Integral l-Adic Sheaves on Local Fields

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Abstract

We study the behavior of integral l-adic sheaves on schemes of finite type over a local field under the six operations and the nearby cycle functor.

1 Introduction

Let R be an excellent henselian discrete valuation ring with finite residue field of characteristic p, and K its field of fractions. Such a field will be called a local field. Let $\eta = \operatorname{Spec} K$.

Let X be a scheme of finite type over η. We denote by $|X|$ the set of its closed points. For $x \in |X|$, the residue field $\kappa(x)$ of X at x is a finite extension of K. We denote by R_x its ring of integers, and x_0 the closed point of $\operatorname{Spec} R_x$. Let $\bar{x}$ be a geometric point of X above x with residue field $\kappa(\bar{x})$ being a separable closure of $\kappa(x)$, $R_{\bar{x}}$ the normalization of R_x in $\kappa(\bar{x})$, and $\bar{x}_0$ the closed point of $\operatorname{Spec} R_{\bar{x}}$. Let $F_x \in \operatorname{Gal}(\kappa(\bar{x}_0)/\kappa(x_0))$ be the geometric Frobenius which sends a to $a^{1/q}$, where $q = \#\kappa(x_0)$.

Fix a prime number $l \neq p$. We denote by $\overline{\mathbb{Q}}_l$ an algebraic closure of $\mathbb{Q}_l$. Let $\mathcal{F}$ be a $\overline{\mathbb{Q}}_l$-sheaf on X. By the local monodromy theorem, the eigenvalues of a lift $\Phi_x \in \operatorname{Gal}(\kappa(\bar{x})/\kappa(x))$ of F_x acting on $\mathcal{F}_{\bar{x}}$ are well-defined up to multiplication by roots of unity [5, 1.7.4].

Recall that $\mathcal{F}$ is said to be integral [7, 0.1] if the eigenvalues of Φ_x are algebraic integers for all $x \in |X|$. This integrality is stable under direct image with proper support [ibid., 0.2]. The proof uses the analog of this result over a finite field [22, XXI 5.2.2].

The purpose of this article is to study, more generally, the behavior of integrality under the usual functors: the six operations and the nearby cycles functor. More precisely, we examine the behavior under these functors of the divisibility of the eigenvalues of Φ_x by powers of q. For this, we introduce a measure of q-divisibility inspired by the "gauges" of Mazur-Ogus. We prove in particular the results expected in [13, 5.5].

In a subsequent work [19], we examine the behavior of rationality and l-independence under the same operations.

The results concerning the six operations are presented in §2. In §3 we treat the crucial case of $Rj_*\mathcal{F}$, for the inclusion $j : U \to X$ of the complement of a normal crossings divisor D in a scheme X smooth over η and a sheaf $\mathcal{F}$ smooth on U and moderately ramified along D. The proofs of the results in §2 are given in §4. The essential ingredient is a theorem of de Jong, thanks to which we reduce to the case treated in §3 by the usual cohomological descent techniques. The main result of §5 is the stability of integrality under the nearby cycles functor $R\Psi$. Again, the key ingredient is a theorem of de Jong, which allows us to reduce to the case of a strictly semi-stable pair and a sheaf smooth on the complement of the divisor D (union of the special fiber and the horizontal components) and moderately ramified along D. The study of this case, more delicate than one might expect, relies on a technical compatibility (5.6 (ii)) generalizing [11, 1.5 (a)]. In §6 we generalize the notion of integrality to algebraic stacks.

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2 Integrality and six operations

We keep the notations from §1. We denote by $\overline{\mathbb{Q}}$ the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$. For $r \in \mathbb{Q}$, we denote by q^r the unique element of $\overline{\mathbb{Q}} \cap \mathbb{R}_{>0}$ satisfying $(q^r)^b = q^a$, where $a, b \in \mathbb{Z}$ are such that $r = a/b$, $b \neq 0$. Let X be a scheme of finite type over η.

Definition 2.1. *Fix an embedding $\iota : \overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_l$. We say that a $\overline{\mathbb{Q}}_l$-sheaf $\mathcal{F}$ on X is r-integral (resp. r-integral inverse) if for every closed point x of X, and every eigenvalue α of Φ_x acting on $\mathcal{F}_{\bar{x}}$, $\alpha/\iota(q^r)$*

(resp. $\iota(q^r)/\alpha$) is an integer over $\mathbb{Z}$, where $q = \#\kappa(x_0)$. This definition does not depend on the choices of Φ_x and ι. We say that $\mathcal{F}$ is integral (resp. integral inverse) if it is 0-integral (resp. 0-integral inverse).

The integral (resp. r-integral, resp. integral inverse, resp. r-integral inverse) $\overline{\mathbb{Q}}_l$-sheaves on X form a thick subcategory [9, 1.11] of $\mathrm{Mod}_c(X, \overline{\mathbb{Q}}_l)$, denoted $\mathrm{Mod}_c(X, \overline{\mathbb{Q}}_l)_{\mathrm{ent}}$ (resp. $\mathrm{Mod}_c(X, \overline{\mathbb{Q}}_l)_{r-\mathrm{ent}}$, resp. $\mathrm{Mod}_c(X, \overline{\mathbb{Q}}_l)_{\mathrm{ent}\text{-}1}$, resp. $\mathrm{Mod}_c(X, \overline{\mathbb{Q}}_l)_{r-\mathrm{ent}\text{-}1}$).

Let K' be a finite extension of K, Z a scheme of finite type over K', $G \in \mathrm{Mod}_c(Z, \overline{\mathbb{Q}}_l)$. Then G is r-integral (resp. r-integral inverse) relative to K' if and only if it is r-integral (resp. r-integral inverse) relative to K.

Recall that for schemes X separated of finite type over a regular scheme S of dimension ≤ 1, and in particular over η, we have, by [6, §6], a triangulated category $D^b_c(X, \overline{\mathbb{Q}}_l)$ and a formalism of six operations: Rf_*, $Rf_!$, f^*, $Rf^!$, $\otimes$, RHom. The category $D^b_c(X, \overline{\mathbb{Q}}_l)$ is the 2-inductive limit of the categories $D^b_c(X, E_\lambda)$, where E_λ runs through the finite extensions of $\mathbb{Q}_l$ contained in $\overline{\mathbb{Q}}_l$. If O_λ is the ring of integers of E_λ, $D^b_c(X, E_\lambda)$ is deduced from the category $D^b_c(X, O_\lambda)$ defined in [ibid.] by scalar extension from O_λ to E_λ. The formalism constructed in [ibid.] for $D^b_c(-, O_\lambda)$ transposes trivially.

This formalism makes sense for schemes of finite type over S (not necessarily separated), and it is only for certain operations ($Rf_!$ and $Rf^!$) that we need a separation hypothesis on the morphisms. For a formalism without a separation hypothesis, see the appendix (§6).

The following definition is inspired by the notion of "gauges" of Mazur-Ogus [2, 8.7].

Definition 2.2. *Let $\epsilon : \mathbb{Z} \to \mathbb{Q}$ be a function. We say that an object $K \in D^b_c(X, \overline{\mathbb{Q}}_l)$ is integral (resp. ϵ-integral, resp. integral inverse, resp. ϵ-integral inverse) if for every $i \in \mathbb{Z}$, $H^i(K)$ is integral (resp. $\epsilon(i)$-integral, resp. integral inverse, resp. $\epsilon(i)$-integral inverse).*

We denote the full subcategory of $D^b_c(X, \overline{\mathbb{Q}}_l)$ formed by the integral (resp. ϵ-integral, resp. integral inverse, resp. ϵ-integral inverse) objects by $D^b_c(X, \overline{\mathbb{Q}}_l)_{\mathrm{ent}}$ (resp. $D^b_c(X, \overline{\mathbb{Q}}_l)_{\epsilon-\mathrm{ent}}$, resp. $D^b_c(X, \overline{\mathbb{Q}}_l)_{\mathrm{ent}\text{-}1}$, resp. $D^b_c(X, \overline{\mathbb{Q}}_l)_{\epsilon-\mathrm{ent}\text{-}1}$).

When ϵ is constant, $D^b_c(X, \overline{\mathbb{Q}}_l)_{\epsilon-\mathrm{ent}}$ and $D^b_c(X, \overline{\mathbb{Q}}_l)_{\epsilon-\mathrm{ent}\text{-}1}$ are triangulated subcategories. We sometimes abbreviate $D^b_c(X, \overline{\mathbb{Q}}_l)$ as $D^b_c(X)$.

We denote by I the inclusion function from $\mathbb{Z}$ into $\mathbb{Q}$.

2.3. Let $r, r_1, r_2 \in \mathbb{Q}$.

For $\mathcal{F} \in \mathrm{Mod}_c(X, \overline{\mathbb{Q}}_l)_{r_1-\mathrm{ent}}$, $G \in \mathrm{Mod}_c(X, \overline{\mathbb{Q}}_l)_{r_2-\mathrm{ent}}$, we have $\mathcal{F} \otimes G \in \mathrm{Mod}_c(X, \overline{\mathbb{Q}}_l)_{(r_1+r_2)-\mathrm{ent}}$.

For $K \in D^b_c(X, \overline{\mathbb{Q}}_l)_{(rI+r_1)-\mathrm{ent}}$, $L \in D^b_c(X, \overline{\mathbb{Q}}_l)_{(rI+r_2)-\mathrm{ent}}$, we have $K \otimes L \in D^b_c(X, \overline{\mathbb{Q}}_l)_{(rI+r_1+r_2)-\mathrm{ent}}$.

The same holds for "integral inverse".

For a smooth sheaf $\mathcal{F} \in \mathrm{Mod}_c(X, \overline{\mathbb{Q}}_l)_{r_1-\mathrm{ent}\text{-}1}$, $G \in \mathrm{Mod}_c(X, \overline{\mathbb{Q}}_l)_{r_2-\mathrm{ent}}$, we have $\mathrm{Hom}(\mathcal{F}, G) \in \mathrm{Mod}_c(X, \overline{\mathbb{Q}}_l)_{(r_2-r_1)-\mathrm{ent}}$.

For a smooth sheaf $\mathcal{F} \in \mathrm{Mod}_c(X, \overline{\mathbb{Q}}_l)_{r_1-\mathrm{ent}}$, $G \in \mathrm{Mod}_c(X, \overline{\mathbb{Q}}_l)_{r_2-\mathrm{ent}\text{-}1}$, we have $\mathrm{Hom}(\mathcal{F}, G) \in \mathrm{Mod}_c(X, \overline{\mathbb{Q}}_l)_{(r_2-r_1)-\mathrm{ent}\text{-}1}$.

Let $f : X \to Y$ be a morphism of schemes of finite type over η. Then f^* preserves ϵ-integral (resp. ϵ-integral inverse) complexes.

Theorem 2.3. *Let $f : X \to Y$ be a separated morphism of schemes of finite type over η, and $\mathcal{F}$ an integral (resp. integral inverse) $\overline{\mathbb{Q}}_l$-sheaf on X. Then for every closed point y of Y, $(Rf_!\mathcal{F})_y$ is integral and $(I-n)$-integral (resp. I-integral inverse and n-integral inverse), where $n = \dim(f^{-1}(y))$. In particular, $Rf_!$ induces*

$$D^b_c(X)_{ent} \to D^b_c(Y)_{ent}, \tag{2.4.1}$$

$$D^b_c(X)_{I-ent} \to D^b_c(Y)_{(I-d_f)-ent}, \tag{2.4.2}$$

$$D^b_c(X)_{I-ent\text{-}1} \to D^b_c(Y)_{I-ent\text{-}1}, \tag{2.4.3}$$

$$D^b_c(X)_{ent\text{-}1} \to D^b_c(Y)_{d_f-ent\text{-}1}, \tag{2.4.4}$$

where $d_f = \max_{y \in |Y|} \dim f^{-1}(y)$ is the relative dimension.

The "integral" case ((2.4.1) and (2.4.2)) of 2.4 is a theorem of Deligne-Esnault [7, 0.2].

Theorem 2.4. *Let $f : X \to Y$ be a separated morphism of schemes of finite type over η, $d_X = \dim X$.*

Then Rf_* induces

$$\begin{aligned}
&(2.5.1) \quad D^b_c(X)_{ent} \to D^b_c(Y)_{ent},\\
&(2.5.2) \quad D^b_c(X)_{I-ent} \to D^b_c(Y)_{(I-d_X)-ent},\\
&(2.5.3) \quad D^b_c(X)_{I-ent\text{-}1} \to D^b_c(Y)_{I-ent\text{-}1},\\
&(2.5.4) \quad D^b_c(X)_{ent\text{-}1} \to D^b_c(Y)_{d_X-ent\text{-}1}.
\end{aligned}$$

Without the separation hypothesis on f, (2.5.1), (2.5.3), and (2.5.4) are still true.

The separation hypothesis is also superfluous for (2.5.2). It can be eliminated either by studying the q-divisibility outside a subscheme of fixed dimension, or by using a theory of $Rf_!$ without a separation hypothesis (see 6.5).

Theorem 2.5. *Let $f : X \to Y$ be a separated morphism of schemes of finite type over η, $d_Y = \dim Y$, $d_f = \max_{y\in|Y|} \dim f^{-1}(y)$. Then $Rf^!$ induces*

$$\begin{aligned}
&(2.6.1) \quad D^b_c(Y)_{ent} \to D^b_c(X)_{-d_f-ent},\\
&(2.6.2) \quad D^b_c(Y)_{I-ent} \to D^b_c(X)_{(I-d_Y)-ent},\\
&(2.6.3) \quad D^b_c(Y)_{I-ent\text{-}1} \to D^b_c(X)_{(I+d_f)-ent\text{-}1},\\
&(2.6.4) \quad D^b_c(Y)_{ent\text{-}1} \to D^b_c(X)_{d_Y-ent\text{-}1}.
\end{aligned}$$

Let X be a scheme of finite type over η, $a_X : X \to \eta$. Recall that $Ra_{X!}\overline{\mathbb{Q}}_l$ is globally defined (no problem in the separated case, in the general case by [1, 3.2.4]). We set $D_X = \mathrm{RHom}(-, Ra'_X\overline{\mathbb{Q}}_l)$.

Theorem 2.6. *Let X be a scheme of finite type over η, $d_X = \dim X$. Then D_X induces*

$$\begin{aligned}
&(2.7.1) \quad D^b_c(X)_{ent\text{-}1} \to D^b_c(X)_{-d_X-ent},\\
&(2.7.2) \quad D^b_c(X)_{I-ent\text{-}1} \to D^b_c(X)_{I-ent},\\
&(2.7.3) \quad D^b_c(X)_{I-ent} \to D^b_c(X)_{(I+d_X)-ent\text{-}1},\\
&(2.7.4) \quad D^b_c(X)_{ent} \to D^b_c(X)_{ent\text{-}1}.
\end{aligned}$$

Moreover, for $K \in \mathrm{Mod}_c(X, \overline{\mathbb{Q}}_l)_{ent\text{-}1}$, $H^a(DK)$ is $(a+1)$-integral, for $-d_X \le a \le -1$.

Theorem 2.7. *Let X be a scheme of finite type over η, $d_X = \dim X$. Then $\mathrm{RHom}_X(-,-)$ induces*

$$\begin{aligned}
&(2.8.1) \quad D^b_c(X)_{ent\text{-}1} \times D^b_c(X)_{ent} \to D^b_c(X)_{ent},\\
&(2.8.2) \quad D^b_c(X)_{I-ent\text{-}1} \times D^b_c(X)_{I-ent} \to D^b_c(X)_{(I-d_X)-ent},\\
&(2.8.3) \quad D^b_c(X)_{I-ent} \times D^b_c(X)_{I-ent\text{-}1} \to D^b_c(X)_{I-ent\text{-}1},\\
&(2.8.4) \quad D^b_c(X)_{ent\text{-}1} \times D^b_c(X)_{ent\text{-}1} \to D^b_c(X)_{d_X-ent\text{-}1}.
\end{aligned}$$

3 Normal crossings divisors

Proposition 3.1. *Let $g : X \to Y$ be a finite morphism of schemes of finite type over η, $L \in D^b_c(X, \overline{\mathbb{Q}}_l)$. Then g_*L is ϵ-integral (resp. ϵ-integral inverse) if and only if L is.*

Proof. We can assume that Y is reduced to a single point y, X is reduced to a single point x and $L = \mathcal{F} \in \mathrm{Mod}_c(X, \overline{\mathbb{Q}}_l)$. Let $G_y = \mathrm{Gal}(\kappa(\bar{y})/\kappa(y))$, $G_x = \mathrm{Gal}(\kappa(\bar{x})/\kappa(x))$. The sheaf $\mathcal{F}$ corresponds to a representation $\rho : G_x \to \mathrm{GL}(\mathcal{F}_{\bar{x}})$. Let K' be a finite quasi-galoisian (i.e., normal) extension of $\kappa(y)$ containing $\kappa(x)$, $x' = \mathrm{Spec}\, K'$. For $s \in G_y$, let $\mathcal{F}_s$ be the sheaf on x' corresponding to the representation $\mathrm{Gal}(\kappa(x')/K') \to \mathrm{GL}(\mathcal{F}_{\bar{x}})$ given by $h \mapsto \rho(s^{-1}hs)$. This sheaf depends, up to isomorphism, only on the image of s in G_y/G_x. By Mackey's formula ([18, 7.3]), we have $(g_*\mathcal{F})_{x'} \simeq \bigoplus_s \mathcal{F}_s$, where s runs through a system of representatives of G_y/G_x. Thus, $g_*\mathcal{F}$ is $\epsilon(0)$-integral $\Leftrightarrow$ $(g_*\mathcal{F})_{x'}$ is $\epsilon(0)$-integral $\Leftrightarrow$ the $\mathcal{F}_s$ are $\epsilon(0)$-integral $\Leftrightarrow$ $\mathcal{F}$ is $\epsilon(0)$-integral. The same applies to the integral inverse case. $\square$

In 3.2 and 3.3, let K be an arbitrary field, $\eta = \mathrm{Spec}\, K$.

We will use the following special case of Gabber's purity theorem [8].

Proposition 3.2. *Let n be an integer invertible on η, $\Lambda = \mathbb{Z}/n\mathbb{Z}$. Let $i : Y \to X$ be a closed immersion of regular schemes of finite type over η of pure codimension c. Then $Ri^!\Lambda \simeq \Lambda(-c)[-2c]$.*

Gabber remarked that this result follows easily from the relative purity theorem [21, XVI 3.7]. Indeed, i is obtained by base change from a closed immersion $i_1 : Y_1 \to X_1$ of schemes of finite type over K_1, where K_1 is a subfield of K which is a finite type extension of a prime field K_0. Then $\operatorname{Spec} K_1$ is the generic point of an integral scheme S_1 of finite type over K_0. After replacing S_1 by an open set, we can assume that i_1 is the generic fiber of a closed immersion $i_2 : Y_2 \to X_2$ of schemes of finite type over S_1. Since X_1 (resp. Y_1) is a regular scheme ([10, 6.5.2 (i)]) and X_2 (resp. Y_2) is of finite type over K_0, and thus in particular excellent, by replacing X_2 and Y_2 with open neighborhoods of their generic fibers, we can assume that X_2 and Y_2 are regular (hence smooth over K_0) and i_2 is of pure codimension c. By the relative purity theorem, $Ri_2^!\Lambda \simeq \Lambda(-c)[-2c]$. We conclude by passing to the limit.

The following lemma is modeled on [22, XXI 5.2.1].

Lemma 3.3. *Let X be a scheme of finite type over η, $a_X : X \to \eta$, l a prime number invertible on η, $G \in \operatorname{Mod}_c(X, \overline{\mathbb{Q}}_l)$. (i) There exists a closed subset Y of dimension 0 in X such that $a_{X*}G \to a_{Y*}(G|Y)$ is injective, where $a_Y : Y \to \eta$. (ii) If X is separated of dimension n, and if U is an open subset of X whose complement Z has dimension $< n$, then there exists a closed subset Y of U of dimension 0 and a surjective map $a_{Y*}(G|Y)(-n) \to R^{2n}a_{X!}G$, where $a_Y : Y \to \eta$.*

Proof. (i) is evident. (ii) By replacing X with X_{red} and shrinking U, we can assume U is regular of pure dimension n and $G|U$ is smooth. Since $\dim Z < n$, we have $0 = R^{2n-1}a_{Z!}(G|Z) \to R^{2n}a_{U!}(G|U) \to R^{2n}a_{X!}G \to R^{2n}a_{Z!}(G|Z) = 0$, where $a_Z : Z \to \eta$. Thus we can assume $X = U$. Applying (i) to $\check{G} = \operatorname{Hom}(G, \overline{\mathbb{Q}}_l)$, we find a closed subset Y of U of dimension 0 such that $a_{U*}\check{G} \to a_{Y*}(\check{G}|Y)$ is injective, hence $D_\eta(a_{Y*}(\check{G}|Y)) \to D_\eta(a_{U*}\check{G})$ is surjective. By the purity theorem 3.2, we have $D_\eta(a_{Y*}(\check{G}|Y)) \simeq a_{Y*}(D_Y(\check{G}|Y)) \simeq a_{Y*}(G|Y)$. On the other hand, we have $D_\eta(a_{U*}\check{G}) = H^0(D_\eta Ra_{U*}\check{G}) \simeq H^0(Ra_{U!}D_U\check{G}) \simeq H^0(Ra_{U!}G(n)[2n]) = R^{2n}a_{U!}G(n)$. The result follows. ☐

We resume the notations of §1.

Corollary 3.4. *Let X be a scheme of finite type over η, $a_X : X \to \eta$, $G \in \operatorname{Mod}_c(X, \overline{\mathbb{Q}}_l)$ integral (resp. integral inverse). (i) $a_{X*}G$ is integral (resp. integral inverse). (ii) If X is separated of dimension n, then $a_{X!}G$ is integral (resp. integral inverse), $R^{2n}a_{X!}G$ is n-integral (resp. n-integral inverse).*

The following proposition is modeled on [22, XXI 5.3 (a)].

Proposition 3.5. *Let $j : X \to Y$ be an open immersion of schemes of finite type over η of dimension 1, $G \in \operatorname{Mod}_c(X, \overline{\mathbb{Q}}_l)$ integral. Then j_*G is integral.*

Proof. We reduce to the case where Y is affine, then projective. Define H by the short exact sequence $0 \to j_!G \to j_*G \to H \to 0$, which gives the exact sequence $a_{X*}G \to a_{Y*}H \to R^1a_{X!}G$, where $a_X : X \to \eta$, $a_Y : Y \to \eta$. According to 3.4 (i), $a_{X*}G$ is integral. By the Deligne-Esnault theorem (2.4.1), $R^1a_{X!}G$ is integral. Thus $a_{Y*}H$ is also integral. But H is supported on a closed subset of Y of dimension 0, hence H is integral. ☐

Proposition 3.6. *Let X be a regular scheme of finite type over η of dimension 1, D a positive regular divisor. Let $U = X - D$, $j : U \to X$. Let G be a smooth, integral $\overline{\mathbb{Q}}_l$-sheaf on U, moderately ramified on X. Then Rj_*G is I-integral.*

Proof. The question is local on X. Let $x \in |D|$. We show that $(Rj_*G)_x$ is I-integral. We have $G \simeq (G_O \otimes_O E) \otimes_E \overline{\mathbb{Q}}_l$, where E is a finite extension of $\mathbb{Q}_l$, O its ring of integers, and G_O a smooth (constructible) O-sheaf on U. By Abhyankar's lemma [20, XIII 5.2], there exists, in a neighborhood of x, a finite covering $g : \tilde{X} \to X$ of the form $\tilde{X} = X[T]/(T^n - t)$ where t is a local equation for x, n is an integer prime to the characteristic exponent of K, such that $(g|U)^*(G_O \otimes_O (O/l^2O))$ extends to a locally constant sheaf on $\tilde{X}$. Since G is a direct factor of $(g|U)_*(g|U)^*G$, we are reduced to proving the lemma for the sheaf $(g|U)_*(g|U)^*G$. As $g^{-1}(D)_{\mathrm{red}}$ is a regular divisor, we can then reduce to proving the lemma for a sheaf G such that $G_O \otimes_O (O/l^2O)$ extends to a locally constant sheaf on X, and then to the case where $G_O \otimes_O (O/l^2O)$ is constant by the projection formula. Let $X_{(x)}$ be the henselization of X at x, $U_{(x)} = X_{(x)} \times_X U$, $j_{(x)} : U_{(x)} \to X_{(x)}$, $H = G|U_{(x)}$. Then $H \simeq (H_O \otimes_O E) \otimes_E \overline{\mathbb{Q}}_l$, with $H_O \otimes_O (O/l^2O)$ constant. We have $(Rj_{(x)*}H)_x = (Rj_*G)_x \in D^b(x, \overline{\mathbb{Q}}_l)$. By 3.5, $(j_{(x)*}H)_x = (j_*G)_x$ is integral. It remains to show that $(R^1j_{(x)*}H)_x$ is 1-integral. We have an exact sequence of groups $1 \to \hat{\mathbb{Z}}'(1) \to \mathcal{G} \to \operatorname{Gal}(\kappa(\bar{x})/\kappa(x)) \to 1$, where $\hat{\mathbb{Z}}'(1) = \prod_{p' \neq \mathrm{char}(K)} \mathbb{Z}_{p'}(1)$, $\mathcal{G} = \pi_1^{\mathrm{mod}}(U_{(x)})$. The sheaf H

corresponds to an l-adic representation of $\mathcal{G}$. By the local monodromy theorem, the restriction of this representation to $\hat{\mathbb{Z}}'(1)$ is quasi-unipotent, hence unipotent. We obtain a finite, increasing filtration M of H, such that each $\mathrm{gr}^M_a H$ extends to a smooth $\overline{\mathbb{Q}}_l$-sheaf G_a on $X_{(x)}$. We show that $(R^1 j_{(x)*}M_a)_x$ is 1-integral by induction on a, which will complete the proof. The assertion is clear for $a \ll 0$. Assume the assertion is established for $a-1$. The short exact sequence $0 \to M_{a-1} \to M_a \to G_a|U_{(x)} \to 0$ gives the distinguished triangle $Rj_{(x)*}M_{a-1} \to Rj_{(x)*}M_a \to Rj_{(x)*}(G_a|U_{(x)}) \to$. By a projection formula, $Rj_{(x)*}(G_a|U_{(x)}) \simeq Rj_{(x)*}\overline{\mathbb{Q}}_l \otimes G_a$. We have $(R^q j_{(x)*}\overline{\mathbb{Q}}_l)_x = \overline{\mathbb{Q}}_l(-q)$ if $q = 0, 1$ and 0 otherwise. Thus we have the exact sequence $(j_{(x)*}M_a)_x \to (G_a)_x \to (R^1 j_{(x)*}M_{a-1})_x \to (R^1 j_{(x)*}M_a)_x \to (G_a)_x(-1)$. Here $(j_{(x)*}M_a)_x$ is a subsheaf of $(j_{(x)*}H)_x$, hence integral. By the induction hypothesis, $(R^1 j_{(x)*}M_{a-1})_x$ is 1-integral. So $(G_a)_x$ is integral, and $(G_a)_x(-1)$ is 1-integral. It follows that $(R^1 j_{(x)*}M_a)_x$ is 1-integral. $\square$

The following lemma is a variant of [22, XXI 5.6.2].

Lemma 3.7. *Let X be a regular noetherian scheme, $D = \sum_{i\in I} D_i$ a strictly normal crossings divisor on X with $(D_i)_{i\in I}$ a finite family of regular divisors, $U = X - D$, n an integer invertible on X, $\Lambda = \mathbb{Z}/n\mathbb{Z}$, $G \in \mathrm{Mod}_c(U, \Lambda)$ locally constant and moderately ramified on X. (i) Let $i \in I$, $U_{(i)} = X - \bigcup_{h\in I-\{i\}} D_h$, $D_{i,U_{(i)}} = D_i \times_X U_{(i)}$, giving a cartesian square*

$$\begin{array}{ccc} D_{i,U_{(i)}} & \xrightarrow{j'_{(i)}} & U_{(i)} \\ \downarrow{\scriptstyle i'_{(i)}} & & \downarrow{\scriptstyle j_{(i)}} \\ D_i & \xrightarrow{i_i} & X \end{array}$$

Then the base change morphism $i_i^* Rj_{(i)*} \to Rj'_{(i)*} i'^*_{(i)}$ is an isomorphism for G, and the sheaves $i'^*_i R^q j_{(i)*}G$ for $q \in \mathbb{Z}$ are locally constant and moderately ramified on D_i. (ii) Let $f : Y \to X$ be a morphism of regular noetherian schemes. Assume that $f^{-1}(D)$ is a normal crossings divisor and that $(f^{-1}(D_i))_{i\in I}$ is a family of regular divisors. Consider the cartesian square:

$$\begin{array}{ccc} U_Y & \xrightarrow{j_Y} & Y \\ \downarrow{\scriptstyle f_U} & & \downarrow{\scriptstyle f} \\ U & \xrightarrow{j} & X \end{array}$$

Then the base change morphism $f^* Rj_* G \to Rj_{Y*} f_U^* G$ is an isomorphism.

Proof. The question is local on X. Let x be a point of X. By Abhyankar's lemma, there exists, in a neighborhood of x, a finite covering $g : \tilde{X} \to X$ of the form $\tilde{X} = X[T_1, \dots, T_r]/(T_1^{n_1} - t_1, \dots, T_r^{n_r} - t_r)$ where the t_i are local equations of the components of D passing through x, and n_i are integers prime to the characteristic exponent of $\kappa(x)$, such that $(g|U)^*G$ extends to a locally constant sheaf on $\tilde{X}$. As G injects into $(g|U)_*(g|U)^*G$ and the quotient G_1 is moderately ramified on X, we can iterate this construction. For any $N \geq 1$, we obtain, after shrinking X, a resolution $G \to (g|U)_*(g|U)^*G \to (g_1|U)_*(g_1|U)^*G_1 \to \cdots \to (g_N|U)_*(g_N|U)^*G_N$. So we are reduced to proving the lemma for the sheaf $(g|U)_*(g|U)^*G$. As $g^{-1}(D)_{\mathrm{red}} = \sum_{i\in I} g^{-1}(D_i)_{\mathrm{red}}$ is a normal crossings divisor with $(g^{-1}(D_i)_{\mathrm{red}})_{i\in I}$ being a family of regular divisors, we can then reduce to proving the lemma for a sheaf G that extends to a locally constant sheaf on X, and then to the case $G = \Lambda_U$ by the projection formula. Point (ii) then follows from [8, §8] and the functoriality of divisor classes [4, Th. finitude, 2.1.1]. For (i), note that $D_{i,U_{(i)}}$ is a regular divisor of $U_{(i)}$, with complement U. For any q, $i'^*_i R^q j_{(i)*}\Lambda_U \simeq \Lambda_{D_{i,U_{(i)}}}(-1)$ if $q = 1$, $\Lambda_{D_{i,U_{(i)}}}$ if $q = 0$, and 0 otherwise. The statement about local constancy and moderate ramification follows. The morphism in (i) is an isomorphism because the base change morphism $i_i^* Rj_{(i)*} \to Rj'_{(i)*} i'^*_{(i)}$ induces isomorphisms on $\Lambda_{U_{(i)}}$ by virtue of (ii) and trivially on $i'^*_i \Lambda_{D_{i,U_{(i)}}}$ in the distinguished triangle. $\square$

Proposition 3.8. *Let X be a scheme of finite type over η, D a normal crossings divisor. Let $U = X - D$, $j : U \to X$. Let G be a smooth, integral $\overline{\mathbb{Q}}_l$-sheaf on U, moderately ramified on X. Then Rj_*G is I-integral.*

Proof. The problem is local for the étale topology near a closed point of D, so we can assume D is strictly normal crossings. Since $\mathrm{Reg}(X)$ is an open subset of X containing D, we can assume X is regular. Let $D = \sum_{i\in I} D_i$ with $(D_i)_{i\in I}$ a finite family of regular divisors. We proceed by induction on $n = |I|$. The case $n = 0$ is trivial. For $n > 0$, we choose an $i \in I$ and apply 3.7 (i), keeping its notation. For any $x \in D_{i,U_{(i)}}$, there exists a regular subscheme of $U_{(i)}$ of dimension 1 whose intersection with $D_{i,U_{(i)}}$ is the scheme x. Thus $Rj_{(i)*}G$ is I-integral, by 3.7 (ii) and 3.6. Note that $\sum_{h\in I-\{i\}} D_h \cap D_i$ is a normal crossings divisor of D_i with complement $D_{i,U_{(i)}}$, and for any q, $i'^*_i R^q j_{(i)*}G$ is a smooth $\overline{\mathbb{Q}}_l$-sheaf on $D_{i,U_{(i)}}$, moderately ramified on D_i by 3.7 (i). Thus $i_i^* Rj_*G \simeq Rj'_{(i)*} i'^*_i Rj_{(i)*}G$ is I-integral, by the induction hypothesis. Since i is arbitrary, we conclude that Rj_*G is I-integral. $\square$

4 Proof of 2.4 to 2.8

The following proposition is a variant of [17, 2.6].

Proposition 4.1. *Let F be a field, X a separated scheme of finite type over* $\mathrm{Spec}(F)$*, and U an open subset of X.*

(i) There exists a proper surjective morphism $r_0 : X'_0 \to X$ with X'_0 regular, and a closed and open subscheme W_0 of X'_0 containing $r_0^{-1}(U)$ such that $r_0^{-1}(U)$ is the complement of a strict normal crossings divisor in W_0.

(ii) For all $n > 0$, there exists a finite radicial extension F' of F and a proper n-truncated s-split hypercovering $r. : X'. \to X_{F'}$ such that X'_m is smooth over $\mathrm{Spec}(F')$ *and $r_m^{-1}(U_{F'})$ is the complement of a strict normal crossings divisor relative to* $\mathrm{Spec}(F')$ *in a closed and open subscheme of X'_m, for $0 \leq m \leq n$.*

Proof. (i) In the case where X is integral and $U \neq \emptyset$, there exists a proper surjective morphism $r_0 : X'_0 \to X$ with X'_0 integral and regular such that $r_0^{-1}(U)$ is the complement of a strict normal crossings divisor, by virtue of [14, 4.1]. We take $W_0 = X'_0$.

The case where X is integral and $U = \emptyset$ follows from this: apply the preceding case to X and the open part X to obtain r_0, and then take $W_0 = \emptyset$.

In the general case, let X_α be the reduced schemes associated with the irreducible components of X, and $\alpha : \coprod X_\alpha \to X$ be the canonical morphism. Then α is finite and surjective. For each α, we apply (i) to X_α and $U \times_X X_\alpha$, obtaining $\phi_\alpha : (X_\alpha)'_0 \to X_\alpha$, a proper surjective morphism with $(X_\alpha)'_0$ regular, and a closed and open subscheme W_α of $(X_\alpha)'_0$ containing the inverse image U_α of U such that U_α is the complement in W_α of a strict normal crossings divisor. We set $X'_0 = \coprod (X_\alpha)'_0$, $r_0 = \alpha \circ \coprod \phi_\alpha$, and $W_0 = \coprod W_\alpha$. Then r_0 and W_0 satisfy the conditions of (i).

(ii) Case where F is perfect. We proceed by induction on n. When $n = 0$, (ii) degenerates to (i). Suppose we are given an n-truncated hypercovering $r. : X'. \to X$ satisfying the conditions of (ii). We apply (i) to the X-scheme $(\mathrm{cosk}_n X')_{n+1}$ and the inverse image of U. We obtain a proper surjective morphism $\beta : N \to (\mathrm{cosk}_n X')_{n+1}$ with N smooth over $\mathrm{Spec}(F)$ and a closed and open subscheme W of N such that the inverse image of U in N is the complement of a strict normal crossings divisor relative to $\mathrm{Spec}(F)$ in W. The proper $(n+1)$-truncated s-split hypercovering associated to the triple (X', N, β) [21, Vbis 5.1.3] verifies the conditions of (ii) for $n+1$.

General case. We take a perfect closure $\bar{F}$ of F and apply (ii) to $\bar{F}$, $X_{\bar{F}}$ and $U_{\bar{F}}$. The truncated hypercovering and the strict normal crossings divisors obtained descend to a finite sub-extension F' of F. □

4.1 Proof of (2.5.1)

We must show that for a constructible, integral $\overline{\mathbb{Q}}_l$-sheaf G on X, Rf_*G is integral.

We proceed by induction on $d = \dim X$. The case $d < 0$ is trivial.

Let $d \geq 1$. We choose an affine open subset $U \overset{j}{\hookrightarrow} X$ such that $G|_U$ is smooth and its complement $Z \overset{i}{\hookrightarrow} X$ has dimension $< d$. The distinguished triangle $i_*Ri_!G \to G \to Rj_*j^*G \to$ induces the distinguished triangle $R(fi)_*Ri_!G \to Rf_*G \to R(fj)_*j^*G \to$.

Taking into account the induction hypothesis, it suffices to see that Rj_*j^*G and $R(fj)_*j^*G$ are integral. It is therefore sufficient to verify the theorem under the additional hypothesis that X is separated and G is smooth.

We have $G \simeq (G_0 \otimes_{\mathcal{O}} E) \otimes_E \overline{\mathbb{Q}}_l$ with G_0 smooth. Let $p : X' \to X$ be a surjective étale covering that trivializes $G_0 \otimes_{\mathcal{O}} (\mathcal{O}/\mathfrak{m})$, where $\mathfrak{m}$ is the maximal ideal of $\mathcal{O}$. The sheaf G is a direct factor of p_*p^*G, so it suffices to see the integrality of $R(fp)_*p^*G$. Thus, it is enough to verify the theorem under the additional hypothesis that X is separated and

$$G \simeq (G_0 \otimes_{\mathcal{O}} E) \otimes_E \overline{\mathbb{Q}}_l \text{ with } G_0 \otimes_{\mathcal{O}} (\mathcal{O}/\mathfrak{m}) \text{ constant.} \quad (1)$$

We factorize f as $X \overset{j}{\hookrightarrow} Z \overset{g}{\to} Y$, where j is an open immersion and g is a proper morphism. Since Rg_* preserves integrality by virtue of the Deligne-Esnault theorem (2.4.1), it suffices to prove the integrality of Rj_*G. We are thus led to prove (2.5.1) for j and G. For this, we can suppose Z is affine, hence separated.

Let $i \geq 0$. We apply 4.1(ii) to j with $n = i+1$. Up to changing notations, we can assume that the radicial extension from loc. cit. is trivial. We obtain a cartesian square (of simplicial schemes that are $(i+1)$-truncated)

$$\begin{tikzcd} X' \arrow[r, "j'"] \arrow[d, "s"] & Z' \arrow[d, "r"] \\ X \arrow[r, "j"] & Z \end{tikzcd}$$

where $r.$ is a proper $(i+1)$-truncated s-split hypercovering, Z'_m is smooth over η, j'_m is an open immersion making X'_m the complement of a normal crossings divisor relative to η in a closed open part of Z'_m, for $0 \leq m \leq i+1$. By cohomological descent,

$$\tau_{\leq i} Rj_* G \simeq \tau_{\leq i} Rj_* Rs_{\bullet,*} s^* G = \tau_{\leq i} Rr_{\bullet,*} Rj'_{\bullet,*} s^* G.$$

Since $Rr_{\bullet,*}$ preserves integrality (2.4.1), it suffices to show the integrality of $Rj'_{m,*} s^*_m G$, $0 \leq m \leq i$. Now $s^*_m G$ still satisfies (1), so it is moderately ramified on Z'_m. It then suffices to apply 3.8. ☐

Proposition 4.2. *Let X be a separated regular scheme of finite type over η, purely of dimension 1, $\alpha_X : X \to \eta$, and G a smooth $\overline{\mathbb{Q}}_l$-sheaf on X, which is inverse integral. Then $R^1\alpha_{X,!}G$ is 1-inverse integral.*

Proof. We can suppose that $G \simeq (G_0 \otimes_{\mathcal{O}} E) \otimes_E \overline{\mathbb{Q}}_l$, with G_0 smooth. For an étale surjective covering $p : X' \to X$ that trivializes $G_0 \otimes_{\mathcal{O}} (\mathcal{O}/\mathfrak{m})$, G is a direct factor of $p_* p^* G$. This allows us to assume $G_0 \otimes_{\mathcal{O}} \mathcal{O}/\mathfrak{m}$ is constant. We have

$$D_\eta(R^1\alpha_{X,!}G) \simeq H^{-1}(D_\eta R\alpha_{X,!}G) \simeq H^{-1}(R\alpha_{X,*}\check{G}(1)[2]) = R^1\alpha_{X,*}\check{G}(1).$$

Let $j : X \to P$ be a regular compactification of X. From $\alpha_X = \alpha_P \circ j$ we deduce a spectral sequence

$$E_2^{p,q} = R^p\alpha_{P,*}R^q j_* \check{G} \Rightarrow R^{p+q}\alpha_{X,*}(\check{G}).$$

From 3.8 and 3.4, $R^p\alpha_{P,*}R^1 j_*\check{G}$ is integral. Furthermore, $R^1\alpha_{P,*}R^0 j_*\check{G}$ is a quotient of $R^1\alpha_{P,*}j_*\check{G}$, so is integral by virtue of 2.4. Thus $R^1\alpha_{X,*}\check{G}$ is integral, so $R^1\alpha_{X,!}G$ is 1-inverse integral. ☐

Proof of 2.4

As already noted, the "integral" case ((2.4.1) and (2.4.2)) of 2.4 is proven in [7, 0.2]. Let us now assume F is inverse integral. We can suppose $Y = \eta$, $f = \alpha_X$.

Let's first treat the case $X = \mathbb{A}^1_\eta$. Let $j : U \to X$ be a dense open set such that $F|_U$ is smooth, and $i : Z \to X$ be the closed complement. The exact sequence $0 \to j_! j^* F \to F \to i_* i^* F \to 0$ gives the distinguished triangle

$$R\alpha_{U,!}(F|_U) \to R\alpha_{X,!}F \to R\alpha_{Z,!}(F|_Z) \to .$$

From 3.4, $R\alpha_{Z,!}(F|_Z)$ is inverse integral, $R\alpha_{U,!}(F|_U)$ is inverse integral, and $R^2\alpha_{U,!}(F|_U)$ is 1-inverse integral. From 4.3, $R^1\alpha_{U,!}(F|_U)$ is 1-inverse integral. Therefore $R\alpha_{X,!}F$ is I-inverse integral and 1-inverse integral.

For the general case, we proceed by induction on n. The case $n \leq 0$ is trivial. Let $n \geq 1$. By the normalization lemma, there exists a dense open set $j : U \hookrightarrow X$ and a morphism $f : U \to Y = \mathbb{A}^n_K$ with fibers of dimension $\leq n-1$. Let $i : Z \hookrightarrow X$ be the complement of U. Then Z has dimension $\leq n-1$. The exact sequence $0 \to j_! j^* F \to F \to i_* i^* F \to 0$ gives the distinguished triangle

$$R\alpha_{U,!}(F|_U) \to R\alpha_{X,!}F \to R\alpha_{Z,!}(F|_Z) \to .$$

By the induction hypothesis, $R\alpha_{Z,!}(F|_Z)$ is I-inverse integral and $(n-1)$-inverse integral. It is therefore sufficient to verify the proposition for $F|_U$. This results from the spectral sequence

$$E_2^{p,q} = R^p\alpha_{Y,!}R^q f_!(F|_U) \Rightarrow R^{p+q}\alpha_{U,!}(F|_U),$$

from the induction hypothesis applied to the fibers of f and the case of an affine line already treated. ☐

The following proposition is an analogue of 3.6.

Proposition 4.3. *Let X be a regular scheme of finite type over η of dimension 1, D a positive regular divisor. Set $U = X - D$, $j : U \to X$. Let G be a smooth $\overline{\mathbb{Q}}_l$-sheaf on U, inverse integral, and moderately ramified on X. Then Rj_*G is I-inverse integral.*

Proof. Since $\check{G}$ is smooth, moderately ramified on X, and integral, $Rj_*\check{G}$ is I-integral by virtue of 3.8. Let $i : D \to X$. From the local duality in dimension 1, $i^*R^1j_*\check{G} \simeq (i^*j_*G)^\vee(-1)$ is 1-integral inverse, and $i^*j_*\check{G} \simeq (i^*R^1j_*G)^\vee(-1)$ is integral inverse. □

The following proposition is an analogue of 3.8.

Proposition 4.4. *Let X be a scheme of finite type over η, D a normal crossings divisor. Set $U = X - D$, $j : U \to X$. Let G be a smooth $\overline{\mathbb{Q}}_l$-sheaf on U, inverse integral, and moderately ramified on X. Then Rj_*G is I-inverse integral.*

One deduces 4.5 from 4.4 in the same way that 3.8 was deduced from 3.6. One deduces (2.5.3) from 4.5 in the same way that (2.5.1) was deduced from 3.8 in 4.2.

Remark 4.5. *Assertion (2.6.1) (resp. (2.6.3)) for f a closed immersion follows from the preceding. Indeed, let $j : Y - X \to Y$ be the complementary open, $K \in D^b_c(Y, \overline{\mathbb{Q}}_l)$ integral (resp. I-integral inverse). We have the distinguished triangle*

$$Rf^!K \to f^*K \to f^*Rj_*j^*K \to .$$

By applying (2.5.1) (resp. (2.5.3)) to j, we obtain that $f^*Rj_*j^*K$ is integral (resp. I-integral inverse), so $f^*Rj_*j^*K[-1]$ is also (resp. $(I-1)$-integral inverse). But f^*K is integral (resp. I-integral inverse). We conclude that $Rf^!K$ is also. This proof gives a little more in the inverse integral case: for $K \in \mathrm{Mod}_c(Y, \overline{\mathbb{Q}}_l)_{ent-1}$, $R^af^!K$ is $(a-1)$-inverse integral, for $a \geq 1$.

Proof of 2.7

We can assume X is reduced. We proceed by induction on d_X. The case $d_X < 0$ is trivial. For $d_X \geq 1$, it suffices to show that for $K \in \mathrm{Mod}_c(X, \overline{\mathbb{Q}}_l)$ integral (resp. inverse integral), DK is inverse integral and $(I + d_X)$-inverse integral (resp. I-integral and $-d_X$-integral and $H^a(DK)$ is $(a+1)$-integral, $-d_X \leq a \leq -1$). Let's take a regular open set $j : U \hookrightarrow X$ purely of dimension d_X such that the complement $i : V \to X$ has dimension $< d_X$ and $K|_U$ is smooth. Then $D(j^*K) \simeq (j^*K)^\vee(d_X)[2d_X]$. We have the distinguished triangle

$$i_*D(i^*K) \to DK \to Rj_*D(j^*K) \to . \tag{2}$$

From (2.5.3) (resp. (2.5.1)), $Rj_*D(j^*K)$ is $(I + d_X)$-inverse integral (resp. $-d_X$-integral). By the induction hypothesis, $i_*D(i^*K)$ is also. Thus DK is $(I + d_X)$-inverse integral (resp. $-d_X$-integral). We have the distinguished triangle

$$j_!D(j^*K) \to DK \to i_*D(Ri^!K) \to .$$

The term $j_!D(j^*K) \in D^{[-2d_X, -2d_X]}$ is inverse integral (resp. I-integral). From 4.6, $Ri^!K$ is integral (resp. $R^ai^!K$ is inverse integral and $R^ai^!K$ is $(a-1)$-inverse integral, $a \geq 1$). By the induction hypothesis, $D(Ri^!K)$ is inverse integral (resp. $H^a(D(Ri^!K))$ is integral and $H^a(D(Ri^!K))$ is $(a+1)$-integral, $a \leq -1$), by virtue of the spectral sequence

$$E_2^{p,q} = H^p(D(R^{-q}i^!K)) \Rightarrow H^{p+q}(DRi^!K).$$

Thus DK is inverse integral (resp. I-integral and $H^a(DK)$ is $(a+1)$-integral, $-d_X \leq a \leq -1$). □

Proof of 2.5

Assertions (2.5.1) and (2.5.3) have already been proven in 4.2 and 4.5. If f is separated, then (2.5.2) and (2.5.4) follow from 2.4 and 2.7: $Rf_* \simeq D_Y Rf_! D_X$ induces

$$D^b_c(X)_{I\text{-ent}} \xrightarrow{D_X} D^b_c(X)_{(I+d_X)\text{-ent}-1} \xrightarrow{Rf_!} D^b_c(Y)_{(I+d_X)\text{-ent}-1} \xrightarrow{D_Y} D^b_c(Y)_{(I-d_X)\text{-ent}},$$
$$D^b_c(X)_{\text{ent}-1} \xrightarrow{D_X} D^b_c(X)_{-d_X\text{-ent}} \xrightarrow{Rf_!} D^b_c(Y)_{-d_X\text{-ent}} \xrightarrow{D_Y} D^b_c(Y)_{d_X\text{-ent}-1}.$$

It remains to show (2.5.4) without assuming f is separated. For this, we take an affine open cover $W \to X$. Let $g. : \mathrm{cosk}_0(W/X). \to X$. Then $Rf_* \simeq Rf_*Rg_{\bullet,*}g_*$, and it suffices to apply the result from the separated case. □

Remark 4.6. *Assertions (2.6.2) and (2.6.4) follow from 2.7: $Rf^! \simeq D_X f^* D_Y$ induces*

$$D_c^b(Y)_{I\text{-}ent} \xrightarrow{D_Y} D_c^b(Y)_{(I+d_Y)\text{-}ent-1} \xrightarrow{f^*} D_c^b(X)_{(I+d_Y)\text{-}ent-1} \xrightarrow{D_X} D_c^b(X)_{(I-d_Y)\text{-}ent},$$
$$D_c^b(Y)_{ent-1} \xrightarrow{D_Y} D_c^b(Y)_{-d_Y\text{-}ent} \xrightarrow{f^*} D_c^b(X)_{-d_Y\text{-}ent} \xrightarrow{D_X} D_c^b(X)_{d_Y\text{-}ent-1}.$$

In the case where f is quasi-finite, one can improve (2.5.2) as follows.

Proposition 4.7. *Let $f : X \to Y$ be a quasi-finite separated morphism of schemes of finite type over η with $d_X = \dim X \geq 1$. Then Rf_* sends $D_c^b(X, \overline{\mathbb{Q}}_l)_{I\text{-}ent}$ into $D_c^b(Y, \overline{\mathbb{Q}}_l)_{(I+1-d_X)\text{-}ent}$.*

Proof. By Zariski's main theorem, we can assume that f is a dominant open immersion. Let $K \in \mathrm{Mod_c}(X, \overline{\mathbb{Q}}_l)$ be integral. We have the distinguished triangle

$$i_* Ri^! f_! K \to f_! K \to Rf_* K \to,$$

where $i : Y - X \to Y$ is the closed complement. It is therefore sufficient to apply (2.6.2). □

The following proposition generalizes [7, 0.4].

Proposition 4.8. *Let $i : X \to Y$ be an immersion of schemes of finite type over η with Y regular, $d_Y = \dim Y$, $d_c = \mathrm{codim}(X, Y)$, $G \in \mathrm{Mod_c}(Y, \overline{\mathbb{Q}}_l)$ integral (resp. inverse integral) smooth. Let $\epsilon : \mathbb{Z} \to \mathbb{Q}$ be a function satisfying*

$$\epsilon(a) = \begin{cases} d_c & \text{if } 2d_c \leq a < d_c + d_Y, \\ a + 1 - d_Y & \text{if } d_c + d_Y \leq a < 2d_Y, \\ d_Y & \text{if } a = 2d_Y. \end{cases}$$

Then $Ri^! G$ is ϵ-integral (resp. $(I - d_c)$-inverse integral).

Proof. We will show that for any closed subset W of X, $Ri_W^! G$ is ϵ-integral (resp. $(I - d_c)$-inverse integral), where $i_W : W \to Y$. Here we have equipped W with the reduced induced scheme structure. We proceed by noetherian induction. The case $W = \emptyset$ is trivial. For $W \neq \emptyset$, we take an irreducible regular open subset $j : U \hookrightarrow W$. Let $i_Z : Z \to W$ be its complement. We have the distinguished triangle

$$i_{Z,*} Ri_Z^! Ri_W^! G \to Ri_W^! G \to Rj_* j^* Ri_W^! G \to .$$

From 3.2, $j^* Ri_W^! G = R(i_W j)^! G \simeq G(-d)[-2d]$, where $d = \mathrm{codim}(U, Y) \geq d_c$. If $d_U = \dim U = 0$, then $Rj_* j^* Ri_W^! G$ is ϵ-integral because $\epsilon(2d) \leq d$; if $d_U \geq 1$, then by (2.5.1) and 4.8, $Rj_* j^* Ri_W^! G$ is d-integral and $(I + 1 - d - d_U)$-integral, hence ϵ-integral, taking into account the fact that $d + d_U \leq d_Y$. (Resp. from (2.5.3), $Rj_* j^* Ri_W^! G$ is $(I - d)$-inverse integral, hence $(I - d_c)$-inverse integral.) By the induction hypothesis, $Ri_Z^! Ri_W^! G = R(i_W i_Z)^! G$ is ϵ-integral (resp. $(I - d_c)$-inverse integral). Thus $Ri_W^! G$ is also. □

Proof of 2.6

Assertions (2.6.2) and (2.6.4) were handled above. For the rest, we proceed by induction on d_Y. The case $d_Y < 0$ is clear. For $d_Y \geq 0$, we can assume Y is reduced. Let $G \in \mathrm{Mod_c}(Y, \overline{\mathbb{Q}}_l)$ be integral (resp. inverse integral). There exists a regular open subset U of Y with complement W of dimension $\leq d_Y - 1$ such that $G|_U$ is smooth. We consider the cartesian square diagram

$$\begin{tikzcd} X_U \arrow[r, "j'"] \arrow[d, "f_U"] & X \arrow[d, "f"] & X_W \arrow[l, "i'"'] \arrow[d, "f_W"] \\ U \arrow[r, "j"] & Y & W \arrow[l, "i"'] \end{tikzcd}$$

We have the distinguished triangle

$$i_* Rf_{W,!} Ri^! G \to Rf^! G \to Rj_* Rf_{U,!} j^* G \to . \tag{3}$$

From 4.6, $Ri^! G$ is integral (resp. I-inverse integral), so $Rf_{W,!} Ri^! G$ is $-d_r$-integral (resp. $(I + d_r)$-inverse integral) by virtue of the induction hypothesis. It remains to consider $Rf_!^!(G|_U)$.

It suffices to show that for Y regular and $G \in \mathrm{Mod_c}(Y, \overline{\mathbb{Q}}_l)$ integral (resp. inverse integral) smooth, $Rf_!^! G$ is $-d_r$-integral (resp. $(I + d_r)$-inverse integral). The problem being local on Y, we can assume Y is irreducible and that f factors as $X \overset{i_X}{\hookrightarrow} \mathbb{A}_Y^n \xrightarrow{p} Y$, where i_X is a closed immersion. $\mathbb{A}_Y^n$ is irreducible [10, 4.5.8], thus bi-equidimensional [10, 5.2.1], so $\mathrm{codim}(X, \mathbb{A}_Y^n) = n + d_Y - d_X \geq n - d_r$, since $d_X \leq d_Y + d_r$. It is therefore sufficient to apply 4.9 to i_X and $Rp^! G[-2n] = p^* G(n)$. □

Proof of 2.8

Assertions (2.8.2) and (2.8.4) follow from 2.7:

$$\mathrm{RHom}_X(-,-) \simeq D_X(- \otimes D_X -)$$

induces

$$\begin{aligned} D_c^b(X)_{I\text{-ent}-1} \times D_c^b(X)_{I\text{-ent}} &\xrightarrow{(\mathrm{id}, D_X)} D_c^b(X)_{I\text{-ent}-1} \times D_c^b(X)_{(I+d_X)\text{-ent}-1} \\ &\to D_c^b(X)_{(I+d_X)\text{-ent}-1} \xrightarrow{D_X} D_c^b(X)_{(I-d_X)\text{-ent}}, \\ D_c^b(X)_{\mathrm{ent}} \times D_c^b(X)_{\mathrm{ent}-1} &\xrightarrow{(\mathrm{id}, D_X)} D_c^b(X)_{\mathrm{ent}} \times D_c^b(X)_{-d_X\text{-ent}} \\ &\to D_c^b(X)_{-d_X\text{-ent}} \xrightarrow{D_X} D_c^b(X)_{d_X\text{-ent}-1}. \end{aligned}$$

For the rest, it suffices to show that for $K, L \in \mathrm{Mod_c}(X, \overline{\mathbb{Q}}_l)$, K inverse integral, L integral (resp. K integral, L inverse integral), $\mathrm{RHom}(K, L)$ is integral (resp. I-inverse integral). By devissage of K, we are reduced to assuming K is of the form $j_!G$, where $j : Y \hookrightarrow X$ is an immersion, $G \in \mathrm{Mod_c}(Y, \overline{\mathbb{Q}}_l)$ is inverse integral (resp. integral) smooth. Then

$$\mathrm{RHom}_X(j_!G, L) \simeq Rj_* \mathrm{RHom}_Y(G, Rj^! L).$$

It suffices to apply (2.6.1) and (2.5.1) (resp. (2.6.3) and (2.5.3)). □

5 Variants and nearby cycles

Variant 5.1. *Let k be a finite field, and l a prime number $\neq \mathrm{char}(k)$. Let X be a scheme of finite type over k. Let $r \in \mathbb{Q}$. We fix an embedding $\iota : \overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_l$. We say that a $\overline{\mathbb{Q}}_l$-sheaf F is r-integral (resp. r-inverse integral) if for every geometric point $\bar{z}$ over a closed point x of X, and for every eigenvalue α of the geometric Frobenius $F_x \in \mathrm{Gal}(\kappa(\bar{z})/\kappa(x))$ acting on $F_{\bar{z}}$, $\alpha/\iota(q^r)$ (resp. $\iota(q^r)/\alpha$) is an integer over $\mathbb{Z}$, where $q = \#\kappa(x)$. This definition does not depend on the choice of ι. We define integrality for $K \in D_c^b(X, \overline{\mathbb{Q}}_l)$ in a manner analogous to 2.2.*

We have similar results for the various operations: 2.4 to 2.8, 3.8, 4.8, 4.9. The integral case of the analogue of 2.4 is a theorem of Deligne [22, XXI 5.2.2]. The proofs of the other results are similar to those given in §3 and 4.

Variant 5.2. *Let R be an excellent Henselian discrete valuation ring with finite residue field $k = \mathbb{F}_{p^\nu}$, K its field of fractions, $S = \mathrm{Spec}\, R$, $\eta = \mathrm{Spec}\, K$, $s = \mathrm{Spec}\, k$. Let X be a scheme of finite type over s. We have a topos $X \times_s \eta$ [22, XIII 1.2.4]. Recall that a sheaf of sets on $X \times_s \eta$ is a sheaf on $X_{\bar{s}}$ equipped with a continuous action [ibid., 1.1.2] of $\mathrm{Gal}(\bar{K}/K)$, compatible with the action of $\mathrm{Gal}(\bar{K}/K)$ on $X_{\bar{s}}$ (via $\mathrm{Gal}(\bar{K}/K) \to \mathrm{Gal}(\bar{k}/k)$). Let us fix a prime number $l \neq p$. Let $F \in \mathrm{Mod_c}(X \times_s \eta, \overline{\mathbb{Q}}_l)$. For $x \in X$, let $\Phi_x \in \mathrm{Gal}(\bar{k}/\kappa(x)) \times_{\mathrm{Gal}(\bar{k}/k)} \mathrm{Gal}(\bar{K}/K)$ be a lifting of the geometric Frobenius $F_x \in \mathrm{Gal}(\bar{k}/\kappa(x))$. According to the local monodromy theorem, the eigenvalues of Φ_x acting on $F_{\bar{x}}$ are well-defined up to multiplication by roots of unity. Let $r \in \mathbb{Q}$. We fix $\iota : \overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_l$. We say that F is r-integral (resp. r-inverse integral) if for all $x \in |X|$ and every eigenvalue α of Φ_x acting on $F_{\bar{x}}$, $\alpha/\iota(q^r)$ (resp. $\iota(q^r)/\alpha$) is an integer over $\mathbb{Z}$, where $q = \#\kappa(x)$. This definition does not depend on the choices of Φ_x and ι. We define integrality for $K \in D_c^b(X \times_s \eta, \overline{\mathbb{Q}}_l)$ in a manner analogous to 2.2.*

Any continuous section σ of $\mathrm{Gal}(\bar{K}/K) \to \mathrm{Gal}(\bar{k}/k)$ induces an exact functor

$$\sigma^* : D_c^b(X \times_s \eta, \overline{\mathbb{Q}}_l) \to D_c^b(X, \overline{\mathbb{Q}}_l).$$

A complex $K \in D_c^b(X \times_s \eta, \overline{\mathbb{Q}}_l)$ is ϵ-integral (resp. ϵ-inverse integral) if and only if $\sigma^* K$ is. Since σ^* commutes with the six operations and with duality, we deduce from 5.1 similar results for these operations: 2.4 to 2.8, 3.8, 4.8, 4.9.

Let S, η, s, l be as in 5.2. The main result of this section is the following.

Theorem 5.3. *Let X be a scheme of finite type over S. The nearby cycle functor*

$$R\Psi_X : D_c^b(X_\eta, \overline{\mathbb{Q}}_l) \to D_c^b(X_s \times_s \eta, \overline{\mathbb{Q}}_l)$$

induces

$$\begin{aligned} D_c^b(X_\eta, \overline{\mathbb{Q}}_l)_{ent} &\to D_c^b(X_s \times_s \eta, \overline{\mathbb{Q}}_l)_{ent}, \\ D_c^b(X_\eta, \overline{\mathbb{Q}}_l)_{I\text{-}ent-1} &\to D_c^b(X_s \times_s \eta, \overline{\mathbb{Q}}_l)_{I\text{-}ent-1}. \end{aligned}$$

5.1

Let $S = \operatorname{Spec} R$ be any Henselian trait, η its generic point, and s its closed point. We will keep these notations until 5.7.

Definition 5.4. *(a) Let X be an S-scheme of finite type, and Z a closed subset containing X_s. We say that the pair (X, Z) is semi-stable if, locally for the étale topology, it has the form*

$$(\operatorname{Spec} R[t_1, \ldots, t_n]/(t_1 \ldots t_r - \pi), Z),$$

where π is a uniformizer of R, and Z is defined by the ideal $(t_1 \ldots t_s)$, with $1 \le r \le s \le n$. The pair is called strictly semi-stable if it is semi-stable and Z is the sum of a finite family of regular divisors of X.

(b) Let X be an S-scheme of finite type. We say that X is strictly semi-stable if (X, X_s) is a strictly semi-stable pair.

Let (X, Z) be a strictly semi-stable pair with $Z = \sum_{i \in I} D_i$, where $(D_i)_{i \in I}$ is a finite family of regular divisors. Then $X_s = \bigcup_{i \in I - J} D_i$, where $J = \{i \in I \mid D_i \not\subset X_s\}$. Let $H = \bigcup_{j \in J} D_j$ be the union of the horizontal components. Then $Z = X_s \cup H$.

We will first establish 5.3 in the semi-stable case. We will need points (ii) and (iii) of the following lemma for this (part (i) is used in the proof of (ii)):

Lemma 5.5. *Let (X, Z) be a strictly semi-stable pair over S, $Z = \sum_{i \in I} D_i$ with $(D_i)_{i \in I}$ a finite family of regular divisors, J as above, $U = X - Z$, u the inclusion $U \hookrightarrow X_\eta$, $\Lambda = \mathbb{Z}/l\mathbb{Z}$ with l invertible on S, $G \in \mathrm{Mod_c}(U, \Lambda)$ locally constant and moderately ramified on X.*

(i) Let $i \in J$, $U_{(i)} = X - \bigcup_{h \in I - \{i\}} D_h$, $D_{i, U_{(i)}} = D_i \times_X U_{(i)}$, whence a diagram with cartesian squares

$$\begin{tikzcd}
D_{i,U_{(i)}} \arrow[r, "j_{(i)}"] \arrow[d, "u'_{(i)}"] & (D_i)_\eta \arrow[r] \arrow[d, "(t_i)_\eta"] & D_i \arrow[r] \arrow[d, "t_i"] & (D_i)_s \arrow[d, "(t_i)_s"] \\
U_{(i)} \arrow[r, "j'_{(i)}"] \arrow[d, "u"] & X_\eta \arrow[r] & X \arrow[r] & X_s \\
U & & &
\end{tikzcd}$$

Then the arrow

$$\alpha : (t_i)_s^* R\Psi_X Rj_{(i),*}((Rj'_{(i)})_* G) \to R\Psi_{D_i} Rj_{(i),*}^{D_i,*}((Rj'_{(i)})_* G) \tag{4}$$

composed of $(t_i)_s^* R\Psi_X(Ru_* G) \to R\Psi_{D_i}(t_i)_\eta^*(Ru_* G)$ [22, XIII (2.1.7.2)] and the base change morphism $R\Psi_{D_i}(t_i)_\eta^* Rj_{(i),*}((Rj'_{(i)})_* G) \to R\Psi_{D_i} Rj_{(i),*}^{D_i,*}(t_i)^*(Rj'_{(i)*} G)$ is an isomorphism.

(ii) Let $i \in I - J$, $U_{(i)}, D_{i,U_{(i)}}$ as above, whence a diagram with cartesian squares

$$\begin{tikzcd}
U \arrow[r, leftrightarrow] \arrow[d, "u"] & U_{(i)} \arrow[d] & D_{i,U_{(i)}} \arrow[l, "j_{(i)}"'] & \\
X_\eta \arrow[r] & X & D_i \arrow[l, "t_i"'] \arrow[r, leftrightarrow, "(t_i)_s"] & X_s
\end{tikzcd}$$

Then $R\Psi_{U_{(i)}} G \simeq \Psi_{U_{(i)}} G \in \mathrm{Mod_c}(D_{i,U_{(i)}} \times_s \eta, \Lambda)$ is smooth, moderately ramified on D_i, and the morphism

$$\beta : (t_i)_s^* R\Psi_X Ru_* G \to Rj_{(i),*} R\Psi_{U_{(i)}} G \tag{5}$$

deduced from $R\Psi_X Ru_* G \to (t_i)_{s,*} Rj_{(i),*} R\Psi_{U_{(i)}} G$ [22, XIII (2.1.7.1)] is an isomorphism.

(iii) Suppose $J = \emptyset$. Let $f : Y \to X$ be a morphism of schemes such that Y is a strictly semi-stable S-scheme with $(f^{-1}(D_i))_{i \in I}$ a family of regular divisors of Y. Then the morphism

$$\gamma : f_s^* R\Psi_X G \to R\Psi_Y f_\eta^* G \tag{6}$$

[22, XIII (2.1.7.2)] is an isomorphism.

Point (ii) is a partial generalization of [11, 1.5(a)].

Proof. We first replace the first statement of (ii) with the assertion that the sheaves $R^q\Psi_{U_{(i)}}G$, for $q \in \mathbb{Z}$, are smooth and moderately ramified on D_i. The vanishing of the higher nearby cycles will result from (iii).

The question is local on X. Let y be a point of X_s. We can assume that $y \in D_j$ for all $j \in I$. Let π be a uniformizer of R. There exists an open set of X containing y which is smooth over $\operatorname{Spec} R[t_j]_{j\in I}/(\prod_{j\in I-J} t_j - \pi)$, with t_j defining D_j. By virtue of Abhyankar's lemma, there exists, in a neighborhood of y, a finite covering $g : \tilde{X} = X[T_j]_{j\in I}/(T_j^n - t_j)_{j\in I} \to X$ where n is an integer prime to the characteristic exponent of s, such that $(g|U)^*G$ extends to a locally constant constructible Λ-module on $\tilde{X}$. As G injects into $(g|U)_*(g|U)^*G$ and the quotient G_1 is moderately ramified on X, we can iterate this construction. For any $N > 1$, we obtain, by shrinking X, a resolution

$$G \to (g|U)_*(g|U)^*G \to (g_1|U)_*(g_1|U)^*G_1 \to \cdots \to (g_N|U)_*(g_N|U)^*G_N.$$

Thus we are reduced to proving the lemma for the sheaf $(g|U)_*(g|U)^*G$. As $g^{-1}(D)_{\mathrm{red}} = \sum_{i\in I} g^{-1}(D_i)_{\mathrm{red}}$ is a normal crossings divisor with $(g^{-1}(D_i)_{\mathrm{red}})_{i\in I}$ a family of regular divisors, we can then reduce to proving the lemma for a sheaf G which extends to a locally constant sheaf on X, and then to the case $G = \Lambda_U$ by the projection formula.

Assertion (iii) then follows from the functoriality of [12, 3.3]. More precisely, we have the commutative diagrams

$$\begin{tikzcd}
\Lambda^q f_{s,*}R\Psi_X\Lambda_{X_\eta} \arrow[r, "\sim"] \arrow[d] & f_{s,*}R^q\Psi_X\Lambda_{X_\eta} \arrow[d] \\
\Lambda^q R\Psi_Y f_{\eta,*}\Lambda_{X_\eta} \arrow[r, "\sim"] & R^q\Psi_Y\Lambda_{Y_\eta}
\end{tikzcd}$$

and

$$\begin{tikzcd}
0 \arrow[r] & f_{s,*}\Lambda_{X_s}(-1) \arrow[r] \arrow[d, "\sim"] & f_{s,*}R^1 j_{X,*}\Lambda_{X_\eta} \arrow[r] \arrow[d] & f_{s,*}R^1\Psi_X\Lambda_{X_\eta} \arrow[r] \arrow[d, "H^1\gamma"] & 0 \\
0 \arrow[r] & \Lambda_{Y_s}(-1) \arrow[r] & i_{Y,*}R^1 j_{Y,*}f_{\eta,*}\Lambda_{X_\eta} \arrow[r] & R^1\Psi_Y f_{\eta,*}\Lambda_{X_\eta} \arrow[r] & 0
\end{tikzcd}$$

where $j_X : X_\eta \to X$, $j_Y : Y_\eta \to Y$, $i_X : X_s \to X$, $i_Y : Y_s \to Y$. The lines of the second diagram are short exact sequences and the square on the left is given by the commutative diagram

$$\begin{tikzcd}
f_{s,*}\Lambda_{X_s}(-1) \arrow[r, "d"] \arrow[d, "\sim"] & f_{s,*}\bigoplus_{i\in I}\Lambda_{D_i}(-1) \arrow[r, "c"] \arrow[d, "\sim"] & f_{s,*}i_{X,*}R^1 j_{X,*}\Lambda_{X_\eta} \\
\Lambda_{Y_s}(-1) \arrow[r, "d"] & \bigoplus_{i\in I}\Lambda_{Y_i}(-1) \arrow[r, "c"] & i_{Y,*}R^1 j_{Y,*}f_{\eta,*}\Lambda_{X_\eta}
\end{tikzcd}$$

where the arrows marked d are diagonals and those marked c are induced by the classes of the regular divisors.

(i) We need to show the following assertion: (A) The morphism of functors

$$(t_i)_s^* R\Psi_X Rj_{(i),*} \to R\Psi_{D_i} Rj^{D_i}_{(i),*} t^*_{i,\eta} \tag{7}$$

induces an isomorphism on $Rj^!_{(i),*}\Lambda_U$.

We have a distinguished triangle

$$(t'_i)^*\Lambda_{D_{i,U_{(i)}}}(-1)[-2] \to \Lambda_{U_{(i)}} \to Rj_{(i),*}\Lambda_U \to .$$

Since (7) trivially induces an isomorphism on the first term, (A) is equivalent to: (B) The morphism (7) on $\Lambda_{U_{(i)}}$

$$(t_i)_s^* R\Psi_X Rj_{(i),*}\Lambda_{U_{(i)}} \to R\Psi_{D_i} Rj^{D_i}_{(i),*}\Lambda_{D_{i,U_{(i)}}} \tag{8}$$

is an isomorphism.

We prove these statements by induction on $\#J \geq 1$. The case $\#J = 0$ is empty. In the general case, we first show that for all $j \in J - \{i\}$, (8) is an isomorphism on $(D_{ij})_s$, where $D_{ij} = D_i \cap D_j$. Let

$U_{(ij)} = X - \bigcup_{h \in I-\{i,j\}} D_h$. We consider the commutative diagram

$$\begin{tikzcd}
D_{i,U_{(i)}} \arrow[r, "j_1"] & D_{i,U_{(ij)}} \arrow[r, "j_{2,i}"] & (D_i)_\eta \arrow[r, "t_{j,i}"] & D_i \\
[ul,"t'_i"]U_{(i)} \arrow[r, "j_1"] & U_{(ij)} \arrow[r, "j_2"] \arrow[u] & X_\eta \arrow[r, "t_j"] \arrow[u, "t_{i,\eta}"] & X \arrow[u, "t_i"] \\
D_{j,U_{(ij)}} \arrow[r, "j_{2,j}"] \arrow[u] & (D_j)_\eta \arrow[r, "t_{i,j}"] \arrow[u] & D_j \arrow[u, "t_{ij}"] &
\end{tikzcd}$$

From [21, XII 4.4(i)], the composite

$$(t_{ij})_s^* R\Psi_X R(j_2 j_1)_* \Lambda_{U_{(i)}} \to (t_{j,i})_s^* R\Psi_{D_i} R(j_{2,i} j_1)_* \Lambda_{D_{i,U_{(i)}}} \xrightarrow{(8)(D_{ij})_s} R\Psi_{D_{ij}} Rj_{2,*}^j ((Rj_1^i \Lambda_{D_{i,U_{(i)}}})|_{D_{ij},U_{(ij)}})$$

is equal to the composite

$$\begin{aligned}
(t_{ij})_s^* R\Psi_X R(j_2 j_1)_* \Lambda_{U_{(i)}} &\xrightarrow{\sim} (t_{i,j})_s^* R\Psi_{D_j} R(j_{2,j})_* ((Rj_1^j \Lambda_{U_{(i)}})|_{D_{j,U_{(ij)}}}) \quad \text{(induction (A))} \\
&\to R\Psi_{D_{ij}} Rj_{2,*}^j ((Rj_1^i \Lambda_{U_{(i)}})|_{D_{ij},U_{(ij)}}) \quad (*) \\
&\xrightarrow{\sim} R\Psi_{D_{ij}} Rj_{2,*}^j ((Rj_1^i \Lambda_{D_{i,U_{(i)}}})|_{D_{ij},U_{(ij)}}) \quad \text{by 3.7(ii)}
\end{aligned}$$

where $(*)$ is a morphism of type (7) applied to $(Rj_1^j \Lambda_{U_{(i)}})|_{D_{j,U_{(ij)}}}$. We have the distinguished triangle

$$\Lambda_{D_{j,U_{(ij)}}}(-1)[-2] \to \Lambda_{D_{j,U_{(ij)}}} \to (Rj_1^j \Lambda_{U_{(i)}})|_{D_{j,U_{(ij)}}} \to .$$

Thus the induction hypothesis (B) implies that $(*)$ is an isomorphism. It follows that (8) restricted to $(D_{ij})_s$ is an isomorphism.

It remains to show that (8) restricted to $(V_i)_s$ is an isomorphism, where $V_i = X - \bigcup_{h \in J-\{i\}} D_h$. As $(V_i)_\eta = U_{(i)}$ and $j_{(i),V_i} = \mathrm{id}_{U_{(i)}}$, this follows from (iii).

(ii) As $U_{(i)}$ is smooth over S, $R\Psi_{U_{(i)}} \Lambda_U \simeq \Lambda_{D_{i,U_{(i)}}}$, so it is moderately ramified on D_i. To show that β is an isomorphism, we first treat two special cases: (a) $\#J = 0$; (b) $\#(I-J) = \#J = 1$.

In case (a), we have $U = X_\eta$, $j_\eta = \mathrm{id}_{X_\eta}$. Let $D = D_i$, $E = \bigcup_{j \in I-\{i\}} D_j$, $D^* = D - D \cap E = D_{i,U_{(i)}}$, which gives a commutative diagram

$$\begin{tikzcd}
U \arrow[r, "j_{(i)}"] & X - E \arrow[d, "u'_i"] & D^* \arrow[l, "t'_{(i)}"'] \\
X_\eta \arrow[u, "j"] \arrow[r] & X \arrow[d, "t_i"] & X_s \arrow[l, "(t_i)_s"'] \\
 & D &
\end{tikzcd}$$

It is enough to show that $H^1\beta$ is an isomorphism. We have the commutative diagram

$$\begin{tikzcd}
\Lambda^q (t_i)_s^* R^1\Psi_X \Lambda_U \arrow[r, "\sim"] \arrow[d, "\Lambda^q H^1\beta"] & (t_i)_s^* \Lambda^q R^1\Psi_X \Lambda_U \arrow[d, "(t_i)_s^* H^1\beta"] \\
\Lambda^q R^1 j_{(i)*} R\Psi_{U_{(i)}} \Lambda_U \arrow[r, "\sim"] & R^q j_{(i)*} R^1\Psi_{U_{(i)}} \Lambda_U
\end{tikzcd}$$

The composite $p_2 r_2$ is induced from the isomorphism $\Lambda_{D^*} \xrightarrow{\sim} R\Psi_{X-E}\Lambda_U$, so is an isomorphism. We have

$$H^1(t_i)_s^* Rj_{(i)*}^! \Lambda_{X-E} = \bigoplus_{j \in I-\{i\}} (t_{j,i})_s^* \Lambda_{D_{ij}},$$

$$H^1(t_i)_s^* Rj_* \Lambda_U = \Lambda_D \oplus \bigoplus_{j \in I-\{i\}} (t_{j,i})_s^* \Lambda_{D_{ij}}.$$

$H^1(t_i)_s^* R\Psi_X \Lambda_U$ is the quotient of $\Lambda_D \oplus \bigoplus_{j \in I - \{i\}} (t_{j,i})_s^* \Lambda_{D_{ij}}$ by Λ_D included diagonally. $H^1 r_1$ is the inclusion into the second factor, $H^1 p_1$ is the projection. Thus $H^1(p_1 r_1)$ is an isomorphism. It follows that $H^1\beta$ is an isomorphism. This proves (a).

In case (b), we have $D_i = X_s$, $(t_i)_s = \mathrm{id}_{X_s}$, $H = D_j$ where j is the element of J. We set $V = U_{(i)} = X - H$, which gives a diagram with cartesian squares

$$\begin{tikzcd}
U \arrow[r, "u"] & V \arrow[r, "i_V"] \arrow[d, "v"] & V_s \\
X_\eta \arrow[u] \arrow[r, "h"] & X \arrow[r, "i"] & X_s \\
H_\eta \arrow[u] & H \arrow[u, hook] \arrow[r] & H_s
\end{tikzcd}$$

We have the distinguished triangle

$$i^* R v_* \Lambda_V \xrightarrow{\beta'} R\Psi_X R u_* \Lambda_U \to R\Phi_X R v_* \Lambda_V \to .$$

We have $R^q v_* \Lambda_V = \begin{cases} \Lambda_X & \text{if } q = 0, \\ h_* \Lambda_H(-1) & \text{if } q = 1, \\ 0 & \text{otherwise.} \end{cases}$ Thus $R\Phi_X R^q v_* \Lambda_V = 0$ for all q. So $R\Phi_X R v_* \Lambda_V = 0$, and β' is an isomorphism. Moreover, we have the commutative diagram

$$\begin{tikzcd}
i^* R v_* \Lambda_V \arrow[r, "\beta'"] & R\Psi_X R u_* \Lambda_U \\
R v_{s,*} i_V^* \Lambda_V \arrow[u, "3.7(\mathrm{ii})\sim"] \arrow[r, "\sim"] & R v_{s,*} R\Psi_V \Lambda_U \arrow[u, "\beta"']
\end{tikzcd}$$

Thus β is an isomorphism. This proves (b).

For the general case, we proceed by induction on $\#J$. The case $\#J = 0$ is case (a) treated above. Assume $\#J \geq 1$. Take $j \in J$, this gives a diagram

$$\begin{tikzcd}
U \arrow[r] \arrow[d] & U_{(i)} \arrow[d] & D_{i,U_{(i)}} \arrow[l] \arrow[d] \\
U_{(j)} \arrow[r] & U_{(ij)} & D_{i,U_{(ij)}} \arrow[l] \\
D_{j,U_{(j)}} \arrow[u] & D_{j,U_{(ij)}} \arrow[l] & D_{ij,U_{(ij)}} \arrow[l] \arrow[u]
\end{tikzcd}$$

We have a commutative diagram in $D_c^b(D_{ij} \times_s \eta, \Lambda)$:

$$\begin{tikzcd}
(t_{i,j})_s^*(t_j)_s^* R\Psi_X R u_* \Lambda_U \arrow[r, "\beta_1"] \arrow[d, "(i)"'] & (t_{i,j})_s^* R j_{(j),*} R\Psi_{U_{(j)}} R u_{(j),*} \Lambda_U \arrow[d, "chg't\ \text{de base}"] & \\
(t_{i,j})_s^* R\Psi_{D_j} R(u|D_j)_* \Lambda_U \arrow[r, "\beta_2"] & R j_{2,*} R\Psi_{D_{j,U_{(ij)}}} R j'_{1,*} \Lambda_U \arrow[r, "3.7(i)"'] & \ldots
\end{tikzcd}$$

where $\beta|_{D_{ij}}$ is the composite of the two arrows of the first line of (†), β_1 is induced by an arrow of type (5.6.2) and β_2 is an arrow of type (5.6.2). The commutativity of the square on the right is clear and that of the square on the left is seen by applying 5.7 to the square

$$\begin{tikzcd}
D_{j,U_{(ij)}} \arrow[r] & U_{(ij)} \\
D_j \arrow[u] \arrow[r] & X
\end{tikzcd}$$

The arrow β_2 is an isomorphism by virtue of the induction hypothesis and the distinguished triangle

$$\Lambda_{D_{j,U_{(ij)}}}(-1)[-2] \to \Lambda_{D_{j,U_{(ij)}}} \to l_{j,*}^* R j_{*,U} \Lambda_U \to .$$

Thus $\beta|_{D_{ij}}$ is an isomorphism. It remains to show that $\beta|_{D_i - D_{ij}}$ is an isomorphism, which results from the induction hypothesis. ☐

Lemma 5.6. *Let*

$$\begin{array}{ccc} X' & \xrightarrow{h} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

be a commutative square of S-schemes, and Λ a ring. Then there is a commutative diagram of functors $D^+(X_\eta, \Lambda) \to D^+(Y'_s \times_s \eta, \Lambda)$

$$\begin{array}{ccc} g_s^* R\Psi_Y Rf_{\eta,*} & \longrightarrow & Rf'_{s,*} Rh_s^* R\Psi_X \\ \downarrow & & \downarrow \\ R\Psi_{Y'} g_\eta^* Rf_{\eta,*} & \longrightarrow & R\Psi_{Y'} Rf'_{\eta,*} h_\eta^* \end{array}$$

where the horizontal arrows are base changes, the upward arrows are [22, XIII (2.1.7.1)], and the downward arrows are [ibid., XIII (2.1.7.2)].

Proof. Let $K = \kappa(\eta)$, $\bar{K}$ a separable closure of K, $\bar{\eta} = \operatorname{Spec} \bar{K}$, and $\bar{S}$ the normalization of S in $\bar{\eta}$. We add a bar over everything for base change $\bar{S} \to S$. We denote $i : \bar{s} \to \bar{S}$, $j : \bar{\eta} \to \bar{S}$. It is enough to show the commutativity of the diagram of functors $D^+(\bar{X}_\eta, \Lambda) \to D^+(\bar{Y}'_s, \Lambda)$

$$\begin{array}{ccc} g_s^* i_Y^* Rf_{\eta,*} j_{X,*} & \xrightarrow{\sim} & f'_{s,*} h_s^* i_X^* j_{X,*} \\ \downarrow{\scriptstyle \sim} & & \downarrow{\scriptstyle \sim} \\ i_{Y'}^* g_\eta^* j_{Y,*} Rf_{\eta,*} & & i_{Y'}^* f'_{\eta,*} h_\eta^* j_{X,*} \\ \downarrow & & \downarrow \\ i_{Y'}^* g_\eta^* Rf_{\eta,*} j_{X,*} & \longrightarrow & i_{Y'}^* Rf'_{\eta,*} h_\eta^* j_{X,*} \end{array}$$

where all the arrows are base changes. The commutativity of the cell at the top (resp. at the bottom) results from [21, XII 4.4 (i)] (resp. [21, XII 4.4 (ii)]). The commutativities of the other two cells are trivial. ☐

Proposition 5.7. *Let S be as in 5.2, (X, Z) a semi-stable pair over S, $U = X - Z$, $u : U \to X_\eta$, $G \in \mathrm{Mod_c}(U, \overline{\mathbb{Q}}_l)$ integral (resp. inverse integral) smooth, moderately ramified on X. Then $R\Psi_X Ru_* G$ is I-integral (resp. I-inverse integral).*

Proof. We can assume that (X, Z) is a strictly semi-stable pair over S. First treat the particular case where Z is a regular divisor. Then $Z = X_s$, $U = X_\eta$, $u = \mathrm{id}$, X is smooth over S. Let $x \in |X_s|$. After an étale base change, we can assume that $X \to S$ admits a section σ such that $\sigma(s) = x$. From 5.6(iii), $(R\Psi_X G)_x = \sigma_s^* R\Psi_X G \simeq R\Psi_S \sigma_\eta^* G$ is I-integral (resp. I-inverse integral), because $R\Psi_S$ is identified with the identity.

The general case follows from 5.6(ii), the special case above, and the variant 5.2 of 3.8 (resp. 4.5) over $s \times_s \eta$. ☐

5.2

For the proof of 5.3, we will need lemma 5.10 below, analogous to 4.1. Let S be an excellent Henselian trait, X a separated S-scheme of finite type, $U \subset X_\eta$ an open subset. For $f : S' \to S$ a finite morphism of traits and $g : Y \to X_s$ a proper morphism of schemes, consider the following condition:

5.9.1 We have $Y = Y_1 \coprod Y_2$, where Y_1 is strictly semi-stable over S', $g^{-1}(U) \subset Y_2$ and $(Y_2, Y_2 - g^{-1}(U))$ is a strictly semi-stable pair over S'.

Lemma 5.8. *(i) There exists a finite morphism of traits $S' \to S$ and a proper morphism $r_0 : X'_0 \to X_s$ of schemes satisfying 5.9.1 (where $Y = X'_0$) and such that $(r_0)_\eta$ is surjective. (ii) For $n \geq 0$, there exists a finite morphism of traits $f : S' \to S$ and an augmentation of n-truncated s-split simplicial schemes $r_. : X'_. \to X_s$ such that for $0 \leq m \leq n$, f and r_m satisfy 5.9.1 (where $Y = X'_m$) and $r_{0,\eta}$ is a proper n-truncated hypercovering.*

Proof. (i) Case where X is integral and X_η is geometrically irreducible. Results from [14, 6.5]. Note that the hypothesis in [ibid.] that S is complete can be replaced by the excellence of S, see [19]. General case. We can assume that the irreducible components of X_η are geometrically irreducible. We proceed by induction on the number n of irreducible components of X_η. If $n = 0$, then X_η is empty. We take $S' = S$, $X'_0 = \emptyset$. (i) is evident. For $n \geq 1$, we take an irreducible component U_1 of X_η. Let X_1 be the closure of U_1 in X. It is an irreducible component of X. Let X_2 be the union of the other irreducible components of X. We equip X_1 and X_2 with the reduced induced scheme structures. $(X_2)_\eta$ has $n-1$ irreducible components, which are geometrically irreducible. We have a finite surjective morphism $X_1 \coprod X_2 \to X$. We apply (i) to X_1 and obtain $S_1 \to S$ and $(X_1)'_0 \to (X_1)_{s_1}$. It is then enough to apply the induction hypothesis to $(X_2)_{s_1}$.

(ii) We proceed by induction on n. When $n = 0$, (ii) degenerates to (i). Suppose we are given $S_n \to S$ and $r^{(n)} : X^{(n)} \to X_{s_n}$ satisfying (ii). We apply (i) to the scheme $(\mathrm{cosk}_n(X^{(n)}/X_{s_n}))_{n+1}$ over S_n (with U replaced by its inverse image) and we obtain a finite morphism of traits $S' \to S_n$ and a morphism $\beta : N \to (\mathrm{cosk}_n((X^{(n)})_{s'}/X_{s'}))_{n+1}$ which is proper with β_n surjective satisfying 5.9.1. Then the $X_{s'}$-simplicial scheme $(n+1)$-truncated s-split $r'. : X'. \to X_{s'}$ associated to the triple $((X^{(n)})_{s'}, N, \beta)$ verifies the conditions of (ii) for $n+1$. □

Proof of 5.3

The proof is parallel to that of (2.5.1). We proceed by induction on $d = \dim X_\eta$. The case $d < 0$ is trivial. Let $d \geq 0$. We must show that for $G \in \mathrm{Mod}_c(X_\eta, \overline{\mathbb{Q}}_l)$ integral (resp. inverse integral), $R\Psi_X G$ is integral (resp. I-inverse integral).

We can assume X_η is reduced. We can assume X is affine, hence separated. We choose a regular open set $j : U \to X_\eta$ such that $G|_U$ is smooth and its complement $Z = X_\eta - U$ has dimension $< d$. Let $\bar{Z}$ be the closure of Z, $i : \bar{Z} \to X$. We have the distinguished triangle

$$R\Psi_{\bar{Z}} i_\eta^* R i_{\eta,!} G \to R\Psi_X G \to R\Psi_X Rj_* j^* G \to .$$

As $R\Psi_{\bar{Z}} i_\eta^* is * R_{\bar{Z}} i_{\eta,!} G$ is integral (resp. I-inverse integral) by the induction hypothesis, it is enough to see that $R\Psi_X Rj_* j^* G$ is integral (resp. I-inverse integral).

Let $H = j^* G$. $H \simeq (H_0 \otimes_{\mathcal{O}} E) \otimes_E \overline{\mathbb{Q}}_l$ with H_0 smooth. Let $p : U' \to U$ be a surjective étale covering that trivializes $H_0 \otimes_{\mathcal{O}} (\mathcal{O}/\mathfrak{m})$, where $\mathfrak{m}$ is the maximal ideal of $\mathcal{O}$. H is a direct factor of $p_* p^* H$, so it suffices to see the integrality (resp. the I-integrality inverse) of $R\Psi_X R(jp)_* p^* H$. We factor the composite $U' \xrightarrow{jp} X_\eta \to X$ into $U' \xrightarrow{j'} X' \xrightarrow{g} X$ where j' is an open immersion and g is proper.

$$R\Psi_X R(jp)_* p^* H \simeq R g_{s,*} R\Psi_{X'} R j'_{n,*} p^* H.$$

So it is enough to verify that for X a separated scheme of finite type over S, $j : U \hookrightarrow X_\eta$ an open set and $G \in \mathrm{Mod}_c(U, \overline{\mathbb{Q}}_l)$ integral (resp. inverse integral), $G \simeq (G_0 \otimes_{\mathcal{O}} E) \otimes_E \overline{\mathbb{Q}}_l$ with $G_0 \otimes_{\mathcal{O}} (\mathcal{O}/\mathfrak{m})$ constant, we have $R\Psi_X Rj_* G$ integral (resp. I-inverse integral).

Let $i \geq 0$. We apply 5.10(ii) to j and $n = i+1$. We obtain a finite morphism of traits $f : S' \to S$ and a cartesian square of simplicial schemes that are $(i+1)$-truncated s-split

$$\begin{array}{ccc} U' & \longrightarrow & X' \\ \Big\downarrow{\scriptstyle s} & & \Big\downarrow{\scriptstyle r} \\ U_{S'} & \longrightarrow & X_{s'} \end{array}$$

where $s.$ is a proper $(i+1)$-truncated hypercovering and r_m satisfies 5.9.1 for $0 \leq m \leq i+1$. We denote changes of base by f again by f.

$$f^* R\Psi_{X/S} Rj_* G \simeq R\Psi_{X_{S'}/S'} f^* Rj_* G \simeq R\Psi_{X_{S'}/S'} Rj'_{s',*} G_{S'},$$

thus

$$\begin{aligned} \tau_{\leq i} f^* R\Psi_{X/S} Rj_* G &\simeq \tau_{\leq i} R\Psi_{X_{S'}/S'} Rj'_{s',*} Rs_{\bullet,*} s^* G_{S'} \\ &= \tau_{\leq i} R\Psi_{X_{S'}/S'} Rr_{\bullet,*} Rj'_{\bullet,*} s^* G_{S'} \simeq \tau_{\leq i} Rr_{\bullet,s,*} R\Psi Rj'_{\bullet,*} s^* G_{S'}. \end{aligned}$$

It suffices to see that $R\Psi_{X'_m/S'} Rj'_{m,*} s_m^* G_{S'}$ is integral (resp. I-inverse integral), for $0 \leq m \leq i$. It is then enough to apply 5.8 (which is licit, since the $s_m^* G_{S'}$ are moderate). □

Variant 5.9. *Let S, η, s, l be as in 5.2. Let X be a scheme of finite type over S, $F \in \mathrm{Mod_c}(X, \overline{\mathbb{Q}}_l)$. For $r \in \mathbb{Q}$, F is said to be r-integral (resp. r-inverse integral) if F_η and F_s are. Same for complexes. We set*

$$\delta(X) = \max\{\dim X_\eta + 1, \dim X_s\},$$

$$D_X = \mathrm{RHom}(-, R\alpha_X^! \overline{\mathbb{Q}}_l(1)[2]), \text{ where } \alpha_X : X \to S.$$

We have analogues of 2.4 to 2.8, 4.8, 4.9, replacing dim by δ, d_r by

$$\max_{y \in |Y_\eta| \cup |Y_s|} \dim f^{-1}(y).$$

We also have an analogue of 3.8 by adding the hypothesis that (X, D) is a semi-stable pair on S.

Indeed, 2.4 for S follows trivially from 2.4 for s and for η. For the analogue of 3.8, let $F \in \mathrm{Mod_c}(U, \overline{\mathbb{Q}}_l)$ be smooth, integral, and moderately ramified on X. From 3.8 for η and 5.8, Ru_*F and $R\Psi_X Ru_*F$ are I-integral, where $u : U \to X_\eta$. Let $I = \mathrm{Ker}(\mathrm{Gal}(\bar\eta/\eta) \to \mathrm{Gal}(\bar s/s))$ be the inertia group. It is an extension of $\mathbb{Z}_{p'}(1)$ by a pro-p-group P. The Hochschild-Serre spectral sequence gives

$$E_2^{p,q} = H^p(I, R^q\Psi_X Ru_*F) \Rightarrow i^* R^{p+q} j_* F,$$

where $i : X_s \to X$. Let $R^q = (R^q\Psi_X Ru_*F)^P$. If σ is a generator of $\mathbb{Z}_{p'}(1)$, we have

$$E_2^{0,q} = \mathrm{Ker}(\sigma - 1, R^q), \quad E_2^{1,q} = \mathrm{Coker}(\sigma - 1, R^q)(-1),$$

and $E_2^{p,q} = 0$ for $p \neq 0, 1$. Thus Rj_*F is I-integral.

The results (2.5.1) and (2.5.3) for S follow from these results for s and for η and from 5.3, by imitating the arguments in [4, Th. finitude, 3.11, 3.12] as follows. The special case $f = j_Y : Y_\eta \to Y$ results from 5.3 and the Hochschild-Serre spectral sequence

$$E_2^{p,q} = H^p(I, R^q\Psi_Y -) \Rightarrow i_Y^* R^{p+q} j_{Y,*} -,$$

where $i_Y : Y_s \to Y$. Let's treat the general case. Let $L \in D_c^b(X, \overline{\mathbb{Q}}_l)$ be integral (resp. I-inverse integral). Let $i_X : X_s \to X, j_X : X_\eta \to X$. We have the distinguished triangles

$$i_{X,*} Ri_X^! L \to L \to Rj_{X,*} j_X^* L \to,$$

$$R(fi_X)_* Ri_X^! L \to Rf_* L \to R(fj_X)_* j_X^* L \to .$$

From the special case, $Rj_{X,*}j_X^*L$ is integral (resp. I-inverse integral), so $Ri_X^!L$ is also. As $fi_X = i_Y f_s$, $fj_X = j_Y f_\eta$, we conclude by applying (2.5.1) (resp. (2.5.3)) for s and for η and the special case.

Once (2.5.1) and (2.5.3) are established for S, we can redo 4.6 to 4.9 and 4.11, giving the proof of 2.5 to 2.8, except for (2.6.1) and (2.6.3). The results (2.6.1) and (2.6.3) for S follow from their analogues for s and for η, by applying 4.6 for S and (4.10.1) to $U = Y_\eta, W = Y_s$. Note that (2.7.3) and (2.7.1) for S can also be deduced from their analogues for s and for η, by applying (4.7.1) to $U = X_\eta, V = X_s$.

Variant 5.10. *One can replace the usual sheaves everywhere by Weil sheaves [5, 1.1.10]. All the results and variants that precede remain valid.*

Variant 5.11. *Let A be an integrally closed subring of $\overline{\mathbb{Q}}_l$. We set $A^{-1} = \{\alpha \in \overline{\mathbb{Q}}_l \mid \alpha^{-1} \in A\}$. We fix an embedding $\iota : \overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_l$. With the notation of §1, for $r \in \mathbb{Q}$, a $\overline{\mathbb{Q}}_l$-sheaf F on a scheme X of finite type over η is called r-A-integral (resp. r-A-inverse integral) if for every $x \in |X|$, the eigenvalues of Φ_x are in $\iota(q^r)A$ (resp. $\iota(q^r)A^{-1}$), where $q = \#\kappa(x)$. This definition does not depend on the choices of Φ_x and ι. We say that F is A-integral (resp. A-inverse integral) if it is 0-A-integral (resp. 0-A-inverse integral). We also define the A-integrality of complexes. All the results and variants remain valid for these notions.*

If A is moreover completely integrally closed [3, V, §1, n° 4, def. 5] (in particular if A is the integral closure of a noetherian integrally closed subring of $\overline{\mathbb{Q}}_l$ [ibid., exerc. 14]), then according to a lemma of Fatou [13, 8.3], F is r-A-integral if and only if for all $x \in |X|$ and all integers $n \geq 1$, $\mathrm{Tr}(\Phi_x^n, F_{\bar x})$ belongs to $\iota(q^{nr})A$. This criterion does not have an analogue for inverse integral sheaves.

If we take for A the integral closure of $\mathbb{Z}$ in $\overline{\mathbb{Q}}_l$, we recover the notion of integrality from what precedes.

Let T be a set of prime numbers. If we take for A the integral closure of $\mathbb{Z}[1/t]_{t \in T}$ in $\overline{\mathbb{Q}}_l$, we recover the notion of T-integrality from [22, XXI 5] and [7].

6 Appendix: Integrality on algebraic stacks

Let K be a finite field of characteristic p or a local field of residual characteristic p, $\eta = \operatorname{Spec} K$, l a prime number $\neq p$. For X an n-algebraic stack [16, 4.1] of finite type, we denote by $\mathrm{Mod_c}(X, \overline{\mathbb{Q}}_l)$ the category of constructible $\overline{\mathbb{Q}}_l$-sheaves on the lisse-étale site of X [ibid., 12.1(i)]. We have, by [15], a triangulated category $D_c(X, \overline{\mathbb{Q}}_l)$ equipped with a t-structure with heart $\mathrm{Mod_c}(X, \overline{\mathbb{Q}}_l)$. We will write $\mathrm{Mod_c}(X)$ for $\mathrm{Mod_c}(X, \overline{\mathbb{Q}}_l)$ and $D_c(X)$ for $D_c(X, \overline{\mathbb{Q}}_l)$. We have a formalism of six operations: for $f : X \to Y$ a morphism of n-algebraic stacks of finite type,

$$\begin{aligned} D_X : D_c(X)^{op} &\to D_c(X), \\ - \otimes - : D_c^-(X) \times D_c^-(X) &\to D_c^-(X), \\ \mathrm{RHom}_X(-,-) : D_c(X)^{op} \times D_c^+(X) &\to D_c^+(X), \\ Rf_* : D_c^+(X) &\to D_c^+(Y), \\ Rf_! : D_c^-(X) &\to D_c^-(Y), \\ f^*, Rf^! : D_c(Y) &\to D_c(X). \end{aligned}$$

If X is an n-Deligne-Mumford stack of finite type, $\mathrm{Mod_c}(X)$ identifies with the category of constructible $\overline{\mathbb{Q}}_l$-sheaves on the étale site of X [16, 12.1(ii)].

Definition 6.1. *Let $\epsilon : \mathbb{Z} \to \mathbb{Q}$ be a function. We say that $L \in D_c(X)$ is ϵ-integral (resp. ϵ-inverse integral) if for any point $i : x \to X$ with $\kappa(x)$ a finite extension of K and any $a \in \mathbb{Z}$, $H^a(i^*L) \in \mathrm{Mod_c}(x, \overline{\mathbb{Q}}_l)$ is $\epsilon(a)$-integral (resp. $\epsilon(a)$-inverse integral) (in the sense of 2.1).*

If X is a scheme of finite type over η and $L \in D_c^b$, then this definition coincides with 2.2.

Let $f : X \to Y$ be a morphism of n-algebraic stacks of finite type. If $M \in D_c(Y)$ is ϵ-integral (resp. ϵ-inverse integral), then so is $f^*M \in D_c(X)$. The converse is true when f is surjective. This gives the following criterion: $L \in D_c(X)$ is ϵ-integral (resp. ϵ-inverse integral) if and only if for any (or for one) surjective morphism $g : \tilde{X} \to X$ with $\tilde{X}$ a scheme of finite type over η, g^*L is ϵ-integral (resp. ϵ-inverse integral) (in the sense of 2.2). For $r \in \mathbb{Q}$, $\otimes$ induces

$$\begin{aligned} D_c(X)_{rI\text{-ent}} \times D_c(X)_{rI\text{-ent}} &\to D_c(X)_{rI\text{-ent}}, \\ D_c(X)_{rI\text{-ent}-1} \times D_c(X)_{rI\text{-ent}-1} &\to D_c(X)_{rI\text{-ent}-1}. \end{aligned}$$

For $F, G \in \mathrm{Mod_c}(X)$ with F smooth, if F is inverse integral (resp. integral) and G is integral (resp. inverse integral), we have

$$\mathrm{Hom}(F, G) \in \mathrm{Mod_c}(X)_{\mathrm{ent}} (\text{resp. } \in \mathrm{Mod_c}(X)_{\mathrm{ent}-1}).$$

We henceforth assume that X is non-empty. Recall that a presentation $P : \tilde{X} \to X$ is a surjective smooth morphism with $\tilde{X}$ an algebraic space. We set $c_X = \min_P \dim P \in \mathbb{N}$, where P runs through presentations $P : \tilde{X} \to X$, $\dim P = \sup_{x \in \tilde{X}} \dim_x P$ [16, p. 98], $d_X = \dim X \in \mathbb{Z}$ [ibid., (11.15)]. By definition, $d_X \geq -c_X$. We have $c_X = 0$ if and only if X is an n-Deligne-Mumford stack.

Let $f : X \to Y$ be a morphism of stacks. On pose $c_r = \min \dim P \in \mathbb{N}$, where the minimum is taken over all systems

$$\begin{tikzcd} \tilde{X} \arrow[r, "P"] & X \times_Y \tilde{Y} \arrow[r] & X \arrow[d, "f"] \\ & \tilde{Y} \arrow[u, "Q"] \arrow[r] & Y \end{tikzcd}$$

where P and Q are presentations and the square is 2-cartesian, $d_r = \dim f = \max_\xi \dim X_\xi \in \mathbb{Z}$, where ξ runs through the points of Y. We have $d_r \geq -c_r$. Recall that f is called relatively Deligne-Mumford [ibid., 7.3.3] if for any affine scheme $\tilde{Y}$ and any morphism $\tilde{Y} \to Y$, the fiber product $\tilde{Y} \times_Y X$ is an n-Deligne-Mumford stack. We have $c_r = 0$ if and only if f is a relatively Deligne-Mumford morphism. We have $c_r \leq c_X \leq c_Y + c_r$, $d_r - c_Y \leq d_X \leq d_Y + d_r$.

Proposition 6.2. *The functor D_X induces*

$$\begin{aligned} D_c(X)_{ent-1} &\to D_c(X)_{-d_X\text{-}ent}, \\ D_c(X)_{I\text{-}ent-1} &\to D_c(X)_{(I-c_X)\text{-}ent}, \\ D_c(X)_{I\text{-}ent} &\to D_c(X)_{(I+d_X)\text{-}ent-1}, \\ D_c(X)_{ent} &\to D_c(X)_{c_X\text{-}ent-1}. \end{aligned}$$

Furthermore, for $F \in \mathrm{Mod}_c(X)_{ent-1}$, $H^a(D_X F)$ is $(a - c_X + 1)$-integral, for $a \le c_X - 1$.

Proof. Take a presentation $P : \tilde{X} \to X$ purely of dimension c_X with $\tilde{X}$ a scheme of finite type over η. We have $\dim \tilde{X} \le d_X + c_X$. For $L \in D_c(X)$, $P^* D_X L \simeq D_{\tilde{X}} RP^! L \simeq (D_{\tilde{X}} P^* L)(-c_X)[-2c_X]$. As the cohomological amplitude of $D_{\tilde{X}}$ is bounded, it is sufficient to apply 2.7. $\square$

Proposition 6.3. *The functor $Rf^!$ induces*

$$\begin{aligned} D_c(Y)_{ent} &\to D_c(X)_{-d_r\text{-}ent}, \\ D_c(Y)_{I\text{-}ent} &\to D_c(X)_{(I-d_Y-c_Y-c_r)\text{-}ent}, \\ D_c(Y)_{I\text{-}ent-1} &\to D_c(X)_{(I+d_r)\text{-}ent-1}, \\ D_c(Y)_{ent-1} &\to D_c(X)_{(d_Y+c_Y+c_r)\text{-}ent-1}. \end{aligned}$$

Proof. Form the 2-cartesian square diagram

$$\begin{tikzcd} \tilde{X} \arrow[r, "P"] & \tilde{Y} \times_Y X \arrow[r, "Q'"] & X \arrow[d, "f"] \\ & \tilde{Y} \arrow[u, "Q"] \arrow[r] & Y \end{tikzcd}$$

where Q is a presentation purely of dimension c_Y, P is a presentation purely of dimension c_r, $\tilde{Y}$ is a quasi-compact scheme, $\tilde{X}$ is an affine scheme (hence separated over $\tilde{Y}$). We have $\dim \tilde{Y} \le d_Y + c_Y$, $\dim(f_Y \circ P) \le d_r + c_r$. For $L \in D_c(Y)$,

$$(Q' \circ P)^* Rf^! L \simeq P^* Rf_Y^! Q^* L \simeq R(f_Y \circ P)^! Q^* L(-c_r)[-2c_r].$$

It is then sufficient to apply 2.6. $\square$

Proposition 6.4. *The functor Rf_* induces*

$$\begin{aligned} D_c^+(X)_{ent} &\to D_c^+(Y)_{ent}, \\ D_c^+(X)_{I\text{-}ent-1} &\to D_c^+(Y)_{I\text{-}ent-1}, \end{aligned}$$

and $Rf_!$ induces

$$\begin{aligned} D_c^-(X)_{I\text{-}ent} &\to D_c^-(Y)_{(I-d_r)\text{-}ent}, \\ D_c^-(X)_{ent-1} &\to D_c^-(Y)_{d_r\text{-}ent-1}. \end{aligned}$$

If f is relatively Deligne-Mumford, Rf_* induces

$$\begin{aligned} D_c^+(X)_{I\text{-}ent} &\to D_c^+(Y)_{(I-d_X-c_Y)\text{-}ent}, \\ D_c^+(X)_{ent-1} &\to D_c^+(Y)_{(d_X+c_Y)\text{-}ent-1}, \end{aligned}$$

and $Rf_!$ induces

$$\begin{aligned} D_c^-(X)_{ent} &\to D_c^-(Y)_{ent}, \\ D_c^-(X)_{I\text{-}ent-1} &\to D_c^-(Y)_{I\text{-}ent-1}. \end{aligned}$$

Proof. Let $\tilde{Y} \to Y$ be a presentation with $\tilde{Y}$ a scheme of finite type over η. For (6.4.1) and (6.4.2), up to replacing Y by $\tilde{Y}$, we can assume that Y is a scheme. We take a smooth hypercovering $P. : X. \to X$ where the X_n are affine schemes (hence separated over Y). For $L \in D_c^+(X)$, $Rf_* \to Rf_* RP_{\bullet,*} P^* L$. It is then sufficient to apply (2.5.1) and (2.5.3).

For the results concerning $Rf_!$, we can suppose that Y is the spectrum of a field. We have $d_X = d_r$. Then (6.4.3) and (6.4.4) follow from the last line and from 6.2: $Rf_! = D_Y Rf_* D_X$ induces

$$\begin{aligned} D_c(X)_{I\text{-ent}} &\xrightarrow{D_X} D_c^+(X)_{(I+d_X)\text{-ent}-1} \xrightarrow{Rf_*} D_c^+(Y)_{(I+d_X)\text{-ent}-1} \xrightarrow{D_Y} D_c(Y)_{(I-d_X)\text{-ent}}, \\ D_c(X)_{\text{ent}-1} &\xrightarrow{D_X} D_c^+(X)_{-d_X\text{-ent}} \xrightarrow{Rf_*} D_c^+(Y)_{-d_X\text{-ent}} \xrightarrow{D_Y} D_c(Y)_{d_X\text{-ent}-1}. \end{aligned}$$

For (6.4.7) and (6.4.8), we are thus reduced to the case where X is an n-Deligne-Mumford stack. We proceed by induction on d_X. There exists a dominant open immersion $j : U \to X$ and a finite étale morphism $\pi : \tilde{U} \to U$, where $\tilde{U}$ is an affine scheme [16, 6.1.1]. Let Z be the closed complement of U in X, $i : Z \to X$. For $L \in D_c(X)$, the distinguished triangle $j_!j^*L \to L \to i_*i^*L \to$ induces the distinguished triangle

$$R(fj)_!j^*L \to Rf_!L \to R(fi)_!i^*L \to .$$

Since j^*L is a direct factor of $\pi_*\pi^*j^*L$, it is sufficient to apply (2.4.1) and (2.4.3) to $R(fj\pi)_!(j\pi)^*L$ and the induction hypothesis to $R(fi)_!i^*L$. Finally, (6.4.5) and (6.4.6) result from the last line and from 6.2: $Rf_* \simeq D_Y Rf_! D_X$ induces

$$D_c^+(X)_{I\text{-ent}} \xrightarrow{D_X} D_c^-(X)_{(I+d_X)\text{-ent}-1} \xrightarrow{Rf_!} D_c^-(Y)_{(I+d_X)\text{-ent}-1} \xrightarrow{D_Y} D_c^+(Y)_{(I-d_X-c_Y)\text{-ent}},$$
$$D_c^+(X)_{\text{ent}-1} \xrightarrow{D_X} D_c^-(X)_{-d_X\text{-ent}} \xrightarrow{Rf_!} D_c^-(Y)_{-d_X\text{-ent}} \xrightarrow{D_Y} D_c^+(Y)_{(d_X+c_Y)\text{-ent}-1}. \qquad \square$$

Corollary 6.5. *Assertion (2.5.2) is true without the separation hypothesis.*

Proof. This is a particular case of (6.4.5). $\square$

Remark 6.6. *(i) The first part of the proof of 6.4 shows that Rf_* sends $\mathrm{Mod_c}(X)_{ent-1}$ into $D_c^+(Y)_{\epsilon\text{-}ent-1}$, where*

$$\epsilon(a) = \begin{cases} a & \text{if } 0 \le a \le d_X + c_X + c_Y, \\ \lfloor \frac{a-(d_X-d_Y)}{c_X+1} \rfloor & \text{if } a > d_X + c_X + c_Y. \end{cases}$$

Here E is the integer part function. When f is not relatively Deligne-Mumford, this slightly improves (6.4.2). One can deduce a slight improvement of (6.4.3).

(ii) If f is a separated, representable, and quasi-finite morphism [16, 3.10.1] with $d_X + c_Y \ge 1$, we have an analogue of 4.8 that improves (6.4.5): Rf_* sends $D_c^+(X)_{I\text{-}ent}$ into $D_c^+(Y)_{(I+1-d_X-c_Y)\text{-}ent}$.

Proposition 6.7. *The functor* $\mathrm{RHom}(-,-)$ *induces*

$$D_c(X)^{op}_{ent-1} \times D_c^+(X)_{ent} \to D_c^+(X)_{ent},$$
$$D_c(X)^{op}_{I\text{-}ent-1} \times D_c^+(X)_{I\text{-}ent} \to D_c^+(X)_{(I-d_X-c_X)\text{-}ent},$$
$$D_c(X)^{op}_{I\text{-}ent} \times D_c^+(X)_{I\text{-}ent-1} \to D_c^+(X)_{I\text{-}ent-1},$$
$$D_c(X)^{op}_{ent} \times D_c^+(X)_{ent-1} \to D_c^+(X)_{(d_X+c_X)\text{-}ent-1}.$$

Proof. We proceed as in 4.11. $\square$

One can also consider integrality on algebraic stacks over an excellent trait with finite residue field, which generalizes 5.11. The results are similar to those exposed in this section, with appropriate modifications of the dimension estimates. The variants 5.12 and 5.13 remain valid.

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