shows that M is congruent modulo $E_n(A)$ to a matrix $M' \in GL_{n-1}(A)$. This matrix being close to the identity, it belongs to $GL_{n-1}(A)^0$ (e.g., because it is the exponential of some matrix). □

Theorem 3.2 (2.10)**.** *[16]. Let A be a commutative Banach algebra. Then the homomorphism $K_2(A) \to K_2^{top}(A) = \pi_1(GL(A))$ has as its image the subgroup $\pi_1(SL(A))$.*

The proof of this proposition is more delicate than the previous one. Note in particular that the homomorphism $K_2(\mathbb{C}) \to K_2^{top}(\mathbb{C})$ is reduced to 0. This is a special case of Chern-Weil theory. Indeed, the Chern classes of a flat bundle being zero, the map $BGL(\mathbb{C}) \to BGL(\mathbb{C})^{top}$ induces 0 in rational cohomology, hence in integer homology (because $H_*(BGL(\mathbb{C})^{top})$ is free) and in homotopy (because the Hurewicz homomorphism $\pi_i(BGL(\mathbb{C})^{top}) \to H_i(BGL(\mathbb{C})^{top})$ is injective). If A denotes the algebra of continuous functions on a compact space

X with real or complex values, we know very little in general about the homomorphism $K_i(A) \to K_i^{top}(A)$ or even its image for $i > 2$.

2.11. The groups $K_i(A)$ have been defined for any unital ring. In the case where A does not necessarily have a unit element but where A is a k-algebra (k a commutative ring with unit element), we can define $K_i(A)_k$ as the kernel of the homomorphism $K_i(A^+) \to K_i(k)$ (cf. 1.2.). It is important to note that, unlike the case of the group K_0, $K_i(A)_k$ depends on k. For example, if A is a Banach algebra, we can choose $k = \mathbb{R}$ or $\mathbb{C}$ (or even $\mathbb{H}$ if A is complex). A priori, the groups $K_i(A)_k$ obtained may be different.

2.12. Let $0 \to A' \to A \to A'' \to 0$ be a short exact sequence of k-algebras. Then, according to Milnor [16], we have a short exact sequence $K_1(A') \to K_1(A) \to K_1(A'') \to K_0(A') \to K_0(A) \to K_0(A'')$ where the $K_i(A) = K_i(A)_k$ are defined above. Unfortunately, we only know how to define a long exact sequence in general $K_i(A') \to K_i(A) \to K_i(A'') \to K_{i-1}(A') \to K_{i-1}(A) \to K_{i-1}(A'')$ for $i \leq 1$ (cf. [14] or [2] and Section 3.2). In the same vein, consider a Cartesian diagram of k-algebras

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

where we assume, for example, that $C \to D$ is surjective. Then we have a long exact sequence, essentially due to Milnor [16], $K_i(A) \to K_i(B) \oplus K_i(C) \to K_i(D) \to K_{i-1}(A) \to K_{i-1}(B) \oplus K_{i-1}(C) \to K_{i-1}(D)$ for $i \leq 1$ (cf. 3.2).

4 THE GROUPS K_i AND K_i^{top} FOR $i \leq 0$

3.1. In this short section, we reproduce definitions essentially given in [11, 14]. For any ring A (possibly without a unit element), consider the set of infinite matrices (a_{ij}), $(i, j) \in \mathbb{N} \times \mathbb{N}$, with coefficients in A. A matrix is said to be of finite type if there exists an integer n such that i) On each row and each column there are at most n non-zero elements. 2) The coefficients of the matrix are chosen from among n elements of A. The set of finite type matrices obviously forms a ring for the usual laws of addition and multiplication of matrices. This ring is the cone CA of the ring A; it is a flasque ring canonically associated with A (other definitions of the cone are possible; this does not alter the definition of the groups K_i for $i < 0$ according to the axiomatic characterization developed in [14]). A matrix is said to be finite if it has a finite number of non-zero coefficients. The set $\mathfrak{F}$ of finite matrices forms an ideal in CA which is isomorphic to $\lim M_n(A)$. The quotient ring $SA = CA/\mathfrak{F}$ is the suspension of A. For $i > 0$, we define $K_{-i}(A) = K_0(S^i A)$. A recurrent definition of the K_{-i} has been proposed by Bass [2] and is equivalent to this one (cf. [11]) $K_{-i}(A) = \text{Coker}[K_{-i+1}(A[t]) \oplus K_{-i+1}(A[t^{-1}]) \to K_{-i+1}(A[t, t^{-1}])]$. The

relationship between the two definitions is made through the homomorphism from $A[t,t^{-1}]$ to SA defined by $\sum a_n t^n \mapsto \begin{pmatrix} \ddots & & & & & \\ & a_0 & a_1 & a_2 & \cdots & \\ & -1 & a_0 & a_1 & a_2 & \cdots \\ & & -1 & a_0 & a_1 & \cdots \\ & & & \ddots & \ddots & \end{pmatrix}$.
This relationship allows us to show, for example, that $K_{-i}(A) = 0$ for regular Noetherian A.

3.2. Recall the axiomatic characterization of the groups K_{-i}: 1) $K_0(A)$ is the usual Grothendieck group. 2) For any short exact sequence of rings $0 \to A' \to A \to A'' \to 0$ we have a long exact sequence $K_{-i}(A') \to K_{-i}(A) \to K_{-i}(A'') \to K_{-i-1}(A') \to K_{-i-1}(A) \to \cdots$ for $i \geq 0$. 3) $K_{-i}(A) = 0$ if A is flasque. 4) The inclusion of A into $\mathfrak{F}$ induces an isomorphism $K_{-i}(A) \to K_{-i}(\mathfrak{F})$. Axiom 4 can be replaced by the (stronger) axiom: $K_{-i}(\lim A_r) \cong \lim K_{-i}(A_r)$. Note also that axiom 2 is not true for the groups K_i with $i > 0$. This is one of the reasons for the difficulty of algebraic K-theory.

3.3. Now suppose that A is a Banach algebra. For any matrix $(a_{ij}) \in CA$, we set $\|M\|_1 = \sup_i \sum_j \|a_{ij}\|$, $\|M\|_2 = \sup_j \sum_i \|a_{ij}\|$, and $\|M\| = \sup(\|M\|_1, \|M\|_2)$. Then the completion $\overline{CA}$ of CA for the norm $M \mapsto \|M\|$ is a topologically flasque Banach algebra canonically associated with A. The closure $\overline{\mathfrak{F}}$ of $\mathfrak{F}$ in $\overline{CA}$ is a closed ideal in $\overline{CA}$ and the quotient algebra $\overline{SA} = \overline{CA}/\overline{\mathfrak{F}}$ is the topological suspension of A. As in the algebraic case, we can then define the groups K_{-i}^{top} by the formula $K_{-i}^{top}(A) = K_0(\overline{S^iA})$. A recurrent definition of the groups K_{-i}^{top} is also possible. We have $K_{-i}^{top}(A) \cong \mathrm{Coker}[K_{-i+1}^{top}(A) \to K_{-i+1}^{top}(A\langle t,t^{-1}\rangle)]$. The relationship between the rings $A\langle t,t^{-1}\rangle$ and $\overline{SA}$ is given by the extension to the completions of the homomorphism $A[t,t^{-1}] \to SA$ explained in 3.1 (cf. [11]).

3.4. Theorem. Let A be a real (resp. complex) Banach algebra. Then we have a natural isomorphism $K_{-i}^{top}(A) \cong \pi_i(GL(A))$ (resp. $K_{-i}^{top}(A) \cong \pi_{2i-1}(GL(A))$). This theorem is proved in [14]. It essentially follows from the axiomatic characterization of the functors K_{-i}^{top} developed in [14]. In the case where A is a complex Banach algebra, a more conceptual proof is possible based on the fact that the rings $A\langle t,t^{-1}\rangle$ and $A(S^1)$ have the same K-theory (cf. 1.9 and [11]).

3.5. As in the case of the groups K_i and K_i^{top} for $i > 0$, we can try to compare the groups $K_{-i}(A)$ and $K_{-i}^{top}(A)$ by the homomorphism $K_{-i}(A) \to K_{-i}^{top}(A)$ induced by the homomorphism $S^iA \to \overline{S^iA}$. The following result will be useful to us in Section 5.

3.6. Theorem. Let A be a complex C*-algebra. Then $K_{-1}(A) \to K_{-1}^{top}(A)$ is surjective. This theorem is essentially proved in [10] although it is not presented in this form. We will give an independent proof. For any Banach algebra, we have isomorphisms $K_{-1}(A) \cong \pi_0^{top}(\overline{SA}) \cong \pi_0(\overline{CA}\langle t\rangle) \cong \lim \pi_0(GL_n(\overline{SA}))$. Suppose now that A is a C^*-algebra. Then every element of $K_1^{\mathrm{top}}(A)$ can be represented by a matrix $a \in GL_r(A)$ such that $\|a\| = 1$ (consider the polar decomposition of a). Consequently, if $\sum_{n=-\infty}^{n=+\infty} a_n u^n$ is a formal series such that $\sum_{n=-\infty}^{n=+\infty} \|a_n\| < +\infty$, the element $\sum_{n=-\infty}^{n=+\infty} a_n u^n$ is well-defined in $M_r(A)$. Therefore, there exists a homomorphism $\gamma : \mathbb{C}\langle u, u^{-1}\rangle \to M_r(A)$ such that $\gamma(u) = a$. We have the commutative diagram

$$K_1^{\mathrm{top}}(\mathbb{C}\langle u, u^{-1}\rangle) \to K_1^{\mathrm{top}}(M_r(A))$$
$$\downarrow \downarrow$$
$$K_{-1}(\mathbb{C}\langle u, u^{-1}\rangle) \to K_{-1}(M_r(A))$$

which is thus associated with a. The element of $K_{-1}(\mathbb{C}\langle u, u^{-1}\rangle) \cong K_0(S\mathbb{C}\langle u, u^{-1}\rangle)$ associated with u can be identified (up to sign) with the topological generator of $K_0(\mathbb{C}(T^2))$ according to the density theorem (T^2 being the 2-torus), i.e., the image of the projector $(J+1)/2$ where J is the involution in A^2 with $A = \mathbb{C}\langle t, u, t^{-1}, u^{-1}\rangle$ defined by the matrix (cf. [12])

$$\begin{pmatrix} x & z \\ z^* & -x \end{pmatrix}$$

where x and z are Laurent polynomials in t and u. Since the transposition $t \leftrightarrow u$ does not change (up to sign) this generator, we can write the same formulas by interchanging the roles of t and u. It follows that the element of $K_0(SA) \cong K_0(SM_r(A))$ associated with a is, up to sign, represented by a Laurent polynomial in t. Therefore, this element actually belongs to the image of the composite homomorphism $K_0(A[t, t^{-1}]) \to K_0(A\langle t, t^{-1}\rangle) \to K_0(SA)$. Thus, every element of $K_0^{\mathrm{top}}(SA) \cong K_1^{\mathrm{top}}(A)$ belongs to the image of the homomorphism $K_0(SA) \to K_0(\overline{SA})$. This completes the proof of Theorem 3.6.

3.7. It is generally false that the homomorphism $K_2(A) \to K_2^{\mathrm{top}}(A)$ is surjective. For example, $K_2(\mathbb{C}) = 0$ while $K_2^{\mathrm{top}}(\mathbb{C}) \cong \mathbb{Z}$. As an exercise, one can verify, however, that the homomorphism $K_1(A) \to K_1^{\mathrm{top}}(A)$ is surjective for $A = C_0(X \times \mathbb{R}^{i-1}, \mathbb{C})$, the algebra of continuous functions on $X \times \mathbb{R}^{i-1}$ with values in $\mathbb{C}$ that tend to 0 at infinity.

5 MULTIPLICATIVE STRUCTURES

4.1. Let A, B, and C be three unital rings. We call a "bimorphism" from $A \times B$ to C a bilinear map $\phi : A \times B \to C$ such that $\phi(aa', bb') = \phi(a, b)\phi(a', b')$ and such that $\phi(1, 1) = 1$. This is equivalent to saying that ϕ induces a ring homomorphism $A \otimes B \to C$. A bimorphism induces a bilinear map $\phi_* : K_0(A) \times K_0(B) \to K_0(C)$ in the following way. If $M \in \mathrm{Ob}(\mathcal{P}(A))$ and $N \in \mathrm{Ob}(\mathcal{P}(B))$, $M \otimes N$ is an $A \otimes B$-module. By scalar extension, $R = (M \otimes N) \otimes_{A \otimes B} C$ is a finitely generated projective C-module. A more "concrete" way to describe the module R is as follows. If $M = \mathrm{Im}(p)$ with $p : A^n \to A^n$ and if $N = \mathrm{Im}(q)$ with $q : B^m \to B^m$ where p and q are two projectors, then $R = \mathrm{Im}(p \otimes q)$ where $p \otimes q$ is the projector defined by the matrix $\phi(p_{ij}, q_{kl})$. The correspondence $(M, N) \mapsto R$ clearly induces the desired bilinear map $K_0(A) \times K_0(B) \to K_0(C)$. This homomorphism enjoys obvious associativity properties that we will not detail.

4.2. Example: If $A = B = C$ is a commutative ring and if ϕ is the bimorphism defined by the product, ϕ_* equips $K_0(A)$ with a commutative ring structure.

4.3. Example: If k is a commutative ring and if A is a k-algebra, the obvious bimorphism $k \times A \to A$ allows us to equip the group $K_0(A)$ with a $K_0(k)$-algebra structure.

4.4. Suppose now that A, B, and C are three k-algebras (k a commutative ring with unit) not necessarily having a unit element. We call a bimorphism $\phi : A \times B \to C$ a k-bilinear map such that $\phi(aa', bb') = \phi(a, b)\phi(a', b')$.

If D denotes the fiber product of A_k^+ and B_k^+ over k, we have the exact sequence $0 \to A \otimes B \to A_k^+ \otimes B_k^+ \to D \to 0$ which induces the exact sequence $0 \to K_0(A \otimes B) \to K_0(A_k^+ \otimes B_k^+) \to K_0(D) \to 0$ (because $K_1(A_k^+ \otimes B_k^+) \to K_1(D)$ is surjective). We deduce a bilinear map $K_0(A) \times K_0(B) \cong \mathrm{Ker}[K_0(A_k^+) \to K_0(k)] \times \mathrm{Ker}[K_0(B_k^+) \to K_0(k)] \to \mathrm{Ker}[K_0(A_k^+ \otimes B_k^+) \to K_0(D)] \cong K_0(A \otimes B)$. The bimorphism ϕ inducing a homomorphism $A \otimes B \to C$, we deduce a bilinear map $K_0(A) \times K_0(B) \to K_0(C)$ which enjoys good formal properties.

4.5. The preceding "cup-product" allows us to define a cup-product $K_i(A) \times K_j(B) \to K_{i+j}(C)$ for i and $j \leq 0$. This cup-product is induced, for example, by the bimorphism $S^i(A) \times S^j(B) \to S^{i+j}(C)$.

4.6. If the rings A, B, and C are unital and if ϕ is a bimorphism such that $\phi(1, 1) = 1$, Loday defined in [15] a cup-product $K_i(A) \times K_j(B) \to K_{i+j}(C)$ for i and $j > 0$, essentially from the tensor product of matrices: $GL_n(A) \times GL_p(B) \to GL_{np}(A \otimes B) \to GL_{np}(C)$. This cup-product also enjoys good formal properties. In the non-unital case, things are not so simple because there is no reason to assert that $K_{i+j}((A \otimes B)_k^+) \cong \mathrm{Ker}[K_{i+j}(A_k^+ \otimes B_k^+) \to K_{i+j}(D)]$. We do not know in general if the diagram $BGL((A \otimes B)_k^+) \to BGL(A_k^+ \otimes B_k^+) \to BGL(D)$ is Cartesian up to homotopy (note the two different meanings of the + sign). However, if $i, j > 0$ with $i + j > 0$, we can define a pairing $K_i(A) \times K_j(B) \to K_{i+j}(C)$.

Conclusion: If $\phi : A \times B \to C$ is a bimorphism, we know how to define a cup-product $K_i(A) \times K_j(B) \to K_{i+j}(C)$ for all values of i and $j \in \mathbb{Z}$ (cf. [11] for the complementary cases) if A, B, and C are unital and if $\phi(1, 1) = 1$. In the general case where A, B, and C are not necessarily unital k-algebras, we only know how to do this if $i, j \leq 0$ or $i, j > 0$. This cup-product enjoys good formal properties (cf. [15]).

4.7. In the case of Banach algebras (assumed complex for definiteness), we will only consider bimorphisms of $\mathbb{C}$-algebras $\phi : A \times B \to C$ such that $\|\phi(a, b)\| \leq \|a\|\|b\|$. If, moreover, A, B, and C are unital and if $\phi(1, 1) = 1$, the same methods as those applied in the algebraic case allow us to define a cup-product $K_i^{\mathrm{top}}(A) \times K_j^{\mathrm{top}}(B) \to K_{i+j}^{\mathrm{top}}(C)$. We can indeed reason with the classifying space of linear groups equipped with their usual topology as well as with topological suspensions instead of algebraic suspensions. Furthermore, the natural diagram

$$\begin{array}{ccc} K_i(A) \times K_j(B) & \longrightarrow & K_{i+j}(C) \\ \downarrow & & \downarrow \\ K_i^{\mathrm{top}}(A) \times K_j^{\mathrm{top}}(B) & \longrightarrow & K_{i+j}^{\mathrm{top}}(C) \end{array}$$

is commutative. If A, B or C does not have a unit element we can reason as in the algebraic framework by choosing $k = \mathbb{C}$ and considering topological tensor products. However, since the excision theorem is true in topological K-theory [14][13], we can define the cup-product

$$\cup : K_i^{top}(A) \times K_j^{top}(B) \to K_{i+j}^{top}(C)$$

without any restriction on the pair (i, j).

6 PERIODICITY OF THE GROUPS K_n OF CERTAIN ALGEBRAS

5.1. Let $\mathcal{K}$ be the ideal of compact operators in a separable complex Hilbert space of infinite dimension H. If we view $\mathcal{K}$ as a C^*-algebra, we can define

$$K_n(\mathcal{K}) = \mathrm{Ker}(K_n(\mathcal{K}^+) \to K_n(\mathbb{C}))$$

where $\mathcal{K}^+$ is the algebra to which we add a unit element. One of the goals of this section is the proof of the following theorem:

5.2. THEOREM. For $n \geq 0$, the groups $K_n(\mathcal{K})$ are $\mathbb{Z}$ for n even and 0 for n odd. The natural homomorphism

$$\phi_n : K_n(\mathcal{K}) \to K_n^{top}(\mathcal{K})$$

is an isomorphism for n even and surjective for n odd.

The proof of this theorem will occupy us for some time and we will need some auxiliary propositions.

5.3. PROPOSITION. The homomorphism

$$K_{-2}(\mathcal{K}) \to K_{-2}^{top}(\mathcal{K})$$

is surjective.

Proof. Let $\mathcal{B}$ be the algebra of bounded operators of H and let $\mathcal{B}/\mathcal{K}$ be the quotient algebra (the Calkin algebra). Since $\mathcal{B}$ is a topologically flasque algebra [14], we have $K_i(\mathcal{B}) = K_i^{top}(\mathcal{B}) = 0$. Furthermore, we have the exact sequences

$$0 = K_{-1}(\mathcal{B}) \to K_{-1}(\mathcal{B}/\mathcal{K}) \to K_{-2}(\mathcal{K}) \to K_{-2}(\mathcal{B}) = 0$$
$$0 = K_{-1}^{top}(\mathcal{B}) \to K_{-1}^{top}(\mathcal{B}/\mathcal{K}) \to K_{-2}^{top}(\mathcal{K}) \to K_{-2}^{top}(\mathcal{B}) = 0$$

Since $\mathcal{B}/\mathcal{K}$ is a C^*-algebra, the homomorphism $K_{-1}(\mathcal{B}/\mathcal{K}) \to K_{-1}^{top}(\mathcal{B}/\mathcal{K})$ is surjective according to 3.6. The same is therefore true for the homomorphism $K_{-2}(\mathcal{K}) \to K_{-2}^{top}(\mathcal{K})$.

5.4. We propose to demonstrate an analogous result for the group K_2. It is convenient for this to introduce for every unital ring A the group $K_i(A;\mathbb{Z}/n)$ [1][3] for $i \geq 2$ which is the $i-1$-th homotopy group of the homotopy fiber of

$$BGL(A)^+ \xrightarrow{\cdot n} BGL(A)^+$$

where the arrow is multiplication by n in the H-space $BGL(A)^+$. Thus we have in particular the exact sequence

$$K_i(A) \xrightarrow{\cdot n} K_i(A) \to K_i(A;\mathbb{Z}/n) \to K_{i-1}(A) \xrightarrow{\cdot n} K_{i-1}(A)$$

Similarly, if A is a Banach algebra we can define $K_i^{top}(A;\mathbb{Z}/n) = \pi_{i-1}(\Omega^{top})$ where Ω^{top} is the homotopy fiber of

$$BGL(A)^{top} \xrightarrow{\cdot n} BGL(A)^{top}$$

5.5. PROPOSITION. The natural homomorphism

$$K_2(\mathbb{C};\mathbb{Z}/n) \to K_2^{top}(\mathbb{C};\mathbb{Z}/n)$$

is an isomorphism.

Proof. The exact sequence

$$K_2(\mathbb{C}) \xrightarrow{\cdot n} K_2(\mathbb{C}) \to K_2(\mathbb{C};\mathbb{Z}/n) \to K_1(\mathbb{C}) \xrightarrow{\cdot n} K_1(\mathbb{C})$$

where $K_2(\mathbb{C})$ is divisible [16] shows that $K_2(\mathbb{C};\mathbb{Z}/n) \cong \operatorname{coker}(K_1(\mathbb{C}) \xrightarrow{n} K_1(\mathbb{C}))$. Since $K_1(\mathbb{C}) = \mathbb{C}^*$, we have $K_2(\mathbb{C};\mathbb{Z}/n) \cong \mathbb{Z}/n$. Similarly, the exact sequence

$$K_2^{top}(\mathbb{C}) \xrightarrow{\cdot n} K_2^{top}(\mathbb{C}) \to K_2^{top}(\mathbb{C};\mathbb{Z}/n) \to K_1^{top}(\mathbb{C}) \xrightarrow{\cdot n} K_1^{top}(\mathbb{C})$$

shows that $K_2^{top}(\mathbb{C};\mathbb{Z}/n) \cong \mathbb{Z}/n$. The isomorphism follows from the commutativity of the diagram relating the algebraic and topological sequences.

5.8. PROPOSITION. The homomorphism

$$K_2(\mathcal{K}) \to K_2^{top}(\mathcal{K})$$

is surjective.

Proof. This can be proven using the multiplicative structures in mod n K-theory developed by Araki and Toda [1] (see also [3]). We consider a cup product $K_{2i-2}(\mathcal{K}) \times K_2(\mathbb{C};\mathbb{Z}/n) \to K_{2i}(\mathcal{K};\mathbb{Z}/n)$ and proceed by induction. The full proof is detailed in the article.

5.9. THEOREM.

The homomorphism $K_{2i}(\mathcal{K}) \to K_{2i}^{\mathrm{top}}(\mathcal{K})$ is surjective for $i \geq 0$.

Now consider a unital C*-algebra A. Then $\mathcal{K} \otimes A$ is an A-bimodule, and we set $K_i(\mathcal{K} \otimes A) = \operatorname{Ker}[K_i((\mathcal{K} \otimes A)_A^+) \to K_i(A)]$.

5.10. THEOREM.

The homomorphism $K_{2i}(\mathcal{K} \otimes A) \to K_{2i}^{\text{top}}(\mathcal{K} \otimes A)$ is surjective and the kernel is a direct summand.

Proof. If we set $B_n = M_n(\mathbb{C}) \otimes A$ and $B = \mathcal{K} \otimes A$, we see that the hypotheses of the density theorem are satisfied for the inductive system of the $B_n \to B$. We therefore have $K_{2i}^{\text{top}}(B) \cong \varinjlim K_{2i}^{\text{top}}(B_n) \cong K_{2i}^{top}(A)$. Moreover, according to Bott periodicity, $K_{2i}^{top}(A) \cong K_0(A)$. Finally, we have the commutative diagram

$$\begin{array}{ccc}
K_{2i}(\mathcal{K}) \times K_0(A) & \longrightarrow & K_{2i}(\mathcal{K} \otimes A) \\
\downarrow & & \downarrow \\
K_{2i}^{\text{top}}(\mathcal{K}) \times K_0^{\text{top}}(A) & \longrightarrow & K_{2i}^{\text{top}}(\mathcal{K} \otimes A)
\end{array}$$

Since the homomorphism $K_{2i}(\mathcal{K}) \to K_{2i}^{\text{top}}(\mathcal{K})$ is surjective, the cup-product by a preimage of the generator defines a subgroup of $K_{2i}(\mathcal{K} \otimes A)$ isomorphic to $K_0(A)$ which surjects onto the topological group.

5.11. THEOREM.

Let A be a unital C*-algebra. Then the homomorphism from $K_i(\mathcal{K} \otimes A)$ to $K_{i+1}(\mathcal{B}/\mathcal{K} \otimes A)$ is an isomorphism and the map $K_{2i+1}(\mathcal{B}/\mathcal{K} \otimes A) \to K_{2i+1}^{\text{top}}(\mathcal{B}/\mathcal{K} \otimes A)$ is surjective for $i \geq 0$ and the kernel is a direct summand.

Proof. This follows from the long exact sequence associated to the fibration of classifying spaces related to the short exact sequence of rings $0 \to \mathcal{K} \otimes A \to \mathcal{B} \otimes A \to \mathcal{B}/\mathcal{K} \otimes A \to 0$, and the fact that $\mathcal{B} \otimes A$ is flasque.

5.12. THEOREM.

Let A be a unital C*-algebra. Then the homomorphisms $K_i(\mathcal{K} \otimes A) \to K_i^{\text{top}}(\mathcal{K} \otimes A)$ and $K_i(\mathcal{B}/\mathcal{K} \otimes A) \to K_i^{\text{top}}(\mathcal{B}/\mathcal{K} \otimes A)$ are surjective for all $i \in \mathbb{Z}$.

5.13. Examples.

Let X be a compact space and $A = C(X)$. Then $\mathcal{K} \otimes A$ (resp. $\mathcal{B}/\mathcal{K} \otimes A$) is identified with the algebra of continuous functions on X with values in $\mathcal{K}$ (resp. $\mathcal{B}/\mathcal{K}$). In this case, we therefore have a surjective homomorphism from the algebraic K_i-groups of these algebras onto the topological K-theory groups $K^{-i}(X)$ [13].

5.14.

From now on, we will try to extend the preceding results to the K_i groups for negative values of i. The tensor product of compact operators defines a bilinear map $\mathcal{K}(H) \times \mathcal{K}(H) \to \mathcal{K}(H \otimes H)$. According to the general considerations developed in Section 4, we deduce a cup-product $K_i(\mathcal{K}(H)) \times K_j(\mathcal{K}(H)) \to K_{i+j}(\mathcal{K}(H \otimes H))$ for $i + j \leq 0$. Since $\mathcal{K}(H \otimes H) \cong M_k(\mathcal{K}(H))$ for some k, by Morita's theorem, we have an isomorphism $K_i(\mathcal{K}(H)) \cong K_i(\mathcal{K}(H \otimes H))$. In particular, if we denote by τ the cup-product $K_i(\mathcal{K}(H)) \times K_0(\mathcal{K}(H)) \to K_i(\mathcal{K}(H \otimes H))$, the homomorphism $x \mapsto \tau(x, \epsilon)$ is an isomorphism if ϵ is a generator of $K_0(\mathcal{K}(H)) \cong \mathbb{Z}$.

5.15. THEOREM.

For all $n \in \mathbb{Z}$, $K_n(\mathcal{K}) \cong \mathbb{Z}$ for n even and $K_n(\mathcal{K}) = 0$ for n odd.
Proof. Let $u_2 \in K_2(\mathcal{K})$ and $u_{-2} \in K_{-2}(\mathcal{K})$ be elements whose images in $K_2^{\text{top}}(\mathcal{K})$ and $K_{-2}^{\text{top}}(\mathcal{K})$ are generators. Then $u_2 \cup u_{-2} \in K_0(\mathcal{K} \otimes \mathcal{K}) \cong K_0(\mathcal{K})$ is a generator of $\mathbb{Z}$. Define then $\beta : K_n(\mathcal{K}) \to K_{n+2}(\mathcal{K})$ for $n \leq -2$ and $\beta' : K_n(\mathcal{K}) \to K_{n-2}(\mathcal{K})$ for $n \geq 2$ by $\beta(x) = x \cup u_2$ and $\beta'(y) = y \cup u_{-2}$. By associativity of the cup-product, $\beta'\beta$ and $\beta\beta'$ are identity maps (up to automorphism). We therefore deduce that $K_n(\mathcal{K}) \cong K_{n-2}(\mathcal{K})$ for all $n \in \mathbb{Z}$. The result follows from knowing the groups for $n = 0$ and $n = 1$. We have $K_0(\mathcal{K}) \cong \mathbb{Z}$. For n odd, we have $K_n(\mathcal{K}) \cong K_{-1}(\mathcal{K})$, which is calculated by the exact sequence $0 = K_0(\mathcal{B}) \to K_0(\mathcal{B}/\mathcal{K}) \to K_{-1}(\mathcal{K}) \to K_{-1}(\mathcal{B}) = 0$. Since $K_0(\mathcal{B}/\mathcal{K}) = 0$ by Calkin's theorem [6], we have $K_{-1}(\mathcal{K}) = 0$, hence $K_n(\mathcal{K}) = 0$ for all odd n.

5.16. The periodicity of the groups $K_n(\mathcal{K})$ for $n \in \mathbb{Z}$ generalizes to a class of $\mathbb{C}$-algebras that we will now define. Let A be a $\mathbb{C}$-algebra. We will say that A is topologically stable if it is equipped with a ring isomorphism $A \cong M_2(A)$ and a bimorphism $\mathcal{K} \times A \to A$ satisfying certain compatibility conditions.

5.17. Examples. a) If B is any $\mathbb{C}$-algebra, $\mathcal{K} \otimes B$ is topologically stable. b) If B is a C*-algebra, the completed tensor product $\mathcal{K} \hat{\otimes} B$ is also topologically stable. c) The algebra $\mathcal{K}$ is topologically stable. d) The quotient $\mathcal{B}/\mathcal{K}$ is topologically stable. e) If A is topologically stable and if B is any $\mathbb{C}$-algebra, $A \otimes B$ is topologically stable.

5.18. THEOREM. Let A be a topologically stable algebra. Then $K_n(A) \cong K_{n-2}(A)$ for all $n \in \mathbb{Z}$. Proof. It suffices to follow the scheme of the proof of Theorem 5.15. The bimorphism $\mathcal{K} \times A \to A$ allows us to define a cup-product

$$K_i(\mathcal{K}) \times K_j(A) \to K_{i+j}(A)$$

which is "associative" up to isomorphism and induces periodicity.

5.19. COROLLARY. Let A be a C*-algebra. Then $K_n(\mathcal{K} \hat{\otimes} A) \cong K_0^{top}(A)$ for n even, and $K_n(\mathcal{K} \hat{\otimes} A) \cong K_1^{top}(A)$ for n odd.

5.20. COROLLARY. Let

$$0 \to A' \to A \to A'' \to 0$$

be an exact sequence of topologically stable C*-algebras. We then have the six-term exact sequence

$$\begin{array}{ccccc} K_0(A') & \longrightarrow & K_0(A) & \longrightarrow & K_0(A'') \\ \uparrow & & & & \downarrow \\ K_1(A') & \longleftarrow & K_1(A) & \longleftarrow & K_1(A'') \end{array}$$

(the groups K_0 and K_1 being defined in a purely algebraic way).

7 STUDY OF CERTAIN RINGS SUCH THAT $K_i(A) \cong K_{i+2}(A)$ FOR $i \in \mathbb{Z}$

6.1. In the previous section, we studied a large class of rings A (topologically stable $\mathbb{C}$-algebras) such that $K_i(A) \cong K_{i-2}(A)$ for $i \leq 0$. We will now define rings A such that $K_i(A) \cong K_{i+2}(A)$ for all $i \in \mathbb{Z}$. Precisely, let $\mathcal{A}_\infty$ be the inductive limit of a system of algebras $\mathcal{A}_i$, where each $\mathcal{A}_i$ is constructed to have periodicity-inducing properties. One can construct a bimorphism

$$\mathcal{A}_\infty \times \mathcal{A}_\infty \to \mathcal{A}_\infty$$

This bimorphism satisfies an associativity property up to isomorphism. It therefore allows us to define a cup-product

$$K_i(\mathcal{A}_\infty) \times K_j(\mathcal{A}_\infty) \to K_{i+j}(\mathcal{A}_\infty)$$

for i and $j \in \mathbb{Z}$ which is associative up to isomorphism.

6.2. THEOREM. For all $i \in \mathbb{Z}$, we have $K_i(\mathcal{A}_\infty) \cong \mathbb{Z}$ for i even and $K_i(\mathcal{A}_\infty) = 0$ for i odd. Proof. This is the same formal proof as that of Theorem 5.15.

6.3. Now consider a C*-algebra A. Then we can consider the inductive limit of the system

$$\mathcal{A}_\infty \otimes A \to \mathcal{A}_\infty \otimes A \to \dots$$

which we will denote by $\mathcal{A}_\infty(A)$.

6.4. THEOREM. For all $i \in \mathbb{Z}$, we have $K_i(\mathcal{A}_\infty(A)) \cong K_i^{top}(A)$. In particular, $K_i(\mathcal{A}_\infty(A)) \cong K_{i+2}(\mathcal{A}_\infty(A))$. Proof. This theorem is proved formally like Theorem 5.18 using the "module" structure of $\mathcal{A}_\infty(A)$ over $\mathcal{A}_\infty$.

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