Grothendieck Duality and D-modules via Diagonally Supported Sheaves

by

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ABSTRACT

We study the !-pullback functor and the theory of (relative) D-modules along morphisms of qcqs truncated schemes $p_X : X \to S$ which are almost of finite presentation and finite tor-amplitude. Key to our approach is the category of quasicoherent sheaves on $X \times_S X$ supported on the diagonal. In particular we indicate that "reduction formulae" can be used as foundations for the theory of Grothendieck Duality. We also set-up the theory of D-modules from scratch using this approach and show that in cases of overlap, it agrees with classical definitions using embeddings or the de Rham stack.

CHAPTER 1

Introduction

For an oriented compact manifold without boundary $\mathcal{M}$, Poincaré duality tells us that the cohomology of $\mathcal{M}$ is (derived) self-dual up to a cohomological shift by the dimension. If the manifold $\mathcal{M}$ was not oriented, the cohomology of $\mathcal{M}$ is instead dual to the cohomology with coefficients in the orientation sheaf, up to the same shift. For a proper variety X over a field, Grothendieck duality analogously equates the dual of the coherent cohomology of X with the cohomology of the dualizing complex of X.

A common approach to proving Poincaré duality starts with the introduction of cohomology with compact support, which allows us to formulate a generalization of Poincaré duality to the setting of non-compact manifolds. In the appendix of [Har66], Deligne employed a similar approach to prove Grothendieck duality, using the theory of pro-coherent sheaves. In [Nee96], Neeman showed that Grothendieck duality also follows from adjoint functor theorems, without needing to modify the usual category of quasicoherent sheaves. However, it is difficult to study the dualizing complex from the latter approach as the global duality does not arise from any local formulation.

In [AILN10], Avramov, Iyengar, Lipman, and Nayak found an interesting formula for the dualizing complex in the local/affine setting, which they refer to as “reduction formulae”.

$$\omega_{A/k} = A \otimes_{A \otimes_k A} \mathrm{Hom}_k(A, A) \tag{1.1}$$

This is Corollary 4.7 in *loc.cit.*[1].

In a paper titled *Grothendieck Duality Made Simple* [Nee20], Neeman related the dualizing complex from the adjoint functor theorem directly with the formula (1.1) above. This is in contrast to the situation prior where the two were related only the construction of the exceptional inverse image functor (!-pullback) in general using inputs from algebraic geometry. This allows us to trade algebro-geometric techniques for categorical techniques and opens the door for generalizations.

[1] The tensor products are implicitly derived

Our thesis starts with the observation that in (1.1), the $*$-pullback of the quasicoherent sheaf $\mathrm{Hom}_k(A, A)$ on $\operatorname{Spec} A \times \operatorname{Spec} A$ to the diagonal is unchanged if we first take the (derived) torsion part of $\mathrm{Hom}_k(A, A)$ with respect to the diagonal. Hence,

$$\omega_{A/k} = A \otimes_{A \otimes_k A} \Gamma_\Delta \mathrm{Hom}_k(A, A) \tag{1.2}$$

This simple observation allows us to simplify the proof of Neeman relating the reduction formula with the dualizing complex from the adjoint functor theorem, as we will explain in the next section.

In [SVdB97], Smith and Van Den Bergh observed that the zeroth cohomology of $\Gamma_\Delta \mathrm{Hom}_k(A, A)$ computes the ring of Grothendieck differential operators on A relative to k, as defined by Grothendieck in [Gro64]. In the same paper, Smith and Van Den Bergh also study the higher cohomologies of $\Gamma_\Delta(\mathrm{Hom}_k(A, A))$– including showing they vanish when A is smooth over k.

In fact, the entire complex $\Gamma_\Delta(\mathrm{Hom}_k(A, A))$ is always an $\mathbb{E}_1$-ring (which is the analogue of an associative ring to the setting of spectra). In this thesis, we study D-modules as defined as modules over this ring–showing that it agrees with other approaches to D-modules for possibly non-smooth varieties. In characteristic zero, this follows from the work of [GR14] using properties of Grothendieck duality.

1.1 Grothendieck Duality

Suppose X is a finite-type, separated, flat scheme over a Noetherian base scheme S, Proposition 3.3 of [Nee18] (building on Theorem 4.6 of [AILN10] and Lemma 3.2.1 of [ILN15]), shows the isomorphism

$$f_c^! \cong \delta^* \pi_1^\times f^* \tag{1.3}$$

where the maps are defined as in the diagram

$$\begin{tikzcd} X \arrow[dr, "\delta"] & & \\ & X \times_S X \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & X \arrow[d, "f"] \\ & X \arrow[r, "f"] & S \end{tikzcd} \tag{1.4}$$

Here, $f_c^!$ denotes the exceptional pullback functor in Grothendieck duality, defined in a classical way, and $\pi_1^\times$ denotes the right adjoint to the pushforward functor $\pi_{1,*}$. This formula has the advantage over classical definitions in that it does not depend on a choice of a compactification of X. Inspired by this, one may ask if it is possible to develop Grothendieck Duality from scratch using this formula, and thus bypassing the issue of compactifications. This was the approach taken by the thesis of Hafiz Khusyairi [Khu17], which proved many properties of (1.3) in the situation of flat morphisms, as above. In [Nee20], Neeman extends this work and gives some indication that (1.3) can be used as a foundation for Grothendieck duality–proving Serre duality without resorting to any existing theory of Grothendieck duality.

One source of complication for developing Grothendieck duality using (1.3) appears in Section 4.2 of [Nee20]. We need to show that the right hand side of (1.3) is local on X. Namely, if we write $f_r^!$ for the right hand side of (1.3) and $u : U \to X$ is an open immersion, we need to show that

$$u^* f_r^! \cong (uf)_r^!$$

The majority of Section 4.2 of [Nee20] is devoted to a proof of this fact. In this thesis, we try to provide a more conceptual framework to understand statements like this one and their proofs, by relying everywhere on the category $\Gamma_\Delta(\mathrm{QCoh}(X \times_S X))$. We note that the category $\Gamma_\Delta(\mathrm{QCoh}(X \times_S X))$ appears in Neeman's writing as well. However, though he makes use of it more sparingly, we aim to use this category whenever possible.

The key point is that the formula $\delta^*\pi_1^\times f^*$ is not inherently local on X, due to the appearance of the (non colimit-preserving) functor $\pi_1^\times$. Additionally, the sheaf of categories $U \mapsto \mathrm{QCoh}(U \times_S U)$ is not a quasicoherent sheaf of categories on X (in a precise sense which we define in Proposition 3.1.3). However, as we will see in Proposition 3.1.3, the sheaf $U \mapsto \Gamma_\Delta(\mathrm{QCoh}(U \times_S U))$ is, where the latter category is the full subcategory of $\mathrm{QCoh}(U \times_S U)$ supported on the diagonal. Additionally, as δ^* only sees the part of the quasicoherent sheaf on $X \times_S X$ that is supported on the diagonal, we can actually rewrite $\delta^*\pi_1^\times f^*$ in a way which bypasses the category $\mathrm{QCoh}(X \times X)$. Namely,

$$\delta^*\pi_1^\times f^* \cong \tilde{\delta}^*\tilde{\pi}_1^\times f^*$$

where

$$\tilde{\delta}^* : \Gamma_\Delta(\mathrm{QCoh}(X \times_S X)) \to \mathrm{QCoh}(X)$$

and

$$\tilde{\pi}_1^\times : \mathrm{QCoh}(X) \to \Gamma_\Delta(\mathrm{QCoh}(X \times_S X))$$

are analogues of δ^* and $\pi_1^\times$ involving only $\Gamma_\Delta(\mathrm{QCoh}(X \times_S X))$ (see Section 3.1 for the precise definitions). Therefore, we achieve a rewriting of $f_r^!$ which is *manifestly* local. All the technical inputs are cleanly packaged into two statements:

1. $U \mapsto \Gamma_\Delta(\mathrm{QCoh}(X \times_S X))$ is quasi-coherent sheaf of categories (Proposition 3.1.3)

2. $\tilde{\pi}_1^\times$ is a quasicoherent map (Proposition 3.1.6)

1.2 D-modules

Modules over the ring of differential operators, or D-modules for short, were first studied following ideas of Mikio Sato. D-modules provide an algebraic framework in which one could study differential equations and constitute a vast generalization of the theory of flat connections on vector bundles. Since then, D-modules have become an invaluable tool in algebraic geometry and representation theory.

For $X = \mathrm{Spec}\, A$, a smooth affine variety over a field k, the Grothendieck ring of differential operators on X relative to k, $D_{X/k}$, is the increasing union

$$D_{X/k} := \bigcup_{n \geq 0} D^{(n)} \subseteq \mathrm{Hom}_k(A, A)$$

where $D^{(n)} \subseteq \mathrm{Hom}_k(A, A)$ is defined inductively by

$$D^{(-1)} = 0$$

and

$$D^{(n)} = \{f \in \mathrm{Hom}_k(A, A) | \forall a \in A, [f, a] \in D^{(n-1)}\}$$

($a \in A$ is thought of as an element $\mathrm{Hom}_k(A, A)$ via multiplication by a)

By a D-module on X, we then refer to a module over the ring $D_{X/k}$. For X a general smooth variety, we can glue this definition Zariski-locally: $D_{X/k}$ becomes a quasicoherent sheaf of algebras (though with two different actions of the structure sheaf–on the left and right), and a $D_{X/k}$-module refers to a quasicoherent sheaf with an action of $D_{X/k}$. If one only studies D-modules on smooth varieties, such a definition will suffice. However, over singular varieties, the same definition will lead to many unpleasant properties.

For this reason, two alternative definitions were proposed for studying D-modules on singular varieties over a field k. The first stems from Kashiwara's equivalence, which says that if Z is embedded in a smooth variety X via a closed immersion, then the category of D-modules on X supported on Z is independent of the choice of X and in the cases where Z is smooth agree with the category of D-modules on Z. Therefore, when Z is singular, one can define D-modules on Z as D-modules on X supported on Z, after a closed embedding $Z \hookrightarrow X$ into a smooth ambient variety X has been chosen.

A more intrinsic definition was given by Grothendieck. Namely, for any variety X, smooth or singular, we can consider the (small) site of infinitesimal thickenings $U \to T$ where U varies over open subsets of X. A crystal (for the infinitesimal site) on X is then (roughly speaking) the data of a quasicoherent $\mathcal{O}_T$ module $\mathcal{F}_T$ for each thickening $U \to T$, such that for any morphism of thickenings in the infinitesimal site, the natural map

$$f^* \mathcal{F}(T') \to \mathcal{F}(T)$$

is an isomorphism. It is possible to show these two definitions agree (in the sense of an equivalence of categories), giving a consistent notion of a D-module on a singular variety. Nevertheless, one may ask whether there is a third approach, more similar to the definition in the smooth setting, where we can explicit construct a quasicoherent sheaf of algebras D_X on a singular variety X such that D_X modules will give the same category of D-modules as the two approachs mentioned above.

In the present thesis, we will show that this is indeed possible, and that the correct definition of D_X will simply the derived version of one of the standard definitions for D_X in the smooth setting. Let us now indicate which definition of D_X we intend to derive. For simplicity, we will assume $X = \operatorname{Spec} A$ is an affine underived Noetherian scheme. In this setting, it is well known that in the case A is smooth, there is an isomorphism

$$D_A \cong \operatorname{colim}_n(\operatorname{Hom}_A((A \otimes_k A)/I_\Delta^n, A))$$

where I_Δ is the kernel of the multiplication map $\mu_A : A \otimes_k A \to A$, and the formula is the same whether we read it in a derived way or not. We should note that it is not extremely clear what the algebra structure on D_A is from this isomorphism. In the case A is singular, we will simply take the same definition, but now require that we read it in a fully derived manner.

We note the following isomorphisms which follow simply from tensor-hom adjunction (all the tensor products are derived)

$$\begin{aligned} \operatorname{colim}_n(\operatorname{Hom}_A((A \otimes_k A)/I_\Delta^n, A)) &\cong \operatorname{colim}_n(\operatorname{Hom}_A((A \otimes_k A) \otimes_{A \otimes_k A} (A \otimes_k A)/I_\Delta^n, A)) \\ &\cong \operatorname{colim}_n(\operatorname{Hom}_{A \otimes_k A}((A \otimes_k A)/I_\Delta^n, \operatorname{Hom}_A(A \otimes A, A)) \\ &\cong \operatorname{colim}_n(\operatorname{Hom}_{A \otimes_k A}((A \otimes_k A)/I_\Delta^n, \operatorname{Hom}_k(A, A))) \\ &\cong \Gamma_\Delta(\operatorname{Hom}_k(A, A)) \end{aligned}$$

where Γ_Δ means taking local-cohomology at the diagonal of $\operatorname{Spec} A$. Note that this presentation because it makes the algebra structure evident. We note that this formula for the ring of differential operators can be found in Section 2.1 of [SVdB97], where they also briefly study the derived ring of differential operators.

It is $\Gamma_\Delta(\operatorname{Hom}_k(A, A))$ that we will take as definition for D_A. Using this ring, we will define the category of D-modules and show that most of the constructions one can do with D-modules in the smooth setting carry over directly. We will also show using Kashiwara's equivalence in our setup that it agrees with classical definitions when they overlap.

Additional discussions on the derived ring of differential operators can be found in [Jef21], though the goals of that paper is markedly different from ours. The derived ring of differential operators is also defined in [GR14], and expanded on in [Yan21]. However we our description of the ring is more explicit in the non-smooth case (we also work in a larger generality).

1.3 Terminology and Conventions

The most general setting in which this thesis applies will be for a map of truncated spectral Deligne-Mumford stack $p_X : X \to S$ which is locally almost of finite presentation and finite tor-amplitude. The reader can find the precise definitions of these terms in [Lur18], however we take this section to give the reader a guide to these assumptions and why we need them (or at least think we need them).

First, we will say nothing about the definition of a spectral Deligne-Mumford stack except that étale locally, it is isomorphic to a spectral affine scheme. In fact this is also the only thing that we will use about them. Our theory extends to spectral Deligne-Mumford stacks formally via étale descent. The rest of the conditions are local, so for the rest of this section we will stick with spectral affine schemes.

A spectral affine scheme is completely determined by a connective $\mathbb{E}_\infty$-ring, just as usual affine schemes are completely determined by a commutative ring. $\mathbb{E}_\infty$-rings are a vast generalization of commutative rings to the realms of homotopy theory. Unlike a commutative ring, which has an underlying abelian group, a connective $\mathbb{E}_\infty$-ring has a underlying connective spectra. Connective spectra are to spaces (homotopy types) what abelian groups are to sets.

The reason that spectral affine schemes shows up in this thesis, even if one only cares about the results in the classical setting, is that we work with the product $X \times_S X$. If $p_X : X \to S$ is not flat, then taking the fibre product in schemes (instead of spectral schemes) will not yield the correct results. One explicit way to see the failure is to note that many base-change results fail if the underived fibre product is taken (this is why the standard push-pull isomorphism for schemes is often stated with tor-independence conditions). However, if the reader is willing to work in the setting of p_X being flat, they are free to ignore this issue. The theory is still interesting in that case–in particular the case of a singular variety over a field will fall within those assumptions. A fair warning that the ring of differential operators can nevertheless be a non-connective ring in that setting (meaning it can have cohomology).

The next thing to explain is the meaning of truncated. This means that our $\mathbb{E}_\infty$-rings are only allowed to have finitely many nonzero homotopy groups. One reason this shows up is that some facts about the quasicoherent sheaves supported on a closed subset become more subtle in the homologically unbounded (non-truncated) setting. For example, even over $\mathbb{Q}$, the free $\mathbb{E}_\infty$-ring on a generator in degree 2, $\mathbb{Q}[t]$, and $\mathbb{Q}$ have the same underlying topological space. However, the pullback map from $\operatorname{Sp\acute{e}t} \mathbb{Q}[t]$ to $\operatorname{Sp\acute{e}t} \mathbb{Q}$ is not conservative. This breaks some arguments in the thesis. Nevertheless, there's no obvious reason why the theory cannot generalize to a setting where the truncated assumption is dropped. It is unclear to the author whether in that situation, the definition of diagonally supported sheaves should change in some way.

The condition that the map is almost of finite presentation is analogous to the usual condition for a map of rings to be finitely presented, which is that it is given by adding finitely many generators and relations. The term almost means (roughly) that we allow infinitely many generators and relations (killing off cells) but only if the dimension of the generators and cells goes to infinity. This condition is useful to obtain finiteness properties of the pushforward maps which occur in the theory.

Finally, a very important condition for us is the finite tor-amplitude condition. We say that a map $k \to A$ of $\mathbb{E}_\infty$-rings is finite tor-amplitude if for any k-module M which is discrete (only having π_0), the tensor product $M \otimes_k A$ has vanishing homotopy groups outside of a uniform bound independent of M. For a discrete ring k, this means that A is isomorphic to a finite complex of flat k-modules. This is done to ensure that the exceptional inverse image functor $p_x^!$ preserves colimits and that the category of D-modules can actually be realized as modules over a ring.

A note on conventions: In this thesis, all categories, unless stated otherwise will be $(\infty, 1)$-categories. All functors, such as Hom, $\otimes$, colim, and lim will be fully derived/done at the ∞-categorical level unless stated otherwise. A stable category will refer to a stable ∞-category. All modules/quasicoherent sheaves will also be assumed to be fully derived. We will aim to follow the terminology of Lurie in [Lur09], [Lur17], and [Lur18].

1.4 Summary of Results

Fix $p_X : X \to S$ a map of qcqs truncated spectral schemes which is almost of finite presentation and finite tor-amplitude. For the first two results, let us initially assume p_X is separated.

We show that the category $\Gamma_\Delta(\mathrm{QCoh}(X \times_S X))$ is a quasicoherent sheaf of categories on X in the following sense (see Proposition 3.1.3 in the main text).

Proposition 1.4.1. *For an étale map $u : U \to X$, we have*

$$\mathrm{QCoh}(U) \otimes_{\mathrm{QCoh}(X)} \Gamma_\Delta(\mathrm{QCoh}(X \times_S X)) \cong \Gamma_\Delta(\mathrm{QCoh}(U \times_S U))$$

where $\mathrm{QCoh}(X)$ *acts on* $\mathrm{QCoh}(U)$ *via* j^* *and* $\mathrm{QCoh}(X)$ *acts on* $\Gamma_\Delta(\mathrm{QCoh}(X \times_S X))$ *via* $\Gamma_\Delta \pi_1^*$.

This in particular shows that $\Gamma_\Delta(\mathrm{QCoh}(X \times_S X))$ satisfies étale descent. Next, we show that the functor $\tilde{\pi}_1^\times$ is a quasicoherent map of categories for formal reasons, see Proposition 3.1.6 in the main text.

Proposition 1.4.2. *For an étale map $u : U \to X$,*

$$\tilde{\pi}_{1,U}^\times : \mathrm{QCoh}(U) \to \Gamma_\Delta(\mathrm{QCoh}(U \times_S U))$$

is $\mathrm{QCoh}(U)$ *linear and agrees with* $\tilde{\pi}_{1,X}^\times$ *for X base changed to U, i.e. tensored with* $\mathrm{QCoh}(U)$ *over* $\mathrm{QCoh}(X)$.

If p_X is not separated, the above results show that we can nevertheless define $\Gamma_\Delta(\mathrm{QCoh}(X \times_S X))$ (and products of more copies of X) by defining it Zariski locally. With this corrected definition, the above two propositions also hold without the separatedness assumption. Henceforth we remove the separateness assumption on p_X.

The two propositions above provide the backbone for our results developing Grothendieck duality using 1.3. We start with the definition (Definition 3.2.1 in the main text),

Definition 1.4.3.

$$p_X^! := \delta^* \pi_1^\times p_X^* : \mathrm{QCoh}(S) \to \mathrm{QCoh}(X)$$

where the maps are as shown in the diagram (1.4).

We prove the exceptional pullback (also referred to as upper shriek) functor defined above satisfies the following properties. The following is contained in Equation (3.2), Corollary 3.2.15, Proposition 3.2.3, and Theorem 3.2.19 in the text. Parts of this theorem are contained in [Nee20], but we take a slightly different approach.

Theorem 1.4.4. *1. $p_X^!$ is colimit-preserving (in fact $\mathrm{QCoh}(S)$-linear)*

2. If p_X is proper, then $p_X^! \cong p_X^{\times}$.

3. If p_X is étale, then $p_X^! \cong p_X^*$.

4. If $g : X' \to X$ is also finite tor-amplitude and locally almost of finite-presentation, then

$$g^! f^! \cong (fg)^!$$

The theorem below can be found in the text in Theorem 4.1.10, Theorem 4.2.1, and Theorem 4.5.10.

Theorem 1.4.5. *There is an object*

$$D_{X/S} \in \Gamma_\Delta(\mathrm{QCoh}(X \times_S X))$$

on X such that if $X = \operatorname{Spec} A$ and $S = \operatorname{Spec} k$, then $D_{X/S} \cong \Gamma_\Delta \operatorname{Hom}_k(A, A)$. This is what we can the (sheaf of) ring of differential operators on X relative to S.

$$\Gamma_\Delta(\mathrm{QCoh}(X \times_S X))$$

is the subcategory of $\mathrm{QCoh}(X \times_S X)$ supported on the diagonal. It acquires a monoidal structure via the isomorphism

$$\mathrm{QCoh}(X \times_S X) \cong \mathrm{Hom}_{\mathrm{QCoh}(S)}(\mathrm{QCoh}(X), \mathrm{QCoh}(X))$$

where on the right hand side the Hom is taken in $\mathrm{QCoh}(S){-}\mathrm{Mod}^L$ (see Appendix A.1). $D_{X/S}$ is an $\mathbb{E}_1$-algebra in $\Gamma_\Delta(\mathrm{QCoh}(X \times_S X))$.

As $\Gamma_\Delta(\mathrm{QCoh}(X \times_S X))$ acts on $\mathrm{QCoh}(X)$, $D_{X/S}$ defines a monad on $\mathrm{QCoh}(X)$ and we can consider the category of $D_{X/S}$ modules

$$D_{X/S}{-}\mathrm{Mod}$$

Additionally, $\Gamma_\Delta(\mathrm{QCoh}(X \times_S X))$ carries a natural involution via swapping the two copies of X. The image of $D_{X/S}$ under this involution is called ${D_{X/S}}^{\mathrm{op}}$. We can also consider the modules under this monad, which we call

$${D_{X/S}}^{\mathrm{op}}{-}\mathrm{Mod}$$

Both $D_{X/S}{-}\mathrm{Mod}$ and ${D_{X/S}}^{\mathrm{op}}{-}\mathrm{Mod}$ satisfies étale (in fact proper finite tor-amplitude) descent with respect to X and fpqc descent with respect to S. Also, we have the following isomorphisms

$$\begin{aligned} {D_{X/S}}^{\mathrm{op}}{-}\mathrm{Mod} &\cong \mathrm{colim}_{\boldsymbol{\Delta_s}^{\mathrm{op}}}(\Gamma_\Delta(\mathrm{QCoh}(X^{n+1})), *) \\ &\cong \mathrm{QCoh}(X) \otimes_{\Gamma_\Delta(\mathrm{QCoh}(X\times X))} \mathrm{QCoh}(X) \end{aligned}$$

$$\begin{aligned} D_{X/S}{-}\mathrm{Mod} &\cong \lim_{\boldsymbol{\Delta_s}}(\Gamma_\Delta(\mathrm{QCoh}(X^{n+1})), *) \\ &\cong \mathrm{Hom}_{\Gamma_\Delta(\mathrm{QCoh}(X\times X))}(\mathrm{QCoh}(X), \mathrm{QCoh}(X)) \end{aligned}$$

where the last Hom is taken in $\mathrm{QCoh}(X \times_S X){-}\mathrm{Mod}^L$.

Of vital importance in D-module theory are the pushforward and pullback functors. We define them in Section 4.3. The following is a rewriting of the beginning of Section 4.3. The last claim below is clear from definitions, see Section 4.3 for details.

Theorem 1.4.6. *Suppose S is a truncated qcqs spectral scheme and $f : X \to Y$ is a map between qcqs schemes which are locally almost of finite presentation and finite tor-amplitude over S. Then, there is a natural pullback functor*

$$f^+ : D_{Y/S}{-}\mathrm{Mod} \to D_{X/S}{-}\mathrm{Mod}$$

that when written as a map

$$f^+ : \lim_{\boldsymbol{\Delta_s}}(\Gamma_\Delta(\mathrm{QCoh}(Y^{n+1})), *) \to \lim_{\boldsymbol{\Delta_s}}(\Gamma_\Delta(\mathrm{QCoh}(X^{n+1})), *)$$

is defined by quasicoherent pullback (upper star) termwise.

There is dually a natural pushforward functor

$$f_+ : {D_{X/S}}^{\mathrm{op}}{-}\mathrm{Mod} \to {D_{Y/S}}^{\mathrm{op}}{-}\mathrm{Mod}$$

that when written as a map

$$f_+ : \lim_{\boldsymbol{\Delta_s}}(\Gamma_\Delta(\mathrm{QCoh}(X^{n+1})), *) \to \lim_{\boldsymbol{\Delta_s}}(\Gamma_\Delta(\mathrm{QCoh}(Y^{n+1})), *)$$

is defined by quasicoherent pushforward (lower-star) termwise.

Both functors compose well, in the sense that if $f : X \to Y$ and $g : Y \to Z$, then

$$f^+g^+ \cong (gf)^+$$

and

$$g_+ f_+ \cong (gf)_+$$

In addition, we have (see Theorem/Definition 4.3.2 for details)

Proposition 1.4.7. *With the same assumptions as above, the functors f_+ and f^+ correspond to the tranfer $(D_{X/S}, D_{Y/S})$-bimodule*

$$\Gamma_f(\mathcal{O}_X \boxtimes \omega_Y) \in \Gamma_f(\mathrm{QCoh}(X \times_S Y))$$

where Γ_f means restricting to sections supported on the graph of X inside $X \times_S Y$.

We also prove a left-right switch for D-modules with our definitions. The following is Theorem 4.2.9 in the main text, combined with the discussion above that Theorem.

Theorem 1.4.8. *There is an isomorphism*

$$D_{X/S}\mathrm{-Mod} \cong D_{X/S}{}^{\mathrm{op}}\mathrm{-Mod}$$

induced by the $(D_{X/S}{}^{\mathrm{op}}, D_{X/S})$-bimodule

$$\Gamma_\Delta(\omega_{X/S} \boxtimes \omega_{X/S})$$

and the inverse is induced by the $(D_{X/S}, D_{X/S}{}^{\mathrm{op}})$-bimodule

$$\Gamma_\Delta(\mathcal{O}_X \boxtimes \mathcal{O}_X)$$

This isomorphism is given by tensoring with the relative dualizing complex on the underlying quasicoherent sheaf.

Lastly in the theory of D-modules, we also prove a form of Kashiwara's equivalence with our definitions–this is Theorem 4.4.2 and Corollary 4.4.3 in the main text.

Theorem 1.4.9. *Let $p_X : X \to S$ be a locally almost of finite presentation, finite tor-amplitude map of truncated qcqs spectral schemes. Suppose $z : Z \to X$ is a finite tor-amplitude, locally almost of finite presentation closed immersion. Then, the functor*

$$z^+ : D_{X/S}\mathrm{-Mod} \to D_{Z/S}\mathrm{-Mod}$$

restricts to an equivalence of categories on $\Gamma_Z(D_{X/S}\mathrm{-Mod})$–the full subcategory supported on Z. Dually, the functor

$$z_+ : D_{Z/S}{}^{\mathrm{op}}\mathrm{-Mod} \to D_{X/S}{}^{\mathrm{op}}\mathrm{-Mod}$$

is an equivalence onto the full subcategory $\Gamma_Z({D_{X/S}}^{\mathrm{op}}\text{−Mod})$ *of the codomain.*

Finally, using the above we prove the following isomorphism (note that the conditions here are more restrictive than above) stated as Theorem 4.6.5 in the text. For the last claim below, see Appendix C.

Theorem 1.4.10. *Let S be an truncated Noetherian scheme and X be a scheme finite-type and finite tor-amplitude over S, then there is a natural isomorphism*

$$QCoh((X/S)_{dR}) \cong D_{X/S}\text{−Mod}$$

The former is also naturally isomorphic to the category of quasi-coherent crystals on the small or big infinitesimal site.

We also prove a decategorification of Proposition 4.2.5 in [Ber19] in the smooth setting. It is (one of) the main results of Section 4.7, and we leave the explanation of the notation to that section.

Proposition 1.4.11. *In the setting of where $X = \operatorname{Spec} A$ is affine and smooth over a base $S = \operatorname{Spec} k$ which is discrete, we have the following isomorphism*

$$D_A \cong \overset{A_{180^\circ}}{\underset{A}{\otimes}} \mathrm{HH}^{\cdot}(A/k) \tag{1.5}$$

Lastly, we provide an application of the theory to recover a main result of Ben-Zvi and Nevins in [BZN04]. See Section 5.1 or [BZN04] for the relevant definitions. The following is the Theorem 5.1.7 in the text and Theorem 1.4 of [BZN04].

Theorem 1.4.12. *Suppose $\tau : \tilde{X} \to X$ is a good cuspidal quotient of good Cohen-Macaulay varieties over a field k, then $D_{\tilde{X}}$ and D_X are concentrated in degree 0 and Morita equivalent.*

CHAPTER 2

Preliminaries

2.1 Support of Quasicoherent Sheaves

In this section we study quasicoherent sheaves supported on a closed subscheme which is locally almost of finite presentation.

We adopt the terminology of [Lur18]. Let $z : Z \to X$ be a closed immersion almost of finite presentation of spectral Deligne-Mumford stacks and $u : U \to X$ be the inclusion of the complement open of Z (u is a quasicompact morphism). We define $\Gamma_Z(\mathrm{QCoh}(X))$ be the fibre of the functor

$$j^* : \mathrm{QCoh}(X) \to \mathrm{QCoh}(U)$$

The map j^* admits a colimit-preserving right adjoint (which is also a section), namely

$$j_* : \mathrm{QCoh}(U) \to \mathrm{QCoh}(X)$$

where we use crucially the quasicompactness of u. Let

$$i_Z : \Gamma_Z(\mathrm{QCoh}(X)) \to \mathrm{QCoh}(X)$$

denote the inclusion functor of that subcategory. Let

$$\Gamma_Z : \mathrm{QCoh}(X) \to \Gamma_Z(\mathrm{QCoh}(X))$$

be the right adjoint of i_Z, or equivalently the left Kan extension of the identity functor on $\Gamma_Z(\mathrm{QCoh}(X))$ to the entirety of $\mathrm{QCoh}(X)$. The fact that j_* preserves colimits implies the same for Γ_Z which in turn shows that i_Z preserves (and reflects) compact objects. The following is a split-exact sequence of categories (in the sense of Definition A.2.3)

$$\Gamma_Z(\mathrm{QCoh}(X)) \to \mathrm{QCoh}(X) \to \mathrm{QCoh}(U) \tag{2.1}$$

Note that $\Gamma_Z(\mathrm{QCoh}(X))$ carries a symmetric monoidal structure with unit $\Gamma_Z(\mathcal{O}_X)$.

If X is an affine spectral scheme, i.e. $X \cong \mathrm{Spét}\, R$. Then $Z \cong \mathrm{Spét}\, S$ where $\pi_0(R) \to \pi_0(S)$ is a surjective with finitely generated kernel I. Let $(t_1, \ldots, t_n)$ denote a sequence (not necessarily regular) in $\pi_0(R)$ generating I and let $R/(t_1, \ldots, t_n)$ denote the (derived) R-module constructed from the Koszul complex on the sequence.

Lemma 2.1.1. *For $X \cong \mathrm{Spét}\, R$ and Z being cut out by finitely many equations $t_1, \ldots, t_n \in \pi_0(R)$, the category $\Gamma_Z(\mathrm{QCoh}(R))$ is compactly generated by $R/(t_1, \ldots, t_n)$.*

Proof. Since $R/(t_1, \ldots, t_n)$ is compact, we only need to show it is a generator. In this situation, it suffices to show if $\mathrm{Hom}_R(R/(t_1, \ldots, t_n), M) \cong 0$ and M is in $\Gamma_Z(\mathrm{QCoh}(X))$ then M is zero. For a general $M \in \mathrm{QCoh}(X)$, by writing j_*j^*M as a finite (Čech) limit, we can calculate that the fibre of the map $M \to j_*j^*M$ is given by

$$\mathrm{colim}_k \mathrm{Hom}_R(R/(t_1^k, \ldots, t_n^k), M)$$

Now suppose $M \in \Gamma_Z(\mathrm{QCoh}(X))$ and $\mathrm{Hom}_R(R/(t_1, \ldots, t_n), M) \cong 0$, then the above colimit is M. However, since each $R/(t_1^k, \ldots, t_n^k)$ is generated under finite colimits from $R/(t_1, \ldots, t_n)$, each term in the colimit is zero. Hence, $M \cong 0$. ■

Proposition 2.1.2. *The category $\Gamma_Z(\mathrm{QCoh}(X))$ is compactly generated for a locally almost of finite presentation closed immersion of qcqs algebraic spaces $z : Z \to X$.*

Proof. By Proposition 8.2.5.1 of [Lur18], we can reduce to showing the full category of connective objects is compactly generated (same as the reduction of Proposition 9.6.1.1 to Proposition 9.6.1.2 in *loc. cit.*). Then, by choosing the scallop decomposition to start with a cover of the complement of Z, the same arguments (of Proposition 9.6.2.1 of [Lur18] which is just a rewording of Proposition 9.6.1.2) carries through completely. ■

The following lemma crucially relies on the truncated-ness of X and will be an important input to the theory of D-modules later.

Lemma 2.1.3. *For a closed immersion $z : Z \to X$ of truncated spectral Deligne-Mumford stacks which is locally almost of finite presentation,*

$$\tilde{z}^* := z^* i_Z : \Gamma_Z(\mathrm{QCoh}(X)) \to \mathrm{QCoh}(Z)$$

is conservative. If X is a qcqs truncated spectral algebraic space,

$$\tilde{z}^\times := z^\times i_Z : \Gamma_Z(\mathrm{QCoh}(X)) \to \mathrm{QCoh}(Z)$$

is conservative.

Proof. We first reduce to the case where X is affine. The first statement reduces immediately, the second reduces using Proposition B.0.2. So we can let $X = \operatorname{Spét} R$ and $Z = \operatorname{Spét} S$.

Because $R/(t_1, \ldots, t_n)$ has finitely many homotopy groups and each homotopy group is a $\pi_0(S)$-module, the localizing subcategory generated by $\pi_0(S)$ contains $R/(t_1, \ldots, t_n)$. Hence the localizing subcategory generated by S does also.

For the first statement, let N be a R-module supported on Z. Consider the collection of R-modules M such that $M \otimes_R N = 0$, this is a localizing subcategory. Therefore if this collection contains S, then it contains $R/(t_1, \ldots, t_n)$ and hence it is zero by Lemma 2.1.1. The second statement follows similarly. ■

Proposition 2.1.4. *Suppose $z : Z \to X$ is a closed immersion of spectral Deligne-Mumford stacks which factors through an étale map $u : U \to X$, then u^* induces an isomorphism*

$$\Gamma_Z(\mathrm{QCoh}(X)) \cong \Gamma_Z(\mathrm{QCoh}(U))$$

Proof. We may assume X is affine by étale descent on X. If u is an open immersion, the statement follow from Zariski descent of quasicoherent sheaves for the covering of X consisting of U and the complement of $|Z|$.

Since étale maps are open, by Zariski descent on U and the analogous result for open immersions, we can reduce to the case where u is affine and surjective. The map $|Z| \to u^{-1}(|Z|)$ (coming from the fact that Z lifts to U) is open (as sections of étale maps are étale by [Lur17] Remark 7.5.1.7), hence without loss of generality we can assume $u^{-1}(|Z|) = |Z|$.

Then the statement follows from taking fibres along the Nisnevich excision square of quasicoherent sheaves ([Lur18] Theorem 3.7.5.1 + Nisnevich descent) .

$$\begin{tikzcd}
U \setminus Z \arrow[r] \arrow[d] & U \arrow[d, "u"] \\
X \setminus Z \arrow[r] & X
\end{tikzcd}$$

■

Remark 2.1.5. *It is possible also to give a direct proof in the affine case where $u^{-1}(|Z|) = |Z|$. Suppose $\mu : R \to R'$ corresponds to the étale cover $U \to X$. Then it suffices to show that $\Gamma_Z(R) \cong \Gamma_Z(R')$. But after tensoring with R' we have*

$$\Gamma_Z(R') \otimes_R R' \cong \Gamma_Z(R' \otimes_R R') \cong \Gamma_Z(R') \cong \Gamma_Z(R) \otimes_R R'$$

where the first isomorphism comes from the fact that u is a homeomorphism over Z and the second equality comes from the fact that R' is an idempotent $R' \otimes R'$-algebra as the map μ is étale.

2.2 Adjoints and Duality in Algebraic Geometry

In this section, we explore two categorical dualities which will be relevant later. The first duality interchanges a dualizable object in a symmetric monoidal category with its dual, which we refer to as up-down duality. The second duality interchanges a dualizable category with its dual (inside $\mathscr{V}-\mathrm{Mod}^L$ for some $\mathscr{V}$), which we refer to as left-right duality. Up-down duality allows us to conjugate compact object preserving functors with taking duals of compact objects to obtain new functors (when compact objects coincide with dualizable objects). Left-right duality produces from a colimit-preserving functor between dualizable categories a functor in the opposite direction on their duals. We will often be in a situation where our categories are in fact self-dual, where left-right duality produces simply a functor in the reverse direction. For up-down duality, we quote extensively from [BDS16].

Let us start with up-down duality. For a compactly generated presentable stable category $\mathscr{X}$, we denote by $\mathscr{X}^c$ the stable subcategory of compact objects. Similarly, if f is a colimit preserving functor between compactly generated presentable stable categories which preserves compact objects, we let f^c be the functor restricted to compact objects. Suppose $f : \mathscr{X} \to \mathscr{Y}$ is a map of compactly generated stable categories with a anti-automorphism on the compact objects which preserves compact objects (for us this will always just be taking the dual of the object in a symmetric monoidal category where all compact objects are dualizable). Then, we can conjugate the functor f^c by the anti-automorphism to get a functor

$$(f^c)^D : (\mathscr{X}^c)^{\mathrm{op}} \to (\mathscr{Y}^c)^{\mathrm{op}}$$

By viewing $(f^c)^D$ as a functor from $\mathscr{X}^c$ to $\mathscr{Y}^c$, we can extend it uniquely to a colimit preserving functor

$$f^D : \mathscr{X} \to \mathscr{Y}$$

We record two lemmas paraphrased from [BDS16] (Lemma 2.6 in loc. cit)

Lemma 2.2.1. *Suppose $f : \mathscr{X} \to \mathscr{Y}$ is a colimit-preserving functor of compactly generated presentable stable categories which preserves compact objects. Then, $f^c : \mathscr{X}^c \to \mathscr{Y}^c$ has a right adjoint if and only if the right adjoint of f preserves compact objects. In which case the right adjoint of f is induced by the right adjoint of f^c.*

Lemma 2.2.2. *Suppose $f : \mathscr{X} \to \mathscr{Y}$ is a colimit-preserving functor of compactly generated presentable stable categories which preserves compact objects. Then, $f^c : \mathscr{X}^c \to \mathscr{Y}^c$ has a left adjoint if and only if f has a left adjoint. In which case the left adjoint of f is induced by the left adjoint of f^c.*

We also record the following proposition from [BDS16]

Proposition 2.2.3. *Suppose $f : \mathscr{X} \to \mathscr{Y}$ is a colimit-preserving functor of compactly generated presentable stable categories (with anti-automorphisms as above) which preserves compact objects and such that $f^D \cong f$. Let g be the right adjoint of f. Then, f preserves limits if and only if g preserves compact objects.*

Proof. We know that f preserves limits if and only if f has a left adjoint. By the second lemma above, f has a left adjoint if and only if f^c has a left adjoint. Now, f^c has a left adjoint if and only if it has a right adjoint because it is invariant under duality. Finally, f^c has a right adjoint if and only if g preserves compact objects by the first lemma above. ■

As a consequence, we have the following lemmas, which can be proven directly.

Lemma 2.2.4. *Suppose $f : \mathscr{X} \to \mathscr{Y}$ is a map of compactly generated stable categories with a anti-automorphism on the compact objects which preserves compact objects. Let g be the right adjoint of f and suppose g preserves compact objects. Then, g^D is the left adjoint of f^D.*

Lemma 2.2.5. *Suppose $f : \mathscr{X} \to \mathscr{Y}$ is a map of compactly generated stable categories with a anti-automorphism on the compact objects which preserves compact objects and limits. Let g be the left adjoint of f. Then, g^D is the right adjoint of f^D.*

Now let us discuss left-right duality. Suppose $\mathscr{X}$ and $\mathscr{Y}$ are dualizable $\mathscr{V}$-categories, in the notation of Appendix A.1. Then for $f : \mathscr{X} \to \mathscr{Y}$ a colimit-preserving $\mathscr{V}$-linear functor, there is a colimit preserving dual functor

$$f^\vee : \mathscr{Y}^\vee \to \mathscr{X}^\vee$$

We refer to this duality as left-right duality. Left-right duality also interchanges adjunctions, namely the following is easily seen.

Proposition 2.2.6. *Suppose $f : \mathscr{X} \to \mathscr{Y}$ is left adjoint to $g : \mathscr{Y} \to \mathscr{X}$ and both are colimit-preserving $\mathscr{V}$-linear functors between $\mathscr{V}$-dualizable categories, then $g^\vee$ is left adjoint to $f^\vee$.*

Corollary 2.2.7. *Suppose $f : \mathscr{X} \to \mathscr{Y}$ is a $\mathscr{V}$-linear colimit-preserving functor between compactly generated $\mathscr{V}$-module categories. Then f preserves compacts if and only if $f^\vee$ is limit preserving.*

Proof. f preserves compact objects if and only if it has a colimit-preserving right adjoint, which is true if and only if $f^\vee$ has a left adjoint, which is equivalent to $f^\vee$ preserving limits. ■

Left-right duality does not change the kernels of Fourier-Mukai transforms. More precisely, the following is also easily checked

Proposition 2.2.8. *Suppose*

$$f : \mathscr{X} \to \mathscr{Y}$$

is given by the Fourier-Mukai transform with kernel

$$K \in \mathscr{X}^\vee \otimes_{\mathscr{V}} \mathscr{Y}$$

(all colimit-preserving $\mathscr{V}$-linear functors are of this form) then

$$f^\vee : \mathscr{Y}^\vee \to \mathscr{X}^\vee$$

is given by the same kernel K inside

$$(\mathscr{Y}^\vee)^\vee \otimes_{\mathscr{V}} \mathscr{X}^\vee \cong \mathscr{X}^\vee \otimes_{\mathscr{V}} \mathscr{Y}$$

Remark 2.2.9. *Suppose $\mathscr{V} \cong k{-}\mathrm{Mod}$ for a commutative ring k, $\mathscr{X} \cong A{-}\mathrm{Mod}$, and $\mathscr{Y} \cong B{-}\mathrm{Mod}$ for some k-algebras A and B. Let $f : A{-}\mathrm{Mod} \to B{-}\mathrm{Mod}$ be given by tensoring over A with some (B, A) bimodule M. In this case $f^\vee : B^{\mathrm{op}}{-}\mathrm{Mod} \to A^{\mathrm{op}}{-}\mathrm{Mod}$ is given by tensoring over B^{op} with the same M, thought of as a $(A^{\mathrm{op}}, B^{\mathrm{op}})$ bimodule.*

In practice we will almost never use the the superscript $^\vee$ to denote left-right duality. We note here that if X is a qcqs spectral algebraic space over S, $\mathrm{QCoh}(X)$ is always self-dual over $\mathrm{QCoh}(S)$ (see [Lur18] 9.4.2.2, 9.4.3.1, 9.4.4.6, and 9.6.1.1). As a consequence of the Proposition 2.2.8, we note that left-right duality switches quasicoherent pullback with quasicoherent pushforward, as they are given by the same Fourier-Mukai kernels. Finally, suppose we are given qcqs spectral algebraic spaces X over S. Let $i_Z : Z \to X$ be a locally almost of finite presentation closed immersion of X. Then

Proposition 2.2.10. *$\Gamma_Z(\mathrm{QCoh}(X))$ is self-dual and left-right duality interchanges i_Z with Γ_Z.*

Proof. We can apply the same argument as the standard proof that $\mathrm{QCoh}(X)$ is self-dual when X is a perfect stack (for example Corollary 4.8 in [BZFN10], though note that they use a stronger than necessary definition of perfect stack). The only difference is that when showing $\mathrm{QCoh}(X)$ is self-dual, the unit and counit maps are given Fourier-Mukai transforms with the kernel

$$\mathscr{O}_\Delta \in \mathrm{QCoh}(X \times_S X)$$

Whereas to show $\Gamma_Z(\mathrm{QCoh}(X))$ is self-dual, we use instead the kernel

$$\Gamma_Z(\mathcal{O}_\Delta) \in \Gamma_Z(\mathrm{QCoh}(X)) \otimes_{\mathrm{QCoh}(S)} \Gamma_Z(\mathrm{QCoh}(X))$$

The rest of the proof proceeds the same way as in [BZFN10].

For the second part of the proposition, simply check that both functors are given by the same Fourier-Mukai kernel, namely,

$$\Gamma_Z(\mathcal{O}_\Delta) \in \Gamma_Z(\mathrm{QCoh}(X)) \otimes_{\mathrm{QCoh}(S)} \mathrm{QCoh}(X)$$

■

CHAPTER 3

Grothendieck Duality

3.1 Diagonally Supported Sheaves

In this section, we introduce the "quasicoherent" sheaf of categories $\Gamma_\Delta(\mathrm{QCoh}(X \times X))$, which is in a sense the main player of the entire thesis.

We adopt the terminology of [Lur18]. Fix a spectral affine scheme S as the base. In this section, let X be a spectral affine scheme with a structure map $p_X : X \to S$ which is almost of finite presentation and finite tor-amplitude. By the results of this section, the theory can be bootstrapped to the case of X a spectral Deligne-Mumford stack using étale descent, such that the map to S is locally almost of finite presentation, finite tor-amplitude. It is also possible to work over a much more general base because of descent of the construction with respect to the fpqc or descendable topology, see Remark 3.1.8.

Let $X \times_S X$ be the pullback of p_X along itself. We define π_1 and π_2 to be the two projection maps of this pullback. Here is a diagram,

$$\begin{tikzcd} X \times_S X \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & X \arrow[d, "p_X"] \\ X \arrow[r, "p_X"] & S \end{tikzcd} \tag{3.1}$$

Let Δ denote the diagonal inside $X \times_S X$ (which is abstractly isomorphic to X). The inclusion $\delta : X \to X \times_S X$ is locally almost of finite presentation by [Lur18] Proposition 4.2.1.6 and [Lur17] Corollary 7.4.3.19. Thus we can consider the subcategory $\Gamma_\Delta(\mathrm{QCoh}(X \times_S X))$ of quasicoherent sheaves on $X \times_S X$ which is supported on the diagonal. Let us denote the inclusion functor by i_Δ and its right adjoint by Γ_Δ.

We write

$$\tilde{\pi}_{1,*} : \Gamma_\Delta(\mathrm{QCoh}(X \times_S X)) \to \mathrm{QCoh}(X)$$

for the composition $\pi_{1,*} i_\Delta$ and

$$\tilde{\pi}_1^\times : \mathrm{QCoh}(X) \to \Gamma_\Delta(\mathrm{QCoh}(X \times_S X))$$

for the right adjoint of $\tilde{\pi}_{1,*}$. Importantly, $\tilde{\pi}_1^\times$ is a colimit-preserving functor. This follows from the following theorem because all the categories in sight are compactly generated (see Lemma 2.1.2).

Theorem 3.1.1. $\tilde{\pi}_{1,*}$ *preserves compact objects.*

Proof. Because i_Δ preserves compact objects, any compact object

$$x \in \Gamma_\Delta(\mathrm{QCoh}(X \times_S X))$$

can be thought of as a compact object $x \in \mathrm{QCoh}(X \times_S X)$ supported at the diagonal. By Proposition 5.6.5.2 in [Lur18], $\pi_{1,*}(x)$ is almost perfect. Because $p_X : X \to S$ is finite tor-amplitude, $\pi_{1,*}(x)$ is also finite tor-amplitude. Therefore, $\pi_{1,*}(x)$ is perfect by [Lur17] Proposition 7.2.4.23, hence compact. ■

Corollary 3.1.2. $\tilde{\pi}_1^\times$ *is a colimit-preserving* $\mathrm{QCoh}(X)$*-linear functor, where* $\mathrm{QCoh}(X)$ *acts on* $\Gamma_\Delta(\mathrm{QCoh}(X \times X))$ *via* $\Gamma_\Delta \pi_1^*$.

Proof. Follows from the theorem above and Theorem A.1.6. ■

Étale descent of the category $\Gamma_\Delta(\mathrm{QCoh}(X \times_S X))$ will follow from the fact that $U \mapsto \Gamma_\Delta(\mathrm{QCoh}(U \times_S U))$ is a "quasicoherent" sheaf of categories on the affine étale site of X[1]. where $\mathrm{QCoh}(U)$ acts on $\Gamma_\Delta(\mathrm{QCoh}(U \times_S U))$ via $\Gamma_\Delta \pi_1^*$ (i.e. it acts by tensoring on the first component).

Proposition 3.1.3. *For an affine étale map* $u : U \to X$*, we have*

$$\mathrm{QCoh}(U) \otimes_{\mathrm{QCoh}(X)} \Gamma_\Delta(\mathrm{QCoh}(X \times_S X)) \cong \Gamma_\Delta(\mathrm{QCoh}(U \times_S U))$$

where $\mathrm{QCoh}(X)$ *acts on* $\mathrm{QCoh}(U)$ *via* j^* *and* $\mathrm{QCoh}(X)$ *acts on* $\Gamma_\Delta(\mathrm{QCoh}(X \times_S X))$ *via* $\Gamma_\Delta \pi_1^*$.

[1] for a non separated U we have to be slightly careful with the definitions of support, but we can avoid this issue by restricting to affine étale maps (the topos is unchanged so there's no loss of generality).

Proof. The left hand side is canonically

$$\mathrm{QCoh}(U \times_S X) \otimes_{\mathrm{QCoh}(X \times_S X)} \Gamma_\Delta(\mathrm{QCoh}(X \times_S X)) \cong \Gamma_\Delta(\mathrm{QCoh}(U \times_S X))$$

because tensor products preserve split-exact sequences of presentable stable categories (see Proposition A.2.7). Now the result follows from the Proposition 2.1.4 applied to the diagonal closed immersion $U \to U \times X$ which factors through the map $U \times U \to U \times U$. ■

Corollary 3.1.4.

$$U \mapsto \Gamma_\Delta(\mathrm{QCoh}(U \times_S U))$$

is a sheaf on the affine étale site of X*.*

Proof. This follows from the above proposition as all quasicoherent sheaves of categories satisfy étale descent (see Remark 10.1.2.10 of [Lur18] or Proposition 3.45 of [Mat16]), though in this case it is easy to check directly that $\Gamma_\Delta(U \times_S X)$ is an étale sheaf directly as well. ■

Remark 3.1.5. *Proposition 3.1.3 and Corollary 3.1.4 admit obvious generalizations to products of more than two terms.*

Next, we show that $\tilde{\pi}_1^\times$ is a "quasicoherent" map of (quasicoherent) sheaves of categories.

Proposition 3.1.6. *For an affine étale map* $u : U \to X$*,*

$$\tilde{\pi}_{1,U}^\times : \mathrm{QCoh}(U) \to \Gamma_\Delta(\mathrm{QCoh}(U \times_S U))$$

is $\mathrm{QCoh}(U)$*-linear, colimit-preserving, and agrees with* $\tilde{\pi}_{1,X}^\times$ *for* X *base changed to* U*, i.e. tensored with* $\mathrm{QCoh}(U)$ *over* $\mathrm{QCoh}(X)$*.*

Proof. The map above is $\mathrm{QCoh}(U)$-linear and colimit-preserving by Corollary 3.1.2. The second claim above follows because tensoring with $\mathrm{QCoh}(U)$ over $\mathrm{QCoh}(X)$ preserves adjoints of colimit-preserving functors and $\tilde{\pi}_{1,*}$ for X tensored to U agrees with $\tilde{\pi}_{1,*}$ for U. ■

For $\mathcal{F} \in \Gamma_\Delta(\mathrm{QCoh}(X \times_S X))$, and an affine étale map $u : U \to X$, we denote by $\mathcal{F}|_U$ the (quasicoherent) pullback of $\mathcal{F}$ in

$$\Gamma_\Delta(\mathrm{QCoh}(U \times_S U))$$

Proposition 3.1.7. *For* $\mathcal{F} \in \mathrm{QCoh}(X)$ *and an affine étale map* $u : U \to X$*,*

$$\tilde{\pi}_{1,X}^\times(\mathcal{F})|_U \cong \tilde{\pi}_{1,U}^\times(\mathcal{F}|_U)$$

Proof. This is a direct consequence of Theorem A.1.4 applied to pullback along $u : U \to X$ and upper cross functor $\tilde{\pi}_1^{\times}$ and the above proposition. ■

Remark 3.1.8. *Note that from étale descent in X, we can define*

$$\Gamma_{\Delta}(\mathrm{QCoh}(X \times_S X))$$

for X a locally almost of finite presentation and finite tor-amplitude spectral Deligne Mumford stack over S. Also because

$$\Gamma_{\Delta}(\mathrm{QCoh}(X \times_S X))$$

admits descent with respect to the fpqc/descendable topology on S, we can even generalize to $p_X : X \to S$ being a locally almost of finite presentation, finite tor-amplitude map of sheaves which is a relative spectral Deligne-Mumford stack. Note that for p_X being a map of sheaves, quasicoherence of this category is not clear to the author.

3.2 Dualizing Complexes and the Upper Shriek Functor

This section is dedicated to defining the upper shriek functor and proving some properties of it. Almost all of the results in this section are, in some form, contained in [ILN15] and [Nee18]. The key differences are the order of presentation–we define the upper shriek functor without compactifications at all and develop its properties from scratch–and the fact that we make heavy use of the category $\Gamma_{\Delta}(\mathrm{QCoh}(X \times_S X))$, which is morally "proper" over X. We are motivated to study this subcategory for its own sake in view of its connections with differential operators.

We begin by defining the upper shriek functor for an almost of finite presentation and finite tor-amplitude map $p_X : X \to S$ between spectral affine schemes. As before, globalization to a more general base S will be immediate by construction and to a more general X will follow from the results proven.

Definition 3.2.1. *The upper shriek functor $p_X^! : \mathrm{QCoh}(S) \to \mathrm{QCoh}(X)$ is defined by*

$$p_X^! := \delta^* \pi_1^{\times} p_X^*$$

where $\delta : X \to X \times_S X$ is the diagonal map.

Remark 3.2.2. *Following [BBST24], it may be possible to give an alternative definition for this functor. Namely, we can define it by*

$$p_X^!(\_) := p_X^*(\_) \otimes \omega_{X/S}$$

and to define $\omega_{X/S}$ be the (necessarily unique if it exists) unit of a symmetric monoidal category of coherent sheaves on X and with cross-pullback as the product. We explore this possibility in Section 3.6.

This formula (often referred to as a reduction formula) for the upper shriek functor appears in many places in the literature, e.g. [Nee18] Proposition 3.3, however here we will take it as a definition. The main property of upper shriek is that it behaves well under composition, that is

$$(fg)^! \cong g^! f^!$$

and that it interpolates between upper-cross pullback in the proper case and upper-star pullback in the étale case. This is what we aim to show in this section.

The pullback functor along the diagonal

$$\delta^* : \mathrm{QCoh}(X \times_S X) \to \mathrm{QCoh}(X)$$

factors through the local cohomology functor Γ_Δ, namely

$$\delta^* \cong \tilde{\delta}^* \Gamma_\Delta$$

Therefore,

$$p_X^! \cong \tilde{\delta}^* \tilde{\pi}_1^\times p_X^* \tag{3.2}$$

From the above we see the upper shriek functor is colimit preserving and QCoh(S)-linear.

Proposition 3.2.3. *Suppose $u : U \to X$ is an affine étale map, then*

$$u^! = u^*$$

Proof. This follows from the Proposition 2.1.4 applied to the closed immersion $U \to U \times_X X$ with a lift to $U \times_X U$. Namely, we know that

$$\tilde{\pi}_{1,*} : \Gamma_\Delta(\mathrm{QCoh}(U \times_X U)) \to \mathrm{QCoh}(U)$$

is an isomorphism and its inverse and adjoint (on both sides) is $\Gamma_\Delta \pi_1^*$ (where $\pi_1 : U \times_X U \to U$ is the projection map to the first component). Therefore

$$\begin{aligned} u^! &\cong \tilde{\delta}^* \tilde{\pi}_1^\times u^* \\ &\cong \tilde{\delta}^* \Gamma_\Delta \pi_1^* u^* \\ &\cong u^* \end{aligned}$$

■

Proposition 3.2.4. *Suppose $u : U \to X$ is an affine étale map, and $p_U : U \to S$ is the structure map, then*

$$p_U^! \cong u^* p_X^! \cong u^! p_X^!$$

Proof. The second isomorphism follows from the previous proposition, the first follows because

$$\begin{aligned} p_U^! \mathcal{F} &\cong \tilde{\delta}_U^* \tilde{\pi}_{1,U}^\times p_U^* \mathcal{F} \\ &\cong \tilde{\delta}_U^* \tilde{\pi}_{1,U}^\times u^* p_X^* \mathcal{F} \\ &\cong \tilde{\delta}_U^* (\tilde{\pi}_{1,X}^\times p_X^* \mathcal{F})|_U \\ &\cong u^* \tilde{\delta}_X^* \tilde{\pi}_{1,X}^\times p_X^* \mathcal{F} \end{aligned}$$

where the third isomorphism uses Proposition 3.1.7. ■

Remark 3.2.5. *The last two propositions allow us to globalize the construction of upper-shriek to a locally almost of finite presentation, finite tor-amplitude map which is a relative Deligne-Mumford stack. Also the propositions bootstrap to allow us to remove the affineness condition on the étale maps in the propositions.*

Definition 3.2.6. *For any map $p_X : X \to S$ of spectral Deligne-Mumford stacks which is locally almost of finite presentation and finite tor-amplitude, we define the relative dualizing complex of X over S to be*

$$\omega_{X/S} := p_X^!(\mathcal{O}_S) \tag{3.3}$$

where $p_X^!$ is defined as in Remark 3.2.5.

Because of the Proposition 3.2.4 and Remark 3.2.5, for an étale $u : U \to X$, we have

$$u^* \omega_{X/S} \cong \omega_{U/S}$$

Also, ω behaves well under base-change with respect to S. Namely, if $q : S' \to S$ is a map of spectral affine schemes, there is an isomorphism

$$\omega_{X \times_S S'/S'} \cong (\mathrm{id} \times q)^*(\omega_{X/S})$$

To be general,

Theorem 3.2.7. *Suppose we have the following pullback diagram of spectral Deligne-Mumford* *stacks where p_Y is locally almost of finite presentation and finite tor-amplitude*

$$\begin{tikzcd} Y_{S'} \cong Y \times_S S' \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & Y \arrow[d, "p_Y"] \\ S' \arrow[r, "p_{S'}"] & S \end{tikzcd}$$

where p_Y is finite tor-amplitude and all maps are almost of finite presentation. Then

$$\pi_1^! p_{S'}^* \cong \pi_2^* p_Y^!$$

Proof. The entire construction base-changes well with respect to S, so this is clear. ■

Because $p_X^!$ is colimit preserving and $\mathrm{QCoh}(S)$-linear, we have

$$p_X^!(\mathcal{F}) \cong \omega_{X/S} \otimes p_X^* \mathcal{F} \tag{3.4}$$

Remark 3.2.8. *Equation (3.4), combined with the analogous statement for the classically defined upper-shriek functor (see [Nee14] Remark 1.22) implies via Corollary 4.7 of [AILN10] that our upper-shriek functor agrees with the classical one for finite tor-amplitude, finite-type, separated morphisms of non-derived Noetherian schemes.*

Remark 3.2.9. *The above statements generalize to relative spectral Deligne-Mumford stacks.*

Proposition 3.2.10. *Suppose $p_X : X \to S$ is a separated map of qcqs algebraic spaces, which is locally almost of finite presentation and finite tor-amplitude. Then there is a natural transformation*

$$p_X^\times \to p_X^!$$

and hence also a natural map

$$\mathrm{Hom}(p_{X,*}\mathcal{F}, \mathcal{G}) \to \mathrm{Hom}(\mathcal{F}, p_X^! \mathcal{G})$$

for $\mathcal{F} \in \mathrm{QCoh}(X)$ and $\mathcal{G} \in \mathrm{QCoh}(S)$.

Proof. Consider the square

$$\begin{tikzcd} X \times_S X \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & X \arrow[d, "p"] \\ X \arrow[r, "p"] & S \end{tikzcd}$$

From the push-pull isomorphism ([Lur18] Proposition 6.3.4.1), there is a map

$$\pi_{1,*}\pi_2^* p_X^\times \cong p_X^* p_{X,*} p_X^\times \to p_X^*$$

hence by adjunction, we have a map

$$\pi_2^* p_X^\times \to \pi_1^\times p_X^*$$

Pulling back along δ gives a map

$$p_X^\times \to \delta^* \pi_1^\times p_X^* \cong p_X^!$$

where we use the fact that Definition 3.2.1 applies for separated, locally almost of finite presentation, finite tor-amplitude maps of relative qcqs algebraic spaces. ■

Remark 3.2.11. *This proposition can be generalized to a more general base S.*

We recall the following proposition from [Lur18], which we will refer to as pull-cross isomorphism. We note that after reduction to the case where S is affine, it follows from the fact ([Lur18] Theorem 6.1.3.2) that pushforward along maps which are locally almost of finite presentation, proper, and finite tor-amplitude preserve compact objects as well as categorical base-change results of Appendix A.1 (Theorem A.1.4) together with the fact that colimit-preserving adjunctions of module categories are preserved under extension of scalars [of categories]).

Proposition 3.2.12. *[[Lur18] Proposition 6.4.2.1] Suppose p_Y is a proper, locally almost of finite presentation, finite tor-amplitude map which is a relative spectral algebraic space. Then, if $p_{S'} : S' \to S$ is any map of spectral Deligne-Mumford stacks,*

$$\pi_1^\times p_{S'}^* \cong \pi_2^* p_X^\times$$

where the notation is as in the diagram

$$\begin{tikzcd} Y_{S'} \cong Y \times_S S' \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & Y \arrow[d, "p_Y"] \\ S' \arrow[r, "p_{S'}"] & S \end{tikzcd}$$

Theorem 3.2.13. *Suppose $p_X : X \to S$ and $g : Y \to X$ are locally almost of finite presentation, finite tor-amplitude maps of qcqs algebraic spaces. Suppose p_X is separated and the composition $p_x \circ g$ is proper. Then (this result is Lemma 3.1 in [Nee18]), the natural transformation $p_X^\times \to p_X^!$ is an isomorphism after post-composition with $g^\times$.*

Proof. Consider the diagram

$$\begin{tikzcd} Y \times_S X \arrow[r, "g\times\mathrm{id}"] \arrow[d, "\pi_1'"] & X \times_S X \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & X \arrow[d, "p_X"] \\ Y \arrow[r, "g"] & X \arrow[r, "p_X"] & S \end{tikzcd}$$

The outer rectangle exhibits pull-cross base-change (Proposition 3.2.12), namely,

$$\pi_1'^* g^\times p_X^\times \cong (g \times \mathrm{id})^\times \pi_2^\times p_X^*$$

The map exhibiting the isomorphism is formed using the pull-cross base-change maps for the two smaller squares. Now we post-compose the above isomorphism with the pullback along the graph of g, $\delta_g : Y \to Y \times_S X$, to get

$$g^\times p_X^\times \cong \delta_g^*(g \times \mathrm{id})^\times \pi_2^\times p_X^*$$

Now looking at the pull-cross base-change for the diagram (since g is also proper)

$$\begin{tikzcd}
Y \arrow[r, "\delta_g"] \arrow[d, "g"] & Y \times_S X \arrow[d, "g\times \mathrm{id}"] \\
X \arrow[r, "\delta"] & X \times_S X
\end{tikzcd}$$

namely,

$$g^{\times}\delta^{*} \cong \delta_g^{*}(g \times \mathrm{id})^{\times}$$

We have,

$$g^{\times}p_X^{\times} \cong \delta_g^{*}(g \times \mathrm{id})^{\times}\pi_2^{\times}p_X^{*} \cong g^{\times}\delta^{*}\pi_2^{\times}p_X^{*} \cong g^{\times}p_X^{!}$$

One checks that the map agrees with the map in the previous proposition post-composed with $g^{\times}$ by staring at the following combined diagram using the fact that the base-change for the left tall rectangle is trivial.

$$\begin{tikzcd}
Y \arrow[r, "g"] \arrow[d, "\delta_g"] & X \arrow[d, "\delta"] & \\
X \times_S Y \arrow[r, "g\times \mathrm{id}"] \arrow[d, "\pi_1'"] & X \times_S X \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & X \arrow[d, "p_X"] \\
Y \arrow[r, "g"] & X \arrow[r, "p_X"] & S
\end{tikzcd}$$

■

Remark 3.2.14. *The base S can be made more general in this proposition by descent.*

Corollary 3.2.15. *Suppose $p_X : X \to S$ is a proper, almost of finite presentation, finite tor-amplitude map of spectral algebraic spaces, then $p_X^{\times} \cong p_X^{!}$.*

Theorem 3.2.16. *Let p_X be a locally almost of finite presentation, finite tor-amplitude map of qcqs algebraic spaces. Suppose Λ is a closed subset of $|X|$ which is proper over S, then*

$$\Gamma_{\Lambda}p_X^{\times} \cong \Gamma_{\Lambda}p_X^{!}$$

Proof. Repeat the argument used to prove Theorem 3.2.13, rephrased in terms of categories of quasicoherent sheaves and then substitute $\Gamma_\Lambda(\mathrm{QCoh}(X))$ wherever $\mathrm{QCoh}(Y)$ appears, using the fact that $p_{X,*}i_\Lambda$ preserves compact objects. This is because i_Λ preserves compacts and $p_{X,*}$ is finite tor-amplitude and sends perfect objects supported on Λ to almost perfect objects (see SAG Proposition 5.6.5.2). ■

Remark 3.2.17. *Note that the generalization of this theorem to S being a stack needs to require that the map from the reduced closed substack Λ to S to be proper.*

Corollary 3.2.18. *The map*

$$\mathrm{Hom}(p_{X,*}\mathcal{F}, \mathcal{G}) \to \mathrm{Hom}(\mathcal{F}, p_X^!\mathcal{G})$$

in Theorem 3.2.10 is an isomorphism if $\mathcal{F}$ is supported on a proper (over S) subscheme.

We conclude by showing that upper shriek composes well.

Theorem 3.2.19. *Suppose $g : Y \to X$ and p_X are almost of finite presentation and finite tor-amplitude maps of spectral Deligne-Mumford stacks. Then,*

$$g^! p_X^! \cong p_Y^!$$

Proof. We immediately reduce to the case of spectral affine schemes.

Consider the diagram

$$\begin{tikzcd} X \times_S Y \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & Y \arrow[d, "p_Y"] \\ X \arrow[r, "p_X"] & S \end{tikzcd}$$

We have

$$\begin{aligned} p_Y^! &\cong \tilde{\delta}_Y^* \tilde{\pi}_2^{(Y\times Y),\times} p_Y^* \\ &\cong \tilde{\delta}_Y^* \widetilde{(g\times \mathrm{id})}^{\times} \tilde{\pi}_2^{(X\times Y),\times} p_Y^* \\ &\cong \tilde{\delta}_Y^* \widetilde{(g\times \mathrm{id})}^{\times} \tilde{\pi}_2^{(X\times Y),!} p_Y^* \\ &\cong \tilde{\delta}_Y^* \widetilde{(g\times \mathrm{id})}^{\times} \tilde{\pi}_1^{(X\times Y),*} p_X^! \end{aligned}$$

where $\tilde{\pi}_2^{(X\times Y),!} := \Gamma_Y \pi_2^{(X\times Y),!}$ and similarly for the $\tilde{\pi}_2^{(X\times Y),\times}$. The last isomorphism follows from Theorem 3.2.7. Now look at the cartesian diagram

$$\begin{tikzcd}
Y \times_X Y \arrow[r, "\phi = \mathrm{id} \times_{p_X} \mathrm{id}"] \arrow[d, "\pi_2"] & Y \times_S Y \arrow[d, "g \times \mathrm{id}"] \\
Y \arrow[r, "\delta_g"] & X \times_S Y
\end{tikzcd}$$

So,

$$\begin{aligned}
p_Y^! &\cong \tilde{\delta}_Y^* \widetilde{(g \times \mathrm{id})}^{\times} \tilde{\pi}_1^{(X \times Y),*} p_X^! \\
&\cong \tilde{\delta}_Y^* \phi^* \widetilde{(g \times \mathrm{id})}^{\times} \tilde{\pi}_1^{(X \times Y),*} p_X^! \\
&\cong \tilde{\delta}_Y^* \tilde{\pi}_2^{(Y \times_X Y),\times} \delta_g^* \tilde{\pi}_1^{(X \times Y),*} p_X^! \\
&\cong \tilde{\delta}_Y^* \tilde{\pi}_2^{(Y \times_X Y),\times} g^* p_X^! \\
&\cong g^! p_X^!
\end{aligned}$$

■

Remark 3.2.20. *The statements of this sections indicates that Grothendieck duality, in the sense of constructing an upper shriek functor satisfies section 2 of [Nee18], can be developed from scratch using Definition 3.2.1 by making ample use of the category* $\Gamma_\Delta(\mathrm{QCoh}(X \times X))$*. In the section on smooth varieties, we follow Neeman and show that we can easily identify* ω_X *with the sheaf of top differential forms (shifted appropriatedly) in the smooth case.*

However, one of the limitations of this thesis is that we do not include a proof of the full homotopy coherence of the upper shriek functor. One approach could be to use the definition indicated in Remark 3.2.2. We leave this for a possible future work.

Remark 3.2.21. *Bhargav Bhatt pointed out that the upper shriek functor is also characterized on separated qcqs algebraic spaces (up to isomorphism) by the following properties.*

1. *There is a map* $p_X^\times \to p_X^!$ *such that the induced map*

$$\mathrm{Hom}(p_{X,*}\mathcal{F}, \mathcal{G}) \to \mathrm{Hom}(\mathcal{F}, p_X^! \mathcal{G})$$

   is an isomorphism when $\mathcal{F}$ *has proper support over* S *(this is the second half of Corollary 3.2.18).*

2. *Theorem 3.2.7 holds.*

This observation can be deduced from the following diagram

$$\begin{tikzcd}
X \arrow[dr, "\delta"] & & \\
& X \times_S X \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & X \arrow[d, "p_X"] \\
& X \arrow[r, "p_X"] & S
\end{tikzcd}$$

Condition (2) implies that

$$p_X^! \cong \delta^* \pi_2^* p_X^! \cong \delta^* \pi_1^! p_X^*$$

Now condition (1) implies

$$\mathrm{id} \cong \delta^\times \pi_1^\times \to \delta^\times \pi_1^!$$

is an isomorphism. Hence

$$\delta^* \pi_1^! p_X^* \cong \delta^* \pi_1^\times p_X^*$$

using Lemma 2.1.3.

Remark 3.2.22. *We work with usual underived schemes in this remark. Suresh Nayak pointed out to us that it is possible to define the upper shriek functor along arbitary maps of finitely presented separated schemes which are finite tor-amplitude over a Noetherian base by factoring such a map*

$$f : X \to Y$$

as the composition of the graph of f

$$\Gamma_f : X \to X \times Y$$

composed with the projection map

$$\pi_Y : X \times Y \to Y$$

Then, we can define $f^!$ by the composition $\Gamma_f^\times \pi_Y^!$ where π_Y is finite tor-amplitude and hence we can define upper shriek along it using the techniques in this paper. However, for such a definition to be compatible with compositions, we must restrict to the subcategory D_{qc}^+ of objects with bounded below cohomology. However, we do not currently know how to adapt the category-theoretic proofs in this paper to this setting.

3.3 Dualizing Complexes and the Lower Shriek Functor

In this section, we introduce the lower shriek functor and prove some Hochschild-type formulas which appear in [Nee18]. We have seen that the upper shriek functor satisfies

$$p^!(\_) \cong p^*(\_) \otimes \omega$$

Now the lower shriek functor will turn out to satisfy an analogous equation, namely,

$$p_!(\_) \cong p_*(\_ \otimes \omega)$$

In fact, these two are simply related by left-right duality. We also caution that our use of the symbol lower shriek is not necessarily standard, in particular it is not analogous to the étale lower shriek. However, this notation is not original either, for example see [Per19]. We insist on this notation because it is consistent with how the rest of our notation behaves under left-right duality. Much of this section is inspired by arguments in [BDS16] and [Nee18].

Suppose $p_X : X \to S$ is a locally almost of finite presentation, finite tor-amplitude map of qcqs algebraic spaces. Generalization to a more general base S is possible, but we ignore this issue in this section. The following theorem is the left-right dual of Theorem 3.1.1.

Theorem/Definition 3.3.1. *Denote by $\tilde{\pi}_1^*$ the functor*

$$\Gamma_\Delta \pi_1^* : \mathrm{QCoh}(X) \to \Gamma_\Delta(\mathrm{QCoh}(X \times_S X))$$

Then, $\tilde{\pi}_1^*$ preserves limits–we denote by $\tilde{\pi}_{1,\times}$ its left adjoint. We remind the reader that if p_X is not separated, the right hand side is defined by descent.

Proof. $\tilde{\pi}_1^*$ is the left-right dual of $\tilde{\pi}_{1,*}$, so the theorem follows from Corollary 2.2.7 applied to Theorem 3.1.1. We implicitly use that $\mathrm{QCoh}(X)$ is self-dual as a $\mathrm{QCoh}(S)$-module category (using the same proof as in [BZFN10]). ■

Remark 3.3.2. *We note that $\tilde{\pi}_{1,\times}$ preserves compact objects because $\tilde{\pi}_1^*$ is colimit-preserving. Also, $\tilde{\pi}_{1,\times}$ is left-right dual to $\tilde{\pi}_1^\times$.*

Let $\delta : X \to X \times_S X$ be the diagonal map. Then,

$$\delta_* : \mathrm{QCoh}(X) \to \mathrm{QCoh}(X \times_S X)$$

factors through

$$i_\Delta : \Gamma_\Delta(\mathrm{QCoh}(X \times_S X)) \to \mathrm{QCoh}(X \times_S X)$$

Namely,

$$\delta_* \cong i_\Delta \tilde{\delta}_*$$

We are now ready to define the lower shriek functor, in analogy to the upper shriek functor.

Definition 3.3.3. *The lower shriek functor* $p_{X,!} : \mathrm{QCoh}(X) \to \mathrm{QCoh}(S)$ *is defined by*

$$p_{X,!} := p_{X,*}\tilde{\pi}_{1,\times}\tilde{\delta}_*$$

Remark 3.3.4. *By comparison with (3.2) it is clear that* $p_{X,!}$ *is the left-right dual of* $p_X^!$*. We implicitly use Proposition 2.2.10.*

We can now take most of the results of section 2 and apply left-right duality to them to obtains new results about lower shriek. For example, we have the following analogue of Proposition 3.2.4, which follows directly from left-right duality.

Proposition 3.3.5. *Suppose* $u : U \to X$ *is an étale map , then*

$$p_{U,!} \cong p_{X,!}u_* \cong p_{X,!}u_!$$

Also, we can take the left-right dual of (3.4) to get

Proposition 3.3.6.

$$p_{X,!}(\mathcal{F}) \cong p_{X,*}(\mathcal{F} \otimes \omega_X)$$

Remark 3.3.7. *We can also show these results directly by arguing with compact objects, however we choose to present the proofs by duality because they are cleaner.*

As a preparation for the next theorem, we need the following result.

Proposition 3.3.8. *For* $\mathcal{F} \in \mathrm{QCoh}(X)$*,*

$$\tilde{\pi}_{1,\times}\tilde{\delta}_*\mathcal{F} \cong \tilde{\delta}^*\tilde{\pi}_1^\times\mathcal{F} \cong \mathcal{F} \otimes \omega_{X/S}$$

as $\mathrm{QCoh}(X)$*-linear colimit-preserving functors.*

Proof. Both the first and second expression are $\mathrm{QCoh}(X)$-linear colimit preserving functors of $\mathcal{F}$ by Corollary 3.1.2 and Corollary 3.3.1. $\mathrm{QCoh}(X)$-linear colimit preserving functors from $\mathrm{QCoh}(X)$ to itself are automatically self-dual because they are simply given by tensoring with a quasicoherent sheaf on X, showing the first equality. In this case, it is easy to see the functor is given by tensoring with $\omega_{X/S}$. This shows the claim. ∎

We are now ready to establish a Hochschild-style formula which is known in some form since [AILN10] and is elaborated on in [Nee18].

Theorem 3.3.9. *Let p_X be a locally of finite presentation, finite tor-amplitude map of qcqs algebraic spaces. For $\mathcal{F} \in \mathrm{QCoh}(S)$ and $\mathcal{G} \in \mathrm{QCoh}(X)$, we have*

$$\delta^\times \pi_1^* \mathscr{H}\!om(p_X^* \mathcal{F}, \mathcal{G}) \cong \mathscr{H}\!om(p_X^! \mathcal{F}, \mathcal{G})$$

where $\mathscr{H}\!om$ denotes internal Hom of quasicoherent sheaves.

Proof. Consider

$$\delta^\times : \mathrm{QCoh}(X \times_S X) \to \mathrm{QCoh}(X)$$

which is right adjoint to

$$\delta_* \cong i_\Delta \Gamma_\Delta \delta_* \cong i_\Delta \tilde{\delta}_*$$

Hence,

$$\delta^\times \cong \tilde{\delta}^\times \Gamma_\Delta$$

where $\tilde{\delta}^\times$ is right adjoint to $\tilde{\delta}_*$.

Therefore, given $\mathcal{H} \in \mathrm{QCoh}(X)$, we have

$$\begin{aligned}
Hom_X(\mathcal{H}, \delta^\times \pi_1^* \mathscr{H}\!om(p_X^* \mathcal{F}, \mathcal{G})) &\cong Hom_X(\mathcal{H}, \tilde{\delta}^\times \tilde{\pi}_1^* \mathscr{H}\!om(p_X^* \mathcal{F}, \mathcal{G})) \\
&\cong Hom_X(\tilde{\pi}_{1,\times} \tilde{\delta}_* \mathcal{H}, \mathscr{H}\!om(p_X^* \mathcal{F}, \mathcal{G})) \\
&\cong Hom_X(\omega_{X/S} \otimes \mathcal{H}, \mathscr{H}\!om(p_X^* \mathcal{F}, \mathcal{G})) \\
&\cong Hom_X(\omega_{X/S} \otimes p_X^* \mathcal{F} \otimes \mathcal{H}, \mathcal{G}) \\
&\cong Hom_X(p_X^! \mathcal{F} \otimes \mathcal{H}, \mathcal{G}) \\
&\cong Hom_X(\mathcal{H}, \mathscr{H}\!om(p_X^! \mathcal{F}, \mathcal{G}))
\end{aligned}$$

∎

Notice that if $X = \operatorname{Spec} A$ and $S = \operatorname{Spec} k$, then this theorem says (in a special case)

$$\mathrm{Hom}_{A \otimes A}(A, A \otimes A) \cong \mathrm{Hom}_A(\omega_A, A)$$

Corollary 3.3.10. *Let p_X be a locally of finite presentation, finite tor-amplitude map of qcqs algebraic spaces.*

$$\mathscr{H}\!om(\omega_{X/S}, \omega_{X/S}) \cong \mathcal{O}_X$$

Proof.

$$\begin{aligned}
\mathscr{H}om(\omega_X, \omega_X) &\cong \tilde{\delta}^\times \tilde{\pi}_2^* \omega_X \\
&\cong \tilde{\delta}^\times \tilde{\pi}_1^\times \mathcal{O}_X \\
&\cong \mathcal{O}_X
\end{aligned}$$

The first isomorphism comes from the theorem above and the second follows from Theorem 3.2.16. ■

Remark 3.3.11. *Both statements above generalize to a more general base S.*

We record here a proposition which is morally dual to Proposition 3.2.10 and Corollary 3.2.18, though we don't know how to show it directly by duality.

Proposition 3.3.12. *Let p_X be a locally of finite presentation, finite tor-amplitude, and separated map of qcqs algebraic spaces. There is a natural map*

$$Hom(p_{X,!}\mathcal{F}, \mathcal{G}) \to Hom(\mathcal{F}, p_X^* \mathcal{G})$$

which is an isomorphism if the support of $\mathcal{F}$ is proper over S.

Proof. The map is constructed as follows

$$\begin{aligned}
Hom(p_{X,!}\mathcal{F}, \mathcal{G}) &\cong Hom(p_{X,*}\tilde{\pi}_{1,\times}\tilde{\delta}_*\mathcal{F}, \mathcal{G}) \\
&\cong Hom(\tilde{\delta}_*\mathcal{F}, \tilde{\pi}_1^* p_X^\times \mathcal{G}) \\
&\to Hom(\tilde{\delta}_*\mathcal{F}, \tilde{\pi}_1^* p_X^! \mathcal{G}) \\
&\cong Hom(\tilde{\delta}_*\mathcal{F}, \Gamma_\Delta \pi_2^! p_X^* \mathcal{G}) \\
&\cong Hom(\mathcal{F}, \tilde{\delta}^\times \Gamma_\Delta \pi_2^! p_X^* \mathcal{G}) \\
&\cong Hom(\mathcal{F}, \tilde{\delta}^\times \Gamma_\Delta \pi_2^\times p_X^* \mathcal{G}) \\
&\cong Hom(\mathcal{F}, \tilde{\delta}^\times \tilde{\pi}_2^\times p_X^* \mathcal{G}) \\
&\cong Hom(\mathcal{F}, p_X^* \mathcal{G})
\end{aligned}$$

where the map in the third line comes from Proposition 3.2.10. The fourth line is base-change for upper shriek (see Theorem 3.2.7). On the sixth line we apply Theorem 3.2.16.

If Z is the support of $\mathcal{F}$, then assuming Z is proper over S, we want to show that the map on line three is an isomorphism. Indeed,

$$\begin{aligned}
Hom(\tilde{\delta}_*\mathcal{F}, \tilde{\pi}_1^* p_X^\times \mathcal{G}) &\cong Hom(\tilde{\delta}_*\mathcal{F}, \Gamma_{Z\times Z}\Gamma_\Delta \pi_1^* p_X^\times \mathcal{G}) \\
&\cong Hom(\tilde{\delta}_*\mathcal{F}, \Gamma_\Delta \Gamma_{Z\times Z} \pi_1^* p_X^\times \mathcal{G}) \\
&\cong Hom(\tilde{\delta}_*\mathcal{F}, \Gamma_\Delta \pi_1^* \Gamma_Z p_X^\times \mathcal{G}) \\
&\cong Hom(\tilde{\delta}_*\mathcal{F}, \Gamma_\Delta \Gamma_{X\times Z} \pi_2^\times p_X^* \mathcal{G}) \\
&\cong Hom(\tilde{\delta}_*\mathcal{F}, \Gamma_{Z\times Z} \Gamma_\Delta \pi_2^\times p_X^* \mathcal{G}) \\
&\cong Hom(\tilde{\delta}_*\mathcal{F}, \Gamma_\Delta \pi_2^\times p_X^* \mathcal{G})
\end{aligned}$$

where the fourth isomorphism follows from Theorem A.1.4 applied to $\mathscr{V} = \mathrm{QCoh}(S)$, $\mathscr{X} = \Gamma_Z(\mathrm{QCoh}(X))$, and $\mathscr{Y} = \mathrm{QCoh}(X)$, where the map $f = \Gamma_Z p_X^\times : \mathscr{V} \to \mathscr{X}$ is the right adjoint of

$$p_{X,*} i_Z : \Gamma_Z(\mathrm{QCoh}(X)) \to \mathrm{QCoh}(S)$$

f is colimit-preserving because $p_{X,*} i_Z$ preserves compact objects (argue as in Theorem 3.1.1). The map $g : \mathscr{V} \to \mathscr{Y}$ is just the quasicoherent pullback. ■

Lastly, we record a theorem about how our functors interact with up-down duality[2].

Theorem 3.3.13. *Let p_X be a locally of finite presentation, finite tor-amplitude map of qcqs algebraic spaces.*

$$(\tilde{\pi}_{1,\times})^D \cong \tilde{\pi}_{1,*}$$

and the isomorphism is étale local.

Proof. It suffices to show there is a natural isomorphism

$$\mathrm{Hom}_{\mathrm{QCoh}(X)}(\tilde{\pi}_{1,\times}(K^\vee), L) \cong \mathrm{Hom}_{\mathrm{QCoh}(X)}((\tilde{\pi}_{1,*}K)^\vee, L)$$

[2]see Section 2.2

for K compact in $\Gamma_\Delta(\mathrm{QCoh}(X \times_S X))$ and L in $\mathrm{QCoh}(X)$. But this follows from

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{QCoh}(X)}(\tilde{\pi}_{1,\times}(K^\vee), L) &\cong \mathrm{Hom}_{\Gamma_\Delta(\mathrm{QCoh}(X\times X))}(K^\vee, \tilde{\pi}_1^* L) \\
&\cong \mathrm{Hom}_{\Gamma_\Delta(\mathrm{QCoh}(X\times X))}(K^\vee, \Gamma_\Delta \pi_1^* L) \\
&\cong \mathrm{Hom}_{\mathrm{QCoh}(X\times X)}(K^\vee, \pi_1^* L) \\
&\cong \mathrm{Hom}_{\mathrm{QCoh}(X\times X)}(\mathcal{O}_{X\times X}, K \otimes_{\mathcal{O}_{X\times X}} \pi_1^* L) \\
&\cong \mathrm{Hom}_{\mathrm{QCoh}(X)}(\mathcal{O}_X, \pi_{1,*}(K \otimes_{\mathcal{O}_{X\times X}} \pi_1^* L)) \\
&\cong \mathrm{Hom}_{\mathrm{QCoh}(X)}(\mathcal{O}_X, \pi_{1,*}K \otimes_{\mathcal{O}_X} L) \\
&\cong \mathrm{Hom}_{\mathrm{QCoh}(X)}((\pi_{1,*}K)^\vee, L) \\
&\cong \mathrm{Hom}_{\mathrm{QCoh}(X)}((\tilde{\pi}_{1,*}K)^\vee, L)
\end{aligned}$$

where by abuse of notation K can also be thought of as an object in $\mathrm{QCoh}(X \times_S X)$. ■

3.4 Grothendieck Differential Operators

In this section, we define the sheaf of Grothendieck differential operators and show that it satisfies étale descent. Let X be a spectral affine scheme which is almost of finite presentation and finite tor-amplitude over S, another spectral affine scheme.

Theorem/Definition 3.4.1. *There is a natural convolution monoidal structure on* $\Gamma_\Delta(\mathrm{QCoh}(X \times_S X))$. $\mathrm{QCoh}(X)$ *has the structure of a left* $\Gamma_\Delta(\mathrm{QCoh}(X \times X))$ *module with respect to this monoidal structure.*[3]

Proof. We construct it by inducing it from $\mathrm{QCoh}(X \times_S X)$. We have the isomorphism

$$\mathrm{QCoh}(X \times_S X) \cong \mathrm{End}_S(\mathrm{QCoh}(X), \mathrm{QCoh}(X))$$

The right hand side is endomorphisms of $\mathrm{QCoh}(X)$ inside $\mathrm{QCoh}(S)-\mathrm{Mod}^L$, and thus has a natural monoidal structure. This gives the desired monoidal structure on $\mathrm{QCoh}(X \times_S X)$. It is easy to see that $\Gamma_\Delta(\mathrm{QCoh}(X \times_S X))$ is closed under this product, and so inherits a convolution monoidal structure. The second part of the theorem is clear from our construction.

To be explicit, given two quasicoherent sheaves F and G on $X \times X$, their convolution is simply

$$F \star G := \pi_{1,3,*}(\pi_{1,2}^* F \otimes \pi_{2,3}^* G)$$

[3] As mentioned in the introduction, with this monoidal product this category is a the categorified ring of differential operators.

where

$$\pi_{i,j} : X \times X \times X \to X \times X$$

are the obvious projection maps. ■

Remark 3.4.2. *If $X = \operatorname{Spec} A$, this tensor product for A-bimodules is simply given by tensoring the two A-bimodules together over A.*

Remark 3.4.3. $\mathrm{QCoh}(X)$ *as a* $\Gamma_\Delta(\mathrm{QCoh}(X \times_S X))$*-module category is related to the enriched category whose objects are quasicoherent sheaves and morphisms are differential operators.*

For U affine étale over X, it's clear from the definition that the pullback map

$$\Gamma_\Delta(\mathrm{QCoh}(X \times_S X)) \to \Gamma_\Delta(\mathrm{QCoh}(U \times_S U))$$

is monoidal with respect to the convolution product.

Definition 3.4.4. *Let $X = \operatorname{Spét} A$ and $S = \operatorname{Spét} k$. The sheaf of Grothendieck differential operator on X over S is defined to be*

$$D_{X/S} := \Gamma_\Delta(\mathrm{Hom}_k(A, A)) \cong \Gamma_\Delta \pi_1^\times \mathcal{O}_X \in \Gamma_\Delta(\mathrm{QCoh}(X \times_S X))$$

where $\pi_1^\times$ is the right adjoint of the pushforward functor $\pi_{1,*}$. Often we will suppress S from the notation and write simply D_X.

Remark 3.4.5. *The functor* $\pi_{1,*} : A \otimes_k A{-}\mathrm{Mod} \to A{-}\mathrm{Mod}$ *is given by the formula*

$$\pi_{1,*}(M) \cong (A \otimes_k A) \otimes_{A \otimes_k A} (M)$$

where A acts on the left A in the tensor and $A \otimes A$ acts on $A \otimes A$ by multiplication inside $A \otimes A$. Therefore its right adjoint is given by the formula

$$\pi_1^\times(M) \cong \mathrm{Hom}_A(A \otimes_k A, M)$$

By adjunction this is the same as $\mathrm{Hom}_k(A, M)$ *where the $A \otimes_k A$ module structure has the left A acting on M (the codomain) and the right A acting on A (the domain). This means the left A acts by postcomposition of the k-linear function with multiplication by an element of A and the right A acts by precomposition. Visually, we have*

$$((a_1 \otimes a_2) f)(x) = a_1 f(a_2 x)$$

for $f \in \mathrm{Hom}_k(A, M)$.

We can directly see that $D_{X/S}$ is an algebra with respect to the convolution monoidal product defined above.

Remark 3.4.6. *Just from the definitions, we can see that $\Gamma_\Delta(\mathrm{QCoh}(X \times_S X))$ looks like a categorification of $D_{X/S}$. Indeed this viewpoint was explored in [Ber21] and [Ber19]. The reason that the Grothendieck-Sato formula involves the dualizing complex and the categorified expression does not is that one categorical level higher, the morphism $p_X : X \to S$ behaves like a 1-proper morphism with trivial 1-dualizing complex. By this I simply mean that the the categorified pushforward and pullback maps of quasicoherent categories (see [Gai15]) are adjoint in both directions. As a fun aside, proper morphisms with trivial dualizing complexes also exist by considering the free loop stack of a smooth proper variety. On such schemes, the left-right switch is literally trivial (as opposed to say for Calabi-Yau varieties where it is a shift). In fact the D-ring on these free loop stacks are simply obtained by taking Hochschild homology of the categorified ring of differential operators of the smooth proper variety.*

It is easily checked that the entire story behaves well with respect to base-change in S. For example, suppose we have a map $q : S' \to S$ of spectral affine schemes, then we can consider $X' = X \times_S S'$ living over S'. The base-change of $D_{X/S}$ to $\Gamma_\Delta(\mathrm{QCoh}(X' \times_{S'} X'))$ is then $D_{X'/S'}$.

Corollary 3.4.7. *For any étale map $u : U \to X$,*

$$D_{X/S}|_U \cong D_{U/S}$$

Proof. Follows from Proposition 3.1.7. ■

The following alternative description of the $D_{X/S}$ is known as the Grothendieck-Sato formula.

Corollary 3.4.8.

$$D_{X/S} \cong \Gamma_\Delta(\pi_2^*(\omega_X)) \cong \Gamma_\Delta(\mathcal{O}_X \boxtimes \omega_X)$$

Proof. By Theorem 3.2.16 we have the isomorphism

$$\Gamma_\Delta \pi_1^\times(\mathcal{O}_X) \cong \Gamma_\Delta \pi_1^!(\mathcal{O}_X)$$

By base-change for upper shriek (Theorem 3.2.7) we have the desired result. ■

If we write $X = \operatorname{Spec} A$ and $S = \operatorname{Spec} k$, then the above implies

$$\omega_{A/k} \cong D_{A/k} \otimes_{A \otimes_k A} A \cong \mathrm{Hom}_k(A, A) \otimes_{A \otimes_k A} A$$

Therefore

$$\omega_{A/k} \cong \tilde{\delta}^* D_{X/S}$$

3.5 Comparison with Classical Definitions for Smooth Varieties

In this section, assume X is a smooth variety over a non-derived base S. In this case, Grothendieck defined the ring (sheaf of rings) of Grothendieck differential operators on X relative to S ([Gro64]). We will show in this section that our definition agrees with this standard definition in this case. Moreover, we will show that the dualizing complex is given by the sheaf of top differential forms homologically shifted by the dimension of the variety, following Neeman [Nee20]. Taken together, this yields a simple and powerful method for deducing Serre duality from scratch.

We begin by showing that the ring of Grothendieck differential operators classically defined agrees with our definition. The following theorem is known, for example see [SVdB97], but we provide a proof here as well.

Theorem 3.5.1. *In the case of X a smooth variety over a discrete base S, our definition of D_X agrees with the classical definition of Grothendieck differential operators (and hence is discrete).*

Proof. We will show that affine locally, there is a canonical isomorphism. This will then imply the global statement.

Suppose we have $X \cong \operatorname{Spec} R$ smooth over $S \cong \operatorname{Spec} k$, both discrete rings. Our definition in this case yields $D_{R/k} \cong \Gamma_\Delta(\operatorname{Hom}_k(R, R))$. We will take as the classical definition of the Grothendieck differential operators

$$\mathcal{D} := \bigcup_{n \geq 0} D^{(n)}$$

the union of the increasing sequence of subspaces $D^{(n)} \subseteq \operatorname{Hom}_k(R, R)$ defined inductively by

$$D^{(-1)} = 0$$

and

$$D^{(n)} = \{f \in \operatorname{Hom}_k(R, R) | \forall r \in R, [f, r] \in D^{(n-1)}\}$$

where $r \in R$ is thought of as an element $\operatorname{Hom}_k(R, R)$ via multiplication by r.

Now let I be the ideal in $R \otimes_k R$ defining the diagonal. Recall that $\mathrm{Hom}_k(R, R)$ has an action of $R \otimes_k R$ via

$$((a_1 \otimes a_2)f)(x) = a_1 f(a_2 x)$$

Therefore, the condition that

$$\forall r \in R, [f, r] \in D^{(n-1)}$$

is equivalent to

$$\forall r \in R, (r \otimes 1)f - (1 \otimes r)f \in D^{(n-1)}$$

which is further equivalent to

$$If \in D^{(n-1)}$$

Therefore, we can conclude that

$$D^{(n)} \cong H^0 \,\mathrm{Hom}_{R \otimes_k R}((R \otimes_k R)/I^n, \mathrm{Hom}_k(R, R))$$

By adjunction we have

$$\begin{aligned}\mathrm{Hom}_{R \otimes_k R}((R \otimes_k R)/I^n, \mathrm{Hom}_k(R, R)) &\cong \mathrm{Hom}_{R \otimes_k R}((R \otimes_k R)/I^n, \mathrm{Hom}_R(R \otimes_k R, R)) \\ &\cong \mathrm{Hom}_R((R \otimes R)/I^n, R)\end{aligned}$$

where the R action is on the first factor of the tensor. However, because R is smooth, we have a noncanonical isomorphism

$$(R \otimes R)/I^n \cong \oplus_{i=0}^{n-1} (\mathrm{Sym}^k(\Omega_{R/k}))$$

and hence

$$D^{(n)} \cong \mathrm{Hom}_{R \otimes_k R}((R \otimes_k R)/I^n, \mathrm{Hom}_k(R, R))$$

because the right hand side is concentrated in degree zero (since $(R \otimes_k R)/I^n$ is projective).

Therefore, as filtered colimits are exact

$$\begin{aligned}\mathcal{D} &\cong \mathrm{colim}_n \,\mathrm{Hom}_{R \otimes_k R}((R \otimes_k R)/I^n, \mathrm{Hom}_k(R, R)) \\ &\cong \Gamma_\Delta(\mathrm{Hom}_k(R, R)) \\ &\cong D_{R/k}\end{aligned}$$

■

Now let us move on to verifying that our definition of the dualizing complex gives the top differential forms in homological degree n in the smooth setting. The idea of the following proof is due to Lipman and is written in [ATJLL14], it is also presented in Section 3.2 of [Nee20].

The intermediate object connecting differential forms with ω_X is Hochschild homology, which can be written as

$$\mathrm{HH}.(X/S) := \delta^*\delta_*\mathcal{O}_X \cong \tilde{\delta}^*\tilde{\delta}_*\mathcal{O}_X$$

Because of the isomorphism

$$\tilde{\pi}_{1,*}\tilde{\delta}_* \cong \mathrm{id}_{\mathrm{QCoh}(X)}$$

there is a natural map (by adjunction)

$$\tilde{\delta}_*\mathcal{O}_X \to \tilde{\pi}_1^{\times}\mathcal{O}_X$$

Therefore by applying $\tilde{\delta}^*$ on both sides there is a natural map

$$\mathrm{HH}.(X/S) \to \omega_X$$

By the HKR isomorphism, in the smooth case, we also have a map

$$\Omega^n_{X/S}[n] \cong \pi_{\geq n}\,\mathrm{HH}.(X/S) \to \mathrm{HH}.(X/S)$$

where $\Omega^i_{X/S}$ is the sheaf of i-forms. These combine to form a natural map

$$\Omega^n_{X/S}[n] \to \omega_X$$

in the smooth case. We wish to show it's an isomorphism.

By étale descent, it is enough to check it for $\mathbb{A}^n$. Now, for X, Y over S satisfying our standing assumptions

$$\Gamma_\Delta(\mathrm{QCoh}((X \times_S Y) \times_S (X \times_S Y))) \cong \Gamma_\Delta(\mathrm{QCoh}(X \times_S X)) \otimes \Gamma_\Delta(\mathrm{QCoh}(Y \times_S Y))$$

and

$$\tilde{\pi}^{(X\times Y)}_{1,*} \cong \tilde{\pi}^{(X)}_{1,*} \boxtimes \tilde{\pi}^{(Y)}_{1,*}$$

therefore also

$$\tilde{\pi}_1^{\times,(X\times Y)} \cong \tilde{\pi}_1^{\times,(X)} \boxtimes \tilde{\pi}_1^{\times,(Y)}$$

Hence we have

$$D_{X\times Y} \cong D_X \boxtimes D_Y$$

Pulling back along the diagonal, we get

$$\omega_{X\times Y} \cong \omega_X \boxtimes \omega_Y$$

We also have similar results for Hochschild homology and $\Omega^n_{X/S}[n]$ compatible with the maps between them. Therefore, it suffices to show the isomorphism for $\mathbb{A}^1$. So the result follows from

Lemma 3.5.2. *Over a base scheme S over* $\operatorname{Spec}\mathbb{Z}$*,*

$$\omega_{\mathbb{A}^1/S} \cong \mathcal{O}_{\mathbb{A}^1/S}[1]$$

and the map

$$\mathcal{O}_{\mathbb{A}^1} \oplus \Omega_{\mathbb{A}^1}[1] \cong \mathrm{HH.}(\mathbb{A}^1) \to \omega_{\mathbb{A}^1}$$

is an isomorphism in degree 1*.*

Proof. By base-change results, we can assume $S \cong \operatorname{Spec}\mathbb{Z}$. We have (by Definition 3.2.6)

$$\omega_{\mathbb{Z}[x]/\mathbb{Z}} \cong D_{\mathbb{Z}[x]/\mathbb{Z}} \otimes_{\mathbb{Z}[x_1,x_2]} \mathbb{Z}[x]$$

where the map

$$\mathbb{Z}[x_1,x_2] \to \mathbb{Z}[x]$$

sends x_1 and x_2 to x. $\mathbb{Z}[x]$ has the following resolution over $\mathbb{Z}[x_1,x_2]$.

$$\mathbb{Z}[x_1,x_2] \xrightarrow{(x_1-x_2)\cdot\_} \mathbb{Z}[x_1,x_2] \to \mathbb{Z}[x]$$

Hence by tensoring with $D_{\mathbb{Z}[x]/\mathbb{Z}}$, we have the following exact triangle in $\mathrm{QCoh}(\mathbb{Z}[x_1,x_2])$.

$$D_{\mathbb{Z}[x]} \xrightarrow{[x,\_]} D_{\mathbb{Z}[x]} \to \omega_{\mathbb{Z}[x]}$$

where the first map is conjugating by multiplication by x.

By a direct computation $D_{\mathbb{Z}[x]/\mathbb{Z}}$ is a free $\mathbb{Z}$ module on the generators $\{\frac{1}{n!}\frac{d}{dx}\}_{n\geq 0}$. Therefore the map

$$[x,\_] : D_{\mathbb{Z}[x]/\mathbb{Z}} \to D_{\mathbb{Z}[x]/\mathbb{Z}}$$

is surjective and the kernel is just $\mathbb{Z}[x]$. Therefore,

$$\omega_{\mathbb{Z}[x]} \cong \mathbb{Z}[x][1]$$

Now, we also have the triangle (by the same resolution of $\mathbb{Z}[x]$ above)

$$\mathbb{Z}[x] \xrightarrow{[x,\_]} \mathbb{Z}[x] \to \mathrm{HH}.(\mathbb{Z}[x])$$

which naturally maps to the triangle

$$D_{\mathbb{Z}[x]} \xrightarrow{[x,\_]} D_{\mathbb{Z}[x]} \to \omega_{\mathbb{Z}[x]}$$

The lemma follows from direct calculation. ■

We have therefore shown

Theorem 3.5.3. *For $p_X : X \to S$ a smooth map of relative dimension n (where S is a discrete scheme), there is a natural isomorphism*

$$\omega_{X/S} \cong \Omega^n_{X/S}[n]$$

3.6 More on Dualizing Complexes

We discuss some properties of dualizing complexes, and compare our definition with the characterization given in [BBST24].

Let $p_X : X \to S$ be a finite tor-amplitude, almost of finite presentation map of spectral affine schemes, i.e. $X = \operatorname{Spec} A$ and $S = \operatorname{Spec} R$ where both rings have only finitely many nonzero homotopy groups. Fix $\bar{X}$ a compactification of X over S which is proper, and almost of finite presentation, which exists by work of Scholze. As stated by Scholze, it is not clear if we can choose $\bar{X}$ to be finite tor-amplitude over S.

In this situation, Lurie in [Lur18] defines the relative dualizing complex on $\bar{X}$.

Definition 3.6.1. *The dualizing complex on $\bar{X}$, in the sense of [Lur18], is defined to be*

$$\omega^{Lurie}_{\bar{X}/S} := p^{\times}_{\bar{X}}(\mathcal{O}_S)$$

If we $*$-pullback the Lurie dualizing complex on $\bar{X}$ to X, it agrees with ours.

Proposition 3.6.2. *Suppose $p_X : X \to S$ is an almost of finite presentation, finite tor-amplitude map of spectral affine schemes. If p_X admits a finite tor-amplitude compactification or S is truncated, then*

$$u^* \omega^{Lurie}_{\bar{X}/S} \cong \omega_{X/S}$$

where $u : X \to \bar{X}$ is the open immersion of X into its compactification $\bar{X}$.

Proof. Consider the diagram

$$\begin{tikzcd} X \times_S \bar{X} \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & \bar{X} \arrow[d, "p_{\bar{X}}"] \\ X \arrow[r, "p_X"] & S \end{tikzcd} \tag{3.5}$$

Then the left hand side in the proposition can be re-written as $\delta_X^* \pi_2^* \omega_{\bar{X}/S}^{Lurie}$ where we abuse notation to write the diagonal map $X \to X \times \bar{X}$ as δ.

First suppose S is truncated. By the Proposition 3.6.3 below, the left hand side is

$$\delta_X^* \pi_2^* p_{\bar{X}/S}^{\times} \mathcal{O}_S \cong \delta_X^* \pi_1^{\times} \mathcal{O}_X$$

which, as X is affine, is simply our definition for the dualizing complex.

Now, suppose X can be compactified by a finite tor-amplitude map. Then, the conclusion of Proposition 3.6.3 holds without the restriction to $\mathrm{QCoh}^+(S)$ (see Remark 3.6.4 below). Hence the same argument applies. ■

Proposition 3.6.3. *Suppose $p_X : X \to S$ is an almost of finite presentation, finite tor-amplitude map of spectral affine schemes.*

$$\pi_2^* p_{\bar{X}}^{\times} \to \pi_1^{\times} p_X^*$$

is an isomorphism when restricted to $\mathrm{QCoh}^+(S)$*, in the terminology of (3.5).*

Proof. The proof of Lemma 0AA8 in [Sta18] applies here verbatim. We repeat the argument here for the benefit of the reader.

Let $\mathcal{M}$ be a compact generator of $\mathrm{QCoh}(\bar{X})$. As p_X is affine, $\pi_2^*\mathcal{M}$ is a compact generator of $\mathrm{QCoh}(X \times_S \bar{X})$.

Fix $\mathcal{F} \in \mathrm{QCoh}^+(S)$. We have

$$\begin{aligned} \mathrm{Hom}_{\mathrm{QCoh}(X \times \bar{X})}(\pi_2^*\mathcal{M}, \pi_2^* p_{\bar{X}}^{\times}\mathcal{F}) &= A \otimes_R \mathrm{Hom}_{\mathrm{QCoh}(\bar{X})}(\mathcal{M}, p_{\bar{X}}^{\times}\mathcal{F}) \\ &= A \otimes_R \mathrm{Hom}_R(p_{\bar{X},*}\mathcal{M}, \mathcal{F}) \\ &= \mathrm{Hom}_A(A \otimes_R p_{\bar{X},*}\mathcal{M}, A \otimes_R \mathcal{F}) \\ &= \mathrm{Hom}_A(\pi_{1,*}\pi_2^*\mathcal{M}, p_X^*\mathcal{F}) \\ &= \mathrm{Hom}_{\mathrm{QCoh}(X \times \bar{X})}(\pi_2^*\mathcal{M}, \pi_1^{\times} p_X^*\mathcal{F}) \end{aligned}$$

where we use Lemma 3.6.5 below in line 3, noting that $p_{\bar{X},*}\mathcal{M}$ is almost perfect as a R-module by [Lur18] Theorem 5.6.0.2. One can directly check that this composition is the same as the one induced by the functor in the proposition. Hence we are done because $\pi_2^*\mathcal{M}$ is a generator. ■

Remark 3.6.4. *If $\bar{X}$ can be chosen to be finite tor-amplitude over S, then the restriction to $\mathrm{QCoh}^+(S)$ can be removed (this follows from Proposition 3.2.12).*

Lemma 3.6.5. *Suppose $R \to R'$ is a finite tor-amplitude map of connective $\mathbb{E}_\infty$ rings. Let B be any connective R algebra. Let M be an almost perfect B module and N be a coconnective B module. Then*

$$\mathrm{Hom}_B(M, N) \otimes_R R' \to \mathrm{Hom}_{B \otimes_R R'}(M \otimes_R R', N \otimes_R R')$$

is an isomorphism.

Proof. The proof of Lemma 0A6A in [Sta18] works here. In particular, the statement is clear if M is a perfect A module. The general case follows by approximating M by perfect A-modules. This process converges because the map $R \to R'$ is finite tor-amplitude. ■

It therefore follows from results of Lurie that the dualizing complex (as defined in this text) is bounded cohomologically.

Proposition 3.6.6. *Suppose $p_X : X \to S$ is a almost of finite presentation map between affine spectral schemes of tor-amplitude at most N such that either S is truncated or p_X admits a finite tor-amplitude compactification, then*

$$\omega_{X/S} \in \mathrm{QCoh}(X)^{(-\infty, N]}$$

Proof. This follows directly from Proposition 3.6.2 and [Lur18] Lemma 6.4.4.3. ■

When S is truncated, the results of Lurie imply that ω is bounded.

Proposition 3.6.7. *If S is M-truncated and $p_X : X \to S$ is a almost of finite presentation map between affine spectral schemes of tor-amplitude at most N. Then*

$$\omega_{X/S} \in \mathrm{QCoh}(X)^{[-(M+cd(X/S)), N]}$$

where $cd(X/S)$ is the cohomological dimension of X relative to S (which is finite by [Lur18] Corollary 3.4.2.3)

Proof. Using Proposition 3.6.2, the claim reduces to the above proposition and [Lur18] Remark 6.4.1.3 (which is a simple consequence of adjunction and the definition of a t-structure). ∎

Remark 3.6.8. *Of course, as the structure sheaf is the dualizing complex of the identity morphism, the dualizing complex can be unbounded homologically in general.*

Following ideas of [BBST24], we can now give an alternative characterization of the dualizing complex and thus for the upper shriek functor for almost of finite presentation and finite tor-amplitude maps between truncated spectral affine schemes. The general case would then follow from results about descent. All results of this section are in direct analogy with those of [BBST24]. We have endeavored to make this section self-contained (as far as possible) to show that the constructions in this section can be used as a foundation for Grothendieck duality independent of the reduction formula.

Recall that $\mathrm{QCoh}^+(X)$ denotes the subcategory of quasicoherent sheaves on X which are cohomologically bounded below.

Proposition 3.6.9. *For $M, N \in \mathrm{QCoh}^+(X)$, $\delta^\times(M \boxtimes N) \in \mathrm{QCoh}^+(X)$.*

Proof. This follows because $A = \Gamma(X, \mathcal{O}_X)$ is in the subcategory generated under colimits by $A \otimes_R A$. Thus if $M, N \in \mathrm{QCoh}^{>-N}(X)$, $\delta^\times(N \boxtimes N) \in \mathrm{QCoh}^{>-N}(X)$. ∎

We can therefore define a non-unital product on $\mathrm{QCoh}^+(X)$ using $\delta^\times$.

Proposition 3.6.10. *$\delta^\times$ defines a symmetric monoidal product on $\mathrm{QCoh}^+(X)$.*

Proof. The same proof as in [BBST24] works here. Namely, it suffices to show that

$$\delta^\times(\mathrm{Hom}_{A^{\otimes I}}(A, \bigotimes_I M_i) \otimes \mathrm{Hom}_{A^{\otimes J}}(A, \bigotimes N_j)) \cong \mathrm{Hom}_{A^{\otimes I \sqcup J}}(A, \bigotimes_I M_i \otimes \bigotimes_J M_j)$$

where the unlabeled tensor products are over R. This follows from the following lemma. ∎

Using the compactification $\bar{X}$, we can show that $\omega_{X/S}$ serves as the unit.

CHAPTER 4

D-Modules

4.1 The Category of ${D_X}^{\mathrm{op}}$-Modules

In this section we define the category of ${D_X}^{\mathrm{op}}$-modules and identify it with the category of modules over a monad on $\mathrm{QCoh}(X)$ corresponding to the "opposite" of the sheaf $D_{X/S}$ defined in 3.4.4. Our approach is somewhat similar to the approach taken in section 5 of the paper *D-modules and Crystals* [GR14] by Gaitsgory and Rozenblyum. However their starting point is de Rham stack and the completion of $X \times X$ at the diagonal (part of what they call the infinitesimal groupoid) is defined in terms of the de Rham stack. In our approach we do the reverse. We view their approach as more stack-theoretic and ours as more category-theoretic. This justifies our choice to give a self-contained presentation of an arguably well-known theory. From a pedagogical perspectively, our presentation also has the benefit of not relying on the theory of stacks and ind-coherent sheaves. However, we do have to limit ourselves to the finite tor-amplitude situation (roughly the eventually coconnective situation in the language of [GR14]).

Let $p_X : X \to S$ be a map between spectral affine schemes which is locally almost of finite presentation and finite tor-amplitude (we can reduce to the affine case in general). Recall that we defined $D_{X/S}$ as (if $X = \mathrm{Spét}\, R$ and $S = \mathrm{Spét}\, k$)

$$D_{X/S} := \Gamma_\Delta(\mathrm{Hom}_k(R, R)) \cong \tilde{\pi}_1^\times \mathcal{O}_X \in \Gamma_\Delta(\mathrm{QCoh}(X \times_S X))$$

in Definition 3.4.4, it is an algebra viewed as an element of

$$\mathrm{QCoh}(X \times_S X) \cong \mathrm{Hom}_{\mathrm{QCoh}(S)-\mathrm{Mod}^L}(\mathrm{QCoh}(X), \mathrm{QCoh}(X))$$

Here the tilde refers to fact that we apply the projection functor Γ_Δ after the $\pi_1^\times$. In general, we use tilde to denote modification of functors which are related to the unmodified version by the functors

$$\mathrm{QCoh}(X') \underset{i_{Z'}}{\overset{\Gamma_{Z'}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\top}}}} \Gamma_{Z'}(\mathrm{QCoh}(X')).$$

relating the category of quasicoherent sheaves supported on a (locally almost finitely presented) closed subscheme Z' of X' with the entire category of quasicoherent sheaves on X'.

We can identify $D_{X/S}$ with a colimit-preserving $\mathrm{QCoh}(S)$-linear endofunctor of $\mathrm{QCoh}(X)$. The "opposite" of $D_{X/S}$ corresponds to the endofunctor which is left-right dual (in the sense of Section 2.2) to the endofunctor of $D_{X/S}$. As an element of $\Gamma_\Delta(\mathrm{QCoh}(X \times X))$, ${D_{X/S}}^{\mathrm{op}}$ is the image of $D_{X/S}$ under the automorphism of $\Gamma_\Delta(\mathrm{QCoh}(X \times X))$, which switches the X's. Hence,

$$D_{X/S}{}^{\mathrm{op}} \cong \tilde{\pi}_2^\times \mathcal{O}_X \in \Gamma_\Delta(\mathrm{QCoh}(X \times_S X))$$

The corresponding endofunctor to ${D_{X/S}}^{\mathrm{op}}$ is $\tilde{\pi}_{1,*}\tilde{\pi}_2^\times$.

We will show in this section that the category of modules over ${D_{X/S}}^{\mathrm{op}}$ is the colimit of the simplicial diagram

$$\ldots \Gamma_\Delta(\mathrm{QCoh}(X \times X \times X)) \substack{\rightarrow\\\rightarrow\\\rightarrow} \Gamma_\Delta(\mathrm{QCoh}(X \times X)) \rightrightarrows \mathrm{QCoh}(X)$$

where the transition maps are (tilde of) quasicoherent pushforward maps. For example, the two maps

$$\Gamma_\Delta(\mathrm{QCoh}(X \times X)) \rightrightarrows \mathrm{QCoh}(X)$$

are simply $\tilde{\pi}_{1,*}$ and $\tilde{\pi}_{2,*}$.

In Remark 4.1.13, we describe how to arrive at the following description from first principles, even though this expression for the category of D-modules is well-known (see [GR14] for instance).

To be more precise, we can consider the simplex category $\mathbf{\Delta}$ consisting of objects $\{[n]\}_{n\geq 0}$ where $[n] = \{0, \ldots, n\}$, and morphisms order-preserving (preserving $\geq$) maps between them. We can define a functor

$$\Delta^{\mathrm{op}} \to \mathrm{QCoh}(S)\mathrm{-Mod}^L$$

by sending

$$[n] \mapsto \Gamma_\Delta \mathrm{QCoh}(X^{n+1})$$

and an order preserving map $[n] \to [m]$ to the functor

$$\tilde{g}_* : \Gamma_\Delta(\mathrm{QCoh}(X^{m+1})) \to \Gamma_\Delta(\mathrm{QCoh}(X^{n+1}))$$

where $g : X^{m+1} \to X^{n+1}$ is defined in the obvious way from the map $[n] \to [m]$. The category which we propose is the category of right D_X modules is then the colimit of this functor, for which we write

$$\operatorname{colim}_{\mathbf{\Delta}^{\mathrm{op}}}(\Gamma_\Delta(\mathrm{QCoh}(X^{n+1})), *)$$

Let us denote by $\mathbf{\Delta_s}$ the subcategory of $\mathbf{\Delta}$ where the morphisms are required to be injective. By [Lur09] 6.5.3.7, the category $\mathbf{\Delta_s}^{\mathrm{op}}$ is cofinal in $\mathbf{\Delta}^{\mathrm{op}}$, and hence our colimit above can be computed over $\mathbf{\Delta_s}^{\mathrm{op}}$ instead, as

$$\operatorname{colim}_{\mathbf{\Delta_s}^{\mathrm{op}}}(\Gamma_\Delta(\mathrm{QCoh}(X^{n+1})), *) \tag{4.1}$$

The advantage of using $\mathbf{\Delta_s}$ is that for any injective morphism $[n] \to [m]$, the transition functor

$$\tilde{g}_* : \Gamma_\Delta(\mathrm{QCoh}(X^{m+1})) \to \Gamma_\Delta(\mathrm{QCoh}(X^{n+1}))$$

described above is compact object preserving, by a mild generalization of Theorem 3.1.1. The proof is identical so we do not repeat it here. Intuitively, when taking the colimit over $\mathbf{\Delta}^{\mathrm{op}}$, one encounters degeneracy maps of simplices which induce functors such as

$$\tilde{\delta}_* : \mathrm{QCoh}(X) \to \Gamma_\Delta(\mathrm{QCoh}(X \times X))$$

which are not compact object preserving if X is not smooth. This problem disappears when we use $\mathbf{\Delta}_s$. We will see the relevance of preserving compact objects shortly.

Let us denote by

$$F_{D_X{}^{\mathrm{op}}} : \mathrm{QCoh}(X) \to \operatorname{colim}_{\mathbf{\Delta_s}}(\Gamma_\Delta(\mathrm{QCoh}(X^{n+1})), *)$$

the inclusion functor into the colimit associated with the object $[0]$ in $\mathbf{\Delta_s}$. Denote by G_{D_X} its right adjoint.

Recall that the underlying category of a colimit in Pr^L_{St} can also be written as a limit in Pr^R_{St}, with the transition functors the right adjoints. This fact is due to Lurie [Lur09], however we find Lemma 1.3.3 in [Gai12] the most convenient reference. The essence is that adjunction provides an anti-equivalence of categories between Pr^L_{St} and Pr^R_{St}. With this in mind, $G_{D_X{}^{\mathrm{op}}}$ can be written as the projection map

$$G_{D_X{}^{\mathrm{op}}} : \lim_{\mathbf{\Delta_s}}(\Gamma_\Delta(\mathrm{QCoh}(X^{n+1})), \times) \to \mathrm{QCoh}(X)$$

where the transition maps are tilde of upper cross functors (the right adjoint of tilde of lower star) and the limit is taken in Pr^R_{St}. We remind the reader that if we are only interested in the underlying category of the limit we can also take the limit in $\widehat{Cat_\infty}$. This functor is also $\mathrm{QCoh}(S)$-linear (see Theorem A.1.6).

Our aim for the rest of the section is to show that adjunction above is monadic, with the monad given by

$$G_{D_X{}^{\mathrm{op}}} F_{D_X{}^{\mathrm{op}}} \cong \tilde{\pi}_{1,*}\tilde{\pi}_2^{\times} \cong D_{X/S}{}^{\mathrm{op}} \otimes_{\mathcal{O}_X} \text{-}$$

However, we will need a few preliminary results

Lemma 4.1.1. *For $m, n \geq 0$, there is a canonical isomorphism*

$$\Gamma_\Delta(\mathrm{QCoh}(X^{m+n+1})) \cong \Gamma_\Delta(\mathrm{QCoh}(X^{m+1})) \otimes_{\mathrm{QCoh}(X)} \Gamma_\Delta(\mathrm{QCoh}(X^{n+1}))$$

where $\mathrm{QCoh}(X)$ acts on the right most copy of X in $\Gamma_\Delta(\mathrm{QCoh}(X^{m+1}))$ and the left most copy of X in $\Gamma_\Delta(\mathrm{QCoh}(X^{n+1}))$ via tilde $*$-pullback.

Proof. Because tensor products preserves split-exact sequences (see Appendix A.2 for the definition of a split-exact sequence), both sides are full subcategories of

$$\mathrm{QCoh}(X^{m+n+1}) \cong \mathrm{QCoh}(X^{m+1}) \otimes_{\mathrm{QCoh}(X)} \mathrm{QCoh}(X^{n+1})$$

It suffices to show they have the same objects. Let us denote by U the complement of the diagonal in X^{m+n+1}. The category $\Gamma_\Delta(\mathrm{QCoh}(X^{m+n+1}))$ can then be characterized as the subcategory of $\mathrm{QCoh}(X^{m+n+1})$ which vanish when restricted to U.

Now let V be the complement of the diagonal in X^{m+1} and W the complement of the diagonal in X^{n+1}. Then, we can express U as a union

$$U = V \times_X X^{n+1} \cup X^{m+1} \times_X W$$

Therefore, vanishing on U is equivalent to vanishing on $V \times_X X^{n+1}$ and $X^{m+1} \times_X W$.

It is then clear that everything in

$$\Gamma_\Delta(\mathrm{QCoh}(X^{m+1})) \otimes_{\mathrm{QCoh}(X)} \Gamma_\Delta(\mathrm{QCoh}(X^{n+1}))$$

vanishes on U. For the reverse, suppose a quasicoherent sheaf $\mathcal{F}$ vanishes on U. It then lives inside $\Gamma_\Delta(\mathrm{QCoh}(X^{m+1})) \otimes_{\mathrm{QCoh}(X)} \mathrm{QCoh}(X^{n+1})$ because it vanishes on $V \times_X X^{n+1}$. Then, because it also vanishes on $X^{m+1} \times_X W$, it is then inside the kernel of the map

$$\Gamma_\Delta(\mathrm{QCoh}(X^{m+1})) \otimes_{\mathrm{QCoh}(X)} \mathrm{QCoh}(X^{n+1}) \to \Gamma_\Delta(\mathrm{QCoh}(X^{m+1})) \otimes_{\mathrm{QCoh}(X)} \mathrm{QCoh}(W)$$

Because tensor product of stable categories preserve split-exact sequences, we see that $\mathcal{F}$ is inside

$$\Gamma_\Delta(\mathrm{QCoh}(X^{m+1})) \otimes_{\mathrm{QCoh}(X)} \Gamma_\Delta(\mathrm{QCoh}(X^{n+1}))$$

■

Remark 4.1.2. *The previous lemma leads to an interesting observation. The simplicial diagram*

$$[n] \mapsto \Gamma_\Delta(\mathrm{QCoh}(X^{n+1}))$$

roughly specifies the data of category internal to $\mathrm{QCoh}(S){-}\mathrm{Mod}^L$ *on the object* $\mathrm{QCoh}(X)$*, relative to the tensor product of categories. This internal category is the categorical analogue of the infinitesimal groupoid on* X*.*

We move to the second preliminary result. Recall that we can equip $\Gamma_\Delta(\mathrm{QCoh}(X \times X))$ with the convolution monoidal structure (Definition 3.4.1). Under this monoidal structure, $\mathrm{QCoh}(X)$ is naturally a left $\Gamma_\Delta(\mathrm{QCoh}(X \times X))$ module. We will give a resolution of $\mathrm{QCoh}(X)$ as a $\Gamma_\Delta(\mathrm{QCoh}(X \times X))$ module. We exhibit this resolution as an augmented simplicial diagram

$$\ldots \Gamma_\Delta(\mathrm{QCoh}(X \times X \times X)) \rightrightarrows \Gamma_\Delta(\mathrm{QCoh}(X \times X)) \to \mathrm{QCoh}(X) \tag{4.2}$$

The augmentation map is

$$\tilde{\pi}_{1,*} : \Gamma_\Delta(\mathrm{QCoh}(X \times X)) \to \mathrm{QCoh}(X)$$

The two maps

$$\Gamma_\Delta(\mathrm{QCoh}(X \times X \times X)) \rightrightarrows \Gamma_\Delta(\mathrm{QCoh}(X \times X))$$

are $\tilde{\pi}_{1,2,*}$ and $\tilde{\pi}_{1,3,*}$. More generally, all the transition maps preserve the left most copy of X. We omit writing down the complete specification of this simplicial diagram and trust that the reader is able to do so if they wish. Importantly, the action of $\Gamma_\Delta(\mathrm{QCoh}(X \times X))$ is always on the left most copy of X which is preserved. The following proposition shows that this is indeed a resolution, i.e. that the geometric realization of the simplicial diagram recovers $\mathrm{QCoh}(X)$.

Proposition 4.1.3. *There is a natural resolution of* $\mathrm{QCoh}(X)$ *as a left* $\Gamma_\Delta(\mathrm{QCoh}(X \times X))$*-module category given by*

$$\ldots \Gamma_\Delta(\mathrm{QCoh}(X \times X \times X)) \rightrightarrows \Gamma_\Delta(\mathrm{QCoh}(X \times X)) \to \mathrm{QCoh}(X) \tag{4.3}$$

where the maps are specified above.

Proof. We apply Lemma 6.1.3.17 from [Lur09]. The augmented simplicial diagram above arises from a simplicial object

$$\dots \Gamma_\Delta(\mathrm{QCoh}(X \times X \times X)) \rightrightarrows \Gamma_\Delta(\mathrm{QCoh}(X \times X)) \rightrightarrows \mathrm{QCoh}(X)$$

by forgetting all the morphisms which do not preserve the left most copy of X. Therefore it is a colimit diagram in $\mathrm{QCoh}(S)-\mathrm{Mod}^L$. Because the forgetful functor from $\Gamma_\Delta(\mathrm{QCoh}(X \times X))-\mathrm{Mod}^L$ to $\mathrm{QCoh}(S)-\mathrm{Mod}^L$ reflects colimits (because it preserves colimits by Corollary 4.2.3.7 of [Lur17] and is conservative), it is also a colimit diagram in $\Gamma_\Delta(\mathrm{QCoh}(X\times X))-\mathrm{Mod}^L$. ■

The above proposition has an important corollary.

Corollary 4.1.4.

$$\mathrm{colim}_{\boldsymbol{\Delta}^{\mathrm{op}}}(\Gamma_\Delta(\mathrm{QCoh}(X^{n+1})), *) \cong \mathrm{QCoh}(X) \otimes_{\Gamma_\Delta(\mathrm{QCoh}(X\times X))} \mathrm{QCoh}(X) \tag{4.4}$$

Proof. Using Proposition 4.1.3, we can write the right hand side as

$$\mathrm{colim}_{\boldsymbol{\Delta_s}^{\mathrm{op}}}(\mathrm{QCoh}(X) \otimes_{\Gamma_\Delta(\mathrm{QCoh}(X\times X))} \Gamma_\Delta(\mathrm{QCoh}(X^{n+2}))$$

Using Lemma 4.1.1, we can write $\Gamma_\Delta(\mathrm{QCoh}(X^{n+2}))$ as

$$\Gamma_\Delta(\mathrm{QCoh}(X \times X)) \otimes_{\mathrm{QCoh}(X)} \Gamma_\Delta(\mathrm{QCoh}(X^{n+1}))$$

Therefore, the right hand side is isomorphic to

$$\mathrm{colim}_{\boldsymbol{\Delta_s}^{\mathrm{op}}}(\Gamma_\Delta(\mathrm{QCoh}(X^{n+1})), *)$$

as desired. ■

Remark 4.1.5. *This expression for the category of right D-modules appears as Equation (4.5) in the proof of Proposition 4.2.5 in [Ber19].*

The adjunction between $F_{D_X{}^{\mathrm{op}}}$ and $G_{D_X{}^{\mathrm{op}}}$ can be described in terms of the isomorphism above. Because of the isomorphism

$$\mathrm{QCoh}(X) \cong \mathrm{QCoh}(X) \otimes_{\Gamma_\Delta(\mathrm{QCoh}(X\times X))} \Gamma_\Delta(\mathrm{QCoh}(X \times X))$$

The map

$$\tilde{\pi}_{1,*} : \Gamma_\Delta(\mathrm{QCoh}(X \times X)) \to \mathrm{QCoh}(X)$$

induces a functor

$$\mathrm{id} \otimes \tilde{\pi}_{1,*} : \mathrm{QCoh}(X) \to \mathrm{QCoh}(X) \otimes_{\Gamma_\Delta(\mathrm{QCoh}(X \times X))} \mathrm{QCoh}(X)$$

Tracing through the proof of Corollary 4.1.4, we see this agrees with functor $F_{D_X{}^{\mathrm{op}}}$, after identifying the two sides of Corollary 4.1.4. Namely,

$$F_{D_X{}^{\mathrm{op}}} \cong \mathrm{id} \otimes \tilde{\pi}_{1,*} \tag{4.5}$$

Now, the right adjoint of $\tilde{\pi}_{1,*}$,

$$\tilde{\pi}_1^\times : \mathrm{QCoh}(X) \to \Gamma_\Delta(\mathrm{QCoh}(X \times X))$$

is also $\Gamma_\Delta(\mathrm{QCoh}(X \times X))$ linear. Hence, by Theorem 3.2.16

$$\tilde{\pi}_1^\times(\mathcal{F}) \cong \Gamma_\Delta(\mathcal{F} \boxtimes \omega_X)$$

Hence, we can construct the functor

$$\mathrm{id} \otimes \tilde{\pi}_1^\times : \mathrm{QCoh}(X) \otimes_{\Gamma_\Delta(\mathrm{QCoh}(X \times X))} \mathrm{QCoh}(X) \to \mathrm{QCoh}(X)$$

Using the unit and counit maps of the adjunction $\tilde{\pi}_{1,*} \dashv \tilde{\pi}_1^\times$, we can see that our $\mathrm{id} \otimes \tilde{\pi}_1^\times$ is right adjoint to $F_{D_X{}^{\mathrm{op}}}$ and hence

$$G_{D_X{}^{\mathrm{op}}} \cong \mathrm{id} \otimes \tilde{\pi}_1^\times \tag{4.6}$$

By examination, or by the involution on

$$\mathrm{colim}_{\mathbf{\Delta_s}^{\mathrm{op}}}(\Gamma_\Delta(\mathrm{QCoh}(X^{n+1})), *)$$

which reverse the order of the X's in X^{n+1} (for all n), we can also arrive at the isomorphism (4.4) through a resolution of the left copy of $\mathrm{QCoh}(X)$ as a right $\Gamma_\Delta(\mathrm{QCoh}(X \times X))$ module (analogously to Proposition 4.1.3) By arriving at the isomorphism this way, we can also express $F_{D_X{}^{\mathrm{op}}}$ as

$$F_{D_X{}^{\mathrm{op}}} \cong \tilde{\pi}_{2,*} \otimes \mathrm{id} \tag{4.7}$$

This expression for $F_{D_X{}^{\mathrm{op}}}$ implies

$$G_{D_X{}^{\mathrm{op}}} \cong \tilde{\pi}_2^\times \otimes \mathrm{id} \tag{4.8}$$

Finally, we can deliver on our promise

Theorem 4.1.6. *The adjunction $F_{D_X^{\mathrm{op}}} \dashv G_{D_X^{\mathrm{op}}}$ is monadic and*

$$G_{D_X^{\mathrm{op}}} F_{D_X^{\mathrm{op}}} \cong \tilde{\pi}_{1,*} \tilde{\pi}_2^{\times} \tag{4.9}$$

Proof. By Lurie-Barr-Beck (Theorem 4.7.3.5 in [Lur17]), to show the adjunction is monadic it is enough to show that $G_{D_X^{\mathrm{op}}}$ is conservative and colimit-preserving. Because all the transition maps used to construct the cosimplicial limit

$$\lim_{\boldsymbol{\Delta_s}} (\Gamma_\Delta(\mathrm{QCoh}(X^{n+1})), \times)$$

are colimit-preserving, $G_{D_X^{\mathrm{op}}}$ is automatically colimit-preserving. To show that $G_{D_X^{\mathrm{op}}}$ is conservative, we need to show that an object

$$\mathcal{F} \in \lim_{\boldsymbol{\Delta_s}} (\Gamma_\Delta(\mathrm{QCoh}(X^{n+1})), \times)$$

is zero if the projection of $\mathcal{F}$ to $\mathrm{QCoh}(X)$ is zero. For this, it is enough to show that the projection to $\Gamma_\Delta(\mathrm{QCoh}(X^{n+1}))$ is zero for any n. But follows from the fact that $[0]$ is weakly initial in $\boldsymbol{\Delta_s}$.

For the second part of the theorem, let us apply Theorem A.1.1 to $\mathscr{X} := \mathrm{QCoh}(X)$, $\mathscr{Y} := \mathrm{QCoh}(X)$, and $\mathscr{V} := \Gamma_\Delta(\mathrm{QCoh}(X \times X))$, with functors $\tilde{\pi}_2^{\times} : \mathscr{X} \to \mathscr{V}$ and $\tilde{\pi}_{1,*} : \mathscr{V} \to \mathscr{Y}$. Then, we have the commutative diagram

$$\begin{tikzcd}
\mathrm{QCoh}(X) \arrow[r, "1 \otimes \tilde{\pi}_{1,*}"] \arrow[d, "\tilde{\pi}_2^{\times}"'] & \mathrm{QCoh}(X) \otimes_{\Gamma_\Delta(\mathrm{QCoh}(X \times X))} \mathrm{QCoh}(X) \arrow[d, "\tilde{\pi}_2^{\times} \otimes 1"] \\
\Gamma_\Delta(\mathrm{QCoh}(X \times X)) \arrow[r, "\tilde{\pi}_{1,*}"] & \mathrm{QCoh}(X)
\end{tikzcd}$$

Now the theorem follows from expressions (4.5) and (4.8). ■

Remark 4.1.7. *We also have*

$$G_{D_X^{\mathrm{op}}} F_{D_X^{\mathrm{op}}} \cong \tilde{\pi}_{2,*} \tilde{\pi}_1^{\times} \tag{4.10}$$

Because of the isomorphism

$$\tilde{\pi}_{2,*} \tilde{\pi}_1^{\times} \cong \tilde{\pi}_{1,*} \tilde{\pi}_2^{\times}$$

which can be simply explained by observing there is a symmetry which switches the order of the two X's in $\Gamma_\Delta(\mathrm{QCoh}(X \times X))$.

Remark 4.1.8. *The functor $F_{D_X{}^{\mathrm{op}}} : \mathrm{QCoh}(X) \to \mathrm{QCoh}(X) \otimes_{\Gamma_\Delta(\mathrm{QCoh}(X\times X))} \mathrm{QCoh}(X)$ can also be arrived at via the monoidal functor*

$$\tilde{\delta_*} : \mathrm{QCoh}(X) \to \Gamma_\Delta(\mathrm{QCoh}(X \times X))$$

which by functoriality induces a functor

$$\mathrm{QCoh}(X) \otimes_{\mathrm{QCoh}(X)} \mathrm{QCoh}(X) \to \mathrm{QCoh}(X) \otimes_{\Gamma_\Delta(\mathrm{QCoh}(X\times X))} \mathrm{QCoh}(X)$$

as desired. The fact that this agrees with the prior definitions can be checked using one of the resolutions above.

Remark 4.1.9. *We can directly check that $D^{\mathrm{op}}_{X/S}$ as an algebra agrees with the algebra structure corresponding to the monad $\tilde{\pi}_{1,*}\tilde{\pi}_2^{\times}$ above.*

Theorem/Definition 4.1.10. *The category of ${D_{X/S}}^{\mathrm{op}}$-modules is defined to be the category of algebras over the monad $\tilde{\pi}_{1,*}\tilde{\pi}_2^{\times}$ corresponding to ${D_{X/S}}^{\mathrm{op}}$ (see Theorem 4.1.6 for why this endofunctor is a monad). We have the isomorphisms*

$${D_{X/S}}^{\mathrm{op}}\mathrm{-Mod} \cong \mathrm{colim}_{\boldsymbol{\Delta_s}^{\mathrm{op}}}(\Gamma_\Delta(\mathrm{QCoh}(X^{n+1})), *)$$

$${D_{X/S}}^{\mathrm{op}}\mathrm{-Mod} \cong \lim_{\boldsymbol{\Delta_s}}(\Gamma_\Delta(\mathrm{QCoh}(X^{n+1})), \times)$$

Additionally, we have the isomorphism

$${D_{X/S}}^{\mathrm{op}}\mathrm{-Mod} \cong \mathrm{QCoh}(X) \otimes_{\Gamma_\Delta(\mathrm{QCoh}(X\times X))} \mathrm{QCoh}(X)$$

Also, ${D_{X/S}}^{\mathrm{op}}\mathrm{-Mod}$ satisfies étale descent with respect to X and fpqc/descendable descent with respect to S.

Proof. All but the last sentence follow directly from Theorem 4.1.6 and Corollary 4.1.4. For the last part, use the isomorphism

$${D_{X/S}}^{\mathrm{op}}\mathrm{-Mod} \cong \lim_{\boldsymbol{\Delta_s}}(\Gamma_\Delta(\mathrm{QCoh}(X^{n+1})), \times)$$

each of the $\Gamma_\Delta(\mathrm{QCoh}(X^{n+1}))$ has étale descent with respect to X (by a variant of Corollary 3.1.4) and all the transition maps base-change correctly (by a variant of Proposition 3.1.6).

The fpqc/descendable descent relative to S follow from fpqc descent of QCoh (Proposition 6.2.3.1 in [Lur18]) and the fact that we can write $\Gamma_Z(\mathrm{QCoh}(X))$ as the kernel in the split-exact sequence

$$\Gamma_Z(\mathrm{QCoh}(X)) \to \mathrm{QCoh}(X) \to \mathrm{QCoh}(U)$$

where U is the complement of Z in X. ■

Remark 4.1.11. *It is also possible to prove Theorem 4.1.6 by giving an explicit description of $G_{D_X{}^{\mathrm{op}}}$ as a functor*

$$G_{D_X{}^{\mathrm{op}}} : \mathrm{colim}_{\boldsymbol{\Delta_s}^{\mathrm{op}}}(\Gamma_\Delta(\mathrm{QCoh}(X^{n+1})), *) \to \mathrm{QCoh}(X)$$

To specify such a functor, it suffices to specify a collection of functors

$$G^{(n)}_{D_X{}^{\mathrm{op}}} : \Gamma_\Delta(\mathrm{QCoh}(X^{n+1}), *) \to \mathrm{QCoh}(X)$$

*together with compatibility isomorphisms. We call the process of constructing $G_{D_X{}^{\mathrm{op}}}$ from the $G^{(n)}_{D_X{}^{\mathrm{op}}}$'s **assembly**. By equation (4.6), we can write $G_{D_X{}^{\mathrm{op}}}$ as the map induced (via colimit over $\boldsymbol{\Delta_s}^{\mathrm{op}}$) by the following map of simplicial diagrams*

$$\begin{tikzcd}
\ldots \Gamma_\Delta(\mathrm{QCoh}(X \times X \times X)) \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[d, "\tilde{\pi}^\times_{1,2,3}"] & \Gamma_\Delta(\mathrm{QCoh}(X \times X)) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "\tilde{\pi}^\times_{1,2}"] & \mathrm{QCoh}(X) \arrow[d, "\tilde{\pi}^\times_{1}"] \\
\ldots \Gamma_\Delta(\mathrm{QCoh}(X \times X \times X \times X)) \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \Gamma_\Delta(\mathrm{QCoh}(X \times X \times X)) \arrow[r, shift left] \arrow[r, shift right] & \Gamma_\Delta(\mathrm{QCoh}(X \times X))
\end{tikzcd}$$

Therefore, we can compute

$$G^{(n)}_{D_X{}^{\mathrm{op}}} \cong \tilde{\pi}_{n+2,*}\tilde{\pi}^\times_{\widehat{n+2}}$$

where

$$\tilde{\pi}^\times_{\widehat{n+2}} : \Gamma_\Delta(\mathrm{QCoh}(X^{n+1})) \to \Gamma_\Delta(\mathrm{QCoh}(X^{n+2}))$$

is defined as tilde upper cross for the projection map $\pi_{\widehat{n+2}}$ to all but the last component of the product.

The above remark is generalized by

Theorem 4.1.12. *The identity functor*

$$\mathrm{colim}_{\boldsymbol{\Delta_s}^{\mathrm{op}}}(\Gamma_\Delta(\mathrm{QCoh}(X^{n+1})), *) \to \lim_{\boldsymbol{\Delta_s}}(\Gamma_\Delta(\mathrm{QCoh}(X^{m+1}), \times))$$

is assembled from the functors

$$\tilde{\pi}_{\{n+2,\dots,n+m+2\},*}\tilde{\pi}^{\times}_{\{1,\dots,n+1\}} : \Gamma_{\Delta}(\mathrm{QCoh}(X^{n+1})) \to \Gamma_{\Delta}(\mathrm{QCoh}(X^{m+1}))$$

where

$$\tilde{\pi}_{\{n+2,\dots,n+m+2\},*} : \Gamma_{\Delta}(\mathrm{QCoh}(X^{n+m+2})) \to \Gamma_{\Delta}(\mathrm{QCoh}(X^{m+1}))$$
$$\tilde{\pi}^{\times}_{\{1,\dots,n+1\}} : \Gamma_{\Delta}(\mathrm{QCoh}(X^{n+1})) \to \Gamma_{\Delta}(\mathrm{QCoh}(X^{n+m+2}))$$

with the obvious transition functors. Therefore, $F_{D_X{}^{\mathrm{op}}}$ is assembled from the functors

$$F^{(n)}_{D_X{}^{\mathrm{op}}} \cong \tilde{\pi}_{\hat{1},*}\tilde{\pi}^{\times}_{1}$$

Proof. Analogous to equation (4.5), the inclusion functor

$$i_m : \Gamma_{\Delta}(\mathrm{QCoh}(X^{m+1}) \to \mathrm{colim}_{\boldsymbol{\Delta_s}{}^{\mathrm{op}}}(\Gamma_{\Delta}(\mathrm{QCoh}(X^{m+1})), *)$$

can be written also as

$$i_m \cong \mathrm{id}\otimes\tilde{\pi}_{1,*} : \mathrm{QCoh}(X)\otimes_{\Gamma_{\Delta}(\mathrm{QCoh}(X\times X))}\Gamma_{\Delta}(\mathrm{QCoh}(X^{m+2})) \to \mathrm{QCoh}(X)\otimes_{\Gamma_{\Delta}(\mathrm{QCoh}(X\times X))}\mathrm{QCoh}(X)$$

Hence, its right adjoint is

$$\mathrm{id}\otimes\tilde{\pi}^{\times}_{1} : \mathrm{QCoh}(X)\otimes_{\Gamma_{\Delta}(\mathrm{QCoh}(X\times X))}\mathrm{QCoh}(X) \to \mathrm{QCoh}(X)\otimes_{\Gamma_{\Delta}(\mathrm{QCoh}(X\times X))}\Gamma_{\Delta}(\mathrm{QCoh}(X^{m+2}))$$

Now, we can resolve the left $\mathrm{QCoh}(X)$ in the tensor

$$\mathrm{QCoh}(X)\otimes_{\Gamma_{\Delta}(\mathrm{QCoh}(X\times X))}\mathrm{QCoh}(X)$$

as a right $\Gamma_{\Delta}(\mathrm{QCoh}(X\times X))$ modules (analogously to Proposition 4.1.3). Using this resolution the right adjoint of i_m can be written as the assembly of

$$(\mathrm{id}\otimes\tilde{\pi}^{\times}_{1})\circ(\tilde{\pi}_{n+1,*}\otimes\mathrm{id})$$

from

$$\Gamma_{\Delta}(\mathrm{QCoh}(X^{n+2}))\otimes_{\Gamma_{\Delta}(\mathrm{QCoh}(X\times X))}\mathrm{QCoh}(X)$$

to

$$\mathrm{QCoh}(X)\otimes_{\Gamma_{\Delta}(\mathrm{QCoh}(X\times X))}\Gamma_{\Delta}(\mathrm{QCoh}(X^{m+2}))$$

By the functoriality of the tensor product, this is also the same as

$$\tilde{\pi}_{\{n+2,\ldots,n+m+2\},*}\tilde{\pi}^{\times}_{\{1,\ldots,n+1\}} : \Gamma_\Delta(\mathrm{QCoh}(X^{n+1})) \to \Gamma_\Delta(\mathrm{QCoh}(X^{m+1}))$$

where the transition isomorphisms are obvious. As taking the right adjoint of i_m yields also the identity functor in the theorem composed with the projection to the m-th component of the limit, we recover the theorem. ■

Remark 4.1.13. *Theorem/Definition 4.1.10 shows an equivalence between right D-modules and costratifications (the name commonly given to the category on the right)*

$$D_X{}^{\mathrm{op}}\text{−Mod} \cong \lim_{\boldsymbol{\Delta}}(\Gamma_\Delta(\mathrm{QCoh}(X^{n+1})), \times)$$

We describe how to arrive at this equivalence naturally. Suppose M is a right D_X module, so that there is a map

$$\tilde{\pi}_{2,*}(\pi_1^{\times}(M)) \to M$$

By adjunction this is the same as a map

$$\phi : \tilde{\pi}_1^{\times} M \to \tilde{\pi}_2^{\times} M$$

which can also be written as

$$\phi : \Gamma_\Delta(M \boxtimes \omega_X) \to \Gamma_\Delta(\omega_X \boxtimes M)$$

Being a right D_X module includes also higher compatibilities. These include things such as the commutativity of the following diagram

$$\begin{tikzcd}
\Gamma_\Delta(\omega_X \boxtimes \omega_X \boxtimes M) \arrow[r, "\tilde{\pi}^{\times}_{2,3}(\phi)"] \arrow[d] & \Gamma_\Delta(\omega_X \boxtimes M \boxtimes \omega_X) \arrow[d, "\tilde{\pi}^{\times}_{1,2}(\phi)"] \\
\Gamma_\Delta(\omega_X \boxtimes \omega_X \boxtimes M) \arrow[r, "\tilde{\pi}^{\times}_{1,3}(\phi)"] & \Gamma_\Delta(M \boxtimes \omega_X \boxtimes \omega_X)
\end{tikzcd}$$

where the left unlabeled map is the identity. All the maps above are also required to be isomorphisms upon cross pullback to $\mathrm{QCoh}(X)$ *along the diagonal map. Because upper cross pullback along the diagonal is conservative (for quasicoherent sheaves supported on the diagonal) all the above maps are isomorphisms. The above discussion explains the first three* *terms of the limit*

$$\mathrm{QCoh}(X) \rightrightarrows \Gamma_\Delta(\mathrm{QCoh}(X \times X)) \;\substack{\rightarrow\\ \rightarrow\\ \rightarrow}\; \Gamma_\Delta(\mathrm{QCoh}(X \times X \times X)) \dots$$

which we showed was equivalent to the category of right D modules.

4.2 D_X-Modules and Left-Right Switch

In this section we discuss left D-modules and the isomorphism between the category of right D-modules and left D-modules, which is called the left-right switch.

Let $p_X : X \to S$ be a map between spectral affine schemes which is locally almost of finite presentation and finite tor-amplitude (we can reduce to the affine case in general). As an algebra in $\Gamma_\Delta(\mathrm{QCoh}(X \times X))$ (with convolutional monoidal product), D_X also defines a monad on $\mathrm{QCoh}(X)$. We refer to modules over this monad as D_X-modules.

If we think of quasicoherent sheaves on $\Gamma_\Delta(\mathrm{QCoh}(X \times X))$ as endofunctors on $\mathrm{QCoh}(X)$, then by Proposition 2.2.8, we know that the involution switching the two copies of X is equivalent to the left-right duality on endofunctors of $\mathrm{QCoh}(X)$ (we remind the reader that $\mathrm{QCoh}(X)$ is naturally self-dual). Hence we see that the monad D_X is the left-right dual to ${D_X}^{\mathrm{op}}$ (meaning they are interchanged by duality in the category $\mathrm{QCoh}(S){-}\mathrm{Mod}^L$ of $\mathrm{QCoh}(S)$-linear categories)[1].

By Corollary A.3.2, we know that left-right duality switches left and right D_X-modules. Namely,

$$({D_X}^{\mathrm{op}}{-}\mathrm{Mod})^\vee \cong D_X{-}\mathrm{Mod}$$

where $^\vee$ denotes duality in $\mathrm{QCoh}(S){-}\mathrm{Mod}^L$. Moreover, by Corollary A.3.3 the adjunction

$$F_{{D_X}^{\mathrm{op}}} \dashv G_{{D_X}^{\mathrm{op}}}$$

becomes, under left-right duality, the adjunction

$$G_{D_X} \vdash F_{D_X}$$

We know from the last section that

$$D_X^{\mathrm{op}}{-}\mathrm{Mod} \cong \mathrm{colim}_{\boldsymbol{\Delta_s}^{\mathrm{op}}}(\Gamma_\Delta(\mathrm{QCoh}(X^{n+1})), *)$$

[1]see 2.2

Applying the 2-functor

$$\mathrm{Hom}_{\mathrm{QCoh}(S)}(\_, \mathrm{QCoh}(S))$$

(the Hom is taken inside $\mathrm{QCoh}(S){-}\mathrm{Mod}^L$) to the equation above, we get the isomorphism

$$D_X{-}\mathrm{Mod} \cong \lim_{\boldsymbol{\Delta}_s}(\Gamma_\Delta(\mathrm{QCoh}(X^{n+1})), *)$$

where the transition maps are tilde of quasicoherent pullbacks. This is because, by Proposition 2.2.10, $\Gamma_\Delta(\mathrm{QCoh}(X^{n+1}))$ is (canonically) self-dual for all n and therefore

$$\mathrm{Hom}_{\mathrm{QCoh}(S)}(\Gamma_\Delta(\mathrm{QCoh}(X^{n+1})), \mathrm{QCoh}(S)) \cong \Gamma_\Delta(\mathrm{QCoh}(X^{n+1}))$$

Let's record our observations in

Theorem 4.2.1. *D_X, as an algebra in the monoidal category $\Gamma_\Delta(\mathrm{QCoh}(X \times_S X))$, corresponds to the monad $\tilde{\pi}_{1,\times}\tilde{\pi}_2^*$ and*

$$\begin{aligned} D_X{-}\mathrm{Mod} &\cong \lim_{\boldsymbol{\Delta_s}}(\Gamma_\Delta(\mathrm{QCoh}(X^{n+1})), *) \\ &\cong \mathrm{colim}_{\boldsymbol{\Delta_s}^{\mathrm{op}}}(\Gamma_\Delta(\mathrm{QCoh}(X^{n+1})), \times) \\ &\cong \mathrm{Hom}_{\Gamma_\Delta(\mathrm{QCoh}(X\times X))}(\mathrm{QCoh}(X), \mathrm{QCoh}(X)) \end{aligned}$$

Also $D_X{-}\mathrm{Mod}$ satisfies étale descent with respect to X and fpqc descent with respect to S.

Proof. ${D_X}^{\mathrm{op}}$ corresponds to the monad $\tilde{\pi}_{2,*}\tilde{\pi}_1^\times$ by Theorem/Definition 4.1.10. Therefore by left-right duality, D_X corresponds to the monad $\tilde{\pi}_{1,\times}\tilde{\pi}_2^*$ [2], the first isomorphism is already proven. The second isomorphism comes from the equivalence between colimits and limits in the form of Lemma 1.3.3 in [Gai12]. For the third isomorphism, we give two proofs.

Proof 1. By the resolution of $\mathrm{QCoh}(X)$ as a left $\Gamma_\Delta(\mathrm{QCoh}(X \times X))$ module category

$$\ldots \Gamma_\Delta(\mathrm{QCoh}(X \times X \times X)) \rightrightarrows \Gamma_\Delta(\mathrm{QCoh}(X \times X)) \to \mathrm{QCoh}(X)$$

(see Proposition 4.1.3), one can directly check that

$$\mathrm{Hom}_{\Gamma_\Delta(\mathrm{QCoh}(X\times X))}(\mathrm{QCoh}(X), \mathrm{QCoh}(X)) \cong \lim_{\boldsymbol{\Delta_s}}(\Gamma_\Delta(\mathrm{QCoh}(X^{n+1})), *)$$

analogously to Corollary 4.1.4.

[2] Tilde lower-cross means the left adjoint of tilde upper-star and is the left-right switch of tilde upper-cross, see Section 3.3

Proof 2.

$$\begin{aligned}
D_X\text{−Mod} &\cong \mathrm{Hom}_{\mathrm{QCoh}(S)}({D_X}^{\mathrm{op}}\text{−Mod}, \mathrm{QCoh}(S)) \\
&\cong \mathrm{Hom}_{\mathrm{QCoh}(S)}(\mathrm{QCoh}(X) \otimes_{\Gamma_\Delta(\mathrm{QCoh}(X\times X))} \mathrm{QCoh}(X), \mathrm{QCoh}(S)) \\
&\cong \mathrm{Hom}_{\Gamma_\Delta(\mathrm{QCoh}(X\times X)}(\mathrm{QCoh}(X), \mathrm{Hom}_{\mathrm{QCoh}(S)}(\mathrm{QCoh}(X), \mathrm{QCoh}(S))) \\
&\cong \mathrm{Hom}_{\Gamma_\Delta(\mathrm{QCoh}(X\times X))}(\mathrm{QCoh}(X), \mathrm{QCoh}(X))
\end{aligned}$$

The descent result is proven identically as in Theorem 4.1.10. ∎

Remark 4.2.2. *The reader is encouraged to compare this result with Remark 1.8.4 in [Ber19]*

Remark 4.2.3. *We can ask for an explicit description of the functor*

$$F_{D_X} : \mathrm{QCoh}(X) \to \lim_{\boldsymbol{\Delta_s}}(\Gamma_\Delta(\mathrm{QCoh}(X^{n+1})), *)$$

as a compatible system of functors

$$F_{D_X}^{(n)} : \mathrm{QCoh}(X) \to \Gamma_\Delta(\mathrm{QCoh}(X^{n+1}), *)$$

In fact, we have

$$F_{D_X}^{(n)} \cong \tilde{\pi}_{\widehat{n+2},\times}\tilde{\pi}^*_{n+2}$$

by left-right duality applied to Remark 4.1.11. Here $\pi_{\widehat{n+2}}$ means projection to all but the $n+2$-th component.

Additonally, using the isomorphism

$$D_X\text{−Mod} \cong \mathrm{Hom}_{\Gamma_\Delta(\mathrm{QCoh}(X\times X))}(\mathrm{QCoh}(X), \mathrm{QCoh}(X))$$

we have the descriptions

$$F_{D_X} \cong \mathrm{Hom}(\tilde{\pi}_1^\times, \mathrm{id})$$

$$G_{D_X} \cong \mathrm{Hom}(\tilde{\pi}_{1,*}, \mathrm{id})$$

which we can prove using left-right duality, the Proof 2. above, and equations (4.5) and (4.6).

Remark 4.2.4. *The limit we gave for the category of D_X-modules*

$$\lim_{\boldsymbol{\Delta}}(\Gamma_\Delta(\mathrm{QCoh}(X^{n+1})), *)$$

can be seen to be the category of quasicoherent crystals on the stratifying site of X. If X is a smooth variety over a field, we can use descent to see that this category is equivalent to the *category of quasicoherent sheaves on the de Rham stack, via the Čech nerve of the map*

$$X \to X_{dR}$$

In fact, D_X-modules are the same as quasicoherent sheaves on X_{dR} in more generality. In characteristic zero this is Proposition 3.4.3 in [GR14]. In our paper, we will show this as a consequence Kashiwara's equivalence in Section 4.6.

Remark 4.2.5. *By expressing the category of D_X-modules as the cosimplicial limit above, we can see that $D_X-\mathrm{Mod}$ is a symmetric monoidal category. In fact, the idemopotents of the D-module category agrees with that of $\mathrm{QCoh}(X)$. This is because the forgetful functor is conservative and any idempotent in fact lifts explicitly to each term of the cosimplicial limit (by explicitly tensoring copies of that idempotent).*

Now it's time to discuss the left-right switch. We can construct an explicit functor Q from ${D_X}^{\mathrm{op}}-\mathrm{Mod}$ to $D_X-\mathrm{Mod}$ as follows. Recall

$${D_X}^{\mathrm{op}}-\mathrm{Mod} \cong \mathrm{QCoh}(X) \otimes_{\Gamma_\Delta(\mathrm{QCoh}(X\times X))} \mathrm{QCoh}(X)$$

and

$$D_X-\mathrm{Mod} \cong \mathrm{Hom}_{\Gamma_\Delta(\mathrm{QCoh}(X\times X))}(\mathrm{QCoh}(X), \mathrm{QCoh}(X))$$

Therefore, the functor

$$\Gamma_\Delta \otimes \mathrm{id} : \mathrm{QCoh}(X \times X) \otimes_{\Gamma_\Delta(\mathrm{QCoh}(X\times X))} \mathrm{QCoh}(X) \to \mathrm{QCoh}(X)$$

which can also be written as

$$\Gamma_\Delta \otimes \mathrm{id} : \mathrm{QCoh}(X) \otimes \mathrm{QCoh}(X) \otimes_{\Gamma_\Delta(\mathrm{QCoh}(X\times X))} \mathrm{QCoh}(X) \to \mathrm{QCoh}(X)$$

is $\Gamma_\Delta(\mathrm{QCoh}(X\times X))$-linear ($\Gamma_\Delta(\mathrm{QCoh}(X\times X))$ acts by convolution on the leftmost $\mathrm{QCoh}(X)$) and therefore induces a functor

$$Q : {D_X}^{\mathrm{op}}-\mathrm{Mod} \to D_X-\mathrm{Mod}$$

Since Q is colimit-preserving, Q can be represented by a $(D_X, {D_X}^{\mathrm{op}})$-bimodule. We can determine which bimodule it is by calculating $G_{D_X} Q F_{{D_X}^{\mathrm{op}}}$. By chasing through the definitions and using equation (4.5) and its left-right dual, we can calculate

$$G_{D_X} Q F_{{D_X}^{\mathrm{op}}} \cong (\tilde{\pi}_2^* \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \tilde{\pi}_{1,*})$$

This has domain

$$\mathrm{QCoh}(X) \cong \mathrm{QCoh}(X) \otimes_{\Gamma_\Delta(\mathrm{QCoh}(X\times X))} \Gamma_\Delta(\mathrm{QCoh}(X \times X))$$

and codomain

$$\mathrm{QCoh}(X) \cong \Gamma_\Delta(\mathrm{QCoh}(X \times X)) \otimes_{\Gamma_\Delta(\mathrm{QCoh}(X\times X))} \mathrm{QCoh}(X)$$

Hence by Theorem A.1.1,

$$G_{D_X} Q F_{D_X{}^{\mathrm{op}}} \cong \tilde{\pi}_{1,*}\tilde{\pi}_2^*$$

and the relevant (D_X, D_X^{op}) bimodule is

$$\tilde{\pi}_1^*\mathcal{O}_X \cong \tilde{\pi}_2^*\mathcal{O}_X \cong \Gamma_\Delta(\mathcal{O}_{X\times X}) \cong \Gamma_\Delta(\mathcal{O}_X \boxtimes \mathcal{O}_X)$$

in $\Gamma_\Delta(\mathrm{QCoh}(X \times_S X))$.

Remark 4.2.6. *Strictly speaking, we have not defined what it means to be a $(D_X, D_X{}^{\mathrm{op}})$-bimodule. $\Gamma_\Delta(\mathrm{QCoh}(X \times_S X))$ is a $\Gamma_\Delta(\mathrm{QCoh}(X \times_S X))$-bimodule category, therefore there is a monad obtained by combining the D_X monad on the left with the $D_X{}^{\mathrm{op}}$ monad on the right. A $(D_X, D_X{}^{\mathrm{op}})$-bimodule is defined to be a module over that monad. $\mathcal{O}_X$ is naturally a D_X module. So $\Gamma_\Delta(\mathcal{O} \boxtimes \mathcal{O})$ has a natural structure of a $(D_X, D_X{}^{\mathrm{op}})$-bimodule.*

Remark 4.2.7. *We can also define Q by assembling the functors*

$$Q^{(m,n)} := \tilde{p}_{1,*}\tilde{p}_2^* : \Gamma_\Delta(\mathrm{QCoh}(X^{m+1})) \to \Gamma_\Delta(\mathrm{QCoh}(X^{n+1}))$$

into the functor

$$Q : \mathrm{colim}_{\mathbf{\Delta}_s^{\mathbf{op}}}(\Gamma_\Delta(\mathrm{QCoh}(X^{m+1})), *) \to \lim_{\mathbf{\Delta}_s}(\Gamma_\Delta(\mathrm{QCoh}(X^{n+1})), *)$$

where p_1, p_2 are the two projection maps of

$$X^{m+n+2} \cong X^{n+1} \times X^{m+1}$$

so that we have the functors

$$\tilde{p}_{1,*} : \Gamma_\Delta(\mathrm{QCoh}(X^{m+n+2})) \to \Gamma_\Delta(\mathrm{QCoh}(X^{n+1}))$$

$$\tilde{p}_2^* : \Gamma_\Delta(\mathrm{QCoh}(X^{m+1})) \to \Gamma_\Delta(\mathrm{QCoh}(X^{m+n+2}))$$

One can see gives the same functor as above for instance by computing the associated bimodule.

Now we would like to construct an inverse to Q. Consider the following functor

$$R : \lim_{\boldsymbol{\Delta}_s}(\Gamma_\Delta(\mathrm{QCoh}(X^{n+1})), *) \to \lim_{\boldsymbol{\Delta}_s}(\Gamma_\Delta(\mathrm{QCoh}(X^{n+1})), \times)$$

which we define by assembling

$$R^{(n)}(\mathcal{F}) := \mathcal{F} \otimes_{\mathcal{O}_{X^{n+1}}} \omega_X^{\boxtimes n+1}$$

This obviously commute with the transition maps by Corollary 3.2.15 because we have thrown away the degeneracy maps (by restricting to $\boldsymbol{\Delta_s}$). We note that R is a colimit-preserving functor with associated bimodule

$$\Gamma_\Delta(\omega_X \boxtimes \omega_X)$$

By inspection of the associated bimodules, we have

Proposition 4.2.8. *R and Q are self-dual under* $\mathrm{QCoh}(S)-\mathrm{Mod}^L$ *duality (left-right duality).*

The left-right switch is the following theorem.

Theorem 4.2.9. *R is the inverse functor of Q, and therefore*

$$D_X-\mathrm{Mod} \cong {D_X}^{\mathrm{op}}-\mathrm{Mod}$$

Proof. We show that $RQ \cong \mathrm{id}$, then the result will follow by duality. We will show this by directly computing RQ using Remark 4.2.7.

Consider RQ as a functor

$$RQ : \mathrm{colim}_{\boldsymbol{\Delta}_s^{\mathrm{op}}}(\Gamma_\Delta(\mathrm{QCoh}(X^{m+1})), *) \to \lim_{\boldsymbol{\Delta}_s}(\Gamma_\Delta(\mathrm{QCoh}(X^{n+1})), \times)$$

then we can regard it as assembled from functors

$$(RQ)^{(m,n)} : \Gamma_\Delta(\mathrm{QCoh}(X^{m+1})) \to \Gamma_\Delta(\mathrm{QCoh}(X^{n+1}))$$

which are defined by

$$(RQ)^{(m,n)}(\mathcal{F}) \cong \omega^{\boxtimes n+1} \otimes_{\mathcal{O}_{X^{n+1}}} \tilde{p}_{1,*}\tilde{p}_2^*\mathcal{F}$$

where

$$\tilde{p}_{1,*} : \Gamma_\Delta(\mathrm{QCoh}(X^{m+n+2})) \to \Gamma_\Delta(\mathrm{QCoh}(X^{n+1}))$$

$$\tilde{p}_2^* : \Gamma_\Delta(\mathrm{QCoh}(X^{m+1})) \to \Gamma_\Delta(\mathrm{QCoh}(X^{m+n+2}))$$

then the claim follows from

$$\omega^{\boxtimes n+1} \otimes_{\mathcal{O}_{X^{n+1}}} \tilde{p}_{1,*}\tilde{p}_2^*\mathcal{F} \cong \tilde{p}_{1,*}\tilde{p}_2^{\times}\mathcal{F}$$

together with Theorem 4.1.12. ■

4.3 Pushforward and Pullback of D-Modules

We discuss in this section how to pushforward and pullback D-modules, both left and right. We take the perspective of defining the functors on the category of D-modules first, and then subsequently defining the transfer bimodules using those functors. Therefore, transfer modules take a back-seat in our story, and we approach these functors as for crystals.

Suppose $f : X \to Y$ is a map of qcqs spectral algebraic spaces over a base spectral affine scheme S, both finite tor-amplitude and locally almost of finite presentation over S. Let us define pullback of D_X modules, using the presentation of $D_X-\mathrm{Mod}$ as a cosimplicial limit. We define the functors

$$f^{+,(n)} : \Gamma_\Delta(\mathrm{QCoh}(Y^{n+1})) \to \Gamma_\Delta(\mathrm{QCoh}(X^{n+1}))$$

by[3]

$$f^{+,(n)} := \Gamma_\Delta(f^*)^{n+1} i_\Delta = \widetilde{(f^*)^{n+1}}$$

where the second equality is just a notation. These functors are obviously compatible with the transition maps, so they assemble into the functor

$$f^+ : \lim_{\boldsymbol{\Delta}_s}(\Gamma_\Delta(\mathrm{QCoh}(Y^{n+1})), *) \to \lim_{\boldsymbol{\Delta}_s}(\Gamma_\Delta(\mathrm{QCoh}(X^{n+1})), *)$$

or equivalently

$$f^+ : D_Y-\mathrm{Mod} \to D_X-\mathrm{Mod}$$

Which is what we call pullback of D_X modules.

Left-right duality takes the pullback functor of D_X modules to the pushforward of ${D_X}^{\mathrm{op}}$ modules, which we can also easily define directly. Consider the functors

$$f_+^{(n)} : \Gamma_\Delta(\mathrm{QCoh}(X^{n+1})) \to \Gamma_\Delta(\mathrm{QCoh}(Y^{n+1}))$$

[3] We define the category $\Gamma_\Delta(\mathrm{QCoh}(X^{n+1}))$ by definining it étale locally on X with the usual definition (same with Y).

defined by

$$f_+^{(n)} := \Gamma_\Delta (f_*)^{n+1} i_\Delta = \widetilde{(f_*)^{n+1}}$$

where as before the second equality is a notation. These assembles into the functor

$$f_+ : \operatorname{colim}_{\mathbf{\Delta}_s^{\mathrm{op}}}(\Gamma_\Delta(\mathrm{QCoh}(X^{n+1})), *) \to \operatorname{colim}_{\mathbf{\Delta}_s^{\mathrm{op}}}(\Gamma_\Delta(\mathrm{QCoh}(Y^{n+1})), *)$$

or equivalently

$$f_+ : {D_X}^{\mathrm{op}}\!-\!\mathrm{Mod} \to {D_Y}^{\mathrm{op}}\!-\!\mathrm{Mod}$$

Now we will define the transfer module to compare with the classical story. As f_+ and f^+ are guaranteed to be colimit-preserving, these functors have corresponding transfer modules. The transfer module for f^+ and the one for f_+ will be the same (up to swapping the order of X and Y) because they are related by left-right duality. As in the section on the left-right switch, we find the transfer module by considering the composition

$$G_{D_X} f^+ F_{D_Y}$$

which simplifies to (by Remark 4.2.3)

$$f^* \tilde{\pi}_{1,\times}^{(Y\times Y)} \tilde{\pi}_2^{(Y\times Y),*}$$

Define

$$\delta_f : X \to X \times_S Y$$

to be the graph of f. Let us denote by Γ_f the local cohomology functor on $\mathrm{QCoh}(X \times_S Y)$ relative to this subset. Consider the split-exact sequence of presentable stable categories

$$\Gamma_\Delta(\mathrm{QCoh}(Y \times Y)) \to \mathrm{QCoh}(Y \times Y) \to \mathrm{QCoh}(U)$$

for the closed subset Δ in $Y \times Y$, where U is the complement of Δ. We can apply the functor $\mathrm{QCoh}(X) \otimes_{\mathrm{QCoh}(Y)} \_$ to the above (where $\mathrm{QCoh}(Y)$ acts on the left) to get the split-exact sequence (see Remark A.2.7)

$$\mathrm{QCoh}(X) \otimes_{\mathrm{QCoh}(Y)} \Gamma_\Delta(\mathrm{QCoh}(Y \times Y)) \to \mathrm{QCoh}(X \times Y) \to \mathrm{QCoh}(V)$$

where V is the complement of the graph of f in $X \times Y$. Therefore, we have the result

Lemma 4.3.1.

$$\Gamma_f(\mathrm{QCoh}(X \times_S Y)) \cong \mathrm{QCoh}(X) \otimes_{\mathrm{QCoh}(Y)} \Gamma_\Delta(\mathrm{QCoh}(Y \times_S Y))$$

where $\mathrm{QCoh}(Y)$ *acts on* $\Gamma_\Delta(\mathrm{QCoh}(Y \times_S Y))$ *via* $\tilde{\pi}_1^*$*.*

With the description of $\Gamma_f(\mathrm{QCoh}(X \times_S Y))$ above, we have (by comparing their right adjoints) [4]

$$\tilde{\pi}_{1,\times}^{X\times Y} \cong \mathrm{id}_{\mathrm{QCoh}(X)} \otimes \tilde{\pi}_{1,\times}^{(Y\times Y)}$$

Consider the diagram

$$\begin{tikzcd}
X \times_S Y \arrow[r, "f\times\mathrm{id}"] \arrow[d, "\pi_1^{(X\times Y)}"] & Y \times_S Y \arrow[d, "\pi_1^{(Y\times Y)}"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

Using Theorem A.1.1, we have

$$\begin{aligned}
f^* \tilde{\pi}_{1,\times}^{(Y\times Y)} \tilde{\pi}_2^{(Y\times Y),*} &\cong \tilde{\pi}_{1,\times}^{(X\times Y)} \widetilde{(f \times \mathrm{id})}^* \tilde{\pi}_2^{(Y\times Y),*} \\
&\cong \tilde{\pi}_{1,\times}^{(X\times Y)} \tilde{\pi}_2^{(X\times Y),*}
\end{aligned}$$

where

$$\tilde{\pi}_{1,\times}^{(X\times Y)} : \Gamma_f(\mathrm{QCoh}(X \times_S Y)) \to \mathrm{QCoh}(X)$$

is defined as before (as the left adjoint of $\tilde{\pi}_1^{(X\times Y),*}$). Hence the bimodule for the pullback functor f^+ is the one corresponding to the functor

$$\tilde{\pi}_{1,\times}^{(X\times Y)} \tilde{\pi}_2^{(X\times Y),*}$$

which is (by the left-right duals of Theorem 3.2.13 and Theorem 3.2.7)

$$\Gamma_f(\mathcal{O}_X \boxtimes \omega_Y)$$

Hence,

[4] Note here both $\tilde{\pi}_1^{\times}$'s are colimit-preserving because their left adjoints are compact object preserving.

Theorem/Definition 4.3.2. *The transfer module $D_{X \to Y}$ for f^+ (and also f_+) is*

$$D_{X \to Y/S} := \Gamma_f(\mathcal{O}_X \boxtimes \omega_Y) \cong \tilde{\pi}_1^{X \times Y, \times} \mathcal{O}_X \in \Gamma_f(\mathrm{QCoh}(X \times_S Y))$$

Corollary 4.3.3.

$$D_{X \to Y/S} \cong \widetilde{(f \times \mathrm{id})}^* D_{Y/S}$$

where

$$\widetilde{(f \times \mathrm{id})}^* : \Gamma_\Delta(\mathrm{QCoh}(Y \times_S Y)) \to \Gamma_f(\mathrm{QCoh}(X \times_S Y))$$

is induced from the pullback functor $(f \times \mathrm{id})^*$.

It is clear that $D_{X \to Y/S}$ naturally carries a left $D_{X/S}$ action and a right $D_{Y/S}$ action. As the plus pullback functors compose well, also the transfer modules must compose well.

Theorem 4.3.4.

$$D_{X \to Z} \cong D_{X \to Y} \star_{D_Y} D_{Y \to Z}$$

Remark 4.3.5. *The star product is used here to recall that the algebra structure on D is with respect to the convolution tensor product; but this can just be thought of as a tensor over D_Y*

Remark 4.3.6. *Suppose additionally that f is finite tor-amplitude. In this situation, we can define the functor*

$$f_\dagger : D_X\mathrm{-Mod} \to D_Y\mathrm{-Mod}$$

as a functor

$$f_\dagger : \mathrm{colim}_{\boldsymbol{\Delta}_s^{\mathbf{op}}}(\mathrm{QCoh}(X^{m+1}), \times) \to \mathrm{colim}_{\boldsymbol{\Delta}_s^{\mathbf{op}}}(\mathrm{QCoh}(Y^{m+1}), \times)$$

which is given simply by assembling

$$f_\dagger^{(n)} \cong \widetilde{f_!^{n+1}} : \mathrm{QCoh}(X^{m+1}) \to \mathrm{QCoh}(Y^{m+1})$$

and dually we can define the functor

$$f^\dagger : {D_Y}^{\mathrm{op}}\mathrm{-Mod} \to {D_X}^{\mathrm{op}}\mathrm{-Mod}$$

as a functor

$$f^\dagger : \lim_{\boldsymbol{\Delta}_s}(\mathrm{QCoh}(Y^{m+1}), \times) \to \lim_{\boldsymbol{\Delta}_s}(\mathrm{QCoh}(X^{m+1}), \times)$$

given simply given by assembling

$$f^{\dagger,(n)} \cong \widetilde{f^{n+1,!}} : \mathrm{QCoh}(Y^{m+1}) \to \mathrm{QCoh}(X^{m+1})$$

If f is in addition proper, then $f_\dagger$ is left adjoint to f^+ and $f^\dagger$ is right adjoint to f_+. The !-functors are as defined in Section 3.2. We note that these constructions are made easier because we restricted to $\mathbf{\Delta}_s$.

Remark 4.3.7. *The dagger functors above are exactly the left-right switches (not dual!) of the plus functors. We must warn the reader here that the pullback and pushforward functors described above differ from the standard presentations, even when the notation is the same! For example, if the reader is comparing to the [HTT08] book, the translation goes as follows for a map $f : X \to Y$ between smooth varieties*

$$\int_f \cong f_+[\dim X - \dim Y]$$

and

$$f^\dagger_{HTT} \cong f^+[\dim X - \dim Y]$$

where the left hand side is the in notation of [HTT08].

4.4 Kashiwara's Equivalence

In this section, we prove a version of Kashiwara's equivalence in our context. In particular it shows that the category of D_Z-modules on a singular variety Z embedded in a smooth variety X is equivalent to the subcategory of D_X-modules supported on Z.

Suppose $z : Z \to X$ is a finite tor-amplitude, locally almost finitely presented closed immersion. For any n, consider the split-exact sequence of presentable stable categories in $\mathrm{QCoh}(S){-}\mathrm{Mod}^L$

$$\Gamma_{\Delta_Z}(\mathrm{QCoh}(X^{n+1})) \xrightarrow{i_Z^{(n)}} \Gamma_{\Delta_X}(\mathrm{QCoh}(X^{n+1})) \xrightarrow{j^{(n),*}} \Gamma_{\Delta_U}(\mathrm{QCoh}(U^{n+1})) \tag{4.11}$$

whose right adjoints $\Gamma_Z^{(n)}$ and $j_*^{(n)}$ (for $(i_Z^{(n)})_*$ and $j^{(n),*}$ respectively) are colimit-preserving.

In fact, using lower star (quasicoherent pushforward) as the transition maps, each of the above categories fits into a simplicial diagram. Each of the functors $i^{(n)}, j^{(n),*}, \Gamma_Z^{(n)}, j_*^{(n)}$ commute with the transition maps, therefore after taking colimits of the simplicial diagrams of categories with n-th objects as described above, we recover a split-exact sequence by Lemma A.2.5.

$$\mathrm{colim}_{\mathbf{\Delta}_s^{\mathrm{op}}} \Gamma_{\Delta_Z}(\mathrm{QCoh}(X^{n+1})) \xrightarrow{i_Z} \mathrm{colim}_{\mathbf{\Delta}_s^{\mathrm{op}}} \Gamma_{\Delta_X}(\mathrm{QCoh}(X^{n+1})) \xrightarrow{j^*} \mathrm{colim}_{\mathbf{\Delta}_s^{\mathrm{op}}} \Gamma_{\Delta_U}(\mathrm{QCoh}(U^{n+1})) \tag{4.12}$$

Note that to check the last condition in the lemma, it suffices to check the exactness for compact objects in $\operatorname{colim}_{\mathbf{\Delta_s}^{\mathrm{op}}} \Gamma_{\Delta}(\mathrm{QCoh}(X^{n+1}))$ where it follows from the split-exactness of (4.11). Let us define

$$\Gamma_Z({D_X}^{\mathrm{op}}-\mathrm{Mod}) := \operatorname{colim}_{\mathbf{\Delta}^{\mathrm{op}}} \Gamma_{\Delta_Z}(\mathrm{QCoh}(X^{n+1}))$$

The above sequence (4.12) can then be written simply as

$$\Gamma_Z({D_X}^{\mathrm{op}}-\mathrm{Mod}) \xrightarrow{i_Z} {D_X}^{\mathrm{op}}-\mathrm{Mod} \xrightarrow{j^*} {D_U}^{\mathrm{op}}-\mathrm{Mod}$$

Therefore, $\Gamma_Z({D_X}^{\mathrm{op}}-\mathrm{Mod})$ is the category of D_X^{op} modules supported on Z (as it is exactly those that vanish after pulling back to U–note that pulling back right D_X modules to open subsets is well-defined).

By modifying the proof of Theorem 4.1.6 we can see that the natural map

$$\Gamma_Z(\mathrm{QCoh}(X)) \to \Gamma_Z({D_X}^{\mathrm{op}}-\mathrm{Mod})$$

is the left adjoint in a monadic adjunction where the monad in question is

$$\tilde{\pi}_{1,*}\tilde{\pi}_2^{\times}$$

The functors

$$\tilde{\pi}_2^{\times} : \Gamma_Z(\mathrm{QCoh}(X)) \to \Gamma_{\Delta_Z}(\mathrm{QCoh}(X \times X))$$

and

$$\tilde{\pi}_{1,*} : \Gamma_{\Delta_Z}(\mathrm{QCoh}(X \times X)) \to \Gamma_Z(\mathrm{QCoh}(X))$$

are defined analogously to before, but restricted only to quasicoherent sheaves supported over Z.

We would like to compare the category $\Gamma_Z({D_X}^{\mathrm{op}}-\mathrm{Mod})$ with

$${D_Z}^{\mathrm{op}}-\mathrm{Mod} \cong \operatorname{colim}_{\mathbf{\Delta}^{\mathrm{op}}}(\Gamma_{\Delta_Z}(\mathrm{QCoh}(Z^{n+1})), *)$$

The functor

$$z_+ : {D_Z}^{\mathrm{op}}-\mathrm{Mod} \to {D_X}^{\mathrm{op}}-\mathrm{Mod}$$

naturally induces (and in fact factors through) the functor

$$\tilde{z}_+ : {D_Z}^{\mathrm{op}}-\mathrm{Mod} \to \Gamma_Z({D_X}^{\mathrm{op}}-\mathrm{Mod})$$

This is the functor which we will shows is an equivalence. By Remark 4.3.6, $\tilde{z}_+$ has a colimit-preserving right adjoint, $\tilde{z}^\dagger$. We wish to show that $\tilde{z}^\dagger\tilde{z}_+ \cong \mathrm{id}$ and $\tilde{z}^\dagger\tilde{z}_+ \cong \mathrm{id}$. First, notice that

Lemma 4.4.1. *The adjunction $\tilde{z}_+ \dashv \tilde{z}^\dagger$ is monadic.*

Proof. By Barr-Beck-Lurie it suffices to show that $\tilde{z}^\dagger$ is conservative and colimit-preserving. The functor is colimit-preserving directly from the definition given in Remark 4.3.6. For conservativeness, notice that

$$G_{D_Z^{\mathrm{op}}}\tilde{z}^\dagger \cong \tilde{z}^\times G_{\Gamma_Z(D_X^{\mathrm{op}})}$$

and both of the functors on the right are conservative (see Lemma 2.1.3). ∎

Therefore we have shown that $\Gamma_Z(D_X^{\mathrm{op}}-\mathrm{Mod})$ is monadic over $D_Z^{\mathrm{op}}-\mathrm{Mod}$ which is itself monadic over $\mathrm{QCoh}(Z)$. Hence the equivalence can be shown by showing that the map of monads

$$G_{D_Z{}^{\mathrm{op}}}F_{D_Z{}^{\mathrm{op}}} \to G_{D_Z{}^{\mathrm{op}}}\tilde{z}^\dagger\tilde{z}_+F_{D_Z{}^{\mathrm{op}}}$$

is an isomorphism. Consider the endofunctor

$$G_{D_Z{}^{\mathrm{op}}}\tilde{z}^\dagger\tilde{z}_+F_{D_Z{}^{\mathrm{op}}}$$

Because of the commutative diagram

$$\begin{tikzcd}
\mathrm{QCoh}(Z) \arrow[r, "\tilde{z}_*"] \arrow[d, "F_{D_Z{}^{\mathrm{op}}}"] & \Gamma_Z(\mathrm{QCoh}(X)) \arrow[d, "F_{\Gamma_Z(D_X{}^{\mathrm{op}})}"] \\
D_Z{}^{\mathrm{op}}-\mathrm{Mod} \arrow[r, "\tilde{z}_+"] & \Gamma_Z(D_X{}^{\mathrm{op}}-\mathrm{Mod})
\end{tikzcd}$$

The above is also

$$\tilde{z}^\times G_{\Gamma_Z(D_X{}^{\mathrm{op}})}F_{\Gamma_Z(D_X{}^{\mathrm{op}})}\tilde{z}_*$$

or

$$\tilde{z}^\times\tilde{\pi}_{1,*}\tilde{\pi}_2^\times\tilde{z}_*$$

By base-changing the split-exact sequence

$$\Gamma_\Delta(\mathrm{QCoh}(X \times X)) \to \mathrm{QCoh}(X) \to \mathrm{QCoh}(U) \tag{4.13}$$

where U is the complement of the diagonal, we can show

$$\Gamma_{\Delta_Z}(\mathrm{QCoh}(X \times Z)) \cong \Gamma_{\Delta_X}(\mathrm{QCoh}(X \times X)) \otimes_{\mathrm{QCoh}(X)} \mathrm{QCoh}(Z)$$

Now because the action of $\mathrm{QCoh}(X)$ on $\mathrm{QCoh}(Z)$ factors through $\Gamma_Z(\mathrm{QCoh}(X))$, we have

$$\Gamma_{\Delta_Z}(\mathrm{QCoh}(X \times Z)) \cong \Gamma_{\Delta_Z}(\mathrm{QCoh}(X \times X)) \otimes_{\Gamma_Z(\mathrm{QCoh}(X))} \mathrm{QCoh}(Z)$$

since

$$\Gamma_{\Delta_X}(\mathrm{QCoh}(X \times X)) \otimes_{\mathrm{QCoh}(X)} \Gamma_Z(\mathrm{QCoh}(X)) \cong \Gamma_{\Delta_Z}(\mathrm{QCoh}(X \times X))$$

Looking at the diagram

$$\begin{tikzcd}
\mathrm{QCoh}(Z) \arrow[r, "\tilde{z}_*"] \arrow[d, "\tilde{\pi}_2^{(X\times Z),\times}"] & \Gamma_Z(\mathrm{QCoh}(X)) \arrow[d, "\tilde{\pi}_2^{\times}"] \\
\Gamma_{\Delta_Z}\mathrm{QCoh}(X \times Z) \arrow[r, "\widetilde{\mathrm{id}\times z_*}"] & \Gamma_{\Delta_Z}(\mathrm{QCoh}(X \times X))
\end{tikzcd}$$

we see that, since the $\tilde{\pi}_2^{\times}$ on the left is the base-change of the $\tilde{\pi}_2^{\times}$ on the right (by comparing their left adjoints), this diagram commutes by Theorem A.1.1. Hence

$$\tilde{\pi}_2^{\times}\tilde{z}_* \cong (\widetilde{\mathrm{id} \times z})_* \tilde{\pi}_2^{(X\times Z),\times}$$

On the other side, we have the analogous commutative diagram

$$\begin{tikzcd}
\Gamma_{\Delta_Z}\mathrm{QCoh}(X \times X) \arrow[r, "(\widetilde{z\times \mathrm{id}})^{\times}"] \arrow[d, "\tilde{\pi}_{1,*}"] & \Gamma_{\Delta_Z}(\mathrm{QCoh}(Z \times X)) \arrow[d, "\tilde{\pi}_{1,*}^{(Z\times X)}"] \\
\Gamma_Z(\mathrm{QCoh}(X)) \arrow[r, "\tilde{z}^{\times}"] & \mathrm{QCoh}(Z)
\end{tikzcd}$$

By Theorem A.1.1, we have (this result is basically right adjoint to above)

$$\tilde{z}^{\times}\tilde{\pi}_{1,*} \cong \tilde{\pi}_{1,*}^{(Z\times X)}(\widetilde{z \times \mathrm{id}})^{\times}$$

Hence

$$\tilde{z}^{\times}\tilde{\pi}_{1,*}\tilde{\pi}_2^{\times}\tilde{z}_* \cong \tilde{\pi}_{1,*}^{(Z\times X)}\widetilde{(z\times \mathrm{id})}^{\times}\widetilde{(\mathrm{id}\times z)}_*\tilde{\pi}_2^{(X\times Z),\times}$$

One can check that the natural map above

$$G_{D_Z{}^{\mathrm{op}}}F_{D_Z{}^{\mathrm{op}}} \to G_{D_Z{}^{\mathrm{op}}}\tilde{z}^{\dagger}\tilde{z}_+F_{D_Z{}^{\mathrm{op}}}$$

is the same as the natural map

$$\tilde{\pi}_{1,*}^{(Z\times X)}\widetilde{(\mathrm{id}\times z)}_*\widetilde{(z\times \mathrm{id})}^{\times}\tilde{\pi}_2^{(X\times Z),\times} \to \tilde{\pi}_{1,*}^{(Z\times X)}\widetilde{(z\times \mathrm{id})}^{\times}\widetilde{(\mathrm{id}\times z)}_*\tilde{\pi}_2^{(X\times Z),\times}$$

coming from adjunction. Now consider the diagram

$$\begin{tikzcd}
\Gamma_{\Delta_Z}\mathrm{QCoh}(X\times Z) \arrow[r, "\widetilde{(z\times\mathrm{id})}^{\times}"] \arrow[d, "\widetilde{(\mathrm{id}\times z)}_*"] & \Gamma_{\Delta_Z}(\mathrm{QCoh}(Z\times Z)) \arrow[d, "\widetilde{(\mathrm{id}\times z)}_*"] \\
\Gamma_Z(\mathrm{QCoh}(X\times X)) \arrow[r, "\widetilde{(z\times\mathrm{id})}^{\times}"] & \Gamma_Z(\mathrm{QCoh}(Z\times X))
\end{tikzcd}$$

Using the fact that Z is a closed immersion, we can show that set-theoretically (working affine locally on X),

$$(Z\times X)\cap(X\times Z) = (Z\times Z)$$

inside $X\times X$. Therefore, we have the categorical isomorphism

$$\Gamma_{\Delta_Z}\mathrm{QCoh}(Z\times Z) \cong \Gamma_{\Delta_Z}\mathrm{QCoh}(X\times Z)\otimes_{\Gamma_{\Delta_Z}\mathrm{QCoh}(X\times X)}\Gamma_{\Delta_Z}\mathrm{QCoh}(Z\times X)$$

(where the action is via pullback not convolution). Now the isomorphism

$$\widetilde{(z\times \mathrm{id})}^{\times}\widetilde{(\mathrm{id}\times z)}_* \cong \widetilde{(\mathrm{id}\times z)}_*\widetilde{(z\times \mathrm{id})}^{\times}$$

follows from the the diagram above via Theorem A.1.1. Hence we have shown

Theorem 4.4.2 (Kashiwara's Equivalence)**.** *Suppose $z : Z \to X$ is a finite tor-amplitude, locally almost finitely presented closed immersion, and p_X, p_Z satisfy our standing assumptions, then*

$$\tilde{z}_+ : D_Z{}^{\mathrm{op}}\mathrm{-Mod} \to \Gamma_Z(D_X{}^{\mathrm{op}}\mathrm{-Mod})$$

is an equivalence of categories with inverse $\tilde{z}^{\dagger}$.

By left-right duality, we also have

Corollary 4.4.3. *Suppose $z : Z \to X$ is a finite tor-amplitude, locally almost finitely presented closed immersion, and p_X,p_Z satisfy our standing assumptions, then*

$$\tilde{z}^+ : \Gamma_Z(D_X\text{−Mod}) \to D_Z\text{−Mod}$$

is an equivalence of categories with inverse $\tilde{z}_\dagger$.

Remark 4.4.4. *In the case that $X = S = \operatorname{Spec} R$ for R a discrete ring, the ring of differential operators on $Z = \operatorname{Spec} R/I$ (relative to X) is simply $D_{Z/X} = Hom_R(R/I, R/I)$ (if R/I is a perfect R module). Theorem 4.4.2 then follows from derived Morita theory since R/I is a compact generator of the category of I-nilpotent R modules (In the sense of Theorem 7.1.1.6 of [Lur18]).*

4.5 Base Change and Fppf Descent

In this section, we prove descent of the category of D-modules with respect to the fppf topology.

Fix a spectral affine scheme S.

We will denote by $X_T^{(n)}$ the n-fold product of X over T for a scheme X over T.

We will need Corollary 4.7.5.3 from [Lur17] which we reproduce here for the benefit of the reader.

Corollary 4.5.1 ([Lur17] Corollary 4.7.5.3)**.** *Let $C^\bullet : N(\Delta_+) \to Cat_\infty$ be an augmented cosimplicial ∞-category, and set $\mathcal{C} = C^{-1}$. Let $G : \mathcal{C} \to \mathcal{C}^0$ be the evident functor. Assume that:*

1. *The ∞-category $\mathcal{C}$ admits geometric realizations of G-split simplicial objects, and those geometric realizations are preserved by G.*
2. *For every morphism $\alpha : [m] \to [n]$ in Δ_+, the diagram*

$$\begin{array}{ccc} C^m & \xrightarrow{d^0} & C^{m+1} \\ \downarrow & & \downarrow \\ C^n & \xrightarrow{d^0} & C^{n+1} \end{array}$$

is left adjointable.

Then the canonical map $\theta : \mathcal{C} \to \varprojlim_{n\in\Delta} C^n$ admits a fully faithful left adjoint. If G is conservative, then θ is an equivalence.

Proposition 4.5.2. *Suppose S is a spectral affine scheme and X and Y are locally almost of finite presentation, finite tor-amplitude affine schemes over S. If there is a map $f : X \to Y$ which is finite tor-amplitude. Then, there is a natural isomorphism*

$$\Gamma_X(\mathrm{QCoh}(X \times_S Y)) \cong \lim_{\boldsymbol{\Delta_s}}(\Gamma_X(\mathrm{QCoh}(X \times_S (X_Y^{(n+1)}))))$$

with the transition maps either upper star or upper cross.

Proof. We will assume the transition maps are upper star, the proof for upper cross is entirely analogous. We will apply [Lur17] Corollary 4.7.5.3. Condition (1) is automatic (for upper cross it follows from the fact that the map $f : X \to Y$ is finite tor-amplitude). For (2), we need to check that, for any map $[m] \to [n]$, the square below is left adjointable. Here the horizontal arrow is upper star along the map induced by projection map to all but the first component for the product over Y and the vertical maps are upper star along the map induced by the map $[m] \to [n]$.

$$\begin{array}{ccc}
\Gamma_X(\mathrm{QCoh}(X \times_S (X_Y^{(m+1)}))) & \xrightarrow{\widetilde{\mathrm{id}\times\pi_{\hat{1}}}^*} & \Gamma_X(\mathrm{QCoh}(X \times_S (X_Y^{(m+2)}))) \\
\Big\downarrow u & & \Big\downarrow v \\
\Gamma_X(\mathrm{QCoh}(X \times_S (X_Y^{(n+1)}))) & \xrightarrow{\widetilde{\mathrm{id}\times\pi_{\hat{1}}}^*} & \Gamma_X(\mathrm{QCoh}(X \times_S (X_Y^{(n+2)})))
\end{array}$$

We need to check the map

$$(\widetilde{\mathrm{id}\times\pi_{\hat{1}}})_\times v \to (\widetilde{\mathrm{id}\times\pi_{\hat{1}}})_\times v(\widetilde{\mathrm{id}\times\pi_{\hat{1}}})^*(\widetilde{\mathrm{id}\times\pi_{\hat{1}}})_\times \cong (\widetilde{\mathrm{id}\times\pi_{\hat{1}}})_\times(\widetilde{\mathrm{id}\times\pi_{\hat{1}}})^* u(\widetilde{\mathrm{id}\times\pi_{\hat{1}}})_\times \to u(\widetilde{\mathrm{id}\times\pi_{\hat{1}}})_\times$$

is an isomorphism. We have the isomorphisms (which can again be verified using split-exact sequences)

$$\begin{aligned}
\Gamma_X(\mathrm{QCoh}(X \times_S (X_Y^{(n+2)}))) &\cong \Gamma_X(\mathrm{QCoh}(X \times_S X \times_Y X_Y^{(n+1)})) \\
&\cong \Gamma_X(\Gamma_X(\mathrm{QCoh}(X \times_S X_Y^{(n+1)})) \otimes_{\mathrm{QCoh}(X\times_S X_Y^{(m+1)})} \mathrm{QCoh}(X \times_S X_Y^{(m+2)})) \\
&\cong \Gamma_X(\mathrm{QCoh}(X \times_S X_Y^{(m+1)})) \otimes_{\Gamma_X(\mathrm{QCoh}(X\times_S X_Y^{(n+1)}))} \Gamma_X(\mathrm{QCoh}(X \times_S X_Y^{(n+2)}))
\end{aligned}$$

using these isomorphisms we have

$$v \cong u \otimes \mathrm{id}$$

and we can therefore easily see the map above is an isomorphism (because we can separate out the vertical and horizontal maps and hence the isomorphism is coming from the adjunction data).

Lastly, we need to check that the map

$$\Gamma_X(\mathrm{QCoh}(X \times_S Y)) \to \Gamma_X(\mathrm{QCoh}(X \times_S X))$$

is conservative, but that follows from Lemma 2.1.3. ■

Corollary 4.5.3. *Suppose S is a spectral affine scheme and X and Y are locally almost of finite presentation, finite tor-amplitude affine schemes over S. If there is a map $f : X \to Y$ which is finite tor-amplitude. Then, there is a natural isomorphism*

$$\Gamma_\Delta(\mathrm{QCoh}(X \times_S X)) \otimes_{\Gamma_\Delta(\mathrm{QCoh}(X \times_Y X))} \mathrm{QCoh}(X) \cong \Gamma_X(\mathrm{QCoh}(X \times_S Y))$$

Proof. Using Proposition 4.1.3, we have the isomorphism

$$\Gamma_\Delta(\mathrm{QCoh}(X \times_S X)) \otimes_{\Gamma_\Delta(\mathrm{QCoh}(X \times_Y X))} \mathrm{QCoh}(X)$$

$$\cong \mathrm{colim}_{\mathbf{\Delta}_s}(\Gamma_\Delta(\mathrm{QCoh}(X \times_S X)) \otimes_{\Gamma_\Delta(\mathrm{QCoh}(X \times_Y X))} \Gamma_X(\mathrm{QCoh}(X_Y^{(n+2)})))$$

By applying Lemma 4.1.1, we also have the isomorphism

$$\Gamma_\Delta(\mathrm{QCoh}(X \times_S X)) \otimes_{\Gamma_\Delta(\mathrm{QCoh}(X \times_Y X)} \Gamma_X(\mathrm{QCoh}(X_Y^{(n+2)}))$$

$$\cong \Gamma_X(\mathrm{QCoh}(X \times_S X)) \otimes_{\mathrm{QCoh}(X)} \Gamma_X(\mathrm{QCoh}(X_Y^{(n+1)})) \cong \Gamma_X(\mathrm{QCoh}(X \times_S X_Y^{(n+1)}))$$

Hence the result follows by switching the colimit to a limit (by taking right adjoints of the transition maps) and applying proposition 4.5.2. ■

Proposition 4.5.4. *Suppose S is a spectral affine scheme and X and Y are locally almost of finite presentation, finite tor-amplitude affine schemes over S. Also suppose that there is a map $f : X \to Y$ which is finite tor-amplitude. Then, there is a natural isomorphism*

$$(\widetilde{\mathrm{id} \times f})_* \left(\Gamma_X(\mathcal{O}_X \boxtimes \omega_{X/S}) \otimes_{D_{X/Y}} \mathcal{O}_X\right) \cong \Gamma_X(\mathcal{O}_X \boxtimes \omega_{Y/S})$$

of objects in $\Gamma_X(\mathrm{QCoh}(X \times_S Y))$ respecting the (D_X, D_Y)-module structure. Note that the left hand side is a bit of an abuse of notation because the object

$$\Gamma_X(\mathcal{O}_X \boxtimes \omega_{X/S}) \otimes_{D_{X/Y}} \mathcal{O}_X$$

has no $\mathcal{O}_X$ action on the right, so the pushforward is really only as Zariski sheaves.

Proof. We can make sense of an element

$$\Gamma_X(\mathcal{O}_X \boxtimes \omega_{X/S}) \otimes_{D_{X/Y}} \mathcal{O}_X \in \Gamma_X(\mathrm{QCoh}(X \times X)) \otimes_{\Gamma_X(\mathrm{QCoh}(X \times_Y X))} \mathrm{QCoh}(X)$$

as follows. The right hand side is the colimit of the simplicial diagram

$$\ldots \Gamma_X(\mathrm{QCoh}(X \times X)) \otimes \Gamma_X(\mathrm{QCoh}(X \times_Y X)) \otimes \mathrm{QCoh}(X) \rightrightarrows \Gamma_X(\mathrm{QCoh}(X \times X)) \otimes \mathrm{QCoh}(X)$$

Therefore, the element

$$\Gamma_X(\mathcal{O}_X \boxtimes \omega_{X/S}) \otimes \mathcal{O}_X \in \Gamma_X(\mathrm{QCoh}(X \times X)) \otimes \mathrm{QCoh}(X)$$

maps to an element in $\Gamma_X(\mathrm{QCoh}(X \times X)) \otimes_{\Gamma_X(\mathrm{QCoh}(X \times_Y X))} \mathrm{QCoh}(X))$ and so does the element

$$\Gamma_X(\mathcal{O}_X \boxtimes \omega_{X/S}) \otimes D_{X/Y} \otimes \mathcal{O}_X \in \Gamma_X(\mathrm{QCoh}(X \times X)) \otimes \Gamma_X(\mathrm{QCoh}(X \times_Y X)) \otimes \mathrm{QCoh}(X)$$

and so on. As $D_{X/Y}$ acts on $\Gamma_X(\mathcal{O}_X \boxtimes \omega_{X/S})$ on the right and $\mathcal{O}_X$ on the right, there is a simplicial diagram

$$\ldots \Gamma_X(\mathcal{O}_X \boxtimes \omega_{X/S}) \otimes D_{X/Y} \otimes \mathcal{O}_X \rightrightarrows \Gamma_X(\mathcal{O}_X \boxtimes \omega_{X/S}) \otimes \mathcal{O}_X$$

whose colimit we call

$$\Gamma_X(\mathcal{O}_X \boxtimes \omega_{X/S}) \otimes_{D_{X/Y}} \mathcal{O}_X \in \Gamma_X(\mathrm{QCoh}(X \times X)) \otimes_{\Gamma_X(\mathrm{QCoh}(X \times_Y X))} \mathrm{QCoh}(X)$$

We can check directly that the image in $\Gamma_X(\mathrm{QCoh}(X \times Y))$ of this element is

$$(\widetilde{\mathrm{id} \times f})_* \left(\Gamma_X(\mathcal{O}_X \boxtimes \omega_{X/S}) \otimes_{D_{X/Y}} \mathcal{O}_X\right)$$

and hence we just need to show this is isomorphic to

$$\Gamma_X(\mathcal{O}_X \boxtimes \omega_{Y/S})$$

Under the isomorphism

$$\Gamma_X(\mathrm{QCoh}(X \times_S Y)) \cong \lim_{\boldsymbol{\Delta_s}}(\Gamma_X(\mathrm{QCoh}(X \times_S (X_Y^{(n+1)}))), \times)$$

the object

$$\Gamma_X(\mathcal{O}_X \boxtimes \omega_{Y/S})$$

is given under the isomorphism by the compatible system

$$\left(\Gamma_X(\mathcal{O}_X \boxtimes \omega_{X_Y^{(n+1)}/S})\right)_n \in \lim_{\boldsymbol{\Delta_s}}(\Gamma_X(\mathrm{QCoh}(X \times_S (X_Y^{(n+1)}))), \times)$$

Now, using the isomorphism

$$\Gamma_\Delta(\mathrm{QCoh}(X \times_S X)) \otimes_{\Gamma_\Delta(\mathrm{QCoh}(X \times_Y X))} \mathrm{QCoh}(X) \cong \lim_{\boldsymbol{\Delta_s}}(\Gamma_X(\mathrm{QCoh}(X \times_S (X_Y^{(n+1)}))), \times)$$

which is induced by the compatible system of functors

$$\mathrm{id} \otimes \tilde{\pi}_1^\times : \Gamma_\Delta(\mathrm{QCoh}(X \times_S X)) \otimes_{\Gamma_\Delta(\mathrm{QCoh}(X \times_Y X))} \mathrm{QCoh}(X) \to \Gamma_X(\mathrm{QCoh}(X \times_S (X_Y^{(n+1)})))$$

the object

$$\Gamma_X(\mathcal{O}_X \boxtimes \omega_{X/S}) \otimes_{D_{X/Y}} \mathcal{O}_X \in \Gamma_X(\mathrm{QCoh}(X \times X)) \otimes_{\Gamma_X(\mathrm{QCoh}(X \times_Y X))} \mathrm{QCoh}(X)$$

corresponds to (following a direct calculation)

$$\left(\Gamma_X(\mathcal{O}_X \boxtimes \omega_{X_Y^{(n+1)}/S})\right)_n \in \lim_{\boldsymbol{\Delta_s}}(\Gamma_X(\mathrm{QCoh}(X \times_S (X_Y^{(n+1)}))), \times)$$

and so the proof is complete. ■

Corollary 4.5.5. *Suppose T and Y are spectral schemes over a base spectral affine scheme S, such that the structure maps p_T and p_Y are finite tor-amplitude, locally almost of finite presentation and separated. Let $g : T \to Y$ be proper, affine, and finite tor-amplitude. Then, the diagram below commutes*

$$\begin{tikzcd} {D_{T/S}}^{\mathrm{op}}\text{–Mod} \arrow[r, "g_+"] \arrow[d, "\Phi_{T\to Y}"] & {D_{Y/S}}^{\mathrm{op}}\text{–Mod} \arrow[d, "G_{{D_Y}^{\mathrm{op}}}"] \\ {D_{T/Y}}^{\mathrm{op}}\text{–Mod} \arrow[r, "g_+"] & {D_{Y/Y}}^{\mathrm{op}}\text{–Mod} \end{tikzcd}$$

in the sense the natural map

$$g_+\Phi_{T\to Y} \to g_+\Phi_{T\to Y} g^\dagger g_+ \cong g_+ g^\dagger G_{{D_Y}^{\mathrm{op}}} g_+ \to G_{{D_Y}^{\mathrm{op}}} g_+$$

is an isomorphism (this is an example of a Beck-Chevalley condition) where

$$\Phi_{T\to Y} : D_{T/S}\text{–Mod} \to D_{T/Y}\text{–Mod}$$

can be defined by assemblying the quasicoherent upper cross maps

$$\Gamma_T(\mathrm{QCoh}(T_S^{(n+1)})) \to \Gamma_T(\mathrm{QCoh}(T_Y^{(n+1)}))$$

using the isomorphisms

$$D_{T/S}{}^{\mathrm{op}}\mathrm{-Mod} \cong \lim_{\mathbf{\Delta}_s}(\Gamma_\Delta(\mathrm{QCoh}(T_S^{(n+1)})), \times)$$

and

$$D_{T/Y}{}^{\mathrm{op}}\mathrm{-Mod} \cong \lim_{\mathbf{\Delta}_s}(\Gamma_\Delta(\mathrm{QCoh}(T_Y^{(n+1)})), \times)$$

Note we have abused notation to use g_+ to denote two maps.

Proof. Now let us express the natural map as a morphism of transfer modules, namely,

$$\mathcal{O}_T \otimes_{D_{T/Y}{}^{\mathrm{op}}} D_{T/S}{}^{\mathrm{op}} \to \mathcal{O}_T \otimes_{D_{T/Y}{}^{\mathrm{op}}} D_{T/S}{}^{\mathrm{op}} \otimes_{D_{T/S}{}^{\mathrm{op}}} \Gamma_T(\omega_{T/S} \boxtimes \mathcal{O}_Y) \otimes_{D_{Y/S}{}^{\mathrm{op}}} \Gamma_T(\omega_{Y/S} \boxtimes \mathcal{O}_T)$$

$$\cong \mathcal{O}_T \otimes_{D_{T/Y}{}^{\mathrm{op}}} \omega_{T/Y} \otimes_{\mathcal{O}_Y} D_{Y/S}{}^{\mathrm{op}} \otimes_{D_{Y/S}{}^{\mathrm{op}}} \Gamma_T(\omega_{Y/S} \boxtimes \mathcal{O}_T) \to D_{Y/S}{}^{\mathrm{op}} \otimes_{D_{Y/S}{}^{\mathrm{op}}} \Gamma_T(\omega_{Y/S} \boxtimes \mathcal{O}_T)$$

which boils down to the above proposition.

■

Proposition 4.5.6. *Suppose S is a spectral affine scheme and X and Y are locally almost of finite presentation, finite tor-amplitude affine schemes over S. Also suppose that there is a map $f : X \to Y$ which is finite tor-amplitude. Then, there is a natural isomorphism*

$$\widetilde{(\mathrm{id} \times f)}_* \Big(\Gamma_X(\omega_{X/S} \boxtimes \mathcal{O}_X) \otimes_{D_{X/Y}{}^{\mathrm{op}}} \omega_{X/Y}\Big) \cong \Gamma_X(\omega_{X/S} \boxtimes \mathcal{O}_Y)$$

of objects in $\Gamma_X(\mathrm{QCoh}(X \times_S Y))$ respecting the right D-module structures. Note that the left hand side is a bit of an abuse of notation as before.

Proof. The element

$$\widetilde{(\mathrm{id} \times f)}_* \Big(\Gamma_X(\omega_{X/S} \boxtimes \mathcal{O}_X) \otimes_{D_{X/Y}{}^{\mathrm{op}}} \omega_{X/Y}\Big)$$

can be expressed by the tensor

$$\Gamma_X(\omega_{X/S} \boxtimes \mathcal{O}_X) \otimes_{D_{X/Y}{}^{\mathrm{op}}} \omega_{X/Y} \in \Gamma_X(\mathrm{QCoh}(X \times X)) \otimes_{\Gamma_X(\mathrm{QCoh}(X \times_Y X))} \mathrm{QCoh}(X)$$

and under the isomorphism of Proposition 4.5.2 is given by the compatible system

$$\Big(\Gamma_X(\omega_{X/S} \boxtimes \omega_{X_Y^{(n+1)}/S})\Big)_n \in \lim_{\mathbf{\Delta}_s}(\Gamma_X(\mathrm{QCoh}(X \times_S (X_Y^{(n+1)}))), \times)$$

The right hand side is given by the element

$$\left(\Gamma_X(\omega_{X/S} \boxtimes \mathcal{O}_{X_Y^{(n+1)}/S})\right)_n \in \lim_{\boldsymbol{\Delta_s}}(\Gamma_X(\mathrm{QCoh}(X \times_S (X_Y^{(n+1)}))), *)$$

which also corresponds to the element

$$\left(\Gamma_X(\omega_{X/S} \boxtimes \omega_{X_Y^{(n+1)}/S})\right)_n \in \lim_{\boldsymbol{\Delta_s}}(\Gamma_X(\mathrm{QCoh}(X \times_S (X_Y^{(n+1)}))), \times)$$

under the isomorphism

$$\lim_{\boldsymbol{\Delta_s}}(\Gamma_X(\mathrm{QCoh}(X \times_S (X_Y^{(n+1)}))), *) \cong \lim_{\boldsymbol{\Delta_s}}(\Gamma_X(\mathrm{QCoh}(X \times_S (X_Y^{(n+1)}))), \times)$$

■

Corollary 4.5.7. *Suppose T and Y are spectral affine schemes over a base spectral affine scheme S, such that the structure maps are finite tor-amplitude and locally almost of finite presentation. Let $f : T \to Y$ be proper, affine, and finite tor-amplitude. Then, the diagram below commutes*

$$\begin{array}{ccc} D_{T/S}\text{−Mod} & \xrightarrow{g_\dagger} & D_{Y/S}\text{−Mod} \\ \downarrow{\scriptstyle \Phi_{T\to Y}} & & \downarrow{\scriptstyle G_{D_Y}} \\ D_{T/Y}\text{−Mod} & \xrightarrow{g_\dagger} & D_{Y/Y}\text{−Mod} \end{array}$$

in the sense the natural map

$$g_\dagger \Phi_{T\to Y} \to g_\dagger \Phi_{T\to Y} g^+ g_\dagger \cong g_\dagger g^+ G_{D_Y} g_\dagger \to G_{D_Y} g_\dagger$$

is an isomorphism (this is an example of a Beck-Chevalley condition) where

$$\Phi_{T\to Y} : D_{T/S}\text{−Mod} \to D_{T/Y}\text{−Mod}$$

can be defined by assemblying the quasicoherent pullback maps

$$\Gamma_T(\mathrm{QCoh}(T_S^{(n+1)})) \to \Gamma_T(\mathrm{QCoh}(T_Y^{(n+1)}))$$

using the isomorphisms

$$D_{T/S}\text{−Mod} \cong \lim_{\boldsymbol{\Delta}_s}(\Gamma_\Delta(\mathrm{QCoh}(T_S^{(n+1)})), *)$$

and

$$D_{T/Y}\text{−Mod} \cong \lim_{\boldsymbol{\Delta}_s}(\Gamma_\Delta(\mathrm{QCoh}(T_Y^{(n+1)})), *)$$

Note we have abused notation to use $g_\dagger$ to denote two maps.

Proof. Now let us express the natural map as a morphism of transfer modules, namely,

$$\omega_{T/Y} \otimes_{D_{T/Y}} D_{T/S} \to \omega_{T/Y} \otimes_{D_{T/Y}} D_{T/S} \otimes_{D_{T/S}} \Gamma_T(\mathcal{O}_T \boxtimes \omega_{Y/S}) \otimes_{D_{Y/S}} \Gamma_T(\mathcal{O}_Y \boxtimes \omega_{T/S})$$

$$\cong \omega_{T/Y} \otimes_{D_{T/Y}} \mathcal{O}_T \otimes_{D_{Y/Y}} D_{Y/S} \otimes_{D_{Y/S}} \Gamma_T(\mathcal{O}_Y \boxtimes \omega_{T/S}) \to D_{Y/S} \otimes_{D_{Y/S}} \Gamma_T(\mathcal{O}_Y \boxtimes \omega_{T/S})$$

which boils down to the above proposition. ∎

Remark 4.5.8. *The core of the two corollaries above is the statement that D-module pushforward is in some sense invariant of the choice of the base S.*

Proposition 4.5.9. *Suppose X, Y, and T are spectral affine schemes over a base spectral Noetherian scheme S, such that the structure maps are finite tor-amplitude and locally almost of finite presentation. Let $g : T \to Y$ be proper, affine and finite tor-amplitude and $f : X \to Y$ be a morphism. Then, the diagram below commutes*

$$\begin{tikzcd}
D_{T/S}\text{−Mod} \arrow[r, "g_\dagger"] \arrow[d, "(f')^+"] & D_{Y/S}\text{−Mod} \arrow[d, "f^+"] \\
D_{T\times_Y X/S}\text{−Mod} \arrow[r, "(g')_\dagger"] & D_{X/S}\text{−Mod}
\end{tikzcd}$$

in the sense that the natural map

$$(g')_\dagger(f')^+ \to (g')_\dagger(f')^+g^+g_\dagger \cong (g')_\dagger(g')^+f^+g_\dagger \to f^+g_\dagger$$

is an isomorphism (this is an example of a Beck-Chevalley condition).

Proof. We can check the natural map is an isomorphism after post-composition with

$$G_{D_X} : D_{X/S}\text{−Mod} \to \mathrm{QCoh}(X)$$

since G_{D_X} is conservative. By Corollary 4.5.7 the following diagram commutes

$$\begin{tikzcd}
D_{T\times_Y X/S}\text{−Mod} \arrow[r, "(g')_\dagger"] \arrow[d, "\Phi_{T\times_Y X}"] & D_{X/S}\text{−Mod} \arrow[d, "G_{D_X}"] \\
D_{T\times_Y X/X}\text{−Mod} \arrow[r, "(g')_\dagger"] & D_{X/X}\text{−Mod}
\end{tikzcd}$$

in the sense the natural map

$$(g')_\dagger \Phi_{T\times_Y X} \to (g')_\dagger \Phi_{T\times_Y X}(g')^+(g')_\dagger \cong (g')_\dagger (g')^+ G_{D_X}(g')_\dagger \to G_{D_X}(g')_\dagger$$

is an isomorphism where

$$\Phi_{T\times_Y X} : D_{T\times_Y X/S} \to D_{T\times_Y X/X}$$

is assembled from the quasicoherent pullback maps

$$\Gamma_{T\times_Y X}(\mathrm{QCoh}((T\times_Y X)_S^{(n+1)})) \to \Gamma_{T\times_Y X}(\mathrm{QCoh}((T\times_Y X)_X^{(n+1)}))$$

using the isomorphisms

$$D_{T\times_Y X/S}-\mathrm{Mod} \cong \lim_{\mathbf{\Delta}_s}(\mathrm{QCoh}(\Gamma_{T\times_Y X}(\mathrm{QCoh}((T\times_Y X)_S^{(n+1)}))), *)$$

and

$$D_{T\times_Y X/X}-\mathrm{Mod} \cong \lim_{\mathbf{\Delta}_s}(\mathrm{QCoh}(\Gamma_{T\times_Y X}(\mathrm{QCoh}((T\times_Y X)_X^{(n+1)}))), *)$$

Hence it suffices to show that the following diagram commutes (with the Beck-Chevalley map)

$$\begin{tikzcd}
D_{T/S}-\mathrm{Mod} \arrow[r, "(g')_\dagger"] \arrow[d, "\Phi_{T\times_Y X}(f')^+"] & D_{Y/S}-\mathrm{Mod} \arrow[d, "G_{D_X} f^+"] \\
D_{T\times_Y X/X}-\mathrm{Mod} \arrow[r, "(g')_\dagger"] & D_{X/X}-\mathrm{Mod}
\end{tikzcd}$$

Since the diagram

$$\begin{tikzcd}
D_{T/S}-\mathrm{Mod} \arrow[r, "(g')_\dagger"] \arrow[d, "\Phi_T"] & D_{Y/S}-\mathrm{Mod} \arrow[d, "G_{D_Y}"] \\
D_{T/Y}-\mathrm{Mod} \arrow[r, "(g')_\dagger"] & D_{Y/Y}-\mathrm{Mod}
\end{tikzcd}$$

commutes (with the Beck-Chevalley maps) by Corollary 4.5.7, it suffices to show that the diagram

$$\begin{tikzcd}
D_{T/Y}-\mathrm{Mod} \arrow[r, "(g')_\dagger"] \arrow[d, "(f')^*"] & D_{Y/Y}-\mathrm{Mod} \arrow[d, "f^*"] \\
D_{T\times_Y X/X}-\mathrm{Mod} \arrow[r, "(g')_\dagger"] & D_{X/X}-\mathrm{Mod}
\end{tikzcd}$$

commutes with the Beck-Chevalley maps. This holds since the entire theory base-changes well with respect to the base. More precisely, there are isomorphisms

$$D_{T\times_Y X/X}-\mathrm{Mod} \cong D_{T/Y}-\mathrm{Mod} \otimes_{\mathrm{QCoh}(Y)} \mathrm{QCoh}(X)$$

$$D_{X/X}-\mathrm{Mod} \cong D_{Y/Y}-\mathrm{Mod} \otimes_{\mathrm{QCoh}(Y)} \mathrm{QCoh}(X)$$

and the functor $(g')_\dagger$ on the bottom is the base-change of the functor $(g')_\dagger$ on top.

■

Theorem 4.5.10. *The category of D-modules satisfies descent along maps which are proper, affine, surjective, and finite tor-amplitude.*

Proof. Suppose $f: T \to X$ is locally almost of finite presentation, proper, finite tor-amplitude, and surjective over some base S so that the situation satisfies the standing assumptions. We wish to show the map

$$D_X-\mathrm{Mod} \to \lim_{\mathbf{\Delta}}(D_{T_X^{(n)}}-\mathrm{Mod})$$

is an isomorphism, where the transition maps are D-module pullback ($^+$-pullback) and $T_X^{(n)}$ is the n-fold (derived) cartesian product of T over X. We will apply Corollary 4.7.5.3 of [Lur17]. We need to check three conditions

1. $D_X-\mathrm{Mod}$ admits geometric realizations of f^+-split simplicial objects and those geometric realizations are preserved by f^+.

2. For every morphism $[m] \to [n]$ in $\mathbf{\Delta}_+$, the diagram

$$\begin{array}{ccc} D_{T_X^{(m)}}-\mathrm{Mod} & \xrightarrow{d^0} & D_{T_X^{(m+1)}}-\mathrm{Mod} \\ \downarrow & & \downarrow \\ D_{T_X^{(n)}}-\mathrm{Mod} & \xrightarrow{d^0} & D_{T_X^{(n+1)}}-\mathrm{Mod} \end{array}$$

   is left-adjointable (see [Lur17] 4.7.4.13) where $d^0 : [N] \to [N+1]$ denotes the map which sends k to $k+1$ for $k \in [N]$.

3. f^+ is conservative.

(1) is automatic. (2) is a direct application of Proposition 4.5.9. (3) follows from Lemma 2.1.3 (reducing to the discrete setting) and [BS17] Theorem 11.12 (h-descent with derived Čech nerve).

■

Corollary 4.5.11. *Suppose S is an underived Noetherian scheme, then the category of D-modules on finite-type, finite tor-amplitude S-schemes (relative to S) satisfies fppf-descent.*

Proof. Follows from étale descent (Theorem 4.2.1) and finite-flat descent (Theorem 4.5.10) together with [Sta18] Lemma 0DET. ∎

4.6 Comparison with the De Rham Stack for Truncated Noetherian Schemes

In this section, we discuss the relationship between D-modules as defined in the previous sections and the more classical story of quasicoherent sheaves on the de Rham stack. The latter is the same thing as quasi-coherent crystals on the (big) infinitesimal site. Over characteristic zero, all the results below appear in [GR14].

Suppose S is a truncated Noetherian spectral affine scheme. Let us denote by $AFF^{ft}_{/S}$ the category of all finite-type truncated spectral affine schemes over S (as always we work in the affine setting and globalization follows from descent properties). For any finite-type morphism $X \to Y$ in $AFF^{ft}_{/S}$, we can define

Definition 4.6.1. *The relative de Rham stack $(X/Y)_{dR}$ is the presheaf on $AFF^{ft}_{/S}$ defined by*

$$(X/Y)_{dR}(U) := \mathrm{Hom}(U_{red}, X) \times_{\mathrm{Hom}(U_{red}, Y)} \mathrm{Hom}(U, Y)$$

where U_{red} is the reduced subscheme U and the Hom's are computed in $AFF^{ft}_{/S}$.

In other words, it is the presheaf of maps from U to Y such that on U_{red} the map lifts to X. This presheaf is in fact a sheaf on the Zariski (or étale) topology. A reminder to the reader that we use the terms presheaf/sheaf to mean presheaf/sheaf of spaces, in the sense of [Lur09]. This is also the shriek pushforward of the relative de Rham stack as a sheaf on $AFF^{ft}_{/Y}$ to $AFF^{ft}_{/S}$.

The (contravariant) functor taking an affine scheme to its category of quasicoherent sheaves

$$QCoh : AFF^{ft^{\mathrm{op}}}_{/S} \to \widehat{Cat_\infty}$$

is a sheaf of categories on $AFF^{ft}_{/S}$, with respect to either the Zariski or étale topology. Hence, we can define $QCoh$ for any presheaf on $AFF^{ft}_{/S}$ by

$$QCoh(\mathcal{F}) := \mathrm{Hom}(\mathcal{F}, QCoh)$$

where the Hom is taken in the category of presheaves of categories on $AFF_{/S}$. Alternatively, we can think of this as defining $QCoh$ via Kan extension. We note that this agrees with the definition given in [Sta18] Tag 0H0H, as the difference in the choice of sites does not matter here. As $(X/Y)_{dR}$ is shriek extended from Y, $QCoh((X/Y)_{dR})$ is independent of S.

Lemma 4.6.2. *For any truncated Noetherian $\mathbb{E}_\infty$-ring R, the map*

$$R \to R_{red}$$

is descendable.

Proof. The map $R \to \pi_0(R)$ is descendable by [Mat16] Proposition 3.32. The map $\pi_o(R) \to R_{red}$ is descendable by [Mat16] Proposition 3.33. Hence the composition is descendable by [Mat16] Proposition 3.23. ■

Lemma 4.6.3. *Suppose R is a truncated $\mathbb{E}_\infty$-ring, then the Koszul quotient of R by a sequence $t_1^N, \ldots, t_n^N$ in $\pi_0(R)$ admits the structure of an $\mathbb{E}_\infty$ ring over R for all sufficiently large N.*

Proof. Suppose R is k-truncated, then the Koszul quotients will be uniformly $k+n$-truncated. For $k+n$-truncated connective R-modules, the data of a $\mathbb{E}_{k+n+2}$-algebra lifts uniquely to a $\mathbb{E}_\infty$-algebra by [Lur17] Corollary 5.1.1.7 (because of connectivity estimates of the $\mathbb{E}_m$-operad). Now the claim follows because by [Bur22] Theorem 5.2, the Koszul quotients will admit R-linear $\mathbb{E}_{k+n+2}$-structures for all $N \gg 0$. ■

Proposition 4.6.4. *Suppose $X \to Y$ is a closed immersion between truncated Noetherian spectral affine schemes, then*

$$QCoh((X/Y)_{dR}) \cong \Gamma_X(\mathrm{QCoh}(Y))$$

Proof. Note first that both sides only depend on the reduced part of X, so we may choose X to be finite tor-amplitude over Y (by taking a large enough Koszul quotient using Lemma 4.6.3).

Let's work in the site $AFF^{ft}_{/Y}$ with the descendable topology. Then the map $X \to (X/Y)_{dR}$ is an effective epimorphism because for any truncated Noetherian ring R, the map $R \to R_{red}$ is descendable (Lemma 4.6.2). Hence the left hand side can be written as

$$\lim(\mathrm{QCoh}(X) \rightrightarrows \mathrm{QCoh}(X \times_Y X) \substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow} \ldots)$$

from a direct computation. But the right hand side can also be written in this form by Theorem 4.4.3. ■

Theorem 4.6.5. *Suppose $X \to Y$ is a finite tor-amplitude map in $AFF^{ft}_{/S}$, then there is a natural isomorphism*

$$D_{X/Y}\text{–Mod} \cong QCoh((X/Y)_{dR})$$

Proof. As $X \to (X/Y)_{dR}$ is an effective epimorphism in the descendable topology, we have

$$QCoh((X/Y)_{dR}) = \lim(QCoh(X) \rightrightarrows QCoh((X/X \times_Y X)_{dR}) \overset{\rightarrow}{\rightrightarrows} \dots)$$

From there the claim follows from Proposition 4.6.4 above and the limit presentation for the category of D-modules. ■

4.7 Relation with Hochschild Cohomology

In this section, we discuss a decategorification of Corollary 4.1.4 in the case $X = \operatorname{Spec} A$ is a smooth affine variety over $S = \operatorname{Spec} k$, which we assume to be affine and discrete (concentrated in π_0). Namely, we will show a result of the form

$$D_A \cong A \otimes_H A$$

for H being the E_2 ring of Hochschild cohomology of A, where D_A is the ring of differential operators on $\operatorname{Spec} A$. Corollary 4.1.4 has been known since the work of Beraldo, in [Ber21] and [Ber19], and we are heavily influenced by those works. We will also allow A to be noncommutative in this section, as it will not affect our proofs and may even be helpful psychologically.

Suppose A is an E_1 ring over k (which no longer needs to be concentrated in π_0), which is compact in the category of A-bimodules, $(A \otimes A^{\mathrm{op}})-\mathrm{Mod}$. This is a condition that we have not assumed in the previous sections and is some sort of generalization of smoothness. In fact, it is equivalent to $A-\mathrm{Mod}$ being a smooth category, using Definition 4.5 in [Per19]. The Hochschild cohomology of A over k is the E_2 ring defined by

$$\mathrm{HH}^{\cdot}(A/k) := \mathrm{Hom}_{\mathrm{End}_k(A-\mathrm{Mod})}(\mathrm{id}, \mathrm{id}) \tag{4.14}$$

where $\mathrm{End}_k(A-\mathrm{Mod})$ is the monoidal category of k-linear endomorphisms of $A-\mathrm{Mod}$. Notice that

$$\mathrm{End}_k(A-\mathrm{Mod}) \cong (A \otimes A^{\mathrm{op}})-\mathrm{Mod}$$

and therefore we also have

$$\mathrm{HH}^{\cdot}(A/k) \cong \mathrm{Hom}_{A \otimes A^{\mathrm{op}}}(A, A)$$

although it is harder to see the E_2 structure this way. We establish a convention for the E_2 ring $\mathrm{HH}^{\cdot}(A/k)$. Using equation (4.14), we call the E_1 algebra structure on $\mathrm{HH}^{\cdot}(A/k)$ induced from the monoidal structure of $\mathrm{End}_k(A{-}\mathrm{Mod})$ the horizontal product–μ_1, and the E_1 algebra structure induced from composition of morphisms in $\mathrm{End}_k(A{-}\mathrm{Mod})$ the vertical product–μ_2. For

$$f, g \in \mathrm{HH}^{\cdot}(A/k)$$

$\mu_2(f,g)$ is the composition fg in $\mathrm{Hom}_{\mathrm{End}_k(A-\mathrm{Mod})}(\mathrm{id},\mathrm{id})$ and will be denoted by f above g. These two E_1 structures are compatible and also they are noncanonically isomorphic. In particular we have the following coherence diagram

$$\begin{tikzcd}
\begin{array}{c} \mathrm{HH}^{\cdot}(A/k)\otimes\mathrm{HH}^{\cdot}(A/k) \\ \otimes \qquad \otimes \\ \mathrm{HH}^{\cdot}(A/k)\otimes\mathrm{HH}^{\cdot}(A/k) \end{array} \arrow[r, "\mu_2\otimes\mu_2"] \arrow[d, "\begin{array}{c}\mu_1\\ \otimes\\ \mu_1\end{array}"] & \mathrm{HH}^{\cdot}(A/k)\otimes\mathrm{HH}^{\cdot}(A/k) \arrow[d, "\mu_1"] \\
\begin{array}{c} \mathrm{HH}^{\cdot}(A/k) \\ \otimes \\ \mathrm{HH}^{\cdot}(A/k) \end{array} \arrow[r, "\mu_2"] & \mathrm{HH}^{\cdot}(A/k)
\end{tikzcd}$$

Let us explain the notation. The vertical tensor product mean the same as horizontal tensor, but the author finds it clearer to reserve writing the tensor product vertically when applying the vertical product. The upper left term is just the tensor product of four copies of $\mathrm{HH}^{\cdot}(A/k)$, denoted as a square for the reasons we just mentioned. Normally, for a E_1 ring, we can define left and right modules over it. Because $\mathrm{HH}^{\cdot}(A/k)$ has vertical multiplication, we can also define up and down modules over it similarly. We denote the category of modules of left modules over $\mathrm{HH}^{\cdot}(A/k)$ by

$$\mathrm{HH}^{\cdot}(A/k)^{left}{-}\mathrm{Mod}$$

and similarly for right, up, and down modules. Each of these is a monoidal category where the monoidal structure is taken in an orthogonal direction. In particular left modules (the module is to the right of the ring) have downwards monoidal products, etc.

Let us think of the multiplication in A, μ_A as being horizontal, so that we can form left modules, right modules, and bimodules over A naturally. Then $A\otimes A^{\mathrm{op}}{-}\mathrm{Mod}$, the category of bimodules over A, is naturally a monoidal category by tensoring over A (we think of the monoidal product as happening in the horizontal direction. Let $\Gamma_\Delta((A\otimes A^{\mathrm{op}}){-}\mathrm{Mod})$ denote the subcategory of $(A\otimes_k A^{\mathrm{op}}){-}\mathrm{Mod}$ generated under colimits by A. We can think of $\mathrm{HH}^{\cdot}(A/k)$ as a one object monoidal category where the endomorphisms of that object is $\mathrm{HH}^{\cdot}(A/k)$ with μ_2 product (and μ_1 is responsible for the monoidal structure). Then this monoidal category naturally maps into $\Gamma_\Delta((A\otimes A^{\mathrm{op}}){-}\mathrm{Mod})$ as a map of monoidal categories, basically by definition, where the object maps to A. This induces a map of monoidal categories

$$\Phi : \mathrm{HH}^{\cdot}(A/k)^{down}\text{−Mod} \to \Gamma_{\Delta}(A \otimes A^{\mathrm{op}}\text{−Mod})$$

which is an isomorphism because A is a compact generator whose ring of endomorphisms is $\mathrm{HH}^{\cdot}(A/k)$ with μ_2 product.

For a down $\mathrm{HH}^{\cdot}(A/k)$ module M, Φ sends M to the $A \otimes A^{\mathrm{op}}$ module given by

$$\begin{matrix} M \\ \otimes \\ A \end{matrix} \mathrm{HH}^{\cdot}(A/k)$$

where the A on the bottom has commutating up $\mathrm{HH}^{\cdot}(A/k)$ action and left and right A actions, i.e. a left $A \otimes A^{\mathrm{op}}$ action. This gives an $A \otimes A^{\mathrm{op}}$-module structure on the tensor product. (The vertical tensor is a normal tensor product over the E_1 ring $\mathrm{HH}^{\cdot}(A/k)$ with the μ_2 product). We can think of the monoidal-ness of the functor Φ as follows. First note A is a up $\mathrm{HH}^{\cdot}(A/k)$ algebra, because the evaluation map

$$A \otimes \mathrm{Hom}_{A \otimes A^{\mathrm{op}}}(A, A) \to A$$

(which we think of as an up action because composition of functions in $\mathrm{HH}^{\cdot}\ A/k$ is visualized upwards) is compatible with the horizontal monoidal product–tensoring over A. Therefore, we have coherence diagrams such as

$$\begin{tikzcd}
\begin{smallmatrix} \mathrm{HH}^{\cdot}(A/k) \otimes \mathrm{HH}^{\cdot}(A/k) \\ \otimes \quad \otimes \\ A \otimes A \end{smallmatrix} \arrow[r] \arrow[d, "\begin{smallmatrix} \mu_1 \\ \otimes \\ \mu_A \end{smallmatrix}"] & A \otimes A \arrow[d, "\mu_A"] \\
\begin{smallmatrix} \mathrm{HH}^{\cdot}(A/k) \\ \otimes \\ A \end{smallmatrix} \arrow[r] & A
\end{tikzcd}$$

where the horizontal maps are the structure maps of A as a up $\mathrm{HH}^{\cdot}(A/k)$ module. Now, suppose M and N are both down $\mathrm{HH}^{\cdot}(A/k)$ modules. We can consider the tensor product

$$\Phi(M) \otimes_A \Phi(n) \cong \left(\begin{matrix} M \\ \otimes \\ A \end{matrix} \mathrm{HH}^{\cdot}(A/k) \right) \otimes_A \left(\begin{matrix} N \\ \otimes \\ A \end{matrix} \mathrm{HH}^{\cdot}(A/k) \right)$$

We rewrite this as

$$\left(\begin{smallmatrix} M \\ \otimes \\ A \end{smallmatrix}\mathrm{HH}^{\cdot}(A/k)\right) \otimes_{\left(\mathrm{HH}^{\cdot}(A/k) \underset{A}{\otimes} \mathrm{HH}^{\cdot}(A/k)\right)} \left(\begin{smallmatrix} N \\ \otimes \\ A \end{smallmatrix}\mathrm{HH}^{\cdot}(A/k)\right)$$

We can instead evaluate this tensor horizontally first to get

$$\left(\underset{A}{\overset{M\otimes_{\mathrm{HH}^{\cdot}} N}{\otimes}} \mathrm{HH}^{\cdot}(A/k)\right)$$

which captures the fact that the functor Φ is monoidal. We note that the horizontal actions of $\mathrm{HH}^{\cdot}(A/k)$ on M and N are coming from the monoidal structure on $\mathrm{HH}^{\cdot}(A/k)^{down}-\mathrm{Mod}$.

In the reverse direction, for an $A \otimes A^{\mathrm{op}}$-module N, the down $\mathrm{HH}^{\cdot}(A/k)$ module corresponding to N is

$$\Psi(N) := \mathrm{Hom}_{A\otimes A^{\mathrm{op}}}(A, N)$$

as this is the right adjoint of Φ. Ψ is also monoidal since A is the unit of the monoidal structure on $A \otimes A^{\mathrm{op}}-\mathrm{Mod}$. Pictorially, we can write an element of $\Psi(N)$ as a vertical map

$$\begin{matrix} N \\ \Big\uparrow \\ A \end{matrix}$$

and the monoidal structure of Ψ is seen by tensoring horizontally over A

$$\left(\begin{matrix} N \\ \Big\uparrow \\ A \end{matrix}\right) \otimes \left(\begin{matrix} N' \\ \Big\uparrow \\ A \end{matrix}\right) \to \left(\begin{matrix} N \otimes_A N' \\ \Big\uparrow \\ A \end{matrix}\right) \tag{4.15}$$

In fact there is also a left and right $\mathrm{HH}^{\cdot}(A/k)$ naturally on $\Psi(N)$, because we can tensor (over A) an A-bimodule map from A to N on the left or right with a A-bimodule map from A to A. The left, down and right actions are compatible, in the sense that any of these actions can induce the others by rotating the E_1 structure on $\mathrm{HH}^{\cdot}(A/k)$, assuming that we never cross the direction which makes the action into an up action. We can represent these actions by the following cartoon.

$$\begin{array}{ccccc} A & & N & & A \\ \uparrow & \circlearrowright & \uparrow & \circlearrowleft & \uparrow \\ A & & A & & A \\ & & & & \\ & & \circlearrowleft & & \\ & & & & \\ & & A & & \\ & & \uparrow & & \\ & & A & & \end{array} \tag{4.16}$$

The fact that the drawn actions are compatible follows from the fact that we can fill in more copies of $\mathrm{HH}^{\cdot}(A/k)$ in the lower left and lower right corners, whose actions on their neighboring $\mathrm{HH}^{\cdot}(A/k)$'s is compatible with the actions on $\mathrm{Hom}_k(A, N)$ indicated in the diagram. This makes is clear that the map in (4.15) is compatible with the actions of $\mathrm{HH}^{\cdot}(A/k)$.

Inside $\Gamma_\Delta(A \otimes A^{\mathrm{op}}-\mathrm{Mod})$, there is the natural ring D_A which we've encountered,

$$D_A := \Gamma_\Delta(\mathrm{Hom}_k(A, A))$$

which is here thought of as a ring with horizontal multiplication. D_A is sent to a down $\mathrm{HH}^{\cdot}\ A/k$ algebra by Ψ. To see which, we compute

$$\begin{aligned} \mathrm{Hom}_{A\otimes A^{\mathrm{op}}}(A, D_A) &\cong \mathrm{Hom}_{A\otimes A^{\mathrm{op}}}(A, \Gamma_\Delta(\mathrm{Hom}(A, A))) \\ &\cong \mathrm{Hom}_{A\otimes A^{\mathrm{op}}}(A, \mathrm{Hom}(A, A)) \\ &\cong \mathrm{Hom}_{A\otimes A^{\mathrm{op}}}(A, \mathrm{Hom}_A(A \otimes A, A)) \end{aligned}$$

where the action of A on $A \otimes A$ in the last line is on the left multiplication on the left A. The right $A \otimes A^{\mathrm{op}}$ action on $A \otimes A$ is via acting on the left A on the right and the right A on the left (which is a right A^{op} action), inducing a left $A \otimes A^{\mathrm{op}}$ module structure on $\mathrm{Hom}_{A^{\mathrm{op}}}(A \otimes A^{\mathrm{op}}, A)$. Therefore,

$$\begin{aligned} \mathrm{Hom}_{A\otimes A^{\mathrm{op}}}(A, D_A) &\cong \mathrm{Hom}_{A\otimes A^{\mathrm{op}}}(A, \mathrm{Hom}_A(A \otimes A, A)) \\ &\cong \mathrm{Hom}_A((A \otimes A) \otimes_{A\otimes A^{\mathrm{op}}} A, A) \\ &\cong \mathrm{Hom}_A(A, A) \\ &\cong A^{\mathrm{op}} \end{aligned}$$

where a direct check shows that the algebra structure on the last line is indeed the opposite of the algebra structure on A. We would like to figure out the down $\mathrm{HH}^{\cdot}(A/k)$ action. But before we do that, let's streamline the computation above to just

$$\begin{aligned}
\mathrm{Hom}_{A\otimes A^{\mathrm{op}}}(A, D_A) &\cong \mathrm{Hom}_{A\otimes A^{\mathrm{op}}}(A, \Gamma_\Delta(\mathrm{Hom}(A, A))) \\
&\cong \mathrm{Hom}_{A\otimes A^{\mathrm{op}}}(A, \mathrm{Hom}(A, A)) \\
&\cong \mathrm{Hom}_A(A \otimes_A A, A) \\
&\cong \mathrm{Hom}_A(A, A) \\
&\cong A^{\mathrm{op}}
\end{aligned}$$

where from line two to three, we think of the isomorphism as the application of a single "enriched" tensor-hom adjunction with the (k, A) bimodule A, both actions are on the left, where the left A actions on both sides of the Hom just come along for the ride. Tensor-hom adjunction in this form is probably well-known, but one can think of the computations above as justification for this "enriched" tensor-hom adjunction as well. Let us draw a picture of this isomorphism.

$$\begin{array}{c} \mathrm{Hom}_k(A \leftarrow {\color{red}A}) \\ {\scriptstyle A}\big\uparrow{\scriptstyle A} \\ A \end{array} \quad \mapsto \quad \begin{array}{c} A \\ {\scriptstyle A}\big\uparrow \\ A \otimes_A {\color{red}A} \end{array}$$

where the left arrow is labeled on both sides to indicate that it is required to be (A, A)-bilinear whereas the right diagram only requires that the map is left A-linear. From this diagram it is clear that the left $\mathrm{HH}^{\cdot}(A/k)$ action will be the most convenient to work with, because it is unfazed by the tensor-hom adjunction. Namely, it is simply the action

$$\begin{array}{c} A \\ {\scriptstyle A}\big\uparrow{\scriptstyle A} \\ A \end{array} \circlearrowleft \begin{array}{c} A \\ {\scriptstyle A}\big\uparrow \\ A \end{array} \tag{4.17}$$

Our diagram therefore shows the left $\mathrm{HH}^{\cdot}\ A/k$ action, in fact it shows a left $\mathrm{HH}^{\cdot}\ A/k$ algebra structure. We would like to drag it to a down $\mathrm{HH}^{\cdot}\ A/k$ algebra and describe it. First, let us start with the standard action of $\mathrm{HH}^{\cdot}(A/k)$ on A, namely A as an up $\mathrm{HH}^{\cdot}(A/k)$ algebra. We can visualize it like so

$$\begin{array}{c} A \\ {}_{A}\big\uparrow_{A} \\ A \\ \circlearrowright \\ A \\ {}_{A}\big\uparrow_{A} \\ A \otimes_k A \end{array}$$

By writing it this way, we see that indeed there are compatible actions, as in the diagram (4.16)

$$\begin{array}{ccccc} & & A & & \\ & & {}_{A}\big\uparrow_{A} & & \\ & & A & & \\ & & \circlearrowright & & \\ A & & A & & A \\ {}_{A}\big\uparrow_{A} & \circlearrowleft & {}_{A}\big\uparrow_{A} & \circlearrowright & {}_{A}\big\uparrow_{A} \\ A & & A \otimes_k A & & A \end{array}$$

Therefore, we can deduce that the action in diagram (4.17) is the standard up $\mathrm{HH}^{\cdot}(A/k)$ algebra A rotated by 90° counterclockwise. We visualize $\mathrm{HH}^{\cdot}(A/k)$ staying still and the module rotating around it. To get to the down $\mathrm{HH}^{\cdot}$ A/k algebra, we further rotate by 90° counterclockwise. Therefore, in total we have

$$\Psi(D_A) \cong \mathrm{Hom}_{A \otimes A^{\mathrm{op}}}(A, D_A) \cong A_{180^\circ} \tag{4.18}$$

meaning that we drag the standard up $\mathrm{HH}^{\cdot}(A/k)$ algebra 180° degrees counterclockwise to obtain a down $\mathrm{HH}^{\cdot}(A/k)$ algebra. Note that doing this naturally reverses the order of multiplication on the ring, making the underlying ring A^{op}. We note that the order of the dragging matters, and we do not even get the same underlying module if we drag in the opposite direction. Using the inverse functor to Ψ, we have

$$D_A \cong \overset{A_{180^\circ}}{\underset{A}{\otimes}} \mathrm{HH}^{\cdot}(A/k) \tag{4.19}$$

We can also define the opposite ring ${D_A}^{\mathrm{op}}$, and by rotating equation (4.19) by 180° clockwise, we can see that

$${D_A}^{\mathrm{op}} \cong \overset{A_{-180^\circ}}{\underset{A}{\otimes}} \mathrm{HH}^{\cdot}(A/k)$$

Since in general D_A and ${D_A}^{\mathrm{op}}$ are not canonically isomorphic even as A-bimodules, we must conclude that dragging A as a down $\mathrm{HH}^{\cdot}(A/k)$ module counterclockwise by one full rotation should genuinely yields a different $\mathrm{HH}^{\cdot}(A/k)$ module in general.

Denote by A_{90° the left $\mathrm{HH}^{\cdot}(A/k)$ algebra and A_{-90° the right $\mathrm{HH}^{\cdot}(A/k)$ algebra obtained by draggin the standard up $\mathrm{HH}^{\cdot}(A/k)$ algebra by the corresponding angles. Then, by rotating the isomorphism (4.19) above by 90° clockwise, we get

$$D_{A,-90^\circ} \cong A_{-90^\circ} \otimes_{\mathrm{HH}^{\cdot}(A/k)} A_{90^\circ}$$

We can categorify the above to get

$${D_A}^{\mathrm{op}}{-}\mathrm{Mod} \cong A^{\mathrm{op}}{-}\mathrm{Mod} \otimes_{\mathrm{HH}^{\cdot}(A/k)^{down}{-}\mathrm{Mod}} A{-}\mathrm{Mod}$$

which was indeed what we intended to decategorify.

CHAPTER 5

Applications

5.1 Universal Homeomorphisms and Relation with [BZN04]

In this section, we discuss an analogous result to Kashiwara's Equivalence for universal homeomorphisms and describe an application of our work to recover some results of [BZN04].

Let S be a truncated Noetherian affine scheme and suppose that there is a universal homeomorphism (on the classical truncation) $\tau : \tilde{X} \to X$ of truncated affine schemes which are finite-type (which implies almost of finite presentation in the Noetherian setting) and finite tor-amplitude over S. By descent, we can clearly globalize X and S (as universal homeomorphisms are always affine).

Let us denote by $\tilde{X}^{(m+1)_X}$ the $(m+1)$-fold (derived) product of $\tilde{X}$ over X. Without the subscript the product will be implied to be over S. We'll need the following lemma.

Lemma 5.1.1. *Any map of truncated Noetherian $\mathbb{E}_\infty$-rings which is a h-cover on π_0 is descendable.*

Proof. Suppose $R \to S$ is such a map. Then $R_{red} \to S_{red}$ is descendable by [BS17] Theorem 11.12 and $R \to R_{red}$ is descendable by the results of [Mat16] (see Lemma 4.6.2). Therefore the result follows from [Mat16] Proposition 3.23. ■

Lemma 5.1.2. *For any $n \geq 0$,*

$$\lim_{[m]\in\mathbf{\Delta}}(\Gamma_\Delta(\mathrm{QCoh}((\tilde{X}^{(m+1)_X})^{(n+1)})), *) \cong \Gamma_\Delta(\mathrm{QCoh}(X^{(n+1)}))$$

Proof. By Lemma 5.1.1 above, we have

$$\lim_{[m]\in\mathbf{\Delta}}(\mathrm{QCoh}((\tilde{X}^{(m+1)_X})^{(n+1)}), *) \cong \mathrm{QCoh}(X^{(n+1)})$$

via the cover $\tilde{X}^{(n+1)} \to X^{(n+1)}$. Now the result follows after tensoring with $\Gamma_\Delta(\mathrm{QCoh}(X^{(n+1)}))$ (which commutes with limits in $\mathrm{QCoh}(X^{(n+1)})-\mathrm{Mod}^L$ because it is dualizable). ■

Theorem 5.1.3. *The functor*

$$\tau^+ : D_X-\mathrm{Mod} \to D_{\tilde{X}}-\mathrm{Mod}$$

is an equivalence of categories.

Proof. We first exhibit a functor in the reverse direction. We start from the isomorphism

$$D_{\tilde{X}}-\mathrm{Mod} \cong \lim_{\mathbf{\Delta}_s}(\Gamma_\Delta(\mathrm{QCoh}(\tilde{X}^{n+1})), *) \tag{5.1}$$

Now, we have a functor

$$\times : \mathbf{\Delta}_s \times \mathbf{\Delta}_s \to \mathbf{\Delta}_s$$

such that

$$\times([m],[n]) = [m] \times [n] \cong [mn+m+n]$$

where we order $[m] \times [n]$ by lexicographic ordering. Hence, we have a functor

$$\lim_{\mathbf{\Delta}_s}(\Gamma_\Delta(\mathrm{QCoh}(\tilde{X}^{n+1})), *) \to \lim_{\mathbf{\Delta}_s \times \mathbf{\Delta}_s}(\Gamma_\Delta(\mathrm{QCoh}(\tilde{X}^{(m+1)(n+1)})), *) \tag{5.2}$$

Because of the natural map

$$\tilde{X}^{(m+1)_X} \to \tilde{X}^{(m+1)}$$

where the product on the right hand side is over S, there is a natural pull-back map

$$\mathrm{QCoh}(\tilde{X}^{(m+1)(n+1)}) \to \mathrm{QCoh}((\tilde{X}^{(m+1)_X})^{(n+1)})$$

This induces a functor

$$\lim_{\mathbf{\Delta}\times\mathbf{\Delta}}(\Gamma_\Delta(\mathrm{QCoh}(\tilde{X}^{(m+1)(n+1)})), *) \to \lim_{\mathbf{\Delta}\times\mathbf{\Delta}}(\Gamma_\Delta(\mathrm{QCoh}((\tilde{X}^{(m+1)_X})^{(n+1)})), *) \tag{5.3}$$

Finally, by Lemma 5.1.2, we have

$$\lim_{\mathbf{\Delta}\times\mathbf{\Delta}}(\mathrm{QCoh}((\tilde{X}^{(m+1)_X})^{(n+1)}), *) \cong \lim_{\mathbf{\Delta}}(\mathrm{QCoh}(X^{(n+1)}), *) \tag{5.4}$$

Combining (5.1), (5.2), (5.3), and (5.4), we can construct a functor

$$\tau_- : D_{\tilde{X}}-\mathrm{Mod} \to D_X-\mathrm{Mod} \tag{5.5}$$

Now we can check that $\tau_- \tau^+ \cong \mathrm{id}$ and $\tau^+ \tau_- \cong \mathrm{id}$ by computing transfer modules of the composites (by computing the image of D_X and $D_{\tilde{X}}$ respectively). ■

Remark 5.1.4. *The transfer module of τ_- is $\Gamma_\Delta(\mathcal{O}_X \boxtimes \omega_{\tilde{X}})$ and the transfer module of τ^+ is $\Gamma_\Delta(\mathcal{O}_{\tilde{X}} \boxtimes \omega_X)$.*

By left-right duality, we also have

Corollary 5.1.5.

$$z_+ : {D_{\tilde{X}}}^{\mathrm{op}}\mathrm{-Mod} \to {D_X}^{\mathrm{op}}\mathrm{-Mod}$$

is an equivalence of categories.

To compare our results with those of [BZN04], let us recall their setup. Assuming for the rest of this section that $S = \operatorname{Spec} k$ where k is a field, X and $\tilde{X}$ are Cohen-Macaulay k-varieties of dimension d, and finally that

$$H^1(\Gamma_\Delta(M \boxtimes \omega_{\tilde{X}})) = 0$$

and

$$H^1(\Gamma_\Delta(M \boxtimes \omega_X)) = 0$$

for all $M \in \mathrm{QCoh}(X)^{[0,0]}$, so that τ is a *good* cuspidal quotient between *good* Cohen-Macaulay varieties in the terminology of *loc. cit.*

Lemma 5.1.6. *In the above situation,*

$$H^i(\Gamma_\Delta(M \boxtimes \omega_{\tilde{X}})) = 0$$

and

$$H^i(\Gamma_\Delta(M \boxtimes \omega_X)) = 0$$

for all $i \neq 0$ and $M \in \mathrm{QCoh}(X)^{[0,0]}$

Proof. Without loss of generality, we can assume that X and $\tilde{X}$ are affine. Namely, $X = \operatorname{Spec} R$ and $\tilde{X} = \operatorname{Spec} \tilde{R}$. Let $\pi_1 : X \times \tilde{X} \to X$ be the projection to the first component. Then, there is an isomorphism (by Theorem 3.2.16)

$$\Gamma_\Delta(M \boxtimes \omega_{\tilde{X}}) \cong \tilde{\pi}_1^{\times} M$$

We can rewrite this as

$$\operatorname{colim}_n \operatorname{Hom}_{R \otimes_k \tilde{R}}((R \otimes_k \tilde{R})/I^n, \operatorname{Hom}_R(R \otimes_k \tilde{R}, M)) \cong \operatorname{colim}_n \operatorname{Hom}_R((R \otimes_k \tilde{R})/I^n, M)$$

where I is the kernel of the surjection $R \otimes_k \tilde{R} \to \tilde{R}$. Hence, we can see that for injective (discrete) M, $\Gamma_\Delta(M \boxtimes \omega_{\tilde{X}})$ is discrete. Using the assumptions, we can then conclude using injective resolutions that for all discrete M, $\Gamma_\Delta(M \boxtimes \omega_{\tilde{X}})$ is discrete. The second claim follows similarly. ∎

Theorem 5.1.7 (Theorem 1.2 in [BZN04]). *In the above situation, there is a Morita equivalence between the (sheaf of) algebras $H^0(D_{\tilde{X}})$ and $H^0(D_X)$ induced by*

$$H^0(D_{\tilde{X}\to X}) \cong H^0(\Gamma_\Delta(\mathcal{O}_{\tilde{X}} \boxtimes \omega_X))$$

and

$$H^0(D_{\tilde{X}\leftarrow X}) := H^0(\Gamma_\Delta(\mathcal{O}_X \boxtimes \omega_{\tilde{X}}))$$

Proof. Without the H^0's, this is simply Theorem 5.1.3. Hence, it suffices to show all the H^0's above are redundant, because the objects are already in degree 0 under our assumptions. But this follows from the Grothendieck-Sato formula (Corollary 3.4.8) and the above lemma. ∎

APPENDIX A

Background in Category Theory

A.1 Tensor Products of Module Categories

In this section we collect some results in category theory which we will use throughout the document.

We denote by Pr^L_{St} the 2-category of presentable stable categories with colimit preserving functors. By section 4.8.3 of [Lur17], there is a tensor product on Pr^L_{St}. Therefore, let $\mathscr{V}$ be a monoidal presentable stable category, $\mathscr{X}$ a right $\mathscr{V}$ module and $\mathscr{Y}$ a left $\mathscr{V}$ module (inside Pr^L_{St}). Then, using section 4.4 of [Lur17], we can form the relative tensor product of $\mathscr{X}$ and $\mathscr{Y}$ over $\mathscr{V}$, namely,

$$\mathscr{X} \otimes_{\mathscr{V}} \mathscr{Y}$$

We record two basic properties of this tensor product here for easy use later

Theorem A.1.1. *Suppose we have a functor $f : \mathscr{X} \to \mathscr{V}$ which is right $\mathscr{V}$-linear and colimit-preserving and $g : \mathscr{V} \to \mathscr{Y}$ which is left $\mathscr{V}$-linear and colimit-preserving. Then, the following diagram commutes*

$$\begin{tikzcd}
\mathscr{X} \cong \mathscr{X} \otimes_{\mathscr{V}} \mathscr{V} \arrow[r, "1\otimes g"] \arrow[d, "f"] & \mathscr{X} \otimes_{\mathscr{V}} \mathscr{Y} \arrow[d, "f\otimes 1"] \\
\mathscr{V} \arrow[r, "g"] & \mathscr{Y} \cong \mathscr{V} \otimes_{\mathscr{V}} \mathscr{Y}
\end{tikzcd}$$

Proof. Follows from functoriality of the relative tensor product. ■

Remark A.1.2. *We must caution the reader that when we write a functor is $\mathscr{V}$-linear and colimit-preserving, we mean that it is a map of $\mathscr{V}$ module categories as defined above. Being*

$\mathscr{V}$-linear refers to preserving tensoring by objects of $\mathscr{V}$, and does not mean $\mathscr{V}$-enriched for us.

Remark A.1.3. *We can think of the above theorem as a category-theoretic analogue of the base-change isomorphism in algebraic geometry.*

Theorem A.1.4. *Suppose $f : \mathscr{V} \to \mathscr{X}$ is right $\mathscr{V}$-linear and colimit-preserving and $g : \mathscr{V} \to \mathscr{Y}$ is left $\mathscr{V}$-linear and colimit-preserving. Then, the following diagram commutes*

$$\begin{tikzcd}
\mathscr{V} \arrow[r, "g"] \arrow[d, "f"] & \mathscr{Y} \cong \mathscr{V} \otimes_{\mathscr{V}} \mathscr{Y} \arrow[d, "f\otimes 1"] \\
\mathscr{X} \cong \mathscr{X} \otimes_{\mathscr{V}} \mathscr{V} \arrow[r, "1\otimes g"] & \mathscr{X} \otimes_{\mathscr{V}} \mathscr{Y}
\end{tikzcd}$$

Proof. Follows from functoriality of the relative tensor product. ■

Theorem A.1.5. *Suppose $f : \mathscr{X} \to \mathscr{V}$ is right $\mathscr{V}$-linear and colimit-preserving and $g : \mathscr{Y} \to \mathscr{V}$ is left $\mathscr{V}$-linear and colimit-preserving. Then, the following diagram commutes*

$$\begin{tikzcd}
\mathscr{X} \otimes_{\mathscr{V}} \mathscr{Y} \arrow[r, "f\otimes 1"] \arrow[d, "1\otimes g"] & \mathscr{Y} \cong \mathscr{V} \otimes_{\mathscr{V}} \mathscr{Y} \arrow[d, "g"] \\
\mathscr{X} \cong \mathscr{X} \otimes_{\mathscr{V}} \mathscr{V} \arrow[r, "f"] & \mathscr{V}
\end{tikzcd}$$

Proof. Follows from functoriality of the relative tensor product. ■

Now let $\mathscr{V}$ be a symmetric monoidal compactly generated stable category, such that the compact objects are the same as the dualizable objects. This will happen whenever $\mathscr{V}$ is the category of quasicoherent sheaves on a qcqs spectral algebraic space by Proposition 6.2.6.2 of [Lur18]. Let us consider the 2-category of presentable stable category $\mathscr{X}$ with a colimit-preserving left action of $\mathscr{V}$ where the morphisms are colimit preserving functors which preserve the $\mathscr{V}$ action. We denote this 2-category by $\mathscr{V}-\mathrm{Mod}^L$ and we call its objects $\mathscr{V}$-module categories. In this setting, there are enriched forms of adjoint functor theorems.

Theorem A.1.6. *Let $\mathscr{X}$ and $\mathscr{Y}$ be compactly generated $\mathscr{V}$-module categories. Suppose $f : \mathscr{X} \to \mathscr{Y}$ is a colimit-preserving functor between $\mathscr{V}$-module categories which preserves* *compact objects, then the right adjoint g can be upgraded to a colimit-preserving $\mathscr{V}$-linear functor. (Please note that the assumptions on $\mathscr{V}$ in this theorem are stronger than the beginning of the section, see the previous paragraph)*

Proof. See A.3.6 in [MGS21] ■

Remark A.1.7. *We can think of the above as a category-theoretic analogue of the projection formula in algebraic geometry.*

Theorem A.1.8. *Suppose $g : \mathscr{X} \to \mathscr{Y}$ is a colimit-preserving functor between $\mathscr{V}$-module categories which preserves limits, then the left adjoint f can be upgraded to a $\mathscr{V}$-linear functor. (Please note that the assumptions on $\mathscr{V}$ in this theorem are stronger than the beginning of the section, see the paragraph before Theorem A.1.6)*

Proof. See A.3.6 in [MGS21] ■

A.2 Exact Sequences of Categories

In this section, we record what it means for a sequence of categories to be exact or split-exact. We use [BGT13] as our reference. Note that our definition for split-exactness differs from theirs. In particular we require the right adjoints to commute with colimits.

Definition A.2.1 (Definition 5.4 of [BGT13])**.** *Let $f : \mathscr{A} \to \mathscr{B}$ be a fully faithful functor of presentable stable categories (this implies that f preserves colimits). The Verdier quotient $\mathscr{B}/\mathscr{A}$ of $\mathscr{B}$ by $\mathscr{A}$ is the cofiber of f in the category Pr^L_{St} of presentable stable categories.*

Definition A.2.2 (Definition 5.8 of [BGT13])**.** *A sequence of presentable stable categories*

$$\mathscr{A} \to \mathscr{B} \to \mathscr{C}$$

is exact if the composite is trivial, $\mathscr{A} \to \mathscr{B}$ is fully faithful, and the map $\mathscr{B}/\mathscr{A} \to \mathscr{C}$ is an equivalence.

Definition A.2.3. *An exact sequence of presentable stable categories*

$$\mathscr{A} \xrightarrow{f} \mathscr{B} \xrightarrow{g} \mathscr{C}$$

is split-exact if the are colimit-preserving right adjoints i and j (to f and g respectively) such that $i \circ f = \mathrm{id}$ and $g \circ j = \mathrm{id}$.

Remark A.2.4. *The reader is warned that this definition differs from that of [BGT13] Definition 5.18.*

Lemma A.2.5. *Suppose*

$$\mathscr{A} \xrightarrow{f} \mathscr{B} \xrightarrow{g} \mathscr{C}$$

is a split-exact sequence of presentable stable categories. Let i and j be the right adjoints of f and g respectively. Then, j is fully faithful and for $\mathcal{M} \in \mathscr{B}$

$$fi\mathcal{M} \to \mathcal{M} \to jg\mathcal{M}$$

is exact in $\mathscr{B}$.

Proof. $g \circ j = \mathrm{id}$ implies j is fully faithful. Consider the fibre K of $\mathcal{M} \to jg\mathcal{M}$. It is easy to see that it is in the kernel of g, and hence the image of f. Hence we have $K \cong fiK$ and therefore K is the fibre of $fi\mathcal{M} \to fijg\mathcal{M}$. Since $ij = 0$, we conclude $K \cong fi\mathcal{M}$. ■

In the reverse direction, we have

Lemma A.2.6. *Suppose*

$$\mathscr{A} \xrightarrow{f} \mathscr{B} \xrightarrow{g} \mathscr{C} \tag{A.1}$$

is a sequence of presentable stable categories which composes to zero, where f and g are colimit-preserving. Let i and j be the right adjoints of f and g respectively. If i and j also preserve colimits and $i \circ f = \mathrm{id}$ and $g \circ j = \mathrm{id}$ and for any $\mathcal{M} \in \mathscr{B}$, the sequence

$$fi\mathcal{M} \to \mathcal{M} \to jg\mathcal{M}$$

is exact in $\mathscr{B}$, then the sequence (A.1) is split-exact.

Proof. $i \circ f = \mathrm{id}$ and $g \circ j = \mathrm{id}$ guarantees that f and j are fully-faithful. It remains to check that

$$\mathscr{B}/\mathscr{A} \to \mathscr{C}$$

is an equivalence. Suppose

$$H : \mathscr{B} \to \mathscr{D}$$

is a colimit-preserving functor which vanishes on $\mathscr{A}$. Then, using the sequence

$$fi\mathcal{M} \to \mathcal{M} \to jg\mathcal{M}$$

one can easily check that

$$H \cong Hjg$$

and hence there is a unique functor $F : \mathscr{C} \to \mathscr{D}$, namely $F \cong Hj$, such that H factors as the projection functor g composed with F. Therefore $\mathscr{C}$ is the desired cofibre. ■

We note that in [HSS17], this is taken as definition for a split-exact sequence.

Remark A.2.7. *The above lemma is very useful as it provides a purely 2-categorical way to check if a sequence is split-exact. For example, it implies that tensoring a split-exact sequence of module categories with another module category gives another split-exact sequence. More generally, after defining the notion of split-exactness in a purely 2-categorical way using the above lemma, it will be preserved under functors between presentable 2-categories, in the language of [Ste20]. We may also sometimes refer to split-exact sequences as Bousfield localization sequences.*

A.3 Dualizability and Monads

In this section, we give sufficent conditions for the category of modules of a colimit preserving monad is dualizable. Let $\mathscr{V}$ be a symmetric monoidal presentable stable category. Let $\mathscr{X}$ be a dualizable category in $\mathscr{V}-\mathrm{Mod}^L$ and

$$T : \mathscr{X} \to \mathscr{X}$$

be a colimit-preserving $\mathscr{V}$-linear monad on $\mathscr{X}$.

Theorem A.3.1. *The functor which takes the pair$(\mathscr{X}, T)$ (of a dualizable $\mathscr{V}$-module and a colimit preserving monad on it) to the category*

$$T-\mathrm{Mod}(\mathscr{X})$$

is symmetric monoidal.

Proof. As in [RV16] and known in some form since [SS86], colimit preserving monads in $\mathscr{V}-\mathrm{Mod}^L$ are given by 2-functors

$$\mathfrak{mnd} \to \mathscr{V}-\mathrm{Mod}^L$$

There is another 2-category $\mathfrak{adj}$, such that 2-functors

$$\mathfrak{adj} \to \mathscr{V}-\mathrm{Mod}^L$$

classify adjunctions. Therefore, as $\mathscr{V}-\mathrm{Mod}^L$ is a symmetric monoidal 2-category, it induces a symmetric monoidal product on monads and adjunctions in $\mathscr{V}-\mathrm{Mod}^L$. Now because of the inclusion

$$\mathfrak{mnd} \to \mathfrak{adj}$$

there is a natural symmetric monoidal functor which associates to an adjunction a monad

$$\mathrm{Hom}(\mathfrak{adj}, \mathscr{V}-\mathrm{Mod}^L) \to \mathrm{Hom}(\mathfrak{mnd}, \mathscr{V}-\mathrm{Mod}^L)$$

This functor has a lax symmetric monoidal right adjoint

$$\mathrm{Hom}(\mathfrak{mnd}, \mathscr{V}-\mathrm{Mod}^L) \to \mathrm{Hom}(\mathfrak{adj}, \mathscr{V}-\mathrm{Mod}^L)$$

which associates to a monad its category of modules (see also Remark 5.7 in [Hau21]). This is the functor we wish to show is symmetric monoidal.

It is obvious the functor preserves units. As there is clearly a map

$$\bigotimes T_i-\mathrm{Mod}(\mathscr{X}_i) \to \left(\bigotimes T_i\right)-\mathrm{Mod}(\bigotimes \mathscr{X}_i)$$

coming from the fact that the functor is lax symmetric monoidal, it suffices to show this map is an isomorphism. By induction we reduce to showing

$$T_1-\mathrm{Mod}(\mathscr{X}_1) \otimes_{\mathscr{V}} T_2-\mathrm{Mod}(\mathscr{X}_2) \xrightarrow{\cong} (T_1 \otimes T_2)-\mathrm{Mod}(\mathscr{X}_1 \otimes_{\mathscr{V}} \mathscr{X}_2)$$

This can be shown by Lurie-Barr-Beck (Theorem 4.7.3.5 in [Lur17]) if we can show that the functor

$$G_1 \otimes G_2 : T_1-\mathrm{Mod}(\mathscr{X}_1) \otimes_{\mathscr{V}} T_2-\mathrm{Mod}(\mathscr{X}_2) \to \mathscr{X}_1 \otimes_{\mathscr{V}} \mathscr{X}_2$$

(where the G_i's are the forgetful functors) is conservative. By Theorem 4.8.4.6 in [Lur17], we have

$$T_1-\mathrm{Mod}(\mathscr{X}_1) \cong T_1-\mathrm{Mod}(\mathrm{Hom}_{\mathscr{V}}(\mathscr{X}_1, \mathscr{X}_1)) \otimes_{\mathrm{Hom}_{\mathscr{V}}(\mathscr{X}_1, \mathscr{X}_1)} \mathscr{X}_1$$

Hence, we have the isomorphism (using Theorem 4.8.5.16 of [Lur17])

$$T_1-\mathrm{Mod}(\mathscr{X}_1) \otimes_{\mathscr{V}} T_2-\mathrm{Mod}(\mathscr{X}_2) \cong (T_1 \otimes \mathrm{id})-\mathrm{Mod}(\mathscr{X}_1 \otimes_{\mathscr{V}} T_2-\mathrm{Mod}(\mathscr{X}_2))$$

So it suffices to check that the functor

$$\mathscr{X}_1 \otimes_{\mathscr{V}} T_2-\mathrm{Mod}(\mathscr{X}_2) \to \mathscr{X}_1 \otimes_{\mathscr{V}} \mathscr{X}_2$$

is conservative. But here we can apply the same argument again [1]. ■

Corollary A.3.2. *If $\mathscr{X}$ is a dualizable $\mathscr{V}$-module category, then for any T a $\mathscr{V}$-linear colimit preserving monad on $\mathscr{X}$,*

$$T\mathrm{-Mod}(\mathscr{X})$$

is dualizable with dual

$$T^\vee\mathrm{-Mod}(\mathscr{X}^\vee)$$

Proof. As T is a colimit preserving $\mathscr{V}$-linear monad on $\mathscr{X}$, we can write T as

$$T \in \mathrm{Hom}_{\mathscr{V}}(\mathscr{X}, \mathscr{X}) \cong \mathscr{X}^\vee \otimes_{\mathscr{V}} \mathscr{X}$$

Clearly T is a (T, T)-bimodule. Equivalently, T is a $(T \otimes T^\vee)$-module, and hence we can write

$$T \in (T \otimes T^\vee)\mathrm{-Mod}(\mathscr{X}^\vee \otimes \mathscr{X}) \cong T^\vee\mathrm{-Mod}(\mathscr{X}^\vee) \otimes_{\mathscr{V}} T\mathrm{-Mod}(\mathscr{X})$$

This defines a map

$$T : \mathscr{V} \to T^\vee\mathrm{-Mod}(\mathscr{X}^\vee) \otimes_{\mathscr{V}} T\mathrm{-Mod}(\mathscr{X})$$

Now, by Theorem 4.8.4.6 in [Lur17], we have the isomorphism

$$T^\vee\mathrm{-Mod}(\mathscr{X}^\vee) \cong T^\vee\mathrm{-Mod}(\mathrm{Hom}_{\mathscr{V}}(\mathscr{X}^\vee, \mathscr{X}^\vee)) \otimes_{\mathrm{Hom}_{\mathscr{V}}(\mathscr{X}^\vee, \mathscr{X}^\vee)} \mathscr{X}^\vee$$

However,

$$\mathrm{Hom}_{\mathscr{V}}(\mathscr{X}^\vee, \mathscr{X}^\vee) \cong \mathscr{X} \otimes \mathscr{X}^\vee \cong \mathrm{Hom}_{\mathscr{V}}(\mathscr{X}, \mathscr{X})$$

is an isomorphism of categories which reverses the monoidal structure and identifies $T^\vee$ with T. Therefore, we also have the isomorphism

$$T^\vee\mathrm{-Mod}(\mathscr{X}^\vee) \cong \mathscr{X}^\vee \otimes_{\mathrm{Hom}_{\mathscr{V}}(\mathscr{X}, \mathscr{X})} T\mathrm{-RMod}(\mathrm{Hom}_{\mathscr{V}}(\mathscr{X}, \mathscr{X})) \cong T\mathrm{-RMod}(\mathscr{X}^\vee)$$

so it is isomorphic to the category of right modules over the monad T on $\mathscr{X}^\vee$ ($T\mathrm{-RMod}$ here means right T modules). Hence, there is a map, coming from tensor product over the monad T,

$$\otimes_T : T^\vee\mathrm{-Mod}(\mathscr{X}^\vee) \otimes_{\mathscr{V}} T\mathrm{-Mod}(\mathscr{X}) \to \mathscr{V}$$

By a standard argument these form unit and counit maps, witnessing the dualizability of $T\mathrm{-Mod}(\mathscr{X})$. ■

[1] This argument is adapted from the proof of Theorem 4.8.5.16 in [Lur17]

Corollary A.3.3. *Suppose*

$$F_T : \mathscr{X} \to T\mathrm{-Mod}(\mathscr{X})$$

is the free T-module functor and

$$G_T : T\mathrm{-Mod}(\mathscr{X}) \to \mathscr{X}$$

is the forgetful functor. Then

$$(F_T)^\vee \cong G_{T^\vee}$$

and

$$(G_T)^\vee \cong F_{T^\vee}$$

Proof. Direct calculation from the unit and counit maps above. ∎

APPENDIX B

Cross-Descent

In this section, we explain how to endow every stack with a category of $\times$-quasicoherent sheaves. This is analogous to the standard definition of quasicoherent sheaves but using $\times$-pullback instead of $*$-pullback.

By [Lur18] Proposition 6.2.4.1, we know that the quasicoherent sheaves is a fpqc sheaf on the site of affine spectral schemes corresponding to a spectral Deligne-Mumford stack agrees with the category of quasicoherent sheaves on its functor of functor of points (defined via Kan extension, see [Lur18] Definition 2.2.2.1. However, it is also possible to define the $\times-$quasicoherent sheaves on a presheaf $\mathscr{X}$ by right Kan extending from spectral affine schemes the category of quasicoherent sheaves and $\times$-pullback. Recall that $\times$-pullback refers to taking the right adjoint to pushforward instead of the left adjoint. There are some set-theoretic issues which will not be important in most geometric applications.

The following proposition is mentioned in Clausen-Scholze's video lectures on Analytic Stacks.

Proposition B.0.1. *Quasicoherent sheaves with $\times$-pullback admit descent along descendable morphisms on spectral affine schemes (see [Mat16] for the definition of a descendable morphism).*

Proof. We imitate the proof of the analogous statement for usual quasicoherent sheaves (see [Mat16] and [Ram24]). Suppose $\mu : R \to R'$ is descendable. We need to show that the functor induced by $\times$-pullback

$$G : \mathrm{QCoh}(R) \to \lim(\mathrm{QCoh}(R') \rightrightarrows \mathrm{QCoh}(R' \otimes_R R') \substack{\to\\\to\\\to} \ldots)$$

(with $\times$-pullback functors) is an equivalence. An element of the codomain of G is a compatible system of modules over tensor products of R' which are compatible under $\times$-pullback. Such a system can be viewed as a simplicial object in R-modules. Taking the colimit of this simplicial object yields a left adjoint F to G.

To show that $FG \cong \mathrm{id}$, we note that $FG(M)$ can be written as

$$\mathrm{colim}(\dots \substack{\rightarrow\\\rightarrow\\\rightarrow} \mathrm{Hom}_R(R' \otimes_R R', M) \rightrightarrows \mathrm{Hom}_R(R', M))$$

Now, because μ is descendable, the limit diagram

$$R \to R' \rightrightarrows R' \otimes_R R' \substack{\rightarrow\\\rightarrow\\\rightarrow} \dots$$

is preserved by any map of stable categories (what is referred to as a Sp-absolute limit in [Ram24]). This immediately implies $FG \cong \mathrm{id}$ by applying $\mathrm{Hom}_R(\_, M)$ to the limit diagram.

The more difficult part is to show that $GF \cong \mathrm{id}$. To do this, consider an element of

$$\lim(\mathrm{QCoh}(R') \rightrightarrows \mathrm{QCoh}(R' \otimes_R R') \substack{\rightarrow\\\rightarrow\\\rightarrow} \dots)$$

By viewing this element as a compatible system of R-modules, we obtain a simplicial diagram, which we call $M_\bullet$.

Consider the simplicial diagram $\mathrm{Hom}_R(R', M_\bullet)$. By adding its colimit, we obtain an augmented simplicial diagram which is a Sp-absolute colimit (because it is a split colimit). Now the collection N's such that $\mathrm{Hom}_R(N, M_\bullet)$ can be completed to a Sp-absolute colimit is a tensor ideal, so it contains R. This means that $M_\bullet$ is a Sp-absolute colimit, so it is preserved by any $\times$-pullback. The finishes the proof because we only need to compute the $\times$-pullback of the colimit of $M_\bullet$ to the tensor product of a nonzero number of copies of R' over R. But after commuting the $\times$-pullback to be inside the colimit, the colimit diagram becomes split (with the correct colimit).

■

Proposition B.0.2. *If X is a qcqs algebraic space, then any quasicoherent sheaf on X will induce a $\times$-quasicoherent sheaf on the sheaf corresponding to X by $\times$-pullback to any affine mapping to X. This functor in fact induces an equivalence of categories beween* $\mathrm{QCoh}(X)$ *and $\times$-quasicoherent sheaves on X.*

Proof. For a qcqs spectral scheme, the statement follows from $\times$-Zariski descent above. In general, We can induct on the length of a minimal scallop decomposition. It therefore reduces to showing that any excision square (see [Lur18] Definition 2.5.2.2) in the étale topos of X,

$$\begin{tikzcd} W \arrow[r] \arrow[d] & U \arrow[d] \\ V \arrow[r] & X \end{tikzcd}$$

where U is affine, the map $V \to X$ is (-1)-truncated, and both W and V are quasicompact, is sent to a pullback square of categories where the transition functors are $\times$-pullback.

It suffices to show two statments. One, given a quasicoherent sheaf $\mathcal{F}$ on X, then $\mathcal{F}$ is a pushout of the pushforward of the $\times$-pullbacks to U, V and W. Now $\mathrm{QCoh}(X)$ is generated by pushforwards of objects in $\mathrm{QCoh}(V)$ and $\mathrm{QCoh}(U)$ by usual Nisnevich excision of quasicoherent sheaves. Hence $\times$-pullback along the maps $V \to X$ and $U \to X$ are jointly conservative. However the square is clearly a pushout after $\times$-pullback to either U or V, so the first statement follows.

The second statement is that given $\times$-pullback compatible sheaves on U, V and W, the pushout of their pushforwards to X has the original sheaves as its $\times$-pullbacks to U, V and W. But this is clear because pushouts are preserved by $\times$-pullback. ■

APPENDIX C

Crystals on Truncated Noetherian Schemes

In this section, we recall the definition of a crystal on the infinitesimal site.

Let S be a truncated Noetherian affine scheme (the theory for a more general base S can be reduced to this case). Denote by $AFF^{ft}_{/S}$ the category of truncated Noetherian affine schemes which are finite-type over S (these are automatically almost of finite presentation because S is Noetherian by [Lur18] Remark 4.2.0.4).

Definition C.0.1. *Suppose $X \in AFF^{ft}_{/S}$, the big infinitesimal site $INF(X/S)$ has as objects diagrams*

$$\begin{tikzcd} U \arrow[r, "u"] \arrow[d, "b"] & X \arrow[d] \\ T \arrow[r] & S \end{tikzcd} \tag{C.1}$$

in $AFF^{ft}_{/S}$ such that b is a thickening–a closed immersion inducing a homeomorphism. Morphisms in $INF(X/S)$ are defined in the obvious way. A family of morphisms in $INF(X/S)$, $\{(U_i \to T_i) \to (U \to T)\}$ is a Zariski (resp. étale) covering if each

$$\begin{tikzcd} U_i \arrow[r] \arrow[d] & U \arrow[d] \\ T_i \arrow[r] & T \end{tikzcd}$$

is a pullback square and the maps $\{T_i \to T\}$ is a Zariski (resp. étale) covering.

The assignment $(U \to T) \mapsto \mathrm{QCoh}(T)$ defines a (Zariski or étale) sheaf of categories on $INF(X/S)$ where the transition maps are given by quasicoherent pullback.

Definition C.0.2. *The small infinitesimal site $Inf(X/S)$ is the full subcategory of $INF(X/S)$*

consisting of those objects such that the map u (in the notation of (C.1)) is an open immersion. It is also endowed with either the Zariski or étale topology induced from the big site.

Definition C.0.3. *A quasicoherent crystal on the big infinitesimal site $Inf(X/S)$ is an object of the category*

$$\lim_{INF(X/S)^{\mathrm{op}}} \mathrm{QCoh}(T)$$

with $*$-pullback transition functors. We will call this category $CRYS(X/S)$. Similarly we can define the category of quasicoherent crystals on the small infinitesimal site

$$Crys(X/S) := \lim_{Inf(X/S)^{\mathrm{op}}} \mathrm{QCoh}(T)$$

Remark C.0.4. *Unwinding the definitions, it is clear that*

$$CRYS(X/S) \cong QCoh((X/S)_{dR})$$

in the notation of Definition 4.6.1

Note that the definition of a quasicoherent crystal does not make use of the topology at all.

Remark C.0.5. *There is an equivalence of categories*

$$Res : CRYS(X/S) \cong Crys(X/S)$$

induced by the natural restriction functor.

This is because in the big topos, the final object is covered by objects in the small site (even in the trivial topology and with only global thickening since we are in the affine case).

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