K-Theory of Adic Spaces

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1 Introduction

Since its invention, K-theory has occupied a special place in mathematics. Informally speaking, it is a collection of topological and algebraic methods for analyzing a suitably defined "category of vector bundles" on a space X. This analysis should provide a sequence of invariants that behaves like a generalized cohomology theory and is fundamentally related to other algebraic invariants. For example, in algebraic topology, K-theory can be related to singular cohomology via the Atiyah-Hirzebruch spectral sequence. In algebraic geometry, the famous Quillen-Lichtenbaum conjecture, proven by Voevodsky and Rost, connects K-theory with étale cohomology via a similar spectral sequence. The goal of this work is to lay the foundations of K-theory in the context of adic geometry and, in particular, to prove an analogue of the Grothendieck-Riemann-Roch theorem. The most important conceptual innovation of this work is undoubtedly the definition of K-theory in the adic context. Once the correct framework is found, the other parts of the work are purely formal. Before presenting the plan of this dissertation, we would therefore like to briefly discuss the development of the definition of K-theory here and explain the motivation for our approach.

The first complete cohomological formalism of this kind was developed in algebraic topology and is now called *topological K-theory*. For a paracompact topological Hausdorff space X, we consider the groupoid[1] $\mathrm{Vect}_X^{\mathbb{R}/\mathbb{C}}$ of finite-dimensional locally free real or complex vector bundles on X, which we will simply call *vector bundles* in the following. The starting point is the following definition by Grothendieck:

Definition 1.1. The group $K_0(X)$ is defined as the group completion of the monoid of isomorphism classes of vector bundles on X, whose monoid structure is given by the direct sum.

From this definition, an integer-indexed sequence of algebraic invariants of X, called *the higher K-groups*, is then constructed using the *suspension isomorphism* and *Bott periodicity*. However, such methods are not available in algebraic or analytic geometry. Therefore, if one wants to define K-theory in these contexts, one must find a different, more abstract definition. The usual modern approach to algebraic K-theory makes essential use of the theory of ∞-categories: Instead of the group completion of the set of isomorphism classes of vector bundles, one considers the *∞-group completion* of the groupoid of vector bundles, viewed as a category enriched over Top. This yields a spectrum in the sense of algebraic topology and allows the following definition of topological K-theory, which agrees with the classical one for finite-dimensional cell complexes:

Definition 1.2. The group $K_i(X)$ is defined as the i-th homotopy group $\pi_i((\mathrm{Vect}_X^{\mathbb{R}/\mathbb{C}})^{\mathrm{grp}})$ of the ∞-group completion of the groupoid of vector bundles on X, whose monoid structure is given by the direct sum.

One of the main advantages of this definition is that it can be directly transferred to the context of commutative algebra. Specifically, if one replaces the category of vector bundles with the category of finite projective modules over a commutative ring R, one obtains the following definition:

Definition 1.3. Let R be a commutative ring. The group $K_i(R)$ is defined as the i-th homotopy group $\pi_i((\mathrm{Proj}_R)^{\mathrm{grp}})$ of the ∞-group completion of the groupoid of finite projective modules over R, whose monoid structure is given by the direct sum.

However, if one wants to go further and define K-theory for all schemes, one must proceed somewhat differently due to some technical problems. The first general construction goes back to Quillen, who introduced the concept of an *exact category* for this purpose. Without going into detail, let it just be mentioned here that his approach is very similar to the one above: The K-groups are given as the homotopy groups of the spectrum obtained by applying the so-called Quillen's Q-construction to the exact category of vector bundles. However, it turns out that this definition is "wrong". The problem is that the K-theory defined in this way generally has poor local-global properties; in other words, it does not satisfy descent with respect to the Zariski topology. In his famous work [TT90], Robert Thomason proposed another definition, now generally accepted, which agrees with Quillen's version in good situations. In particular, Thomason's "correct" K-theory agrees with Quillen's in the affine case. In modern language, the definition is as follows:

Definition 1.4. Let X be a scheme and $\mathrm{Cat}_\infty^{\mathrm{perf}}$ the ∞-category of small idempotent complete stable ∞-categories, whose morphisms are exact functors. We denote by $\mathrm{Perf}(X)$ the derived ∞-category of

[1] i.e., a category whose morphisms are all invertible

perfect complexes on X and by $K : \mathrm{Cat}_\infty^{\mathrm{perf}} \to \mathrm{Sp}$ the K-theory functor, which maps a small stable ∞-category to a spectrum. Then the K-theory spectrum $K(X)$ is defined as $K(\mathrm{Perf}(X))$.

We would like to make some remarks about this definition. Let $\mathcal{D}_{\mathrm{qc}}(X)$ denote the derived ∞-category of quasi-coherent sheaves on X. One can think of the derived ∞-category of perfect complexes on X as the full ∞-subcategory of $\mathcal{D}_{\mathrm{qc}}(X)$ that is "locally" generated by vector bundles under shifts and finite colimits. This description has the advantage that the transition from Quillen's definition to Thomason's, at least from today's perspective, does not seem so surprising: one follows Grothendieck and passes to the "derived level". However, there is another description that is much more important for many conceptual reasons. Namely, the derived ∞-category $\mathcal{D}_{\mathrm{qc}}(X)$ of quasi-coherent sheaves on a quasi-compact quasi-separated scheme X is *compactly generated*, and the ∞-subcategory $\mathrm{Perf}(X)$ is precisely the *∞-subcategory of compact objects* of $\mathcal{D}_{\mathrm{qc}}(X)$. Using this characterization, Thomason succeeded in reducing many geometric questions about the K-theory of schemes to formal properties of compactly generated categories and the K-theory functor. Since the latter is a universal "algebraic" invariant of small stable ∞-categories, this reformulation creates a formal framework in which many otherwise very complicated arguments become clear.

As already mentioned, the main advantage of Thomason's definition is that his version of K-theory satisfies descent with respect to the Zariski topology. In fact, it also satisfies descent with respect to the so-called *Nisnevich topology*, which is finer than the Zariski topology but coarser than the étale topology. More precisely, Thomason proved the following theorem for a specific variant of $K : \mathrm{Cat}_\infty^{\mathrm{perf}} \to \mathrm{Sp}$:

Theorem 1.5 ([TT90, Theorem 8.4])**.** *Let X be a quasi-compact quasi-separated scheme and U any quasi-compact étale scheme over X. Let $\mathbb{K} : \mathrm{Cat}_\infty^{\mathrm{perf}} \to \mathrm{Sp}$ denote the non-connective K-theory functor. Then the assignment $U \mapsto \mathbb{K}(\mathrm{Perf}(U))$ defines a sheaf of spectra with respect to the Nisnevich topology.*

Before the publication of [TT90], there were several ad hoc attempts to prove Zariski descent for K-theory under certain assumptions on X. The most important point in Thomason's work is that he was able to prove the optimal descent theorem. In the present work, we investigate his ideas in the context of adic geometry. There, under various assumptions on the adic space considered, there are also non-trivial definitions of K-theory and associated descent results. However, our abstract approach allows us, as in the case of schemes, to prove the optimal descent theorem in this situation. Obviously, however, we are not the first to have thought about applying Thomason's methods in adic geometry. We now briefly describe two problems in the adic case that have hindered progress for years and made a naive transfer of the theory outlined above to the adic case impossible. Their solution is based on some theoretical foundations that have only recently become available.

The first problem can be traced back to the fact that before the invention of condensed mathematics by Clausen and Scholze, there was no sufficiently flexible algebraic formalism for sheaves of modules on analytic spaces (both non-archimedean and complex) without finiteness conditions, i.e., there was no theory of quasi-coherent sheaves in these contexts. One might hope that this degree of generality is actually superfluous for the purposes of K-theory, because at first glance, all definitions in the algebraic case only use coherent sheaves. Furthermore, under certain Noetherian assumptions, which are satisfied by rigid analytic spaces, there is a good theory of coherent sheaves. Unfortunately, this is not sufficient for our purposes for several reasons. The most important of these is that the K-theory that would use the derived category of perfect complexes as formal input would not satisfy descent: In rigid geometry, there is no analogue of Thomason's open-closed sequence for perfect complexes, which plays the key role in his proof. Another reason is that we are looking for such a formalism that also works for a certain class of non-Noetherian adic spaces, namely the so-called *perfectoid* spaces. Therefore, before we can turn to K-theory, we must clarify this fundamental question. In [And21], we introduced a definition of the derived ∞-category of quasi-coherent sheaves for general analytic adic spaces. As already mentioned, our approach is based on the ideas of Clausen-Scholze and uses condensed mathematics in an essential way. Without going into too much detail, we would nevertheless like to state precisely some of the most important results of that work.

Theorem 1.6 ([And21], Theorem 3.28, Proposition 3.34, Theorem 5.9, Lemma 5.10)**.** *Let* CAff *denote the category of complete Huber pairs and* AnRing *the category of complete commutative analytic animated condensed rings in the sense of Clausen-Scholze. Then there exists a fully faithful functor*

$$\mathrm{CAff} \to \mathrm{AnRing}, \ (A, A^+) \mapsto (A, A^+)_\blacksquare.$$

In particular, for every affinoid adic space $X = \mathrm{Spa}(A, A^+)$, one obtains a stable ∞-category $\mathcal{D}_\blacksquare(X)$, which is defined as the derived ∞-category $\mathcal{D}((A, A^+)_\blacksquare)$ of solid modules over $(A, A^+)_\blacksquare$. Furthermore, *there exists a fully faithful embedding of the usual algebraic derived ∞-category $\mathcal{D}(A)$ into $\mathcal{D}_{\blacksquare}(X)$, which is compatible with base change.*

Theorem 1.7 ([And21], Theorem 4.1)**.** *Let X be an analytic adic space and $U = \mathrm{Spa}(A, A^+)$ any affinoid open subset of X. Then the functor that maps U to the derived ∞-category $\mathcal{D}((A, A^+)_{\blacksquare})$ defines a sheaf of ∞-categories on X with respect to the analytic topology. In particular, one obtains by gluing a stable ∞-category $\mathcal{D}_{\blacksquare}(X)$, which we call the derived ∞-category of solid sheaves on X.*

It should be mentioned that the analogue of this statement, where $\mathcal{D}((A, A^+)_{\blacksquare})$ is replaced by $\mathcal{D}(A)$, is false. Therefore, it is important to work with condensed modules instead of the usual "discrete" modules. However, a key role in the present work is played not by the ∞-category $\mathcal{D}((A, A^+)_{\blacksquare})$, but by its full ∞-subcategory of so-called *nuclear modules*, which also satisfies descent and can be regarded as an analogue of the category of quasi-coherent sheaves for our purposes. The reason for this is that although $\mathcal{D}((A, A^+)_{\blacksquare})$ is compactly generated, its full subcategory of compact objects is too large, which would lead to the K-theory of X defined with its help being trivial. The category of nuclear sheaves, on the other hand, has "the right size", which is particularly evident in the discrete case: For a discrete Huber pair (A, A^+), it is equivalent to the derived category $\mathcal{D}(A)$. Furthermore, an "open-closed sequence" can be constructed for nuclear sheaves, which allows the application of Thomason's methods. However, the transition to this category leads to the second problem: In general, the ∞-category of nuclear sheaves is not compactly generated, but only dualizable. However, there is a purely categorical solution to this problem according to Efimov, which removes the last obstacle to transferring Thomason's approach to the adic case:

Theorem 1.8 ([Hoy18], Theorem 10)**.** *Let $\mathrm{Cat}_\infty^{\mathrm{st,cg}}$ (resp. $\mathrm{Cat}_\infty^{\mathrm{st,dual}}$) denote the ∞-category of stable compactly generated ∞-categories (resp. the ∞-category of stable dualizable ∞-categories), whose morphisms are compact (resp. dualizable) functors. Consider the functor*

$$\mathrm{Cat}_\infty^{\mathrm{st,cg}} \xrightarrow{\mathcal{C} \mapsto \mathcal{C}^\omega} \mathrm{Cat}_\infty^{\mathrm{perf}} \xrightarrow{\mathbb{K}} \mathrm{Sp},$$

where $\mathcal{C}^\omega$ denotes the full ∞-subcategory of compact objects in $\mathcal{C}$. Then it possesses a unique extension to a functor $\mathbb{K}_{\mathrm{cont}} : \mathrm{Cat}_\infty^{\mathrm{st,dual}} \to \mathrm{Sp}$ that maps Verdier sequences to fiber sequences.

The present dissertation is structured as follows. In Section 2, we establish the necessary formal categorical framework for our work. We recall the concepts of compactly generated and dualizable categories and investigate their relationship. Our most important result is Theorem **??**, which deals with the descent properties of dualizable ∞-categories. In Section 3, we recall the definition of a nuclear module over an analytic condensed ring according to Clausen and Scholze. Then, for a Huber pair (A, A^+), we investigate the ∞-category of nuclear modules over $(A, A^+)_{\blacksquare}$, where $(A, A^+)_{\blacksquare}$ denotes the analytic ring constructed via the functor $\mathrm{CAff} \to \mathrm{AnRing}$, $(A, A^+) \mapsto (A, A^+)_{\blacksquare}$. Our main result is the fact that this ∞-category is dualizable for a large class of Huber pairs. In particular, it is dualizable for all sheafy Tate Huber pairs. In Section 4, we generalize this statement using the results from Section 2 to the case of general analytic adic spaces. Specifically, we introduce the ∞-category of nuclear sheaves on an analytic adic space and prove that it is also dualizable for quasi-compact quasi-separated spaces. The next logical step is found in the Appendix. In its first part, we define the Nisnevich topology for analytic adic spaces and then show that it possesses the expected properties. In particular, we prove that it can be given by a *cd*-structure in the sense of Voevodsky. Then, in the next section of the Appendix, we investigate the relationship between sheaves and hypersheaves with respect to the analytic, Nisnevich, and étale topologies of an analytic adic space. In doing so, we follow the approach of [CM21] in the schematic case and obtain similar results. In other words, the goal of the Appendix is to show that the properties of the Nisnevich and étale topologies for schemes and analytic adic spaces are analogous. In Section 5, we use our previous results to analyze the descent properties of localizing invariants on analytic adic spaces. In doing so, we follow the ideas of Thomason in their modern form as in [Cla+20], [BCM20], and [CM21]. In particular, we prove the analogue of Theorem **??** and investigate the étale descent properties of localizing invariants after chromatic localization. In the last section of this work, we formulate and prove an analogue of the Grothendieck-Riemann-Roch theorem, the importance of which requires no further explanation.

Reading Suggestions:

Are you a:	then read the work in this order:
Dedicated reader	§2, §3, §4, Appendix A and B, §5 and §6;
Daring cheat	§2, §3, §4, §5 and §6 keeping the analogy between schemes and adic spaces in mind;
Adrenaline junkie	§4, §5 and §6.

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2 Dualizable Categories and their Gluing

One of the most important mathematical achievements of Robert Thomason is undoubtedly that he found the "right"perspective on the K-theory of schemes. Before the publication of his groundbreaking work [TT90], the K-theory of a scheme X was defined by applying the so-called Quillen Q-construction to the category of vector bundles on X. A major reason why this approach remained conceptually unsatisfactory was the fact that the K-theory defined in this way generally did not satisfy Zariski descent. Thomason's main insight, which he attributed to the Grothendieck school, was the idea of considering so-called *perfect complexes* instead of vector bundles. By definition, these are those objects of the derived category of quasi-coherent sheaves that can be written "locallyäs finite colimits of shifts of vector bundles. One could say this is another manifestation of the idea, now ubiquitous in algebraic geometry, that one should consistently work at the "derived level"when investigating many questions. Indeed, the perfect complexes on a scheme X are precisely given by the compact objects in the derived category of quasi-coherent sheaves on X, which provides an abstract characterization of them. Accordingly, the concept of a *compactly generated category* plays a special role in Thomason's approach to K-theory, and the main part of the work [TT90] consists in investigating the descent properties of such categories. We recall the definition:

Definition 2.1 ([Lur09, Definition 5.5.7.1])**.** Let $\mathcal{C}$ be an accessible ∞-category[2], closed under small filtered colimits.

(i) An object $c \in \mathcal{C}$ is called *compact* if the functor $\hom_{\mathcal{C}}(c, -)$ commutes with small filtered colimits. We write $\mathcal{C}^{\omega}$ for the full ∞-subcategory of compact objects in $\mathcal{C}$.

(ii) The ∞-category $\mathcal{C}$ is called *compactly generated* if $\mathrm{Ind}(\mathcal{C}^{\omega}) \xrightarrow{\sim} \mathcal{C}$ holds. This is equivalent to the existence of a small ∞-category $\mathcal{D}_0$ such that $\mathcal{C} \cong \mathrm{Ind}\,\mathcal{D}_0$.

Let X be a quasi-compact quasi-separated scheme. Let $\mathcal{D}_{qc}(X)$ denote the derived ∞-category of quasi-coherent sheaves on X. Its full ∞-subcategory of compact objects is called the *derived ∞-category of perfect complexes on* X and is denoted by $\mathrm{Perf}(X)$. According to Thomason, the K-theory of X is defined by applying the K-theory functor $K : \mathrm{Cat}_{\infty}^{\mathrm{st}} \to \mathrm{Sp}$ to $\mathrm{Perf}(X)$. One of the main advantages of this definition is that the Zariski and Nisnevich descent of the K-theory of X are now a formal consequence of the fact that the ∞-category $\mathcal{D}_{qc}(X)$ is compactly generated and the compact objects satisfy the respective descent. To see that $\mathcal{D}_{qc}(X)$ is indeed compactly generated, we cite the following theorem:

Theorem 2.2 ([BB03, Proof of Theorem 3.1.1])**.** *Let X be a quasi-compact quasi-separated topological space with a basis $\mathcal{B}$ consisting of quasi-compact subsets. Let $U \mapsto \mathcal{C}_U$ be a sheaf of stable ∞-categories on X whose value on each subset $B \in \mathcal{B}$ is compactly generated. Assume that for all $V \in \mathcal{B}$ and all quasi-compact open subsets $V' \subset V$, the restriction functor $\mathcal{C}_V \to \mathcal{C}_{V'}$ is a left Bousfield localization that preserves compact objects and whose fiber is compactly generated. Then the category $\mathcal{C}_X$ is compactly generated.*

From this theorem, the desired property of $\mathcal{D}_{qc}(X)$ can be easily derived, because for every affine scheme $X = \mathrm{Spec}\, R$, $\mathcal{D}_{qc}(X)$ is equivalent to the derived ∞-category $\mathcal{D}(R)$ and thus compactly generated.[3] If we want to apply Thomason's ideas to the definition of K-theory in rigid geometry, we thus need a "category of quasi-coherent sheaves"for adic spaces. In Section 4, we introduce such a category, which we call the category of *nuclear sheaves*, and investigate its properties. The main problem we address in the present section is the fact that the category of nuclear sheaves, even in the affinoid case, is not compactly generated, but only *dualizable*. Therefore, we must first investigate the descent properties of dualizable categories. Since these are a very natural generalization of compactly generated categories, the situations are very similar. At the end of this section, we prove an analogue of Theorem **??** for dualizable categories, which we will then apply in Section 4 to demonstrate the dualizability of the category of nuclear sheaves on an analytic adic space. We begin with a reminder of all necessary categorical concepts, especially the definition and basic properties of dualizable categories.

Notations and Terminology.

[2] I.e., $\mathcal{C} \cong \mathrm{Ind}_{\kappa}(\mathcal{C}^0)$ for a small ∞-category $\mathcal{C}^0$ and a regular cardinal κ, see [Lur09, Definition 5.4.2.1].

[3] For the proof that the restriction functors for quasi-coherent sheaves satisfy the technical condition of the theorem, we refer to [BN93, Proposition 6.1].

(i) We denote by $\widehat{\mathrm{Cat}}_\infty^\omega$ the ∞-category of accessible ∞-categories closed under small filtered colimits, whose morphisms are functors preserving small filtered colimits.

(ii) We denote by $\mathcal{P}r^{\mathrm{L}}$ the ∞-category of presentable ∞-categories, whose morphisms are functors preserving (small) colimits.

(iii) We denote by $\mathcal{P}r^{\mathrm{St}}$ the full ∞-subcategory of $\mathcal{P}r^{\mathrm{L}}$ of stable presentable ∞-categories.

(iv) We denote by $\mathrm{Cat}_\infty^{\mathrm{perf}}$ the ∞-category of small idempotent complete stable ∞-categories, whose morphisms are exact functors.

We first recall the definition of a *Verdier sequence* of stable ∞-categories.

Definition 2.3. Let $\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{G} \mathcal{E}$ be a sequence in $\mathrm{Cat}_\infty^{\mathrm{perf}}$ (resp. $\mathcal{P}r^{\mathrm{St}}$) with $G \circ F \cong 0$. It is called a *Verdier sequence* if it is a cofiber sequence and the functor F is fully faithful.

In the following, Verdier sequences where the functor G is a left Bousfield localization are of particular importance. In this case, the functor F also possesses a right adjoint functor, as the following two lemmas show.

Lemma 2.4 ([Cal+21, Lemma A.2.5])**.** *For a sequence $\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{G} \mathcal{E}$ in $\mathrm{Cat}_\infty^{\mathrm{perf}}$ with $G \circ F \cong 0$, the following are equivalent:*

(i) The sequence $\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{G} \mathcal{E}$ is a cofiber sequence and the functor G possesses a fully faithful right or left adjoint functor.

(ii) The sequence $\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{G} \mathcal{E}$ is a fiber sequence and the functor F is fully faithful and possesses a right or left adjoint functor.

One proves in a similar manner the following analogue of the above lemma for Verdier sequences in $\mathcal{P}r^{\mathrm{St}}$.

Lemma 2.5. *For a sequence $\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{G} \mathcal{E}$ in $\mathcal{P}r^{\mathrm{St}}$ with $G \circ F \cong 0$, the following are equivalent:*

(i) The sequence $\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{G} \mathcal{E}$ is a cofiber sequence and the functor G possesses a fully faithful right adjoint functor that commutes with colimits.

(ii) The sequence $\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{G} \mathcal{E}$ is a fiber sequence and the functor F is fully faithful and possesses a right adjoint functor that commutes with colimits.

Definition 2.6. Let $\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{G} \mathcal{E}$ be a Verdier sequence in $\mathrm{Cat}_\infty^{\mathrm{perf}}$ (resp. $\mathcal{P}r^{\mathrm{St}}$). It is called *right splitting* if the functor G possesses a fully faithful right adjoint functor (resp. a fully faithful colimit-preserving right adjoint functor).

We now introduce the concept of dualizability for stable ∞-categories and recall their basic properties.

Lemma 2.7. *For an ∞-category $\mathcal{C}$, the following are equivalent:*

(i) The category $\mathcal{C}$ is a retract of a compactly generated category in the category $\widehat{\mathrm{Cat}}^\omega$.

(ii) The category $\mathcal{C}$ is a retract of a compactly generated category in the category $\mathcal{P}r^{\mathrm{L}}$.

(iii) The category $\mathcal{C}$ is presentable and the functor $\mathrm{colim} : \mathrm{Ind}\,\mathcal{C} \to \mathcal{C}$ possesses a left adjoint functor.

If $\mathcal{C}$ is moreover stable, then the above conditions are equivalent to the following conditions:

(iv) The category $\mathcal{C}$ is a retract of a compactly generated category in the category $\mathcal{P}r^{\mathrm{St}}$.

(v) The category $\mathcal{C}$ is a dualizable object in $\mathcal{P}r^{\mathrm{St}}$ with respect to the Lurie tensor product.

Beweis. Combine [Lur18, Theorem 21.1.2.10], [Lur18, Corollary 21.1.2.18] and [Lur18, Proposition D.7.3.1]. □

Definition 2.8. An ∞-category (resp. stable ∞-category) $\mathcal{C}$ is called *compactly crafted* (resp. *dualizable*) if it satisfies one of the equivalent conditions (i) – (iii) (resp. (i) – (v)) of Lemma **??**.

Remark 2.9. From Lemma **??** it follows that, in particular, every stable compactly generated ∞-category $\mathcal{C}$ is a dualizable object in $\mathcal{P}r^{\mathrm{St}}$. One can show that the dual of $\mathcal{C}$ is given by $\mathrm{Ind}((\mathcal{C}^{\omega})^{\mathrm{op}})$.

Often it is necessary not to consider general functors in $\mathcal{P}r^{\mathrm{St}}$, but only those that preserve compact objects. A functor between compactly generated ∞-categories that possesses this property is called *compact*. One reason for restricting to this class of functors is the fact that the K-theory of compactly generated ∞-categories is functorial only with respect to such morphisms in $\mathcal{P}r^{\mathrm{St}}$. Another reason, which is particularly important for our current goal, is that this class of functors appears in the formulation of Theorem **??**. However, the same definition is not fine enough for dualizable ∞-categories, as they do not possess enough compact objects. To find a suitable class of functors in the case of dualizable ∞-categories, we note the following equivalent description of compact functors, which is better suited for our purposes:

Lemma 2.10. *Let $\mathcal{C}$ and $\mathcal{D}$ be presentable ∞-categories and $F : \mathcal{C} \to \mathcal{D}$ a left adjoint functor. Let G denote the right adjoint functor of F. If G commutes with filtered colimits, then F preserves compact objects. If the category $\mathcal{C}$ is compactly generated, then G commutes with filtered colimits if and only if F preserves compact objects.*

Beweis. Let X be a compact object in $\mathcal{C}$. Then

$$\mathrm{Hom}_{\mathcal{D}}(F(X), \operatorname{colim} Y_i) \cong \mathrm{Hom}_{\mathcal{C}}(X, \operatorname{colim} G(Y_i)) \cong \operatorname{colim} \mathrm{Hom}_{\mathcal{D}}(F(X), Y_i).$$

Assume $\mathcal{C}$ is compactly generated and F preserves compact objects. Let X be a compact object in $\mathcal{C}$. We perform the following elementary calculation:

$$\begin{aligned}\mathrm{Hom}_{\mathcal{C}}(X, G(\operatorname{colim} Y_i)) &\cong \mathrm{Hom}_{\mathcal{D}}(F(X), \operatorname{colim} Y_i) \cong \operatorname{colim} \mathrm{Hom}_{\mathcal{D}}(F(X), Y_i) \\ &\cong \operatorname{colim} \mathrm{Hom}_{\mathcal{C}}(X, G(Y_i)) \cong \mathrm{Hom}_{\mathcal{C}}(X, \operatorname{colim} G(Y_i)).\end{aligned}$$
□

Definition 2.11. Let $\mathcal{C}$ and $\mathcal{D}$ be dualizable ∞-categories and $F : \mathcal{C} \to \mathcal{D}$ a left adjoint functor. Let G denote the right adjoint functor of F. The functor F is called *dualizable* if G commutes with filtered colimits.

Let $\mathcal{F}$ be a sheaf of stable ∞-categories on a topological space X. If we want to show using Theorem **??** that the value of $\mathcal{F}$ on X is compactly generated, it is not sufficient to know that $\mathcal{F}$ is "locallyä sheaf of compactly generated ∞-categories. The sheaf $\mathcal{F}$ must satisfy another important condition: The fibers of the localization functors must also be compactly generated. However, the analogous condition for dualizable categories is automatically satisfied, as the following lemma shows.

Lemma 2.12. *Let $\mathcal{C}$ and $\mathcal{D}$ be stable presentable ∞-categories and $F : \mathcal{C} \to \mathcal{D}$ a left Bousfield localization whose right adjoint functor G commutes with colimits. If $\mathcal{C}$ is dualizable, then the fiber of F is also dualizable.*

Beweis. Let $\mathcal{F}$ denote the fiber of F. Since the functor G commutes with colimits, Lemma **??** yields the following right splitting Verdier sequence:

$$\mathcal{F} \underset{G'}{\overset{F'}{\rightleftarrows}} \mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$$

The ∞-category $\mathcal{F}$ is thus a retract of $\mathcal{C}$ in $\mathcal{P}r^{L}$ and hence dualizable. □

Let $\mathcal{C}$ and $\mathcal{D}$ be stable ∞-categories and $F : \mathcal{C} \to \mathcal{D}$ a left Bousfield localization. The only essential difficulty in the proof of Theorem **??** lies in the lifting of compact objects in $\mathcal{D}$ to $\mathcal{C}$. In the case of schemes, this is equivalent to the question of when a perfect complex on an open subset U of a scheme X can be extended to a perfect complex on all of X. As a final step before we turn to the analogue of Theorem **??** for dualizable categories, we give a different (and somewhat more compact) proof of the following classical theorem, whose core lies in the so-called *Thomason Trick*.

Theorem 2.13 ([Nee92, Theorem 2.1])**.** *Let $\mathcal{C}$ and $\mathcal{D}$ be stable ∞-categories and $F : \mathcal{C} \to \mathcal{D}$ a left Bousfield localization whose right adjoint functor G commutes with colimits. Then the following hold:*

(i) If the ∞-category $\mathcal{C}$ is dualizable, then the ∞-category $\mathcal{D}$ is also dualizable.

(ii) *Assume the ∞-category $\mathcal{C}$ is compactly generated. Then the category $\mathcal{D}$ is also compactly generated. Let X be a compact object in $\mathcal{D}$, Y an arbitrary object in $\mathcal{C}$, and f a morphism $f : X \to F(Y)$. Then there exists a compact object $\tilde{X}$ in $\mathcal{C}$ and a morphism $\tilde{f} : \tilde{X} \to Y$ such that $F(\tilde{f})$ factors as follows:*

$$F(\tilde{X}) \xrightarrow{\sim} X \oplus X' \xrightarrow{\pi_X} X \xrightarrow{f} F(Y),$$

where X' is a compact object in $\mathcal{D}$. In particular, every compact object in $\mathcal{D}$ is a retract of a compact object in the essential image of F.

(iii) *Thomason's Trick: Assume the ∞-category $\mathcal{C}$ and the fiber of F are both compactly generated. Let X be a compact object in $\mathcal{C}$, Y an arbitrary object in $\mathcal{C}$, and f a morphism $f : F(X) \to F(Y)$. Then there exists a compact object X' in $\mathcal{C}$ and morphisms $\phi : X' \to X$ and $\psi : X' \to Y$ such that $F(\phi)$ is an isomorphism and f factors as follows:*

$$f : F(X) \xrightarrow{F(\phi)^{-1}} F(X') \xrightarrow{F(\psi)} F(Y).$$

In other words, every morphism in $\mathcal{D}$ of the above form possesses a lift to $\mathcal{C}$. In particular, a compact object in $\mathcal{D}$ lies in the essential image of F if and only if its class in $K_0(\mathcal{D})$ lies in $\operatorname{Im} K_0(\mathcal{C})$.

Beweis. (i) From the assumption it follows directly that $\mathcal{D}$ is a retract of $\mathcal{C}$ and thus dualizable.

(ii) As one can easily see, the first statement follows from the second. Write the object $G(X) \in \mathcal{C}$ as a filtered colimit $\operatorname{colim} X_i$ of compact objects in $\mathcal{C}$. Since G is fully faithful and F commutes with colimits, $X \cong \operatorname{colim} F(X_i)$ holds. The isomorphism $X \cong \operatorname{colim} F(X_i)$ factors through some $F(X_i)$, because X is compact. The object X is thus a direct summand of $F(X_i)$. Let p_i denote the canonical morphism $F(X_i) \to X$ and f' the morphism $X_i \to GF(Y)$, which is the adjoint morphism to $f \circ p_i : F(X_i) \to F(Y)$. We now consider the following commutative diagram:

$$\begin{tikzcd} & Y \arrow[d] \\ X_i \arrow[r, "f'"] \arrow[dr, dashed] & GF(Y) \arrow[d] \\ & \operatorname{cofib}(Y \to GF(Y)) \end{tikzcd}$$

Write $\operatorname{cofib}(Y \to GF(Y))$ as a filtered colimit $\operatorname{colim} Z_j'$ of compact objects in $\mathcal{C}$. Since X_i is compact, the map $X_i \to \operatorname{cofib}(Y \to GF(Y))$ factors through a morphism $g : X_i \to Z_j$. Furthermore, we may assume that the morphism $F(g)$ is trivial. We thus obtain the following commutative diagram:

$$\begin{tikzcd} \operatorname{fib}(X_i \to Z_j) \arrow[r] \arrow[d] & Y \arrow[d] \\ X_i \arrow[r, "f'"] \arrow[d, "g"] & GF(Y) \arrow[d] \\ Z_j \arrow[r] & \operatorname{cofib}(Y \to GF(Y)) \end{tikzcd}$$

One now sees without difficulty that the pair $(\operatorname{fib}(X_i \to Z_j), \operatorname{fib}(X_i \to Z_j) \to Y)$ forms a desired pair $(\tilde{X}, \tilde{f})$, because $\operatorname{fib}(X_i \to Z_j)$ is compact and $F(\operatorname{fib}(X_i \to Z_j)) \cong F(X_i) \oplus F(Z_j[-1])$ holds.

(iii) Consider the diagram above and replace X_i by X. One then easily sees that we may assume that $F(Z_j) = 0$ holds for every Z_j, because the fiber of F is compactly generated. From this it follows immediately that one can lift morphisms. Let now K be a compact object in $\mathcal{D}$ whose class lies in $\operatorname{Im} K_0(\mathcal{C})$, and $\tilde{K}$ a compact object in $\mathcal{C}$ with $[F(\tilde{K})] = [K] \in K_0(\mathcal{D})$. One easily checks that there exists a compact object $K' \in \mathcal{D}$ such that $K \oplus K' \cong F(\tilde{K}) \oplus K'$ holds. By part (ii) of the theorem, we may assume that K' lies in the essential image of F, because every compact object in $\mathcal{D}$ is a retract of a compact object in the essential image of F. It thus suffices to show that for every fiber sequence $K_1 \to K_2 \to K_3$ of compact objects in $\mathcal{D}$, for which two out of three terms lie in the essential image of F, its third term also lies in the essential image of F. But this follows immediately from the first half of (iii).

$\square$

We have now made all preparations for the formulation and proof of the analogue of Theorem **??** for dualizable ∞-categories. We will obtain it as a formal consequence of the compactly generated case. We therefore treat both cases simultaneously and also prove Theorem **??** below.

Theorem 2.14. *Let X be a quasi-compact quasi-separated topological space with a basis $\mathcal{B}$ consisting of quasi-compact subsets. Let $U \mapsto \mathcal{C}_U$ be a sheaf of stable ∞-categories on X, whose value on each subset $B \in \mathcal{B}$ is compactly generated or dualizable, respectively. Assume that for all $V \in \mathcal{B}$ and all quasi-compact open subsets $V' \subset V$, the restriction functor $C_V \to C_{V'}$ is a compact or dualizable left Bousfield localization, respectively, with compactly generated or dualizable fiber, respectively (in the dualizable case, the last condition is automatically satisfied according to Lemma **??** and Theorem **??**). Then the category $\mathcal{C}_X$ is compactly generated or dualizable, respectively.*

Beweis. We first consider the compactly generated case. We argue by induction on the minimal number of open subsets in a cover $\mathcal{U} \subset \mathcal{B}$ of X. Let $U_1, \ldots, U_n$ be the elements of $\mathcal{U}$. Let V (resp. U) denote the subset $\overset{n-1}{\underset{i=1}{\cup}} U_i$ (resp. U_n). Then the following diagram is cartesian:

$$\begin{tikzcd} \mathcal{C}_X \arrow[r] \arrow[d] & \mathcal{C}_U \arrow[d] \\ \mathcal{C}_V \arrow[r] & \mathcal{C}_{U\cap V} \end{tikzcd}$$

Let M be an object in $\mathcal{C}_X$. Let $M|_U$ (resp. $M|_V$ resp. $M|_{V\cap U}$) denote the image of M in $\mathcal{C}_U$ (resp. $\mathcal{C}_V$ resp. $\mathcal{C}_{V\cap U}$). To prove that $\mathcal{C}_X$ is compactly generated, it suffices to show that for every $M \in \mathcal{C}_X$ and every compact $N \in \mathcal{C}_U$ (resp. $N \in \mathcal{C}_V$) together with a morphism $f : N \to M|_U$ (resp. $f : N \to M|_V$), there exists a compact object $\tilde{N} \in \mathcal{C}_X$ and a morphism $\tilde{f} : \tilde{N} \to M$ such that $\tilde{f}|_U$ (resp. $\tilde{f}|_V$) factors as follows:

$$\tilde{N}|_U \xrightarrow{\sim} N \oplus L \xrightarrow{\pi_N} N \xrightarrow{f} M|_U \text{ (resp. } \tilde{N}|_V \xrightarrow{\sim} N \oplus L \xrightarrow{\pi_N} N \xrightarrow{f} M|_V),$$

where L is a compact object in $\mathcal{C}_U$ (resp. $\mathcal{C}_V$). We only treat the case of U, because the case of V is completely analogous. Consider the morphism $f : N|_{V\cap U} \to M|_{V\cap U}$. As one easily checks, the functor $\mathcal{C}_V \to \mathcal{C}_{U\cap V}$ is a compact left Bousfield localization. Therefore – by Theorem **??**(ii) – there exists a compact object $\hat{N} \in \mathcal{C}_V$ and a morphism $\hat{f} : \hat{N} \to M|_V$ such that $\hat{f}|_{U\cap V}$ factors as follows:

$$\hat{N}|_{U\cap V} \xrightarrow{\sim} N|_{U\cap V} \oplus N' \xrightarrow{\pi_N|_{U\cap V}} N \xrightarrow{f|_{U\cap V}} M|_{U\cap V},$$

where N' is a compact object in $\mathcal{C}_{U\cap V}$. Let now N'' be a lift of $N' \oplus N'[1]$ to U, which exists by Theorem **??**(iii). Consider the following two morphisms:

$$\hat{N} \oplus \hat{N}[1] \xrightarrow{\pi_{\hat{N}}} \hat{N} \xrightarrow{\hat{f}} M|_V$$

in C_V and

$$N \oplus N[1] \oplus N'' \xrightarrow{\pi_N} N \xrightarrow{f} M|_U$$

in C_U. One easily convinces oneself that they are compatible on $U \cap V$, hence they glue together to a pair $(\tilde{N}, \tilde{f})$ with the desired properties.

We now treat the dualizable case. We first recall the following theorem; for the proof we refer to [NS18, Proposition I.3.5].

Theorem 2.15 (Thomason-Neeman)**.** *Let $\mathcal{C}' \xrightarrow{f} \mathcal{C} \xrightarrow{g} \mathcal{C}''$ be a sequence in $\mathrm{Cat}_\infty^{\mathrm{perf}}$ with $g \circ f = 0$. Then it is a Verdier sequence if and only if $\operatorname{Ind}\mathcal{C}' \xrightarrow{\operatorname{Ind} f} \operatorname{Ind}\mathcal{C} \xrightarrow{\operatorname{Ind} g} \operatorname{Ind}\mathcal{C}''$ is a Verdier sequence in $\mathcal{P}r^{\mathrm{St}}$.*

Let $\mathcal{C}$ now be a dualizable ∞-category. Let $\hat{y} : \mathcal{C} \to \operatorname{Ind}\mathcal{C}$ denote the left adjoint functor to colim : $\operatorname{Ind}\mathcal{C} \to \mathcal{C}$. One easily checks that there exists a sufficiently large regular cardinal κ such that the fully faithful functor $\hat{y} : \mathcal{C} \to \operatorname{Ind}\mathcal{C}$ factors through $\operatorname{Ind}(\mathcal{C}^\kappa) \to \operatorname{Ind}\mathcal{C}$, where $\mathcal{C}^\kappa$ denotes the full ∞-subcategory of κ-compact objects[4] in $\mathcal{C}$. The full ∞-subcategory $\mathcal{C}^\kappa$ is small according to [Lur09, Remark 5.4.2.13],

[4] see [Lur09, Definition 5.3.4.5]

hence the ∞-category $\mathrm{Ind}(\mathcal{C}^\kappa)$ is compactly generated. For the rest of the proof, we implicitly choose a sufficiently large cardinal κ such that all considered dualizable ∞-categories possess such a factorization, and write simply $\mathrm{Ind}\,\mathcal{C}$ for $\mathrm{Ind}(\mathcal{C}^\kappa)$.

We consider the following diagrams:

$$\begin{tikzcd}
\mathcal{C}_X \arrow[r, "F'_U"] \arrow[d, "F'_V"] & \mathcal{C}_U \arrow[d, "F_U"] \\
\mathcal{C}_V \arrow[r, "F_V"] & \mathcal{C}_{U\cap V}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{Ind}\,\mathcal{C}_V \underset{\mathrm{Ind}\,\mathcal{C}_{U\cap V}}{\times} \mathrm{Ind}\,\mathcal{C}_U \arrow[r, "\pi_U"] \arrow[d, "\pi_V"] & \mathrm{Ind}\,\mathcal{C}_U \arrow[d, "\mathrm{Ind}\,F_U"] \\
\mathrm{Ind}\,\mathcal{C}_V \arrow[r, "\mathrm{Ind}\,F_V"] & \mathrm{Ind}\,\mathcal{C}_{U\cap V},
\end{tikzcd}$$

where F_U, F_V, F'_U, F'_V denote the localization functors and G_U, G_V, G'_U, G'_V their right adjoint functors. By the theorem of Thomason-Neeman, the fiber of $\mathrm{Ind}\,F_U$ (resp. $\mathrm{Ind}\,F_V$) is equivalent to $\mathrm{Ind}(\mathrm{fib}F_U)$ (resp. $\mathrm{Ind}(\mathrm{fib}F_V)$). The fiber product $\mathrm{Ind}\,\mathcal{C}_V \underset{\mathrm{Ind}\,\mathcal{C}_{U\cap V}}{\times} \mathrm{Ind}\,\mathcal{C}_U$ is thus compactly generated according to the first half of the proof. Therefore, it suffices to show that C_X is a retract of $\mathrm{Ind}\,\mathcal{C}_V \underset{\mathrm{Ind}\,\mathcal{C}_{U\cap V}}{\times} \mathrm{Ind}\,\mathcal{C}_U$.

Consider the following pairs of adjoint functors:

$$\begin{tikzcd}
\mathcal{C}_U \arrow[r, shift left, "L_U"] \arrow[r, phantom, "\perp"] & \mathrm{Ind}\,\mathcal{C}_U, \arrow[l, shift left, "\mathrm{colim}_U"]
\end{tikzcd}
\quad
\begin{tikzcd}
\mathcal{C}_V \arrow[r, shift left, "L_V"] \arrow[r, phantom, "\perp"] & \mathrm{Ind}\,\mathcal{C}_V, \arrow[l, shift left, "\mathrm{colim}_V"]
\end{tikzcd}
\quad
\begin{tikzcd}
\mathcal{C}_{U\cap V} \arrow[r, shift left, "L_{U\cap V}"] \arrow[r, phantom, "\perp"] & \mathrm{Ind}\,\mathcal{C}_{U\cap V}, \arrow[l, shift left, "\mathrm{colim}_{U\cap V}"]
\end{tikzcd}$$

We claim that the following diagrams are commutative:

$$\begin{tikzcd}
\mathrm{Ind}\,\mathcal{C}_V \underset{\mathrm{Ind}\,\mathcal{C}_{U\cap V}}{\times} \mathrm{Ind}\,\mathcal{C}_U \arrow[r, "\pi_U"] \arrow[d, "\pi_V"] & \mathrm{Ind}\,\mathcal{C}_U \arrow[d, "\mathrm{Ind}\,F_U"] \arrow[dr, "\mathrm{colim}_U"] & \\
\mathrm{Ind}\,\mathcal{C}_V \arrow[r, "\mathrm{Ind}\,F_V"] \arrow[dr, "\mathrm{colim}_V"] & \mathrm{Ind}\,\mathcal{C}_{U\cap V} \arrow[dr, "\mathrm{colim}_{U\cap V}"] & \mathcal{C}_U \arrow[d, "F_U"] \\
& \mathcal{C}_V \arrow[r, "F_V"] & \mathcal{C}_{U\cap V}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{C}_X \arrow[r, "F'_U"] \arrow[d, "F'_V"] & \mathcal{C}_U \arrow[d, "F_U"] \arrow[dr, "L_U"] & \\
\mathcal{C}_V \arrow[r, "F_V"] \arrow[dr, "L_V"] & \mathcal{C}_{U\cap V} \arrow[dr, "L_{U\cap V}"] & \mathrm{Ind}\,\mathcal{C}_U \arrow[d, "\mathrm{Ind}\,F_U"] \\
& \mathrm{Ind}\,\mathcal{C}_V \arrow[r, "\mathrm{Ind}\,F_V"] & \mathrm{Ind}\,\mathcal{C}_{U\cap V}
\end{tikzcd}$$

We first consider the diagrams

$$\begin{tikzcd}
\mathcal{C}_V \arrow[r, "F_V"] \arrow[d, "L_V"] & \mathcal{C}_{U\cap V} \arrow[d, "L_{U\cap V}"] \\
\mathrm{Ind}\,\mathcal{C}_V \arrow[r, "\mathrm{Ind}\,F_V"'] & \mathrm{Ind}\,\mathcal{C}_{U\cap V}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{C}_U \arrow[r, "L_U"] \arrow[d, "F_U"] & \mathrm{Ind}\,\mathcal{C}_U \arrow[d, "\mathrm{Ind}\,F_U"] \\
\mathcal{C}_{U\cap V} \arrow[r, "L_{U\cap V}"'] & \mathrm{Ind}\,\mathcal{C}_{U\cap V}
\end{tikzcd}$$

Since all functors in the two diagrams are left adjoint, it suffices according to Lemma **??** to show that the diagrams

$$\begin{tikzcd}
\mathcal{C}_V & \mathcal{C}_{U\cap V} \arrow[l, "G_V"'] \\
\mathrm{Ind}\,\mathcal{C}_V \arrow[u, "\mathrm{colim}_V"] & \mathrm{Ind}\,\mathcal{C}_{U\cap V} \arrow[l, "\mathrm{Ind}\,G_V"] \arrow[u, "\mathrm{colim}_{U\cap V}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{C}_U & \mathrm{Ind}\,\mathcal{C}_U \arrow[l, "\mathrm{colim}_U"'] \\
\mathcal{C}_{U\cap V} \arrow[u, "G_U"] & \mathrm{Ind}\,\mathcal{C}_{U\cap V} \arrow[l, "\mathrm{colim}_{U\cap V}"] \arrow[u, "\mathrm{Ind}\,G_U"]
\end{tikzcd}$$

are commutative. Let y_U (resp. y_V resp. $y_{U\cap V}$) denote the Yoneda embedding $\mathcal{C}_U \to \mathrm{Ind}\,\mathcal{C}_U$ (resp. $\mathcal{C}_V \to \mathrm{Ind}\,\mathcal{C}_V$ resp. $\mathcal{C}_{U\cap V} \to \mathrm{Ind}\,\mathcal{C}_{U\cap V}$). One checks directly that

$$\mathrm{colim}_V \circ \mathrm{Ind}\,G_V \circ y_{U\cap V} \cong G_V \circ \mathrm{colim}_{U\cap V} \circ y_{U\cap V} \text{ and } \mathrm{colim}_U \circ \mathrm{Ind}\,G_U \circ y_{U\cap V} \cong G_U \circ \mathrm{colim}_{U\cap V} \circ y_{U\cap V}$$

holds. The desired commutativity now follows from the universal property of $\mathrm{Ind}(-)$. The commutativity of the diagrams

$$\begin{tikzcd}
\mathrm{Ind}\,\mathcal{C}_V \arrow[r, "\mathrm{Ind}\,F_V"] \arrow[d, "\mathrm{colim}_V"] & \mathrm{Ind}\,\mathcal{C}_{U\cap V} \arrow[d, "\mathrm{colim}_{U\cap V}"] \\
\mathcal{C}_V \arrow[r, "F_V"] & \mathcal{C}_{U\cap V}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{Ind}\,\mathcal{C}_U \arrow[r, "\mathrm{colim}_U"] \arrow[d, "\mathrm{Ind}\,F_U"] & \mathcal{C}_U \arrow[d, "F_U"] \\
\mathrm{Ind}\,\mathcal{C}_{U\cap V} \arrow[r, "\mathrm{colim}_{U\cap V}"] & \mathrm{Ind}\,\mathcal{C}_{U\cap V}
\end{tikzcd}$$

is proven completely analogously.

The universal property of the fiber product thus yields functors

$$\Phi : \mathcal{C}_X \to \operatorname{Ind}\mathcal{C}_V \underset{\operatorname{Ind}\mathcal{C}_{U\cap V}}{\times} \operatorname{Ind}\mathcal{C}_U \text{ and } \Psi : \operatorname{Ind}\mathcal{C}_V \underset{\operatorname{Ind}\mathcal{C}_{U\cap V}}{\times} \operatorname{Ind}\mathcal{C}_U \to \mathcal{C}_X.$$

$\Psi \circ \Phi \cong \mathrm{Id}_{\mathcal{C}_X}$ also holds, which one again proves using the universal property of the fiber product. One easily convinces oneself that the functors Ψ and Φ preserve filtered colimits, which is why the ∞-category $\mathcal{C}_X$ is a retract of the fiber product $\operatorname{Ind}\mathcal{C}_V \underset{\operatorname{Ind}\mathcal{C}_{U\cap V}}{\times} \operatorname{Ind}\mathcal{C}_U$ in the ∞-category $\mathcal{P}r^{\mathrm{St}}$. □

Lemma 2.16. *Let*

$$\begin{tikzcd} \mathcal{C} \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, shift left, "G"] \end{tikzcd}$$

be adjoint functors between small ∞-categories. Then the functors

$$\begin{tikzcd} \operatorname{Ind}\mathcal{C} \arrow[r, shift left, "\operatorname{Ind}F"] \arrow[r, phantom, "\perp"] & \operatorname{Ind}\mathcal{D} \arrow[l, shift left, "\operatorname{Ind}G"] \end{tikzcd}$$

are adjoint. If G is fully faithful here, then $\operatorname{Ind}G$ is also fully faithful.

Beweis. Let Cat_∞ (resp. $\widehat{\mathrm{Cat}}_\infty$) denote the $(\infty,2)$-category of small ∞-categories (resp. the $(\infty,2)$-category of all ∞-categories). Then the desired statement follows quite formally from the fact that the functor $\operatorname{Ind} : \mathrm{Cat}_\infty \to \widehat{\mathrm{Cat}}_\infty$ is a functor of $(\infty,2)$-categories. □

3 Nuclear Modules

In the modern approach to K-theory of schemes, one defines it by applying a purely categorical construction to the derived category of quasi-coherent sheaves. To proceed analogously in our situation, we accordingly need such a category for analytic adic spaces. In the next section, we introduce the ∞-category of *nuclear sheaves* and investigate its properties. The goal of the present section is to prepare this discussion. Specifically, we treat the affinoid case here by briefly recalling the definition of the ∞-category of *nuclear modules* over a Huber ring and analyzing its properties. Our main result is the proof of the fact that this ∞-category is dualizable, see Theorem **??**. In the following, we assume familiarity with the basics of condensed mathematics. For a brief summary of the required definitions and theorems, we refer to [And21, Section 2]; more details can be found in [Sch19] and [Sch20].

The main component of our argument is the technical concept of *weak proregularity* for ideals of a ring, which can be thought of as the analogue of the Noetherian condition for non-Noetherian rings. We briefly recall the definition and its most important properties, but we will omit further details here and refer instead to [Yek21].

Definition 3.1 ([Yek21, Section 2 and Definitions 3.1 and 3.2])**.** Let A be a (commutative) ring and $\boldsymbol{a} = (a_1, \cdots, a_n)$ a finite sequence of elements of A. For $i \in \mathbb{N}$, let $\boldsymbol{a}^i$ denote the sequence $(a_1^i, \ldots, a_n^i)$.

(i) Let a be an element of A. By the *Koszul complex* associated to a, we mean the complex of A-modules

$$K(A; a) = (\cdots \to 0 \to A \xrightarrow{\cdot a} A \to 0 \to \ldots),$$

which is concentrated in homological degrees 0 and 1. For each pair of natural numbers $i, j \in \mathbb{N}$ with $j \geq i$, we denote by μ_{ji} the map $K(A; a^j) \to K(A; a^i)$ of complexes over A, given by the identity morphism in degree 0 and multiplication by a^{j-i} in degree 1.

(ii) By the *Koszul complex* of A-modules associated to $\boldsymbol{a}$, we mean the tensor product $K(A; \boldsymbol{a}) = K(A; a_1) \underset{A}{\otimes} \cdots \underset{A}{\otimes} K(A; a_n)$. The morphisms μ_{ji} defined above induce in an obvious way morphisms $K(A; \boldsymbol{a}^j) \to K(A; \boldsymbol{a}^i)$, which we also denote by μ_{ji}.

(iii) The sequence $\boldsymbol{a}$ is called *weakly proregular* if for every $q > 0$ the inverse system $\{H_q(K(A; \boldsymbol{a}^i))\}_{i \in \mathbb{N}}$ of A-modules is pro-trivial, i.e., if for every $i \geq 0$ there exists a $j \geq i$ such that the map $H_q(\mu_{ji}) : H_q(K(A; \boldsymbol{a}^j)) \to H_q(K(A; \boldsymbol{a}^i))$ is trivial.

(iv) An ideal $I \subset A$ is called *weakly proregular* if there exists a finite weakly proregular sequence whose elements generate I.

Theorem 3.2 ([Yek21, Theorem 3.3])**.** *Let A be a Noetherian ring. Then every ideal of A is weakly proregular.*

The weak proregularity of an ideal is indeed independent of the choice of the generating sequence, as the following lemma shows.

Lemma 3.3 ([Yek21, Corollary 3.5])**.** *Let A be a ring and $I \subset A$ an ideal. If I is weakly proregular, then every finite sequence that generates the ideal I is weakly proregular.*

For our analysis of the ∞-category of nuclear modules, we need the following version of the concept of weak proregularity for Huber rings.

Definition 3.4. A complete Huber ring A is called *weakly proregular* if it possesses a pair of definition (A_0, I) with I weakly proregular in A_0.

Just as for ideals, the proregularity of a Huber ring is independent of the choice of the pair of definition, as the following lemma shows.

Lemma 3.5. *Let A be a complete weakly pro-regular Huber ring and (A_0, I) an arbitrary pair of definition of A. Then the ideal I is weakly proregular in A_0.*

Beweis. Since the product of two rings of definition is also a ring of definition, it suffices to prove the following two statements.

(i) Assume that the ideal $I \subset A_0$ is weakly proregular. Then every ideal of definition $J \subset A_0$ is also weakly proregular.

(ii) Let A_0' be a ring of definition of A with $A_0 \subset A_0'$. Then the ideal I is weakly proregular if and only if the ideal $I \cdot A_0'$ is weakly proregular.

The first statement follows as a direct consequence of [Yek21, Theorem 3.4], since the radical of every ideal of definition of A_0 is equal to the intersection $A_0 \cap A^{\circ\circ}$. We now prove the second statement. Let $\boldsymbol{a}$ denote a finite sequence of elements of A_0 that generates the ideal I. The Koszul complexes $K(A_0; \boldsymbol{a}^i)$ are in an obvious way subcomplexes of the Koszul complexes $K(A_0'; \boldsymbol{a}^i)$. One now checks directly that for $q > 0$ the inverse system $\{H_q(K(A_0; \boldsymbol{a}^i))\}_{i \in \mathbb{N}}$ is pro-trivial if and only if the system $\{H_q(K(A_0'; \boldsymbol{a}^i))\}_{i \in \mathbb{N}}$ is pro-trivial, because there exists a $k \geq 0$ such that $I^k \cdot A_0' \subset A_0$ holds. □

The main application of the concept of weak proregularity is that it allows one to determine the relationship between different notions of completeness for complexes of A-modules. Although we omit a complete discussion of weak proregularity here, this section can still be read without impairing understanding: In the following, we only need one very specific statement, for which we provide a reference at the appropriate point.

In the following, let (A, A^+) be a fixed complete weakly proregular Huber pair and (A_0, I) its fixed pair of definition. Let $(f_1, \ldots, f_n)$ be a finite system of elements of A_0 that generates the ideal I. Let $(R, \mathfrak{m})$ denote the ring $\mathbb{Z}[[x_1, \ldots, x_n]]$ together with the ideal $(x_1, \ldots, x_n) \subset R$. In this section, we equip R with the $\mathfrak{m}$-adic topology and denote the associated analytic condensed ring by $R_\blacksquare$, see [And21, Theorem 3.28] and the discussion preceding the theorem. We equip A_0, and thus also A, with the structure of a module over R, which is given by the map $R \to A_0,\ x_i \mapsto f_i$ for $i = 1, \ldots, n$. The notation $(\mathcal{A}, \mathcal{M})$, when it appears without comment to the contrary, always denotes the analytic condensed ring $(A, A^+)_\blacksquare$, see again [And21, Theorem 3.28]. We use the underscore to denote the condensed modules associated with usual topological modules; however, if the considered module M is discrete, we omit the underscore and also denote the associated condensed module by M.

We briefly recall the general definition of nuclearity for analytic condensed rings and its equivalent description in the context of adic geometry; for details, consult [Sch20, Lecture 13] and [And21, Section 5.3].

Definition 3.6 ([Sch20, Definition 13.10])**.** Let $(\mathcal{A}, \mathcal{M})$ be an arbitrary analytic animated condensed ring. An object $C \in \mathcal{D}(\mathcal{A}, \mathcal{M})$ is called *nuclear* if for every extremally disconnected space S the natural map $(\mathcal{M}[S]^\vee \underset{(\mathcal{A},\mathcal{M})}{\otimes} C)(*) \to C(S)$ in $\mathcal{D}(\mathrm{Ab})$ is an isomorphism, where $\mathcal{M}[S]^\vee$ denotes the derived internal dual $\mathrm{R}\underline{\mathrm{Hom}}_{(\mathcal{A},\mathcal{M})}(\mathcal{M}[S], \mathcal{A})$.

Lemma 3.7 ([And21, Proposition 5.35])**.** *Let (A, A^+) be an arbitrary complete Huber pair. Let $(\mathcal{A}, \mathcal{M})$ denote the analytic condensed ring $(A, A^+)_\blacksquare$. An object $C \in \mathcal{D}(\mathcal{A}, \mathcal{M})$ is nuclear if and only if for every profinite set S the natural map*

$$\underline{C(S, A)} \overset{L}{\underset{(\mathcal{A},\mathcal{M})}{\otimes}} C \to \mathrm{R}\underline{\mathrm{Hom}}_{(\mathcal{A},\mathcal{M})}(\mathcal{M}[S], C)$$

is an isomorphism, where $C(S, A)$ denotes the module of continuous functions on S with values in A.

We now begin the analysis of the nuclearity condition for modules over the analytic condensed ring $(\mathcal{A}, \mathcal{M})$.

Lemma 3.8. *If S is a profinite set, then the module $\underline{C(S, R)}$ of continuous functions on S with values in R is nuclear over $R_\blacksquare$.*

Beweis. We first perform the following elementary calculation:

$$\underline{C(S, R)} \cong \varprojlim C(S, R/\mathfrak{m}^n) \cong \varprojlim \bigoplus_J R/\mathfrak{m}^n \cong \underset{\substack{\tilde{J} \subset J \\ \text{countable}}}{\operatorname{colim}} \underline{R\langle T \rangle}.$$

Here T is a formal variable and $R\langle T \rangle$ is the Tate algebra. In this calculation, we used the fact that the module of continuous functions on S with values in a discrete module is free, see [Sch19, Theorem 5.4].

Since the ∞-category of nuclear modules is closed under colimits, it suffices to show that the module $\underline{R\langle T\rangle}$ is nuclear. Let S' be a profinite set. Then we have

$$R_\blacksquare[S']^\vee \overset{L}{\underset{R_\blacksquare}{\otimes}} \underline{R\langle T\rangle} \cong \underline{C(S',R)} \overset{L}{\underset{R_\blacksquare}{\otimes}} \underline{R\langle T\rangle} \cong \underset{\substack{\tilde{J}\subset J\\ \text{countable}}}{\operatorname{colim}} (\underline{R\langle T'\rangle} \overset{L}{\underset{R_\blacksquare}{\otimes}} \underline{R\langle T\rangle}),$$

$$\underline{C(S', R\langle T\rangle)} \cong \varprojlim C(S, R/\mathfrak{m}^n[T]) \cong \varprojlim \bigoplus_J R/\mathfrak{m}^n[T] \cong \underset{\substack{\tilde{J}\subset J\\ \text{countable}}}{\operatorname{colim}} \underline{R\langle T,T'\rangle},$$

where T' is also a formal variable. Therefore, it suffices to show that the fixed tensor product $\underline{R\langle T'\rangle} \overset{L}{\underset{R_\blacksquare}{\otimes}} \underline{R\langle T\rangle}$ is isomorphic to the Tate algebra $\underline{R\langle T,T'\rangle}$.

For a natural number $k \geq 0$, let $\mathfrak{m}_k$ denote the ideal of R generated by $x_1^k, \dots, x_n^k$. As one easily checks, the Tate algebra $\underline{R\langle T\rangle}$ can be written as the colimit

$$\underset{\substack{f:\mathbb{N}\to\mathbb{N}\\ f(n)\to+\infty}}{\operatorname{colim}} \prod_{i\in\mathbb{N}} \underline{\mathfrak{m}}_{f(i)} T^i$$

in the category of condensed modules, where $f:\mathbb{N}\to\mathbb{N}$ runs through the set of non-negative sequences converging to $+\infty$. We now claim that each $\prod_{i\in\mathbb{N}} \underline{\mathfrak{m}}_{f(i)}$ is compact in $\mathcal{D}(R_\blacksquare)$. Indeed, $\mathfrak{m}_{f(i)}$ is isomorphic to the complex $\operatorname{fib}(R \to K(R; x_1^{f(i)}, \dots, x_n^{f(i)}))$, where $K(R; x_1^{f(i)}, \dots, x_n^{f(i)})$ denotes the Koszul complex associated to the sequence $(x_1^{f(i)}, \dots, x_n^{f(i)})$. From this it follows immediately that the condensed module $\prod_{i\in\mathbb{N}} \underline{\mathfrak{m}}_{f(i)}$ is quasi-isomorphic to a finite complex whose terms are all of the form $\prod \underline{R}$. Since for every pair of sets I, I' the tensor product $\prod_I \underline{R} \overset{L}{\underset{R_\blacksquare}{\otimes}} \prod_J \underline{R}$ is isomorphic to the product $\prod_{I\times J} \underline{R}$, we have

$$\prod_{i\in\mathbb{N}} \underline{\mathfrak{m}}_{f(i)} \overset{L}{\underset{R_\blacksquare}{\otimes}} \prod_{j\in\mathbb{N}} \underline{\mathfrak{m}}_{g(j)} \cong \prod_{i,j\in\mathbb{N}} (\underline{\mathfrak{m}}_{f(i)} \overset{L}{\underset{R_\blacksquare}{\otimes}} \underline{\mathfrak{m}}_{g(j)}),$$

where $g:\mathbb{N}\to\mathbb{N}$ is a non-negative sequence converging to $+\infty$. Let $\mathfrak{m}_{f(i),g(j)}$ denote the ideal of R generated by $\{x_k^{f(i)} x_l^{g(j)}\}_{k,l=1}^n$. One checks directly, for example using the Koszul resolution, that the tensor product $\underline{\mathfrak{m}}_{f(i)} \overset{L}{\underset{R_\blacksquare}{\otimes}} \underline{\mathfrak{m}}_{g(j)}$ is isomorphic to $\underline{\mathfrak{m}}_{f(i),g(j)}$, because the sequence $(x_1^{f(i)}, \dots, x_n^{f(i)})$ is $\mathfrak{m}_{g(j)}$-regular. One then verifies without difficulty that the Tate algebra $\underline{R\langle T,T'\rangle}$ can be written as the colimit

$$\underset{\substack{f,g:\mathbb{N}\to\mathbb{N}\\ f(n),g(n)\to+\infty}}{\operatorname{colim}} \prod_{i,j\in\mathbb{N}} \underline{\mathfrak{m}}_{f(i),g(j)} T^i \cdot (T')^j$$

. $\square$

We now introduce a version of the *(idealistic) derived completion functor*. For a discussion of the classical, non-condensed case, we refer to [Yek21, Section 1].

Definition 3.9. Let $\mathcal{D}(R)$ be the algebraic derived ∞-category of the ring R. The *derived completion functor* $\mathbb{L}\Lambda_\mathfrak{m} : \mathcal{D}_{\geq 0}(R) \to \mathcal{D}_{\geq 0}(R_\blacksquare)$ is given by

$$(\cdots \to \bigoplus_J R \to \bigoplus_{J'} R \to 0 \to \cdots) \mapsto (\cdots \to \varprojlim \bigoplus_J R/\mathfrak{m}^n \to \varprojlim \bigoplus_{J'} R/\mathfrak{m}^n \to 0 \to \cdots).$$

Note that for any index set J, the condensed module $\varprojlim \bigoplus_J R/\mathfrak{m}^n$ is isomorphic to the module $\underset{\substack{\tilde{J}\subset J\\ \text{countable}}}{\operatorname{colim}} \underline{R\langle T\rangle}$. Therefore, as a consequence of the above lemma, we obtain the following theorem:

Theorem 3.10. *The functor* $\mathbb{L}\Lambda_\mathfrak{m} : \mathcal{D}_{\geq 0}(R) \to \mathcal{D}_{\geq 0}(R_\blacksquare)$ *is monoidal.*

Let R_{disk} (resp. $(A_0)_{\text{disk}}$) denote the ring R (resp. A_0) equipped with the discrete topology. We first prove the following lemma.

Lemma 3.11. *If S is a profinite set, then $\mathbb{L}\Lambda_{\mathfrak{m}}(C(S,(A_0)_{\mathrm{disk}})) \cong \underline{C(S,A_0)}$. In particular, the condensed module $\mathbb{L}\Lambda_{\mathfrak{m}}((A_0)_{\mathrm{disk}})$ is isomorphic to the module $\underline{A_0}$.*

Beweis. The module $C(S,(A_0)_{\mathrm{disk}})$ is isomorphic to a direct sum $\bigoplus_J (A_0)_{\mathrm{disk}}$ according to [Sch19, Theorem 5.4]. Let $P_\bullet$ be a free resolution of $(A_0)_{\mathrm{disk}}$ (in the category Mod_R). We perform the following calculation:

$$\mathbb{L}\Lambda_{\mathfrak{m}}(C(S,(A_0)_{\mathrm{disk}})) \cong \varprojlim \bigoplus_J P_\bullet/\mathfrak{m}^n P_\bullet \cong \underset{\substack{\tilde{J}\subset J\\ \text{countable}}}{\mathrm{colim}} \varprojlim (P_\bullet/\mathfrak{m}^n P_\bullet[T]) \overset{\dagger}{\cong} \underset{\substack{\tilde{J}\subset J\\ \text{countable}}}{\mathrm{colim}} \underline{A_0\langle T\rangle} \cong \underline{C(S,A_0)}.$$

We need to prove the isomorphism ($\dagger$). Since the ring A_0 is classically I-complete, it is also derived I-complete. The sequence

$$\cdots \xrightarrow{\phi_2} \varprojlim P_1/\mathfrak{m}^n P_1 \xrightarrow{\phi_1} \varprojlim P_0/\mathfrak{m}^n P_0 \xrightarrow{\phi_0} A_0\langle T\rangle \to 0$$

is exact in the category Mod_R according to [Yek21, Theorem 3.11], because the ideals $\mathfrak{m}\subset R$ and $I\subset A_0$ are weakly proregular. It remains to show that it is also exact in the category of condensed modules over $\underline{R}$. From the definition and the exactness of the upper sequence, it follows immediately that the kernel of ϕ_i is isomorphic to the I-adic completion of the kernel of the map $P_i \to P_{i-1}$, which is why the maps $\phi_i : \varprojlim P_i/\mathfrak{m}^n P_i \to \mathrm{Im}\,\phi_i$ are all open. The desired exactness now follows from [And21, Lemma 3.1]. ☐

We now want to show that the module $\underline{C(S,A)}$ is nuclear over the analytic condensed ring $(\mathcal{A},\mathcal{M})$ for every profinite set S. We first prove the following lemma.

Lemma 3.12. *For every $\mathfrak{m}^\infty$-torsion module M over R, we have $M \overset{L}{\underset{R_{\mathrm{disk}}}{\otimes}} R_\blacksquare \cong M$.*

Beweis. Since the functor $- \overset{L}{\underset{R_{\mathrm{disk}}}{\otimes}} R_\blacksquare$ commutes with colimits, it suffices to show that for every $i\geq 0$ the canonical map

$$R/(x_1^i,\cdots,x_n^i) \overset{L}{\underset{R_{\mathrm{disk}}}{\otimes}} R_\blacksquare \to R/(x_1^i,\cdots,x_n^i)$$

is an isomorphism. This follows directly from the Koszul resolution. ☐

Theorem 3.13. *The ring $\underline{A}$ is nuclear as a condensed module over $R_\blacksquare$.*

Beweis. Consider the following exact sequence of condensed modules:

$$0 \to \underline{A_0} \to \underline{A} \to A/A_0 \to 0.$$

As one easily verifies, the module A/A_0 is a discrete torsion module over R due to the continuity of multiplication. From this it now follows, using Lemma **??**, that the module A/A_0 is nuclear over $R_\blacksquare$. Therefore, we may assume that $A = A_0$, because the full ∞-subcategory of nuclear modules is closed under colimits. We now perform the following easy calculation:

$$A_{\mathrm{disk}} \overset{L}{\underset{R_{\mathrm{disk}}}{\otimes}} C(S,R_{\mathrm{disk}}) \cong A_{\mathrm{disk}} \overset{L}{\underset{R_{\mathrm{disk}}}{\otimes}} (\bigoplus_J R_{\mathrm{disk}}) \cong \bigoplus_J A_{\mathrm{disk}} \cong C(S,A_{\mathrm{disk}}).$$

From Theorem **??** and Lemma **??** it therefore follows

$$\underline{C(S,R)} \overset{L}{\underset{R_\blacksquare}{\otimes}} \underline{A} \cong \mathbb{L}\Lambda_{\mathfrak{m}}(C(S,R_{\mathrm{disk}}) \overset{L}{\underset{R_{\mathrm{disk}}}{\otimes}} A_{\mathrm{disk}}) \cong \mathbb{L}\Lambda_{\mathfrak{m}}(C(S,A_{\mathrm{disk}})) \cong \underline{C(S,A)}.$$ ☐

Corollary 3.14. *For every profinite set S, the module $\underline{C(S,A)}$ is nuclear over the analytic condensed ring $(\mathcal{A},\mathcal{M})$.*

Beweis. Let S' be a profinite set. From Lemma **??** and Theorem **??** it follows

$$\underline{C(S,A)} \overset{L}{\underset{(\mathcal{A},\mathcal{M})}{\otimes}} \underline{C(S',A)} \cong \underline{A} \overset{L}{\underset{R_\blacksquare}{\otimes}} \underline{C(S,R)} \overset{L}{\underset{R_\blacksquare}{\otimes}} \underline{C(S',R)} \cong \underline{A} \overset{L}{\underset{R_\blacksquare}{\otimes}} \underline{\mathrm{Hom}}_{R_\blacksquare}(R_\blacksquare[S],\underline{C(S',R)}) \cong \underline{C(S\times S',A)}.$$

☐

Corollary 3.15. *Let S and S' be profinite sets. Then*

$$\underline{C(S,A)} \overset{L}{\underset{(\mathcal{A},\mathcal{M})}{\otimes}} \underline{C(S',A)} \cong \underline{C(S\times S',A)}.$$

We now come to the proof of the fact that the ∞-category of nuclear modules over $(A,A^+)_{\blacksquare}$ is dualizable. For this, we need a generally valid statement for arbitrary analytic animated condensed rings. Let $(\mathcal{A},\mathcal{M})$ be such a ring. Let $\mathcal{C}$ denote the derived ∞-category $\mathcal{D}(\mathcal{A},\mathcal{M})$ of modules over $(\mathcal{A},\mathcal{M})$. Consider the equivalence $\mathcal{C} \xrightarrow{\sim} \mathrm{Fun}^{\lim}(\mathcal{C}^{\mathrm{op}},\mathrm{Sp})$, which is induced by the Yoneda embedding. Here $\mathrm{Fun}^{\lim}(\mathcal{C}^{\mathrm{op}},\mathrm{Sp})$ denotes the full ∞-subcategory of limit-preserving functors. Since $\mathcal{C}$ is compactly generated, this equivalence induces, by restriction to the full ∞-subcategory $\mathcal{C}^{\omega}$ of compact objects in $\mathcal{C}$, an equivalence $\mathcal{C} \xrightarrow{\sim} \mathrm{Fun}^{\mathrm{ex}}((\mathcal{C}^{\omega})^{\mathrm{op}},\mathrm{Sp})$, where $\mathrm{Fun}^{\mathrm{ex}}((\mathcal{C}^{\omega})^{\mathrm{op}},\mathrm{Sp})$ is the full ∞-subcategory of exact functors. Let $(-)^{\vee}:\mathcal{C}\to\mathcal{C}$ denote the internal dual $\mathrm{R}\underline{\mathrm{Hom}}_{(\mathcal{A},\mathcal{M})}(-,\mathcal{A})$. Under these equivalences, the functor

$$\mathcal{C}\to \mathrm{Fun}^{\mathrm{ex}}((\mathcal{C}^{\omega})^{\mathrm{op}},\mathrm{Sp}),\ M\mapsto M^{\mathrm{tr}} := ((-)^{\vee} \overset{L}{\underset{(\mathcal{A},\mathcal{M})}{\otimes}} M)(*)$$

defines a functor $\mathcal{C}\to\mathcal{C}$, which we call the *trace functor*. Now let M be an object of $\mathcal{C}$. The trace functor equipped with a natural transformation $(-)^{\mathrm{tr}}\to\mathrm{id}$ of endofunctors of $\mathcal{C}$, which is induced by the natural transformation $((-)^{\vee} \overset{L}{\underset{(\mathcal{A},\mathcal{M})}{\otimes}} M)(*) \to \mathrm{Hom}_{\mathcal{C}}(-,M)$. From the definitions, it follows directly that the morphism $M^{\mathrm{tr}}\to M$ for $M\in\mathcal{C}$ is an isomorphism if and only if M is nuclear. To prove the dualizability of the ∞-category of nuclear modules, we need the following alternative description of the trace functor.

Theorem 3.16. *Let $(\mathcal{A},\mathcal{M})$ be an arbitrary analytic animated condensed ring. Then the trace functor is given by*

$$M\in\mathcal{C}\mapsto M^{\mathrm{tr}} = \operatorname*{colim}_{P,\ \mathcal{A}\to P\otimes M} P^{\vee}$$

where P runs through the compact objects of $\mathcal{C}$ together with a morphism $\mathcal{A}\to P\otimes M$.

Beweis. Given an object $M\in\mathcal{C}$. Since the ∞-category $\mathcal{C}$ is compactly generated, we have $M^{\mathrm{tr}} = \operatorname*{colim}_{Q\to M^{\mathrm{tr}}} Q$, where Q runs through the compact objects of $\mathcal{C}$ together with a morphism $Q\to M^{\mathrm{tr}}$. As we saw in the discussion above, the morphism spectrum $\mathrm{Hom}_{\mathcal{C}}(N,M^{\mathrm{tr}})$ for every compact object $N\in\mathcal{C}$ is isomorphic to the spectrum $(N^{\vee} \overset{L}{\underset{(\mathcal{A},\mathcal{M})}{\otimes}} M)(*)$. From this it follows that $\mathrm{Hom}_{\mathcal{C}}(Q,M^{\mathrm{tr}})$ is isomorphic to $\operatorname*{colim}_{P\to Q^{\vee}} (P \underset{(\mathcal{A},\mathcal{M})}{\otimes} M)(*)$, where P runs through the compact objects of $\mathcal{C}$ together with a morphism $P\to Q^{\vee}$. Therefore, we have

$$M^{\mathrm{tr}} \cong \operatorname*{colim}_{\substack{Q,\ P\\ 1\to P\otimes M\\ P\to Q^{\vee}}} Q \cong \operatorname*{colim}_{\substack{Q,\ P\\ 1\to P\otimes M\\ Q\to P^{\vee}}} Q \cong \operatorname*{colim}_{P,\ 1\to P\otimes M} P^{\vee}.$$

□

We can now prove the dualizability of the ∞-category of nuclear modules over $(\mathcal{A},\mathcal{M}) = (A,A^+)_{\blacksquare}$.

Theorem 3.17. *Let $\mathcal{D}^{\mathrm{nuk}}(\mathcal{A},\mathcal{M})$ denote the ∞-category of nuclear modules over $(\mathcal{A},\mathcal{M})$. The trace functor $\mathcal{D}(\mathcal{A},\mathcal{M})\to\mathcal{D}(\mathcal{A},\mathcal{M})$ commutes with colimits and factors through the natural embedding $\mathcal{D}^{\mathrm{nuk}}(\mathcal{A},\mathcal{M})\hookrightarrow\mathcal{D}(\mathcal{A},\mathcal{M})$. In particular, $\mathcal{D}^{\mathrm{nuk}}(\mathcal{A},\mathcal{M})$ is a retract of a compactly generated ∞-category and thus dualizable.*

Beweis. The fact that the trace functor preserves colimits follows directly from the definition. Given an object $M\in\mathcal{D}(\mathcal{A},\mathcal{M})$. According to Theorem **??**, the object M^{tr} is isomorphic to a colimit of objects of the form $P^{\vee}$ with P compact. Such an object P is isomorphic to a retract of a finite complex whose terms are all of the form $\mathcal{M}[S]$ with S profinite. From our arguments above, it follows that each $\mathcal{M}[S]^{\vee}$ is isomorphic to a colimit of Tate algebras over A. I.e., the object $P^{\vee}$ lies in the full stable ∞-subcategory of $\mathcal{D}(\mathcal{A},\mathcal{M})$ which is generated by $A\langle T\rangle$ and is closed under colimits. From this it also follows that the ∞-category $\mathcal{D}^{\mathrm{nuk}}(\mathcal{A},\mathcal{M})$ lies within the compactly generated full ∞-subcategory $\mathcal{D}(\mathcal{A},\mathcal{M})_{\kappa}$ for an uncountable strong limit cardinal κ, see [Sch19, Appendix to Lecture II]. Since for every $N\in\mathcal{D}^{\mathrm{nuk}}(\mathcal{A},\mathcal{M})$ the natural map $N^{\mathrm{tr}}\to N$ is an isomorphism, $\mathcal{D}^{\mathrm{nuk}}(\mathcal{A},\mathcal{M})$ is thus a retract of $\mathcal{D}(\mathcal{A},\mathcal{M})_{\kappa}$. □

We obtain the following corollary as a direct consequence of the proof and the fact that for every pair of profinite sets S,S' the tensor product $\underline{C(S,A)} \underset{(\mathcal{A},\mathcal{M})}{\otimes} \underline{C(S',A)}$ is isomorphic to $\underline{C(S\times S',A)}$, and thus independent of the choice of the ring A^+.

Corollary 3.18. *The following diagram is commutative:*

$$\begin{tikzcd}
\mathcal{D}((A, A^\circ)_\blacksquare) \arrow[r, phantom, "\subseteq"] \arrow[d, "X \mapsto X^{\mathrm{tr}}"] & \mathcal{D}((A, A^+)_\blacksquare) \arrow[r, phantom, "\subseteq"] \arrow[d, "X \mapsto X^{\mathrm{tr}}"] & \mathcal{D}((A, \mathbb{Z})_\blacksquare) \arrow[d, "X \mapsto X^{\mathrm{tr}}"] \\
\mathcal{D}^{\mathrm{nuk}}((A, A^\circ)_\blacksquare) \arrow[r, phantom, "="] & \mathcal{D}^{\mathrm{nuk}}((A, A^+)_\blacksquare) \arrow[r, phantom, "="] & \mathcal{D}^{\mathrm{nuk}}((A, \mathbb{Z})_\blacksquare)
\end{tikzcd}$$

In other words, the trace functor and the full ∞-subcategory $\mathcal{D}^{\mathrm{nuk}}((A, A^+)_\blacksquare) \subset \mathcal{D}(\underline{A})$ are independent of A^+. Furthermore, the monoidal structure on $\mathcal{D}^{\mathrm{nuk}}((A, A^+)_\blacksquare)$ is also independent of A^+.

Because of this corollary, in the following sections we simply write $\mathrm{Nuk}(A)$ for the ∞-category of nuclear modules $\mathcal{D}^{\mathrm{nuk}}((A, A^+)_\blacksquare)$. Furthermore, the following corollary results from the proof of the above theorem.

Corollary 3.19. *Let (R, R^+) be a discrete Huber pair. Then the ∞-category of nuclear modules over $(R, R^+)_\blacksquare$ is equivalent to the algebraic derived ∞-category of the ring R.*

4 Nuclear Sheaves

In the preceding section, we introduced the ∞-category of nuclear modules and investigated its properties. Our main result, which will play a key role in the further course of this work, is the fact that this category is dualizable. In this section, we explain the more general definition of the ∞-category of *nuclear sheaves* on an analytic adic space and then show, using our results on dualizable categories, that it is also dualizable. As already emphasized several times, this fact is the most important component of our argument in the following section, where we define the K-theory of an analytic adic space and analyze its descent properties. We begin with the proof of the following lemma, which allows us to apply the results of the preceding section to analytic adic spaces.

Lemma 4.1. *Every complete Tate Huber ring A is weakly proregular.*

Beweis. Let ϖ be a uniformizer of A. According to [Yek21, Proposition 5.6], the weak proregularity of the ring A is equivalent to the ϖ-torsion of a ring of definition of A being bounded above, which obviously holds in our situation, because ϖ is invertible. $\square$

In particular, the ∞-category of nuclear modules over a complete Tate Huber ring is dualizable. We now want to define the ∞-category of nuclear sheaves by formally gluing the "local"categories of nuclear modules on open affinoid subsets. For this, we need the following descent theorem for nuclear modules.

Theorem 4.2 ([And21, Theorem 5.42])**.** *Let (A, A^+) be a complete sheafy analytic Huber pair and $U \subset \mathrm{Spa}(A, A^+)$ an arbitrary affinoid subset. Let A_U denote the ring $\mathcal{O}_{\mathrm{Spa}(A,A^+)}(U)$. Then the assignment $U \mapsto \mathrm{Nuk}(A_U)$ defines a sheaf of ∞-categories on $\mathrm{Spa}(A, A^+)$ with respect to the analytic topology.*

We must explain the implicit structure of the sheaf in the above theorem, namely what the corresponding restriction functors look like. Let U, V be open affinoid subsets of $\mathrm{Spa}(A, A^+)$ with $U \subset V$. Let A_U and A_U^+ (resp. A_V and A_V^+) denote the rings $\mathcal{O}_{\mathrm{Spa}(A,A^+)}(U)$ and $\mathcal{O}^+_{\mathrm{Spa}(A,A^+)}(U)$ (resp. $\mathcal{O}_{\mathrm{Spa}(A,A^+)}(V)$ and $\mathcal{O}^+_{\mathrm{Spa}(A,A^+)}(V)$). The restriction functor $\mathrm{Nuk}(V) \to \mathrm{Nuk}(U)$ is given by the pullback of nuclear modules: The functor

$$- \underset{(A_V, A_V^+)_\blacksquare}{\otimes} (A_U, A_U^+)_\blacksquare : \mathcal{D}((A_V, A_V^+)_\blacksquare) \to \mathcal{D}((A_U, A_U^+)_\blacksquare)$$

induces a functor $\mathrm{Nuk}(A_V) \to \mathrm{Nuk}(A_U)$, as the proof of [And21, Theorem 5.42] shows. In the following, we denote this functor by j^*, where $j : U \to V$ is the corresponding open embedding.

Based on Theorem **??**, we can now define the ∞-category of *nuclear sheaves* on an analytic adic space by formally gluing the corresponding categories on the affinoid subsets.

Definition 4.3. Let X be an analytic adic space. The ∞-category $\mathrm{Nuk}(X)$ *of nuclear sheaves on* X is defined as the limit $\varprojlim \mathrm{Nuk}(\mathcal{O}_X(U))$, where U runs through the affinoid open subsets of X.

Since all restriction functors for affinoid open subsets commute with colimits, the ∞-category defined in this way is automatically presentable. As we will see below, it is also dualizable. Using Theorem **??**, one easily checks that the assignment $U \mapsto \mathrm{Nuk}(U)$ for $U \subset X$ open defines a sheaf of ∞-categories with respect to the analytic topology. The corresponding restriction functors are also constructed by gluing the restriction functors on the affinoid subsets, which we will also call *localizations* or *pullback functors* in the following. The first name is justified by the fact that these functors are left Bousfield localizations under milder assumptions. Indeed, let U, V be open subsets of X with $j : U \subset V$. Assume that both subsets are quasi-compact and quasi-separated. If they are also affinoid, the forgetful functor $\mathcal{D}((\mathcal{O}_X(U), \mathcal{O}_X^+(U))_\blacksquare) \to \mathcal{D}((\mathcal{O}_X(V), \mathcal{O}_X^+(V))_\blacksquare)$ defines a functor $\mathrm{Nuk}(\mathcal{O}_X(U)) \to \mathrm{Nuk}(\mathcal{O}_X(V))$, as Lemmata **??** and **??** below show. In the following, we denote this functor by j_* and call it the *pushforward functor*. Since the forgetful functor for all solid modules is fully faithful and right adjoint to the functor

$$- \underset{(\mathcal{O}_X(V), \mathcal{O}_X^+(V))_\blacksquare}{\otimes} (\mathcal{O}_X(U), \mathcal{O}_X^+(U))_\blacksquare : \mathcal{D}((\mathcal{O}_X(V), \mathcal{O}_X^+(V))_\blacksquare) \to \mathcal{D}((\mathcal{O}_X(U), \mathcal{O}_X^+(U))_\blacksquare)$$

according to [And21, Proposition 4.12], its restriction to the ∞-category of nuclear modules defines a fully faithful functor which is right adjoint to the pullback functor j^*. If U and V are now arbitrary open quasi-compact quasi-separated subsets of X, one formally constructs the pushforward functor

$$j_* : \mathrm{Nuk}(U) \to \mathrm{Nuk}(V).$$

As one easily verifies, the functor defined in this way is fully faithful and right adjoint to the pullback functor j^*. Furthermore, due to the assumption, it commutes with colimits, because it is automatically exact and the categories $\mathrm{Nuk}(U)$ and $\mathrm{Nuk}(V)$ are isomorphic to *finite* limits of ∞-categories of nuclear modules.

Lemma 4.4. *Let $f : (A, A^+) \to (B, B^+)$ be a morphism of complete Tate Huber pairs. Then $\underline{B}$ is nuclear over $(A, A^+)_\blacksquare$.*

Beweis. Let ϖ be a uniformizer of A. We equip A with the structure of a module over $\mathbb{Z}[[x]]$, given by $\mathbb{Z}[[x]] \to A$, $x \mapsto \varpi$. Then, according to Theorem **??**, $\underline{A}$ and $\underline{B}$ are nuclear over $\mathbb{Z}[[x]]_\blacksquare$. Let S be a profinite set. To prove the desired nuclearity, we perform the following calculation:

$$\underline{C(S,A)} \overset{L}{\underset{(A,A^+)_\blacksquare}{\otimes}} \underline{B} \cong \underline{C(S,\mathbb{Z}[[x]])} \overset{L}{\underset{\mathbb{Z}[[x]]_\blacksquare}{\otimes}} \underline{A} \overset{L}{\underset{(A,A^+)_\blacksquare}{\otimes}} \underline{B} \cong \underline{C(S,\mathbb{Z}[[x]])} \overset{L}{\underset{\mathbb{Z}[[x]]_\blacksquare}{\otimes}} \underline{B} \cong \underline{C(S,B)}. \qquad \square$$

Lemma 4.5. *Let $(A, A^+) \to (B, B^+)$ be a morphism of weakly proregular Huber pairs such that $\underline{B}$ is nuclear over $(A, A^+)_\blacksquare$. Then an object $M \in \mathcal{D}((B, B^+)_\blacksquare)$ is nuclear over $(B, B^+)_\blacksquare$ if and only if it is nuclear over $(A, A^+)_\blacksquare$. In particular, the forgetful functor $\mathcal{D}((B, B^+)_\blacksquare) \to \mathcal{D}((A, A^+)_\blacksquare)$ preserves nuclear objects.*

Beweis. Assume M is nuclear over $(A, A^+)_\blacksquare$. Let S be a profinite set. We perform the following calculation:

$$\underline{C(S,B)} \overset{L}{\underset{(B,B^+)_\blacksquare}{\otimes}} M \cong \underline{C(S,A)} \overset{L}{\underset{(A,A^+)_\blacksquare}{\otimes}} \underline{B} \overset{L}{\underset{(B,B^+)_\blacksquare}{\otimes}} M \cong \mathrm{R}\underline{\mathrm{Hom}}_{(A,A^+)_\blacksquare}((A,A^+)_\blacksquare[S], M)$$

$$\cong \mathrm{R}\underline{\mathrm{Hom}}_{(B,B^+)_\blacksquare}((B,B^+)_\blacksquare[S], M).$$

If M is now nuclear over $(B, B^+)_\blacksquare$, a completely analogous calculation shows that it is also nuclear over $(A, A^+)_\blacksquare$. $\square$

We have now proven all required formal properties of the ∞-category of nuclear modules and can formally deduce the following theorem from Theorem **??**.

Theorem 4.6. *Let X be a quasi-separated quasi-compact analytic adic space. Then the ∞-category $\mathrm{Nuk}(X)$ is dualizable.*

In the following sections, we will need the following version of the category of nuclear sheaves.

Definition 4.7. Let X be an analytic adic space and Y a closed subset of X. Let U denote the complement of Y in X. The ∞-category $\mathrm{Nuk}(X \text{ on } Y)$ of *nuclear sheaves on X with support in Y* is defined as the fiber of the restriction functor $\mathrm{Nuk}(X) \to \mathrm{Nuk}(U)$.

Corollary 4.8. *Let X be a quasi-compact quasi-separated analytic adic space and Y a closed subset of X. Assume the complement of Y is quasi-compact. Then the category $\mathrm{Nuk}(X \text{ on } Y)$ is dualizable.*

Beweis. This follows from Lemma **??**. $\square$

We now prove the following further properties of the ∞-category of nuclear sheaves, which we will use to analyze the étale descent properties of K-theory.

Theorem 4.9. *Let $A \to B$ be a finite étale morphism of complete Tate Huber rings. Then*

$$\mathrm{Nuk}(B) \cong \mathrm{Nuk}(A) \underset{\mathrm{Perf}(A)}{\otimes} \mathrm{Perf}(B),$$

holds, where $\mathrm{Perf}(A)$ (resp. $\mathrm{Perf}(B)$) denotes the ∞-category of perfect complexes over A (resp. B).

Beweis. By assumption, the ∞-category $\mathrm{Perf}(B)$ is equivalent to the ∞-category $\mathrm{Mod}_B(\mathrm{Perf}(A))$, and thus $\mathrm{Nuk}(A) \underset{\mathrm{Perf(A)}}{\otimes} \mathrm{Perf}(B)$ is equivalent to $\mathrm{Mod}_B(\mathrm{Nuk}(A))$. The ∞-category $\mathcal{D}((A,\mathbb{Z})_\blacksquare)$ (resp. $\mathcal{D}((B,\mathbb{Z})_\blacksquare)$) is by definition the ∞-category $\mathrm{Mod}_{\underline{A}}(\mathcal{D}(\mathbb{Z}_\blacksquare))$ (resp. $\mathrm{Mod}_{\underline{B}}(\mathcal{D}(\mathbb{Z}_\blacksquare))$). Accordingly, it suffices to show that an object $M \in \mathcal{D}((B,\mathbb{Z})_\blacksquare)$ is nuclear over $(B,\mathbb{Z})_\blacksquare$ if and only if it is nuclear over $(A,\mathbb{Z})_\blacksquare$. But this follows immediately from Lemma **??**. $\square$

Corollary 4.10 (Étale descent theorem for nuclear sheaves)**.** *Let X be an analytic adic space. Then the assignment $U \mapsto \mathrm{Nuk}(U)$ for U étale over X defines a sheaf of ∞-categories with respect to the étale topology on X.*

Beweis. According to Theorem **??** and the analogue of Theorem **??** for sheaves of ∞-categories, it suffices to show that for every surjective finite étale map $f : \mathrm{Spa}(B, B^+) \to \mathrm{Spa}(A, A^+)$ between Tate affinoid adic spaces, the ∞-category $\mathrm{Nuk}(A)$ is isomorphic to the limit $\varprojlim \mathrm{Nuk}(B \underset{A}{\otimes} \dots \underset{A}{\otimes} B)$. For a ring R, let $\mathcal{D}(R)$ denote the algebraic derived ∞-category of R. From Theorem **??**, one can easily deduce

$$\mathrm{Nuk}(B \underset{A}{\otimes} \dots \underset{A}{\otimes} B) \cong \mathrm{Nuk}(A) \underset{\mathcal{D}(A)}{\otimes} \mathcal{D}(B \underset{A}{\otimes} \dots \underset{A}{\otimes} B).$$

As one easily checks, the morphism $\mathrm{Spec}\, B \to \mathrm{Spec}\, A$ is faithfully flat. Therefore,

$$\mathcal{D}(A) \xrightarrow{\sim} \mathrm{Tot}\Big(\mathcal{D}(B) \rightrightarrows \mathcal{D}(B \underset{A}{\otimes} B) \overset{\rightarrow}{\rightrightarrows} \dots\Big)$$

holds. It thus suffices to show that the above totalization commutes with the tensor product $\mathrm{Nuk}(A) \underset{\mathcal{D}(A)}{\otimes} -$. This follows from [Mat16, Corollary 3.42]. $\square$

Finally, we want to prove the following auxiliary lemma, with the help of which we will show in the following section that our definition of K-theory agrees with the old one.

Theorem 4.11. *Let A be a Tate Huber ring, A_0 a ring of definition of A, and $\varpi \in A_0$ a uniformizer. Let $\mathrm{Tors}(\varpi^\infty)$ denote the ∞-category of ϖ-torsion modules in the algebraic derived ∞-category $\mathcal{D}(A_0)$ of the ring A_0. Then the fiber of the localization $\mathrm{Nuk}(A_0) \to \mathrm{Nuk}(A)$ is equivalent to the ∞-category $\mathrm{Tors}(\varpi^\infty)$.*

Beweis. We consider the algebraic derived ∞-category $\mathcal{D}(A_0)$ via the *condensing functor* as a full ∞-subcategory of $\mathcal{D}((A_0, \mathbb{Z})_\blacksquare)$, see [And21, Theorem 5.9]. We must show that every nuclear module C over A_0 that becomes trivial after inverting ϖ lies within $\mathcal{D}(A_0)$. One checks directly that for every profinite set S, $\underline{C(S, A_0)}[1/\varpi] \cong \underline{C(S, A)}$ holds. It thus suffices to show that the cone

$$\mathrm{cone}\left(\underline{C(\mathbb{N} \cup \{\infty\}, A_0)} \to \underline{C(\mathbb{N} \cup \{\infty\}, A)}\right)$$

lies in $\mathcal{D}(A_0)$, because the ∞-category $\mathrm{Nuk}(A_0)$ is generated by $\underline{C(\mathbb{N} \cup \{\infty\}, A_0)}$ and the full ∞-subcategory $\mathcal{D}(A_0) \subset \mathcal{D}((A_0, \mathbb{Z})_\blacksquare)$ is closed under colimits. But this follows directly from the exact sequence of condensed modules

$$0 \to \underline{A_0\langle T\rangle} \to \underline{A\langle T\rangle} \to \bigoplus_{\mathbb{N}} A/A_0 \to 0.$$

$\square$

5 K-Theory, Localizing Invariants, and Descent

The notion of a *localizing invariant* is a very precise and powerful interpretation of the idea of algebraic invariants of categories, which has long been a focus of mathematical research. We recall the definition:

Definition 5.1. Let $\mathcal{D}$ be a stable ∞-category. Let $\mathrm{Cat}_\infty^{\mathrm{perf}}$ denote the ∞-category of small Karoubian[5] stable ∞-categories. By a *localizing invariant with values in* $\mathcal{D}$ we mean a functor $F : \mathrm{Cat}_\infty^{\mathrm{perf}} \to \mathcal{D}$ that maps final objects to final objects and Verdier sequences to fiber sequences.[6]

A particularly important example of a localizing invariant is *non-connective K-theory*. This is a functor $\mathbb{K} : \mathrm{Cat}_\infty^{\mathrm{perf}} \to \mathrm{Sp}$ with the following universal property: It is the initial localizing invariant with values in Sp equipped with a map $\mathbb{S}[-] \circ \mathrm{core} \to \mathbb{K}$. Here, core denotes the functor that maps an ∞-category $\mathcal{C}$ to the maximal subanima of $\mathcal{C}$. The connective version of K-theory is a functor $K : \mathrm{Cat}_\infty^{\mathrm{perf}} \to \mathrm{Sp}_{\geq 0}$. It can be given by an analogous universal property, but we simply set $K \overset{\mathrm{def}}{=} \Omega^\infty \mathbb{K}$. However, the connective version does not constitute a localizing invariant if one considers $\mathrm{Sp}_{\geq 0}$ as an ∞-subcategory of Sp: It is only a so-called *additive invariant*.

Using localizing invariants, numerous different questions and objects from many areas of mathematics have been successfully and thoroughly investigated. They particularly enrich algebraic geometry, where they are closely linked to its most difficult and profound problems. Of particular importance for this work is the following definition:

Definition 5.2. Let X be a quasi-compact quasi-separated scheme and $\mathcal{D}$ be a stable ∞-category. By a *localizing invariant on X with values in* $\mathcal{D}$ we mean a presheaf

$$\mathcal{F} : \{\text{quasi-compact open subsets of } X\} \to \mathcal{D},\ U \mapsto F(\mathrm{Perf}(U)),$$

where $F : \mathrm{Cat}_\infty^{\mathrm{perf}} \to \mathcal{D}$ is a localizing invariant with values in $\mathcal{D}$. Here, $\mathrm{Perf}(U)$ denotes the ∞-category of perfect complexes on U.

One of the most important abstract properties of schematic localizing invariants is Nisnevich descent. According to Thomason-Trobaugh [TT90] (see [Cla+20, Appendix A] for a modern exposition), this follows automatically from the definition:

Theorem 5.3. *Let X be a quasi-compact quasi-separated scheme and $\mathcal{F}$ be a localizing invariant on X with values in $\mathcal{D}$. Then $\mathcal{F}$ forms a sheaf with values in $\mathcal{D}$ with respect to the Nisnevich topology.*

Much more subtle is the question of whether a given localizing invariant on X satisfies the stronger *étale* descent: As the example of K-theory shows, this is generally not always the case.[7] In the same work [TT90], however, Thomason proved – albeit under some unnecessary assumptions – that after *chromatic localization*, every such invariant satisfies étale hyperdescent. His theorem in modern form reads:

Theorem 5.4 ([CM21, Theorem 7.14])**.** *Let X be a quasi-compact quasi-separated scheme of finite Krull dimension and p a fixed prime number. We denote by L_n^f the left Bousfield localization with respect to the spectrum $T(0) \oplus \cdots \oplus T(n)$, where $T(i)$ is the telescope of a p-local finite spectrum of type i. Assume that the virtual p-local Galois cohomological dimensions of the residue fields of X are bounded above. Then every localizing invariant $\mathcal{F}$ with values in L_n^f-local spectra forms an étale hypersheaf on X.*

In this section, we use our previous results to adapt the methods of [TT90] (in their modern form as in [Cla+20] and [CM21]) to adic spaces and to formulate and prove the analogues of the above theorems. In particular, we define the *continuous connective* and *non-connective K-theory*, respectively, for analytic adic spaces. In the following, we assume familiarity with the material in the appendix.

Notations and Terminology.

(i) We denote by $\mathrm{St}^{\mathrm{cg}}$ (resp. $\mathrm{St}^{\mathrm{dual}}$) the ∞-category of stable compactly generated (resp. dualizable) categories, whose morphisms are compact (resp. dualizable) functors.

(ii) We assume that all Huber rings are complete.

[5] i.e., closed under retracts

[6] The definition does not require such a functor to commute with colimits.

[7] For a concrete counterexample, we refer to https://mathoverflow.net/questions/239393/simplest-example-of-failure-of-finite-galois-descent-in-algebraic-k-theory.

The original techniques of [TT90] are very powerful: The only obstacle in our situation is essentially the definition itself; the other components of the proof also work in the adic context. In the schematic case, one implicitly uses the equivalence

$$\mathrm{Cat}_\infty^{\mathrm{perf}} \xrightarrow{\sim} \mathrm{St}^{\mathrm{cg}}, \ \mathcal{C} \mapsto \mathrm{Ind}(\mathcal{C}).$$

Indeed, the derived category of quasi-coherent sheaves on a quasi-compact quasi-separated scheme X is compactly generated according to Theorem **??**, and we have $\mathcal{D}_{qc}(X) \cong \mathrm{Ind}(\mathrm{Perf}(X))$. In our situation, however, the category of nuclear sheaves is only dualizable. Therefore, we first need to extend localizing invariants to dualizable categories. This is possible thanks to the following theorem by Efimov:

Theorem 5.5 ([Hoy18, Theorem 10])**.** *Let $\mathcal{D}$ be a stable ∞-category. By a localizing invariant of dualizable categories with values in $\mathcal{D}$ we mean a functor $F : \mathrm{St}^{\mathrm{dual}} \to \mathcal{D}$ that maps final objects to final objects and Verdier sequences to fiber sequences. Then the functor*

$$\mathrm{Fun}(\mathrm{St}^{\mathrm{dual}}, \mathcal{D}) \to \mathrm{Fun}(\mathrm{Cat}_\infty^{\mathrm{perf}}, \mathcal{D}), \ \ F \mapsto F \circ \mathrm{Ind}$$

yields an equivalence between the full subcategories of localizing invariants on both sides.

In other words, every localizing invariant $F : \mathrm{Cat}_\infty^{\mathrm{perf}} \to \mathrm{D}$ can be uniquely extended to a localizing invariant on $\mathrm{St}^{\mathrm{dual}}$. This extension is called the *continuous extension of* F and is denoted by F_{stet}. It also possesses the following property:

Theorem 5.6 ([Hoy18, Lemma 14])**.** *A localizing invariant F commutes with (filtered) colimits if and only if its continuous extension F_{stet} commutes with (filtered) colimits.*

As a consequence, we obtain a colimit-preserving functor $\mathbb{K}_{\mathrm{stet}} : \mathrm{St}^{\mathrm{dual}} \to \mathrm{Sp}$, which we call the *continuous non-connective K-theory*. Its connective analogue is defined as $K_{\mathrm{stet}} \overset{\mathrm{def}}{=} \Omega^\infty \mathbb{K}_{\mathrm{stet}}$. Using these functors, we can now define the continuous connective (resp. non-connective) K-theory of a weakly proregular Huber ring.

Definition 5.7. Let A be a weakly proregular Huber ring. Then the continuous connective (resp. non-connective) K-theory of A is defined as $K_{\mathrm{stet}}(\mathrm{Nuk}(A))$ (resp. $\mathbb{K}_{\mathrm{stet}}(\mathrm{Nuk}(A))$).

Now let A be an I-adically complete ring with I finitely generated. In this situation, one can define a "continuous K-theory of A"without the formalism of condensed mathematics, namely as $\varprojlim K(A/I^n)$. This definition plays a central role in the classical approach to the K-theory of rigid analytic spaces (see e.g. [Mor16]). If I is also weakly proregular, which is always the case in rigid geometry, our definition and the classical definition coincide according to a theorem by Efimov (see [Efi22]):

Theorem 5.8 (Efimov's Continuity Theorem)**.** *Let A be an I-adically complete ring, where I is a finitely generated weakly proregular ideal of A. Then the natural maps*

$$K_{\mathrm{stet}}(A) \to K_{\mathrm{stet}}(A/I^n) \cong K(A/I^n) \ (\textit{resp. } \mathbb{K}_{\mathrm{stet}}(A) \to \mathbb{K}_{\mathrm{stet}}(A/I^n) \cong \mathbb{K}(A/I^n))$$

for $n \geq 0$ yield an isomorphism

$$K_{\mathrm{stet}}(A) \xrightarrow{\sim} \varprojlim K(A/I^n) \ (\textit{resp. } \mathbb{K}_{\mathrm{stet}}(A) \xrightarrow{\sim} \varprojlim \mathbb{K}(A/I^n)).$$

Now let A be a Tate ring with a ring of definition A_0 and a uniformizer ϖ. In the classical approach, one defines the continuous K-theory of A via the pushout square

$$\begin{tikzcd} K(A_0) \arrow[r] \arrow[d] & K(A_0[1/\varpi]) \arrow[d] \\ K_{\mathrm{stet}}(A_0) \arrow[r] \arrow[ur, phantom, "\ulcorner"] & K'_{\mathrm{stet}}(A_0[1/\varpi]) \end{tikzcd}$$

One then globalizes the K-theory of Tate rings defined in this way to rigid analytic spaces via the so-called *pro-cdh descent.* The obvious conceptual problem of this approach, namely the fact that all constructions are somewhat ad hoc, makes the proofs elusive in the non-Noetherian case. In particular, it is not clear how to prove descent for general analytic spaces. With our more abstract definition, however, we can prove the optimal descent theorem. First, we show that the definitions coincide in the affinoid case. Consider the following diagram:

$$\begin{tikzcd}
\mathrm{Tor}(\varpi^\infty) \arrow[r] \arrow[d] & \mathcal{D}(A_0) \arrow[r, "L"] \arrow[d] & \mathcal{D}(A_0[1/\varpi]) \arrow[d] \\
\mathrm{Tor}^{\mathrm{nuk}}(\varpi^\infty) \arrow[r] & \mathrm{Nuk}(A_0) \arrow[r, "L'"] & \mathrm{Nuk}(A_0[1/\varpi])
\end{tikzcd}$$

Here, L and L' denote the canonical localization functors, and $\mathrm{Tor}(\varpi^\infty)$ and $\mathrm{Tor}^{\mathrm{nuk}}(\varpi^\infty)$ their kernels. Then the functor $\mathrm{Tors}(\varpi^\infty) \to \mathrm{Tor}^{\mathrm{nuk}}(\varpi^\infty)$ is an equivalence by Theorem **??**, i.e., the diagram

$$\begin{tikzcd}
\mathbb{K}(A_0) \arrow[r] \arrow[d] & \mathbb{K}(A[1/\varpi]) \arrow[d] \\
\mathbb{K}_{\mathrm{stet}}(A_0) \arrow[r] & \mathbb{K}_{\mathrm{stet}}(A[1/\varpi])
\end{tikzcd} \tag{5.9}$$

is (co-)Cartesian, which implies $K_{\mathrm{stet}}(A[1/\varpi]) \cong K'_{\mathrm{stet}}(A[1/\varpi])$. Let us now adapt Definition **??** to general analytic adic spaces using the results of the preceding section:

Definition 5.10. Let X be a quasi-compact quasi-separated analytic adic space and $\mathcal{D}$ be a stable ∞-category. By a *localizing invariant on X with values in $\mathcal{D}$* we mean a presheaf

$$\mathcal{F}_{\mathrm{stet}} : \{\text{quasi-compact open subsets of } X\} \to \mathcal{D},\ U \mapsto \mathcal{F}_{\mathrm{stet}}(\mathrm{Nuk}(U)),$$

where $\mathcal{F}$ is a localizing invariant with values in $\mathcal{D}$. Henceforth, we simply write $\mathcal{F}_{\mathrm{stet}}(U)$ for $\mathcal{F}_{\mathrm{stet}}(\mathrm{Nuk}(U))$.

For our analysis of the descent properties of localizing invariants, we need the following elementary lemma, which, just as in the schematic case, will play a key role in our subsequent arguments.

Lemma 5.11. *Let X be a quasi-compact quasi-separated analytic adic space and U a quasi-compact open subset of X. Let Y denote the complement of U in X. Let Z be a closed subset of X with quasi-compact complement. Then the sequences*

$$\mathrm{Nuk}(X \text{ on } Y) \to \mathrm{Nuk}(X) \to \mathrm{Nuk}(U),$$

$$\mathrm{Nuk}(X \text{ on } Y \cap Z) \to \mathrm{Nuk}(X \text{ on } Z) \to \mathrm{Nuk}(U \text{ on } U \cap Z)$$

are right split Verdier sequences.

Beweis. Let j denote the open embedding $U \hookrightarrow X$. We consider the following right split Verdier sequences:

$$\begin{tikzcd}
\mathcal{D}(X \text{ on } Y) \arrow[r, shift left] & \mathcal{D}(X) \arrow[l, shift left] \arrow[r, shift left, "j^*"] & \mathcal{D}(U) \arrow[l, shift left, "j_*"] \\
\mathcal{D}(X \text{ on } Y \cap Z) \arrow[u, hook] \arrow[r, shift left] & \mathcal{D}(X \text{ on } Z) \arrow[u, hook] \arrow[l, shift left] \arrow[r, shift left, "j^*"] & \mathcal{D}(U \text{ on } U \cap Z) \arrow[u, hook] \arrow[l, shift left, "j_*"]
\end{tikzcd}$$

Here, j^* (resp. j_*) denotes the pullback functor (resp. the pushforward functor) along j. The rightward arrows on the left side of the diagram represent the natural embeddings of the kernels of the functors j^* and $j^*|_{\mathcal{D}(X \text{ on } Z)}$. The leftward arrows correspond to the functors provided by the functor j_* according to Lemma **??**. We claim that one obtains the desired sequences by restricting all considered functors to nuclear sheaves. In other words, we claim that the functors j_* and j^* preserve nuclear objects. This follows immediately from the discussions after Theorem **??** and Definition **??**. $\square$

Using this lemma, we now show that localizing invariants in our situation also automatically satisfy Nisnevich descent:

Theorem 5.12. *Let X be a quasi-compact quasi-separated adic space. Then every localizing invariant $\mathcal{F}_{\mathrm{stet}}$ on X satisfies Nisnevich descent. In particular, $\mathbb{K}_{\mathrm{stet}}(-)$ forms a Nisnevich sheaf on X.*

Beweis. By Corollary **??**, it suffices to show that for every elementary Nisnevich cover $U \hookrightarrow X \xleftarrow{f} V$, the diagram

$$\begin{array}{ccc} \mathcal{F}_{\mathrm{stet}}(X) & \longrightarrow & \mathcal{F}_{\mathrm{stet}}(U) \\ \downarrow & & \downarrow \\ \mathcal{F}_{\mathrm{stet}}(V) & \longrightarrow & \mathcal{F}_{\mathrm{stet}}(U \underset{X}{\times} V) \end{array}$$

is Cartesian. We denote the complement of U in X by Z. The rows of the diagram

$$\begin{array}{ccccc} \mathcal{F}_{\mathrm{stet}}(X \text{ on } Z) & \longrightarrow & \mathcal{F}_{\mathrm{stet}}(X) & \longrightarrow & \mathcal{F}_{\mathrm{stet}}(U) \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{F}_{\mathrm{stet}}(U \underset{X}{\times} V \text{ on } f^{-1}(Z)) & \longrightarrow & \mathcal{F}_{\mathrm{stet}}(V) & \longrightarrow & \mathcal{F}_{\mathrm{stet}}(U \underset{X}{\times} V) \end{array}$$

are fiber sequences by Lemma **??**. Therefore, it is sufficient to show that the natural functor

$$\mathrm{Nuk}(X \text{ on } Z) \to \mathrm{Nuk}(U \underset{X}{\times} V \text{ on } f^{-1}(Z))$$

is an equivalence. This follows immediately from the étale descent theorem for nuclear sheaves (Corollary **??**). □

Remark 5.13. In the situation of Theorem **??**, using the second part of Lemma **??**, one can show that the presheaf $U \mapsto \mathcal{F}_{\mathrm{stet}}(\mathrm{Nuk}(U \text{ on } Z' \cap U))$ also forms a Nisnevich sheaf, where Z' is a closed subset of X with quasi-compact complement.

In contrast, étale descent for localizing invariants on analytic adic spaces does not hold in general. As in the case of schemes, we apply chromatic localization to fix this problem:

Theorem 5.14 (Main Descent Theorem)**.** *Let X be a quasi-compact quasi-separated analytic adic space of finite Krull dimension and p a fixed prime number. Assume that the virtual p-local cohomological dimensions of the Galois sites of X are bounded above. Then every localizing invariant $\mathcal{F}_{\mathrm{stet}}$ with values in L_n^f-local spectra forms an étale hypersheaf on X.*

Beweis. Let x be a point of X. Consider the presheaf $x^*\mathcal{F}_{\mathrm{stet}}$ on the Galois site $\mathcal{T}_x$, given by the restriction to $\mathcal{T}_x$ of the Nisnevich pullback of $\mathcal{F}$ along the canonical map $\iota_x : \mathrm{Spa}(\kappa_h(x), \kappa_h^+(x)) \to X$ (see the discussion under Definition **??**). A morphism in $\mathcal{T}_x$ corresponds to a finite étale map $f : A' \to A''$ of finite étale algebras over A, where $A \cong \operatorname*{colim}_{x \in U_i} A_i$ is the non-completed Nisnevich stalk of the structure sheaf at x. Here, U_i runs through the open affinoid Nisnevich neighborhoods of x, and A_i denotes the ring $\mathcal{O}_X(U_i)$. We write the map f as a colimit $\mathrm{colim}(f_i : A'_i \to A''_i)$ of finite étale maps between finite étale algebras over A_i. By the usual argument, we may also assume that the map f is induced by f_0, i.e., $f_i = f_0 \underset{A'_0}{\otimes} A'_i$ for every neighborhood U_i.

Let F denote the localizing invariant on $\mathrm{Cat}_\infty^{\mathrm{perf}}$ with values in Sp corresponding to the localizing invariant $\mathcal{F}_{\mathrm{stet}}$ on X, and let $\mathrm{Cat}_{\infty, A'_0}^{\mathrm{perf}}$ be the ∞-category $\mathrm{Mod}_{\mathrm{Perf}(A'_0)}(\mathrm{Cat}_\infty^{\mathrm{perf}})$. The presheaf $x^*\mathcal{F}_{\mathrm{stet}}$ takes the value $\mathrm{colim}\, F_{\mathrm{stet}}(\mathrm{Nuk}(A'_i))$ (resp. $\mathrm{colim}\, F_{\mathrm{stet}}(\mathrm{Nuk}(A''_i))$) on A' (resp. A''). As one easily checks, the functor

$$\mathrm{Cat}_{\infty, A'_0}^{\mathrm{perf}} \to \mathrm{Sp},\ \mathcal{C} \mapsto \operatorname*{colim}_{i,\ x \in U_i} F_{\mathrm{stet}}(\mathcal{C} \underset{\mathrm{Perf}(A'_0)}{\otimes} \mathrm{Nuk}(A'_i))$$

defines a localizing invariant with values in L_n^f-local spectra. Applying [Cla+20, Theorem 5.1 and Proposition 5.4] to the finite étale map $f_0 : A'_0 \to A''_0$, one sees that the spectrum $\mathrm{colim}\, F_{\mathrm{stet}}(\mathrm{Nuk}(A'_i))$ is isomorphic to the totalization

$$\mathrm{Tot}\Big(\mathrm{colim}\, F_{\mathrm{stet}}(\mathrm{Perf}(A''_0) \underset{\mathrm{Perf}(A'_0)}{\otimes} \mathrm{Nuk}(A'_i)) \rightrightarrows \mathrm{colim}\, F_{\mathrm{stet}}(\mathrm{Perf}(A''_0 \underset{A'_0}{\otimes} A''_0) \underset{\mathrm{Perf}(A'_0)}{\otimes} \mathrm{Nuk}(A'_i)) \substack{\to\\[-1ex]\to\\[-1ex]\to} \dots\Big).$$

For every finite étale algebra B over A_0, we have $\mathrm{Nuk}(B \underset{A'_0}{\otimes} A''_i) \cong \mathrm{Perf}(B) \underset{\mathrm{Perf}(A'_0)}{\otimes} \mathrm{Nuk}(A'_i)$, which can be easily shown using Theorem **??**. Thus, the presheaf $x^*\mathcal{F}_{\mathrm{stet}}$ forms a sheaf on $\mathcal{T}_x$. The argument just carried out also shows that $\mathcal{F}_{\mathrm{stet}}$ satisfies Galois descent on X. I.e., the Nisnevich hypersheaf $\mathcal{F}_{\mathrm{stet}}$ forms an étale sheaf on X by Theorem **??**.

Let B be a finite étale algebra over A. We write this as a colimit $\mathrm{colim}\, B_i$ of finite étale algebras over A_i. By [CM21, Theorem 7.14], the presheaf $G : B \mapsto \mathrm{colim}\, F(\mathrm{Perf}(B_i))$ on $\mathcal{T}_x$ is a hypercomplete sheaf. One immediately sees that the étale sheaf $x^*\mathcal{F}_{\mathrm{stet}}$ is a module over G. Since hypercompletion on $\mathcal{T}_x$ is a tensor localization by Theorem **??**, $x^*\mathcal{F}_{\mathrm{stet}}$ is thus hypercomplete (see Lemma **??**). The hypercompleteness of the étale sheaf $\mathcal{F}_{\mathrm{stet}}$ therefore follows from the local-global principle for hypercompleteness (Theorem **??**) and Theorem **??**. $\square$

We now turn to a more specific situation. For the remainder of this section, let X be a quasi-compact quasi-separated analytic adic space over $\mathrm{Spa}(\mathbb{Z}[1/p], \mathbb{Z}[1/p])$[8] of finite Krull dimension. Furthermore, we assume that the virtual p-local cohomological dimensions of the Galois sites of X are bounded above. We follow the approach of Thomason-Trobaugh in the schematic case in its modern form (see [BCM20, Section 3]) and explicitly describe the étale hypersheaf $L_{K(1)}K_{\mathrm{stet}}(-)$ on X[9], where $K(1)$ denotes the first p-local Morava K-theory. Let $B\mathbb{Z}_p^\times$ denote the ∞-topos $\mathrm{Sh}(\mathcal{T}_{\mathbb{Z}_p^\times})$ (see Definition **??**), where $\mathbb{Z}_p^\times$ is equipped with the p-adic topology. We consider the full subcategory Zykl_p of the small étale site $\acute{\mathrm{E}}\mathrm{t}_X$ of X, consisting of objects of the form $\coprod_{i=1}^{k} \mathrm{Spec}\,\mathbb{Z}[1/p, \zeta_{p^{n_i}}]$. The embedding $\mathrm{Zykl}_p \subset \acute{\mathrm{E}}\mathrm{t}_X$ and the morphism $X \to \mathrm{Spec}\,\mathbb{Z}[1/p]$ of locally ringed spaces yield the morphisms of ∞-topoi

$$\begin{tikzcd} X_{\acute{e}t} \arrow[r] \arrow[rr, bend left, "\pi_X"] & (\mathrm{Spec}\,\mathbb{Z}[1/p])_{\acute{e}t} \arrow[r, "\pi"] & B\mathbb{Z}_p^\times. \end{tikzcd}$$

Let $KU_p^\wedge$ denote the p-complete hypercomplete sheaf of spectra on $\mathcal{T}_{\mathbb{Z}_p^\times}$ constructed in [BCM20, Lemma 3.8]. Then we define the étale hypersheaf $KU_{p,X}^\wedge$ on X as the p-completion of the hypercompletion of the pullback $\pi_X^*(KU_p^\wedge)$. As one can easily see, the hypercompleteness of $KU_{p,X}^\wedge$ follows from Lemma **??**.

We briefly recall the construction of the morphism $\pi^*(KU_p^\wedge) \to L_{K(1)}K(-)$ of étale sheaves on $\mathrm{Spec}\,\mathbb{Z}[1/p]$; for further details, we refer to [BCM20, Construction 3.7 and Theorem 3.9]. From the map $\mu_{p^\infty} \to K_1(\mathbb{Z}[\zeta_{p^\infty}])$, we obtain the morphism of spectra $\mathbb{S}[B\mu_{p^\infty}] \to K(\mathbb{Z}[\zeta_{p^\infty}])$, which yields a morphism $\mathbb{S}[B\mu_{p^\infty}]_p^\wedge \to L_{K(1)}K(\mathbb{Z}[\zeta_{p^\infty}])$. Let $\beta \in \pi_2(\mathbb{S}[B\mu_{p^\infty}])$ denote the Bott element. Since the image of β in $\pi_2(L_{K(1)}K(\mathbb{Z}[\zeta_{p^\infty}]))$ is invertible (see [BCM20, Proposition 3.5]), the above morphism induces a morphism $(\mathbb{S}[B\mu_{p^\infty}]_p^\wedge[\beta^{-1}])_p^\wedge \to L_{K(1)}K(\mathbb{Z}[\zeta_{p^\infty}])$. By Snaith's theorem, there is an isomorphism of spectra $(\mathbb{S}[B\mu_{p^\infty}]_p^\wedge[\beta^{-1}])_p^\wedge \xrightarrow{\sim} KU_p^\wedge$[10], from which the desired morphism $\pi^*(KU_p^\wedge) \to L_{K(1)}K(-)$ of étale sheaves on $\mathbb{Z}[1/p]$ arises.

Now let A be a Tate ring. The natural embedding $\mathcal{D}(A) \hookrightarrow \mathrm{Nuk}(A)$ induces a morphism $K(A) \to K_{\mathrm{stet}}(A)$. Together with the morphism $\pi^*(KU_p^\wedge) \to L_{K(1)}K(-)$, this yields a morphism $\mathrm{rgl}^{-1} : KU_{p,X}^\wedge \to L_{K(1)}K_{\mathrm{stet}}(-)$ of étale hypersheaves on X.

Theorem 5.15. *The morphism* $\mathrm{rgl}^{-1} : KU_{p,X}^\wedge \to L_{K(1)}K_{\mathrm{stet}}(-)$ *is an isomorphism.*

Beweis. Since the sheaves $KU_{p,X}^\wedge$ and $L_{K(1)}K(-)$ are hypercomplete, it suffices to prove the statement stalkwise. Furthermore, due to p-completeness, it is sufficient to show that the stalks of $KU_{p,X}^\wedge$ and $L_{K(1)}K(-)$ agree after p-completion. Let x be a point of X and consider the geometric point $\bar{\iota}_x : \mathrm{Spa}(\bar{\kappa}(x), \bar{\kappa}^+(x)) \to X$. As one easily checks, the p-completion of the étale stalk $(\bar{\iota}_x)^*_{\acute{e}t}(KU_{p,X}^\wedge)$ is isomorphic to the spectrum $KU_p^\wedge$. Since the functor $L_{K(1)}$ is equivalent to the composition $(-)_p^\wedge \circ L_{KU}$ and the Bousfield localization $L_{KU} : \mathrm{Sp} \to \mathrm{Sp}$ commutes with colimits, we have

$$\big(\operatorname*{colim}_{x\in U} L_{K(1)}K(\mathcal{O}_X(U))\big)_p^\wedge \cong L_{K(1)}\big(\operatorname*{colim}_{x\in U} K(\mathcal{O}_X(U))\big), \tag{$*$}$$

where U runs through the affinoid étale neighborhoods of x.

Now let V be a Tate affinoid neighborhood of x. Let A_V denote the Tate ring $\mathcal{O}_X(V)$ and let ϖ be a uniformizer of A_V. By Efimov's Continuity Theorem (Theorem **??**), the continuous K-theory $K_{\mathrm{stet}}(A_{V,0})$ is isomorphic to $\varprojlim K(A_{V,0}/\varpi^n)$, where $A_{V,0}$ is a ring of definition of A_V with $1/p, \varpi \in A_V$. It follows from Gabber's rigidity theorem

$$L_{K(1)}K(A_{V,0}) \xrightarrow{\sim} L_{K(1)}K(A_{V,0}/\varpi) \xleftarrow{\sim} L_{K(1)}K(A_{V,0}/\varpi^n) \xleftarrow{\sim} L_{K(1)}K_{\mathrm{stet}}(A_{V,0}).$$

[8] In other words, p is invertible in $\mathcal{O}_S^+$.

[9] By the telescope conjecture for height 1, it is isomorphic to the étale hypersheaf $L_{T(1)}K_{\mathrm{stet}}(-)$.

[10] Here, $KU_p^\wedge$ does not denote the aforementioned sheaf on $\mathcal{T}_{\mathbb{Z}_p^\times}$, but the p-complete complex K-theory spectrum.

Thus, we have $L_{K(1)}K(A_V) \xrightarrow{\sim} L_{K(1)}K_{\mathrm{stet}}(A_V)$, because diagram **??** is pushout. Using the isomorphism $(*)$, it therefore follows

$$\big(\operatorname*{colim}_{x\in U} L_{K(1)}K_{\mathrm{stet}}(U)\big)^\wedge_p \cong L_{K(1)}K(\operatorname*{colim}_{x\in U} \mathcal{O}_X(U)),$$

where U runs through the Tate affinoid étale neighborhoods of x. Applying Gabber's rigidity theorem again, one sees that $L_{K(1)}K(\operatorname*{colim}_{x\in U} \mathcal{O}_X(U))$ is isomorphic to $L_{K(1)}K(k)$, where $k = (\operatorname*{colim}_{x\in U} \mathcal{O}_X(U))^\wedge$ denotes the completion of the étale stalk of the structure sheaf at x. The theorem now follows from the computation of the p-adic K-theory of separably closed fields according to Suslin, see [BCM20, Theorem 3.9]. $\square$

6 Grothendieck-Riemann-Roch

Against Grothendieck's will, we do not change our course, but let our thirst for knowledge and discovery lead us ever deeper into logical delirium: The goal of this section is to formulate and prove the Grothendieck-Riemann-Roch theorem for analytic adic spaces. Throughout this section, we assume that all considered analytic adic spaces are quasi-compact and quasi-separated, and thus all morphisms are quasi-compact. In the following, let S always be a fixed Noetherian analytic adic space over $\mathrm{Spa}(\mathbb{Z}[1/p], \mathbb{Z}[1/p])$ for a prime number p. Furthermore, we assume that it satisfies the conditions of Theorem **??**. We consider a map $f : X \to Y$ between Noetherian analytic adic spaces, smooth and proper over S.

Theorem 6.1 (Grothendieck-Riemann-Roch Theorem)**.** *The following diagram of étale sheaves of spectra on S is commutative:*

$$\begin{tikzcd} \mathbb{K}_{X,S}(-) \arrow[r, "f_!"] \arrow[d, "\mathrm{Td}(X)\,\mathrm{ch}(-)"'] & \mathbb{K}_{Y,S}(-) \arrow[d, "\mathrm{ch}(-)\,\mathrm{Td}(Y)"] \\ \mathcal{H}_{\text{ét}}(X, \mathbb{Q}_p) \arrow[r, "f_*"] & \mathcal{H}_{\text{ét}}(Y, \mathbb{Q}_p) \end{tikzcd}$$

Before we can turn to the proof, we must explain all notations and relevant concepts. We first recall, for the sake of completeness, the definitions of the Noetherian condition, properness, and smoothness in rigid geometry.

Definition 6.2 ([Hub96, Section 1.1])**.**

(i) Let A be an analytic Huber ring. A is called *strictly Noetherian* if for every $n \geq 0$ the Tate algebra $A\langle T_1, \ldots, T_n\rangle$ is Noetherian.

(ii) An analytic adic space X is called *Noetherian* if there exists an affinoid cover $X = \bigcup_{i \in I} \mathrm{Spa}(A_i, A_i^+)$ with A_i strictly Noetherian.

Definition 6.3 ([Hub96, Definitions 1.2.1, 1.3.1 and 1.3.2])**.** Let $f : X \to Y$ be a map between analytic adic spaces.

(i) A map $r : A \to B$ between complete analytic Huber rings is called *topologically of finite type* if it factors through a continuous open surjective map $A\langle T_1, \ldots, T_n\rangle \to B$ for some $n \geq 0$.

(ii) The map f is called *weakly of finite type* if for every point $x \in X$ there exists an affinoid subset $x \in U \subset X$ and an affinoid subset $V \subset Y$ with $f(U) \subset V$ such that the map $\mathcal{O}_Y(V) \to \mathcal{O}_X(U)$ is topologically of finite type.

(iii) The map f is called $^+$*weakly of finite type* if for every point $x \in X$ there exists an affinoid subset $x \in U \subset X$, an affinoid subset $V \subset Y$ with $f(U) \subset V$, and a finite set $E \subset \mathcal{O}_X^+(U)$ such that the map $\mathcal{O}_Y(V) \to \mathcal{O}_X(U)$ is topologically of finite type and the ring $\mathcal{O}_X^+(U)$ coincides with the integral closure of $\mathcal{O}_Y^+[E \cup \mathcal{O}_X(U)^{\circ\circ}]$.

(iv) The map f is called *separated* if it is weakly of finite type and the diagonal $\Delta(X)$ is closed in $X \times_Y X$.

(v) The map f is called *universally closed* if it is weakly of finite type and the base change $Y' \times_Y X \to Y'$ is closed for every morphism $Y' \to Y$ of adic spaces.

(vi) The map f is called *proper* if it is $^+$weakly of finite type, separated, and universally closed.

Definition 6.4 ([Hub96, Corollary 1.6.10])**.** Let $f : X \to Y$ be a map between Noetherian adic spaces. f is called *smooth* if for every affinoid open subset $\mathrm{Spa}(A, A^+) \subset Y$ there exists an open subset $U \subset X$ with $f(U) \subset \mathrm{Spa}(A, A^+)$ such that the restriction $f|_U : U \to \mathrm{Spa}(A, A^+)$ can be written as a composition

$$\begin{tikzcd}
U \arrow[dd, "f|_U"'] \arrow[dr, "g"] & \\
& \mathrm{Spa}(A\langle T_1,\dots,T_n\rangle, A^+\langle T_1,\dots,T_n\rangle) \arrow[dl, "h"] \\
\mathrm{Spa}(A,A^+) &
\end{tikzcd}$$

for some $n \geq 0$, where g is étale and h is the natural projection.

Let $\mathbb{Z}/p^k\mathbb{Z}(n)$ be the discrete abelian group $\mathbb{Z}/p^k\mathbb{Z}$ together with the continuous $\mathbb{Z}_p^\times$-action given by $(\alpha, x) \in (\mathbb{Z}_p^\times, \mathbb{Z}/p^k\mathbb{Z}) \mapsto \alpha^n \cdot x$. For an analytic adic space X over $\operatorname{Spec}\mathbb{Z}[1/p]$, let $\underline{\mathbb{Z}/p^k\mathbb{Z}}$ denote the pullback of $\mathbb{Z}/p^k\mathbb{Z}(n)$ to $X_{ét}$, where $\mathbb{Z}/p^k\mathbb{Z}(n)$ is viewed as a sheaf on $\mathcal{T}_{\mathbb{Z}_p^\times}$ (see the discussion after Theorem **??**). Then we define the étale hypersheaf $\underline{\mathbb{Z}_p}(n)$ (resp. $\underline{\mathbb{Q}_p}(n)$) on X as the étale hypersheaf $\varprojlim_{k\in\mathbb{N}} \underline{\mathbb{Z}/p^k\mathbb{Z}}(n)$ (resp. $(\varprojlim_{k\in\mathbb{N}} \underline{\mathbb{Z}/p^k\mathbb{Z}}(n))[1/p]$), where the limit is taken in the category of étale sheaves of spectra.

We now define the *pushforward morphism* at the level of K-theory for a proper local complete intersection morphism $f : X \to Y$ between Noetherian analytic adic spaces. For this, we use the six-functor formalism for solid sheaves on analytic adic spaces developed in [Sch23, Lecture IX]. Let $\mathcal{D}_\blacksquare(X)$ (resp. $\mathcal{D}_\blacksquare(Y)$) denote the ∞-category of solid sheaves on X (resp. Y). First, we note that the functor $f_! : \mathcal{D}_\blacksquare(X) \to \mathcal{D}_\blacksquare(Y)$ preserves nuclearity. Indeed, let $N \in \mathcal{D}_\blacksquare(X)$ be a nuclear sheaf on X and M a compact object in $\mathcal{D}_\blacksquare(Y)$. For an arbitrary object $L \in \mathcal{D}_\blacksquare(Y)$, we perform the following calculation:

$$\mathrm{Hom}(L, f_!N \otimes M^\vee) \cong \mathrm{Hom}(L, f_!(N \otimes f^*(M^\vee))) \cong \mathrm{Hom}(f^*L, N \otimes f^*(M^\vee)) \cong \mathrm{Hom}(f^*L, \underline{\mathrm{Hom}}(f^*M, N))$$

$$\cong \mathrm{Hom}(f^*(L \otimes M), N) \cong \mathrm{Hom}(L \otimes M, f_!N) \cong \mathrm{Hom}(L, \underline{\mathrm{Hom}}(M, f_!N)).$$

Here $M^\vee$ denotes the dual $\underline{\mathrm{Hom}}(M, \mathcal{O}_Y)$ and we implicitly use the equivalence $f_! \xrightarrow{\sim} f_*$ and the isomorphism $f^*(M^\vee) \cong (f^*M)^\vee$. We now note that the functor $f^! : \mathcal{D}_\blacksquare(X) \to \mathcal{D}_\blacksquare(Y)$ also preserves nuclearity and commutes with colimits. Indeed, up to a twist by an invertible sheaf, it is equivalent to the functor f^*. In other words, the functor $f_!$ by restriction yields a dualizable functor $f_! : \mathrm{Nuk}(X) \to \mathrm{Nuk}(Y)$ and thus induces a pushforward morphism $f_! : \mathbb{K}_{\mathrm{stet}}(X) \to \mathbb{K}_{\mathrm{stet}}(Y)$. Let $\mathbb{K}_X(-)$ (resp. $\mathbb{K}_Y(-)$) denote the étale presheaf on X (resp. on Y) defined by continuous non-connective K-theory. From our considerations, we thus obtain a morphism $f_! : f_*\mathbb{K}_X(-) \to \mathbb{K}_Y(-)$ of étale presheaves on Y. We now assume that the morphism f is a morphism of spaces over S, and denote the structure maps $X \to S$ and $Y \to S$ by p_X and p_Y. In this case, we set $\mathbb{K}_{X,S}(-) = p_{X*}\mathbb{K}_X(-)$ and $\mathbb{K}_{Y,S}(-) = p_{Y*}\mathbb{K}_Y(-)$ and note that the morphism $f_! : f_*\mathbb{K}_X(-) \to \mathbb{K}_Y(-)$ induces in an obvious way a morphism $\mathbb{K}_{X,S}(-) \to \mathbb{K}_{Y,S}(-)$, which we also denote by $f_!$.

Now let $f : X \to Y$ be a proper map between Noetherian analytic adic spaces, smooth over S of relative dimensions d_X and d_Y. We again denote the structure map $X \to S$ (resp. $Y \to S$) by p_X (resp. p_Y). We define the relative étale cohomology $\mathcal{H}_{\text{ét}}(X, \mathbb{Q}_p)$ (resp. $\mathcal{H}_{\text{ét}}(Y, \mathbb{Q}_p)$) as the pushforward $p_{X*}(\bigoplus_{n\in\mathbb{Z}} \underline{\mathbb{Q}_p}(n)[2n])$ (resp. $p_{Y*}(\bigoplus_{n\in\mathbb{Z}} \underline{\mathbb{Q}_p}(n)[2n])$). As in the classical case, we use Poincaré duality to define the pushforward morphism f_* on the étale cohomology $\mathcal{H}_{\text{ét}}(-, \mathbb{Q}_p)$:

Theorem 6.5 (Huber)**.** *Let $p_X : X \to S$ be a Noetherian analytic adic space, smooth over S of relative dimension d_X. Then for every $k \geq 0$*

$$p_X^!(\underline{\mathbb{Z}/p^k\mathbb{Z}}) \cong \underline{\mathbb{Z}/p^k\mathbb{Z}}(d_X)[2d_X].$$

Beweis. For the proof, we refer to [Zav23b, Theorem 6.1.6]. □

Let $k > 0$ be a natural number. For each $n \in \mathbb{Z}$, by applying the counit $f_!f^! \to \mathrm{id}$ to the identity morphism $p_Y^!(\underline{\mathbb{Z}/p^k\mathbb{Z}}(n)[2n]) \xrightarrow{\sim} p_Y^!(\underline{\mathbb{Z}/p^k\mathbb{Z}}(n)[2n])$ of étale sheaves on Y, we obtain a morphism $f_*p_X^!(\underline{\mathbb{Z}/p^k\mathbb{Z}}(n)[2n]) \to p_Y^!(\underline{\mathbb{Z}/p^n\mathbb{Z}}(n)[2n])$. Then we define the pushforward morphism $f_* : \mathcal{H}_{\text{ét}}(X, \mathbb{Q}_p) \to \mathcal{H}_{\text{ét}}(Y, \mathbb{Q}_p)$ via the diagrams

$$\begin{tikzcd}
p_{X*}p_X^!(\underline{\mathbb{Z}/p^k\mathbb{Z}}(n)[2n]) \arrow[r] \arrow[d, "\wr"] & p_{Y*}p_Y^!(\underline{\mathbb{Z}/p^k\mathbb{Z}}(n)[2n]) \arrow[d, "\wr"] \\
p_{X*}p_X^*(\underline{\mathbb{Z}/p^k\mathbb{Z}}(n+d_X)[2n+2d_X]) \arrow[r] & p_{Y*}p_Y^*(\underline{\mathbb{Z}/p^k\mathbb{Z}}(n+d_Y)[2n+2d_Y])
\end{tikzcd}$$

for all $k > 0$, $n \in \mathbb{Z}$.

We now come to the definition of the *Chern character* and the *Todd class*.

Definition 6.6 ([Hub96, Sequence 2.2.6])**.** Let X be an analytic adic space over $\operatorname{Spec}\mathbb{Z}[1/p]$. Let $\mathbb{G}_{m,X}$ denote the étale sheaf $U \mapsto \mathcal{O}_X^\times(U)$ of abelian groups on X. By the *Kummer sequence* we mean the exact sequence

$$1 \to \mu_{p^n} \to \mathbb{G}_{m,X} \xrightarrow{\cdot p^n} \mathbb{G}_{m,X} \to 1$$

of étale sheaves of abelian groups on X.

Under the identification $\mu_{p^n} = \underline{\mathbb{Z}/p^n\mathbb{Z}(1)}$ for all $n \geq 0$, we obtain from this sequence a morphism $\mathbb{G}_{m,X}[1] \to \underline{\mathbb{Z}_p}(1)[2]$, which we use to define the *first Chern class*.

Lemma 6.7 ([Hub96, Theorem 2.2.7])**.** *Let X be a Noetherian analytic adic space. Then the étale cohomology group $H^1_{\acute{e}t}(X, \mathbb{G}_{m,X})$ is isomorphic to the Picard group of X (with respect to the analytic topology).*

Definition 6.8. Let X be a Noetherian analytic adic space over $\operatorname{Spec}\mathbb{Z}[1/p]$ and $\mathcal{L}$ a line bundle on X. Then the *first Chern class* $c_1(\mathcal{L})$ of $\mathcal{L}$ is defined as the image of the class of $\mathcal{L}$ in $\operatorname{Pic}(X)$ under the map

$$\operatorname{Pic}(X) \xrightarrow{\sim} H^1_{\acute{e}t}(X, \mathbb{G}_{m,X}) \to H^2_{\acute{e}t}(X, \underline{\mathbb{Z}_p}(1)).$$

Exactly as in the case of schemes and complex manifolds, we now define the Chern character and the Todd class through this. Let X be an analytic adic space X over $\operatorname{Spec}\mathbb{Z}[1/p]$. In the following, we denote by $H^*_{\acute{e}t}(X, \mathbb{Z}_p)$ (resp. $H^*_{\acute{e}t}(X, \mathbb{Q}_p)$) the direct sum of cohomology groups $\bigoplus_{n\in\mathbb{Z}} H^{2n}(X, \underline{\mathbb{Z}_p}(n))$ (resp. $\bigoplus_{n\in\mathbb{Z}} H^{2n}(X, \underline{\mathbb{Q}_p}(n))$).

Theorem 6.9. *There exists a unique functorial assignment $\mathcal{V} \mapsto c(\mathcal{V}) \in H^*_{\acute{e}t}(X, \mathbb{Z}_p)[[t]]$, where t is a formal variable and $\mathcal{V}$ is a vector bundle on a Noetherian analytic adic space X over* $\operatorname{Spec}\mathbb{Z}[1/p]$*, which has the following properties:*

(i) The assignment $\mathcal{V} \mapsto c(\mathcal{V})$ commutes with pullback.

(ii) For every exact sequence of vector bundles $0 \to \mathcal{V}' \to \mathcal{V} \to \mathcal{V}'' \to 0$, we have $c(\mathcal{V}) = c(\mathcal{V}') \cdot c(\mathcal{V}'')$.

(iii) For a line bundle $\mathcal{L}$, we have $c(\mathcal{L}) = 1 + c_1(\mathcal{L})t$.

Beweis. As with schemes and complex manifolds, the theorem follows from the *splitting principle*, which also holds in our situation, because one can also construct the flag space of a vector bundle $\mathcal{V}$ in the adic context.[11] For details, consult for example [Gro58]. □

Theorem 6.10. *There exists a unique functorial assignment $\mathcal{V} \mapsto \operatorname{ch}(\mathcal{V}) \in H^*(X, \mathbb{Q}_p)$, where $\mathcal{V}$ is a vector bundle on a Noetherian analytic adic space X over* $\operatorname{Spec}\mathbb{Z}[1/p]$*, which has the following properties:*

(i) The assignment $\mathcal{V} \mapsto \operatorname{ch}(\mathcal{V})$ commutes with pullback.

(ii) For every exact sequence of vector bundles $0 \to \mathcal{V}' \to \mathcal{V} \to \mathcal{V}'' \to 0$, we have $\operatorname{ch}(\mathcal{V}) = \operatorname{ch}(\mathcal{V}') + \operatorname{ch}(\mathcal{V}'')$.

(iii) For a line bundle $\mathcal{L}$, we have $\operatorname{ch}(\mathcal{L}) = e^{c_1(\mathcal{L})}$.

Furthermore, the Künneth formula $\operatorname{ch}(\mathcal{V} \otimes \mathcal{W}) = \operatorname{ch}(\mathcal{V}) \cdot \operatorname{ch}(\mathcal{W})$ holds for vector bundles $\mathcal{V}$ and $\mathcal{W}$ on a Noetherian analytic adic space X.

Beweis. The existence, uniqueness, and the formula $\operatorname{ch}(\mathcal{V} \otimes \mathcal{W}) = \operatorname{ch}(\mathcal{V}) \cdot \operatorname{ch}(\mathcal{W})$ are proven exactly as in classical cases, see for example [HBS66]. □

Finally, the following theorem is proven in a similar way:

Theorem 6.11. *There exists a unique functorial assignment $\mathcal{V} \mapsto \operatorname{Td}(\mathcal{V}) \in H^*(X, \mathbb{Q}_p)$, where $\mathcal{V}$ is a vector bundle on a Noetherian analytic adic space X over* $\operatorname{Spec}\mathbb{Z}[1/p]$*, which has the following properties:*

(i) The assignment $\mathcal{V} \mapsto \operatorname{Td}(\mathcal{V})$ commutes with pullback.

[11] For a discussion of the construction of $\mathbb{P}(\mathcal{V})$ for a vector bundle $\mathcal{V}$ in adic geometry, we refer to [Zav23a, Section 6].

(ii) For every exact sequence of vector bundles $0 \to \mathcal{V}' \to \mathcal{V} \to \mathcal{V}'' \to 0$*, we have* $\mathrm{Td}(\mathcal{V}) = \mathrm{Td}(\mathcal{V}') \cdot \mathrm{Td}(\mathcal{V}'')$.

(iii) For a line bundle $\mathcal{L}$*, we have* $\mathrm{Td}(\mathcal{L}) = \frac{c_1(\mathcal{L})}{1-e^{-c_1(\mathcal{L})}}$.

Let X be a Noetherian analytic adic space, smooth over S. Let $\mathcal{T}_X$ denote the relative tangent bundle of X over S. To simplify notation, we simply write $\mathrm{Td}(X)$ for $\mathrm{Td}(\mathcal{T}_X)$.

For a complete explanation of the formulation of the Grothendieck-Riemann-Roch theorem, we only lack the *refined Chern character* $\mathrm{ch} : \mathbb{K}_{X,S}(-) \to \mathcal{H}_{ét}(X, \mathbb{Q}_p)$, where X is a Noetherian analytic adic space over S which satisfies the conditions of Theorem **??**.[12] We define this via the isomorphism $KU^{\wedge}_{p,X} \xrightarrow{\sim} L_{K(1)}K_X(-)$ from the previous section using the following theorem.

Theorem 6.12. *Let X be an analytic adic space over* $\operatorname{Spec}\mathbb{Z}[1/p]$ *which satisfies the conditions of Theorem* **??**. *Then the étale sheaf* $KU^{\wedge}_{p,X}$ *becomes isomorphic to the étale sheaf* $\bigoplus_{n\in\mathbb{Z}} \underline{\mathbb{Q}_p}(n)[2n]$ *as a sheaf of* $\mathbb{E}_\infty$*-rings after inverting* p.[13]

Beweis. For the proof, we pass to the pro-étale site of X.[14] More concretely, we consider the following diagram of geometric morphisms of ∞-topoi (see the discussion under Theorem **??**):

$$\begin{tikzcd}
X_{proét} \arrow[r] \arrow[d, "\alpha"] \arrow[rr, bend left, "\tilde{\pi}_X"] & (\operatorname{Spec}\mathbb{Z}[1/p])_{proét} \arrow[r, "\tilde{\pi}"] \arrow[d, "\beta"] & (B\mathbb{Z}_p^\times)_{proét} \arrow[d, "\gamma"] \\
X_{ét} \arrow[r] \arrow[rr, bend right, "\pi_X"'] & (\operatorname{Spec}\mathbb{Z}[1/p])_{ét} \arrow[r, "\pi"] & B\mathbb{Z}_p^\times
\end{tikzcd}$$

To clarify our arguments, we slightly change our notations for the course of this proof. For a natural number $k \geq 0$ and an integer $n \in \mathbb{Z}$, we denote by $\mathbb{Z}/p^k\mathbb{Z}(n)$ the discrete spectrum $\mathbb{Z}/p^k\mathbb{Z}$ together with the $\mathbb{Z}_p^\times$-action given by $(\alpha, x) \in (\mathbb{Z}_p^\times, \mathbb{Z}/p^k\mathbb{Z}) \mapsto \alpha^n \cdot x$. In the following, we identify it with the corresponding sheaf on $B\mathbb{Z}_p^\times$ under the Yoneda embedding. Let $\mathbb{Z}_{p,B\mathbb{Z}_p^\times}(n)$ denote the sheaf of spectra on $B\mathbb{Z}_p^\times$ given as $\varprojlim \mathbb{Z}/p^k\mathbb{Z}(n)$. Here the limit is taken in the category of sheaves of spectra on $B\mathbb{Z}_p^\times$. The sheaf $KU_p^\wedge$ on $B\mathbb{Z}_p^\times$ constructed in [BCM20, Lemma 3.8] will henceforth be denoted by $KU^\wedge_{p,B\mathbb{Z}_p^\times}$. We now consider the pullbacks of these sheaves to $X_{ét}$. Let $\mathbb{Z}_{p,X_{ét}}$ (resp. $KU^\wedge_{p,X_{ét}}$) denote the p-completion of the hypercompletion of the pullback $\pi_X^*\mathbb{Z}_{p,B\mathbb{Z}_p^\times}$ (resp. $\pi_X^*KU^\wedge_{p,B\mathbb{Z}_p^\times}$). Note that the sheaf $\mathbb{Z}_{p,X_{ét}}$ defined this way coincides with the sheaf $\underline{\mathbb{Z}_p}(n)$ defined under Definition **??**, because the sheaves $\pi_X^*\mathbb{Z}/p^k\mathbb{Z}(n)$ for $k \geq 0$ are all hypercomplete. Furthermore, we denote by $\mathbb{Q}_{p,X_{ét}}$ the sheaf $\mathbb{Z}_{p,X_{ét}}[1/p]$.

We now define analogous sheaves on the pro-étale sites. Since the pro-étale ∞-topos of $\mathbb{Z}_p^\times$ is equivalent to the ∞-category of condensed anima with $\mathbb{Z}_p^\times$-action (see [BS15, Lemma 4.3.2]), where $\mathbb{Z}_p^\times$ is equipped with the p-adic topology, it is hypercomplete. For $n \in \mathbb{Z}$, let $\mathbb{Z}_{p,(B\mathbb{Z}_p^\times)_{proét}}(n)$ (resp. $\mathbb{Q}_{p,(B\mathbb{Z}_p^\times)_{proét}}(n)$) denote the topological abelian group $\mathbb{Z}_p$ (resp. $\mathbb{Q}_p$) together with the $\mathbb{Z}_p^\times$-action given by $(\alpha, x) \in (\mathbb{Z}_p^\times, \mathbb{Z}_p) \mapsto \alpha^n \cdot x$ (resp. $(\alpha, x) \in (\mathbb{Z}_p^\times, \mathbb{Q}_p) \mapsto \alpha^n \cdot x$). Henceforth, we identify them with the corresponding sheaves of abelian groups on $(B\mathbb{Z}_p^\times)_{proét}$. As one checks directly, for example using the fact that the pushforward functor γ_* commutes with limits and filtered colimits in $\mathrm{Sh}((B\mathbb{Z}_p^\times)_{proét}, \mathrm{Sp}_{\leq k})$ for every fixed $k \in \mathbb{Z}$ and preserves representable objects, the canonical maps

$$\mathbb{Z}_{p,B\mathbb{Z}_p^\times}(n) \to \gamma_*\mathbb{Z}_{p,(B\mathbb{Z}_p^\times)_{proét}}(n) \quad \text{and} \quad \mathbb{Q}_{p,B\mathbb{Z}_p^\times}(n) \to \gamma_*\mathbb{Q}_{p,(B\mathbb{Z}_p^\times)_{proét}}(n)$$

are isomorphisms. We now define the sheaf $KU^\wedge_{p,(B\mathbb{Z}_p^\times)_{proét}}$ on $(B\mathbb{Z}_p^\times)_{proét}$ as

$$(\mathbb{S}[B^2\mathbb{Z}_{p,(B\mathbb{Z}_p^\times)_{proét}}(1)]^\wedge_p[\beta^{-1}])^\wedge_p,$$

[12] Note that they are automatically satisfied as soon as X is smooth over S.

[13] For the definition of the structure of an $\mathbb{E}_\infty$-ring on $\bigoplus_{n\in\mathbb{Z}} \underline{\mathbb{Q}_p}(n)[2n]$, use the fact that the p-completion of the tensor product $\underline{\mathbb{Z}_p}(n) \otimes \underline{\mathbb{Z}_p}(m)$ is isomorphic to $\underline{\mathbb{Z}_p}(n+m)$.

[14] For a discussion of the pro-étale topology and in particular the pro-étale site of a profinite group, we refer to [BS15].

where β denotes the Bott element. One easily convinces oneself that the canonical map $KU^{\wedge}_{p,B\mathbb{Z}_p^\times} \to \gamma_* KU^{\wedge}_{p,(B\mathbb{Z}_p^\times)_{pro\acute{e}t}}$ is an isomorphism, because the functor γ_* commutes with Postnikov limits. Let

$$\mathbb{Z}_{p,X_{pro\acute{e}t}} \quad \text{resp.} \quad \mathbb{Q}_{p,X_{pro\acute{e}t}} \quad \text{resp.} \quad KU^{\wedge}_{p,X_{pro\acute{e}t}} \quad \text{resp.} \quad KU^{\wedge}_{p,X_{pro\acute{e}t}}[1/p]$$

denote the hypercompletion of the pullback

$$\tilde{\pi}_X^* \mathbb{Z}_{p,(B\mathbb{Z}_p^\times)_{pro\acute{e}t}}(n) \quad \text{resp.} \quad \tilde{\pi}_X^* \mathbb{Q}_{p,(B\mathbb{Z}_p^\times)_{pro\acute{e}t}}(n) \quad \text{resp.} \quad \tilde{\pi}_X^* KU^{\wedge}_{p,(B\mathbb{Z}_p^\times)_{pro\acute{e}t}} \quad \text{resp.} \quad \pi_* KU^{\wedge}_{p,X_{pro\acute{e}t}}[1/p].$$

Since the pullback functor $\tilde{\pi}_X^*$ commutes with limits[15], the sheaves $\mathbb{Z}_{p,X_{pro\acute{e}t}}$ and $KU^{\wedge}_{p,X_{pro\acute{e}t}}$ defined this way are moreover p-complete. One now easily checks that the canonical maps

$$\mathbb{Z}_{p,X_{\acute{e}t}}(n) \to \alpha_* \mathbb{Z}_{p,X_{pro\acute{e}t}}(n), \quad \mathbb{Q}_{p,X_{\acute{e}t}}(n) \to \alpha_* \mathbb{Q}_{p,X_{pro\acute{e}t}}(n),$$
$$KU^{\wedge}_{p,X_{\acute{e}t}} \to \alpha_* KU^{\wedge}_{p,X_{pro\acute{e}t}} \quad \text{and} \quad KU^{\wedge}_{p,X_{\acute{e}t}}[1/p] \to \alpha_* KU^{\wedge}_{p,X_{pro\acute{e}t}}[1/p]$$

are also isomorphisms.

From our arguments above, it follows that it suffices to construct an isomorphism

$$KU^{\wedge}_{p,(B\mathbb{Z}_p^\times)_{pro\acute{e}t}} \xrightarrow{\sim} \bigoplus_{n\in\mathbb{Z}} \mathbb{Q}_{p,(B\mathbb{Z}_p^\times)_{pro\acute{e}t}}(n)[2n]$$

of condensed ring spectra with $\mathbb{Z}_p^\times$-action. We consider two cases.

Case $p \neq 2$. Let $g \in \mathbb{Z} \subset \mathbb{Z}_p$ be a topological generator of the multiplicative group $\mathbb{Z}_p^\times$. We consider the geometric morphism $\nu : (B\mathbb{Z})_{pro\acute{e}t} \to (B\mathbb{Z}_p^\times)_{pro\acute{e}t}$ of ∞-topoi, which is induced by the map $\mathbb{Z} \to \mathbb{Z}_p^\times$, $1 \mapsto g$. Let $\mathbb{S}_p$ denote the condensed spectrum $\varprojlim \mathbb{S}/p^k$, where $\mathbb{S}/p^k$ is viewed as a discrete condensed spectrum and the limit is taken in the ∞-category of condensed spectra. We equip it henceforth with the trivial $\mathbb{Z}_p^\times$-action. We claim that the restriction of the pullback functor ν^* to solid[16] $\mathbb{S}_p$-modules with $\mathbb{Z}_p^\times$-action is fully faithful. As one can easily see, our claim is equivalent to the tensor product $\mathbb{S}_p[\mathbb{Z}_p^\times]^{\blacksquare} \otimes^{\blacksquare}_{\mathbb{S}_p[g^{\mathbb{Z}}]} \mathbb{S}_p[\mathbb{Z}_p^\times]^{\blacksquare}$ being isomorphic to the solid spectrum $\mathbb{S}_p[\mathbb{Z}_p^\times]^{\blacksquare}$. Here $(-)^{\blacksquare}$ denotes the solidification functor. Let $\mu_{p-1} \subset \mathbb{Z}_p^\times$ denote the subgroup of $(p-1)$-th roots of unity. We perform the following calculation:

$$\mathbb{S}_p[\mathbb{Z}_p^\times]^{\blacksquare} \overset{\text{def}}{=} \varprojlim_{k\geq 1} \mathbb{S}_p[\mu_{p-1} \times ((1+p\mathbb{Z}_p)/(1+p^k\mathbb{Z}_p))] \overset{\log}{\cong} \varprojlim_{k\geq 1} \mathbb{S}_p[\mu_{p-1} \times \mathbb{Z}/p^{k-1}\mathbb{Z}] \cong \mathbb{S}_p[\mu_{p-1}] \otimes \mathbb{S}_p[[T]],$$

where T is a formal variable and g acts on $\mathbb{S}_p[[T]]$ by $\cdot(1+T)$. In other words, we must show that

$$\mathbb{S}_p[[1-T]] \otimes^{\blacksquare}_{\mathbb{S}_p[T]} \mathbb{S}_p[[1-T]] \cong \mathbb{S}_p[[1-T]]^{\blacksquare}$$

holds. Since the analytic condensed ring spectrum $(\mathbb{S}_p[T], \mathbb{S}_p)_{\blacksquare}$ is isomorphic to the analytic condensed ring spectrum $\mathbb{S}_p[T]_{\blacksquare}$, the left side is isomorphic to $\mathbb{S}_p[[1-T]] \otimes_{\mathbb{S}_p[T]_{\blacksquare}} \mathbb{S}_p[[1-T]]$. We write $\mathbb{S}_p[[1-T]]$ as the fiber

$$\mathrm{fib}\,\Big(\prod_{k\in\mathbb{N}} \mathbb{S}_p[T]/(1-T)^k \to \prod_{k\in\mathbb{N}} \mathbb{S}_p[T]/(1-T)^k\Big).$$

Note here that

$$\mathbb{S}_p[T]/(1-T)^k \cong \mathrm{cofib}\,(\mathbb{S}_p[T] \xrightarrow{\cdot(1-T)^k} \mathbb{S}_p[T]) \quad \text{and} \quad \prod_{k\in\mathbb{N}} \mathbb{S}_p[T] \cong \mathbb{S}_p[T]_{\blacksquare}[\mathbb{N} \cup \{\infty\}].$$

The desired isomorphism is now verified by direct calculation.

Let $\mathbb{Q}_p(n)$ (resp. $KU_p^{\wedge}$) denote the underlying condensed spectrum of $\mathbb{Q}_{p,(B\mathbb{Z}_p^\times)_{pro\acute{e}t}}(n)$ (resp. $KU^{\wedge}_{p,(B\mathbb{Z}_p^\times)_{pro\acute{e}t}}$) equipped with its $g^{\mathbb{Z}}$-action. From the paragraph above, it follows that it suffices to construct an isomorphism

$$KU_p^{\wedge} \to \bigoplus_{n\in\mathbb{Z}} \mathbb{Q}_p(n)[2n]$$

[15]This follows from the properties of the pro-étale topology.

[16]A condensed spectrum with $\mathbb{Z}_p^\times$-action is called *solid* if its underlying condensed spectrum is solid.

of condensed ring spectra with $g^{\mathbb{Z}}$-action. Let $\mathbb{Q}(n)$ denote the discrete topological group $\mathbb{Q}$ together with the $g^{\mathbb{Z}}$-action given by $(g^k, x) \in g^{\mathbb{Z}} \times \mathbb{Q} \mapsto g^{nk} \cdot x$. As one easily checks, $\mathbb{Q}_p(n) \cong \mathbb{Q}(n) \otimes \mathbb{S}_p$. At the level of the underlying condensed ring spectra, i.e., without the $g^{\mathbb{Z}}$-action, we also have $KU_p^\wedge \cong KU \otimes \mathbb{S}_p$, where KU is viewed as a discrete condensed ring spectrum. Indeed, this follows easily from the isomorphisms $ku_p^\wedge \cong ku \otimes \mathbb{S}_p$ and $KU_p^\wedge \cong ku_p^\wedge[\beta^{-1}]$, where β denotes the Bott element. In particular, $\psi : (KU \otimes \mathbb{Q}) \otimes \mathbb{S}_p \cong KU_p^\wedge[1/p]$ holds. We now want to equip the left side with a $g^{\mathbb{Z}}$-action such that ψ becomes a $g^{\mathbb{Z}}$-equivariant isomorphism of condensed ring spectra. Snaith's theorem states that KU is isomorphic to the spectrum $\mathbb{S}[B^2\mathbb{Z}][\beta^{-1}]$. The endomorphisms $B^2\mathbb{Z} \xrightarrow{\cdot g^k} B^2\mathbb{Z}$ for $k \geq 0$ induce in an obvious way endomorphisms $\phi_k : KU \to KU$. Let Φ denote the canonical map $KU \to KU_p^\wedge$. As one can easily see, $\Phi \circ \phi_k = g^k \cdot \Phi$ holds for every $k \geq 0$, where $g^k\cdot$ on the right side of the equation denotes the action of g^k on $KU_p^\wedge$. One now easily convinces oneself that the endomorphisms ϕ_k induce automorphisms of $KU \otimes \mathbb{Q}$ and thus an action of $g^{\mathbb{Z}}$ on $KU \otimes \mathbb{Q}$, such that the map ψ becomes a $g^{\mathbb{Z}}$-equivariant isomorphism.

In other words, we have reduced the problem to the following: We must construct a $g^{\mathbb{Z}}$-equivariant isomorphism $KU \otimes \mathbb{Q} \xrightarrow{\sim} \bigoplus_{n\in\mathbb{Z}} \mathbb{Q}(n)[2n]$ of ring spectra. This, however, follows directly from the isomorphism $KU \cong \mathbb{S}[B^2\mathbb{Z}][\beta^{-1}]$.

Case $p = 2$. This case is actually completely analogous to the one above. The only difference is that the multiplicative group $\mathbb{Z}_2^\times$ does not possess a dense cyclic subgroup. However, there exists an integer g such that the subgroup $\{\pm 1\} \times g^{\mathbb{Z}}$ is dense in $\mathbb{Z}_2^\times$. We therefore replace the group $g^{\mathbb{Z}}$ in the proof above with this subgroup $\{\pm 1\} \times g^{\mathbb{Z}}$ and then argue as above. □

Let X be a Noetherian analytic adic space over S which satisfies the conditions of Theorem ??. We define the refined Chern character $\mathrm{ch} : \mathbb{K}_{X,S}(-) \to \mathcal{H}_{\text{ét}}(X, \mathbb{Q}_p)$ as the morphism obtained by applying the pushforward functor p_{X*} to the composition

$$\mathbb{K}_X(-) \to L_{K(1)}\mathbb{K}_X(-) \cong KU^\wedge_{X,p} \to \bigoplus_{n\in\mathbb{Z}} \underline{\mathbb{Q}_p}(n)[2n]$$

The Chern character defined this way induces in particular, by taking π_0, a map $K_0(\mathrm{Vect}_X) \to H^*(X, \mathbb{Q}_p)$, where Vect_X denotes the groupoid of vector bundles on X. We claim that this map coincides with the Chern character $\mathrm{ch} : K_0(\mathrm{Vect}_X) \to H^*(X, \mathbb{Q}_p)$ defined above using the Kummer sequence. We consider the following diagram:

$$\begin{tikzcd}
\mathbb{S}[B\mathbb{G}_{m,X}] \arrow[dr] \arrow[drrrrr, bend left] & & & & & \\
\mathbb{S}[B\mu_{p^\infty}] \arrow[u] \arrow[r] & \mathbb{K}_X(-) \arrow[r] & L_{K(1)}\mathbb{K}_X(-) & KU^\wedge_{p,X} \arrow[l, "\sim"'] \arrow[r, "\sim"] & \bigoplus_{n\in\mathbb{Z}} \underline{\mathbb{Q}_p}(n)\beta^n[2n] \arrow[r] & \underline{\mathbb{Q}_p}(1)\beta[2]
\end{tikzcd}$$

The morphism $\mathbb{S}[B\mu_{p^\infty}] \to \mathbb{K}_X(-)$ (resp. $\mathbb{S}[B\mathbb{G}_{m,X}] \to \mathbb{K}_X(-)$) in the diagram arises by adjunction from the maps $\mu_{p^\infty}(R) \to K_1(R)$ (resp. $R^\times \to K_1(R)$) for an arbitrary ring R. The slanted arrow above represents the morphism which corresponds by adjunction to the morphism $\mathbb{G}_{m,X}[1] \to \mathbb{Q}_p[2]$ defined by the Kummer sequence. The other morphisms are as in the discussion above. The question of whether the two defined Chern characters coincide is exactly equivalent to whether the right triangle is commutative. This, however, follows immediately from the definition of the morphism $KU^\wedge_{p,X} \to L_{K(1)}K_X(-)$.

We now come to the actual proof of the Grothendieck-Riemann-Roch theorem. After all our work, the theorem can be proven in exactly the same way as in the situation of [CS22, Lecture 15]. We briefly explain the most important steps. We first consider the following abstract framework (see [CS22, Definition 15.6]). Let Man_S be the category of Noetherian analytic adic spaces, smooth and proper over S, and let $(K, \otimes)$ be a symmetric monoidal 1-category, closed under finite products.

(i) By a cohomology theory on Man_S with values in K, we mean a contravariant symmetric monoidal functor $H^* : \mathrm{Man}_S^{\mathrm{op}} \to K$ that maps disjoint unions to products and has the following property: For every $X \in \mathrm{Man}_S$ and every vector bundle $\mathcal{V}$ on X of positive rank, the map $H^*(X) \to H^*(\mathbb{P}(\mathcal{V}))$ is a monomorphism.

(ii) Let H^* be a cohomology theory on Man_S with values in K. Let $\mathrm{Man}_S^{\cong}$ denote the maximal subgroupoid of Man_S. By a *pushforward structure* on H^*, we mean a covariant functor $H_* : \mathrm{Man}_S \to K$ with an identification $H_*|_{\mathrm{Man}_S^{\cong}} \cong H^*|_{\mathrm{Man}_S^{\cong}}$, which has the following properties.

(a) Let $h : Z \to Z'$ be a morphism in Man_S. Let h^* (resp. h_*) denote the map $H^*(h)$ (resp. $H_*(h)$). For every transverse Cartesian diagram

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

in Man_S, we have $g^* f_* \cong f'_* g'^* : H^*(X) \to H^*(Y')$.

(b) Let $f : X \to Y$ be a morphism in Man_S and Z an object in Man_S. We have

$$f_* \otimes \mathrm{id}_{H^*(Z)} \cong (f_* \otimes \mathrm{id}_Z)_* : H^*(X \times_S Z) \cong H^*(X) \otimes H^*(Z) \to H^*(Y) \otimes H^*(Z) \cong H^*(Y \times_S Z).$$

(c) Let $\mathcal{L}$ be a line bundle on a space $X \in \mathrm{Man}_S$. The map

$$H^*(\mathbb{P}(\mathcal{L} \oplus \mathcal{O}_X)) \xrightarrow{(p_*, \infty^*)} H^*(X) \times H^*(X)$$

is a monomorphism. Here $p : \mathbb{P}(\mathcal{L} \oplus \mathcal{O}_X) \to X$ denotes the canonical projection and $\infty : X \to \mathbb{P}(\mathcal{L} \oplus \mathcal{O}_X)$ the ∞-section.

In our situation, we take the derived category $D_{ét}(S, \mathbb{Z})$ of étale sheaves on S as the category K and the assignment $X \mapsto p_{X*} L_{K(1)} K_X[1/p] \cong \mathcal{H}_{ét}(X, \mathbb{Q}_p)$ as the cohomology theory H^*, where p_X denotes the structure map $X \to S$. The Grothendieck-Riemann-Roch theorem can now be interpreted as follows: It states that the two defined pushforward structures, namely those defined by the functor $f_* : \mathbb{K}_{X,S} \to \mathbb{K}_{Y,S}$ and by Poincaré duality, respectively, coincide up to multiplication by the Todd class.

For the proof of the theorem, we introduce, as in [CS22, Lecture 15], the concept of the *Euler class* of a line bundle. For a space $X \in \mathrm{Man}_S$ and a line bundle $\mathcal{L}$ on X, let $0 : X \to \mathbb{P}(\mathcal{L}^* \oplus \mathcal{O}_X)$ denote the zero section. For a cohomology theory (H^*, H_*) together with a pushforward structure, we define $e(\mathcal{L}) \in H^*(X)$ as $e(\mathcal{L}) = 0^* 0_*(1)$. As one easily checks (see [CS22, Example 15.10]), in our situation the Euler class of $\mathcal{L}$ with respect to the K-theoretic pushforward structure is equal to $1 - e^{c_1(\mathcal{L})}$ and with respect to the other is equal to $c_1(\mathcal{L})$.

Now let H_* be a pushforward structure on $\mathcal{H}_{ét}(X, \mathbb{Q}_p)$ and $f : X \to Y$ a morphism in Man_S. As one checks directly, the formula $\mathrm{Td}(Y)^{-1} H_*(f) \mathrm{Td}(X)$ also defines a pushforward structure (for a general definition of the twist of a pushforward structure, we refer to [CS22, Remark 15.11]). In particular, to prove the Grothendieck-Riemann-Roch theorem, it suffices to show that two pushforward structures on a cohomology theory coincide if the corresponding theories of Euler classes coincide. The proof of this statement is literally the same as in the situation of [CS22, Theorem 15.12].

A The Nisnevich Topology

In this section of the appendix, we give a definition of the Nisnevich topology of analytic adic spaces and prove some of its properties. Throughout the section, we assume that all considered analytic adic spaces are quasi-compact and quasi-separated, and thus all morphisms are quasi-compact. When a Huber pair appears without comment to the contrary, it is assumed to be complete.

Notations and Terminology.

(i) By a *topos*, we mean a 1-*topos*.

(ii) Let X be an analytic adic space. We denote by $(\kappa(x), \kappa^+(x))$ the complete residue field of X at $x \in X$.

Definition A.1 ([Sch12, Definition 7.1], [SW20, Definition 7.5.1])**.**

(i) A morphism $f : X \to Y$ of analytic adic spaces is called *finite étale* if for every affinoid open subset $U = \mathrm{Spa}(R, R^+) \subset Y$, the preimage $f^{-1}(U) \cong \mathrm{Spa}(\tilde{R}, \tilde{R}^+)$ is also affinoid such that the map $R \to \tilde{R}$ is finite étale and $\tilde{R}^+$ is the integral closure of R^+.

(ii) A morphism $f : X \to Y$ of analytic adic spaces is called *étale* if every point $x \in X$ has an open neighborhood U such that there exists an open subset $V \subset Y$ with $f(U) \subset V$ for which the restriction $f : U \to V$ can be factored as an open immersion followed by a finite étale morphism.

In the definition of finite étale morphisms, it suffices to require only the existence of a Zariski cover with the above property, because finite projective modules satisfy Zariski descent according to [And21, Theorem 1.4].

Lemma A.2. *An étale morphism of analytic adic spaces is open.*

Beweis. It suffices to show that every finite étale morphism is open. In this case, the proof is almost verbatim the same as in the situation of [Hub96, Lemma 1.7.9]: Ignore its first paragraph and note that the map $X_B \to X_A$ is open, because $X_B = \mathrm{Spv}(f)^{-1}(X_A)$. □

In the following, we make ample use of the theory of valued fields. Although we cannot provide an introduction to this topic here, we nevertheless recall some fundamental definitions, primarily to fix the notation.

Definition A.3. (i) An (possibly non-complete) Huber pair (K, K^+) is called an *analytic affinoid field* if K is a non-archimedean field[17] and K^+ is its open valuation ring.

(ii) An analytic affinoid field (K, K^+) is called *henselian* if the ring K^+ is henselian.

Lemma A.4. *Let K be a (possibly non-complete) non-archimedean field. Then:*

(i) If K is complete, then the affinoid field (K, K°) is henselian.

(ii) If K is separably closed, then every affinoid field of the form (K, K^+) is henselian.

Beweis. (i) See [Bou72, Exercise VI.§8.6(b)].

(ii) This follows directly from the definition.

□

Definition A.5. Let k be a field and $\mathcal{O}$ a valuation ring of k. The *henselization* of $(k, \mathcal{O})$ is given by $(k_h, \mathcal{O}_h)$, where $\mathcal{O}_h$ is the henselization of $\mathcal{O}$ and k_h is the field of fractions of $\mathcal{O}_h$.

Lemma A.6. *Let k be a field and $\mathcal{O}$ a valuation ring of k. Consider the separable closure k^{sep} of k and a valuation ring $\mathcal{O}^{\mathrm{sep}}$ with $\mathcal{O}^{\mathrm{sep}} \cap k = \mathcal{O}$. If we set*

$$G = \{\sigma \in \mathrm{Gal}(k^{\mathrm{sep}}/k) | \sigma(\mathcal{O}^{\mathrm{sep}}) = \mathcal{O}^{\mathrm{sep}}\},$$

then there exists a unique k-isomorphism

$$(k_h, \mathcal{O}_h) \cong ((k^{\mathrm{sep}})^G, (\mathcal{O}^{\mathrm{sep}})^G).$$

[17] i.e., a topological field whose topology is generated by a rank 1 valuation

Beweis. See [EP05, Theorem 5.2.2]. □

From the perspective of rigid geometry, there is no difference between the henselization of an affinoid field and its completion, as the following trivial lemma shows.

Lemma A.7. *Let (K, K^+) be a (possibly non-complete) affinoid field.*

(i) If (K, K^+) is henselian, then $(\hat{K}, \hat{K}^+)$ is also henselian.

(ii) The completion of the henselization of (K, K^+) is isomorphic to the completion of the henselization of $(\hat{K}, \hat{K}^+)$.

Beweis. (i) This follows directly from the definitions (note that $\hat{K}^+/\hat{\mathfrak{m}}^+ \cong K^+/\mathfrak{m}^+$).

(ii) This follows directly from the universal properties of henselization and completion.

□

Definition A.8. Let (K, K^+) be a complete affinoid field.

(i) By the *henselization of* (K, K^+), we mean the completion of the henselization of the pair (K, K^+) in the sense of Definition **??**. In the following, (K_h, K_h^+) will always denote this completed henselization.

(ii) Let $\bar{K}$ denote the completion of a separable closure of K, which is also separably closed, and let $\bar{K}^+$ be any open valuation ring of $\bar{K}$ with $\bar{K}^+ \cap K = K^+$.[18] We call the affinoid field $(\bar{K}, \bar{K}^+)$ a *separable closure of* (K, K^+).

Definition A.9. Let X be an analytic adic space.

(i) The underlying category of the *small étale site* ét_X of X is defined as the category of étale spaces over X. An *étale cover* is a family $\{Y_i \to Y\}_{i \in I}$ of étale morphisms with the property that the map $\coprod Y_i \to Y$ is surjective.

(ii) The underlying category of the *small Nisnevich site* Nis_X of X is also defined as the category of étale spaces over X. A Nisnevich cover is an étale cover $\{Y_i \to Y\}_{i \in I}$ with the property that for every henselian affinoid field (K, K^+), the map

$$\coprod Y_i(K, K^+) \to Y(K, K^+)$$

is surjective.

From Lemma **??**, it is easy to deduce that every étale cover contains a finite subcover. Since spectral spaces are quasi-compact with respect to the constructible topology, it follows from Theorem **??** below that the same holds for the Nisnevich topology.

Lemma A.10. *A base change of a Nisnevich cover is a Nisnevich cover.*

Beweis. This follows directly from the definition, since a base change of an étale morphism is also étale. □

Definition A.11. Let X be an analytic adic space.

(i) By a *geometric point of* X, we mean an affinoid field of the form $(\bar{\kappa}(x), \bar{\kappa}^+(x))$ for $x \in X$ together with the canonical map

$$\bar{\iota}_x : \mathrm{Spa}(\bar{\kappa}(x), \bar{\kappa}^+(x)) \to X.$$

Such a point defines a point of the étale topos $X_{\text{ét}}$ via the functor

$$X_{\text{ét}} \to \mathrm{Sets},\ \mathcal{F} \mapsto (\bar{\iota}_x)^*_{\text{ét}}(\mathcal{F})(\mathrm{Spa}(\bar{\kappa}(x), \bar{\kappa}^+(x))).$$

[18] Such rings are conjugate according to [EP05, Theorem 3.2.14].

(ii) By a *henselian point of X*, we mean an affinoid field (K, K^+) with a map

$$\iota_x : \mathrm{Spa}(K, K^+) \to \mathrm{Spa}(\kappa_h(x), \kappa_h^+(x)) \to X,$$

where $x \in X$, K is a separable extension of $\kappa_h(x)$, and K^+ is the integral closure of $\kappa_h^+(x)$. Such a point defines a point of the Nisnevich topos X_{Nis} via the functor

$$X_{\mathrm{Nis}} \to \mathrm{Sets},\ \mathcal{F} \mapsto (\iota_x)^*_{Nis}(\mathcal{F})(\mathrm{Spa}(K, K^+)).$$

Theorem A.12. *Let X be an analytic adic space. Then:*

(i) The points of the étale topos of X form a conservative family and are precisely the geometric points of X.

(ii) The points of the Nisnevich topos of X form a conservative family and are precisely the henselian points of X.

Beweis. This is proved analogously to [SGA 4, Theorem VIII.7.9]. □

As with schemes, étale descent for analytic adic spaces is the combination of Nisnevich and Galois descent.

Definition A.13. Let X be an analytic adic space and $\mathcal{F}$ an étale presheaf of animes on X. We say that $\mathcal{F}$ *satisfies Galois descent* if for every affinoid open subset $\mathrm{Spa}(A, A^+) \subset X$ and every Galois extension $(A, A^+) \to (B, B^+)$ with Galois group G[19] the natural map $\mathcal{F}(\mathrm{Spa}(A, A^+)) \to \mathcal{F}(\mathrm{Spa}(B, B^+))^G$[20] is an isomorphism.

Theorem A.14. *Let X be an analytic adic space. Then an étale presheaf $\mathcal{F}$ of animes on X is an étale sheaf if and only if it is a Nisnevich sheaf and satisfies Galois descent.*

Beweis. This is proved analogously to [Lur18, Theorem B.7.6.1]. □

In the following, we want to investigate the Nisnevich topology of an analytic adic space in more detail.

Definition A.15. Let $f : Y \to X$ be a morphism of analytic adic spaces, and let $y \in Y$, $x = f(y)$. Consider the cartesian diagram

$$\begin{tikzcd} Y \arrow[d, "f"] & U \arrow[l, "\phi"'] \arrow[d, "\psi"] \\ X & \mathrm{Spa}(C, C^+) \arrow[l] \end{tikzcd}$$

where (C, C^+) is a (complete) separably closed affinoid field, with closed point $\bar{x}$ mapping to x. The *degree* $d(x/y)$ is given by the cardinality of the set $\{\tilde{y} \in U | \phi(\tilde{y}) = y, \psi(\tilde{y}) = \bar{x}\}$.

Lemma A.16. *If the morphism f in the situation of Definition* **??** *is étale, then the degree is well-defined. That is, it is independent of (C, C^+) and finite.*

Beweis. We may assume without loss of generality that $f : Y \to X$ is finite étale and X and Y are affinoid. We consider a separable closure $(C, C^+) = (\bar{\kappa}(x), \bar{\kappa}^+(x))$. As one easily checks, the base change U is a finite disjoint union of affinoid spaces of the form $\mathrm{Spa}(C, \tilde{C}^+)$ with $C^+ \subset \tilde{C}^+$: The degree $d(y/x)$ is thus finite. The independence of (C, C^+) follows from the fact that every morphism $\mathrm{Spa}(C, C^+) \to X$ factors through $(\bar{\kappa}(x), \bar{\kappa}^+(x)) \to X$. □

Lemma A.17. *Let*

$$\begin{tikzcd} Y \arrow[d, "f"] & Y' \arrow[l, "g'"'] \arrow[d, "f'"] \\ X & X' \arrow[l, "g"] \end{tikzcd}$$

[19] I.e., the map $A \to B$ is a Galois extension with Galois group G and B^+ is the integral closure of A^+.

[20] Here $(-)^G$ denotes the homotopy fixed points.

be a cartesian diagram with f and g étale. If $x \in X$, $y \in Y$, $x' \in X'$ with $f(y) = x$, $g(x') = x$, then

$$d(y/x) = \sum_{\substack{y' \in (f')^{-1}(x') \\ g'(y') = y}} d(y'/x').$$

Beweis. The formula is derived directly from the definition by a simple calculation. $\square$

Lemma A.18. *Let $f : Y \to X$ be a separated étale morphism of analytic adic spaces. Then there exists a surjective étale map $X' \to X$ such that the morphism $Y \underset{X}{\times} X' \xrightarrow{f'} X'$ is isomorphic to a morphism of the form*

$$\coprod_{k=1}^{n} U_i \xrightarrow{\coprod j_k} X'$$

where $j_k : U_k \hookrightarrow X'$ are quasi-compact open.

Beweis. Without loss of generality, we may assume that X is affinoid and Tate. Due to separatedness, for every pair of points in Y with the same image, there exists an affinoid subset of Y containing both points. Therefore, we may also assume that Y is affinoid. Finally, one can assume that f is finite étale, because we can pass to a smaller open subset of X if necessary.

Let $x \in X$ and consider $\mathrm{Spa}(\bar\kappa(x), \bar\kappa^+(x)) \to X$. One easily sees that the base change of f to $\mathrm{Spa}(\bar\kappa(x), \bar\kappa^+(x))$ has the desired property. Now write $\mathrm{Spa}(\bar\kappa(x), \bar\kappa^+(x))$ as $\varprojlim_{x \in U} U$ for $U \to X$ étale. Then [GR03, Proposition 5.4.53] and the étale analogue of [Mor19, Proposition III.6.3.7] provide an étale neighborhood $U(x) \to X$ of x for which the base change $Y \underset{X}{\times} U(x) \to U(x)$ is of the desired form. Since X is quasi-compact, it is covered by finitely many such neighborhoods. $\square$

Theorem A.19. *Let X be an analytic adic space. An étale cover $\coprod Y_i \to X$ is a Nisnevich cover if and only if for every $x \in X$ there exists a point $y \in \coprod Y_i$ with $d(y/x) = 1$.*

Beweis. Given $x \in X$ and $y \in \coprod Y_i$ with $f(y) = x$. As one easily checks, the condition that there exists a lift of $\mathrm{Spa}(\kappa_h(x), \kappa_h^+(x)) \to X$ mapping the closed point of $\mathrm{Spa}(\kappa_h(x), \kappa_h^+(x))$ to y is equivalent to the existence of a commutative diagram

$$\begin{tikzcd}
(\kappa(x), \kappa^+(x)) \arrow[r] \arrow[d] & (\kappa_h(x), \kappa_h^+(x)) \\
(\kappa(y), \kappa^+(y)) \arrow[ur, dashed, "\psi"] &
\end{tikzcd}$$

We set $(C, C^+) = (\bar\kappa(x), \bar\kappa^+(x))$ and consider the tensor product

$$(\kappa(y), \kappa^+(y)) \underset{(\kappa(x), \kappa^+(x))}{\otimes} (C, C^+)$$

of analytic Huber pairs. By direct calculation, one verifies that it is isomorphic to a finite direct product of Huber pairs of the form $(C, \tilde{C}^+)$ with $C^+ \subset \tilde{C}^+$. We now claim that the condition above is in turn equivalent to this tensor product containing exactly one copy of (C, C^+).

Assume there exists a commutative diagram as above. One immediately sees that

$$(\kappa(y), \kappa^+(y)) \underset{(\kappa(x), \kappa^+(x))}{\otimes} (\kappa_h(x), \kappa_h^+(x)) \cong \prod_{\sigma \in G} (\kappa_h(x), \kappa_h^+(x) + \sigma(\kappa^+(y))),$$

where G denotes the set of $\kappa(x)$-embeddings of $\kappa(y)$ into $\kappa_h(x)$. We claim that the image of $\kappa^+(y)$ under every $\sigma \in G \setminus \{\psi\}$ is not entirely contained in $\kappa_h^+(x)$. Indeed, if there exists a $\sigma \in G \setminus \{\psi\}$ with $\sigma(\kappa^+(y)) \subset \kappa_h^+(x)$, then one can construct a non-trivial continuous automorphism of $(\kappa_h(x), \kappa_h^+(x))$, which contradicts the universal property of henselization.

Now let $\kappa(y)$ be a separable extension of $\kappa(x)$ that does not lie entirely in $\kappa_h(x)$. Here we have implicitly chosen an embedding of $(\kappa(y), \kappa^+(y))$ into (C, C^+). We set

$$(\kappa_{h,x}(y), \kappa_{h,x}^+(y)) = (\kappa(y), \kappa^+(y)) \cap (\kappa_h(x), \kappa_h^+(x))$$

and write $\kappa(y)$ as $\kappa_{h,x}(y)(\alpha)$ for $\alpha \in C \setminus \kappa_h(x)$. By a similar calculation as above, one verifies that

$$(\kappa(y), \kappa^+(y)) \underset{(\kappa(x),\kappa^+(x))}{\otimes} (\kappa_h(x), \kappa_h^+(x)) \cong (\kappa_h(x)(\alpha), \overline{\kappa_h^+(x)}) \times \prod (\kappa_h(x)(\alpha), \overline{\tilde{\kappa}_h^+(x)}),$$

where $\kappa_h^+(x) \subsetneq \tilde{\kappa_h}^+(x)$ and $\overline{(-)}$ denotes the integral closure. Since the valuation $\nu : \kappa_h(x) \to \Gamma \sqcup \{0\}$ corresponding to the ring $\kappa_h^+(x)$ has a unique extension to C, one easily checks that

$$(\kappa_h(x)(\alpha), \overline{\kappa_h^+(x)}) \underset{(\kappa_h(x),\kappa_h^+(x))}{\otimes} (C, C^+) \cong \prod_{i=1}^{n} (C, C^+), \text{ where } n = [\kappa(y) : \kappa_{h,x}(y)].$$

□

Definition A.20. Let $f : Y \to X$ be an étale morphism of analytic adic spaces and W an arbitrary subset of Y. We say $f|_W$ is *of degree* 1 if $d(y/f(y)) = 1$ for all $y \in W$.

Theorem A.21. *Let $f : Y \to X$ be an étale morphism of analytic adic spaces. If $x \in X$, $y \in Y$ with $f(y) = x$ and $d(y/x) = 1$, then there exist constructible subsets $x \in K \subset X$ and $y \in L \subset Y$ with $\phi(L) \subset K$, such that $f|_L : L \to K$ is a homeomorphism of degree* 1.

Beweis. After shrinking Y, we may assume that $f^{-1}(x) = y$. By the proof of Lemma **??**, there exists a quasi-compact étale neighborhood $g : X' \to X$ of x with $g^{-1}(x) = \{x'\}$, such that the base change

$$\begin{array}{ccc} Y & \xleftarrow{g'} & Y' \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} \\ X & \xleftarrow[g]{} & X' \end{array}$$

is a finite union of quasi-compact open immersions. We then write Y' as $\coprod_{i=1}^{n} U_i$ for $U_i \subset X'$ quasi-compact open. From Lemma **??**, it follows that $x' \in X'$ has exactly one preimage $y' \in Y'$. [Translator's note: The original German text says $y' \in Y$, which seems incorrect based on the diagram and Lemma 4.14; it likely means $y \in Y$ has a unique preimage $y' \in Y'$ under g' such that $f'(y') = x'$, or that $x' \in X'$ has a unique preimage $y' \in Y'$ under f' such that $g'(y') = y$. Assuming the latter interpretation fits the context $d(y/x) = 1$.] Without loss of generality, we may assume $y' \in U_1$. [Translator's note: The original German text says $y \in U_1$, which seems incorrect as $y \in Y$ and $U_1 \subset Y'$. Assuming y' is the unique preimage mentioned before.] The subset $g'(U_i)$ is open for each $i = 1, \dots, n$ by Lemma **??**. It is also quasi-compact, because the image of a quasi-compact subset is also quasi-compact. We now set $K = g(U_1) \setminus g(\bigcup_{i=2}^{n} U_i)$ and $L = f^{-1}(K)$. One sees immediately that $L \subset g'(U_1) \setminus g'(\bigcup_{i=2}^{n} U_i)$. We claim that $f|_L : L \to K$ is a homeomorphism of degree 1. The map is obviously surjective. Let $\tilde{x} \in K$ and let $\tilde{y}_1, \tilde{y}_2 \in L$ be its preimages under f. [Translator's note: Corrected Y to L based on context.] As one easily verifies, for example using the universal property of the fiber product, for every $\tilde{x}' \in X'$ with $g(\tilde{x}') = \tilde{x}$ there exist points $\tilde{y}_1', \tilde{y}_2' \in Y'$ with $f'(\tilde{y}_1') = f'(\tilde{y}_2') = \tilde{x}'$ and $g'(\tilde{y}_1') = \tilde{y}_1, g'(\tilde{y}_2') = \tilde{y}_2$. Since $\tilde{y}_1', \tilde{y}_2'$ lie in $U_1 \subset Y'$ and $f'|_{U_1} : U_1 \to X'$ is injective [Translator's note: Corrected $U_1 \to X$ to $f'|_{U_1} : U_1 \to X'$ based on U_1 being an open immersion into X' via f'], $\tilde{y}_1'$ and $\tilde{y}_2'$ are equal, hence also $\tilde{y}_1$ and $\tilde{y}_2$. From Lemma **??** it further follows that the map $f|_L : L \to K$ is of degree 1, because every point $x' \in U_1 \setminus \bigcup_{i=2}^{n} U_i$ has $d(y'/x') = 1$ for its unique preimage $y' \in Y'$ under f'. [Translator's note: Reinterpreted the confusing German sentence "denn jeder Punkt in $U_1 \setminus \bigcup_{i=2}^{n} U_i$ besitzt genau ein Urbild in Y'"based on the context and Lemma 4.14.] It thus remains to check that the inverse map is continuous. This is clear, because $f^{-1}(K) = L$ and f is open. □

Theorem A.22. *Let $f : Y \to X$ be a Nisnevich cover. Then there exists a finite sequence*

$$\emptyset = Z_n \subset Z_{n-1} \subset \cdots \subset Z_1 \subset Z_0 = X$$

of closed subsets of X with quasi-compact complements such that for each $i \in \{0, \dots, n-1\}$ the restriction

$$f : \tilde{Z}_i \setminus \tilde{Z}_{i+1} \to Z_i \setminus Z_{i+1}$$

is splitting, where $\tilde{Z}_i$ denotes the preimage $f^{-1}(Z_i)$. That is, there exists an open quasi-compact subset $Z_i' \subset \tilde{Z}_i \setminus \tilde{Z}_{i+1}$ (in the topology of $\tilde{Z}_i \setminus \tilde{Z}_{i+1}$) such that $f : Z_i' \to Z_i \setminus Z_{i+1}$ is a homeomorphism of degree 1.

Beweis. By Theorem **??**, every point $x \in X$ has a constructible neighborhood $K = U \cap W$ with U quasi-compact open and W closed with quasi-compact complement, such that there exists a constructible subset $L \subset Y$ with $f(L) \subset K$ for which the restriction $f|_L : L \to K$ is a homeomorphism of degree 1. In the proof of Theorem **??**, we constructed L within a quasi-compact open subset Y' of Y for which $f^{-1}(K) \cap Y' = L$ holds. Therefore, we may assume that L is quasi-compact open in the topology of $f^{-1}(K)$.

Since spectral spaces are quasi-compact with respect to the constructible topology, X is covered by finitely many subsets $K_1 = U_1 \cap W_1, \dots, K_n = U_n \cap W_n$ of this form. We now prove the theorem by induction on n. The case $n = 1$ is trivial. Assume the claim holds for $n - 1$. Consider

$$Y \setminus f^{-1}(W_1) \xrightarrow{f} X \setminus W_1 \text{ and } f^{-1}(X \setminus U_1) \xrightarrow{f} X \setminus U_1.$$

By the induction hypothesis, there exist sequences

$$\emptyset = S_n \subset S_{n-1} \subset \cdots \subset S_1 \subset S_0 = X \setminus W_1,$$

$$\emptyset = R_m \subset R_{m-1} \subset \cdots \subset R_1 \subset R_0 = X \setminus U_1$$

of closed subsets of $X \setminus W_1$ and $X \setminus U_1$ respectively, with quasi-compact complements having the desired property. Since $K_1 = U_1 \cap W_1 = W_1 \setminus (X \setminus U_1)$ possesses a desired splitting [Translator's note: Corrected $K_1 = W_1 \setminus (X \setminus U_1)$ to $K_1 = U_1 \cap W_1$ based on definition], this yields the sequences

$$W_1 \subset W_1 \cup S_{n-1} \subset \cdots \subset W_1 \cup S_1 \subset W_1 \cup S_0 = X,$$

$$\emptyset = R_m \subset R_{m-1} \subset \cdots \subset R_1 \subset R_0 = X \setminus U_1 \subset W_1,$$

which have the desired property. [Translator's note: The logic combining the sequences seems slightly off in the German text. The idea is likely to refine the stratification using K_1.] □

Definition A.23. Let X be an analytic adic space. A Nisnevich cover

$$\{i : U \to X,\ f : V \to X\}$$

is called *elementary Nisnevich* if i is an open immersion and $f^{-1}(X \setminus U) \to X \setminus U$ is a homeomorphism of degree 1.

Theorem A.24. *Let X be an analytic adic space. Then the Nisnevich topology is generated by elementary Nisnevich covers.*

Beweis. Verbatim the same proof as in the situation of [MV99, Proposition 1.4]. □

Definition A.25. Let X be an analytic adic space. A cartesian diagram of analytic adic spaces

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & V \arrow[d] \\ U \arrow[r] & X \end{tikzcd}$$

is called an *elementary Nisnevich square* if $\{U \to X,\ V \to X\}$ forms an elementary Nisnevich cover.

Let X be an analytic adic space. Then a presheaf $\mathcal{F}$ of animes or spectra on the Nisnevich site Nis_X is a sheaf if and only if it maps elementary Nisnevich squares to cartesian diagrams and $\mathcal{F}(\varnothing) = *$ holds.

Beweis. This follows from a theorem of Voevodsky, see [AHW17, Theorem 3.2.5]. □

B Sheaves and Hypersheaves

In this section of the appendix, we investigate the relationship between sheaves and hypersheaves of anima resp. spectra on the Zariski, Nisnevich, and étale sites of an analytic adic space. It is one of our main goals to find necessary and sufficient conditions that a hypersheaf on the Nisnevich site must satisfy to be an étale hypersheaf. In doing so, we follow the approach of Clausen and Mathew (see [CM21]) in the case of schemes, which can be transferred one-to-one to the situation of analytic adic spaces. First, we summarize generally valid definitions and techniques from Sections 2 and 3 of [CM21], which we will then apply in our situation.

Notations and Terminology.

(i) By a *site* we mean a 1-category together with a Grothendieck topology. We assume that all sites contain a final object.

(ii) Let $\mathcal{T}$ be a site. We denote by $\mathrm{Sh}(\mathcal{T})$ (resp. $\mathrm{Sh}(\mathcal{T},\mathrm{Sp})$) the ∞-category of sheaves of anima (resp. spectra) on $\mathcal{T}$.

(iii) Let X be a scheme or an analytic adic space. We denote by X_{Zar} (resp. X_{Nis} resp. $X_{\text{ét}}$) the Zariski (resp. Nisnevich resp. étale) ∞-topos of X.

(iv) Let $\mathcal{P}$ be a fixed subset of prime numbers. A spectrum X is called *$\mathcal{P}$-local* if multiplication by every prime number $q \notin \mathcal{P}$ is invertible on X. The full ∞-subcategory of $\mathcal{P}$-local spectra is denoted by $\mathrm{Sp}_{\mathcal{P}}$. In the case $\mathcal{P} = \{\text{all prime numbers}\}$, the condition is empty.

(v) Let k be a field. For a prime number p, we denote by $\mathrm{cd}_p(k)$ (resp. $\mathrm{vcd}_p(k)$) the Galois p-local cohomological (resp. virtual cohomological) dimension of k. The *Galois $\mathcal{P}$-local cohomological* (resp. *virtual cohomological*) *dimension of k* is defined as $\sup_{p\in\mathcal{P}} \mathrm{cd}_p(k)$ (resp. $\sup_{p\in\mathcal{P}} \mathrm{vcd}_p(k)$).

We first recall the definitions of hyper- and Postnikov completeness, which underlie the entire section.

Definition B.1 ([CM21, Definition 2.4])**.** Let $\mathcal{T}$ be a site and $\mathcal{F}$ a sheaf of anima (resp. spectra) on $\mathcal{T}$. We call $\mathcal{F}$

(i) *acyclic* if $\pi_n(\mathcal{F}) = 0$ for all $n \in \mathbb{N}$ (resp. $n \in \mathbb{Z}$);

(ii) *hypercomplete* or a *hypersheaf* if $\mathrm{Hom}_{\mathrm{Sh}(\mathcal{T})}(\mathcal{G},\mathcal{F}) = 0$ resp. $\mathrm{Hom}_{\mathrm{Sh}(\mathcal{T},\mathrm{Sp})}(\mathcal{G},\mathcal{F}) = 0$ for all acyclic sheaves $\mathcal{G}$ on $\mathcal{T}$;

(iii) *Postnikov complete* if the morphism $\mathcal{F} \to \varprojlim_n \tau_{\leq n}\mathcal{F}$ is an isomorphism.

The category $\mathrm{Sh}(\mathcal{T})$ is called *hypercomplete* if every sheaf of anima on $\mathcal{T}$ is hypercomplete.

Definition **??**(ii) is equivalent to the usual definition via hypercovers. Furthermore, a sheaf of spectra is hypercomplete if and only if its underlying sheaf of anima is hypercomplete. For proofs of both statements, we refer to [CM21, Example 2.5] and the references cited therein.

Lemma B.2 ([CM21, Lemma 2.6])**.**

(i) The full subcategory of hypercomplete sheaves of anima resp. spectra is closed under limits.

(ii) An upward bounded[21] *sheaf of anima resp. spectra is hypercomplete.*

(iii) A Postnikov complete sheaf is hypercomplete.

Although the concepts of hyper- and Postnikov completeness are generally not equivalent (see e.g. [MV99, Example 1.30]), they coincide according to [MR22] for a large class of ∞-topoi. In this work, however, we manage without the results of [MR22] and use a classical criterion based on cohomological dimension.

Definition B.3 ([CM21, Definition 2.8])**.** Let $\mathcal{T}$ be a site and $\mathcal{F}$ a sheaf of connective $\mathcal{P}$-local spectra on $\mathcal{T}$.

[21] I.e., the homotopy groups π_i vanish for all i above a certain number.

(i) Let $\mathcal{A}$ be a sheaf of abelian groups on $\mathcal{T}$. The *i-th cohomology group of $\mathcal{F}$ with coefficients in $\mathcal{A}$* is given by

$$H^i(\mathcal{F};\mathcal{A}) = \pi_0 \mathrm{Hom}_{\mathrm{Sh}(\mathcal{T},\mathrm{Sp})}(\mathcal{F}, \Sigma^i \mathcal{A}).$$

(ii) The sheaf $\mathcal{F}$ is said to be *of cohomological dimension $\leq d$* if $H^i(\mathcal{F};\mathcal{A}) = 0$ for every sheaf of $\mathcal{P}$-local abelian groups $\mathcal{A}$ and every $i \geq d$.

(iii) We say that $\mathrm{Sh}(\mathcal{T}, \mathrm{Sp}_{\mathcal{P},\geq 0})$ *has enough objects of $\mathcal{P}$-local cohomological dimension $\leq d$* if for every $\mathcal{G} \in \mathrm{Sh}(\mathcal{T}, \mathrm{Sp}_{\mathcal{P},\geq 0})$ there exists a morphism $f : \mathcal{H} \to \mathcal{G}$ in $\mathrm{Sh}(\mathcal{T}, \mathrm{Sp}_{\mathcal{P},\geq 0})$ such that $\pi_0(f)$ is an epimorphism and $\mathcal{H}$ is of $\mathcal{P}$-local cohomological dimension $\leq d$.

Theorem B.4 ([CM21, Proposition 2.10])**.** *Let $\mathcal{T}$ be a site and $\mathcal{F} \in \mathrm{Sh}(\mathcal{T}, \mathrm{Sp}_{\mathcal{P}})$. Assume that $\mathrm{Sh}(\mathcal{T}, \mathrm{Sp}_{\mathcal{P},\geq 0})$ has enough objects of $\mathcal{P}$-local cohomological dimension $\leq d$. Then $\mathcal{F}$ is hypercomplete if and only if it is Postnikov complete.*

These completeness concepts play a special role in the course of this work. The most important properties of these concepts for our purposes are summarized in the following two theorems.

Theorem B.5. *Let $\mathcal{T}$ be a site.*

(i) A morphism $\mathcal{F} \to \mathcal{G}$ of hypercomplete sheaves of anima resp. spectra is an isomorphism if and only if the induced morphism $\pi_i(\mathcal{F}) \to \pi_i(\mathcal{G})$ between the homotopy groups is an isomorphism for all $i \in \mathbb{N}$ resp. $i \in \mathbb{Z}$.

(ii) Assume that the 1-topos of sheaves of sets on $\mathcal{T}$ has enough points. Let x be a point of $\mathrm{Sh}(\mathcal{T}, \mathrm{Set})$. Then for a sheaf $\mathcal{F}$ of anima resp. spectra

$$(\pi_i(\mathcal{F}))_x \cong \pi_i(\mathcal{F}_x).$$

In particular, a morphism of hypercomplete sheaves is an isomorphism if and only if it is an isomorphism stalkwise.

Beweis. The first part follows directly from the definition. For the second part, note that the truncation functors $\tau_{\leq n}$ and $\tau_{\geq n}$ commute with pullback for every n. □

Theorem B.6 ([CM21, Proposition 2.13])**.** *Let $\mathcal{T}$ be a site. Let $\mathcal{F}$ resp. $\mathcal{G}$ be a sheaf of spectra resp. connective spectra. Then there exists a conditionally convergent spectral sequence*

$$E_2^{p,q} = H^p(\mathcal{G};\pi_q(\mathcal{F})) \implies \pi_{q-p}\mathrm{Hom}_{\mathrm{Sh}(\mathcal{T},\mathrm{Sp})}(\mathcal{G}, \varprojlim_n \tau_{\leq n}\mathcal{F}).$$

Hyper- and Postnikov completeness are local properties in the following sense:

Lemma B.7 (Local-Global Principle for Hyper- and Postnikov Completeness, [CM21, Proposition 2.25])**.** *Let $\mathcal{T}$ be a site and $\{X_i\}_{i\in I}$ a cover of the final object of $\mathcal{T}$. Then a sheaf $\mathcal{F}$ of anima or spectra is hyper- resp. Postnikov complete if and only if its restriction to the site $\mathcal{T}_{/X_i}$ of objects over X_i is hyper- resp. Postnikov complete for every $i \in I$.*

Let $\mathcal{T}$ be a site. Let $\mathcal{C}$ (resp. $\mathcal{C}^h$) denote the ∞-category of sheaves of $\mathcal{P}$-local spectra on $\mathcal{T}$ (resp. the full ∞-subcategory of hypercomplete sheaves of $\mathcal{P}$-local spectra on $\mathcal{T}$). Then $\mathcal{C}^h$ is a left Bousfield localization of $\mathcal{C}$, see [CM21, Proposition 2.14]. I.e., the natural embedding functor $\mathcal{C}^h \to \mathcal{C}$ possesses an accessible left adjoint functor $\mathcal{C} \to \mathcal{C}^h$, which we call the *hypercompletion functor*. Of particular importance is the following case, in which the full ∞-subcategory $\mathcal{C}^h$ is also closed under colimits.

Lemma B.8 ([CM21, Lemma 2.23])**.** *Let the situation be as above. Let 1^h denote the hypercompletion of the monoidal unit of $\mathcal{C}$. Then the following conditions are equivalent:*

(i) The full ∞-subcategory $\mathcal{C}^h \subset \mathcal{C}$ is closed under colimits and the tensor product functor $-\otimes X$ for every $X \in \mathcal{C}$.

(ii) For every $X \in \mathcal{C}$, the object $1^h \otimes X$ is hypercomplete.

(iii) For every $X \in \mathcal{C}$, the object $1^h \otimes X$ is isomorphic to the hypercompletion of X.

(iv) The forgetful functor $\mathrm{Mod}_{1^h}(\mathcal{C}) \to \mathcal{C}$ is fully faithful with essential image $\mathcal{C}^h$.

(v) *Every object $X \in \mathcal{C}$ which possesses the structure of a module over an algebra $A \in \mathrm{Alg}(\mathcal{C}^h)$ is hypercomplete.*

Definition B.9 ([CM21, Definition 2.24])**.** Let $\mathcal{T}$ be a site. If one of the equivalent conditions from Lemma **??** is satisfied, we call the hypercompletion functor a *tensor localization.*

One of the most important results of [CM21] for us is a precise analysis of hypercompleteness in terms of diagrams. We first recall the following auxiliary definitions.

Definition B.10 ([CM21, Definition 2.29])**.** Let $\mathcal{T}$ be a site, closed under finite limits. $\mathcal{T}$ is called *finitistic* if every cover in $\mathcal{T}$ possesses a finite subcover.

Definition B.11 ([CM21, Definition 2.30])**.** Let $\mathcal{T}$ be a finitistic site. $\mathcal{T}$ is said to be *of $\mathcal{P}$-local cohomological dimension* $\leq d$ if the $\mathcal{P}$-localization of the representable sheaf $\Sigma^\infty_+ h_x$[22] is of $\mathcal{P}$-local cohomological dimension $\leq d$ for every $x \in \mathcal{T}$.

Definition B.12 ([CM21, Definition 2.33])**.** Let $m \geq 0$ be a natural number.

(i) A filtered spectrum $\cdots \to X_{-1} \to X_0 \to X_1 \to \ldots$ is called *m-nilpotent* (resp. *weakly m-nilpotent*) if for every $i \in \mathbb{Z}$ the map $X_i \to X_{i+m+1}$ is null-homotopic (resp. the induced map on homotopy groups is trivial).

(ii) We call an augmented cosimplicial spectrum $X^\bullet \in \mathrm{Fun}(\Delta^+, \mathrm{Sp})$ *m-rapidly convergent* (resp. *weakly m-rapidly convergent*) if the tower

$$\{\mathrm{cofib}(X^{-1} \to \mathrm{Tot_n(X^\bullet)})\}_n$$

is m-nilpotent (resp. weakly m-nilpotent).

The following theorem explains the relationship between nilpotence and hypercompleteness and will help us later in the investigation of étale hypersheaves.

Theorem B.13 ([CM21, Proposition 2.35])**.** *Let $\mathcal{T}$ be a finitistic site of $\mathcal{P}$-local cohomological dimension $\leq d$. For a sheaf of $\mathcal{P}$-local spectra $\mathcal{F}$ on $\mathcal{T}$, the following are equivalent:*

(i) $\mathcal{F}$ is hypercomplete (or equivalently, Postnikov complete).

(ii) For every truncated hypercover $y_\bullet$ in $\mathcal{T}$ of an object $x \in \mathcal{T}$, the augmented cosimplicial spectrum

$$\mathcal{F}(x) \to \mathcal{F}(y_\bullet)$$

is d-rapidly convergent.

As a further application of the concept of nilpotence, Clausen and Mathew give an explicit condition for the commutation of filtered colimits and totalizations:

Lemma B.14 ([CM21, Lemma 2.34])**.** *Let $(X_i^\bullet)_{i\in I}$ be a filtered system of augmented cosimplicial spectra, where I is a partially ordered set. Assume there exists an $m \geq 0$ such that $X_i^\bullet$ is weakly m-rapidly convergent for every $i \in I$. Then the augmented cosimplicial spectrum $\operatorname{colim}_{i\in I} X_i^\bullet$ is weakly m-rapidly convergent and the morphism*

$$\operatorname{colim}_{i\in I} \mathrm{Tot}(X_i^\bullet) \to \mathrm{Tot}(\operatorname{colim}_{i\in I} X_i^\bullet)$$

is an isomorphism.

Our first goal is to show that the Nisnevich site of a finite-dimensional analytic adic space is hypercomplete. In fact, we show more: The Nisnevich site of such a space has finite *homotopy dimension.*

Definition B.15 ([Lur09, Proposition 6.5.1.12, Definition 7.2.1.1])**.** Let $\mathcal{T}$ be a site and $n \geq -1$.

(i) A sheaf of anima $\mathcal{F} \in \mathrm{Sh}(\mathcal{T})$ is called *n-connective* if the truncation $\tau_{n-1}\mathcal{F}$ is a final object of $\mathrm{Sh}(\mathcal{T})$.

(ii) $\mathrm{Sh}(\mathcal{T})$ is said to be *of homotopy dimension* $\leq n$ if every n-connective sheaf $\mathcal{F} \in \mathrm{Sh}(\mathcal{T})$ admits a morphism $* \to \mathcal{F}$ from the final object.

[22] i.e., the $\mathcal{P}$-localization of the sheafification of the naive presheaf $y \mapsto \Sigma^\infty_+ \mathrm{Hom}_{\mathcal{T}}(y, x)$

We recall the following important criterion for hypercompleteness:

Theorem B.16 ([Lur09, Corollaries 7.2.1.12 and 7.2.2.30]). *Let $\mathcal{T}$ be a site. If the ∞-category $\mathrm{Sh}(\mathcal{T})$ is of homotopy dimension $\leq n$, then it is hypercomplete and of cohomological dimension $\leq n$. In particular, every sheaf on $\mathcal{T}$ is Postnikov complete.*

In our proofs, we use the hypercompleteness of the given sheaves and reduce the desired theorems to certain statements at the level of stalks. In general, we can extend a presheaf on the category of étale Huber pairs over an analytic Huber pair (A, A^+) by applying Kan extension to the category of *ind-étale* Huber pairs over (A, A^+).

Definition B.17 ([SW20, Definition 8.2.1]). A morphism $f : (A, A^+) \to (B, B^+)$ of analytic Huber pairs is called *ind-étale* if (B, B^+) is isomorphic to the completion of the colimit $(\operatorname{colim}(A_i, A_i^+))^\wedge$ of a filtered system of étale Huber pairs over (A, A^+).

[cf. [CM21, Construction 4.30]] Let $X = \mathrm{Spa}(A, A^+)$ be an affinoid analytic adic space and $\mathcal{F}$ a presheaf on the category of étale spaces over X. For an ind-étale Huber pair $(B, B^+) \cong (\operatorname{colim}(A_i, A_i^+))^\wedge$, we set

$$\mathcal{F}(B, B^+) = \operatorname{colim} \mathcal{F}(A_i, A_i^+).$$

Strictly speaking, the values of the extension given above are only defined up to isomorphism. In the "correct"definition, the system $(A_i, A_i^+)_{i\in I}$ is replaced by the system of all étale Huber pairs over (A, A^+) with a map to (B, B^+).

Lemma B.18. *Let $(B, B^+) = (\operatorname{colim}(A_i, A_i^+))^\wedge$ be an ind-étale Huber pair over a Tate Huber pair (A, A^+). Then Construction **??** defines for every sheaf of anima (resp. spectra) on the Nisnevich site of $\mathrm{Spa}(A, A^+)$ a Nisnevich sheaf of anima (resp. spectra) on $\mathrm{Spa}(B, B^+)$. In other words, no sheafification is needed in the construction of the pullback along $\mathrm{Spa}(B, B^+) \to \mathrm{Spa}(A, A^+)$.*

Beweis. By Corollary **??**, it suffices to check that the extension sends elementary Nisnevich squares to cartesian diagrams. From [GR03, Proposition 5.4.53] and the proof of [Mor19, Proposition III.6.3.7], it follows that the category of étale Huber pairs over (B, B^+) and the category of étale Huber pairs over $\operatorname{colim}(A_i, A_i^+)$ are isomorphic. As one easily checks, every elementary Nisnevich square for $\operatorname{colim}(A_i, A_i^+)$ is induced by such a square for some (A_i, A_i^+). $\square$

As in the schematic case, the étale analogue of Lemma **??** does not hold.

As a final preparation before we turn to the first proof, we recall the following important result from [CM21]:

Theorem B.19 ([CM21, Theorems 3.14 and 3.30]). *Let X be a quasi-compact quasi-separated scheme of finite Krull dimension. For a point $x \in X$, denote by i_x resp. ι_x the canonical map $\operatorname{Spec} \mathcal{O}_{X,x} \to X$ resp. $\operatorname{Spec} \mathcal{O}^h_{X,x} \to X$. Here $\mathcal{O}^h_{X,x}$ denotes the henselization of $\mathcal{O}_{X,x}$. Let $n \geq 0$ be a natural number.*

(i) Let $\mathcal{F}$ be a sheaf of anima on the Zariski site of X. If the Zariski stalk

$$\Gamma(\operatorname{Spec} \mathcal{O}_{X,x}, (i_x)^*_{\mathrm{Zar}}\mathcal{F})$$

is $(\dim\overline{\{x\}} + n)$-connective for every point $x \in X$, then $\mathcal{F}(X)$ is n-connective.

(ii) Let $\mathcal{F}$ be a sheaf of anima on the Nisnevich site of X. If the Nisnevich stalk

$$\Gamma(\operatorname{Spec} \mathcal{O}^h_{X,x}, (\iota_x)^*_{Nis}\mathcal{F})$$

is $(\dim\overline{\{x\}} + n)$-connective for every point $x \in X$, then $\mathcal{F}(X)$ is n-connective.

In particular, the homotopy dimension of the Zariski resp. Nisnevich site is less than or equal to $\dim X$.

And here now is the first promised proof. Since every spectral topological space is isomorphic to the spectrum of a commutative ring, the first part of the theorem also holds for quasi-compact quasi-separated analytic adic spaces. In fact, the second part also holds in this situation, as the following theorem shows:

Theorem B.20. *Let X be a quasi-compact quasi-separated analytic adic space of finite Krull dimension and $\mathcal{F}$ a sheaf of anima on the Nisnevich site of X. We denote by ι_x the canonical map $(\kappa_h(x), \kappa_h^+(x)) \to X$ for the henselization of the residue field of X at x. Let $n \geq 0$ be a natural number. If the Nisnevich stalk*

$$\Gamma(\mathrm{Spa}(\kappa_h(x), \kappa_h^+(x)), (\iota_x)^*_{Nis}\mathcal{F})$$

is $(\dim\overline{\{x\}} + n)$-connective for every point $x \in X$, then $\mathcal{F}(X)$ is n-connective.[23] *In particular, the homotopy dimension is less than or equal to $\dim X$.*

Beweis. We prove the theorem by induction on $\dim X$. The case $\dim X = -1$ is trivial. Now assume the claim holds for $\dim X - 1$. By Theorem **??**, it suffices to show that the Zariski stalk of $\mathcal{F}$ at x is $(\dim\overline{\{x\}} + n)$-connective for every point $x \in X$. Let $j : U \hookrightarrow X$ be a quasi-compact open neighborhood of x. Consider the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Spa}(\kappa_h(x), \kappa_h^+(x))_{\mathrm{Nis}} \arrow[r] \arrow[d] & \mathrm{Spa}(\kappa(x), \kappa^+(x))_{\mathrm{Nis}} \arrow[r, "i_{\mathrm{Nis}}"] \arrow[d] & U_{\mathrm{Nis}} \arrow[r, "j_{\mathrm{Nis}}"] \arrow[d, "\beta"] & X_{\mathrm{Nis}} \arrow[d, "\alpha"] \\
\mathrm{Spa}(\kappa_h(x), \kappa_h^+(x))_{\mathrm{Zar}} \arrow[r] & \mathrm{Spa}(\kappa(x), \kappa^+(x))_{\mathrm{Zar}} \arrow[r, "i_{\mathrm{Zar}}"] & U_{\mathrm{Zar}} \arrow[r, "j_{\mathrm{Zar}}"] & X_{\mathrm{Zar}}
\end{tikzcd}$$

Here the vertical arrows represent the canonical morphisms of corresponding ∞-topoi, which map a Nisnevich sheaf to the underlying Zariski sheaf. With the notation of this diagram, the connectivity we want to prove is apparently equivalent to the anima

$$\Gamma(\mathrm{Spa}(\kappa(x), \kappa^+(x)), (j \circ i)^*_{\mathrm{Zar}}\alpha_*(\mathcal{F}))$$

being $(\dim\overline{\{x\}} + k)$-connective. One sees immediately that the natural transformation $j^*_{\mathrm{Zar}}\alpha_* \to \beta_* j^*_{\mathrm{Nis}}$ is an isomorphism. Since $\mathrm{Spa}(\kappa(x), \kappa^+(x)) \cong \varprojlim_{x \in U} U$, where U runs through the open neighborhoods of x, it suffices, due to the previous claim and Lemma **??**, to show the following: Let (K, K^+) be an affinoid field of finite Krull dimension and $\mathcal{F}$ a sheaf on its Nisnevich site. If the conditions of the theorem are satisfied, then the anima $\mathcal{F}(\mathrm{Spa}(K, K^+))$ is n-connective. In other words, it suffices to prove the theorem in the case $X = \mathrm{Spa}(K, K^+)$.

We now consider the preceding case. We denote by x the closed point of $\mathrm{Spa}(K, K^+)$. By assumption, there exists a Nisnevich neighborhood of x[24] of the form $\mathrm{Spa}(\tilde{K}, \tilde{K}^+)$ for an affinoid field $(\tilde{K}, \tilde{K}^+)$, such that the anima $\mathcal{F}(\mathrm{Spa}(\tilde{K}, \tilde{K}^+))$ is non-empty. Moreover, the anima $\mathcal{F}(\mathrm{Spa}(K, K^+) \setminus \{x\})$ and $\mathcal{F}(\mathrm{Spa}(\tilde{K}, \tilde{K}^+) \underset{\mathrm{Spa}(K,K^+)}{\times} (\mathrm{Spa}(K, K^+) \setminus \{x\}))$ are $n+1$-connective by the induction hypothesis. Since the pair

$$\{\mathrm{Spa}(K, K^+) \setminus \{x\}, \mathrm{Spa}(\tilde{K}, \tilde{K}^+)\}$$

forms an elementary Nisnevich cover, the diagram

$$\begin{tikzcd}
\mathcal{F}(\mathrm{Spa}(K, K^+)) \arrow[r] \arrow[d] & \mathcal{F}(\mathrm{Spa}(K, K^+) \setminus \{x\}) \arrow[d] \\
\mathcal{F}(\mathrm{Spa}(\tilde{K}, \tilde{K}^+)) \arrow[r] & \mathcal{F}(\mathrm{Spa}(\tilde{K}, \tilde{K}^+) \underset{\mathrm{Spa}(K,K^+)}{\times} (\mathrm{Spa}(K, K^+) \setminus \{x\}))
\end{tikzcd}$$

is cartesian, which is why $\mathcal{F}(\mathrm{Spa}(K, K^+))$ is non-empty. The higher connectivity is checked completely analogously. $\square$

We now begin the analysis of hypercompleteness on the étale site of an analytic adic space X. Our first goal is a hypercompleteness criterion in terms of points of X. We first recall the following definition.

Definition B.21 ([CM21, Definition 4.1])**.** Let G be a profinite group. The underlying category of the *site $\mathcal{T}_G$ of continuous G-sets* is defined as the category of finite sets with continuous G-action. A family $\{M_i \to M\}_{i \in I}$ of morphisms forms a cover if the map $\coprod M_i \to M$ is surjective.

[23] The condition is only imposed on the henselizations of the points of X: In this case, it is automatically satisfied for the other henselian points.

[24] I.e., the degree of the map at the closed point is equal to 1.

For a quasi-compact quasi-separated analytic adic space X and a point $x \in X$, we call the site of continuous $\mathrm{Gal}(\kappa_h(x))$-sets the *Galois site of X at x*. We denote it by $\mathcal{T}_x$. In the terminology of pseudo-adic spaces (see [Hub96, Section 1.10]), the corresponding (∞-)topos is equivalent, according to [Hub96, Proposition 2.3.10], to the étale (∞-)topos of the pseudo-adic space $(\mathrm{Spa}(\kappa(x), \kappa^+(x)), \tilde{x})$, where $\tilde{x}$ denotes the closed point.

The criterion that we will prove below holds for analytic adic spaces for which the étale cohomological dimensions of the spaces in the corresponding étale sites are bounded above. Therefore, we first investigate the relationship between these "global"dimensions and the "local"dimensions of the Galois sites.

Theorem B.22 (cf. [CM21, Corollary 3.29])**.** *Let X be a quasi-compact quasi-separated analytic adic space of dimension d. Let $\mathrm{cd_x}$ denote the $\mathcal{P}$-local cohomological dimension of the Galois site of X at x. Then:*

$$\sup_{x \in X} \mathrm{cd}_x \leq \sup_{U \in \acute{E}t_X} \mathrm{CohDim}_{\mathcal{P}}(U_{\acute{e}t}) \leq d + \sup_{x \in X} \mathrm{cd}_x.$$

Beweis. We first prove the first inequality. Let x be a point of X. As one easily checks, the direct image functor $r_* : \mathcal{T}_x \to \mathrm{Spa}(\kappa_h(x), \kappa_h^+(x))$ is exact, thus

$$\mathrm{cd}_x \leq \mathrm{CohDim}_{\mathcal{P}}(\mathrm{Spa}(\kappa_h(x), \kappa_h^+(x))_{\acute{e}t}).$$

Since the adic space $\mathrm{Spa}(\kappa_h(x), \kappa_h^+(x))$ is isomorphic to the inverse limit of the Nisnevich neighborhoods of x, the statement thus follows from the adic analogue of [Stacks, Tag 03Q4].

The second inequality can be easily derived from [Hub96, Proposition 2.8.1]. □

Let $X = \mathrm{Spa}(A, A^+)$ be an affinoid analytic adic space. By the *affinoid étale site of X* we mean the category of affinoid étale spaces over X. A cover in this site is by definition a family of maps between affinoid étale spaces over X that forms a cover in the usual étale site of X. The following lemma, which is proved in the same way as in [CM21], will allow us in the following to reduce the proof of the desired criterion to the analogue for the affinoid site.

Lemma B.23 (cf. [CM21, Proposition 4.34])**.** *Let X be a quasi-compact quasi-separated analytic adic space and $\mathcal{F}$ a Nisnevich sheaf of anima resp. spectra on X. If one of the following statements holds for every étale map $f : U \to X$, with U affinoid Tate, then its analogue holds for $\mathcal{F}$ on the usual étale site:*

(i) the pullback $f^*\mathcal{F}$ is a sheaf on the affinoid étale site of U;

(ii) the pullback $f^*\mathcal{F}$ is a hypercomplete sheaf on the affinoid étale site of U;

(iii) the pullback $f^*\mathcal{F}$ is a Postnikov complete sheaf on the affinoid étale site of U;

(iv) the pullback $f^*\mathcal{F}$ is isomorphic to the trivial sheaf on the affinoid étale site of U;

As a final step before formulating the criterion, we recall the analysis from [CM21] of hypercompleteness on the site of continuous sets of a profinite group.

Definition B.24 ([CM21, Definition 4.8])**.** Let K be a finite group and X a spectrum with K-action. The K-action is called *weakly m-nilpotent* if the augmented cosimplicial spectrum

$$X^{hK} \to X \rightrightarrows \prod_K X \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} \ldots$$

which computes the homotopy fixed points, is weakly m-rapidly convergent.

Theorem B.25 ([CM21, Proposition 4.16])**.** *Let G be a profinite group of $\mathcal{P}$-local cohomological dimension d. For a sheaf of $\mathcal{P}$-local spectra $\mathcal{F}$ on $\mathcal{T}_G$, the following are equivalent:*

(i) The sheaf $\mathcal{F}$ is hypercomplete.[25]

(ii) There exists an $m \geq 0$ such that for every open normal subgroup $N \subset H$ of an open subgroup of G, the H/N-action on the spectrum $\mathcal{F}(G/N)$ is weakly m-nilpotent.

(iii) For every open normal subgroup $N \subset H$ of an open subgroup of G, the H/N-action on the spectrum $\mathcal{F}(G/N)$ is weakly d-nilpotent.

[25] Resp. Postnikov complete, which in this case is equivalent to the hypercompleteness of $\mathcal{F}$ by Theorem **??**.

Theorem B.26 ([CM21, Proposition 4.17])**.** *Let G be a profinite group of $\mathcal{P}$-local virtual cohomological dimension d. Then the hypercompletion functor for sheaves of $\mathcal{P}$-local spectra on $\mathcal{T}_G$ is a tensor localization.*

Finally, we are ready to formulate and prove the promised criterion:

Theorem B.27 (cf. [CM21, Theorem 4.36])**.** *Let X be a quasi-compact quasi-separated analytic adic space such that the $\mathcal{P}$-local étale cohomological dimensions of $\{U \to X\} \in \acute{\mathrm{E}}\mathrm{t}_X$ are bounded above. If $\mathcal{F}$ is a hypercomplete Nisnevich sheaf of $\mathcal{P}$-local spectra, then it is a hypercomplete étale sheaf if and only if for all points $x \in X$ the Nisnevich pullback along $\mathrm{Spa}(\kappa_h(x), \kappa_h^+(x)) \to X$ defines a hypercomplete sheaf on $\mathcal{T}_x$.*

Beweis. Our proof is identical to the proof of [CM21, Theorem 4.36]. Let d be an upper bound for the $\mathcal{P}$-local étale cohomological dimensions of $\{U \to X\} \in \acute{\mathrm{E}}\mathrm{t}_X$. Assume $\mathcal{F}$ is a hypercomplete étale sheaf. Let $\kappa_h(x) \to k \to k'$ be finite separable field extensions and k^+ (resp. k'^+) the integral closure of $\kappa_h(x)^+$ in k (resp. in k'). We write $(\kappa_h(x), \kappa_h^+(x))$ as $(\operatorname{colim}_U(\mathcal{O}_X(U), \mathcal{O}_X^+(U)))^\wedge$, where U runs through the Nisnevich neighborhoods of x. As one easily checks using [GR03, Proposition 5.4.53], the map $(k, k^+) \to (k', k'^+)$ can be written as a filtered colimit

$$\left(\operatorname*{colim}_U (B_U, B_U^+) \to (B'_U, B'^+_U)\right)^\wedge$$

of fully faithful[26] étale maps of étale Huber pairs over $(\mathcal{O}_X(U), \mathcal{O}_X^+(U))$. By Theorem **??**, the Čech nerve

$$\mathcal{F}(B_U, B_U^+) \to \mathcal{F}(B'_U, B'^+_U) \rightrightarrows \cdots$$

is d-rapidly convergent, which is why the cosimplicial diagram

$$\mathcal{F}(k, k^+) \to \mathcal{F}(k', k'^+) \rightrightarrows \cdots$$

is weakly d-rapidly convergent due to Lemma **??**. The Nisnevich pullback of $\mathcal{F}$ thus defines an étale sheaf on $\mathcal{T}_x$, which is moreover hypercomplete by Theorem **??**.

Conversely, let $\mathcal{F}$ be a Nisnevich sheaf such that its Nisnevich pullback along the map $\mathrm{Spa}(\kappa_h(x), \kappa_h^+(x)) \to X$ defines a hypercomplete sheaf on $\mathcal{T}_x$ for every point $x \in X$. Let $\mathcal{F} \to \tilde{\mathcal{F}}$ be the étale hypercompletion of $\mathcal{F}$. We consider the diagram

$$\begin{tikzcd}
\mathrm{Sh}(\mathcal{T}_x, \mathrm{Sp}) \arrow[r] \arrow[d] & \mathrm{Spa}(\kappa_h(x), \kappa_h^+(x))_{\acute{e}t} \arrow[r, "\iota_{\acute{e}t}"] \arrow[d, "\beta"] & X_{\acute{e}t} \arrow[d, "\alpha"] \\
\mathrm{PSh}_{\prod}(\mathcal{T}_x, \mathrm{Sp}) \arrow[r] & \mathrm{Spa}(\kappa_h(x), \kappa_h^+(x))_{\mathrm{Nis}} \arrow[r, "\iota_{\mathrm{Nis}}"] & X_{\mathrm{Nis}}
\end{tikzcd}$$

where $\mathrm{PSh}_{\prod}(\mathcal{T}_x, \mathrm{Sp})$ denotes the full ∞-subcategory of those presheaves on $\mathcal{T}_x$ which map disjoint sums of $\mathrm{Gal}(\kappa_h(x))$-sets to products. By Theorem **??** and Lemma **??**, it suffices to show that the map

$$\iota_{\mathrm{Nis}}^* \mathcal{F} \to \iota_{\mathrm{Nis}}^* \alpha_* \tilde{\mathcal{F}}$$

is an isomorphism. The Nisnevich sheaf $\iota_{\mathrm{Nis}}^* \mathcal{F}$ (resp. $\iota_{\mathrm{Nis}}^* \alpha_* \tilde{\mathcal{F}}$) defines by assumption (resp. due to the first half of the proof) a hypercomplete sheaf on $\mathcal{T}_x$. Since hypercompletion does not change stalks, they define the same sheaf on $\mathcal{T}_x$. In other words, the map

$$\iota_{\mathrm{Nis}}^* \mathcal{F}(\mathrm{Spa}(k, k^+)) \to \iota_{\mathrm{Nis}}^* \alpha_* \tilde{\mathcal{F}}(\mathrm{Spa}(k, k^+))$$

is an isomorphism for every affinoid field $\mathrm{Spa}(k, k^+)$, where k is a finite separable extension of $\kappa_h(x)$ and k^+ is the integral closure of $\kappa_h(x)^+$ in k. By considering open subsets of $\mathrm{Spa}(\kappa_h(x), \kappa_h^+(x))$[27], we obtain the desired statement for open subsets of such affinoid fields. □

From this, just as in the schematic case, the following corollary can be derived.

[cf. [CM21, Corollary 4.40]] Let X be a quasi-compact quasi-separated analytic adic space of finite Krull dimension such that the virtual $\mathcal{P}$-local cohomological dimensions of the Galois sites of X are bounded above. Then the hypercompletion of $\mathcal{P}$-local spectra is a tensor localization.

[26] I.e., the induced map between adic spectra is surjective.
[27] They correspond to the generalizations of x in X.

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