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SOLUTIONS OF EQUATIONS WITH COEFFICIENTS IN A HENSELIAN RING

By Renée Elkik

The beginning of this article is devoted to the proof of a theorem which generalizes a result due to Tougeron [10] and establishes the existence of solutions to polynomial equations with coefficients in a Henselian ring, based on the existence of approximate solutions, under certain smoothness hypotheses on the system of equations.

The proof proceeds in two steps. In the first part, we consider only complete rings. We then approximate formal solutions by Henselian solutions. In particular, we will demonstrate the following result (corollary of Theorem 2):

Let $(A, \mathfrak{J})$ be a Noetherian Henselian pair, X an affine scheme of finite type over A, assumed to be smooth over Spec A outside the closed set $V(\mathfrak{J})$. Let (E_n) denote the projective system of approximate sections modulo $\mathfrak{J}^n$ of X, let $\hat{E}$ be its limit, and let E be the set of A-sections of X.

Then the projective system (E_n) satisfies a uniform Mittag-Leffler condition and E is dense in $\hat{E}$.

We then apply these results to certain algebraization problems. In the fourth part, in particular, we show that the versal formal deformation of an isolated singularity is algebraizable, and then that there exist versal Henselian deformations of isolated singularities.

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0. Preliminaries

All rings considered in the following are assumed to be Noetherian.

0.1. Henselian Pairs

We begin by recalling some properties of Henselian pairs that we will use.

Definition 0.1. Let A be a ring, $\mathfrak{J}$ an ideal of A, and B an étale A-algebra. We say that B is an **étale neighborhood** of $\mathfrak{J}$ in A [or that Spec B is an étale neighborhood of $V(\mathfrak{J})$] if the induced morphism

$$A/\mathfrak{J} \to B/\mathfrak{J}B$$

is an isomorphism.

Definition 0.2. Let A be a ring, and $\mathfrak{J}$ an ideal contained in the radical of A. We say that the pair $(A, \mathfrak{J})$ is **Henselian** if for every étale A-algebra B that is an étale neighborhood of $\mathfrak{J}$ in A, there exists an A-morphism $B \to A$.

To a pair $(A, \mathfrak{J})$, we universally associate a Henselian pair. First, let's say that given two pairs $(A, \mathfrak{J})$, (B, J), a morphism from A to B is a morphism of pairs if the image of I is contained in J.

Definition 0.3. Let $(A, \mathfrak{J})$ be a pair. We call the **$\mathfrak{J}$-adic Henselization** of A a Henselian pair $(\tilde{A}, \tilde{\mathfrak{J}})$ equipped with a morphism of pairs

$$i : (A, I) \to (\tilde{A}, \tilde{I}),$$

such that for any Henselian pair (B, J) and any morphism of pairs

$$v : (A, I) \to (B, J),$$

there exists a unique morphism

$$\tilde{v} : (\tilde{A}, \tilde{I}) \to (B, J),$$

such that $\tilde{v} \circ i = v$.

All these definitions appear in [7]. The existence of the Henselization is proven and its construction given. The Henselization of a pair (A, I) is obtained as the inductive limit of the étale neighborhoods of I in A. If I is contained in the radical of A, $\tilde{A}$ is faithfully flat over A. It is clear that a separated ring that is complete for the $\mathfrak{J}$-adic topology is Henselian for this topology, and that a ring and its Henselization have the same $\mathfrak{J}$-adic completion.

We will use M. Artin's approximation theorem [1] several times in the last part of this article.

0.2. Some Notations

Consider the following data:

- A, a Noetherian ring;
- B, a finitely generated A-algebra equipped with a presentation

$$B = A[X_1, \dots, X_N]/J, \quad J = (f_1, \dots, f_q), \quad f_i \in A[X_1, \dots, X_N].$$

For any integer p and any multi-index

$$(\alpha) = (\alpha_1, \dots, \alpha_p), \quad 1 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_p \leq q,$$

we will denote by $F_{(\alpha)}$ the ideal of $A[X_1, \dots, X_N]$ generated by the f_{α_i}, $i \in [1, p]$, and by $\Delta_{(\alpha)}$ the ideal of $A[X_1, \dots, X_N]$ generated by the p-th order minors of the Jacobian matrix

$$M_{(\alpha)} = (\partial f_{\alpha_i} / \partial X_j)_{i=1,\dots,p; j=1,\dots,N}.$$

For any multi-index (α), let $K_{(\alpha)}$ be the ideal of $A[X_1, \dots, X_N]$, the conductor of J into $F_{(\alpha)}$:

$$K_{(\alpha)} = \{a \in A[X_1, \dots, X_N] \mid aJ \subseteq F_{(\alpha)}\}.$$

The image in B of the ideal $\sum_{(\alpha),p} K_{(\alpha)} \Delta_{(\alpha)}$ has as its support the singular locus of B over A.

Whenever we consider such data in what follows, H_B will denote an ideal of $A[X_1, \dots, X_N]$ having the property of being contained in $\sum_{(\alpha),p} K_{(\alpha)} \Delta_{(\alpha)}$. Note that this property is stable under base change since $K_{(\alpha)}$ can only increase after base extension. We will use this several times without explicitly mentioning it. Furthermore, an ideal H_B chosen this way is such that $\mathrm{Spec}\,(B) - V(H_B)$ is smooth over $\mathrm{Spec}\,(A)$.

I. The Complete Case

Throughout this part, A denotes a ring that is complete for the topology defined by an ideal $\mathfrak{J}$, and B is a finitely generated A-algebra admitting a presentation

$$B = A[X_1, \dots, X_N]/J, \quad J = (f_1, \dots, f_q), \quad f_i \in A[X_1, \dots, X_N].$$

We begin by treating the case where $\mathfrak{J}$ is principal. The introduction of the ideal I in the following lemma is for a technical reason in the sequel.

Lemma 0.4. *Suppose A is complete for the topology defined by a principal ideal generated by t. Let I be any ideal of A, let Λ be the ideal of A formed by elements annihilated by a power of t, and let k be an integer such that*

$$\Lambda \cap (t^k) = (0) \quad \textit{(Artin-Rees)}.$$

Let (h, n) be a pair of integers such that $n > \sup(2h, h+k)$ and let $\mathbf{a} = (a_1, \ldots, a_N) \in A^N$ be such that

$$H_B(\mathbf{a}) \subset t^h,$$

$$J(\mathbf{a}) \subset t^n I.$$

Then there exists $\mathbf{a}' = (a'_1, \ldots, a'_N) \in A^N$ congruent to $\mathbf{a}$ modulo $t^{n-h}I$ and such that $J(\mathbf{a}') = 0$.

Proof. We proceed by successive approximations, showing that we can find $\mathbf{y} = (y_1, \ldots, y_N) \in A^N$ satisfying

$$\begin{cases} y_i \equiv 0 \pmod{t^{n-h}I}, \quad \forall i \in [1, N], \\ J(\mathbf{a} - \mathbf{y}) \subset t^{2n-2h}I, \end{cases} \tag{1}$$

where $(\mathbf{a} - \mathbf{y})$ denotes the element of A^N, $(a_1 - y_1, a_2 - y_2, \ldots, a_N - y_N)$. The choice of n strictly greater than $2h$ ensures $2n - 2h > n$. We can thus construct a sequence of elements in A^N that converges to $\mathbf{a}'$.

Let M be the Jacobian matrix $(\partial f_i / \partial X_j)_{i=1,\ldots,q; j=1,\ldots,N}$, and let $M(\mathbf{a})$ be its value at $\mathbf{a}$. For any $\mathbf{y} = (y_1, \ldots, y_N) \in A^N$, we can write a Taylor expansion:

$$\begin{pmatrix} f_1(\mathbf{a} - \mathbf{y}) \\ \vdots \\ f_q(\mathbf{a} - \mathbf{y}) \end{pmatrix} = \begin{pmatrix} f_1(\mathbf{a}) \\ \vdots \\ f_q(\mathbf{a}) \end{pmatrix} - M(\mathbf{a}) \begin{pmatrix} y_1 \\ \vdots \\ y_N \end{pmatrix} + \sum_{i,j} y_i y_j Q_{ij}(\mathbf{a}, \mathbf{y}), \tag{2}$$

where Q_{ij} is a q-column vector whose components are polynomials in $\mathbf{y}$ and $\mathbf{a}$. To ensure (1), it suffices to find $(\mathbf{y})$ congruent to 0 modulo $t^{n-h}I$ and such that

$$\begin{pmatrix} f_1(\mathbf{a}) \\ \vdots \\ f_q(\mathbf{a}) \end{pmatrix} \equiv M(\mathbf{a}) \begin{pmatrix} y_1 \\ \vdots \\ y_N \end{pmatrix} \pmod{t^{2n-2h}I}; \tag{3}$$

the terms $y_i y_j Q_{ij}$ from (2) will then all be congruent to 0 modulo $t^{2n-2h}I$ and (1) will be satisfied.

Let p be an integer $\in (1, q)$, (α) a multi-index $1 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_p \leq q$, δ_α a minor of order p of the Jacobian matrix associated with the system of equations $(f_{\alpha_1}, \ldots, f_{\alpha_p})$, and k_α an element of K_α. By definition, H_B is generated by elements of the form $\delta_\alpha k_\alpha$. We will show that for any element $k_\alpha \delta_\alpha$ of the preceding type, we can find $(\mathbf{z}) = (z_1, \ldots, z_N) \in A^N$ such that $z_i \in t^n I$ and

$$k_\alpha(\mathbf{a}) \delta_\alpha(\mathbf{a}) \begin{pmatrix} f_1(\mathbf{a}) \\ \vdots \\ f_q(\mathbf{a}) \end{pmatrix} \equiv M(\mathbf{a}) \begin{pmatrix} z_1 \\ \vdots \\ z_N \end{pmatrix} \pmod{t^{2n}I}. \tag{4}$$

Let's first show how this implies the lemma. Since by hypothesis t^h belongs to the ideal $H_B(\mathbf{a})$, we can deduce from (5) that there exists... $\square$

$(\mathbf{v}) = (v_1, \ldots, v_N) \in A^N$, where $v_i \in t^n I$, such that

$$t^h \begin{pmatrix} f_1(\mathbf{a}) \\ \vdots \\ f_q(\mathbf{a}) \end{pmatrix} \equiv M(\mathbf{a}) \begin{pmatrix} v_1 \\ \vdots \\ v_N \end{pmatrix} \pmod{t^{2n} I},$$

and since n was chosen large enough so that

$$\Lambda \cap (t^{n-h}) = (0),$$

we can deduce

$$\begin{pmatrix} f_1(\mathbf{a}) \\ \vdots \\ f_q(\mathbf{a}) \end{pmatrix} \equiv M(\mathbf{a}) \begin{pmatrix} y_1 \\ \vdots \\ y_N \end{pmatrix} \pmod{t^{2n-h} I},$$

where y_i belongs to $t^{n-h}I$, $t^h y_i = v_i$. This is what was announced in (3). It only remains to establish (4).

To fix the notations, let $(\alpha) = (1, \ldots, p)$ and let δ_α be the minor of order p of the matrix M corresponding to the first p equations and the first p variables. We will simply write δ for δ_α, and k for k_α. By definition of k, we can find polynomials λ_{ij} in $A[X_1, \ldots, X_N]$ such that

$$k f_j = \sum_{i=1}^{p} \lambda_{ij} f_i, \quad \forall j \in [p+1, q],$$

which, after differentiating with respect to X_l, leads to

$$k \frac{\partial f_j}{\partial X_l} = \sum_{i=1}^{p} \lambda_{ij} \frac{\partial f_i}{\partial X_l} \pmod{J}, \quad \forall j \in [p+1, q], \quad \forall l \in [1, N];$$

evaluating at the vector $\mathbf{a}$, we obtain for all $j \in [p+1, q]$ and all $l \in [1, N]$:

$$k(\mathbf{a}) f_j(\mathbf{a}) = \sum_{i=1}^{p} \lambda_{ij}(\mathbf{a}) f_i(\mathbf{a}), \tag{5}$$

$$k(\mathbf{a}) \frac{\partial f_j}{\partial X_l}(\mathbf{a}) = \sum_{i=1}^{p} \lambda_{ij}(\mathbf{a}) \frac{\partial f_i}{\partial X_l}(\mathbf{a}) \pmod{t^n}, \tag{6}$$

since $J(\mathbf{a}) \subset t^n I \subset t^n$.

This last system of relations implies that, for any vector $\begin{pmatrix} g_1 \\ \vdots \\ g_q \end{pmatrix}$ in the image of $M(\mathbf{a})$, we have

$$k(\mathbf{a}) g_j = \sum_{i=1}^{p} \lambda_{ij}(\mathbf{a}) g_i \pmod{t^n}, \quad \forall j \in [p+1, q].$$

and more precisely if

$$\begin{pmatrix} g_1 \\ \vdots \\ g_q \end{pmatrix} = M(\mathbf{a}) \begin{pmatrix} h_1 \\ \vdots \\ h_N \end{pmatrix}$$

where the h_i belong to an ideal $\mathfrak{A}$ of A, we have

$$\forall j > p, \quad k(\mathbf{a})g_j = \sum_{i=1}^{p} \lambda_{ij} g_i \pmod{t^n \mathfrak{A}}. \tag{7}$$

Let M_0 be the matrix $(\partial f_i/\partial X_j(\mathbf{a}))_{1\leq i,j\leq p}$ and N_0 be the (p,p) matrix such that

$$M_0 N_0 = N_0 M_0 = \delta(\mathbf{a})\mathrm{Id}_p,$$

where Id_p represents the identity matrix of order p. Let N_0' be the (N,p) matrix whose upper square is N_0 and which we extend by 0. Let's define $u_{p+1}, \dots, u_q$ by the formula

$$MN_0' \begin{pmatrix} f_1(\mathbf{a}) \\ \vdots \\ f_p(\mathbf{a}) \end{pmatrix} = \begin{pmatrix} \delta(\mathbf{a})f_1(\mathbf{a}) \\ \vdots \\ \delta(\mathbf{a})f_p(\mathbf{a}) \\ u_{p+1} \\ \vdots \\ u_q \end{pmatrix}.$$

From (6), where we take $\mathfrak{A} = t^n I$, we have

$$\forall j > p, \quad k(\mathbf{a})u_j = \delta(\mathbf{a}) \sum_{i=1}^{p} \lambda_{ij}(\mathbf{a}) f_i(\mathbf{a}) \pmod{t^{2n} I}.$$

But from (5):

$$\forall j > p, \quad k(\mathbf{a})f_j(\mathbf{a}) = \sum_{i=1}^{p} \lambda_{ij}(\mathbf{a}) f_i(\mathbf{a}).$$

Therefore

$$k(\mathbf{a})u_j = k(\mathbf{a})\delta(\mathbf{a})f_j(\mathbf{a}) \pmod{t^{2n} I}.$$

Therefore

$$MN_0' k(\mathbf{a}) \begin{pmatrix} f_1(\mathbf{a}) \\ \vdots \\ f_p(\mathbf{a}) \end{pmatrix} = k(\mathbf{a})\delta(\mathbf{a}) \begin{pmatrix} f_1(\mathbf{a}) \\ \vdots \\ f_q(\mathbf{a}) \end{pmatrix} \pmod{t^{2n} I},$$

which is what we aimed to establish.

Theorem 0.5. *Let A be a Noetherian ring complete for the topology defined by an ideal $\mathfrak{J}$, and B an A-algebra of finite type admitting a presentation*

$$B = A[X_1, \dots, X_N]/J, \quad J = (f_1, \dots, f_q).$$

Then for any integer h, there exists a pair of integers (n_0, r) such that if n is an integer $> n_0$ and if $(\mathbf{a}^0) = (a_1^0, \ldots, a_N^0) \in A^N$ is such that

$$H_B(\mathbf{a}^0) \supset \mathfrak{J}^h,$$

$$J(\mathbf{a}^0) \subset \mathfrak{J}^n,$$

there exists $(\mathbf{a}) = (a_1, \ldots, a_N) \in A^N$, congruent to $\mathbf{a}^0$ modulo $\mathfrak{J}^{n-r}$ and such that

$$J(\mathbf{a}) = 0.$$

Proof. We reason by induction on the minimal number of elements of a generating system for a definition ideal of the $\mathfrak{J}$-adic topology. Let l be this number; for $l = 0$, the result is trivial. Suppose it is proven for $l - 1$ and let $t_1, \ldots, t_l$ be elements of A generating a definition ideal. We can further suppose that the t_i belong to $\mathfrak{J}$. Let T be the ideal of A formed by elements annihilated by a power of t_l and let k be an integer such that

$$T \cap (t_l^k) = (0) \quad \text{(Artin-Rees)}.$$

Fix an integer s satisfying $s > \sup(2h, h + k)$ and let

$$A_1 = A/t_l^s A.$$

By the induction hypothesis applied to A_1, there exist n_0 and s' such that for any n greater than n_0 and any $(\mathbf{a}) = (a_1, \ldots, a_N) \in A_1^N$ satisfying $J(\mathbf{a}) \subset \mathfrak{J}^n$, we can find $(\mathbf{a}') = (a_1', \ldots, a_N') \subset A_1^N$ satisfying $(\mathbf{a}') \equiv (\mathbf{a}) \pmod{\mathfrak{J}^{n-r}}$ and $(\mathbf{a}') \subset t_l A_1^s$, such that $J(\mathbf{a}') \subset t_l^s$. In reality, we have

$$J(\mathbf{a}') \subset t_l^s \mathfrak{J}^{n-r'}.$$

By the Artin-Rees lemma, there exists a pair of integers (n_1, λ) such that for $n - r' > n_1$ we have

$$J(\mathbf{a}') \subset t_l^s \cdot \mathfrak{J}^{n-r'-\lambda}.$$

On the other hand,

$$H_B(\mathbf{a}) \supset \mathfrak{J}^h \quad \text{and} \quad H_B(\mathbf{a}) \subset H_B(\mathbf{a}') + \mathfrak{J}^{n-r'}.$$

If we fix n large enough so that $n - r' > h$, we can deduce by Nakayama's lemma,

$$H_B(\mathbf{a}') \supset \mathfrak{J}^h,$$

thus

$$H_B(\mathbf{a}') \supset t_l^h.$$

□

The preceding lemma then implies that we can find

$$(\mathbf{b}) = (b_1, \ldots, b_N) \in A^N,$$

such that $J(\mathbf{b}) = 0$ and such that for all $i \in [1, N]$:

$$b_i - a'_i \in (t_l^{s-h}\mathfrak{J}^{n-r'-\lambda}) \subset (\mathfrak{J}^{n+s-(r'+\lambda+h)}),$$

which establishes the theorem.

Remark 0.6. Instead of assuming that B is a finite type A-algebra, we could take B to be a formal finite type A-algebra, i.e., a quotient of an algebra of restricted power series with coefficients in A. The same proof would apply.

II. The Case of Henselian Pairs

The purpose of this section is to generalize the results of the first part to Henselian pairs.

Theorem 0.7. *Let $(A, \mathfrak{J})$ be a Noetherian Henselian pair and h an integer. Then there exists a pair of integers (n_0, r) such that if B is a finite type A-algebra,*

$$B = A[X_1, \ldots, X_N]/J,$$

n is an integer greater than n_0 and $(\mathbf{a}^0) = (a_1^0, \ldots, a_N^0) \in A^N$ such that

$$J(\mathbf{a}^0) \subset \mathfrak{J}^n,$$

$$H_B(\mathbf{a}^0) \supset \mathfrak{J}^h,$$

(where H_B is an ideal of $A[X_1, \ldots, X_N]$ having the properties indicated in 0), then there exists $(\mathbf{a}) = (a_1, \ldots, a_N) \in A^N$ congruent to $(\mathbf{a}^0)$ modulo $\mathfrak{J}^{n-r}$ and such that $J(\mathbf{a}) = 0$.

This theorem is an immediate consequence of Theorem 1 and of:

Theorem 0.8 (2 bis)**.** *Let $(A, \mathfrak{J})$ be a Henselian pair, $\hat{A}$ the $\mathfrak{J}$-adic completion of A, B a finite type A-algebra, $\hat{B} = B \otimes_A \hat{A}$. Let V be a smooth open set of Spec B over Spec A and $\hat{V}$ its inverse image in Spec $\hat{B}$. Then for any integer n and any $\hat{A}$-section $\hat{s}$ of Spec $\hat{B}$ whose restriction over* Spec $(\hat{A}) - V(\mathfrak{J}\hat{A})$ *factors through $\hat{V}$, there exists an A-section s of Spec B congruent to $\hat{s}$ modulo $\mathfrak{J}^n$ and whose restriction over* Spec $(A) - V(\mathfrak{J})$ *factors through V.*

Before proving this theorem, let's add a remark here:

Remark 0.9. It is sufficient to prove that for every n there exists a section s of Spec B congruent to $\hat{s}$ modulo $\mathfrak{J}^n$. Indeed, let I be an ideal of B such that $V = \operatorname{Spec}(B) - V(I)$. The hypothesis on $\hat{s}$ means that there exists an integer α such that

$$\hat{s}^*(I) \supset \mathfrak{J}^\alpha \hat{A}.$$

Then any section s of B congruent to $\hat{s}$ modulo $\mathfrak{J}^n$ (n $> \alpha$) will be such that

$$s^*(I) \supset \mathfrak{J}^\alpha$$

and will therefore factor through V over $\operatorname{Spec}(A) - V(\mathfrak{J})$.

The following lemma due to Tougeron [10] gives more precise results than those of Theorem 2 in the case where we also assume that the maximal rank minors of the Jacobian are invertible in Spec B outside of $V(\mathfrak{J})$.

Lemma 0.10. *Let $(A, \mathfrak{J})$ be a Henselian pair, B a finite type A-algebra equipped with a presentation*

$$B = A[X_1, \ldots, X_N]/J, \quad J = (f_1, \ldots, f_q), \quad f_i \in A[X_1, \ldots, X_N].$$

Let Δ be the ideal generated by the minors of order q of the Jacobian matrix

$$M = (\partial f_i / \partial X_j)_{i=1,\ldots,q; j=1,\ldots,N}.$$

Let n and h be two integers such that $n > 2h$ and $\mathbf{a} = (a_1, \ldots, a_N) \in A^N$ such that

$$J(\mathbf{a}) \subset \mathfrak{J}^n \quad \textit{and} \quad \Delta(\mathbf{a}) \supset \mathfrak{J}^h.$$

Then there exists an element $(\mathbf{a}') = (a'_1, \ldots, a'_N) \in A^N$ satisfying $(\mathbf{a}') \equiv \mathbf{a} \pmod{\mathfrak{J}^{n-h}}$ and $J(\mathbf{a}') = 0$.

Proof. Let $M(\mathbf{a})$ be the matrix $(\partial f_i / \partial X_j(\mathbf{a}))_{i=1,\ldots,q; j=1,\ldots,N}$. Let $(t_1, \ldots, t_r)$ be a system of generators for $\mathfrak{J}^h$. We can find for each $i \in [1, r]$ a matrix (N, q), say N_i, such that

$$M(\mathbf{a}) N_i = t_i \mathrm{Id}_q,$$

where Id_q denotes the identity matrix of rank q. We seek for each $i \in [1, r]$ vectors $\mathbf{u}_i = (u_{i,1}, \ldots, u_{i,N}) \in A^N$ satisfying $u_{i,j} \in \mathfrak{J}^{n-2h}$ for all $j \in [1, N]$ and such that for all $k \in [1, q]$:

$$f_k\left(\mathbf{a} + \sum_{i=1}^r t_i \mathbf{u}_i\right) = 0.$$

Let's write a Taylor expansion:

$$f_k\left(\mathbf{a} + \sum t_i \mathbf{u}_i\right) = f_k(\mathbf{a}) + \sum_{i=1}^r t_i M(\mathbf{a}) \begin{pmatrix} u_{i,1} \\ \vdots \\ u_{i,N} \end{pmatrix} + \sum_{i,j} t_i t_j Q_{ij},$$

□

where Q_{ij} is a column vector whose components are polynomials in $\mathbf{a}$ and $\mathbf{u}_i$, whose minimal degree term in the $\mathbf{u}_i$ is of degree ≥ 2. We therefore seek to solve

$$0 = \begin{pmatrix} f_1(\mathbf{a}) \\ \vdots \\ f_q(\mathbf{a}) \end{pmatrix} + \sum_i t_i M(\mathbf{a}) \begin{pmatrix} u_{i,1} \\ \vdots \\ u_{i,N} \end{pmatrix} + \sum_{i,j} t_i t_j Q_{ij}. \tag{8}$$

We can write

$$\begin{pmatrix} f_1(\mathbf{a}) \\ \vdots \\ f_q(\mathbf{a}) \end{pmatrix} = -\sum_{i,j} t_i t_j \begin{pmatrix} l_{ij,1} \\ \vdots \\ l_{ij,q} \end{pmatrix} \quad \text{with} \quad l_{ij,k} \in \mathfrak{J}^{n-2h}.$$

Equation (1) can be rewritten as

$$0 = \sum_i t_i M(\mathbf{a}) \left(\sum_j N_j \begin{pmatrix} l_{ij,1} \\ \vdots \\ l_{ij,q} \end{pmatrix} \right) + \sum_i t_i M(\mathbf{a}) \begin{pmatrix} u_{i,1} \\ \vdots \\ u_{i,N} \end{pmatrix} + \sum_i t_i M(\mathbf{a}) \left(\sum_j N_j Q_{ij} \right).$$

To solve this, it is sufficient to solve each of the systems of equations in the u_{ij}:

$$\begin{pmatrix} u_{i,1} \\ \vdots \\ u_{i,N} \end{pmatrix} + \sum_j N_j Q_j + \begin{pmatrix} l_{ij,1} \\ \vdots \\ l_{ij,q} \end{pmatrix} = 0.$$

Now, 0 is an approximate solution modulo $\mathfrak{J}^{n-2h}$ of this system, and furthermore, the Jacobian matrix at the origin is equal to the identity matrix of order N. This system of equations is therefore étale in the neighborhood of the section modulo $\mathfrak{J}^{n-2h}$ defined by $u_{ij} = 0$, and this solution can thus be lifted to a true solution.

Let u_i be this solution. The element $(\mathbf{a}') = (\mathbf{a}) + \sum t_i(\mathbf{u}_i)$ defines the solution of the system of equations (f) announced.

Lemma 0.11. *Let A be a Noetherian ring, B a finite type A-algebra admitting a presentation $B = A[X_1, \dots, X_N]/J$, $J = (f_1, \dots, f_q)$. Let V be a smooth open set of Spec B over Spec A. Let C be the symmetric algebra of the B-module J/J^2 and V' the inverse image of V in Spec C. Then V' is smooth over A of constant relative dimension N. There exists a plunging of Spec C into the affine space of dimension $2N + q$ over A such that on any affine open set of Spec C contained in V', the conormal sheaf of this immersion is free of rank $N + q$ (i.e., in the neighborhood of such an open set, the ideal defining C in the considered affine space can be generated by $N + q$ equations).*

Proof. Let r be the structural morphism

$$\text{Spec}\,(B) \to \text{Spec}\,(A),$$

let f be the canonical morphism Spec $(C) \to$ Spec (B), and let us set $g = f \circ r$. □

In the neighborhood of a point in V where the relative dimension of B over A is d, J/J^2 is locally free of rank $N-d$, so Spec C above such a point is smooth over B of relative dimension $N-d$, and therefore smooth over A of dimension N. Spec C is plunged into the affine space of dimension q over B. Let I/I^2 be the conormal sheaf of this immersion

$$\mathrm{Spec}\,(C) \to \mathrm{Spec}\,(B)[Y_1,\ldots,Y_q].$$

We can also consider Spec C as plunged into $\mathrm{Spec}\,(A)[X_1,\ldots,X_N,Y_1,\ldots,Y_q]$. Let K/K^2 be the conormal sheaf of this immersion. Let U be an affine open set of Spec C contained in V'. We can write over U the following exact sequences, which are exact sequences of locally free modules over an affine scheme, hence split:

$$0 \to f^*(J/J^2)|_U \to f^*(\Omega_{A[X]/A} \otimes B)|_U \to f^*(\Omega_{B/A})|_U \to 0 \tag{9}$$

where $f^*(\Omega_{A[X]/A} \otimes B)$ is isomorphic to C^N,

$$0 \to f^*(\Omega_{B/A})|_U \to \Omega_{C/A}|_U \to \Omega_{C/B}|_U \to 0. \tag{10}$$

Since Spec C is the vector bundle associated with the B-module J/J^2, we have $\Omega_{C/B} \simeq f^*(J/J^2)$. It follows from this identification and from sequences (1) and (2) that $\Omega_{C/A}$ is globally free of rank N over U. We have indeed:

$$\Omega_{C/A}|_U \simeq f^*(J/J^2)|_U \oplus f^*(\Omega_{B/A})|_U \simeq C^N|_U.$$

On the other hand, considering Spec C as plunged into the affine space over A of dimension $N+q$, $\mathrm{Spec}\,[X_1,\ldots,X_N,Y_1,\ldots,Y_q]$, we have the exact sequences

$$0 \to (K/K^2)|_U \to \Omega_{A[X,Y]/A} \otimes C|_U \to \Omega_{C/A}|_U \to 0.$$

Thus, taking into account the preceding isomorphisms, we have an exact sequence

$$0 \to (K/K^2)|_U \to C^{N+q}|_U \to C^N|_U \to 0. \tag{11}$$

We then plunge Spec C into $\mathrm{Spec}\,(A)[X_1,\ldots,X_N,Y_1,\ldots,Y_q,T_1,\ldots,T_N]$ in which it is defined by the additional equations $T_i = 0$ for $i \in [1,N]$. Let K'/K'^2 be the conormal sheaf associated with this new immersion. This sheaf is globally free over U. We have in effect

$$(K'/K'^2)|_U \simeq (K/K^2)|_U \oplus C^N|_U \simeq C^{N+q}|_U \quad \text{by (3)}.$$

There are thus $N+q$ elements of $A[X_1,\dots,X_N,Y_1,\dots,Y_q,T_1,\dots,T_N]$ generating the ideal defining C in a neighborhood of U in

$$\mathrm{Spec}\,(A)[X_1,\dots,X_N,Y_1,\dots,Y_q,T_1,\dots,T_N].$$

We now prove Theorem 2 bis in the case where $\mathfrak{J}$ is a principal ideal.

Lemma 0.12. *Let $(A,\mathfrak{J})$ be a Henselian pair and assume $\mathfrak{J}$ is principal, generated by t. Let $\hat{A}$ be the $\mathfrak{J}$-adic completion of A, B a finite type A-algebra, V a smooth open set of Spec B over Spec A. Let*

$$\hat{B} = B \otimes_A \hat{A}$$

and let $\hat{V}$ be the inverse image of V in Spec $\hat{B}$. Then if $\hat{s}$ is an $\hat{A}$-section of Spec $\hat{B}$ that factors over $\mathrm{Spec}\,(\hat{A}) - V(\mathfrak{J})$ *through $\hat{V}$, there exists for any $n\in\mathbb{N}$ a section s of Spec B congruent to $\hat{s}$ modulo t^n that factors over* $\mathrm{Spec}\,(A) - V(\mathfrak{J})$ *through V.*

Proof. Given an embedding of Spec B in an affine space over A, we can replace Spec B by the vector bundle of the conormal sheaf of this immersion. Indeed, let's denote this bundle by Q. The embedding of Spec B into Q defined by the zero section allows us to lift $\hat{s}$ to $Q\otimes_A \hat{A}$. On the other hand, a section of Q defines, thanks to the structural morphism $Q\to \mathrm{Spec}\,(B)$, a section of Spec B. This amounts to saying, by virtue of Lemma 3, that we can identify Spec B with a closed subscheme of an affine space $\mathrm{Spec}\,(A)[X_1,\dots,X_N]$, defined by an ideal J, that V is smooth of constant relative dimension equal to d, and that in the neighborhood of any affine open set of V in $\mathrm{Spec}\,(A)[X_1,\dots,X_N]$, J can be generated by $N-d$ elements. Let

$$\begin{aligned} j &: \mathrm{Spec}\,(B) \to \mathrm{Spec}\,(A)[X_1,\dots,X_N],\\ \hat{j} &: \mathrm{Spec}\,(\hat{B}) \to \mathrm{Spec}\,(\hat{A})[X_1,\dots,X_N],\\ \hat{s} &: \mathrm{Spec}\,(\hat{A}) \to \mathrm{Spec}\,(B), \end{aligned}$$

be the given morphisms. Let's first show that there exists an affine open set of $\hat{V}$ whose inverse image in V contains $\hat{s}(\mathrm{Spec}\,(\hat{A}) - V(\mathfrak{J}))$. □

Let H be an ideal of $A[X_1, \ldots, X_N]$ whose image in B defines the complement of V, and K the ideal of $\hat{A}[X_1, \ldots, X_N]$ defining the closed set $\hat{j} \circ \hat{s}(\hat{A})$. By hypothesis, there exists $z \in \mathbb{N}$ such that

$$H\hat{A}[X_1, \ldots, X_N] + K \supset t^z \hat{A}[X_1, \ldots, X_N].$$

Let's write

$$t^z = \tilde{h} + k, \quad \text{where } \tilde{h} \in H\hat{A}[X, \ldots, X_N] \text{ and } k \in K.$$

And let's replace $\tilde{h}$ by an element h of $A[X_1, \ldots, X_N]$ satisfying

$$h \in H \quad \text{and} \quad h \equiv \tilde{h} \pmod{t^\beta}, \beta > z.$$

We can write

$$h + k = t^z(1 + ty)$$

for some y belonging to $\hat{A}[X_1, \ldots, X_N]$. Since $1 + ty$ is invertible at the points of $\hat{s}$, the open set of Spec B where h is invertible answers the question. (This open set is non-empty if t is not nilpotent; the case t nilpotent is trivial.) Let $(f_1, \ldots, f_{N-d}) = (f)$ be elements of J generating J in the neighborhood of Spec B_h in $A[X_1, \ldots, X_N]$. Up to modifying h (without changing the open set Spec B_h), we can assume that $(f_1, \ldots, f_{N-d})$ generate J on the open set $\operatorname{Spec} A[X_1, \ldots, X_N]_h$. And up to replacing h by one of its powers, we can assume on the other hand that we have $hJ \subset (f)$. Furthermore, there exists an integer δ such that

$$(j \circ \hat{s})^*(h) \supset t^\delta \hat{A}.$$

Let Δ be the ideal generated by the minors of order $N - d$ of the Jacobian matrix associated with (f). We can find an integer γ such that

$$(j \circ \hat{s})^*(\Delta) \supset t^\gamma \hat{A}.$$

Lemma 2 allows us to assert that for any integer n, we can find a section σ of $\operatorname{Spec} A[X_1, \ldots, X_N]$ congruent to $\hat{s}$ modulo t^n and such that

$$\sigma^*(f_1, \ldots, f_{N-d}) = 0.$$

And for $n > \delta$ we will still have

$$\sigma^*(h) \supset t^\delta,$$

thus

$$t^\delta \sigma^*(J) = 0.$$

We will also have, since $\sigma \equiv \hat{s} \pmod{t^n}$,

$$\sigma^*(J) \subset t^n.$$

Let Λ be the ideal of A formed by elements with support in $V(t)$. If n is chosen large enough so that we have

$$\Lambda \cap (t^\delta) = 0 \quad \text{(Artin-Rees)},$$

we will then have

$$\sigma^*(J) = 0.$$

This completes the proof of Lemma 4.

Proof of Theorem 2 bis. — We thus consider the following diagram:

$$\begin{array}{ccc} \mathrm{Spec}\,(B) = \mathrm{Spec}\,(A[X_1, \ldots, X_N]/J) & \longrightarrow & \mathrm{Spec}\,(\hat{B}) \\ \downarrow & & \downarrow\hat{s} \\ \mathrm{Spec}\,(A) & \longrightarrow & \mathrm{Spec}\,(\hat{A}) \end{array}$$

and an open set V of Spec B, smooth over Spec A, with inverse image $\hat{V}$ in Spec $\hat{B}$ and such that

$$\hat{s}(\mathrm{Spec}\,(\hat{A}) - V(\mathfrak{J})) \subset \hat{V}.$$

Let n be an arbitrary integer. We want to show that we can find a section

$$s : \mathrm{Spec}\,(A) \to \mathrm{Spec}\,(B),$$

$$s \equiv \hat{s} \pmod{\mathfrak{J}^n}.$$

We reason by recurrence on the minimal number of elements in a generating system of a definition ideal of the $\mathfrak{J}$-adic topology. Let l be this number. For $l = 0$ the result is trivial. Suppose it is proven for $l - 1$ and let $(z_1, \ldots, z_l)$ be a system of generators of a definition ideal contained in $\mathfrak{J}$. Let A_1 be the z_1-adic completion of A and let $B_1 = B \otimes_A A_1$. Let H_B be an ideal of $A[X_1, \ldots, X_N]$ as stated in 0.2 and such that $V(H_B) \cap \hat{s}(\mathrm{Spec}\,(\hat{A})) \subset V(\mathfrak{J})$. Let

$$j : \mathrm{Spec}\,(B) \to \mathrm{Spec}\,(A)[X_1, \ldots, X_N].$$

The hypothesis made on $\hat{s}$ means that there exists an integer h such that

$$(j \circ \hat{s})^*(H_B) \supset \mathfrak{J}^h.$$

Let's also consider

$$j_1 : \mathrm{Spec}\,(B_1) \to \mathrm{Spec}\,(A_1)[X_1, \ldots, X_N] \quad \text{and} \quad H_1 = H_B \otimes_A A_1.$$

Then, according to Theorem 1, we can find a pair of integers (n_0, r) such that for any $n > n_0$ and any approximate section of Spec B_1, say

$$s' : \mathrm{Spec}\,(A_1/z_1^n) \to \mathrm{Spec}\,(B_1)$$

satisfying

$$(j_1 \circ s')^*(H_1) \supset z_1^h,$$

there exists

$$s'' : \mathrm{Spec}\,(A_1) \to \mathrm{Spec}\,(B_1),$$

congruent to s' modulo z_1^{n-r}.

Fix an integer $n > \sup(n_0, m+r, h+r)$ and apply the recurrence hypothesis to the pair $(A/z_1^m = A_1/z_1^m, \mathfrak{J}/z_1^m)$. There thus exists a section $\bar{s}_1$ approximate modulo z_1^m of B_1 over A_1/z_1^m and congruent to $\hat{s}$ modulo $\mathfrak{J}^n$. Moreover, since n is greater than h, we still have

$$\bar{s}_1^*(H_1) \supset \mathfrak{J}^h A_1.$$

Given the choice of n, there thus exists a section

$$s_1 : \mathrm{Spec}\,(A_1) \to \mathrm{Spec}\,(B_1)$$

congruent to $\bar{s}_1$ modulo z_1^{n-r} which ensures in particular

$$s_1 \equiv \hat{s} \pmod{\mathfrak{J}^m}.$$

And we still have

$$(j \circ s_1)^*(H_1) \supset \mathfrak{J}^h.$$

But this also means that the image of $(\mathrm{Spec}\,(A_1) - V(z_1))$ by s' is contained in the smoothness open set of B_1. It now follows from Lemma 4 that we can approximate s_1 by an A-section s of B congruent to s' modulo z_1^n. We will then have

$$s \equiv \hat{s} \pmod{\mathfrak{J}^m}.$$

Corollary 0.13. *Let $(A, \mathfrak{J})$ be a Henselian pair, T an A-scheme, affine, and $X = \mathrm{Spec}\,(B)$ a T-scheme, affine of finite type, smooth over T outside the closed set $V(\mathfrak{J})$. For any section $\sigma : \mathrm{Spec}\,(A) \to T$, let us denote by X_σ the A-scheme deduced from X by base change.*

$$\begin{array}{ccc}
\mathrm{Spec}\,(B) = X & & \\
\downarrow & & \\
T & \xrightarrow{\sigma} & X_\sigma = \mathrm{Spec}\,(B_\sigma) \\
\uparrow \sigma & & \uparrow \\
\mathrm{Spec}\,(A) & \xrightarrow{Id} & \mathrm{Spec}\,(A)
\end{array}$$

Then there exist two integers n_0 and r such that for any section $\sigma : \operatorname{Spec}(A) \to T$ and any integer $n > n_0$ and any A-morphism

$$s_n : \operatorname{Spec}(A/\mathfrak{J}^n) \to X_\sigma,$$

there exists a section

$$s : \operatorname{Spec}(A) \to X_\sigma,$$

congruent to s_n modulo $\mathfrak{J}^{n-r}$. Indeed, let $B = T[X_1, \ldots, X_N]/J$, $J = (f_1, \ldots, f_q)$, be a presentation of B over T and H_B an ideal of $T[X_1, \ldots, X_N]$, defined as previously, and such that

$$V(H_B) \cap \operatorname{Spec}(B) \subset V(JB)$$

and let H_{B_σ} be the image of H_B in B_σ. We will have for any σ,

$$V(H_{B_\sigma}) \subset V(JB_\sigma).$$

More precisely, there exists $z \in \mathbb{N}$ such that

$$H_B B \supset J^z B$$

and we will have for any section σ:

$$H_{B_\sigma} B_\sigma \supset J^z B_\sigma.$$

It is then sufficient to apply Theorem 2 to conclude. We have as an immediate consequence of what precedes the following theorem:

Theorem 0.14. *Let $(A, \mathfrak{J})$ be a Henselian pair, B a finite type A-algebra, smooth over A, let n be an integer and s_n an A-morphism* $\operatorname{Spec}(A/\mathfrak{J}^n) \to \operatorname{Spec}(B)$. *Then there exists a section* $s : \operatorname{Spec}(A) \to \operatorname{Spec}(B)$ *lifting s_n.*

We could have, of course, proven this result with less effort; Lemma 3 allows one to suppose B is a global relative complete intersection over A, and the result is then a consequence of Lemma 2.

III. Various Applications

1. Applications to Modules

Let A be a ring, M an A-module, p and q two integers. We will say of a presentation of the module M of the form

$$A^q \to A^p \to M \to 0$$

that it is a presentation of type (p, q).

Theorem 0.15. *Let $(A, \mathfrak{J})$ be a Henselian pair, $\hat{A}$ the $\mathfrak{J}$-adic completion of A and $\hat{M}$ a finite type $\hat{A}$-module, locally free over the open set* $\operatorname{Spec}(\hat{A}) - V(\mathfrak{J}\hat{A})$. *Then there exists a finite type A-module M, locally free over the open set* $\operatorname{Spec}(A) - V(\mathfrak{J})$ *and such that $M \otimes_A \hat{A}$ is $\hat{A}$-isomorphic to $\hat{M}$.*

Let

$$\hat{A}^q \xrightarrow{\hat{L}} \hat{A}^p \to \hat{M} \to 0, \quad \hat{L} = (\hat{l}_{ij})_{i=1,\dots,p;j=1,\dots,q}$$

be a presentation of $\hat{M}$. We actually prove the following result: For any integer k, we can find a presentation

$$A^q \xrightarrow{L} A^p \to M \to 0$$

of an A-module M such that we have $L \equiv \hat{L} \pmod{\mathfrak{J}^k}$ and an isomorphism of $\hat{M}$ to $M \otimes_A \hat{A}$ induced by automorphisms of $\hat{A}^p$ and $\hat{A}^q$ congruent to the identity modulo $\mathfrak{J}^k$.

Proof. (a) We begin by showing that we can approximate the presentation of $\hat{M}$ so as to define an A-module locally free on $\operatorname{Spec}(A) - V(\mathfrak{J})$. Indeed, we can classify A-modules equipped with a presentation of type (p, q) by (p, q) matrices, hence by a polynomial algebra in pq variables over A, say $A[X] = A[(X_{ij})_{i=1,\dots,p;j=1,\dots,q}]$ and let's also denote by $X = (X_{ij})$ the universal (p, q) matrix. Let Δ^k be the ideal of $A[X_{ij}]$ generated by the family of polynomials corresponding to the minors of rank k of the matrix X. To say that a module, admitting a presentation of type (p, q), is free of rank r in the neighborhood of a point is equivalent to saying that the minors of rank $q - r + 1$ of the associated matrix are zero at this point and that one of the minors of rank $q - r$ is invertible. The first condition means that the rank of the module at this point is at least equal to r.

Let's denote by U the open set $\operatorname{Spec}(A) - V(\mathfrak{J})$, by $U_1, \dots, U_s$ its different connected components, and let $I_1, \dots, I_s$ be ideals of A such that

$$V(I_j) = \operatorname{Spec}(A) - U_j.$$

Let U' be the inverse image of U in $\operatorname{Spec}(\hat{A})$, and $U'_1, \dots, U'_s$ those of the U_j. We know [4] that they form a decomposition into connected components of U'. □

For any $j \in [1,l]$, let r_j be the rank of the module $\hat{M}$ on U'_j, β_j an integer such that any element of A with support in $V(I_j)$ is annihilated by $I_j^{\beta_j}$, and let $t_{j1},\ldots,t_{jk}$ be a system of generators for $I_j^{\beta_j}$. Let B be the quotient of $A[X_{11},\ldots,X_{pq}]$ defined by all the equations

$$t_{ju}\Delta^{q-r_j+1}=0,$$

which express that the minors of rank $q-r_j+1$ are zero over U_j; the A-algebra B classifies A-modules with a presentation of type (p,q) and which are of rank at least equal to r_j on U_j; the $\hat{A}$-algebra $B\otimes_A\hat{A}$ classifies $\hat{A}$-modules of the same type and of rank at least equal to r_j on U'_j ($\forall j\in[1,l]$). Let V_i be the open set of Spec B over U_i, where the ideal generated by (Δ^{q-r_j}) is the unit ideal, $V=\bigcup_{i=1}^{l}V_i$, and V' the inverse image of V in Spec$\,(B\otimes_A\hat{A})$. The matrix $\hat{L}$ is defined by an $\hat{A}$-section $\hat{\varepsilon}$ of $B\otimes_A\hat{A}$ such that $\hat{\varepsilon}(U')\subset V'$. Let's show that V is smooth over Spec A: Let's place ourselves in the neighborhood of a point in V_j and let δ be one of the minors of order $q-r_j$ invertible at this point. Under this hypothesis, the conditions expressing the nullity of the minors of order $q-r_j+1$ obtained by bordering δ with a row and a column are obviously smooth. Furthermore, they are sufficient to imply the nullity of all minors of order $q-r_j+1$. On the other hand, we have for $k\neq j$, $V(I_kI_j)=V(0)$. Thus I_kI_j is nilpotent and, according to the choice of β_j, $I_j^{\beta_j}$ is zero where I_k is invertible. The conditions that translate to $I_j^{\beta_j}$ annihilating Δ^{q-r_j+1} are therefore trivial at any point of V_j, which establishes the smoothness of V over Spec A.

We can therefore (according to Theorem 2), find for any $n\in\mathbb{N}$, a section ε of B congruent to $\hat{\varepsilon}$ modulo $\mathfrak{J}^n$, factoring over U through V, and thus a matrix L congruent to $\hat{L}$ modulo $\mathfrak{J}^n$ such that the module M defined by

$$A^q\xrightarrow{L}A^p\to M\to 0$$

is locally free of rank r_i over U_i ($\forall i\in[1,l]$).

(b) We will use, to complete the proof of the theorem, the following lemma:

Lemma 0.16. *Let $(A,\mathfrak{J})$ be a Noetherian Henselian pair, p,q,r,α be integers. There then exists a pair of integers (n_0,s) with the following property:*

Let M and M' be two A-modules, locally free of rank r outside the closed set $V(\mathfrak{J})$, admitting presentations

$$A^q \xrightarrow{L} A^p \to M \to 0,$$

$$A^q \xrightarrow{L'} A^p \to M' \to 0,$$

and such that the Fitting ideal of rank r of M contains $\mathfrak{J}^\alpha$. Then if we have for an integer $n > n_0$ a commutative diagram

$$\begin{array}{ccc} A^p/\mathfrak{J}^n & \xrightarrow{L_n} & A^q/\mathfrak{J}^n \\ \downarrow u_n & & \downarrow v_n \\ A^p/\mathfrak{J}^n & \xrightarrow{L'_n} & A^q/\mathfrak{J}^n \end{array}$$

where u_n and v_n are automorphisms of $A^p/\mathfrak{J}^n$ and $A^q/\mathfrak{J}^n$, M is isomorphic to M' and there exists a commutative diagram

$$\begin{array}{ccc} A^p & \xrightarrow{L} & A^q \\ \downarrow u & & \downarrow v \\ A^p & \xrightarrow{L'} & A^q \end{array}$$

where u and v are automorphisms of A^p and A^q congruent respectively to u_n and v_n modulo $\mathfrak{J}^{n-s}$.

Proof. We can first note that if n is greater than α, which we assume in the following, the hypothesis implies that the Fitting ideal of rank r of M' also contains $\mathfrak{J}^\alpha$. Let's consider the A-algebra T classifying the following data:

- a pair of (p, q) matrices with coefficients in A, say $X = (X_{ij})$ and $X' = (X'_{ij})$ such that
  - their minors of rank $q - r + 1$ are zero outside of $V(\mathfrak{J})$;
  - the minors of rank $q - r$ of X (resp. of X') generate the ideal $\mathfrak{J}^\alpha$ in A.

Let's briefly describe T: it is obviously the tensor product of two algebras, one relative to X, the other to X'. Let $A[X_{ij}]$ classify the p-matrices $(p, q) : X = (X_{ij})_{i=1,\ldots,p; j=1,\ldots,q}$. For any s, let Δ_s be the family of minors of order $q - s$ of X. Let finally $(t) = (t_1, \ldots, t_u)$ be a system of generators of $\mathfrak{J}^\alpha$, λ an integer such that $\mathfrak{J}^\lambda$ annihilates any element of A with support in $V(\mathfrak{J})$, and $(u) = (u_1, \ldots, u_d)$ a system of generators of $\mathfrak{J}^\lambda$. The algebra T can then be written as a tensor product of two algebras of the form

$$A[(X_{ij}), (Y_{ij})]/(t_l - \sum_j Y_{ij}\delta_j)_{l=1,\ldots,\mu}$$

□

Let M and M' be two A-modules, locally free of rank r outside the closed set $V(\mathfrak{J})$, admitting presentations

$$A^p \xrightarrow{L} A^q \to M \to 0,$$

$$A^p \xrightarrow{L'} A^q \to M' \to 0$$

and such that the Fitting ideal of rank r of M contains $\mathfrak{J}^\alpha$. Then if we have for an integer $n > n_0$ a commutative diagram

$$\begin{array}{ccc} A^p/\mathfrak{J}^n & \xrightarrow{L_n} & A^q/\mathfrak{J}^n \\ \downarrow u_n & & \downarrow v_n \\ A^p/\mathfrak{J}^n & \xrightarrow{L'_n} & A^q/\mathfrak{J}^n \end{array}$$

where u_n and v_n are automorphisms of $A^p/\mathfrak{J}^n$ and $A^q/\mathfrak{J}^n$, M is isomorphic to M' and there exists a commutative diagram

$$\begin{array}{ccc} A^p & \xrightarrow{L} & A^q \\ \downarrow u & & \downarrow v \\ A^p & \xrightarrow{L'} & A^q \end{array}$$

where u and v are automorphisms of A^p and A^q congruent respectively to u_n and v_n modulo $\mathfrak{J}^{n-s}$.

Proof. We can first note that if n is greater than α, which we assume in the following, the hypothesis implies that the Fitting ideal of rank r of M' also contains $\mathfrak{J}^\alpha$.

Consider then the A-algebra T classifying the following data:

- a pair of matrices (p, q) with coefficients in A, let them be $X = (X_{ij})$ and $X' = (X'_{ij})$ such that
- their minors of rank $q - r + 1$ are zero outside of $V(\mathfrak{J})$;
- the minors of rank $q - r$ of X (resp. of X') generate the ideal $\mathfrak{J}^\alpha$ in A.

Let's briefly describe T: it is obviously the tensor product of two algebras, one relative to X, the other to X'. Let $A[\mathbf{X}_{ij}]$ be the algebra classifying the p-matrices (p, q): $X = (X_{ij})_{i=1,...,p;j=1,...,q}$. For any s, let Δ_s be the family of minors of order $q - s$ of X. Finally, let $(t) = (t_1, ..., t_\mu)$ be a system of generators of $\mathfrak{J}$, λ be an integer such that $\mathfrak{J}^\lambda$ annihilates every element of A with support in $V(\mathfrak{J})$ and $(u) = (u_1, ..., u_d)$ be a system of generators of $\mathfrak{J}^\alpha$. The algebra T can then be written as a tensor product of two algebras of the form

$$A[(X_{ij})_{i=1,...,p;j=1,...,q}, (Y_{ij})_{i=1,...,k;j=1,...,\mu}]/(t_l\Delta_1, u_l - \sum_{j=1}^{k_l} Y_{lj}\delta_j)_{l=1,...,\mu}$$

□

Let B be the T-algebra classifying pairs of square matrices,

$$U \text{ of order } p \text{ and } V \text{ of order } q$$

such that

$$UX = X'V.$$

We can easily verify that the algebra B is smooth over T outside of $V(\mathfrak{J})$. Indeed, let's localize at a point x of $\mathrm{Spec}\,(T)$ above $\mathrm{Spec}\,(A) - V(\mathfrak{J})$ and let's denote its local ring by T_x. We then have a direct sum decomposition

$$T_x \cong \ker X_x \oplus \mathrm{Coim}\ X_x = \ker X'_x \oplus \mathrm{Coim}\ X'_x,$$

$$T_x^q \cong \text{Im } X_x \oplus \text{Coker } X_x = \text{Im } X'_x \oplus \text{Coker } X'_x.$$

and the preceding assertion follows easily from this remark.

On the other hand, the data of M and M' with their presentations correspond to an A-section of T. It then suffices to use corollary 1 of theorem 2 to conclude the proof of lemma 5.

End of the proof of Theorem 3. — Lemma 5 combined with the first part of the proof allows to immediately terminate the demonstration of Theorem 3 in the case where the rank of M is constant on $\text{Spec}\,(\hat{A}) - V(\mathfrak{J}\hat{A})$. The general case results for example from the following lemma, analogous to Lemma 5.

Lemma 0.17. *Let $(A, \mathfrak{J})$ be a noetherian henselian pair, M an A-module locally free outside the closed set $V(\mathfrak{J})$ endowed with a presentation*

$$A^p \xrightarrow{L} A^q \to M \to 0.$$

Then there exists a pair of integers (n_0, s) having the following property: let n be an integer greater than n_0 and M' an A-module locally free and of the same rank as M at every point of $\text{Spec}\,(A) - V(\mathfrak{J})$ *and endowed with a presentation*

$$A^p \xrightarrow{L'} A^q \to M' \to 0$$

such that there exists a commutative diagram

$$\begin{array}{ccc} A^p/\mathfrak{J}^n & \xrightarrow{L_n} & A^q/\mathfrak{J}^n \\ \downarrow u_n & & \downarrow v_n \\ A^p/\mathfrak{J}^n & \xrightarrow{L'_n} & A^q/\mathfrak{J}^n \end{array}$$

[where L_n and L'_n are the reductions of L and L' and where u_n and v_n denote automorphisms of $A^p/\mathfrak{J}^n$ and $A^q/\mathfrak{J}^n$], then there exist automorphisms u and v of A^p and A^q respectively, congruent to u_n and v_n modulo $\mathfrak{J}^{n-s}$ and such that $vL = uL'$.

Outline of the proof. — Let $\bar{A}$ be the finite A-algebra obtained by separating the connected components of $\mathcal{U}$. We have an exact sequence of A-modules.

$$0 \to H \to A \to \bar{A} \to K \to 0.$$

And j is an isomorphism outside of $V(\mathfrak{J})$ so there exists an integer h such that we have

$$\mathfrak{J}^h K = 0 \quad \text{and} \quad \mathfrak{J}^h \cap H = 0.$$

So let $n > h$ and

$$u_n : A^p/\mathfrak{J}^n \cong A^p/\mathfrak{J}^n,$$

an automorphism admitting a lifting

$$u : A^p \to A^p.$$

We can easily show that there exists a unique $u : A^p \to A^q$ such that

$$\bar{u} = u \otimes_{\bar{A}} \text{id}.$$

The result follows immediately. And this completes the proof of theorem 3.

Corollary 0.18. *Let $\mathcal{U}$ be the open complement of $V(\mathfrak{J})$ in* $\text{Spec}\,(A)$*, $\mathcal{U}'$ its inverse image in* $\text{Spec}\,(\hat{A})$ *and let P be a locally free $\mathcal{U}$-module of finite type. Then there exists a locally free module of finite type on $\mathcal{U}$ whose inverse image on $\mathcal{U}'$ is isomorphic to P. It suffices to extend P to a finite type $\hat{A}$-module and apply theorem 3.*

Lemma 0.19. *With the notations of corollary 1, let P and Q be two locally free $\mathcal{U}$-modules of finite type whose inverse images $\hat{P}$ and $\hat{Q}$ on $\mathcal{U}'$ are isomorphic. Then P and Q are isomorphic. Indeed, let M be a finite type $\hat{A}$-module extending $\hat{P}$ or $\hat{Q}$. According to a result from [4], one can find a finite type A-module M (resp. N) whose completion is isomorphic to $\hat{M}$ and whose restriction to $\mathcal{U}$ is isomorphic to P (resp. Q). M and N are isomorphic since they have the same completion, hence P and Q are isomorphic.*

Corollary 0.20. *The application*

$$Pic\,\mathcal{U} \to Pic\,\mathcal{U}'$$

is bijective, and A is parafactorial [EGA IV, 21-13] if and only if $\hat{A}$ is.

This is what the two preceding results express when applied to invertible modules. This result was demonstrated by J. Boutot [3] in the case where the ring A is local of dimension 2 or when it verifies the conditions of application of M. Artin's approximation theorem [1].

The following corollary allows to descend blow-ups of $\hat{A}$ with support in $V(\mathfrak{J})$.

Corollary 0.21. *Let $\hat{J}$ be an ideal of $\hat{A}$ locally free of rank 1 over $\mathcal{U}'$. Then for all n there exists an ideal J of A congruent to $\hat{J}$ modulo $\mathfrak{J}^n$ and such that the $\hat{A}$-modules $J \otimes \hat{A} = \hat{J} \otimes \hat{A}$ and J are isomorphic.*

Let

$$A^p \xrightarrow{L} A^q \to M \to 0, \quad L = (l_{ij}) \quad (i \in [1,p], j \in [1,q]),$$

be a presentation of an A-module M, locally free of rank 1 over $\mathcal{U}$ and whose completion is isomorphic to $\hat{J}$. We want to realize M as an ideal of A. The morphism $M \otimes_A \hat{A} \to \hat{J} \to \hat{A}$ defines a section $\bar{\varepsilon}$ over $\hat{A}$ of the vector bundle of $M \otimes_A \hat{A}$. Let

$$B = A[Y_1, ..., Y_q]/(\sum_{j=1}^{q} l_{ij} Y_j)_{i=1,...,p}$$

be the symmetric algebra of this A-module, and let $(y_i)_{i=1,...,q}$ be the system of generators of $\hat{J}$ in $\hat{A}$ defined by

$$\bar{y}_i = \bar{\varepsilon}(Y_i).$$

Let, moreover, s be an integer sufficiently large so that we have in $\hat{A}$:

$$\mathfrak{J}^{s+k}\hat{A} \cap \hat{J} \subset \mathfrak{J}^k \hat{J}, \quad \forall k \geq 0 \quad \text{(Artin-Rees)}$$

and let us fix $n > s$. We can approximate $\bar{\varepsilon}$ by an A-section, ε'; of Spec$\,(B)$, congruent to $\bar{\varepsilon}$ modulo $\mathfrak{J}^n$. From this we deduce a complex

$$(\star) \quad A^p \xrightarrow{L} A^q \xrightarrow{\varepsilon} A.$$

The image of A^q in A is an ideal J whose system of generators is constituted by the elements

$$y_i = \varepsilon(Y')$$

and we have

$$y_i = \bar{y}_i \pmod{\mathfrak{J}^n}.$$

Let's tensorize $(\star)$ by $\hat{A}$; we obtain a complex.

$$\hat{A}^p \xrightarrow{\hat{L}} \hat{A}^q \xrightarrow{\hat{\varepsilon}} \hat{A}.$$

As $\bar{J}$ is isomorphic to the cokernel of $\hat{L}$, we have a surjection of $\hat{A}$-modules

$$J \otimes_A \hat{A} \to \bar{J}.$$

And it remains to show that it is an isomorphism. Consider the diagram

$$\begin{tikzcd}
 & \hat{A}^q \arrow[r, "\bar{\pi}"] \arrow[d, "\pi"] \arrow[dl, "\hat{L}"'] & J \arrow[d, "f"] \\
\hat{A}^p \arrow[r, "\hat{\varepsilon}"] & J \otimes_A \hat{A} \arrow[r] & \hat{A}
\end{tikzcd}$$

in which $\bar{\pi}$ (resp. π) is the application defined by $\bar{\pi}(e_i) = \bar{y}_i$ [resp. $\pi(e_i) = y_i$], if $(e_1, ..., e_q)$ is the canonical basis of A^q, and f the linear application defined by $f(\bar{y}_i) = y_i$. We have for any element j of $\bar{J}$, $f(j) = j$ modulo $\hat{A}\mathfrak{J}^n$. We thus have

$$\ker f \subset \bar{J} \cap \mathfrak{J}^n,$$

thus

$$\ker f \subset \bar{J} \cap \mathfrak{J}^{n-s} \quad \text{(Artin-Rees)}$$

But let α be an integer and $j \in \mathfrak{J}^\alpha \bar{J} \cap \ker f$. We can write

$$j = \sum_{i=1}^{q} m_i \bar{y}_i = \sum m_i \bar{\pi}(e_i), \quad m_i \in \mathfrak{J}^\alpha.$$

We thus have in $\hat{A}$:

$$j = (\bar{\pi} - \pi)(\sum m_i e_i) = \sum m_i(\bar{y}_i - y_i),$$

thus $j \in \mathfrak{J}^{n+\alpha-s} J$. It follows that we have in fact

$$\ker f \subset J \cap (\cap \mathfrak{J}^\alpha).$$

Thus $\ker f = 0$ and f is an isomorphism of the $\hat{A}$-modules J and $J \otimes \hat{A}$, which establishes the corollary.

2. APPLICATIONS TO FINITE ALGEBRAS

We maintain the previous notations. The goal of this paragraph is the following theorem:

Theorem 0.22. *Let $\bar{C}$ be a finite $\hat{A}$-algebra, assumed to be a local complete intersection [EGA IV, 19.3.6] over $\mathcal{U}'$. Then for any n, there exists a finite A-algebra C, a local complete intersection over the open set $\mathcal{U}$, congruent to $\bar{C}$ modulo $\mathfrak{J}^n$ and such that the isomorphism of algebras of $C/\mathfrak{J}^n$ onto $\bar{C}/\mathfrak{J}^n\bar{C}$ lifts to an isomorphism of the $\hat{A}$-modules $C \otimes \hat{A}$ and $\bar{C}$.*

Definition 0.23. Let

$$A^p \xrightarrow{L} A^q \to M \to 0, \quad L = (l_{ij})_{i=1,...,p;j=1,...,q}$$

be a presentation of an A-module M. A commutative A-algebra structure on M equipped with a rigidification associated with the presentation consists of the data of the following A-linear applications:

$$\begin{aligned}
m &: A^q \otimes A^q \to A^q \quad \text{symmetric} \\
\varphi &: A^p \otimes A^q \to A^p, \\
\psi &: A^q \otimes A^p \to A^p, \\
\varepsilon &: A \to A^q, \\
\gamma &: A^p \otimes A^q \otimes A^q \otimes A^p \to A^p,
\end{aligned}$$

satisfying the relations:

(a) $m(L \otimes 1) = L \circ \varphi$, this relation expresses that m passes to the quotient and allows to define a symmetric bilinear application on M.

(b) $m(1 \otimes m) - m(m \otimes 1) = L \circ \psi$ (associativity).

(c) $m(1 \otimes \varepsilon) - \mathrm{Id}/A^q = L\gamma$, this relation translates that $\pi \circ \varepsilon(1)$ defines a neutral element for the multiplication on M and $\pi \circ \varepsilon$ endows M with a structure of an A-algebra.

Lemma 0.24. *(i) The functor of rigidified algebra structures on M is representable by a finite type affine A-scheme X.*

(ii) Suppose M is locally free and let $\mathcal{A}$ be the universal $\mathcal{O}_X$-algebra underlying $M \otimes_A \mathcal{O}_X$. The subfunctor of X which makes $\mathcal{A}$ a relative complete intersection over X is representable by an open set V of X smooth over A.

(i) The proof is very simple.

Let indeed $(e_i)_{i=1,\dots,q}$ be a basis of A^q, $(\bar{e}_j)_{j=1,\dots,p}$ a basis of A^p. We can classify the application m (resp. $\varphi, \psi, \varepsilon, \gamma$) by variables $(m_{ijk})_{i,j,k\in[1,q]}$ (resp. $\varphi_{ij}, \psi_{ijk}, \varepsilon_i, \gamma_{ij}$) where m_{ij}^k represents the component on e_k of $m(e_i \otimes e_j)$. Let's then only make the relation (b) explicit as an example $\forall i, j, k, \lambda \in [1, q]$:

$$\left(\sum_{\mu=1}^{q} m_{\mu k}^{\lambda} m_{ij}^{\mu}\right) - \left(\sum_{\mu=1}^{q} m_{i\mu}^{\lambda} m_{jk}^{\mu}\right) = \sum_{\nu=1}^{p} \psi_{ijk}^{\nu} l_{\nu\lambda}$$

(ii) It is easy to see that this subfunctor is representable by an open set of X, let it be V. Verifying the smoothness of V amounts to verifying the following property. Let B be a local ring, B_0 a quotient of B by a nilpotent ideal

$$B \to B_0 \to 0$$

a presentation of a free B-module inducing

$$B_0^p \to B_0^q \to M_0 = M \otimes_B B_0 \to 0.$$

Suppose given on M_0 a structure of a rigidified B_0-algebra which is a complete intersection. Then this structure lifts. Let's first lift the algebra structure. We know that there is no obstruction to lifting a complete intersection. We then verify without difficulty that the different introduced rigidifications lift.

Proof of the theorem. Let

$$A^p \xrightarrow{L} A^q \to M \to 0$$

be a presentation of an A-module M whose completion is $\hat{A}$-isomorphic to $\bar{C}$. We know that this is possible thanks to theorem 3. Let's consider the scheme X over A defined in the lemma and let V be the open set of X over $\mathcal{U}$ over which $\mathcal{A}$ is a local complete intersection. Let $\hat{X} = X \otimes \hat{A}$ and $\hat{V}$ be the inverse image of V in $\hat{X}$. The algebra $\bar{C}$ corresponds to an $\hat{A}$-section $\bar{\varepsilon}$ of $\hat{X}$ such that $\bar{\varepsilon}(\mathcal{U}') \subset \hat{V}$. Theorem 4 then results immediately from theorem 2 bis. □

Theorem 0.25. *The application $B \to B \otimes_A \hat{A}$ is an equivalence of categories between the category of finite A-algebras, inducing an étale covering of $\mathcal{U}$ and the corresponding category over $\hat{A}$ and $\mathcal{U}'$.*

(a) Let's first show that a finite A-algebra, étale over $\mathcal{U}'$ descends to A.

The set of points where an algebra is étale being an open set contained in the set of points where it is a relative complete intersection, the reasoning that allowed to establish theorem 4 applies here and allows to approximate the finite $\hat{A}$-algebras, étale over $\mathcal{U}'$ by finite A-algebras, étale over $\mathcal{U}$.

Let $\bar{C}$ be a finite $\hat{A}$-algebra, étale over $\mathcal{U}'$, M an A-module such that $M \otimes \hat{A}$ is isomorphic to $\bar{C}$ and endowed with a presentation

$$A^p \to A^q \to M \to 0.$$

Let X be the scheme defined in the previous paragraph and $\mathcal{A}$ the universal algebra. Let $j : \operatorname{Spec} \mathcal{A} \to \operatorname{Spec} X[Y_1, ..., Y_N]$ be a plunging of $\mathcal{A}$ into an affine space over X and H_a be an ideal defining the singular locus of $\mathcal{A}$ [the ideal H was defined in paragraph 0]. Let G be the open set of X over which $\mathcal{A}$ is étale. The algebra $\bar{C}$ corresponds to a section $\bar{\varepsilon}$ of $X \otimes \hat{A}$ which factors over $\mathcal{U}'$ through G. If $\bar{H}_{\bar{\varepsilon}}$ denotes the inverse image of H_a by $\bar{\varepsilon}$, there exists an integer h such that

$$\bar{H}_{\bar{\varepsilon}} \supset \mathfrak{J}^h \bar{C}.$$

According to theorem 1, there exists a pair (n_0, r) such that for any $n > n_0$ and any approximate section σ_n modulo $\mathfrak{J}^n C$ of a $\bar{C}$-algebra C' such that

$$\sigma_n^*(H_\varepsilon) \supset \mathfrak{J}^h \bar{C}.$$

We can find a section

$$\sigma : \operatorname{Spec} C \to \operatorname{Spec} C'$$

congruent to σ_n modulo $\mathfrak{J}^{n-r}$. Let's fix n greater than n_0 and h and let C be an A-algebra defined by a section ε of X, congruent to $\bar{\varepsilon}$ modulo $\mathfrak{J}^n$ and factoring through G over $\mathcal{U}'$. It results from the choice of n that we will still have $H_\varepsilon \supset \mathfrak{J}^h C$ and that the isomorphism of $C/\mathfrak{J}^n C$ onto $\bar{C}/\mathfrak{J}^n \bar{C}$ is congruent mod $\mathfrak{J}^{n-r}$ to a morphism of $\hat{A}$-algebras

$$\hat{\sigma} : C \otimes_A \hat{A} \to \bar{C}.$$

We can easily see then that $\hat{\sigma}$ is an isomorphism.

In effect, this application is surjective. Moreover, the isomorphism $C/\mathfrak{J}^n C \to \bar{C}/\mathfrak{J}^n \bar{C}$ comes by construction from an isomorphism of modules C_n and $\bar{C}_n$, so that $\hat{\sigma}$, congruent modulo $\mathfrak{J}^{n-r}$ to an isomorphism of modules, is an isomorphism of modules and therefore of algebras.

(b) Let B and B' be two finite A-algebras, étale outside of $V(\mathfrak{J})$. To finish the proof of theorem 5 it suffices to show that the application

$$\operatorname{Hom}_{A\text{-alg}}(B, B') \to \operatorname{Hom}_{\hat{A}\text{-alg}}(B \otimes_A \hat{A}, B' \otimes_A \hat{A})$$

is bijective.

A change of base $A \to B'$ allows us to reduce to the case where $B' = A$. Let thus B be finite over A étale over $\mathcal{J}$ and

$$\bar{\varepsilon} : B \otimes_A \hat{A} \to \hat{A}.$$

We know (theorem 2 bis) that for any $n \in \mathbb{N}$ there exists an A-section ε of B congruent to $\bar{\varepsilon}$ modulo $\mathfrak{J}^n$. It suffices then to show the following proposition: There exists an integer $n \in \mathbb{N}$ such that any section of $\bar{B} = B \otimes_A \hat{A}$ congruent to $\bar{\varepsilon}$ modulo $\mathfrak{J}^n$ is equal to $\bar{\varepsilon}$.

The proof of this proposition appears in M. Artin [1]. We will recall it nevertheless. Let T be the ideal of A formed by the elements with support in $V(\mathfrak{J})$, A_0 the quotient of A by T and $B_0 = B \otimes_A A_0$. Let n be an integer large enough so that any A_0-section of B_0 congruent to $\bar{\varepsilon}_0$ modulo $\mathfrak{J}^n$ is equal to $\bar{\varepsilon}_0$ and so that we have

$$T \cap \mathfrak{J}^n = (0).$$

It is clear that then ε congruent to $\bar{\varepsilon}$ modulo $\mathfrak{J}^n$ implies $\varepsilon = \bar{\varepsilon}$. We can thus suppose A without section with support in $V(\mathfrak{J})$. Under this hypothesis the kernel of a section $\varepsilon : \bar{B} \to \hat{A}$ is entirely determined by its data over $\mathcal{U}'$ and since $\operatorname{Spec} B/\mathcal{U}'$ is étale over $\mathcal{U}'$ it is determined by the open and closed part of $\operatorname{Spec} B/\mathcal{U}'$, underlying $\varepsilon(\mathcal{U}')$. There exists thus a finite number of sections, which concludes the proof of the proposition and thus of the theorem.

Corollary 0.26. *With the preceding notations, suppose $\mathcal{U}$ (thus $\mathcal{U}'$) connected. We have*

$$\pi_1(\mathcal{U}) = \pi_1(\mathcal{U}').$$

3. APPROXIMATIONS OF SMOOTH ALGEBRAS.

Theorem 0.27. *Let $(A, \mathfrak{J})$ be a noetherian henselian pair. Then any smooth $A/\mathfrak{J}$-algebra $\bar{B}$ lifts to a smooth A-algebra B.*

Proof. Let's set

$$A/\mathfrak{J} = \bar{A}.$$

(a) Suppose first that B is a relative complete intersection over $\bar{A}$, that is to say, suppose that it admits a presentation

$$\bar{B} = \bar{A}[X_1, ..., X_N]/\bar{J}, \quad \bar{J} = (\bar{f}_1, ..., \bar{f}_m),$$

and that the ideal generated by the minors of rank m of the jacobian matrix $(\partial \bar{f}_i/\partial X_j)_{i=1,...,m; j=1,...,N}$ is the unit ideal in $\bar{B}$. In this case it suffices to lift the $\bar{f}_i$ to f_i in $[X_1, ..., X_N]$. The algebra $B = A[X_1, ..., X_N]/(f_i)$ is smooth in the neighborhood of the closed set $V(J)$ and a localization of B answers the question.

(b) In the general case let

$$\bar{B} = \bar{A}[X_1, ..., X_N]/J, \quad J = (f_1, ..., f_q),$$

be a presentation of $\bar{B}$. The $\bar{B}$-module J/J^2 is locally free on $\bar{B}$. Let $B^p \to B^q \to 0$ be a presentation of this $\bar{B}$-module and let C be its symmetric algebra. $\bar{C}$ is smooth over $\bar{A}$ of relative dimension N and a global complete intersection according to a previous lemma. There exists thus a smooth A-algebra C inducing $\bar{C}$ on $\bar{A}$. Let's consider the following diagram in which the morphisms are the canonical morphisms

$$\begin{tikzcd}
\operatorname{Spec} \bar{C} \arrow[r] & \operatorname{Spec} C \\
\operatorname{Spec} \bar{B} \arrow[u, "f"] & \\
\operatorname{Spec} A/\mathfrak{J} \arrow[u] \arrow[r] & \operatorname{Spec} A
\end{tikzcd}$$

Let $\bar{M} = f^*(J/J^2)$. It is a locally free $\bar{C}$-module endowed with a presentation

$$\bar{C}^p \xrightarrow{\bar{L}} \bar{C}^q \to \bar{M} \to 0.$$

□

If we denote by $\tilde{C}$ the $\mathfrak{J}$-adic henselization of C, there exists a locally free $\tilde{C}$-module M admitting a presentation of the same type as $\bar{M}$ and lifting the $\bar{C}$-module $\bar{M}$ (endowed with its presentation). Up to replacing C by an étale neighborhood of $\mathfrak{J}$ in C, we can suppose M defined on C and endowed with the presentation.

$$C^p \xrightarrow{L} C^q \to M \to 0.$$

So let

$$D = C[Y_1, ..., Y_q]/(\sum l_{ij} Y_j)$$

be the symmetric algebra of this module, and let's consider

$$\begin{tikzcd}
 & & \operatorname{Spec} D \otimes_C \bar{C} = \operatorname{Spec} \bar{C} \otimes_{\bar{B}} \bar{C} \arrow[d] \\
\operatorname{Spec} D \arrow[r] & \operatorname{Spec} C \arrow[r] & \operatorname{Spec} \bar{C} \arrow[d, "f"] \\
 & & \operatorname{Spec} \bar{B}
\end{tikzcd}$$

Spec D is smooth over C so that, up to replacing again C by an étale neighborhood of $\mathfrak{J}$ in C, we can suppose that the diagonal section

$$\operatorname{Spec} C \to \operatorname{Spec} C \otimes_B \bar{C} = \operatorname{Spec} \bar{C} \otimes_C D$$

lifts to a section $\Delta : \operatorname{Spec} C \to \operatorname{Spec} D$. Let K be the ideal of C inverse image by Δ of the ideal $(Y_1, ..., Y_q)$ of D. The quotient of C by the ideal K is a smooth A-algebra lifting B.

We answer in particular the question of Monsky who asked in [11] if there existed a "weakly complete" lifting of a smooth algebra defined on $A/\mathfrak{J}$. This is in effect implied by the existence of an algebraic lifting.

ALGEBRAIZATION OF SMOOTH FORMAL ALGEBRAS OUTSIDE A CLOSED SET. — Let A be a complete ring for a $\mathfrak{J}$-adic topology. Let again $\hat{B}$ be a formal $\hat{A}$-algebra of finite type

$$\hat{B} = \hat{A}\{X_1, ..., X_N\}/J, \quad J = (f_1, ..., f_m).$$

We will say that $\hat{B}$ is formally smooth over $\hat{A}$ outside of $V(J)$ if at every point x of $\operatorname{Spec} \hat{B} - V(J\hat{B})$ there exists an integer d and a family $f_1, ..., f_d$,

elements of $\mathfrak{J}$ generating this ideal in a neighborhood of x in $\operatorname{Spec} \hat{A}\{X_1, ..., X_N\}$ and such that the minors of order d of the jacobian $(\partial f_i/\partial X_j)_{i=1,...,d; j=1,...,N}$ generate the unit ideal at this point.

We propose to demonstrate the following theorem:

Theorem 0.28. *Let $(A, \mathfrak{J})$ be a noetherian henselian pair, $\hat{A}$ the $\mathfrak{J}$-adic completion of A and suppose $\mathfrak{J}$ principal, $\mathfrak{J} = (a)$. Let $\hat{B}$ be a formal $\hat{A}$-algebra of finite type, formally smooth over $\hat{A}$ outside of $V(\mathfrak{J})$. Then there exists a finite type A-algebra B' smooth over* Spec A *outside of $V(\mathfrak{J})$ whose formal completion is isomorphic to $\hat{B}$.*

We first demonstrate the following lemma (which does not suppose $\mathfrak{J}$-principal).

Lemma 0.29. *Let $\hat{B} = \hat{A}\{X_1, ..., X_N\}/J$, $J = (f_1, ..., f_m)$, be a formal $\hat{A}$-algebra smooth outside of $V(\mathfrak{J})$. Let's give a presentation of the $\hat{A}\{X\}$ module J (we write $\hat{A}\{X\}$ for $\hat{A}\{X_1, ..., X_N\}$):*

$$\hat{A}\{X\}^q \xrightarrow{L} \hat{A}\{X\}^m \to J \to 0.$$

Then there exists a pair n_0, r having the following property. If n is an integer greater than n_0 and B' a quotient of $\hat{A}\{X_1, ..., X_N\}$ defined by equations $f'_1, ..., f'_m$ satisfying $f' = f_i(\mathfrak{J}^n)$, $i \in [1, m]$, and a system of relations $L'f' = 0$ in which L' is a matrix (q, m) with coefficients in $\hat{A}\{X_1, ..., X_N\}$ congruent to L modulo $\mathfrak{J}^n$, then there exists an $\hat{A}$-automorphism of $\hat{A}\{X_1, ..., X_N\}$ congruent to the identity modulo $\mathfrak{J}^{n-r}$ and inducing an isomorphism of B onto B'.

Proof. Let's associate to B an ideal H_B as in paragraph 1 [generated by products of minors of L and of $(\partial f_i/\partial X_j)$]. There exists $h \in \mathbb{N}$ such that

$$H_B + J \supset \mathfrak{J}^h \hat{A}\{X\}.$$

For $n > h$ we will still have B' formally smooth over $\operatorname{Spec} \hat{A}$ outside of $V(\mathfrak{J})$ and

$$H_{B'} + J' \supset \mathfrak{J}^h \hat{A}\{X\}.$$

This is clear from the definition of H, since we can take

$$H_{B'} = H_B \pmod{\mathfrak{J}^n}.$$

□

We suppose therefore $n > h$ in the following. We then look for a section over $\hat{B}$ of

$$\operatorname{Spec}\hat{B} \times_{\operatorname{Spec}\hat{A}} \operatorname{Spec}\hat{B}' \to \operatorname{Spec}\hat{B}$$

which is congruent to the diagonal modulo $\mathfrak{J}^{n-r}$. According to theorem 1 and the remark that follows it, there exists a pair of integers (n_1, s) such that for $n > n_1$ we can find a morphism $\sigma : \hat{B}' \to \hat{B}$ congruent modulo $\mathfrak{J}^{n-s}$ to the isomorphism given modulo $\mathfrak{J}^n$. It remains to show that for n large enough, σ is an isomorphism.

We can find an $\hat{A}$-morphism

$$\varphi : \hat{A}\{X\} \to \hat{A}\{X\}$$

congruent to the identity modulo $\mathfrak{J}^{n-s}$, and such that the following diagram is commutative

$$\begin{tikzcd}
0 \arrow[r] & J' \arrow[r] & \hat{A}\{X\} \arrow[r] & \hat{B}' \arrow[r] \arrow[d, "\sigma"] & 0 \\
0 \arrow[r] & J \arrow[r] & \hat{A}\{X\} \arrow[r] \arrow[u, "\varphi"] & \hat{B} \arrow[r] & 0
\end{tikzcd}$$

We want to show that for n large enough, we have

$$\varphi(J') = J.$$

We obviously have $\varphi(J') \subset J$ since the diagram commutes. Moreover

$$J \subset \varphi(J') + \mathfrak{J}^{n-s} \cap J, \quad \text{because } J = J'(\mathfrak{J}^n) \text{ and } \varphi = \mathrm{id}_{\hat{A}\{X\}}(\mathfrak{J}^{n-s}).$$

We deduce from these two inclusions that

$$J \subset \varphi(J') + \mathfrak{J}^{n-s-s'} \cap J.$$

According to the Artin-Rees theorem, this ensures that for n large enough (not depending on J) we will have

$$\mathfrak{J}^{n-s-s'} \cap J \subset \mathfrak{J}J,$$

thus

$$J \subset \varphi(J') + \mathfrak{J}J,$$

thus

$$J \subset \varphi(J')$$

which proves the lemma. We now prove the theorem in a particular case.

Lemma 0.30. *Let $\hat{B}$ be as in Theorem 7. We make the supplementary hypothesis that the system of equations $(f_1, ..., f_d)$ generates J in the neighborhood of* $\operatorname{Spec}\hat{B} - V(\mathfrak{J})$ *in* $\operatorname{Spec}\hat{A}\{X_1, ..., X_N\}$ *(d is the codimension of $\hat{B}$ in $\hat{A}\{X_1, ..., X_N\}$ at these points). Then there exists an affine A-scheme of finite type smooth outside of $V(a)$ whose formal completion is A-isomorphic to Spf$\hat{B}$.*

Proof. Let's consider the closed set $V(f_1, ..., f_d)$ in $\operatorname{Spec}\hat{A}\{X_1, ..., X_N\}$. Over the open set $\operatorname{Spec}\hat{A}_a$, $\operatorname{Spec}\hat{B} \otimes \hat{A}_a$ is an open and closed part of $\operatorname{Spec}\hat{A}\{X_1, ..., X_N\}/(f_1, ..., f_d) \otimes \hat{A}_a$. It corresponds to an idempotent e. We can write in $\hat{A}\{X\} \otimes \hat{A}_a$:

$$e(1-e) = \sum_{i=1}^{d} \mu_i f_i.$$

After multiplication by a suitable power of a, we obtain in $\hat{A}\{X_1, ..., X_N\}$ a relation that we put in the form

$$(1) \quad ke = \sum_{i=1}^{d} \lambda_i f_i,$$

retaining that the ideal generated in $\hat{A}\{X_1, ..., X_N\}$ by k and e contains a power of a, say a^h. Moreover $\hat{B} \otimes \hat{A}_a$ is defined by $(f_1, ..., f_d, e)$ in $\hat{A}\{X_1, ..., X_N\} \otimes \hat{A}_a$. Let $\hat{\Delta}$ be the ideal generated in $\hat{B}$ by the minors of order d of the jacobian $(\partial f_i / \partial X_j)_{i=1,...,d; j=1,...,N}$. There exists an integer h_1 such that we have in $\hat{B}$, $a^h \hat{B} \subset \hat{\Delta}$. Fix an integer h greater than $h_0 + h_1$, and consider over $A[X_1, ..., X_N]$, the $\mathfrak{J}$-adic henselization of $A[X_1, ..., X_N]$, the algebra R defined by

$$R = A[X_1, ..., X_N][F_1, ..., F_d, E, K, \Lambda_1, ..., \Lambda_d]/(\sum \Lambda_i F_i - KE).$$

Let σ be the section

$$\operatorname{Spec} \hat{A}\{X_1, ..., X_N\} \to \operatorname{Spec} R \otimes_{A[X_1,...,X_N]} \hat{A}\{X_1, ..., X_N\}$$

defined by (k, e, λ_i, f_i). It factors over the open set

$$\operatorname{Spec} \hat{A}\{X_1, ..., X_N\} - V(a)$$

through a smooth open set since we remarked that the ideal generated by (k, e) contains a power of a. We can thus approximate this solution over $A[X_1, ..., X_N]$. □

Let n be an integer greater than h, ε_n an approximate section modulo $\mathfrak{J}^n$ of σ defined by $(k_n, e_n, \lambda_{in}, f_{in})$ in $A[X_1, ..., X_N]$ and let C_n be the algebra $A[X_1, ..., X_N]/(f_{in}, e_n)$, $\hat{C}_n$ its formal completion. C is formally smooth over $\operatorname{Spec} \hat{A}$ outside of $V(\mathfrak{J})$. It is clear in effect thanks to the imposed congruences, that the minors of the jacobian $(\partial f_i / \partial X_j)$ generate a power of a and the equations (f_n) suffice to generate the definition ideal of C_n in the neighborhood of $\operatorname{Spec} C_n - V(a)$ in $\operatorname{Spec} \hat{A}\{X\}$. This is what the equation

$$k_n e_n = \sum \lambda_{in} f_{in}$$

ensures.

According to lemma 1, there exists a pair (n_0, s) such that for $n > n_0$, there exists an automorphism φ of $\hat{A}\{X_1, ..., X_N\}$ congruent to the identity modulo $\mathfrak{J}^{n-s}$ and inducing an isomorphism of $\hat{C}_n$ onto $\hat{C} = \hat{A}\{X\}/(f_1, ..., f_d, e)$. We have

$$\varphi^{-1}((f_{1n}, ..., f_{dn}, e_n)) = (f_{1n}, ..., f_{dn}, e_n).$$

Let $\bar{J}_n$ be the image of J by φ_n^{-1}, $\bar{J}_n$ defines a quotient B_n of $\hat{A}\{X_1, ..., X_N\}$ isomorphic to B. Moreover this ideal coincides with the ideal $(f_{1n}, ..., f_{dn}, e_n)$ in the open set $\operatorname{Spec} \hat{A}\{X\} - V(\mathfrak{J})$. It results then from proposition 2 of [4] that there exists an ideal J_0 of $A[X_1, ..., X_N]$ coinciding with $(f_{1n}, ..., f_{dn}, e_n)$ in $\operatorname{Spec} A[X_1, ..., X_N] - V(\mathfrak{J})$ and generating $\bar{J}_n$ in $\hat{A}\{X_1, ..., X_N\}$. The formal completion of the quotient B_0 of $A[X_1, ..., X_N]$ by J_0 is isomorphic to B.

On the other hand, one can find a quotient B of an étale neighborhood $\tilde{A}$ of J in $A[X_1, ..., X_N]$ such that

$$B \otimes_A A[X_1, ..., X_N] \text{ is } B_0.$$

This A-algebra B fulfills the required conditions and this demonstrates lemma 7.

End of the proof of Theorem 7. — We now resume the general case, the end of the proof is quite analogous to that of Theorem 6. Let

$$\hat{B} = \hat{A}\{X_1, ..., X_N\}/J, \quad J = (f_1, ..., f_m) \quad \text{and} \quad \hat{A}\{X\}^q \xrightarrow{L} \hat{A}\{X\}^m \to J \to 0$$

be a presentation of J. And let's consider $\hat{C}$ the formal completion of the symmetric algebra of the conormal sheaf J/J^2 on $\hat{B}$.

$\hat{C}$ verifies the conditions of application of lemma 7 if we consider it as plunged into a formal power series algebra with $2N + m$ variables over $\hat{A}$. (We extend without difficulty lemma 4 of theorem 2 bis to the formal case). So let C_0 be an algebraization of $\hat{C}$ given by lemma 7. We have the diagram

$$\begin{tikzcd} \operatorname{Spec} C_0 \arrow[r, "\phi"] & \operatorname{Spec} \hat{C} & \operatorname{Spec} B \arrow[l, "\varepsilon"'] \\ \operatorname{Spec} A \arrow[u] \arrow[r] & \operatorname{Spec} \hat{A} \arrow[u] & \end{tikzcd}$$

where ε is the unit section and where ϕ defines an isomorphism of $\hat{C}_0$ onto $\hat{C}$ which allows to identify these two algebras in the following. Let M be the $\hat{C}$-module $f^*(J/J^2)$ locally free outside of $V(\mathfrak{J}C)$. For any n we can find, up to replacing C_0 by an étale neighborhood of $V(\mathfrak{J}C_0)$, a C_0-module M_0 whose completion $M_0 \otimes_{C_0} \hat{C}$ is isomorphic to M and endowed with a presentation

$$C_0^q \xrightarrow{L_0} C_0^m \to M_0 \to 0.$$

verifying $L_0 = L(\mathfrak{J}^n)$ in $\hat{C}$.

Suppose n fixed (we will specify it later) and M_0 thus chosen. We then construct D_0 the symmetric algebra of M_0 over C_0:

$$D_0 = C_0[Z_1, ..., Z_m] / \sum C_{ij} Z_j.$$

Its completion $\hat{D}_0$ is $\hat{C}$-isomorphic to the product $\hat{C} \times_B \hat{C}$ and for any $p \in \mathbb{N}$, up to replacing C_0 by an étale neighborhood, we can approximate modulo $\mathfrak{J}^p$ the diagonal section of $\hat{C} \times_B \hat{C}$ into a section

$$\delta : \operatorname{Spec} C_0 \to \operatorname{Spec} D_0$$

(thanks to theorem 2 bis).

The ideal generated in C_0 by $\delta^*(Z_1, ..., Z_m)$ allows to define a quotient B' of C_0. Its formal completion $\hat{B}'$ can then be identified with a closed subscheme of $\hat{C}$ defined by equations congruent to the definition equations of B in $\hat{C}$, $(Y_1, ..., Y_m)$, modulo $\mathfrak{J}^{\sup(n,p)}$ and linked by relations defined by L_0 thus congruent to the relations between the Y_i modulo $(\mathfrak{J}^n)$.

It then results from lemma 6 that for n and p sufficiently large, the formal completion $\hat{B}'$ of B' is isomorphic to B, which concludes the demonstration.

Remark 0.31. One might think that theorem 7 is still true when we no longer suppose $\mathfrak{J}$ monogenous. A recurrence comparable to the one we used previously would probably allow to extend it.

Remark 0.32. Some of these results, in particular theorems 3 and 6, are to be compared with those of Hironaka [5].

Remark 0.33. Where we eliminate some noetherian hypotheses: (a) Let's first consider lemma 1 of I. The hypothesis A noetherian is only used when we use Artin-Rees to define an integer k such that (with the notations of the lemma): $A \cap (t^k) = (0)$. The result thus extends without change if instead of supposing A noetherian, we only suppose the existence of such an integer. Outside of the trivial case where the ring A has no section with support in $V(t)$, this happens if we suppose for example A flat over a noetherian ring A_0, and t image of an element t_0 of A_0.

(b) Generalizations of the same type apply to theorem 2 bis. Let's take for $(A, \mathfrak{J})$ a not necessarily noetherian henselian pair but let's suppose one or the other of the following hypotheses satisfied (i) $\mathfrak{J}$ is generated by an element a whose a fixed power annihilates every element of A with support in $V(a)$. (ii) A is flat over a noetherian ring A_0, and there exists an ideal $\mathfrak{J}_0$ of A_0 such that $\mathfrak{J} = \mathfrak{J}_0 A$. We always suppose moreover that B is an A-algebra of finite presentation. Then theorem 2 bis remains valid and the demonstration extends without modification. In lemmas 2 and 3 the hypothesis A noetherian does not intervene. In lemma 3 the introduced algebra C is not necessarily of finite presentation but only of finite type. However it is locally of finite presentation over the open set $\operatorname{Spec} A - V(\mathfrak{J})$ that we consider. Lemma 4 and the end of the demonstration of theorem 2 bis then extend for the reasons invoked in remark (a) above. Under these hypotheses we can briefly say that most of the approximation results of formal structures by henselian structures remain valid. In the case (ii), they are however insufficient to obtain a descent result since we do not have in general theorem 1.

Let's take the example of theorem 3. Under hypotheses (i) or (ii) one can, given an $\hat{A}$-module of finite presentation M_0, locally free of rank r over $\operatorname{Spec} \hat{A} - V(\mathfrak{J}\hat{A})$ find an A-module M locally free of rank r over $\operatorname{Spec} A - V(\mathfrak{J})$ and endowed with a presentation congruent to that of M modulo an arbitrary power of $\mathfrak{J}$.

But we do not know how to ensure that its completion is isomorphic to M unless we can apply to $\hat{A}$ theorem 1, for example in case (i) or when $\hat{A}$ is noetherian since we then know how to construct the automorphisms u and v of lemma 5.

(c) Let's now consider the following case. Let's denote by A_0 a valuation ring of height 1, by $\mathfrak{m}$ its maximal ideal and let a be an element of strictly positive valuation. Let again A be the a-adic henselian of a flat A_0-algebra of finite presentation and $\hat{A}$ its completion (complete always means separated complete and A is separated). It results from the preceding remarks that theorems 1 and 2 can be applied to A and $\hat{A}$ (by replacing however the hypotheses of finite type by those of finite presentation). The proofs of theorems 3 and 4 also adapt to this situation without important modification.

We can likewise approximate the algebra $\bar{C}$ of theorem 5 by a finite A-algebra C, étale over $\operatorname{Spec} A - V(aA)$ and if we suppose C separated a sufficient approximation will provide an A-algebra whose completion is isomorphic to $\bar{C}$. We are now interested in theorem 7 (of which theorem 6 is a particular case). We will suppose that the algebra $\hat{B}$, separated complete is of finite presentation over $\hat{A}$,

$$\hat{B} = \hat{A}\{X_1, ..., X_N\}/J, \quad J = (f_1, ..., f_m).$$

And we will suppose that J is itself an $A\{X\}$ module of finite presentation. To be able to trace the demonstration of theorem 7 and find an A-algebra B' whose formal completion is isomorphic to B it suffices to verify that the morphism σ constructed in lemma 6 is still an isomorphism despite the absence of noetherian hypotheses. It suffices for this that we can still write an Artin-Rees formula:

$$J \cap (a^n) \subset a^{n-r} J \quad \text{for } n > 0.$$

But let I be an ideal of a power series ring $A_0\{T_1, ..., T_q\}$. Its saturated I', is by definition the smallest ideal of $A_0\{T\}$ containing I, and such that the quotient $A_0\{T\}/I$ is A_0-flat. We have

$$I = \{x \in A_0\{T\} \text{ such that } \exists p \in \mathbb{N}/a^p x \in I\}.$$

Then I' is of finite type. This implies that there exists an integer n_0 such that

$$\forall x \in I', a^{n_0} x \in I.$$

We thus have

$$\forall n > n_0, \quad I' \cap (a^n) \subset a^{n-n_0} I$$

and this ends the demonstration.

IV DEFORMATIONS OF ISOLATED SINGULARITIES

1. Let A be a complete noetherian local ring, k its residue field, X_0 a k-scheme of finite type and equidimensional presenting one (or several) isolated singularities. $\mathcal{C}$ denotes the category of artinian A algebras having for residue field k. Following Schlessinger [9], for any $A \in \mathcal{C}$ we

call deformation of X_0 on A a pair formed by a scheme X of finite type, flat over A and an isomorphism $X \otimes_A k \cong X_0$. Two such deformations X and X' are said to be equivalent if there exists an A-isomorphism $X \to X'$ inducing the identity on X_0. We then define a functor $F : \mathcal{C} \to$ Ens, associating to each object A of $\mathcal{C}$ the set of equivalence classes of deformations of X_0 over A.

We can extend F to the category $\hat{\mathcal{C}}$ of complete local A-algebras $\hat{A}$ such that $\hat{A}/M^n \in \mathcal{C}$ for all n (M denotes the maximal ideal of $\hat{A}$) by setting

$$F(\hat{A}) = \lim_{\leftarrow n} F(\hat{A}/M^n).$$

An element of $F(\hat{A})$ can then be represented by a pair formed by a formal scheme of finite type flat over $\hat{A}$, and an isomorphism of its closed fiber with X_0. We will say that an element x of $F(\hat{A})$ is effective if it can be represented by an algebraizable $\hat{A}$-scheme, that is to say if there exists an $\hat{A}$-scheme of finite type whose formal completion defines x.

On the other hand, we say that there exists an algebraic representative of a pair $(\hat{A}, x)$ if there exists a local A-algebra essentially of finite type A_1, whose completion is isomorphic to $\hat{A}$, and a scheme of finite type X over A_1 whose formal completion defines $x \in F(\hat{A})$.

A pair (S, x), $S \in \hat{\mathcal{C}}$, $x \in F(S)$ is called formal versal if it possesses the following property: (1) Let $A' \to A$ be a surjection in $\mathcal{C}$ and $y' \in F(A')$ inducing y on A. Then for any morphism

$$u : S \to A \quad \text{such that} \quad F(u)(x) = y,$$

there exists

$$\bar{u} : S \to A' \quad \text{such that} \quad F(\bar{u})(x) = y' \quad \text{and} \quad j \circ \bar{u} = u.$$

We say that it is semi-universal when it has property (1) and the property: (2) The lifting $\bar{u}$ defined in (1) is unique when we take $A = k$ and $A' = k[\varepsilon]$, algebra of dual numbers. We know from Schlessinger [9] that there exists a semi-universal formal deformation of X_0 (unique up to non-unique isomorphism).

Remark 0.34. What we call semi-universal deformation is called versal by Schlessinger who does not consider versal deformations in the sense we understand them here.

2. AN EXTENSION OF VERSAL PROPERTIES.

Let $S \in \hat{\mathcal{C}}$, X a formal deformation of X_0 over S such that the pair (S, X) represents the semi-universal deformation of X_0. Let $A' \to A$ be a surjection in $\mathcal{C}$ and Y' a deformation of X over A' inducing $Y = Y' \otimes_A A$ over A. Suppose a commutative diagram is given

$$\begin{array}{ccc} X & \xleftarrow{\theta} & Y \\ \downarrow & & \downarrow \\ \operatorname{Spec} \bar{S} & \xrightarrow{\sigma} & \operatorname{Spec} A \end{array}$$

such that θ defines an isomorphism $Y \to \sigma^*(X)$. Then we can find a commutative diagram

$$\begin{array}{ccc} X & \xleftarrow{\theta'} & Y' \\ \downarrow & & \downarrow \\ \operatorname{Spec} \bar{S} & \xrightarrow{\sigma'} & \operatorname{Spec} A' \end{array}$$

such that σ' lifts σ, θ' lifts θ, and defines an isomorphism of deformations $Y' \to \sigma'^*(X)$.
Rim ([cf. 8]) proves this result. Moreover, it is this property [in place of (1)] that he uses to define versal elements.

Proof. — We can assume that the kernel of the surjection A' → A → 0 is an ideal of square zero in A', hence a k-vector space, and that it has dimension 1 over k.

Let $S_n = S/M^n$ where M is the maximal ideal of S and consider the commutative diagram

$$\begin{tikzcd}
 & & Y' \arrow[d, "j'"] \\
X \arrow[r] \arrow[d] & X_n = X \otimes_S S_n \arrow[r] \arrow[d] & Y \arrow[d, "i"] \\
\operatorname{Spec} S \arrow[r] & \operatorname{Spec} S_n \arrow[r, "\sigma_n"] & \operatorname{Spec} A' \arrow[d] \\
 & & \operatorname{Spec} A
\end{tikzcd}$$

(n is chosen so that σ factors through the artinian quotient S_n of S).

Let $B = S_n \times_A A'$ and let Z be the deformation $X_n \cup_{Y'} Y'$ over B so that we have the following commutative diagrams (cf. [9]):

$$\begin{tikzcd}
 & Z \arrow[dl, "\pi_1'"'] \arrow[dr, "\pi_2'"] & \\
X_n & & Y'
\end{tikzcd}
\quad \text{over} \quad
\begin{tikzcd}
 & \operatorname{Spec} B \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
\operatorname{Spec} S_n & & \operatorname{Spec}
\end{tikzcd}$$

in which $\pi_1, \pi_2, \pi_1', \pi_2'$ are the canonical maps; π_1' (resp. π_2') defines an isomorphism $X_n \xrightarrow{\sim} \pi_1'^{-1}(Z)$ [resp. $Y' \xrightarrow{\sim} \pi_2'^{-1}(Z)$].

To establish the result, it suffices to show that one can find a commutative diagram

$$\begin{tikzcd}
 & Z \arrow[dl, "\phi"'] \arrow[d, "\pi_1'"] \\
X \arrow[d] & \operatorname{Spec} B \arrow[l, "s"] \\
\operatorname{Spec} S &
\end{tikzcd}$$

such that ϕ defines an isomorphism from Z to $s^*(X)$.

It will then suffice to take
$$\sigma' = s \circ \pi_2 \quad \text{and} \quad \theta' = \psi \circ \pi_2'.$$
We know from the definition of S that we can find

$$t : \operatorname{Spec} B \to \operatorname{Spec} S,$$

lifting the canonical morphism
$$\operatorname{Spec} S_n \to \operatorname{Spec} S,$$
and such that $t^*(X)$ is isomorphic to Z. Let ζ be an isomorphism

$$Z \xrightarrow{\sim} t^*(X)$$

and t' the canonical map
$$t' : t^*(X) \to X.$$
Let us then consider the amalgamated sums

$$\begin{tikzcd}
 & T & \\
t^*(X) \arrow[ur, "\alpha_1'"] \arrow[dr, "\pi_1''"'] & & Z \arrow[ul, "\alpha_2'"'] \arrow[dl, "\pi_1'"] \\
 & X_n &
\end{tikzcd}
\quad \text{over} \quad
\begin{tikzcd}
 & \operatorname{Spec} B \times_{S_n} B & \\
\operatorname{Spec} B \arrow[ur, "\alpha_1"] \arrow[dr, "\pi_1"'] & & \operatorname{Spec} B \arrow[ul, "\alpha_2"'] \arrow[dl, "\pi_1"] \\
 & \operatorname{Spec} S_n &
\end{tikzcd}$$

where T is the deformation over $B \times_{S_n} B : t^*(X) \cup_{X_n} Z$, where $\alpha_1, \alpha_1', \alpha_2, \alpha_2'$ denote the canonical maps and where π_1 is the canonical map

$$\pi_1''^*(t^*(X)) = X_n \to t^*(X).$$

Let us still denote by π_1 the canonical ring morphism

$$B = S_n \times_A A' \to S_n.$$

Its kernel J is a k-vector space of dimension 1 (like the kernel of the surjection $A' \to A$). Let us choose an isomorphism $J \simeq k$. We then define an isomorphism

$$B \times_{S_n} B \xrightarrow{\sim} B \times_k k[\varepsilon]$$

by
$$(x, y) \to [x, \bar{x} + \varepsilon(y - x)],$$
where $\bar{x}$ is the reduction of x in k and where $y - x$ which belongs to J is identified with an element of k.

Thanks to this isomorphism, T defines a deformation T' over $B \times_k k[\varepsilon]$ which induces by projection $t^*(X)$ on B and a deformation T" on $k[\varepsilon]$.

According to the definition of S, we can find a commutative diagram

$$\begin{tikzcd} X \arrow[r, "b'"] \arrow[d] & T'' \arrow[d] \\ \operatorname{Spec} S \arrow[r, "b"] & \operatorname{Spec} k[\varepsilon] \end{tikzcd}$$

where b' induces an isomorphism $T'' \to b^*(X)$, and thus a commutative diagram

$$\begin{tikzcd} X \arrow[r, "i \cup b'"] \arrow[d] & t^*(X) \cup_{X_n} T'' \arrow[d] \\ \operatorname{Spec} S \arrow[r, "t \cup b"'] & \operatorname{Spec} (B \times_k k[\varepsilon]) \end{tikzcd}$$

Moreover, $t^*(X) \cup_{X_n} T''$ is isomorphic to T' because we have a bijection

$$F(B \times_k k[\varepsilon]) \simeq F(B) \times F(k(\varepsilon)) \quad \text{(cf. [9])}.$$

This is therefore equivalent to being given morphisms:

$$\operatorname{Spec} B \times_{S_n} B \xrightarrow{\psi} \operatorname{Spec} S,$$

$$T \xrightarrow{\phi} X,$$

such that the diagrams

$$\begin{tikzcd} X & & T \arrow[ll, "\phi'"'] \\ & t^*(X) \arrow[ul, "t'"] \arrow[ur, "\alpha_1'"'] & \end{tikzcd}$$

Above

$$\begin{tikzcd} \operatorname{Spec} \overline{S} & & \operatorname{Spec} B \times_{S_n} B \arrow[ll, "s'"'] \\ & \operatorname{Spec} B \arrow[ul, "t"] \arrow[ur, "\alpha_1"'] & \end{tikzcd}$$

are commutative.

It then suffices to take for s, $s' \circ \alpha_2$, and for ψ, $\psi' \circ \alpha_2'$, to define the commutative diagrams

$$\begin{tikzcd} & X \\ Z \arrow[ur, "\phi"] \arrow[dr, "\pi_1'"'] & \\ & X_n \end{tikzcd} \quad \text{over} \quad \begin{tikzcd} & \operatorname{Spec} S \\ \operatorname{Spec} T \arrow[ur, "s"] \arrow[dr, "\pi_1"'] & \\ & \operatorname{Spec} S_n \end{tikzcd}$$

which proves the announced result.

It admits as an immediate corollary:

Corollary 0.35. *Let $A' \xrightarrow{j} A$ be a surjection in $\hat{C}$, Y' a formal scheme over A' defining an element of $F(A')$ and Y its reduction over A. Let there also be a morphism $\sigma : \operatorname{Spec} A \to \operatorname{Spec} \bar{S}$ and an A-isomorphism $\theta : Y \to \sigma^*(X)$. Then there exists a morphism $\sigma' : \operatorname{Spec} A' \to \operatorname{Spec} \bar{S}$ lifting σ, and an A'-isomorphism θ' lifting θ of Y' to $\sigma'^*(X)$.*

Proof. For the moment, we only know this result if we further assume that the kernel of j contains a power of the maximal ideal M of A'. For all $n \in \mathbb{N}$, let us set

$$A'_n = A'/M^n, \quad A_n = A/M^n.$$

We will denote in the same way, j_n, σ_n, θ_n the reduction of the morphisms j, σ, θ. For a given n we know that we can find a morphism

$$\sigma'_n : \operatorname{Spec} A'_n \to \operatorname{Spec} \overline{S},$$

such that the diagram

$$\begin{tikzcd} \operatorname{Spec} A'_n \arrow[r, "\sigma'_n"] & \operatorname{Spec} \overline{S} \\ \operatorname{Spec} A_n \arrow[u, "j_n"] \arrow[ur, "\sigma_n"] & \end{tikzcd}$$

is commutative, and an isomorphism

$$\theta'_n : Y'_n \to \sigma'^*_n(X) \quad \text{lifting } \theta_n.$$

Consider now

$$\begin{tikzcd} \operatorname{Spec} A_{n+1} \arrow[d] & \operatorname{Spec} A_n \arrow[l] \arrow[d, "j_n"] \\ \operatorname{Spec} B = \operatorname{Spec} A_{n+1} \times_{A_n} A'_n & \operatorname{Spec} A'_n \arrow[l] \end{tikzcd}$$

The two maps σ'_n and σ_{n+1} induce

$$\sigma_B : \operatorname{Spec} A_n \to \operatorname{Spec} S.$$

□

We thus have a factorization

$$V : \operatorname{Spec} B \to \operatorname{Spec} S.$$

of these two maps. We have in the same way a canonical morphism

$$\varphi : \operatorname{Spec} B \to \operatorname{Spec} A_{n+1}.$$

Since we were careful to ensure that θ'_n lifts θ_n, the two deformations $v^*(X)$ and $\varphi^*(Y_{n+1})$ over B, are isomorphic and more precisely there exists an isomorphism

$$\gamma : \varphi^*(Y_{n+1}) \to V^*(X),$$

which induces θ_{n+1} and θ_n. φ being a closed immersion we can find (thanks to the versal properties of S) a map

$$\sigma'_{n+1} : \operatorname{Spec} A_{n+1} \to \operatorname{Spec} S,$$

lifting v and such that there exists an isomorphism

$$\theta'_{n+1} : Y_{n+1} \to \sigma'^*_{n+1}(X),$$

inducing γ over B. We thus construct a coherent system of morphisms

$$\sigma'_n : \operatorname{Spec} A'_n \to \operatorname{Spec} S,$$

inducing

$$\sigma_n : \operatorname{Spec} A_n \to \operatorname{Spec} S$$

and a coherent system of isomorphisms θ'_n lifting θ_n. By passing to the limit we thus define

$$\sigma' : \operatorname{Spec} A' \to \operatorname{Spec} S \quad \text{lifting } \sigma$$

and an A'-isomorphism

$$\theta' : Y' \to \sigma'^*(X) \quad \text{lifting } \theta.$$

Theorem 0.36. *There exists an effective representative of the semi-universal deformation of X_0.*

Proof. Let $S \in \hat{C}$ and X be a formal scheme over S such that (S, X) is a semi-universal formal deformation of X_0. We will approximate X by an algebraic scheme over S. Consider the morphism

$$X \to \operatorname{Spec} S.$$

Let d be the dimension of X_0 (assumed equidimensional) and let Z_X be the relative singular locus of X. It is finite over S. By lifting a suitable Noether presentation of X_0, we can therefore find a finite morphism

$$X \to \operatorname{Spec} S\{Y_1, \dots, Y_d\} = \operatorname{Spec} S\{Y\},$$

such that the inverse image of the closed set defined by $(Y) = (Y_1, \dots, Y_d)$ contains Z_X ($S\{Y_1, \dots, Y_d\}$ denotes the ring of restricted formal power series in d variables over S). Let $S[Y] = S[Y_1, \dots, Y_d]$ be the henselization for the $Y \cdot M$-adic topology of the polynomial ring $S[Y_1, \dots, Y_d]$ (M denotes the maximal ideal of S and Y the ideal generated by the Y_i). Its completion for this topology is also its M-adic completion $S\{Y\}$. Over the open set $\operatorname{Spec} S\{Y\} - V(Y \cdot M)$ the finite scheme X is a relative local complete intersection. It follows from Theorem 4 that one can approximate it as much as one wants by a finite scheme $\tilde{X}$ over $S[Y]$, which is a relative local complete intersection outside of $\operatorname{Spec} S[Y] - V(Y \cdot M)$ in such a way that the sheaves of $S\{Y\}$-modules $\mathcal{O}_{\tilde{X}} \otimes_{S[Y]} S\{Y\}$ and $\mathcal{O}_X$ are isomorphic. The sheaf $\mathcal{O}_{\tilde{X}}$ will in particular be flat over S since it is so after faithfully flat extension. Assume (with n chosen greater than or equal to 2), $\tilde{X}$ is congruent to X modulo M^n. By passing to the inductive limit one can find an S-scheme $\tilde{X}$, flat of finite type, finite over an étale neighborhood of $V(Y \cdot M)$ in $S[Y]$, and defining X over $S[Y]$. It follows from the definition of (S, X) that one can find a morphism

$$\varphi : \operatorname{Spec} S \to \operatorname{Spec} S,$$

lifting the identity modulo M^n, thus an automorphism of S, such that for all p we have

$$X_p = X \otimes_S S/M^p \simeq \varphi^*(X)_p = \varphi^*(X) \otimes_S S/M^p$$

and this implies that (S, X) is a semi-universal formal deformation of X_0. □

4. Existence of Henselian versal deformations

We now take for A a Henselian local algebra, essentially of finite type over a field or an excellent discrete valuation ring (i.e., verifying the conditions of application of M. Artin's approximation theorem [1]) and having residue field k. (By essentially Henselian of finite type we mean henselization of an algebra essentially of finite type.) Let $\check{C}$ be the category of Henselian local A-algebras essentially of finite type having residue field k. We call deformation of X_0 over $A \in \check{C}$ a pair formed by an A-scheme X flat of finite type and an isomorphism of $X \otimes_A k$ with X_0. We now say that two A-deformations X and X' are equivalent if there exists an A-isomorphism between the henselizations X^h and X'^h of X and X' along their closed fiber inducing the identity on X_0. We thus extend to $\check{C}$ the functor F previously defined on the subcategory C of $\check{C}$ by associating to each element A of $\check{C}$ the set of classes of deformations of X_0 over A. We say of a pair (S, x), $S \in \check{C}$, $x \in F(S)$ that it is Henselian versal (resp. semi-universal) if it possesses property (1) [resp. (1) and (2)] of section 1 when we now assume in (1) that $A' \to A$ is a surjection in $\check{C}$.

Theorem 0.37. *There exists a pair (S, x), $S \in \check{C}$, $x \in F(S)$ whose completion is isomorphic to a semi-universal formal deformation of X_0.*

Mike Artin has indeed shown in [2] that this theorem is a consequence of the existence of an effective representative of the semi-universal formal deformation. We now propose to establish the following theorem:

Theorem 0.38. *Let (S, x) be a pair, $S \in \check{C}$, $x \in F(S)$, representing the semi-universal formal deformation of X_0. Then it is a Henselian versal pair.*

It is clear that it is sufficient to show that it has property (1). Let then $A' \to A \to 0$ be a surjection in $\check{C}$, Y' an A'-scheme of finite type, defining an A'-deformation of X_0 and inducing Y over A. Let

$$\sigma : \operatorname{Spec} A \to \operatorname{Spec} S,$$

be a morphism and

$$\tilde{\theta} : \sigma^*(X)^h \to Y^h,$$

be an A-isomorphism of the henselizations of $\sigma^*(X)$ and Y along their closed fiber.

Up to replacing Y' by an étale neighborhood of its closed fiber one can assume that $\tilde{\theta}$ comes from an isomorphism

$$\theta : \sigma^*(X)_{et} \to Y,$$

where $\sigma^*(X)_{et}$ denotes a suitable étale neighborhood of $\sigma^*(X)$. Let $\tilde{\sigma}$ be the map deduced from σ,

$$\tilde{\sigma} : \operatorname{Spec} \hat{S} \to \operatorname{Spec} \hat{A}$$

and $\tilde{\theta}$ the morphism deduced from θ:

$$\tilde{\sigma}^*(\hat{X})_{et} = \sigma^*(X)_{et} \otimes_A \hat{A} \to Y \otimes_A \hat{A}.$$

Let I be the kernel of the surjection $A' \to A \to 0$. And let's set

$$\hat{A} = \hat{A}' = \hat{A}'/I, \quad A_n = A'/I^{n+1}, \quad \forall n \in \mathbb{N}.$$

Let's also set

$$Y_n = Y' \otimes_{A'} A_n.$$

We first construct step by step a morphism

$$\hat{\sigma}' : \operatorname{Spec} \hat{A}' \to \operatorname{Spec} \hat{S},$$

lifting $\tilde{\sigma}$ and an $\hat{A}'$-isomorphism $\hat{\theta}'$ of the formal I-adic completions of $\hat{\sigma}'^*(X)_{et}$ and $Y' \otimes_{A'} \hat{A}'$. We will then finish the proof by an approximation technique, using M. Artin's theorem [1] [the meaning of the notation $\sigma^*(X)_{et}$ a bit abusive here will be clarified in what follows]. Let's assume therefore found for $n \geq 0$:

$$\hat{\sigma}_n : \operatorname{Spec} \hat{A}_n \to \operatorname{Spec} S \quad (\hat{\sigma}_0 = \tilde{\sigma})$$

and an algebraic $\hat{A}_n$-isomorphism

$$\hat{\theta}_n : \hat{\sigma}_n^*(X)_{et} \to Y_n \otimes_{A_n} \hat{A}'$$

lifting $\theta = \theta_0$, and where $\hat{\sigma}_n^*(X)_{et}$ denotes an étale neighborhood of the closed fiber of $\hat{\sigma}_n^*(X)$ lifting $\sigma^*(X)_{et}$. Thanks to the formal versal properties of $\hat{S}$, and in particular to corollary 2 of paragraph 2 above, we can find

$$\hat{\sigma}_{n+1} : \operatorname{Spec} \hat{A}_{n+1} \to \operatorname{Spec} S$$

lifting $\hat{\sigma}_n$ and a formal A_{n+1}-isomorphism

$$\theta_{n+1,formal} : \hat{\sigma}_{n+1}^*(X)^\wedge \to Y_{n+1}^\wedge,$$

such that $\hat{\theta}_{n+1}$ and $\hat{\theta}_n$ induce the same formal isomorphism of the completions along their closed fiber of $\hat{\sigma}_n^*(X)$ and Y_n. Let $\hat{\sigma}_{n+1}^*(X)_{et}$ be an étale neighborhood of the closed fiber of $\hat{\sigma}_{n+1}(X)$ lifting $\hat{\sigma}_n(X)_{et}$. It then follows from the following proposition (Illusie) that one can approximate $\theta_{n+1,formal}$ by an algebraic $\hat{A}_{n+1}$-isomorphism

$$\hat{\theta}_{n+1} : \hat{\sigma}_{n+1}^*(X)_{et} \to Y_{n+1} \otimes_{A_{n+1}} \hat{A}'$$

lifting $\hat{\theta}_n$.

Lemma 0.39 (Illusie)**.** *Let $S' = \operatorname{Spec} A'$ be the spectrum of a complete local ring, I its maximal ideal; $S = \operatorname{Spec} A$ a closed subscheme of S' defined by an ideal of square null I, X', Y' two affine schemes flat of finite type over S' whose relative singular locus is finite over S', $\hat{X}'$, $\hat{Y}'$ their J-adic formal completions, X, Y their reductions over S, $\hat{X}, \hat{Y}$ the J-adic completions of X, Y. Let f be an S-morphism $X \to Y$, $\hat{f}$ the map obtained by completion $\hat{X} \to \hat{Y}$. Then there exists an S'-morphism $f' : X' \to Y'$ lifting f if and only if $\hat{f}$ lifts to a morphism of formal schemes $\hat{f}' : \hat{X}' \to \hat{Y}'$.*

Proof. Let $L_{Y/S}$ be the cotangent complex of Y over S. The obstruction to lifting f to an f' is a class $O(f)$ in $\operatorname{Ext}^1(f^* L_{Y/S}, I_X)$ where I_X is the inverse image of the S-module I on X. Let us denote by S_n, (resp. X_n, Y_n, f_n, I_n) the truncations at order n of S (resp. X, Y, f, I). The complex $L_{Y/S}$ induces the cotangent complex L_{Y_n/S_n}. And the obstruction to lifting f_n to an

$$f_n' : X_n \to Y_n$$

is a class $O(f_n)$ in $\operatorname{Ext}^1(f_n^* L_{Y_n/S_n}, I_{X_n})$ which is none other than the image of $O(f)$ by the canonical morphism

$$\operatorname{Ext}^1(f^* L_{Y/S}, I_X) \to \operatorname{Ext}^1(f_n^* L_{Y_n/S_n}, I_{X_n}).$$

The existence of $\hat{f}'$ ensures $O(f_n) = 0$, $\forall n$. We want to deduce $O(f) = 0$. The complex $f^* L_{Y/S}$ has coherent cohomology hence is equivalent to a complex L' of free modules of finite type over X and we have

$$\operatorname{RHom}(f^* L_{X/Y}, I_X) = \operatorname{Hom}^{\cdot}(L', I_X) = L'^{\vee} \otimes I_X.$$

And

$$\operatorname{RHom}(f_n^* L_{X_n/Y_n}, I_{X_n}) = \operatorname{Hom}^{\cdot}(L_n', I_{X_n}) = L_n'^{\vee} \otimes I_{X_n}$$

where L_n' denotes the truncation at order n of L'. ☐

Since we have assumed the relative singular loci of X and Y are finite over S

$$\operatorname{Ext}^1(f^* L_{Y/S}, I_X) = H^1(L'^{\vee} \otimes I_X)$$

is an S module of finite type and

$$\begin{aligned}\operatorname{Ext}^1(f^* L_{Y/S}, I_X) &= \varprojlim_n H^1(L'^{\vee} \otimes I_X)_n \\ &= \varprojlim_n H^1(L_n'^{\vee} \otimes I_{X_n}) \quad \text{(Artin-Rees)} \\ &= \varprojlim_n \operatorname{Ext}^1(f_n^* L_{Y_n/S_n}, I_{X_n})\end{aligned}$$

which establishes the lemma. By passing to the limit over n we thus obtain

$$\hat{\sigma}' : \operatorname{Spec} \hat{A}' \to \operatorname{Spec} \hat{S} \quad \text{lifting } \tilde{\sigma}$$

and an $\hat{A}'$-isomorphism

$$\lim_n \hat{\theta}_n : \lim_n \hat{\sigma}_n^*(X)_{et} \to \lim_n Y_n \otimes_{A_n} \hat{A}'.$$

$\lim_n \hat{\sigma}_n^*(X)_{et}$ is an étale neighborhood of the closed fiber of the I-adic completion of $\tilde{\sigma}'^*(X)$; it is therefore algebrizable, we denote this $\hat{A}'$ algebraic scheme by $\sigma'^*(X)_{et}$. It induces $\sigma^*(X)_{et}$ modulo I. $\lim_n \hat{\theta}_n$ is an $\hat{A}'$-isomorphism of formal I-adic schemes. It follows from the following lemma that it can be approximated by an algebraic isomorphism lifting θ of étale neighborhoods of $V(I)$ in $\sigma^*(X)_{et}$ and $Y' \otimes_A \hat{A}'$.

Lemma 0.40. *Let $A \in \hat{C}$, I an ideal of A, X and Y, two algebraic deformations of X_0 over A. We denote by $\hat{X}_I$ (resp. $\hat{Y}_I$) the formal I-adic completion of X (resp. Y), by $\tilde{X}_I$ (resp. $\tilde{Y}_I$) the I-adic henselization of X (resp. of Y). Let*

$$\hat{\theta} : \hat{X}_I \to \hat{Y}_I$$

be a formal A-isomorphism. Then for any integer n, one can find an A-isomorphism $\tilde{\theta}$:

$$\tilde{\theta} : \tilde{X}_I \to \tilde{Y}_I$$

congruent to $\hat{\theta}$ modulo I^n.

Proof. Let

$$\begin{aligned}\Gamma &= \hat{A}[X_1, \ldots, X_N]/(f_1, \ldots, f_m) &\quad& \text{such that } X = \operatorname{Spec} \Gamma, \\ \Lambda &= \hat{A}[Y_1, \ldots, Y_p]/(g_1, \ldots, g_q) && \text{such that } Y = \operatorname{Spec} \Lambda.\end{aligned}$$

Let d be the dimension of the irreducible components of X_0 and let Δ_X (resp. Δ_Y) be the ideal of Γ (resp. Λ) generated by the minors of rank $N-d$ (resp. $p-d$) of the Jacobian matrix $(\partial f_i/\partial X_j)$ [resp. $(\partial g_i/\partial Y_j)$]. Δ_X (resp. Δ_Y) defines the relative singular locus Z_X (resp. Z_Y) of X (resp. of Y) over A. We can note that Z_X (resp. Z_Y), being finite over A the I-adic completion of Γ (resp. of Λ) is also its $I\Delta_X$-adic (resp. $I\Delta_Y$-adic) completion. Let $\tilde{X}_I$ (resp. $\tilde{Y}_I$) be the henselization of X (resp. Y) along the closed set defined by $I\Delta_X$ (resp. $I\Delta_Y$) (or by I). Consider the diagram

$$\begin{array}{ccc}\tilde{X}_I \times_A \tilde{Y}_I & \xrightarrow{p_2} & \tilde{Y}_I \\ \downarrow{\scriptstyle p_1} & & \downarrow \\ \tilde{X}_I & \longrightarrow & \operatorname{Spec} A\end{array}$$

Since $\hat{\theta}$ is an $\hat{A}$-isomorphism we have in particular

$$\hat{\theta}^*(Z_Y) = Z_X$$

so that the section

$$\mathrm{Id} \times_A \hat{\theta} : \hat{X}_I \to \hat{X}_I \times_{\hat{A}} \hat{Y}_I$$

factors, outside of $V(I \cdot \Delta_X)$, through the smoothness locus of $\tilde{X}_I \times_A \tilde{Y}_I$ over $\tilde{X}_I$. It then suffices to use theorem 2 to find for any n a morphism

$$V : \tilde{X}_I \to \tilde{X}_I \times_A \tilde{Y}_I \quad \text{such that} \quad V = (\mathrm{Id} \times_A \hat{\theta}), \qquad \mathrm{mod}\ (I\Delta_X)^n$$

and from it we deduce a morphism

$$\theta' : \tilde{X} \to \tilde{Y}$$

congruent to $\hat{\theta}$ modulo I^n. It remains to verify that for n suitably chosen θ' is an isomorphism. The induced map on the I-adic completions is congruent to $\hat{\theta}$ modulo I^n. Let's take $n \geq 2$. We thus have a closed immersion

$$\hat{X} \to \hat{Y}.$$

Moreover $\hat{\theta}^{-1}\hat{\theta}'$ is an automorphism of $\hat{X}_I$ since it is an endomorphism of an affine noetherian scheme inducing a surjection on the rings of global sections. Thus $\hat{\theta}'$ is an isomorphism, and θ' makes $\tilde{Y}_I$ an étale neighborhood of $\tilde{X}_I$, so it is an isomorphism. □

End of the proof of the theorem. Consider the functor G defined on the A'-algebras in the following way. Let B' be an A'-algebra

$$\varphi' : \operatorname{Spec} B' \to \operatorname{Spec} A'.$$

Let B and φ be the restrictions of (B', φ') over A. Let

$$\sigma_B = \varphi \circ \sigma : \operatorname{Spec} B \to \operatorname{Spec} S.$$

Let's also set

$$\sigma_B^*(X)_{et} = \varphi^*(\sigma^*(X)_{et})$$

and let's denote by

$$\theta_B : \sigma_B^*(X)_{et} \to Y_B = Y \otimes_A B.$$

the inverse image by φ of θ. Let's set then

$$G(B') = \left\{ \begin{array}{l} \text{[morphism } G_{B'} : \operatorname{Spec} B' \to \operatorname{Spec} S \text{ lifting } G_B + \text{étale neighborhood of } G_{B'}^*(X) \\ \text{(resp. of } Y' \otimes_{A'} B') \text{ lifting } \sigma_B^*(X)_{et} \text{(resp. } Y_B) + B' \text{ isomorphism } \theta_{B'} \\ \text{of these étale neighborhoods lifting } \theta_B]. \end{array} \right\}$$

This functor commutes with inductive limits. It follows then from M. Artin's approximation theorem [1] that the element of $G(\hat{A}')$ previously exhibited is approximated by an element of $G(A')$. And this proves a slightly more precise result than what was announced:

Theorem 0.41. *Let $A' \to A \to 0$ be a surjection in $\check{C}$, Y' an A' deformation of X_0 inducing Y over A. Let σ be a morphism $\operatorname{Spec} A \to \operatorname{Spec} S$, Y_{et} an étale neighborhood of the closed fiber of Y, $\sigma^*(X)_{et}$ an étale neighborhood of the closed fiber of $\sigma^*(X)$ and θ an A-isomorphism $\sigma^*(X)_{et} \simeq Y_{et}$. Then there exists $\sigma' : \operatorname{Spec} A' \to \operatorname{Spec} S$ lifting σ, Y'_{et} [resp. $\sigma'^*(X)_{et}$] an étale neighborhood of Y' [resp. $\sigma'^*(X)$] lifting Y_{et} [resp. $\sigma^*(X)_{et}$] and an A'-isomorphism*

$$\theta' : \sigma'^*(X)_{et} \to Y'_{et}$$

lifting θ.

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