A Finiteness Theorem for Monodromy

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0. Introduction

Let S be a smooth (i.e., without singularity) connected complex algebraic variety and let $(X_s)_{s \in S}$ be an algebraic family, parametrized by S, of smooth projective varieties $X_s \subset \mathbb{P}^N(\mathbb{C})$. By definition, this is the data of a projective and smooth morphism $f : X \to S$, equipped with a factorization

$$\begin{array}{ccc} X & \longrightarrow & \mathbb{P}^N \times S \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{pr}_2} \\ S & = & S \end{array}$$

and the X_s are the fibers of $f : X_s := f^{-1}(s)$. Whatever i, the cohomology groups $\mathrm{H}^i(X_s, \mathbb{Z})$ form a local system on S. If we choose a base point $o \in S$, the fundamental group $\pi_1(S, o)$ thus acts on $\mathrm{H}^i(X_o, \mathbb{Z})$. This is the monodromy representation

$$\sigma_i : \pi_1(S, o) \to \mathrm{Aut}(\mathrm{H}^i(X_o, \mathbb{Z})).$$

The index $\mathbb{Z}$ reminds us that $\pi_1(S, o)$ acts on a $\mathbb{Z}$-module of finite type. We will mostly consider the corresponding representation on the $\mathbb{Q}$-vector space $\mathrm{H}^i(X_o, \mathbb{Q}) = \mathrm{H}^i(X_o, \mathbb{Z}) \otimes \mathbb{Q}$, also called the monodromy representation, and simply denoted by

$$\sigma : \pi_1(S, o) \to \mathrm{GL}(\mathrm{H}^i(X_o, \mathbb{Q})).$$

We know almost nothing about which groups can be fundamental groups of algebraic varieties, and which linear representations can be obtained as monodromy. For example, we do not know the answer to Serre's question: can $\pi_1(S, o)$ be non-trivial, and have no non-trivial finite quotient? A necessary condition for a group to be the fundamental group of an algebraic variety was given by J. Morgan [8].

The Zariski closure $\bar{M}$ of the monodromy group

$$M := \mathrm{Im}(\sigma) \quad \text{in the linear group} \quad \mathrm{GL} := \mathrm{GL}(\mathrm{H}^i(X_o, \mathbb{Q}))$$

is often easier to compute than $\mathrm{Im}(\sigma)$. We know that it is a semi-simple algebraic group ([2], 4.2.9a)). By abuse of notation, let us denote by $M(\mathbb{Z})$ the subgroup of $M(\mathbb{Q})$ which respects the integer lattice $\mathrm{H}^i(X_o, \mathbb{Z})/\mathrm{torsion} \subset \mathrm{H}^i(X_o, \mathbb{Q})$. We have $M \subset M(\mathbb{Z})$. We know examples where M has infinite index in $M(\mathbb{Z})$ (Mostow-Deligne [3], M. Nori [9]). In Nori's example, the group M is not finitely presented. The fact that, initially, the examples one could compute all provided groups M of finite index in $M(\mathbb{Z})$ may stem from the lack of a criterion to recognize the contrary. We do not know if, for $\gamma \in M(\mathbb{Z})$, the question of whether $\gamma \in M$ is decidable.

This article presents some finiteness results on monodromy representations. The most striking is the following:

Theorem 0.1 (0.1). *Fix S, $o \in S$, and an integer N. The monodromy representations σ of $\pi_1(S, o)$ (for variable i and X) which are of dimension N form only a finite number of isomorphism classes.*

We will also prove the more precise result:

Variant 0.2 (0.2). *Fix S, $o \in S$, and an integer N. The rational representations of dimension N of $\pi_1(S, o)$, which are direct factors of a monodromy representation, form only a finite number of isomorphism classes.*

These statements are for the rational representations provided by monodromy. The following well-known lemmas allow us to also treat representations on integers.

Lemma 0.3 (0.3). *Let $\sigma : \Gamma \to \mathrm{GL}(V)$ be a semi-simple representation of a group Γ on a finite-dimensional rational vector space V. The representations $\sigma' : \Gamma \to \mathrm{Aut}(L)$, for $L \subset V$ a stable integer lattice under Γ, form only a finite number of isomorphism classes.*

The proof is recalled in 2.6.

Lemma 0.4 (0.4). *Let H be a $\mathbb{Z}$-module of finite type, T its torsion subgroup, $L = H/T$ and $\sigma : \Gamma \to \mathrm{Aut}(L)$ be a representation of a group Γ on L. If Γ is of finite type, there are only a finite number of lifts of σ to an action of Γ on H.*

Proof. The kernel of $\mathrm{Aut}(H) \to \mathrm{Aut}(L)$ is finite, as it is contained in the finite set $1 + \mathrm{Hom}(H, T)$. For each generator γ of Γ, $\sigma(\gamma)$ thus has only a finite number of lifts in $\mathrm{Aut}(H)$. $\square$

If $f : X \to S$ is a projective and smooth morphism, we have for each X_s a Hodge decomposition

$$\mathrm{H}^i(X_s, \mathbb{C}) = \mathrm{H}^i(X_s, \mathbb{C}) = \bigoplus_{p+q=i} \mathrm{H}_s^{p,q}.$$

The subspace $\mathrm{H}_s^{p,q}$ of $\mathrm{H}^i(X_s, \mathbb{C})$ is the space of cohomology classes representable by a closed form of type (p, q). For any Kähler structure on X_s, it is equivalent to require that the harmonic representative of $c \in \mathrm{H}^i(X_s, \mathbb{C})$ be of type (p,q). When s varies, the $\mathrm{H}^i(X_s, \mathbb{C})$ form a local system on S, and the Hodge decomposition varies in a real analytic way. P.A. Griffiths has identified the essential properties of how the Hodge decomposition varies with s. If we take them as axioms, we obtain the notion of a "polarizable variation of Hodge structures of weight i": a local system of $\mathbb{Z}$-modules of finite type, whose complexifications are equipped with a "Hodge" decomposition varying continuously and satisfying suitable axioms ([6], [12]). We will mainly consider the local system of $\mathbb{Q}$-vector spaces of finite dimension deduced from the local system of $\mathbb{Z}$-modules of finite type by tensoring with $\mathbb{Q}$. We will say that it underlies the variation.

Theorem 0.1 and its variant 0.2 result from the following:

Theorem 0.5 (0.5)**.** *Fix S and an integer N. The local systems on S of $\mathbb{Q}$-vector spaces of dimension N, which are direct factors of a local system underlying a polarizable variation of Hodge structures, form only a finite number of isomorphism classes.*

The proof is given in paragraph 2. The reader interested only in 0.1, or who does not care in 0.5 about being able to take direct factors, can omit the end of paragraph 1, from 1.11 onwards.

The present work is the result of an effort to understand article [4] by G. Faltings. In paragraph 3, we briefly explain how, in the case where S is a curve, where X is an abelian scheme over S and where $i = 1$, theorem 0.1 can be obtained by the methods of [4]. We also obtain a "uniform in S" theorem for S a curve of a given topological type (3.10). I do not know if this theorem holds for polarizable variations of Hodge structures of a more general type than $\{(0,1),(1,0)\}$.

1 Complex Variations

1.1

Let S be a smooth complex analytic manifold. A **complex variation of weight 0** on S is the data of a complex local system V on S, with fibers V_s ($s \in S$) equipped with a decomposition

$$V_s = \bigoplus_{p \in \mathbb{Z}} V_s^p,$$

satisfying the following axioms:

(V.C.1) The subspaces $F^p V_s := \bigoplus_{i \geq p} V_s^i$ (resp. the $\bar{F}^q V_s := \bigoplus_{i \leq q} V_s^i$) of V_s vary holomorphically (resp. anti-holomorphically) with s.

(V.C.2) If a local differentiable section v of V is at each point in $F^p V_s$ (resp. $\bar{F}_s^q$), its derivative with respect to a vector field on S is in $F^{p-1} V_s$ (resp. $\bar{F}_s^{q-1}$).

If $\mathcal{V}$ is the sheaf of C^∞ sections of V and if ∇ is the connection, seen as a morphism of sheaves $\nabla : \mathcal{V} \to \Omega_S^1(V)$, the decompositions of the V_s into the V_s^p provide a decomposition $\mathcal{V} = \bigoplus \mathcal{V}^p$ of $\mathcal{V}$, and (V.C.1), (V.C.2) are equivalent to ∇ sending $\mathcal{V}^i$ into $\Omega_S^{1,0}(\mathcal{V}^{i-1}) \oplus \Omega_S^{0,1}(\mathcal{V}^{i+1})$.

In what follows, we will only consider complex variations of weight 0 and will omit specifying "of weight 0".

For S connected, the **Hodge numbers** h^p are the dimensions $h^p := \dim V_s^p$, independent of $s \in S$. Similarly, the dimension of V is that of the V_s.

A **polarization** of the complex variation V is a horizontal Hermitian form ψ on V such that at each point $s \in S$ the decomposition $V_s = \bigoplus V_s^p$ is orthogonal, and the restriction of ψ to V_s^p is positive definite for p even and negative definite for p odd. We set $\Psi = \sum (-1)^p \psi|_{V^p}$. This is a positive definite Hermitian form on the vector bundle V.

1.2 Example

Let H be a variation of Hodge structures of weight w on S and V be the complexification of the local system of rational vector spaces $H_{\mathbb{Q}}$ underlying H. By definition, it is equipped with decompositions

$$V_s = \bigoplus_{p+q=w} V_s^{p,q} \quad (s \in S).$$

If we set $V_s^p := V_s^{p,w-p}$, V, equipped with these decompositions, is a complex variation. A polarization ψ_0 of H is a horizontal bilinear form on $H_{\mathbb{Q}}$ with values in $(2\pi i)^{-w}\mathbb{R}$ satisfying suitable axioms. If ψ is the sesquilinear extension of ψ_0 to V, these axioms are equivalent to $(2\pi i)^w \psi$ being a polarization of the complex variation V.

1.3

Let us specialize 1.1 to the case where S is reduced to a point: a **complex Hodge structure** on a complex vector space H is a decomposition $H = \bigoplus_{p \in \mathbb{Z}} H^p$; its Hodge numbers are $h^p := \dim H^p$; a **polarization** is a Hermitian form ψ for which the H^p are pairwise orthogonal and whose restriction to H^p is positive definite for p even, negative definite for p odd.

Let H be a complex vector space, ψ a Hermitian form on H and (h^p) a family of integers ≥ 0. We assume that $\dim H = \sum h^p$ and that $\mathrm{sgn}(\psi) = \sum (-1)^p h^p$. We will denote by $M(H, \psi, (h^p))$ or simply M the space of complex Hodge structures on H, of Hodge numbers h^p, polarized by ψ. The application which associates to a point of M the Hodge filtration of H by the

$$F^p := \bigoplus_{i \geq p} H^i$$

is an injection of M into a flag variety of H. Indeed, $\bar{F}^q := \bigoplus_{i \leq q} H^i$ is the orthogonal of F^{1-q} and $H^p = F^p \cap \bar{F}^p$. This injection identifies M with an open subset of this flag variety. We equip M with the induced complex structure.

At each point of M, $\mathrm{End}(H)$ inherits a Hodge decomposition, and a corresponding Hodge filtration, with

$$F^p\mathrm{End}(H) = \{f : H \to H | f(F^i) \subset F^{i+p}\}.$$

The tangent space to M (i.e., to the flag variety) identifies with $\mathrm{End}(H)/F^0\mathrm{End}(H)$, and inherits the filtration image of F. This is a holomorphic filtration F of the tangent bundle, with $F^0 = 0$.

Let U be the unitary group $U(H, \Psi)$. It acts transitively on M. If $m \in M$ corresponds to a decomposition $H = \bigoplus H^p$ of H, the stabilizer of m is the compact subgroup

$$K := \prod U(H^p, \psi|_{H^p}) \subset U.$$

The compactness of K ensures the existence on $M \simeq U/K$ of Riemannian (or even Hermitian) U-invariant metrics.

1.4 Remark

If (H', ψ') is isomorphic to (H, ψ), a U-invariant metric on M determines one on $M' := M(H', \psi', (h^p))$: transport the metric given by the isomorphism $M \to M'$ induced by an isomorphism $u : (H, \psi) \to (H', \psi')$. The U-invariance ensures that the metric obtained on M' does not depend on the choice of u.

1.5 Remark

For any U-invariant Riemannian metric d on M, the Riemannian manifold (M, d) is homogeneous; it is therefore complete and its closed balls are compact. If $o \in M$ and C is a constant, the set of $g \in U$ such that $d(g \cdot o, o) \leq C$ is compact.

1.6

Let (V, ψ) be a polarized complex variation on S, assumed to be connected, and with Hodge numbers (h^p). Let us choose a base point $o \in S$ and let $(\tilde{S}, o)$ be the universal cover of (S, o). Let V_o be the fiber of the local system V at o and ψ_o the Hermitian form on V_o induced by ψ. The inverse image on $\tilde{S}$ of the complex variation V is a complex variation on $\tilde{S}$. The underlying local system is constant. It identifies with the constant local system with fibers V_o and this identification allows us to view the variation as a family, parametrized by $\tilde{S}$, of complex Hodge structures on the fixed vector space V_o. They are polarized by ψ_o and have Hodge numbers (h^p). We thus attach to (V, ψ) an application

$$P : \tilde{S} \to M(V_o, \psi_o, (h^p)).$$

Axiom (V.C.1) means that P is a holomorphic application, and (V.C.2) means that dP takes its values in the sub-bundle F^{-1}(tangent) of the tangent bundle (cf. 1.3).

We have an action of $\pi_1(S, o)$ on $\tilde{S}$ and on V_o (monodromy) – thus on $M := M(V_o, \psi_o, (h^p))$ – and the application P is equivariant. The application P determines the complex variation inverse image of (V, ψ) on $\tilde{S}$, and the complex variation (V, ψ) on S is deduced by passing to the quotient by $\pi_1(S, o)$.

Our essential tool is the following theorem of P.A. Griffiths ([5], 10.1).

Theorem 1.1 (1.7, P.A. Griffiths)**.** *Suppose that S is the unit disk, and let us equip it with its Poincaré metric (constant curvature -1). Then, for a suitable U-invariant Riemannian metric on M, depending only on the Hodge numbers h^p (cf. 1.4), the application P decreases distances.*

Let us resume the hypotheses and notations of 1.6, and let $\sigma : \pi_1(S,o) \to \mathrm{GL}(V_o)$ be the monodromy representation. We will say that a basis e of V_o is **adapted** to the Hodge decomposition $P(o) : V_o = \bigoplus V_o^p$ of V_o if it is the union of orthonormal bases for $(-1)^p\psi_o$ of the V_o^p.

Corollary 1.2 (1.8)**.** *Fix (S,o) and the Hodge numbers (h^p). For each $\gamma \in \pi_1(S,o)$, there exists C such that, for any polarized variation V on S, of Hodge numbers (h^p), and for any adapted basis e of V_o, the coefficients of the matrix $\sigma(\gamma)$, in the basis e, are bounded in absolute value by C.*

Proof. Let $H := \bigoplus \mathbb{C}^{h^p}$, ψ be the Hermitian form on H sum of the Hermitian forms $(-1)^p|z_i|^2$ on the $\mathbb{C}^{h^p}$, and $M := M(H, \psi, (h^p))$ be the point of M corresponding to the decomposition of H into the $\mathbb{C}^{h^p}$. For any variation V as in 1.7, the choice of an adapted basis of V_o identifies V_o with H and allows us to view the application P defined by V as an application

$$P_V : \tilde{S} \to M$$

verifying $P_V(o) = o$.

Let d_K be the Kobayashi pseudo-metric on $\tilde{S}$: the largest pseudo-metric d such that for any holomorphic application ϕ from the unit disk into $\tilde{S}$, $d(\phi(x), \phi(y))$ is less than or equal to the Poincaré distance from x to y. Since $\tilde{S}$ is connected, $d_K(x,y)$ is finite for all x and y. For M equipped with a suitable U-invariant Riemannian metric d_M, independent of (S,o) and of V, it follows from 1.7 that

$$d_M(\sigma(\gamma)o, o) = d_M(P_V(\gamma \cdot o), P_V(o)) \leq d_K(\gamma \cdot o, o).$$

From 1.5, $\sigma(\gamma)$ is thus in a compact set of U independent of V, which proves 1.8. □

Corollary 1.3 (1.9)**.** *Fix (S,o) and an integer N. For each $\gamma \in \pi_1(S,o)$, there exists C such that, for any polarized variation V on S, of dimension N, and any adapted basis e of V_o, the coefficients of the matrix $\sigma(\gamma)$, in the basis e, are bounded in absolute value by C.*

Proof. We proceed by recurrence on N. The case $N = 0$ is trivial. If p is such that $h^p = 0$, let $V_{<p}$ (resp. $V_{>p}$) be the sum of the V^i for $i < p$ (resp. $i > p$). It follows from (V.C.1) and (V.C.2) that the sub-bundles $V_{\leq p}$ and $V_{>p}$ are horizontal. If there exists $i < p < j$ with $h^i \neq 0, h^p = 0, h^j \neq 0$, we thus obtain a decomposition of the complex variation V into the sum of two complex variations of strictly smaller dimensions than that of V. An adapted basis of V_o is the union of adapted bases of $(V_{<p})_o$ and $(V_{>p})_o$, and we conclude by recurrence.

It remains to treat the case where the i such that $h^i \neq 0$ form an interval. Let $V(n)$ be the complex variation with the same underlying local system as V, deduced from V by renumbering: $(V(n))_s^p := V_s^{p+n}$. It is polarized by $(-1)^n\psi$, an adapted basis of V_o is also an adapted basis of $V(n)_o$, and the coefficients of the matrix $\sigma(\gamma)$ are the same for V and $V(n)$. We do not restrict generality by only treating the V such that $\{p | h^p \neq 0\}$ is an interval starting at 0. There are then only a finite number of possibilities for the system of the h^p, and we conclude by 1.8. □

Corollary 1.4 (1.10)**.** *Fix (S,o) and an integer N. For each $\gamma \in \pi_1(S,o)$, there exists C such that, for any polarizable complex variation V on S, of dimension N, we have*

$$|\mathrm{Tr}(\sigma(\gamma))| < C.$$

1.7

In the end of this paragraph, we suppose that S is the complement, in a compact analytic manifold $\bar{S}$, of an analytic subspace. This hypothesis implies that on S every plurisubharmonic function bounded above is constant. Let V be a polarizable complex variation of Hodge structures on S. If U is an open set of S, and v a horizontal section of V on U, W. Schmid [10] shows that $\psi(v, v)$ is bounded on the trace on U of any compact subset of $\bar{S}$. A word of caution: the framework in which W. Schmid works is different from ours, but his proofs adapt without difficulty: he works with "real variations"; each complex variation defines a real variation of double dimension, and we apply [10] to the latter; he assumes local monodromies are quasi-unipotent, but does not really use this hypothesis. This being acquired, the arguments of [6], 7.1 (cf. [10], §7) show that if v is a global section of V on S, the components of v in the V^p are still horizontal.

1.8

Let V_o be a semi-simple complex local system on S. We suppose V_o admits a decomposition

$$(1.12.1) \quad V_o = \bigoplus_{i \in I} S_i \otimes W_i$$

where the S_i are irreducible local systems, pairwise non-isomorphic, and where the W_i are non-zero complex vector spaces. The hypothesis that V_o is semi-simple is automatically fulfilled if V_o underlies a polarizable variation of Hodge structures (by 1.11, the arguments of [2], 4.2.6 apply). M. Nori (unpublished) has shown that it is also true if V_o underlies a complex polarizable variation and if $\bar{S}$ is Kähler or algebraic (or more generally if each class $a \in \mathrm{H}^1(\bar{S}, \mathbb{R})$ is representable by a 1-form $\alpha + \bar{\alpha}$ with α holomorphic).

Proposition 1.5 (1.13)**.** *Under the hypotheses of 1.11 and 1.12, if V_o underlies a complex polarizable variation,*

(i) Each S_i underlies a complex polarizable variation, unique up to a renumbering $p \mapsto p + n$.

(ii) Let us choose on each S_i a complex polarizable variation. Via (1.12.1), complex Hodge structures on the W_i then provide a complex polarizable variation on V_o. Any complex polarizable variation on V_o is thus obtained.

Proof. The decomposition (1.12.1) provides an isomorphism

$$(1.13.1) \quad \mathrm{End}(V_o) = \prod \mathrm{End}(W_i).$$

The vector space $\mathrm{End}(V_o)$ is the space of global horizontal sections of the local system $\mathrm{End}(V_o)$. For V_o underlying a polarizable variation V, $\mathrm{End}(V_o)$ inherits the Hodge decomposition of the $\mathrm{End}(V_s)$ ($s \in S$) (1.11):

$$\mathrm{End}(V_o) = \bigoplus_p \mathrm{End}(V_o)^p.$$

Lemma 1.6 (1.14)**.** *Any graduation of $\prod \mathrm{End}(W_i)$, compatible with the algebra structure, comes from graduations of the W_i.*

Proof. Let's view a graduation as an action of the multiplicative group $\mathbb{G}_m$, $\lambda \in \mathbb{G}_m(\mathbb{C}) = \mathbb{C}^*$ acting by multiplication by λ^n on the component of degree n (SGA3 14.7.3). The neutral component of the group of automorphisms of the algebra $\prod \mathrm{End}(W_i)$ is the quotient of the group $\prod \mathrm{GL}(W_i)$ by its center $\prod \mathbb{G}_m$. Any central extension of $\mathbb{G}_m$ by $\prod \mathbb{G}_m$ being trivial (SGA3 IX 8.2), a morphism $\mathbb{G}_m \to \mathrm{Aut}(\prod \mathrm{End}(W_i))$ lifts to a morphism of $\mathbb{G}_m$ into $\prod \mathrm{GL}(W_i)$, i.e., to a graduation of the W_i. □

Proof of 1.13 (continued). For V_o underlying a polarizable variation V, let's choose (1.14) graduations of the W_i such that the isomorphism (1.13.1) is compatible with the graduations. If the line $L_i \subset W_i$ is homogeneous, it is the image of a projector $e \in \mathrm{End}(W_i)$ homogeneous of degree 0 and $S_i \otimes L_i \subset V_o$, isomorphic to S_i, is the image of a projector $e \in \mathrm{End}(V_o) = \mathrm{End}(V_o)^0$. It is therefore a sub-variation of V, a direct factor in fact, and we deduce the existence of a complex polarizable variation to which S_i is underlying.

Let us choose on each S_i a complex polarizable variation. For any polarizable variation V on V_o,

$$W_i = \mathrm{Hom}(S_i, V_o)$$

inherits from the polarizable variation $\mathrm{Hom}(S_i, V)$ a Hodge decomposition (1.11 applied to $\mathrm{Hom}(S_i, V)$). The isomorphism

$$\bigoplus S_i \otimes \mathrm{Hom}(S_i, V_o) \to V_o$$

respects the Hodge structures, and (ii) follows. The uniqueness assertion in (i) is (ii) for $V_o = S_i$. □

Corollary 1.7 (1.15)**.** *Let S be a Zariski open set of a compact $\bar{S}$, $o \in S$ and an integer N. For every $\gamma \in \pi_1(S, o)$ there exists C such that, for any polarizable variation of Hodge structures W on S and any direct factor V of the underlying rational local system, if $\dim(V) = N$, the monodromy $\sigma(\gamma)$ verifies $|\mathrm{Tr}(\sigma(\gamma))| < C$.*

Results from 1.10 and 1.13.

2 Proof of theorem 0.5

We will rely on the following classical theorem.

Theorem 2.1 (2.1)**.** *Let Γ be a finitely generated group and N an integer. There exists a finite part F of Γ such that if two linear representations of dimension N of Γ over a field k of characteristic 0, of characters χ_1 and χ_2, verify $\chi_1(\gamma) = \chi_2(\gamma)$ for $\gamma \in F$, then $\chi_1 = \chi_2$.*

Proof. A finitely generated group is finitely generated as a monoid. It is therefore sufficient to prove 2.1 in the more general case where Γ is a finitely generated monoid and where the representations take values in $N \times N$ matrices, not necessarily invertible. Let T be a finite part of Γ that generates Γ. We do not restrict generality by assuming that Γ is the free monoid generated by T. In this case, the data of a representation of Γ is equivalent to that of a family of $N \times N$ matrices indexed by T: to ρ, attach the family of $\rho(t)$ $(t \in T)$.

Let $X^t_{i,j}$ $(i, j \in [1, N], t \in T)$ be indeterminates and A be the algebra of polynomials in $N^2|T|$ variables $\mathbb{Q}[\{X^t_{i,j}\}]$. This is the algebra of polynomial functions on the algebraic variety (an affine space) which parametrizes the families indexed by T of $N \times N$ matrices. Let τ be the representation $\Gamma \to M_N(A)$ for which $\tau(t) = (X^t_{i,j})_{i,j\in[1,N]}$. The action by $X \mapsto gXg^{-1}$ of the linear group GL_N on the matrices $N \times N$ provides an action on the $\mathbb{Q}$-algebra A of the algebraic group GL_N (on $\mathbb{Q}$). Let A^{GL_N} be the algebra of invariants. For any vector space V of dimension N on k, the k-algebra $A^{\mathrm{GL}_N} \otimes_{\mathbb{Q}} k$ identifies with the algebra of $\mathrm{GL}(V)$-invariant polynomial functions of $|T|$ endomorphisms of V. The elements $\mathrm{Tr}(\tau(\gamma))$ $(\gamma \in \Gamma)$ of A are invariants.

Lemma 2.2 (2.2, C. Procesi [13]). *] The algebra of invariants of* GL_N *in* A *is generated by the* $\mathrm{Tr}(\tau(\gamma))$ *(*$\gamma \in \Gamma$*).*

The proof consists in reducing by polarization to the study of multilinear invariants of n endomorphisms, to interpret them as multilinear invariants of n vectors and n covectors and to use the description that H. Weyl gives of them.

Proof of 2.1 (end). By Hilbert, the algebra of invariants of GL_N in A is of finite type. There thus exists a finite part F of Γ such that every invariant is a polynomial in the $\mathrm{Tr}(\tau(f))$ for $f \in F$. In particular, for any $\gamma \in \Gamma$, there exists a polynomial with rational coefficients P_γ in indeterminates x_f $(f \in F)$, such that

$$\mathrm{Tr}\tau(\gamma) = P_\gamma((\mathrm{Tr}\tau(f))_{f\in F}).$$

For any field k of characteristic 0 and any representation of Γ in $M_N(k)$, of character χ, we have by specialization,

$$\chi(\gamma) = P_\gamma((\chi(f))_{f\in F}).$$

The theorem follows. $\square$

Remark 2.3 (2.3). Let $s = s(N)$ be the smallest integer t such that any associative $\mathbb{Q}$-algebra without unit verifying the identity $z^N = 0$ also verifies the identity $z_1 \cdots z_t = 0$. In [13], C. Procesi shows that the algebra A^{GL_N} of 2.2 is generated by the $\mathrm{Tr}(\tau(\gamma))$ for γ of length $\leq s$, and that this bound is optimal: if $|T| \geq s$ and $\pi \in \Gamma$ is a product of s distinct generators, $\mathrm{Tr}(\tau(\pi))$ is not in the algebra generated by the $\mathrm{Tr}(\tau(\gamma))$ for γ of length $< s$. In 2.1, one can therefore take for F the set of words of length $\leq s$ in the elements of a symmetric generating system T. The "symmetric" restriction is in fact useless since a character on a group Γ is determined by its restriction to a sub-monoid that generates Γ.

Highman showed that $s \leq 2^N - 1$. For a very short proof, see N. Jacobson, Structure of rings (2nd ed.), p. 274. This bound was improved by Yu. P. Razmislov to $s \leq N^2$ (Izvestia A.N. 38 4 (1974), p. 756).

2.1 Proof of 0.5

Fix $o \in S$. If V is a polarizable variation of Hodge structures on S, and W is a direct factor of dimension N of the corresponding monodromy representation σ, the rational representation σ_W of $\pi_1(S, o)$ on W verifies (A)(B)(C) below.

(A) For all $\gamma \in \pi_1(S, o)$, $\mathrm{Tr}(\sigma_W(\gamma)) \in \mathbb{Z}$. *Proof.* The representation σ respects by hypothesis an integer lattice. The sub-representation σ_W also.

(B) For all $\gamma \in \pi_1(S, o)$ there exists $C(\gamma, N)$ such that $|\mathrm{Tr}(\sigma_W(\gamma))| < C(\gamma, N)$. This is an application of 1.15.

(C) The representation σ_W is semi-simple. Indeed, σ is (cf. [2] 4.2.6, whose method applies via 1.11).

For a fixed N, by (A) and (B), there is only a finite number of possibilities for the value of each $\mathrm{Tr}(\sigma_W(\gamma))$. According to 2.1, there is only a finite number of possibilities for the character of σ_W, hence, according to (C), for its isomorphism class.

Remark 2.4 (2.5)*.* In 0.5, we suppose that S is an algebraic variety. The proof still applies if S is a Zariski open set of a compact analytic variety. If we only suppose that S is an analytic variety whose fundamental group is finitely generated, the same arguments still give that the polarizable variations of Hodge structures of dimension N on S only give rise to a finite number of characters of $\pi_1(S, o)$.

2.2 Proof of Lemma 0.3

The following statement must be proven.

Lemma 2.5. *Let $H = \mathbb{Z}^N$, Γ a group, σ a representation $\Gamma \to \mathrm{Aut}(H)$ and suppose that the action of Γ on $H_{\mathbb{Q}} := H \otimes \mathbb{Q}$ is completely reducible. Let G be the group of automorphisms of $H_{\mathbb{Q}}$ which commute with the action of Γ. Then, the lattices $H' \subset H_{\mathbb{Q}}$ stable by Γ form only a finite number of G-orbits.*

Proof. Let $A \subset \mathrm{End}(H)$ be the sub-algebra generated by the $\sigma(\gamma)$ $(\gamma \in \Gamma)$, and $A_{\mathbb{Q}} = A \otimes \mathbb{Q}$. The complete reducibility of $H_{\mathbb{Q}}$ is equivalent to $A_{\mathbb{Q}}$ being a semi-simple algebra. Let $A_1 \subset A_{\mathbb{Q}}$ be a maximal order containing A. For any prime number l, a local study shows that $G(\mathbb{Q}_l)$ acts transitively on the $\mathbb{Z}_l$-lattices of $H_{\mathbb{Q}_l}$ stable by $A_1 \otimes \mathbb{Z}_l$.

Let A_f be the ring of finite adeles, restricted product of the $\mathbb{Q}_l$. We know that for any open subgroup K of $G(A_f)$, the set of double classes $K \backslash G(A_f)/G(\mathbb{Q})$ is finite (A. Borel, Some finiteness properties of adèle groups over number fields. Publ. Math. IHES 16 (1963), p. 5-30, theorem 5.1). This finiteness implies that $G(\mathbb{Q})$ has only a finite number of orbits in the set of lattices $H' \subset H_{\mathbb{Q}}$ stable under A_1. Any A-stable lattice H' is contained in an A_1-stable lattice H'' with an index $[H'' : H']$ which divides $[A_1 : A]^N$: take $H'' = A_1 H'$, and we conclude by observing that a lattice has only a finite number of sub-lattices of a given index. □

3 Relation with G. Faltings [4]

3.1

The present article was inspired by reading G. Faltings [4].

One can view [4] as being another proof of theorem 0.1 in the particular case where S is a curve and where one only considers abelian schemes X over S, of a fixed relative dimension $n = N/2$, and the monodromy representation on the H^1 of the fibers. The restriction to the case of curves is unimportant, because for any smooth algebraic variety S there exists a smooth curve C traced on S such that, for $o \in C$, $\pi_1(C, o)$ maps onto $\pi_1(S, o)$: for U a dense open set of S, embeddable in a projective space P, it is sufficient to take for C the intersection of U with a sufficiently general linear subspace of dimension $\dim P - \dim S + 1$; we will have $\pi_1(C) \twoheadrightarrow \pi_1(U)$ (Bertini) and $\pi_1(U) \twoheadrightarrow \pi_1(S)$.

3.2

Let $\bar{S}$ be a smooth projective curve, T a finite set of points of $\bar{S}$ and $S := \bar{S} - T$. Let X be an abelian scheme on S which we suppose extends to a semi-abelian scheme $\bar{X}$ over $\bar{S}$ (semi-stable reduction). Let n be the relative dimension of X over S, e be the zero section and let us set $\omega := e^*\Omega^n_{\bar{X}/\bar{S}}$. This is the invertible sheaf on $\bar{S}$ whose fiber at $s \in \bar{S}$ is the maximal exterior power of the dual of the Lie algebra of the fiber $\bar{X}_s$ of $\bar{X}/\bar{S}$. In [4], G. Faltings begins by bounding the degree of ω, independently of X. If we simplify his argument by a reference to S. Zucker [11], we obtain the following estimate.

Lemma 3.1 (3.2). *With the preceding notations, if $-\chi(S) \geq 0$, we have*

$$\deg \omega \leq \frac{1}{2} n(-\chi(S)).$$

In this formula, $\chi(S)$ is the topological Euler-Poincaré characteristic: for $\bar{S}$ of genus g, $-\chi(S) = 2g - 2 + |T|$.

Proof. Let H be the variation of Hodge structures on S of fibers the $\mathrm{H}^1(X_s)$ ($s \in S$). It gives rise to a complex local system $H_{\mathbb{C}}$, and to a complex vector bundle $\mathcal{H}$ equipped with a connection ∇, of which $H_{\mathbb{C}}$ is the local system of horizontal sections. On $\mathcal{H}$, we have the Hodge filtration F, reduced here to a sub-bundle $F^1(\mathcal{H})$ of $\mathcal{H}$. The fiber $\mathrm{H}^0(X_s, \Omega^1_X)$ of $F^1(\mathcal{H})$ at s identifies with the dual of the Lie algebra of X_s. The hypothesis that X has semi-stable reduction is equivalent to the unipotence of the local monodromy of $H_{\mathbb{C}}$ at each $t \in T$. Let $\mathcal{H}_{\mathrm{can}}$ be the canonical extension ([7], 5.2) of the vector bundle $\mathcal{H}$ to $\bar{S}$ and let $F^1(\mathcal{H}_{\mathrm{can}})$ be the locally direct factor sub-bundle of $\mathcal{H}_{\mathrm{can}}$ which extends $F^1\mathcal{H}$. Any polarization of X (hence of H) induces a perfect duality between $\mathcal{H}/F^1\mathcal{H}$ and $F^1\mathcal{H}$, and this duality extends to a perfect duality between $\mathcal{H}_{\mathrm{can}}/F^1\mathcal{H}_{\mathrm{can}}$ and $F^1\mathcal{H}_{\mathrm{can}}$. We will admit from the theory of Néron models that

$$e^*\Omega^1_{\bar{X}/\bar{S}} = F^1\mathcal{H}_{\mathrm{can}}.$$

We will first treat the case where X/S has no fixed part. This is equivalent to $\mathrm{H}^0(S, H^i_{\mathbb{C}}) = 0$ and implies that $\mathrm{H}^i(S, H^i_{\mathbb{C}}) = 0$ for $i \neq 1$, whence.

$$\text{(3.2.1)} \quad \dim \mathrm{H}^1(S, H_{\mathbb{C}}) = -\chi(S, H_{\mathbb{C}}) = -\mathrm{rank}(H_{\mathbb{C}}) \cdot \chi(S) = -2n\chi(S).$$

The cohomology $\mathrm{H}^*(S, H_{\mathbb{C}})$ can be calculated as the hypercohomology, on $\bar{S}$, of the De Rham complex

$$\mathcal{H}_{\mathrm{can}} \xrightarrow{\nabla} \Omega^1_{\bar{S}}(T) \otimes \mathcal{H}_{\mathrm{can}}.$$

This complex, denoted K, is filtered by the sub-complexes

$$F^1K := (F^1\mathcal{H}_{\mathrm{can}} \to \Omega^1_{\bar{S}}(T) \otimes F^{1-1}\mathcal{H}_{\mathrm{can}}).$$

and, according to S. Zucker [11], §13, the corresponding spectral sequence degenerates at E_1. In particular,

$$(3.2.2) \quad \dim E_1^{0,1} + \dim E_1^{1,0} \leq \dim \mathrm{H}^1(S, H).$$

We have $E_1^{1,0} = \mathrm{H}^0(\bar{S}, \Omega^1_{\bar{S}}(T) \otimes F^0\mathcal{H}_{\mathrm{can}})$ and $E_1^{0,1} = \mathrm{H}^1(\mathcal{H}_{\mathrm{can}}/F^1\mathcal{H}_{\mathrm{can}})$, dual by Serre to $\mathrm{H}^0(\bar{S}, \Omega^1_{\bar{S}} \otimes (F^1\mathcal{H}_{\mathrm{can}})^*)$. If $F^1\mathcal{H}_{\mathrm{can}}$ is of degree d, i.e. if $d := \deg(\omega)$, we have then

$$\begin{aligned}\dim E_1^{0,1} + \dim E_1^{1,0} &= \chi_{\bar{S}}(\mathcal{H}_{\mathrm{can}}/F^1\mathcal{H}_{\mathrm{can}}) + \chi_{\bar{S}}(\Omega^1_{\bar{S}}(T) \otimes F^1\mathcal{H}_{\mathrm{can}}) \\ &= [n(g-1) - d] + [n(g-1) + |T|) - d] \\ &= -n\chi(S) - 2d.\end{aligned}$$

Combined with (3.2.1) and (3.2.2), this gives

$$-n\chi(S) - 2d \leq -2n\chi(S),$$

i.e. the assertion of 3.2.

An abelian scheme X over S is isogenous to the sum of a constant abelian scheme $S \times B$ and an abelian scheme Y without fixed part. If X has semi-stable reduction, Y does too. We have ω for $X = (\omega$ for $S \times B) \otimes (\omega$ for $Y)$, and the first factor is trivial. This reduces 3.2 for X to 3.2 for Y, and finishes the proof of 3.2. □

Remark 3.2 (3.3). If $-\chi(S) \leq 0$, the fundamental group of S is abelian and any abelian scheme over S becomes constant over a finite étale cover of S. If it has semi-stable reduction, it has good reduction, becomes constant over a finite étale cover of S and we have $\deg(\omega) = 0$.

Remark 3.3 (3.4). In the case where X is a family of elliptic curves ($n = 1$), we have a "Kodaira-Spencer" morphism

$$(3.3.1) \quad \omega^{\otimes 2} \to \Omega^1_S(T).$$

If it is trivial, the modular invariant j is constant and $\deg(\omega) = 0$ (X is always assumed to have semi-stable reduction, hence, in this case, good reduction). If it is non-trivial, its existence disproves 3.2.

This implies that $2\deg(\omega) \leq \deg(\Omega^1_S(T))$ and we have $-\chi(S) = \deg(\Omega^1_S(T))$. For X a modular family, (3.3.1) is an isomorphism and we have equality in (3.2).

Remark 3.4 (3.5). Suppose that $\chi(S) < 0$, i.e., that the universal cover of S is the unit disk D. This is the interesting case (cf. 3.3). A polarization of the abelian scheme X/S defines a polarization of the variation of Hodge structures H (H^1 of the fibers), and in particular a Hermitian metric on the vector bundle $F^1\mathcal{H} = e^*\Omega^1_{X/S}$. The variation H gives rise to an application of periods $P : D \to M$ as in 1.7. The theorem of Griffiths 1.7 that P decreases distances entails an upper bound for the curvature tensor of the metrized bundle $e^*\Omega^1_{X/S}$, and in particular of its 2-form of Chern. A local study at infinity shows that the integral of this 2-form is the degree of ω on $\bar{S}$ (C. Peters [12]). Integrating the upper bound above, we recover 3.2.

Remark 3.5 (3.6). Let H be a complex variation of Hodge structures polarizable on a curve $S = \bar{S} - T$, (H, ∇) the corresponding vector bundle with connection, $\mathcal{H}_{\mathrm{can}}$ the canonical extension of H, and F the Hodge filtration of $\mathcal{H}_{\mathrm{can}}$: $F^i\mathcal{H}_{\mathrm{can}}$ is the locally direct factor sub-bundle of $\mathcal{H}_{\mathrm{can}}$ which extends the sub-bundle $F^i\mathcal{H}$ of H. Griffiths' theorem 1.7 and a local study at infinity ([12]) still allow to bound the degrees of $\mathrm{Gr}^i_F\mathcal{H}_{\mathrm{can}}$ in terms of the genus of S, of $|T|$ and of the Hodge numbers of the variation H.

3.3

Let n be an integer ≥ 3 and $M_{d,n}$ the moduli space of principally polarized abelian varieties of dimension d equipped with a level n structure. This is a finite moduli space: the data of a principally polarized abelian scheme $X \to S$, of relative dimension d, equipped with a level n structure, is equivalent to that of a morphism $f : S \to M_{d,n}$. For $S = \bar{S} - T$ as in 3.1, this morphism extends to a morphism $\bar{f}$ of $\bar{S}$ into the Satake compactification $\bar{M}_{d,n}$ of $M_{d,n}$. Faltings deduces from 3.2 that the graph $\Gamma(\bar{f}) \subset \bar{S} \times \bar{M}_{d,n}$ of $\bar{f}$ has bounded degree independently of X/S, for a suitable projective embedding of $\bar{S} \times \bar{M}_{d,n}$. From this it follows that the abelian schemes of the type considered are parametrized by a finite type scheme Y: there exists an abelian scheme $\mathcal{X}_Y$ over $S \times Y$ such that each abelian scheme of the said type over S is isomorphic to the restriction $\mathcal{X}_y$ of $\mathcal{X}_Y$ to $S \times \{y\} \simeq S$. The monodromy being a discrete invariant, the isomorphism class of the monodromy representation of $\pi_1(S)$ defined by $\mathcal{X}_y$ depends only on the connected component of Y where y is found. This reproves theorem 0.1, restricted to the particular case of abelian schemes of relative dimension d, principally polarized and equipped with a level n structure.

3.4

Let Z be a connected scheme of finite type and $u : S_Z \to Z$ a family of curves parametrized by Z. We suppose that $\bar{S}_Z \to Z$ with $\bar{S}_Z$ proper and smooth over Z, with fibers curves of genus g, and with T_Z finite étale over Z, with fibers sets of t points. For $z \in Z$, let us set $S_z := u^{-1}(z)$. For $x \in S_Z$, the fundamental groups $\pi_1(S_{u(x)}, x)$ of the fibers of u form a local system on S_Z. A path from x to y thus defines an isomorphism of $\pi_1(S_{u(x)}, x)$ to $\pi_1(S_{u(y)}, y)$. If, for $i = 1, 2$, S_i is the complement in a curve $\bar{S}_i$ of genus g of a set T_i of t points, and that $x_i \in S_i$, an isomorphism of $\pi_1(S_1, x_1)$ with $\pi_1(S_2, x_2)$ will be called **permissible** if there exists a family as above, of which the S_i are fibers and for which the isomorphism is deduced from a path from x_1 to x_2. The moduli space of curves of genus g being connected, there always exists a permissible isomorphism of $\pi_1(S_1, x_1)$ with $\pi_1(S_2, x_2)$.

3.5

The arguments of 3.6 still apply with parameters. For a family $u : S_Z \to Z$ as in 3.7 and $d \geq 0, n \geq 3$, there exists a scheme of finite type Y over Z and an abelian scheme X over $S_Y := S_Z \times_Z Y$ having the following property: for every $z \in Z$ and every principally polarized abelian scheme of dimension d, A, over S_z, equipped with a level n structure, there exists $y \in Y$ over z such that A is the inverse image of X by $S_z \to S_y : s \mapsto (s, y)$. Taking a universal family S_Z, we deduce the following result.

Proposition 3.6 (3.10)**.** *Let Γ be the fundamental group of a curve of genus g with t punctures. Fix $n \geq 3$. For any curve of genus g with t punctures S, for any abelian scheme A over S, principally polarized, of dimension d, equipped with a level n structure, and for any permissible isomorphism γ of Γ with $\pi_1(S, o)$, the monodromy representation σ of $\pi_1(S, o)$ on $\mathrm{H}^1(A_o, \mathbb{Z})$ provides a representation σ_γ of Γ. The isomorphism classes of representations of Γ thus obtained form only a finite number of orbits under the permissible automorphisms of Γ.*

Remark 3.7 (3.11). The analogue of 3.9, without imposed level structure, is still true. Any abelian scheme of dimension d over a curve S acquires a level 3 structure on a covering of S of degree $< 3^{4d^2}$, and one describes an abelian scheme on S by the data of a covering S' of S, of degree $< 3^{4d^2}$, an abelian scheme with level 3 structure on S', and a descent data from S' to S for this abelian scheme.

Remark 3.8 (3.12). We deduce that the analogue of 3.10, without imposed level structure, is still true. One can also bypass polarizations by using Zarhin's theorem that for any polarizable abelian scheme A, there exists a principal polarization on $(A \times A^{\mathrm{dual}})^4$ (see for example Astérisque 127 (seminar on arithmetic pencils: the Mordell conjecture, directed by L. Szpiro) VII 1.).

Question 3.9 (3.13). If we replace "abelian scheme" by "variation of Hodge structures", does statement 3.10 remain valid?

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