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Weights in the Cohomology of Algebraic Varieties

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1 Introduction

Let X be a complex algebraic variety (i.e., a separated scheme of finite type over $\mathbb{C}$). We also denote by X the usual topological space $X(\mathbb{C})$ underlying X. In this exposition, we describe a remarkable filtration of the rational cohomology groups of X, the *weight filtration*, and we provide a summary of results about its properties. Its definition will be given in §10. For the proofs, we refer to the works cited in the bibliography where the theorems are often proved in more general frameworks; working over $\mathbb{C}$ allows us to simultaneously use Hodge theory, Galois group actions, and resolution of singularities.

The weight filtration is an increasing finite filtration. We will denote it by W. It is also defined in relative situations (or in cohomology with proper support). It depends not only on the topological space X, but also on how it is realized as an algebraic variety. It is compatible with Künneth isomorphisms (i.e., via the isomorphism $H^*(X \times Y) \cong H^*(X) \otimes H^*(Y)$, we have

$$W_j(H^*(X \times Y)) = \sum_{j'+j''=j} W_{j'}(H^*(X)) \otimes W_{j''}(H^*(Y))$$

and is functorial for algebraic morphisms. More precisely, if $f : X \to Y$ is algebraic, then $f^* : H^*(Y) \to H^*(X)$ is *strictly compatible* with the weight filtrations of $H^*(Y)$ and $H^*(X)$: If the class $x \in H^i(X)$ is in the image of f^*, it is of filtration $\leq i$ if and only if it is the image of a class of filtration $\leq i$. More generally, any natural map is *strictly compatible* with weight filtrations.

The weight filtration is a discrete invariant; it is invariant under deformation of the algebraic structure of X.

2 Deformation Invariance and Examples

More precisely, we have the following theorem.

Theorem 2.1 (Deformation Invariance)**.** *Let $f : X \to S$ be a morphism. For $t \in S$, let $X_t = f^{-1}(t)$. If the sheaf $R^i f_*\mathbb{Q}$ is locally constant (a local system), there exists a filtration by weight W of $R^i f_*\mathbb{Q}$ by local subsystems, such that the maps $r_t : (R^i f_*\mathbb{Q})_t \to H^i(X_t)$ are strictly compatible with the weight filtrations, and in particular where r_t is an isomorphism, W induces the weight filtration of $H^i(X_t)$.*

2.1 General Idea and Examples

Roughly speaking, the weight filtration expresses how the cohomology of X can be built in terms of the cohomology of non-singular projective varieties. Here are some examples.

Example 2.2 (Smooth Proper Case)**.** *If X is proper (= compact, for example projective) and smooth (= non-singular), then $H^i(X)$ (defined as $H^i(X, \mathbb{Q})$) is purely of weight i: $Gr^j_W(H^i(X)) = 0$ for $j \neq i$. In other terms, $W_{i-1}(H^i(X)) = 0$ and $W_i(H^i(X)) = H^i(X)$.*

Example 2.3 (Point Cohomology and Thom-Gysin)**.** *Let X be proper, smooth, connected, of dimension d and P a point of X. Among the cohomology groups with support $H^i_{\{P\}}(X) = H^i(X \text{ mod } (X \setminus \{P\}))$, only the one with index $2d$ is non-zero, and*

$$H^{2d}_{\{P\}}(X) \simeq H^{2d}(X) \cong \mathbb{Q}. \tag{1}$$

According to our principles, $H^{2d}_{\{P\}}(X)$ is thus purely of weight $2d$. The inverse of the isomorphism (1) can be viewed as a Thom-Gysin isomorphism

$$\mathbb{Q} = H^0(P) \xrightarrow{\sim} H^{2d}_{\{P\}}(X);$$

we note that it does not respect the weights. The general situation is the following: for Y a smooth subvariety purely of codimension d in a smooth variety X, the Thom-Gysin isomorphism $H^i(Y) \xrightarrow{\sim} H^{i+2d}_Y(X)$ transforms W_k into W_{k+2d}. Denoting by (n) a shift of $2n$ for W $((W(n))_k = W_{k-2n})$, this can be written as a filtered isomorphism

$$H^i(Y)(-d) \simeq H^{i+2d}_Y(X).$$

(Here $H^{i+2d}_Y(X)$ denotes cohomology with support in Y.)

Example 2.4 (Complement of a Subvariety)**.** *Let X be proper and smooth, and Y a smooth (closed) subvariety purely of codimension d. We have an exact sequence (Gysin sequence):*

$$\cdots \to H^i_Y(X) \to H^i(X) \to H^i(X - Y) \to H^{i+1}_Y(X) \to \cdots .$$

According to Examples 2.2 and 2.3, we thus have $Gr^j_W(H^i(X - Y)) = 0$ for $j \neq i, i+1$; $W_i(H^i(X - Y))$ is the image of $H^i(X)$.

Example 2.5 (Blow-up of a Point)**.** *Let X be proper, and smooth except for an isolated singular point P. Suppose that the variety $\tilde{X}$ deduced from X by blowing up P is smooth, and that the exceptional divisor D (inverse image of P) is smooth. X is deduced from $\tilde{X}$ (proper and smooth) by contracting the divisor D (proper and smooth) to a point. The space X has the homotopy type of $[\tilde{X} \cup (\text{cone on } D)]$, whose cohomology is calculated by Mayer-Vietoris; we find an exact sequence*

$$\cdots \to H^{i-1}(D) \xrightarrow{\delta} H^i(X) \to H^i(\tilde{X}) \oplus H^i(\{P\}) \to H^i(D) \to \cdots$$

(where $H^i(\{P\}) = 0$ for $i \neq 0$ and $H^0(\{P\}) = \mathbb{Q}$). This shows that $Gr^j_W(H^i(X)) = 0$ for $j \neq i-1, i$. $W_{i-1}(H^i(X))$ is the image of δ. For $i \neq 0$, $W_{i-1}(H^i(X))$ is also the kernel of $H^i(X) \to H^i(D)$, and $Gr^i_W(H^i(X))$ is the image of $H^i(\tilde{X})$ in $H^i(X)$.

Example 2.6 (Flag Varieties and Groups)**.** *Flag varieties are proper and smooth varieties. The weight filtration of their cohomology is therefore given by Example 2.2. If G is a connected reductive group, the same result holds for the cohomology of BG [1, III]. This allows calculation of the weight filtration of the cohomology* *of G, linked to that of BG by transgression. We find that $W_i H^i(G) = 0$, and that $W_{i+1} H^i(G)$, zero for i even, is equal to the primitive part of the cohomology of degree i of G [1, III]. If $f : G \to H$ is an algebraic map between varieties of reductive groups, the reciprocal image of a primitive rational cohomology class of H is therefore still primitive. For other corollaries, see [1, III].*

3 Grading and Compatibility

The weight filtration is gradable in a very strong sense. There exist gradings $\mathbf{W}$ of $H^i(X)$, which decompose W:

$$W_n(H^i(X)) = \bigoplus_{j \leq n} \mathbf{W}_j(H^i(X)) \tag{2}$$

and which are compatible with the cup-product and with higher operations derived from the cup-product (Massey products...). These latter not being defined everywhere, the sense of "compatible with a graduation" must be specified. The simplest is to see a graduation as an action of the group $\mathbb{G}_m$, i.e., an action of $\mathbb{Q}^*$ given by algebraic formulas: to a graduation $\mathbf{W} = \bigoplus \mathbf{W}_j$ we associate the action where $\lambda \in \mathbb{Q}^*$ acts on $\mathbf{W}_j$ by multiplication by λ^j. The "compatibility" is that $\mathbb{Q}^*$ acts by automorphisms of $H^*(X)$ equipped with its graduation by degree, of the cup-product, and of higher operations derived from the cup-product.

4 Minimal Models

Suppose for simplicity X connected, and let $\mathcal{M}$ be the minimal model of the rational homotopy type of X, in the sense of Sullivan [7]. It is a differential graded algebra (DGA) with degrees ≥ 0, connected ($\mathcal{M}^0 = \mathbb{Q}$), (anti) commutative free as a graded algebra, and generated by its indecomposable elements (i.e., $d_{\mathcal{M}} \subset (\mathcal{M}^{>0})^2$). We have $H^*(\mathcal{M}) \cong H^*(X)$, and if X is simply connected, $\mathcal{M}^{>0}/(\mathcal{M}^{>0})^2 \cong (\pi_*(X) \otimes \mathbb{Q})^\vee$.

A more precise and more convenient statement than that given in §3 is that there exists a grading $\mathbf{W}$ (on cohomology) verifying (2), deduced from a grading W of $\mathcal{M}$ (with non-negative degrees, sum of gradings of the $\mathcal{M}^i$), such that $d_{\mathcal{M}}$ and the product are homogeneous of degree 0; in other terms, $\mathbb{Q}^*$ acts by automorphisms of $\mathcal{M}$.

5 Homotopy Type Restrictions

The mere existence of W (the filtration) and $\mathbf{W}$ (the grading) imposes no restriction on the homotopy type of X: One can always assume that $\mathcal{M}$ is entirely of weight 0. Any finite polyhedron indeed has the same homotopy type as an algebraic variety.

Let S be a finite set, equipped with a set $\mathcal{S}$ of subsets (the simplexes). We suppose that every subset of an element of $\mathcal{S}$ is still in $\mathcal{S}$. Identify S with the set of basis vectors of $\mathbb{R}^S$, and for $\sigma \in \mathcal{S}$, let $|\sigma|$ be the simplex spanned by the $s \in \sigma$. Let $|S| = \cup_{\sigma\in\mathcal{S}}|\sigma|$. We show that $|S|$ has the homotopy type of an algebraic variety X. If we set $|\sigma|_{\mathbb{C}}$ = complex affine space spanned by $\sigma \subset \mathbb{C}^S$, it suffices to take $X = \cup_{\sigma\in\mathcal{S}}|\sigma|_{\mathbb{C}}$, of which $|S|$ is a deformation retract. We can verify that in this example $H^*(X)$ is purely of weight 0 filtration.

However, as soon as the weights are non-zero, constraints appear on the homotopy type of X (cf. §8). The following rules help to locate the weights.

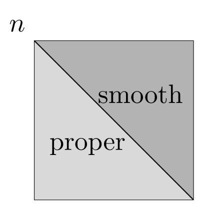
6 Constraints on Weights (Hodge Numbers)

For a cohomology group H of type n (i.e., $H = H^n$), each rule will consist of describing a subset $\mathcal{E}$ of $\mathbb{Z}\times\mathbb{Z}$. The weights will be controlled by $\mathcal{E}$ in the following sense: If $\mathrm{Gr}_k^W(H) \neq 0$, there exists $(p,q) \in \mathcal{E}$, with $p+q=k$. This way of expressing it, which seems artificial here, is not: Other information than the possible weights are controlled by the same region $\mathcal{E}$, cf. §12.

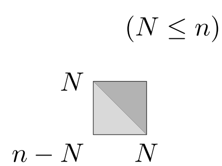
(7.1) $H^n(X)$ is controlled by the square $[0,n]\times[0,n]$.

(7.2) If X is proper, $H^n(X)$ is controlled by the part of this square below or on the second diagonal, i.e., $\{(p,q)\in[0,n]\times[0,n] \mid p+q\le n\}$. If X is smooth, $H^n(X)$ is controlled by the part of this square above or on this diagonal, i.e., $\{(p,q)\in[0,n]\times[0,n] \mid p+q\ge n\}$.

(7.3) If $N = \dim X \le n$, the square $[0,n]\times[0,n]$ can be replaced by the square $[n-N,N]\times[n-N,N]$.



Case (7.2)



Case (7.3)

7 Variants

(8.1) The cohomology group with proper support $H_c^n(X)$ is controlled by the part of the square $[0,n]\times[0,n]$ below or on the second diagonal. For $N=\dim X\le n$, one can still replace this square by $[n-N,N]\times[n-N,N]$. In particular, $H_c^{2N}(X)$ is purely of weight $2N$.

(8.2) The "duality" between the proper and smooth cases can be deduced from Poincaré duality: for X smooth purely of dimension N,

$$H^n(X) = \mathrm{Hom}(H_c^{2N-n}(X), H_c^{2N}(X)) \cong (H_c^{2N-n}(X)^{\mathrm{dual}})(-N).$$

This argument allows extending the results given for X smooth to the case where X is a "rational homology manifold". For such an X, the image of $H_c^n(X)$ in $H^n(X)$ is purely of weight n.

(8.3) For X only assumed normal, it remains true that $H^1(X)$ is controlled by $\{(0,1),(1,0),(1,1)\}$.

8 Relation to the Fundamental Group

Let X be a connected variety, $x \in X$, and $\Pi^{(n)}$ the largest torsion-free nilpotent quotient of length n of $\pi_1(X,x)$. We know that the subgroups H of finite index in $\Pi^{(n)}$ and small enough have the following property.

(*) There exists a nilpotent Lie algebra of length n, $(\mathcal{H},[,])$, with $\mathcal{H}$ a free $\mathbb{Z}$-module, such that the Campbell-Hausdorff formula

$$x \circ y = x + y + \frac{1}{2}[x,y] + \frac{1}{12}[x,[x,y]] + \frac{1}{12}[y,[y,x]] + \dots$$

makes $\mathcal{H}$ (viewed locally via exp) a group, isomorphic to H.

Moreover, the Lie algebra $\mathcal{L}^{(n)} = \mathcal{H} \otimes \mathbb{Q}$ depends only on $\Pi^{(n)}$, not on H.

Theorem 8.1 (Morgan, cf. [5]). *] If X is normal, $\mathcal{L}^{(n)}$ admits a grading W, with degrees < 0, such that $\mathcal{L}^{(n)}$ is generated by elements of degree -1 and -2, these being subject only to relations of degree -2, -3 and -4, plus the nullity of n-times iterated commutators.*

It is more convenient to work with the projective system $\mathcal{L}^{(\infty)} = \varprojlim \mathcal{L}^{(n)}$. Defining $H^i(\mathcal{L}^{(\infty)}) = \varprojlim H^i(\mathcal{L}^{(n)})$ (Lie algebra cohomology), we have

$$H^1(\mathcal{L}^{(\infty)}) \cong H^1(X), \quad H^2(\mathcal{L}^{(\infty)}) \hookrightarrow H^2(X).$$

Moreover, a grading W of $\mathcal{M}$ as in §4 defines compatible gradings of $\mathcal{L}^{(\infty)}$, of its cohomology, and of that of X. According to (8.3) (resp. (7.1)), $H^1(X)$ has weights 1 and 2 and $H^2(X)$ has weight ≤ 4. Therefore, $H_1(\mathcal{L}^{(\infty)})$ (the dual of the group $\mathcal{L}^\infty/[\mathcal{L}^\infty,\mathcal{L}^\infty]$ of generators of $\mathcal{L}^\infty$) has weights 1 and 2, and $H_2(\mathcal{L}^{(\infty)})$ (the dual of the group of relations between generators) has weight ≤ 4. This verifies the theorem.

9 Proper Smooth Case and Rational Homotopy

For a proper and smooth variety, the weight filtration reduces to little (Example 2.2). The fact that it is gradable in the sense of §3 implies the nullity of all Massey products. That it is so in the sense of §4 implies even that the rational homotopy type of X can be read from its cohomology algebrasee [3].

10 Hodge Theory Definition

The cohomology group $H^i(X)$ of any complex algebraic variety is equipped with a mixed Hodge structure (MHS) (W, F) (W is a filtration of $H^i(X, \mathbb{Q})$, and F a filtration of $H^i(X) \otimes \mathbb{C} = H^i(X, \mathbb{C})$, W and F satisfying compatible axiomssee [1, II]). The weight filtration is by definition the filtration W.

11 l-adic Theory

An algebraic variety X is defined in terms of a finite number of polynomial equations. The coefficients of these equations generate a finite type subring R of $\mathbb{C}$, with fraction field K, and X is deduced by extension of scalars to $\mathbb{C}$ from a scheme over K, possibly over R. The statements that follow become true when R is replaced by $R[1/f]$ with $f \in R$ sufficiently divisible.

(13.1) Let l be a prime number. The group $H^i(X, \mathbb{Q}) \otimes \mathbb{Q}_l$ is identified with the group of l-adic étale cohomology $H^i_{\text{ét}}(X, \mathbb{Q}_l)$. The latter is defined in a purely algebraic way, thus equipped by transport of structure with an action of $\mathrm{Aut}(\mathbb{C}/K)$.

(13.2) Let $\bar{K}$ be the algebraic closure of K in $\mathbb{C}$. The action factors through $\mathrm{Gal}(\bar{K}/K)$. It is unramified over $R[1/l]$, i.e., it factors through the Galois group of the largest extension $K^{\mathrm{ur},R[1/l]}$ of K unramified over $R[1/l]$.

(13.3) Let $\mathfrak{m}$ be a maximal ideal of $R[1/l]$, and $N(\mathfrak{m}) = \#R/\mathfrak{m}$ ($R/\mathfrak{m}$ is a finite field). To $\mathfrak{m}$ corresponds a conjugacy class of Frobenius substitutions $\phi_{\mathfrak{m}} \in \mathrm{Gal}(K^{\mathrm{ur},R[1/l]}/K)$, whose inverses are the geometric Frobenius $F_{\mathfrak{m}} = \phi_{\mathfrak{m}}^{-1}$.

Theorem 11.1 (Weight Filtration via Frobenius Eigenvalues)**.** *For f sufficiently divisible, we have*

(i) For all l and $\mathfrak{m}$, the eigenvalues α of $F_{\mathfrak{m}}$ acting on $H^i_{\acute{e}t}(X_{\bar{K}}, \mathbb{Q}_l)$ (in a suitable finite extension of $\mathbb{Q}_l$) are algebraic integers. For each α, there exists an integer $w(\alpha)$ such that all complex conjugates of α have absolute value $N(\mathfrak{m})^{w(\alpha)/2}$.

(ii) Choose a geometric Frobenius $F_{\mathfrak{m}}$, and let ${}_{\mathfrak{m}}W_j$ be the sum of generalized eigenspaces corresponding to eigenvalues α of $F_{\mathfrak{m}}$ such that $w(\alpha) = j$. The filtration defined by $W_k = \bigoplus_{j \le k} {}_{\mathfrak{m}}W_j$ is independent of $\mathfrak{m}$ and the choice of $F_{\mathfrak{m}}$; it is rational (defined over $\mathbb{Q}$) and the induced filtration on $H^i(X, \mathbb{Q})$ is independent of l; it is the weight filtration W.

Principle of Proof. We express (via a spectral sequence) the cohomology of X in terms of the cohomology of proper and smooth varieties, as in [1, II]. The spectral sequence leads to a filtration W of $H^*(X)$, which is by definition the weight filtration in Hodge theory (§10). This spectral sequence has an l-adic analogue; the l-adic completion $W \otimes \mathbb{Q}_l$ of W is thus stable under the Galois group action. We deduce from Weil's conjectures (proved in [2, I]) that the eigenvalues of $F_{\mathfrak{m}}$ on $\mathrm{Gr}_W^j H^i(X_{\bar{K}}, \mathbb{Q}_l)$ are algebraic integers of complex absolute values $N(\mathfrak{m})^{j/2}$, and the theorem follows. $\square$

12 Further Control by $\mathcal{E}$

In the cases considered in §6 (7.1), (7.2), and (7.3), the region $\mathcal{E}$ described controls not only the weights, but also the Hodge numbers and the divisibility of the eigenvalues of $F_{\mathfrak{m}}$: in (7.1), (7.2), and (7.3), a region $\mathcal{E}$ has been assigned to a group $H = H^n(X)$, and:

(a) The non-zero Hodge numbers $h^{p,q}$ of the Hodge structure on $\mathrm{Gr}_W^k(H)$ satisfy $(p,q) \in \mathcal{E}$ (where $p+q=k$).

(b) (Useful only in case (7.3).) If the pairs $(p,q) \in \mathcal{E}$ with $p+q=k$ satisfy $p,q \geq s$, then (always for f sufficiently divisible), the eigenvalues of $F_{\mathfrak{m}}$ with complex absolute values $N(\mathfrak{m})^{k/2}$ are divisible by $N(\mathfrak{m})^s$ (as algebraic integers).

We hope that this is a particular case of a general principle, cf. [4].

13 Remarks on Proofs and Functoriality

The following result is clear from the point of view of Hodge theory.

Proposition 13.1. *For j odd,* $\dim Gr_W^j H^i(X)$ *is even.*

However, I do not know how to prove Theorem 2.1 (Section 2) and the results of §4 except by l-adic methods. For §4, the method of §11 (Theorem 11.1) allows obtaining an l-adic grading; a trick of Sullivan [7] allows deducing the existence of rational gradings of the desired type.

The functoriality properties of the weight filtration are evident from the l-adic point of view (because Galois commutes with everything that can be defined algebraically). It is nevertheless useful to prove them from the point of view of Hodge theory, to obtain analogous properties for the Hodge filtration F. Morgan [5] has obtained numerous results in this directionenough to prove a slightly weaker result than §8 (Morgan's Theorem) by Hodge theory.

14 Homotopy Groups and Vanishing Cycles

As §4 suggests, a weight filtration also exists on homotopy groups (tensorized by $\mathbb{Q}$) under simple connectivity hypotheses.

It also exists on evanescent cycle groups (cf. [6]), and I hope they will help us better understand the latter.

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