Fourier Transform and Majoration of Exponential Sums

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Résumé

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1 Introduction

Let X be an affine scheme, of finite type over $\mathbb{Z}$, which is smooth, purely of relative dimension $m \geq 0$ over $\mathbb{Z}[1/N]$, for some integer $N \geq 1$, and let $f_1, \ldots, f_r$ ($r \geq 1$) be functions on X which define a finite morphism to the affine space $\mathbb{A}^r_{\mathbb{Z}[1/N]}$,

$$f = (f_1, \ldots, f_r) : X \to \mathbb{A}^r_{\mathbb{Z}[1/N]}$$

(for example, if $h \in \mathbb{Z}[x_1, \ldots, x_n]$ is such that the Jacobian ideal

$$\left(h, \frac{\partial h}{\partial x_1}, \ldots, \frac{\partial h}{\partial x_n}\right)$$

in $\mathbb{Z}[x_1, \ldots, x_n]$ is principal, generated by an integer $N \geq 1$, and if $k_1 \geq 1, \ldots, k_r \geq 1$ are integers, one can take for X the hypersurface of $\mathbb{A}^n_{\mathbb{Z}[1/N]}$ with equation $h = 0$ and take $f_1 = x_1^{k_1}, \ldots, f_r = x_r^{k_r}$).

Given X and $f_1, \ldots, f_r$ as above, for each r-tuple $(a_1, \ldots, a_r) \in \mathbb{Z}^r$ and for each prime number p not dividing N, one can form the trigonometric sum

$$S(p; a_1, \ldots, a_r) \overset{\text{dfn}}{=} \sum_{x \in X(\mathbb{F}_p)} \exp\left(\frac{2\pi i}{p} \sum_{j=1}^{r} a_j f_j(x)\right).$$

More generally, for each function g on X which is invertible on X and for any multiplicative character of the field $\mathbb{F}_p$

$$\chi_p : \mathbb{F}_p^\times \to \mathbb{C}^\times$$

one can form the trigonometric sum

$$S(p; a_1, \ldots, a_r; g, \chi_p) \stackrel{\text{dfn}}{=} \sum_{x \in X(\mathbb{F}_p)} \exp\left(\frac{2\pi i}{p} \sum_{j=1}^{r} a_j f_j(x)\right) \chi_p(g(x)).$$

Then, we prove (cf. (5.2), (5.2.1)) the existence of a non-zero polynomial

$$F(y_1, \ldots, y_r) \in \mathbb{Z}[y_1, \ldots, y_r]$$

and the existence of a constant d (which only depends on the algebraic topology of the complex variety $X_{\mathbb{C}}$ equipped with the application $f_{\mathbb{C}} : X_{\mathbb{C}} \to \mathbb{A}^r_{\mathbb{C}}$ and which is easily computable) having the following property : if

$$F(a_1, \ldots, a_r) \not\equiv 0 \pmod p$$

then, for any invertible function g on X and for any character χ_p of $\mathbb{F}_p^\times$, we have the estimate

$$|S(p; a_1, \ldots, a_r; g, \chi_p)| \leq d \cdot (\sqrt{p})^m$$

(we state and prove in fact (cf. (5.2), (5.2.1)) a slightly stronger result, where $\mathbb{Z}[1/N]$ is replaced by any subring $R \subset \mathbb{C}$ which is of finite type over $\mathbb{Z}$ and (in (5.7)) where X is no longer assumed to be smooth, this last case following a suggestion by Deligne).

As one can expect, such estimates are obtained as a consequence of a precise analysis of the l-adic cohomology which gives rise to such sums and of the fundamental results of Deligne on the "Weil conjectures". The possibility of performing this cohomological analysis rests on two ideas.

The first is that of the *Fourier transform in l-adic cohomology* (cf. Section 2), an operation which exists "separately" in each characteristic $p > 0$. Thanks to the cohomological interpretation of exponential sums and to the base change theorem for cohomology with proper support, this operation is directly related to trigonometric sums. A recent discovery, due to Brylinski, Deligne, Verdier and the second author of this article, is that, although it may seem rather strange *a priori*, the Fourier transform commutes with duality (cf. (2.1.3) and (2.1.5)). Once armed with this unexpected result, it suffices to combine it with the fundamental results of Deligne (cf. [8]), through the formalism of perversity (cf. [1], [6] and Section 1), to control the situation in each characteristic $p \gg 0$ quite well (cf. (5.5)).

The second idea is the following : although one cannot hope to group the Fourier transforms for characteristics $p > 0$ into a single Fourier transform over $\mathbb{Z}$ (over $\mathbb{F}_p$, there are $p-1$ Fourier transforms, one for each non-trivial additive character $\psi : \mathbb{F}_p \to \overline{\mathbb{Q}}_\ell^\times$ and there is no reasonable way to choose in a compatible manner such a ψ for each $p > 0$), everything happens from a topological point of view as if such a transform over $\mathbb{Z}$ existed (all the ψ for all $p > 0$ have an identical topological behavior, what distinguishes them is of a purely arithmetic nature). More precisely, we demonstrate (cf. (4.1)) a Riemann-Roch type statement for "the" Fourier transform by using a delicate argument of balancing between wild ramification and moderate ramification (this argument is, by the way, already at the basis of the results of the course at Orsay by the first author, cf. [11]). As an intermediate step to the proof of (4.1), we also obtain a Riemann-Roch type theorem for $f : X \to Y$ a morphism of finite type between schemes of finite type over $\mathbb{Z}$ and for $\overline{\mathbb{Q}}_\ell$-sheaves (cf. (3.1.2)) (*).

Another transform, this one in characteristic 0, also deserves the name of Fourier transform (cf. [5] and (7.1)) : it is a transform for $\mathcal{D}$-Modules. This transform is quite similar to the l-adic Fourier transforms considered above (just as $\psi : \mathbb{F}_p \to \overline{\mathbb{Q}}_\ell^\times$ is similar to the character $t \mapsto e^{2\pi i t}$ of $\mathbb{R}$, just as Gauss sums are similar to the values of the $\Gamma(s)$ function at $s \in \mathbb{Q}$, ...). This similarity and the "Riemann-Hilbert" dictionary between regular singular holonomic $\mathcal{D}$-Modules and constructible sheaves (in analytic geometry) naturally leads us to conjecture a Riemann-Roch type statement common to Fourier transforms in characteristic $p > 0$ and in characteristic 0. Unfortunately, to correctly formulate such a statement, we currently lack a fundamental tool, namely the notion of characteristic cycle for a $\overline{\mathbb{Q}}_\ell$-sheaf, analogous to the characteristic cycle of a $\mathcal{D}$-Module ; also rather than stating a general conjecture in a somewhat restricted form, we limit ourselves to formulating its "practical" consequences in determining the constant d and an "optimal" polynomial $F(y_1, \dots, y_r)$ that makes the above estimates of trigonometric sums work (cf. Conjecture (7.4.2)).

We also treat a second kind of exponential sums in Section 6. We still consider an affine scheme, of finite type over $\mathbb{Z}$, X_0, which is smooth, purely of relative dimension m over $\mathbb{Z}[1/N]$, for a certain integer $N \geq 1$, and we consider functions $f_1, \dots, f_r$ ($r \geq 1$) on X and an invertible function g on X such that the morphism

$$(f; g) = (f_1, \dots, f_r; g) : X \to \mathbb{A}^r_{\mathbb{Z}[1/N]} \times \mathbb{G}_{m,\mathbb{Z}[1/N]}$$

is finite (for example, if we start from a situation $(X_0, f_1, \dots, f_{r+1})$ as at the beginning of the introduction, i.e. with X_0 affine, smooth and purely of relative dimension m over $\mathbb{Z}[1/N]$ and with $(f_1, \dots, f_{r+1}) : X_0 \to \mathbb{A}^{r+1}_{\mathbb{Z}[1/N]}$ finite, then we can take

$$X = X_0[1/f_{r+1}],$$

i.e. the open set of invertibility of f_{r+1} and

$$(f_1, \dots, f_r; g) := (f_1, \dots, f_r; f_{r+1}).$$

(*) In a spirit already envisioned by A. Grothendieck twenty years ago.

2 Duality, perversity and relative purity

(1.0) Throughout this article, by "scheme", we will always mean a separated and noetherian scheme ; moreover, we will say that a scheme is good if there exists a morphism of finite type, $X \to S$, with S a regular scheme of dimension ≤ 1 (for any integer $N \geq 1$, a good $\mathbb{Z}[1/N]$-scheme is therefore a good scheme on which N is invertible, which does not mean that this scheme is of finite type over $\mathbb{Z}[1/N]$; same remark for a good $\mathbb{F}_p$-scheme).

Let us fix a prime number ℓ and an algebraic closure $\overline{\mathbb{Q}}_\ell$ of $\mathbb{Q}_\ell$. For any good $\mathbb{Z}[1/\ell]$-scheme, we have at our disposal (cf. [8] (1.1.3) and [18]) the derived category $D_c^b(X, \overline{\mathbb{Q}}_\ell)$, equipped with the internal operations $\mathrm{R\,Hom}$ and $\otimes^L$. For any morphism of finite type

$$f : X \to Y$$

between good $\mathbb{Z}[1/\ell]$-schemes, we have at our disposal (cf. [8] (1.1.2)) the functors

$$Rf_!, Rf_* : D_c^b(X, \overline{\mathbb{Q}}_\ell) \to D_c^b(Y, \overline{\mathbb{Q}}_\ell)$$
$$f!, f^* : D_c^b(Y, \overline{\mathbb{Q}}_\ell) \to D_c^b(X, \overline{\mathbb{Q}}_\ell)$$

which satisfy the usual formalism of duality (cf. [SGA 4], XVIII, § 3).

(1.1) Let S be a good $\mathbb{Z}[1/\ell]$-scheme, for any S-scheme of finite type

$$\pi : X \to S$$

we define the relative dualizing complex $K_{X/S}$ as being the object

$$K_{X/S} = \pi!\overline{\mathbb{Q}}_\ell$$

of $D_c^b(X, \overline{\mathbb{Q}}_\ell)$; the functor

$$D_{X/S}(-) = \mathrm{R\,Hom}(-, K_{X/S}) : D_c^b(X, \overline{\mathbb{Q}}_\ell)^{\mathrm{op}} \to D_c^b(X, \overline{\mathbb{Q}}_\ell)$$

is, by definition, the relative dualizing functor. One can easily verify the following properties of $K_{X/S}$ and $D_{X/S}$:

(1.1.1) $D_{X/S}(K \otimes^L M) = \mathrm{R\,Hom}(K, D_{X/S}(M))$

and, if $f : X \to Y$ is an S-morphism between S-schemes of finite type,

(1.1.2) $K_{X/S} = f!K_{Y/S}$

(1.1.3) $Rf_* D_{X/S}(K) = D_{Y/S}(Rf_! K)$

(1.1.4) $f!D_{Y/S}(N) = D_{X/S}(f^*N)$

($K, M \in \mathrm{ob}\, D_c^b(X, \overline{\mathbb{Q}}_\ell)$ and $N \in \mathrm{ob}\, D_c^b(Y, \overline{\mathbb{Q}}_\ell)$; (1.1.3) is none other than the duality of Verdier). On the other hand, for $\pi : X \to S$ and a given K, it is not true in general that the canonical arrow

(1.1.5) $K \to D_{X/S} \circ D_{X/S}(K)$

is an isomorphism. We have, however, the following results :

Proposition 2.1 (1.1.6)**.** *Let $\pi : X \to S$ be a morphism of finite type, let $K \in \mathrm{ob}\, D_c^b(X, \overline{\mathbb{Q}}_\ell)$ and $K' = D_{X/S}(K)$. If the formation of $D_{X/S}(K)$ and $D_{X/S}(K')$ commutes with any base change $S' \to S$, with S' good, the canonical arrow (1.1.5) is an isomorphism.*

Démonstration. Indeed, for S the spectrum of a field, $D_{X/S}$ is none other than the absolute dualizing functor and in this case (1.1.5) is an isomorphism (cf. [SGA 4½], [Th. Finitude] (4.3)). □

Proposition 2.2 (1.1.7)**.** *Let $\pi : X \to S$ be a morphism of finite type and let $K \in \operatorname{ob} D_c^b(X, \overline{\mathbb{Q}}_\ell)$. There exists a dense open set $U \subset S$ such that the formation of $D_{X/S}(K)$ commutes with any base change $S' \to S$ over U, with S' good (U depends on π and K).*

Démonstration. Indeed, this results from [SGA 4½], [Th. Finitude] (2.9) and (2.10). □

Definition 2.3 (1.1.8)**.** Let $\pi : X \to S$ be a morphism of finite type. We will say that an object K of $D_c^b(X, \overline{\mathbb{Q}}_\ell)$ is *reflexive relative to* S if it satisfies the hypotheses and thus the conclusion of (1.1.6).

Remark 2.4 (1.1.9)**.** (i) By definition, the property of being reflexive relative to S is stable under duality relative to S and by any base change $S' \to S$, with S' good. (ii) According to (1.1.7), for any object K of $D_c^b(X, \overline{\mathbb{Q}}_\ell)$, there exists a dense open set $U \subset S$ such that $K|\pi^{-1}(U)$ is reflexive relative to U ; in particular, if S is the spectrum of a field, any object of $D_c^b(X, \overline{\mathbb{Q}}_\ell)$ is reflexive relative to S. (iii) If $\pi : X \to S$ is smooth and if the cohomology sheaves of K are smooth on X, K is reflexive relative to S (for a smooth $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{F}$ on X, the formation of $\mathrm{R}\operatorname{Hom}(\mathcal{F}, \overline{\mathbb{Q}}_\ell) = \operatorname{Hom}(\mathcal{F}, \overline{\mathbb{Q}}_\ell)[0]$ commutes with any base change $S' \to S$, with S' good). (iv) If $f : X \to Y$ is a smooth S-morphism, purely of relative dimension d, between S-schemes of finite type, it results from (1.1.4) that $f^*(-)[d] = f!(-)[-d](-d)$ preserves reflexivity relative to S. (v) Let $f : X \to Y$ be an S-morphism between S-schemes of finite type and let K be an object of $D_c^b(X, \overline{\mathbb{Q}}_\ell)$ which is reflexive relative to S. Suppose that, for any base change $S' \to S$, with S' good, transforming $(f : X \to Y, K)$ into $(f' : X' \to Y', K')$, the forgetful maps of supports

$$Rf'_! K' \to Rf'_* K'$$
$$Rf'_! D_{X'/S'}(K') \to Rf'_* D_{X'/S'}(K')$$

are isomorphisms. Then, it results from (1.1.3) and the base change theorem for a proper morphism (cf. [SGA 4] XII (5.1)) that $Rf_! K \simeq Rf_* K$ is an object of $D_c^b(Y, \overline{\mathbb{Q}}_\ell)$ that is reflexive relative to S. In particular, if f is a proper morphism, Rf_* preserves reflexivity relative to S. (vi) Any direct factor of an object reflexive relative to S in $D_c^b(X, \overline{\mathbb{Q}}_\ell)$ is also reflexive relative to S.

(1.2) Let S be a good $\mathbb{Z}[1/\ell]$-scheme and let $\pi : X \to S$ be an S-scheme of finite type. For any closed set Z of X, we call the dimension of Z relative to S and we denote by $\dim_{\mathrm{S}}(Z)$ the maximum of the dimensions of the geometric fibers of $\pi|Z : Z \to S$.

Definition 2.5 (1.2.1)**.** We will say that an object K of $D_c^b(X, \overline{\mathbb{Q}}_\ell)$ is *perverse relative to* S if it satisfies the following conditions :

(i) K is reflexive relative to S,

(ii) for any integer i, we have
 (A) $\dim_{\mathrm{S}}(\operatorname{Supp} \mathcal{H}^i(K)) \leq -i$

(B) $\dim_S(\operatorname{Supp}\mathcal{H}^i(D_{X/S}(K))) \leq -i$

where, for any $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{F}$ on X, $\operatorname{Supp}\mathcal{F}$ denotes its support.

Remark 2.6 (1.2.2)**.** (i) By definition, the property of perversity relative to S is stable under duality relative to S and by any base change $S' \to S$, with S' good. (ii) If S is the spectrum of a field, an object K of $D_c^b(X, \overline{\mathbb{Q}}_\ell)$ is perverse relative to S if, and only if, it is a perverse sheaf for the intermediate perversity in the sense of [6] (2.3) or [1] §4. Consequently, for S again a good $\mathbb{Z}[1/\ell]$-scheme, an object K of $D_c^b(X, \overline{\mathbb{Q}}_\ell)$, which is reflexive relative to S, is perverse relative to S if, and only if, its restriction to each geometric fiber of $\pi : X \to S$ is a perverse sheaf for the intermediate perversity. (iii) If $\pi : X \to S$ is smooth, purely of relative dimension d, and if K is an object of $D_c^b(X, \overline{\mathbb{Q}}_\ell)$ of the form $\mathcal{F}[d]$, where $\mathcal{F}$ is a smooth $\overline{\mathbb{Q}}_\ell$-sheaf on X, K is perverse relative to S (cf. (1.1.9) (iii)). (iv) If $f : X \to Y$ is a smooth S-morphism, purely of relative dimension d, between S-schemes of finite type, the functor $f^*(-)[d] = f!(-)[-d](-d)$ preserves perversity relative to S (cf. (1.1.9) (iv)). (v) Let $f : X \to Y$ be an affine S-morphism between S-schemes of finite type and let K be an object of $D_c^b(X, \overline{\mathbb{Q}}_\ell)$ perverse relative to S. Suppose that, for any base change $S' \to S$, with S' good, transforming $(f : X \to Y, K)$ into $(f' : X' \to Y', K')$, the forgetful maps of supports

$$Rf'_! K' \to Rf'_* K'$$
$$Rf'_! D_{X'/S'}(K') \to Rf'_* D_{X'/S'}(K')$$

are isomorphisms. Then $Rf_! K \simeq Rf_* K$ is an object of $D_c^b(Y, \overline{\mathbb{Q}}_\ell)$ perverse relative to S (indeed, according to (1.1.9) (v)), $Rf_! K \simeq Rf_* K$ is reflexive relative to S and according to (1.2.2) (ii), we are reduced to the case where S is the spectrum of an algebraically closed field, in which case the conclusion results from [1] (4.1.1) and (4.1.2)). In particular, if f is finite (affine and proper), the functor $Rf_* = f_*$ preserves perversity relative to S. (vi) Any direct factor of an object perverse relative to S in $D_c^b(X, \overline{\mathbb{Q}}_\ell)$ is also perverse relative to S.

(1.3) Let us now suppose that S is a scheme of finite type over $\mathbb{Z}[1/\ell]$ and let $\pi : X \to S$ be an S-scheme of finite type.

Definition 2.7 (1.3.1)**.** Let m be an integer. An object K of $D_c^b(X, \overline{\mathbb{Q}}_\ell)$ will be called *pure, of weight m, relative to S* if it satisfies the following conditions :

(i) K is reflexive relative to S,

(ii) for any integer i, the cohomology sheaf

(A) $\mathcal{H}^i(K)$ is mixed of punctual weights $\leq i + m$

(B) $\mathcal{H}^i(D_{X/S}(K))$ is mixed of punctual weights $\leq i - m$.

Remark 2.8 (1.3.2)**.** (i) By definition, the notion of purity, of weight m, relative to S is stable by duality relative to S and by any base change $S' \to S$ with S' of finite type over $\mathbb{Z}[1/\ell]$. (ii) If S is the spectrum of a finite field, an object K of $D_c^b(X, \overline{\mathbb{Q}}_\ell)$ is pure, of weight m, relative to S if, and only if, it is pure, of weight m, in the sense of [8] (6.2.4). Consequently, for S again an arbitrary scheme of finite type over $\mathbb{Z}[1/\ell]$, if K is an object of $D_c^b(X,\overline{\mathbb{Q}}_\ell)$ which is reflexive relative to S and which is mixed as well as $D_{X/S}(K)$, then K is pure, of weight m, relative to S if, and only if, its restriction to the fiber of $\pi : X \to S$, at each closed point of S, is pure, of weight m (we will pay attention to the fact that "mixed" (cf. [8] (1.2.2) and (6.2.2)) is not tested fiber by fiber but is an absolute property of K over X). (iii) If $\pi : X \to S$ is smooth, purely of relative dimension d, and if K is an object of $D_c^b(X,\overline{\mathbb{Q}}_\ell)$ of the form $\mathcal{F}[d]$, where $\mathcal{F}$ is a smooth $\overline{\mathbb{Q}}_\ell$-sheaf on X, punctually pure of weight m, K is pure, of weight $m+d$, relative to S (cf. (1.1.9) (iii)). (iv) If $f : X \to Y$ is a smooth S-morphism, purely of relative dimension d, between S-schemes of finite type, then the functor $f^*(-)[d] = f!(-)[-d](-d)$ transforms pure objects, of weight m, relative to S into pure objects, of weight $m+d$, relative to S (cf. (1.1.9) (iv)). (v) Let $f : X \to Y$ be an S-morphism between S-schemes of finite type and let K be a pure object of $D_c^b(X,\overline{\mathbb{Q}}_\ell)$, of weight m, relative to S. Suppose that, for any base change $S' \to S$ with S' of finite type over $\mathbb{Z}[1/\ell]$, transforming $(f : X \to Y, K)$ into $(f' : X' \to Y', K')$, the forgetful maps of supports

$$Rf'_!K' \to Rf'_*K'$$
$$Rf'_!D_{X'/S'}(K') \to Rf'_*D_{X'/S'}(K')$$

are isomorphisms. Then $Rf_!K \simeq Rf_*K$ is an object of $D_c^b(Y,\overline{\mathbb{Q}}_\ell)$ pure, of weight m, relative to S (indeed, according to (1.1.9) (v)), $Rf_!K \simeq Rf_*K$ is reflexive relative to S and, according to [8] (3.3.1) and (6.2.3), $Rf_!K$ and $D_{Y/S}(Rf_*K) = Rf_!D_{X/S}(K)$ are mixed, thus, according to (1.3.2) (ii), we are reduced to the case where S is the spectrum of a finite field ; in this case the conclusion results from [8] (6.2.3)). In particular, if f is proper, the functor Rf_* preserves purity, of weight m, relative to S. (vi) Any direct factor of a pure object, of weight m, relative to S of $D_c^b(X,\overline{\mathbb{Q}}_\ell)$ is also pure, of weight m, relative to S.

(1.4) Let S be a good $\mathbb{Z}[1/\ell]$-scheme.

Proposition 2.9 (1.4.1)**.** *Let $\pi : X \to S$ be a smooth morphism, of finite type, purely of relative dimension d, and let K be an object of $D_c^b(X,\overline{\mathbb{Q}}_\ell)$ whose cohomology sheaves are smooth on X and which we suppose perverse relative to S. Then K is of the form $\mathcal{F}[d]$, where $\mathcal{F}$ is a smooth $\overline{\mathbb{Q}}_\ell$-sheaf on X ; moreover $D_{X/S}(K) = \mathrm{Hom}(\mathcal{F},\overline{\mathbb{Q}}_\ell)[d](d)$. If we suppose in addition that S is of finite type over $\mathbb{Z}$ and that K is pure, of weight m, relative to S, then $\mathcal{F}$ is punctually pure of weight $m-d$ on X.*

Démonstration. Indeed, the hypotheses of perversity relative to S of K and of smoothness of the $\mathcal{H}^i(K)$ on X ensure that $\mathcal{H}^i(K) = 0$ if $i > -d$. On the other hand, since π is smooth, purely of relative dimension d, $K_{X/S} = \overline{\mathbb{Q}}_\ell[2d](d)$ and thus

$$\mathcal{H}^i(D_{X/S}(K)) = \mathcal{H}^i(\mathrm{R}\,\mathrm{Hom}(K, K_{X/S})) = \mathrm{Hom}(\mathcal{H}^{-i-2d}(K),\overline{\mathbb{Q}}_\ell)(d).$$

We deduce from what precedes that the $\mathcal{H}^i(D_{X/S}(K))$ are smooth and the hypothesis of perversity relative to S of K ensures then that $\mathcal{H}^i(D_{X/S}(K)) = 0$

if $i > -d$. Consequently, $\mathcal{H}^j(K) = 0$ if $-j - 2d > -d$, i.e. if $j < -d$ and the cohomology of K is concentrated in degree $-d$, thus $K = \mathcal{H}^{-d}(K)[d]$. The other assertions are now easy. $\square$

3 The Fourier transform for $\overline{\mathbb{Q}}_\ell$-sheaves

(2.0) Let $\mathbb{F}_q$ be a finite field of characteristic p and let ℓ be a prime number $\ell \neq p$. For any non-trivial additive character

$$\psi : \mathbb{F}_q \to \overline{\mathbb{Q}}_\ell^\times$$

we denote by $\mathcal{L}_\psi$ the Artin-Schreier $\overline{\mathbb{Q}}_\ell$-sheaf on $\mathbb{A}^1_{\mathbb{F}_q}$ associated with ψ (cf. [SGA $4\frac{1}{2}$], [Sommes trig.] (1.7)) ; if x is the coordinate of $\mathbb{A}^1_{\mathbb{F}_q}$, it is the rank 1 smooth $\overline{\mathbb{Q}}_\ell$-sheaf on $\mathbb{A}^1_{\mathbb{F}_q}$, deduced from the $\mathbb{F}_p$-torsor on $\mathbb{A}^1$ of equation

$$t^p - t = x$$

by extension of the structural group via ψ^{-1}. For any good $\mathbb{F}_q$-scheme $S \to \mathrm{Spec}(\mathbb{F}_q)$, we still denote by $\mathcal{L}_\psi$ the rank 1 smooth $\overline{\mathbb{Q}}_\ell$-sheaf on $\mathbb{A}^1_S$ deduced from $\mathcal{L}_\psi$ by pullback via $(\mathrm{id}, a) : \mathbb{A}^1_S \to \mathbb{A}^1_{\mathbb{F}_q}$. Let $\pi : E \to S$ be a vector bundle of constant rank $r \geq 1$ on a good $\mathbb{F}_q$-scheme S ; we denote $\pi^\vee : E^\vee \to S$ its dual vector bundle,

$$\begin{tikzcd} E \times_S E^\vee \arrow[r, "\mathrm{pr}_{E^\vee}"] \arrow[d, "\mathrm{pr}_E"'] & E^\vee \arrow[d, "\pi^\vee"] \\ E \arrow[r, "\pi"] & S \end{tikzcd}$$

the two projections and

$$\mu : E \times_S E^\vee \to \mathbb{A}^1_S$$

the evaluation map (for $E = \mathbb{A}^r_S$ of coordinates $x_1, \ldots, x_r$ and $E^\vee = \mathbb{A}^r_S$ of coordinates $x_1^\vee, \ldots, x_r^\vee$, $\mu(x, x^\vee) = \sum x_i x_i^\vee$).

(2.1) Deligne associated to $\pi : E \to S$ and to ψ as above, two functors worthy of the name 'Fourier transform' :

$$\mathcal{F}_{!,\psi} \text{ and } \mathcal{F}_{*,\psi} : D^b_c(E, \overline{\mathbb{Q}}_\ell) \to D^b_c(E^\vee, \overline{\mathbb{Q}}_\ell),$$

defined by

$$\mathcal{F}_{!,\psi}(-) = R\,\mathrm{pr}_{E^\vee,!}(\mathrm{pr}_E^*(-) \otimes^L \mu^*\mathcal{L}_\psi)[r] \tag{2.1.1}$$

$$\mathcal{F}_{*,\psi}(-) = R\,\mathrm{pr}_{E^\vee,*}(\mathrm{pr}_E^*(-) \otimes^L \mu^*\mathcal{L}_\psi)[r]. \tag{2.1.2}$$

In fact, a fundamental result for our purpose is that there is only one Fourier transform ; more precisely, we have (cf. the Appendix to this number for a proof) :

Theorem 3.1 (2.1.3)**.** *The natural "forget supports" arrow*

$$\mathcal{F}_{!,\psi}(-) \to \mathcal{F}_{*,\psi}(-)$$

is an isomorphism of functors.

We will therefore simply denote

$$\mathcal{F}_{\psi}(-) = \mathcal{F}_{!,\psi}(-) = \mathcal{F}_{*,\psi}(-) \tag{2.1.4}$$

the Fourier transform functor associated with $\pi : E \to S$ and the non-trivial additive character ψ.

Corollary 3.2 (2.1.5)**.** *(i) The formation of $\mathcal{F}_{\psi}(-)$ commutes with any base change $S' \to S$, with S' good. (ii) Let $\bar{\psi} = \psi^{-1}$ be the inverse character of ψ, we have*

$$D_{E^{\vee}/S} \circ \mathcal{F}_{!,\psi}(-) = R\,\mathrm{pr}_{E^{\vee},*}\,\mathrm{R\,Hom}(\mu^{*}\mathcal{L}_{\psi}, \mathrm{pr}_{E}^{*}\,D_{E/S}(-))[-r]$$

(iii) If K is an object of $D_c^b(E, \overline{\mathbb{Q}}_{\ell})$ that is reflexive relative to S (resp. perverse relative to S), $\mathcal{F}_{\psi}(K)$ is an object of $D_c^b(E^{\vee}, \overline{\mathbb{Q}}_{\ell})$ which has the same property.

Démonstration. Indeed, property (i) is verified thanks to the base change theorem for a proper morphism (cf. [SGA 4] XII (5.1)) on the expression $\mathcal{F}_{!,\psi}$ of Fourier. For part (ii), it follows easily from (1.1.1), (1.1.3) and (1.1.4) that

$$D_{E^{\vee}/S} \circ \mathcal{F}_{!,\psi}(-) = R\,\mathrm{pr}_{E^{\vee},*}\,\mathrm{R\,Hom}(\mu^{*}\mathcal{L}_{\psi}, \mathrm{pr}_{E}^{*}(-)) = \mathrm{pr}_{E}^{*}(-) \otimes^{L} \mu^{*}\mathcal{L}_{\bar{\psi}}[2r](r)$$

but $\mu^{*}\mathcal{L}_{\psi}$ is a smooth $\overline{\mathbb{Q}}_{\ell}$-sheaf on $E \times_S E^{\vee}$, of dual $\mathrm{Hom}(\mu^{*}\mathcal{L}_{\psi}, \overline{\mathbb{Q}}_{\ell}) = \mu^{*}\mathcal{L}_{\bar{\psi}}$ and pr_E is smooth, purely of relative dimension r, so that

$$\mathrm{R\,Hom}(\mu^{*}\mathcal{L}_{\psi}, \mathrm{pr}_{E}^{!}(-)) = \mathrm{pr}_{E}^{*}(-) \otimes^{L} \mu^{*}\mathcal{L}_{\bar{\psi}}[2r](r)$$

whence the conclusion, taking into account (2.1.3). For part (iii), we note that the functor $\mathrm{pr}_{E}^{*}(-)[r]$ respects reflexivity and perversity relative to S (cf. (1.1.9) (iv) and (1.2.2) (iv)) and the same is true for the functor $(-) \otimes^{L} \mu^{*}\mathcal{L}_{\psi}$. Now, the conclusion follows from (1.1.9) (v) and (1.2.2) (v) : indeed, $\mathrm{pr}_{E^{\vee}}$ is affine and, thanks to (2.1.3) and the equality

$$D_{E \times_S E^{\vee}/S}(\mathrm{pr}_{E}^{*}(-) \otimes^{L} \mu^{*}\mathcal{L}_{\psi}[r]) = \mathrm{pr}_{E}^{*}(D_{E/S}(-)) \otimes^{L} \mu^{*}\mathcal{L}_{\bar{\psi}}[r](r)$$

proved above, we see that the hypotheses of (1.1.9) (v) and (1.2.2) (v) are satisfied. □

(2.2) Suppose now that S is of finite type over $\mathbb{F}_q$. If we combine the fundamental result of Deligne [8] (6.2.3) and theorem (2.1.3), we obtain :

Theorem 3.3 (2.2.1)**.** *If K is an object of $D_c^b(E, \overline{\mathbb{Q}}_{\ell})$ that is pure of weight m relative to S, then $\mathcal{F}_{\psi}(K)$ is an object of $D_c^b(E^{\vee}, \overline{\mathbb{Q}}_{\ell})$ that is pure of weight $m + r$ relative to S.*

(2.3) Let S be a good $\mathbb{F}_q$-scheme. For any object K of $D^b_c(E,\overline{\mathbb{Q}}_\ell)$, there exists a dense open set $U^\vee \subset E^\vee$ above which the cohomology sheaves of $\mathcal{F}_\psi(K)$ are smooth (simply because $\mathcal{F}_\psi(K) \in \operatorname{ob} D^b_c(E^\vee,\overline{\mathbb{Q}}_\ell)$). We will retain especially, from the results above relating to Fourier, the following scholie :

Scholie 3.4 (2.3.1)**.** *Let K be an object of $D^b_c(E,\overline{\mathbb{Q}}_\ell)$ that is perverse relative to S and let $U^\vee \subset E^\vee$ be an open set above which the cohomology sheaves of $\mathcal{F}_\psi(K)$ are smooth. Then :*

(i) the cohomology of $\mathcal{F}_\psi(K)|U^\vee$ and of $\mathcal{F}_{\bar\psi}(D_{E/S}(K))|U^\vee$ is concentrated in degree $-r$;

(ii) $\mathcal{H}^{-r}(\mathcal{F}_\psi(K)|U^\vee)$ and $\mathcal{H}^{-r}(\mathcal{F}_{\bar\psi}(D_{E/S}(K))|U^\vee)$ are smooth $\overline{\mathbb{Q}}_\ell$-sheaves of the same rank ; moreover, if S is of finite type over the finite field $\mathbb{F}_q$ and if K is pure of weight m relative to S, these $\overline{\mathbb{Q}}_\ell$-sheaves are punctually pure of weight $m+r$ and $-m-r$ respectively ;

(iii) these two smooth $\overline{\mathbb{Q}}_\ell$-sheaves on $U^\vee$ are in duality : we have a perfect pairing

$$\mathcal{H}^{-r}(\mathcal{F}_\psi(K)|U^\vee) \times \mathcal{H}^{-r}(\mathcal{F}_{\bar\psi}(D_{E/S}(K))|U^\vee) \to \overline{\mathbb{Q}}_\ell$$

of smooth $\overline{\mathbb{Q}}_\ell$-sheaves on $U^\vee$.

This follows immediately from (1.4.1), (2.1.5) and (2.2.1).

Appendix (2.4) : proof of the equality $\mathcal{F}_{!,\psi} = \mathcal{F}_{*,\psi}$

In this appendix, we will work with sheaves of torsion rather than with $\overline{\mathbb{Q}}_\ell$-sheaves. The statement (2.1.3) that we want to demonstrate results, by the standard arguments of passing to the limit, from the theorem (2.4.1) below.

Let S be an $\mathbb{F}_q$-scheme, $E \to S$ a vector bundle of constant rank $r \geq 1$, A a finite commutative local ring of residual characteristic ℓ, $\ell \neq p$, and $\psi : \mathbb{F}_q \to A^\times$ a non-trivial additive character. We define $\mathcal{L}_\psi, \mathcal{F}_{!,\psi}$ and $\mathcal{F}_{*,\psi}$ as in (2.0) ; now $\mathcal{L}_\psi$ is a locally constant constructible sheaf of free A-modules of rank 1 on $\mathbb{A}^1_S$ and $\mathcal{F}_{!,\psi}, \mathcal{F}_{*,\psi}$ are functors from $D^b_c(E,A)$ to $D^b_c(E^\vee,A)$.

Theorem 3.5 (2.4.1)**.** *The "forget supports" arrow*

$$\mathcal{F}_{!,\psi}(-) \to \mathcal{F}_{*,\psi}(-)$$

is an isomorphism of functors.

Démonstration. Let's prove (2.4.1). Let us first remark that the statement (2.4.1) is local for the Zariski topology on S, which allows us to suppose $E = \mathbb{A}^r_S$. On the other hand, inspired by the relation

$$\sum_{x_1,\ldots,x_r \in \mathbb{F}_q} f(x_1,\ldots,x_r)\psi\left(\sum_{i=1}^r x_i x_i'\right)$$
$$= \sum_{x_r \in \mathbb{F}_q} \psi(x_r x_r') \sum_{x_1,\ldots,x_{r-1} \in \mathbb{F}_q} f(x_1,\ldots,x_r)\psi\left(\sum_{i=1}^{r-1} x_i x_i'\right)$$

we can easily check that the arrow $\mathcal{F}_{!,\psi}(-) \to \mathcal{F}_{*,\psi}(-)$ is computed variable by variable, which reduces us by recurrence on r, to the case where $r = 1$. We will therefore prove (2.4.1) under the supplementary hypotheses : $r = 1$ and $E = \mathbb{A}^1_S$. Fix the notations by the diagram

$$\begin{tikzcd}
 & \mathbb{A}^1_S \times_S \mathbb{A}^1_S \arrow[dl, "\mathrm{pr}_1"'] \arrow[d, "\mu"] \arrow[dr, "\mathrm{pr}_2"] & \\
\mathbb{A}^1_S \arrow[d, "\pi_1"'] & \mathbb{A}^1_S \arrow[d, "\pi"] & \mathbb{A}^1_S \arrow[d, "\pi_2"] \\
S & S & S
\end{tikzcd}$$

and let x (resp. y) be the coordinate of the first (resp. second) copy of $\mathbb{A}^1_S$ in $\mathbb{A}^1_S \times_S \mathbb{A}^1_S$ (so that $\mu(x, y) = xy$). Since

$$R\,\mathrm{pr}_{2!} = R\,\mathrm{pr}_{2*} \circ j_!$$
$$R\,\mathrm{pr}_{2*} = R\,\mathrm{pr}_{2*} \circ Rj_*$$

it is sufficient to show that the canonical functor arrow

$$(a) \qquad j_!(\mathrm{pr}_1^*(-) \otimes^L_{\mathbb{A}^1_S \times_S \mathbb{A}^1_S} \mu^*\mathcal{L}_\psi) \to Rj_*(\mathrm{pr}_1^*(-) \otimes^L_{\mathbb{A}^1_S \times_S \mathbb{A}^1_S} \mu^*\mathcal{L}_\psi)$$

is an isomorphism. Above $\mathbb{A}^1_S \times_S \mathbb{A}^1_S$ this is trivially the case, there is no problem except above $\infty_S \times_S \mathbb{A}^1_S$ where ∞_S is the section at infinity of $\mathbb{A}^1_S$. We will now see that it is sufficient to show that (a) is an isomorphism above $\infty_S \times_S \mathbb{G}_{m,S} \to \infty_S \times_S \mathbb{A}^1_S$. Note indeed, for any $a \in \mathbb{A}^1(S)$,

$$[a] : \mathbb{A}^1_S \times_S \mathbb{A}^1_S \to \mathbb{A}^1_S \times_S \mathbb{A}^1_S$$

the translation by a on the second factor ($(x, y) \mapsto (x, y + a)$) and

$$\langle a\rangle : \mathbb{A}^1_S \to \mathbb{A}^1_S$$

the multiplication by a, then

$$[a]^*\mu^*\mathcal{L}_\psi = \mu^*\mathcal{L}_\psi \otimes_{\mathbb{A}^1_S \times_S \mathbb{A}^1_S} \mathrm{pr}_1^*\langle a\rangle^*\mathcal{L}_\psi$$

(by virtue of the additive character of $\mathcal{L}_\psi$ and the equality $x(y+a) = xy+xa$) ; thus, if (a) is an isomorphism above $\infty_S \times_S \mathbb{G}_{m,S}$, it is also an isomorphism above $\infty_S \times_S 0_S$, where 0_S is the null section of $\mathbb{A}^1_S$ (take for a the constant section of value 1, 1_S, of $\mathbb{A}^1_S$ to reduce what happens on $\infty_S \times_S 0_S$ to what happens on $\infty_S \times_S 1_S$). To show that (a) is an isomorphism above $\infty_S \times_S \mathbb{G}_{m,S}$, consider the Cartesian square

$$\begin{tikzcd}
\mathbb{A}^1_S \times_S \mathbb{G}_{m,S} \arrow[r, hook, "j"] \arrow[d, "\mathrm{pr}_1"'] & \mathbb{P}^1_S \times_S \mathbb{G}_{m,S} \arrow[d, "\tilde{\mathrm{pr}}_1"] \\
\mathbb{A}^1_S \arrow[r, hook, "i"] & \mathbb{P}^1_S
\end{tikzcd}$$

from which we deduce a commutative square of morphisms of functors

$$\begin{tikzcd}
\tilde{\mathrm{pr}}_1^* i_!(-) \otimes^L \mu^*\mathcal{L}_\psi \arrow[r, "(b)"] \arrow[d] & \tilde{\mathrm{pr}}_1^* Ri_*(-) \otimes^L \mu^*\mathcal{L}_\psi \arrow[d, "(c)"] \\
j_!(\mathrm{pr}_1^*(-) \otimes^L \mu^*\mathcal{L}_\psi) \arrow[r, "(d)"] & Rj_*(\mathrm{pr}_1^*(-) \otimes^L \mu^*\mathcal{L}_\psi)
\end{tikzcd}$$

where the horizontal arrows are the forgetful maps of supports and where the vertical arrows are base change arrows relative to the Cartesian square above. It results from the projection formula for j,

$$j_! j^*(-) \otimes^L_{\mathbb{A}^1_S \times_S \mathbb{A}^1_S} \mu^*\mathcal{L}_\psi = (-) \otimes^L_{\mathbb{P}^1_S \times_S \mathbb{A}^1_S} j_!\mu^*\mathcal{L}_\psi$$

that (b) and (c) are isomorphisms of functors. It remains to show that (d) is an isomorphism above $\infty_S \times_S \mathbb{G}_{m,S}$. For this, let us consider the Artin-Schreier covering

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & \mathbb{A}^1_S \times_S (\mathbb{A}^1_S \times_S \mathbb{G}_{m,S}) \\
{} & \mathbb{A}^1_S \times_S \mathbb{G}_{m,S}
\end{tikzcd}$$

of equation $t^p - t = xy$ and its extension

$$\begin{tikzcd}
X \arrow[r, "\tilde{f}"] & \tilde{X} \arrow[r] & \mathbb{P}^1_S \times_S (\mathbb{P}^1_S \times_S \mathbb{G}_{m,S}) \\
\mathbb{A}^1_S \times_S \mathbb{G}_{m,S} \arrow[u] \arrow[r] & \mathbb{P}^1_S \times_S \mathbb{G}_{m,S} \arrow[u] &
\end{tikzcd}$$

of equation $X_0 T_1^p - X_1 T_0 T_1^{p-1} = X_0 X_1' y$, with $T_1/T_0 = t$, $X_1/X_0 = x$; $\tilde{f}$ is finite, Galois of group $\mathbb{F}_p$, totally ramified above $\infty_S \times_S \mathbb{G}_{m,S}$ and étale above $\mathbb{A}^1_S \times_S \mathbb{G}_{m,S}$; moreover, $\tilde{X}$ is smooth over S.

Lemma 3.6 (2.4.2)**.** *We have a canonical isomorphism*

$$\tilde{\mathrm{pr}}_1^*(-)|_{\mathbb{P}^1_S \times_S \mathbb{G}_{m,S}} \otimes^L \big(\bigoplus_{\psi' \neq 1} j_!\mu^*\mathcal{L}_{\psi'}\big) \simeq \tilde{f}_* \mathbb{A}_{\tilde{X}}$$

where $\psi' : \mathbb{F}_p \to A^\times$ runs through the non-trivial additive characters.

Démonstration. Indeed, since $\tilde{f}$ is finite, étale, Galois of group $\mathbb{F}_p$, we have the decomposition

$$\tilde{f}_* \mathbb{A}_{\tilde{X}}|_{\mathbb{A}^1_S \times_S \mathbb{G}_{m,S}} \otimes (\mu^*\mathcal{L}_\psi) \simeq \bigoplus_{\psi' \neq 1} \tilde{f}_* \mathbb{A}_{\tilde{X}}.$$

On the other hand, it is easy to define the arrow and, to show the lemma, it remains to verify that this arrow is an isomorphism above $\infty_S \times_S \mathbb{G}_{m,S}$, which results immediately from the fact that $\tilde{f}$ is totally ramified above $\infty_S \times_S \mathbb{G}_{m,S}$. □

According to (2.4.2) the arrow (d) is a direct factor of the base change arrow

$$(d') \qquad \tilde{\mathrm{pr}}_1^* Ri_*(-) \otimes^L_{\mathbb{P}^1_S} \tilde{f}_* A_{\tilde{X}} \to Rj_*(\mathrm{pr}_1^*(-) \otimes^L_{\mathbb{A}^1_S} \tilde{f}_* A_{\tilde{X}}),$$

arrow which can be rewritten, thanks to the projection formulas for f and $\tilde{f}$,

$$(d') \qquad \tilde{f}_* \tilde{f}^* \tilde{\mathrm{pr}}_1^* Ri_*(-) \to \tilde{f}_* Rj'_* \tilde{f}^* \,\mathrm{pr}_1^*(-).$$

Consequently, to finish the proof of 2.4.1, it is sufficient to show that the base change arrow

$$(d'') \qquad (\mathrm{pr}_1 \circ f)^* Ri_*(-) \to Rj'_*(\mathrm{pr}_1 \circ f)^*(-),$$

relative to the Cartesian square

$$\begin{array}{ccc} X & \xrightarrow{\tilde{f}} & \tilde{X} \\ {\scriptstyle \mathrm{pr}_1 \circ f}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}_1 \circ \tilde{f}} \\ \mathbb{A}^1_S & \xrightarrow{i} & \mathbb{P}^1_S \end{array}$$

is an isomorphism. If $\mathrm{pr}_1 \circ \tilde{f}$ were smooth, the conclusion would immediately result from the base change theorem for a smooth morphism (cf. [SGA 4] XVI (1.2)). In fact, this is not the case, but we will see that, above a neighborhood U of $\infty_S \times_S \mathbb{G}_{m,S}$ in $\mathbb{P}^1_S \times_S \mathbb{G}_{m,S}$, one can factor $\mathrm{pr}_1 \circ \tilde{f}$ into a surjective radicial morphism followed by a smooth morphism and, as the étale topology is insensitive to surjective radicial morphisms, this is enough to prove that (d'') is an isomorphism. Let us take as neighborhood U of $\infty_S \times_S \mathbb{G}_{m,S}$, the chart $X_1 \neq 0$ of $\mathbb{P}^1_S \times_S \mathbb{G}_{m,S}$, with coordinates $(\xi = X_0/X_1, y)$; on $\tilde{f}^{-1}(U) \subset \tilde{X}$, $T_1 \neq 0$ and

$$(\tau = T_0/T_1, y) : \tilde{f}^{-1}(U) \to \mathbb{A}^1_S \times_S \mathbb{G}_{m,S}$$

is an open immersion whose closed complement has for equation $\tau^p - \tau^{-1} = 1$. Then $\mathrm{pr}_1 \circ \tilde{f}|_{\tilde{f}^{-1}(U)}$ has for expression

$$\mathrm{pr}_1 \circ \tilde{f}|_{\tilde{f}^{-1}(U)} : (\tau, y) \mapsto \xi = \frac{\tau^p y}{1 - \tau^{p-1}}$$

and factors into a surjective radicial morphism,

$$(\tau, y) \mapsto (\tau', y') = \left(\tau^p, \frac{y}{1 - \tau^{p-1}}\right),$$

followed by a smooth morphism $(y' \neq 0)$,

$$(\tau', y') \mapsto \xi = \tau' y',$$

whence the conclusion.

Remark 3.7 (2.4.3)**.** Recall that a morphism $f : X \to S$ is said to be strongly locally acyclic relative to an object K of $D^b_{\mathrm{ctf}}(X, A)$ if for any geometric point $\bar{x}$ of X, with image $\bar{s}$ in S, any specialization $t \to \bar{s}_{(\bar{s})}$ and any A-module M, the restriction map

$$K_{\bar{s}} \otimes_A M \to R\Gamma(X_{(t)}, (K|X_{(t)}) \otimes_A M)$$

is an isomorphism (cf. [SGA 4½], [Th. Finitude] (A.2.9)). We will say that f is universally strongly locally acyclic relative to K if f is strongly locally acyclic relative to K and if it remains so after any base change $S' \to S$. The arguments developed above show in fact the stronger result :

Theorem 3.8 (2.4.4)**.** *The projection* $\mathrm{pr}_{\mathbb{A}^1_{\mathbb{F}_q}} : \mathbb{P}^1_{\mathbb{F}_q} \times_{\mathbb{F}_q} \mathbb{A}^1_{\mathbb{F}_q} \to \mathbb{P}^1_{\mathbb{F}_q}$ *is universally strongly locally acyclic relative to* $\mu^* \mathcal{L}_\psi$ *extended by 0 to* $\mathbb{P}^1_{\mathbb{F}_q} \times_{\mathbb{F}_q} \mathbb{A}^1_{\mathbb{F}_q}$ *entirely.*

□

4 Stratifications and uniformity theorem for the functors $Rf_!$, $D_{X/S}$ and Rf_*

(3.0) *Stratifications of schemes.* If X is a scheme, a stratification $\mathcal{X} = \{X_\alpha\}$ of X is a finite partition

$$X_{\mathrm{red}} = \coprod_\alpha X_\alpha$$

of X_{red} by locally closed reduced subschemes of X (the symbol $\coprod$ is taken here in the set-theoretic sense and not in the schematic sense ; the X_α are not assumed to be irreducible nor even connected and can possibly be empty). If $f : X' \to X$ is a morphism of schemes and if $\mathcal{X} = \{X_\alpha\}$ is a stratification of X, the inverse image stratification of $\mathcal{X}$ by f is the stratification, denoted $\mathcal{X}_{X'}$, or $f^*\mathcal{X}$ or even simply $\mathcal{X}'$, of X' defined by

$$(X')_{\mathrm{red}} = \coprod_\alpha (f^{-1}(X_\alpha))_{\mathrm{red}}.$$

Let X be a good scheme and $\mathcal{X}$ a stratification of X. We will say that a constructible function

$$\varphi : X \to \mathbb{Z}$$

is adapted to $\mathcal{X}$ if it is constant on each stratum X_α ; for any good X-scheme $f : X' \to X$, the constructible function

$$\varphi \circ f : X' \to \mathbb{Z},$$

still denoted $\varphi_{X'}$, or $\varphi \circ f$ or even simply φ', is adapted to $f^*\mathcal{X}$. Let there be in addition ℓ a prime number and $\overline{\mathbb{Q}}_\ell$ an algebraic closure of $\mathbb{Q}_\ell$. We will say that an object K of $D^b_c(X[1/\ell], \overline{\mathbb{Q}}_\ell)$ is adapted to $\mathcal{X}$ if all its cohomology sheaves $\mathcal{H}^i(K)$ are smooth on each stratum $X_\alpha[1/\ell]$. We will denote

$$\chi(K) \text{ and } \|K\| : X[1/\ell] \to \mathbb{Z}$$

the constructible functions defined by

$$\chi(K)(x) = \sum_i (-1)^i \dim_{\overline{\mathbb{Q}}_\ell} [\mathcal{H}^i(K)]_{\bar{x}}$$
$$\text{and } \|K\|(x) = \sum_i \dim_{\overline{\mathbb{Q}}_\ell} [\mathcal{H}^i(K)]_{\bar{x}},$$

where $\bar{x}$ is an arbitrary geometric point above $x \in X[1/\ell]$; we will say that K is χ-adapted to $\mathcal{X}$ if K is adapted to $\mathcal{X}$ and if the constructible function $\chi(K) : X[1/\ell] \to \mathbb{Z}$ is adapted to the stratification $\mathcal{X}[1/\ell]$ of $X[1/\ell]$.

Remark 4.1 (3.0.1)**.** If K is adapted to $\mathcal{X}$, the functions $\chi(K)$ and $\|K\|$ are constant on the connected components of the strata $X_\alpha[1/\ell]$, as well as each function $x \mapsto \dim_{\overline{\mathbb{Q}}_\ell} [\mathcal{H}^i(K)]_{\bar{x}}$, but, for technical reasons, we have not assumed the strata to be connected, so we must distinguish between "adapted" and "χ-adapted".

(3.0.2) For any $n \in \mathbb{Z}$, we have

$$\chi(K[n]) = (-1)^n \chi(K) \qquad \text{and} \qquad \|K[n]\| = \|K\|;$$

for any "triangle" (cf. [8] (1.1.2))

$$K' \to K \to K'' \to K'[1]$$

in $D^b_c(X[1/\ell], \overline{\mathbb{Q}}_\ell)$, we have

$$\chi(K) = \chi(K') + \chi(K'') \qquad \text{and} \qquad \|K\| \leq \|K'\| + \|K''\|;$$

finally, we have

$$\|K\| = 0 \iff K = 0.$$

(3.1) *Stratifications and* $Rf_!$. Let Y be a good scheme and $f : X \to Y$ a morphism of finite type. An !-*stratification* of $f : X \to Y$ is a quadruple $(\mathcal{X}, \mathcal{Y}, C, f_!)$, composed of a stratification $\mathcal{X} = \{X_\alpha\}$ of X, a stratification $\mathcal{Y} = \{Y_\beta\}$ of Y, an integer $C \geq 1$ and an additive application

$$f_! : \begin{Bmatrix} \text{constructible functions} \\ X \to \mathbb{Z} \text{ adapted to } \mathcal{X} \end{Bmatrix} \to \begin{Bmatrix} \text{constructible functions} \\ Y \to \mathbb{Z} \text{ adapted to } \mathcal{Y} \end{Bmatrix}$$

possessing the following property : for any prime number ℓ, for any $K \in \operatorname{ob} D^b_c(X[1/\ell], \overline{\mathbb{Q}}_\ell)$ adapted to $\mathcal{X}$, for any good Y-scheme $g : Y' \to Y$ and for any direct factor L of $g^*(K)$ in $D^b_c(X'[1/\ell],\overline{\mathbb{Q}}_\ell)$, where we have formed the Cartesian square

$$\begin{array}{ccc} X' & \xrightarrow{g_X} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[g]{} & Y \end{array}$$

we have

(i) the object $Rf'_!L$ of $D^b_c(Y'[1/\ell],\overline{\mathbb{Q}}_\ell)$ is adapted to $g^*\mathcal{Y}$ and we have the inequality
$$\|Rf'_!(L)\|(y') \leq C \cdot \sup_{f'(x')=y'}(\|L\|(x')),$$

(ii) if L is χ-adapted to $g_X^*\mathcal{X}$, then $Rf'_!L$ is χ-adapted to $g^*\mathcal{Y}$ and
$$\chi(Rf'_!L) = f_!\chi(L)$$
in the following sense : for any constructible function $\varphi : X \to \mathbb{Z}$ adapted to $\mathcal{X}$ inducing $\chi(L)$ on $X'[1/\ell]$, $f_!\varphi : Y \to \mathbb{Z}$ induces $\chi(Rf'_!L)$ on $Y'[1/\ell]$.

Remark 4.2 (3.1.1)**.** The application $f_!$, if it exists, is uniquely determined by the morphism $f : X \to Y$; moreover, if $(\mathcal{X},\mathcal{Y},C,f_!)$ is an !-stratification of f, $f_!$ is given by the following "recipe" : choose, for each β, an arbitrary geometric point $\bar{y}_\beta$ of Y_β, then, for each α such that $f^{-1}(\bar{y}_\beta)\cap X_\alpha$ is not empty, choose an arbitrary geometric point $\bar{x}_{\alpha,\beta}$ of $f^{-1}(\bar{y}_\beta)\cap X_\alpha$ and finally choose a prime number ℓ distinct from the residual characteristics of the $\bar{y}_\beta$, then $f_!\varphi$ is the function whose constant value on Y_β is given by
$$(f_!\varphi)(\bar{y}_\beta) = \sum_{\substack{\alpha \text{ such that} \\ f^{-1}(\bar{y}_\beta)\cap X_\alpha \neq \emptyset}} \varphi(\bar{x}_{\alpha,\beta})\chi_{\alpha,\beta}$$
with
$$\chi_{\alpha,\beta} = \sum_i (-1)^i \dim_{\overline{\mathbb{Q}}_\ell}[\mathcal{H}^i_c(f^{-1}(\bar{y}_\beta)\cap X_\alpha,\overline{\mathbb{Q}}_\ell)].$$

For $f : X \to Y$ as above, we will say that an !-stratification $(\mathcal{X},\mathcal{Y},C,f_!)$ is *universal* if, for any good Y-scheme $g : Y' \to Y$ which is flat, the morphism $f' : X' \to Y'$ deduced from f by base change admits $(g_X^*\mathcal{X}, g^*\mathcal{Y}, C, g^*f_!)$ as an !-stratification.

Theorem 4.3 (3.1.2)**.** *Let Y be a good scheme, $f : X \to Y$ a morphism of finite type and $\mathcal{X} = \{X_\alpha\}$ a stratification of X. Then, there exists an integer $N \geq 1$, a stratification $\mathcal{Y} = \{Y_\beta\}$ of $Y[1/N]$, an integer $C \geq 1$ and an additive application $f_!$ such that $(\mathcal{X}[1/N],\mathcal{Y},C,f_!)$ is a universal !-stratification of the morphism*
$$f : X[1/N] \to Y[1/N]$$
deduced from $f : X \to Y$ by inverting N.

The proof of this theorem will be given in Appendix to this number (cf. (3.4.2)).

(3.2) Stratifications and relative duality.

Theorem 4.4 (3.2.1)**.** *Let S be a good scheme, $f : X \to S$ a morphism of finite type and $\mathcal{X}$ a stratification of X. Then, there exists an integer $N \geq 1$, a dense open set $U \subset S[1/N]$ and a stratification $\tilde{\mathcal{X}}$ of X_U having the following property : for any prime number ℓ, for any étale morphism $X' \to X_U[1/\ell]$ and for any object K of $D^b_c(X', \overline{\mathbb{Q}}_\ell)$ which is adapted to the stratification $\mathcal{X}_{X'}$, one has :*

(i) the object $D_{X'/U}(K)$ of $D^b_c(X', \overline{\mathbb{Q}}_\ell)$ is adapted to the stratification $\tilde{\mathcal{X}}_{X'}$,

(ii) the formation of $D_{X'/U}(K)$ commutes with any base change $T \to U[1/\ell]$, with T a good scheme.

The proof of this theorem will be given in Appendix to this number (cf. (3.4.3)).

Corollary 4.5 (3.2.2)**.** *Let S be a good scheme, $f : X \to S$ a morphism of finite type and $\mathcal{X} = \{X_\alpha\}$ a stratification of X. Then there exists an integer $N \geq 1$, a dense open set $U \subset S[1/N]$ and a stratification $\tilde{\mathcal{X}}$ of X_U having the following property : for any prime number ℓ, for any étale morphism $X' \to X_U[1/\ell]$ and for any object K of $D^b_c(X', \overline{\mathbb{Q}}_\ell)$ adapted to $\mathcal{X}_{X'}$, K is reflexive relative to U and $D_{X'/U}(K)$ is adapted to the stratification $\tilde{\mathcal{X}}_{X'}$.*

Démonstration. Indeed, we apply (3.2.1) twice : first to $(f : X \to S, \mathcal{X})$, which produces an integer $N_1 \geq 1$, a dense open set $U_1 \subset S[1/N_1]$ and a stratification $\tilde{\mathcal{X}}_1$ of X_{U_1}, then to $(f_{U_1} : X_{U_1} \to U_1, \tilde{\mathcal{X}}_1)$, which produces an integer $N_2 \geq 1$, a dense open set $U_2 \subset U_1[1/N_2]$ (and a stratification that we forget). Then $N = N_1 N_2$, $U = U_2$ and $\tilde{\mathcal{X}} = (\tilde{\mathcal{X}}_1)_U$ work (3.2.2). ☐

(3.3) Stratifications and Rf_*. Let S be a good scheme, X and Y be S-schemes of finite type,

$$f : X \to Y$$

an S-morphism, $\mathcal{X} = \{X_\alpha\}$ a stratification of X and $\mathcal{Y} = \{Y_\beta\}$ a stratification of Y. We will say that $(\mathcal{X}, \mathcal{Y})$ is a **-stratification of f relative to S* if, for any étale morphism $Y' \to Y$ providing the Cartesian square

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & X \arrow[d] \\ Y' \arrow[r] & Y \\ & S \arrow[u] \end{tikzcd}$$

the pair of stratifications $(\mathcal{X}_{X'}, \mathcal{Y}_{Y'})$ has the following property : for any prime number ℓ and for any object K of $D^b_c(X'[1/\ell], \overline{\mathbb{Q}}_\ell)$ adapted to $\mathcal{X}_{X'}$, the object

Rf_*K of $D^b_c(Y'[1/\ell], \overline{\mathbb{Q}}_\ell)$ is adapted to $\mathcal{Y}_{Y'}$, and its formation commutes with any base change $S' \to S$, with S' a good scheme.

Remark 4.6 (3.3.1)**.** It results from [13] (1.1) that the χ functions of the objects Rf_*K and Rf'_*K', for K as above, coincide (make base changes $S' \to S$, with S' the spectrum of a field).

Theorem 4.7 (3.3.2)**.** *Let S be a good scheme, X and Y be S-schemes of finite type, $f : X \to Y$ an S-morphism and $\mathcal{X} = \{X_\alpha\}$ a stratification of X. Then, there exists an integer $N \geq 1$, a dense open set U of $S[1/N]$ and a stratification $\mathcal{Y}$ of Y_U such that $(\mathcal{X}_{X_U}, \mathcal{Y})$ is a *-stratification of $f_U : X_U \to Y_U$ relative to U.*

Démonstration. Indeed, let us show that (3.3.2) results from (3.1.2), (3.2.1) and (3.2.2). By (3.2.2), there exists an integer $N_1 \geq 1$, a dense open set $U_1 \subset S[1/N_1]$ and a stratification $\tilde{\mathcal{X}}_1$ of X_{U_1}, such that, for any prime number ℓ, for any étale morphism $Y' \to Y_{U_1}[1/\ell]$ and for any object K of $D^b_c(X', \overline{\mathbb{Q}}_\ell)$ adapted to $\mathcal{X}_{X'}$, K is reflexive and $D_{X'/U_1}(K)$ is adapted to $(\tilde{\mathcal{X}}_1)_{X'}$; thanks to the reflexivity of K, we deduce from (1.1.3) that

$$Rf'_*K = Rf'_*(D_{X'/U_1}(D_{X'/U_1}(K))) = D_{Y'/U_1}(Rf'_!(D_{X'/U_1}(K))).$$

We then apply (3.1.2) to $(f_{U_1} : X_{U_1} \to Y_{U_1}, \tilde{\mathcal{X}}_1)$ which produces an integer $N_2 \geq 1$ and a stratification $\mathcal{Y}_2$ of $Y_{U_1}[1/N_2]$ such that $Rf'_!(D_{X'/U_1}(K))$ is adapted to $\mathcal{Y}_2$, as soon as $Y' \to Y_{U_1}[1/\ell]$ factors through $Y_{U_1}[1/N_2\ell] \to Y_{U_1}[1/\ell]$; finally we apply (3.2.1) to $Y_{U_1}[1/N_2] \to U_1[1/N_2]$ and to $\mathcal{Y}_2$, which produces an integer $N_3 \geq 1$, a dense open set $U_3 \subset U_1[1/N_2N_3]$ and a stratification $\mathcal{Y}_3$ of Y_{U_3}. Now, it is clear that $N = N_1N_2N_3$, $U = U_3$ (seen as a dense open set of $S[1/N]$) and $\mathcal{Y} = \mathcal{Y}_3$ work (3.3.2). □

Taking a common refinement of the stratifications produced by (3.1.2) and (3.3.2), we finally obtain :

Corollary 4.8 (3.3.3)**.** *Let $S, X, Y, f : X \to Y$ and $\mathcal{X}$ be as in (3.3.2). Then, there exists an integer $N \geq 1$, a dense open set U of $S[1/N]$, a stratification $\mathcal{Y}$ of Y_U, an integer $C \geq 1$ and an additive application $f_!$ such that :*

(i) $(\mathcal{X}_{X_U}, \mathcal{Y}, C, f_!)$ *is a universal* !*-stratification of* $f : X_U \to Y_U$ *;*

(ii) $(\mathcal{X}_{X_U}, \mathcal{Y})$ *is a *-stratification of* $f : X_U \to Y_U$*.*

Appendix (3.4) : proof of theorems (3.1.2) and (3.2.1)

(3.4.1) *Lemmas on the existence of stratifications.*

Lemma 4.9 (3.4.1.1)**.** *Let Y be a good-scheme and $\mathcal{X}$ a stratification of $\mathbb{A}^1_Y = Y[T]$. Then, there exists an integer $N \geq 1$, a stratification $\mathcal{Y} = \{Y_\beta\}$ of $Y[1/N]$, with strata Y_β normal, connected and flat over $\mathbb{Z}$, and, for each β, a decomposition of $\mathbb{A}^1_{Y_\beta}$ into*

$$\mathbb{A}^1_{Y_\beta} = (\mathbb{A}^1_{Y_\beta} - D_\beta) \cup D_\beta, \quad D_\beta = \coprod_{1 \leq \nu \leq n_\beta} D_{\beta,\nu},$$

with D_β (resp. $D_{\beta,\nu}$) a divisor of $\mathbb{A}^1_{Y_\beta}$, finite étale over Y_β, of degree d_β (resp. $d_{\beta,\nu}$), defined by a unitary polynomial $F_\beta(T)$ (resp. $F_{\beta,\nu}(T)$) of degree d_β (resp. $d_{\beta,\nu}$), with coefficients in $\Gamma(Y_\beta, \mathcal{O}_{Y_\beta})$, whose discriminant is invertible on Y_β, and with

$$F_\beta(T) = \prod_{\nu=1}^{n_\beta} F_{\beta,\nu}(T),$$

such that the stratification of $\mathbb{A}^1_{Y[1/N]}$,

$$\mathbb{A}^1_{Y[1/N]} = \coprod_\beta \{(\mathbb{A}^1_{Y_\beta} - D_\beta) \cup (\coprod_{1\le\nu\le n_\beta} D_{\beta,\nu})\},$$

is finer than the stratification $\mathcal{X}[1/N]$.

Démonstration. Indeed, this results, by noetherian induction on Y, from [11] (4.2). $\square$

We will say that a morphism $f : X \to Y$ is *very elementary* if it is Y-isomorphic either to $D \to Y$, or to $\mathbb{A}^1_Y - D \to Y$, where $D \subset \mathbb{A}^1_Y$ is either empty or finite étale of degree $d \geq 1$ over Y and defined in $\mathbb{A}^1_Y$ by a unitary polynomial of degree d, with coefficients in $\Gamma(Y, \mathcal{O}_Y)$ whose discriminant is invertible on Y. We will say that a morphism $f : X \to Y$ is a *very elementary fibration* (of length $\leq n$) if it admits a factorization into a finite sequence (of at most n) very elementary morphisms.

Lemma 4.10 (3.4.1.2)**.** *Let Y be a good scheme, $f : X \to Y$ a morphism of finite type and $\mathcal{X}$ a stratification of X. Then, there exists an integer $N \geq 1$, a stratification $\mathcal{Y} = \{Y_\beta\}$ of $Y[1/N]$, with strata Y_β flat over $\mathbb{Z}$, and, for each β, a stratification*

$$f^{-1}(Y_\beta)_{red} = \coprod_{1\le\nu\le n_\beta} X_{\beta,\nu}$$

of $f^{-1}(Y_\beta)$, such that

(i) the stratification of $X[1/N]$,

$$X[1/N]_{red} = \coprod_{\beta,\nu} X_{\beta,\nu},$$

is finer than the stratification $\mathcal{X}[1/N]$,

(ii) for any β and any ν, $1 \leq \nu \leq n_\beta$, the morphism $f : X_{\beta,\nu} \to Y_\beta$ is a very elementary fibration.

Démonstration. Indeed, one can suppose that f is the restriction of the canonical projection $\mathbb{A}^n_Y \to Y$ to a closed subscheme $X \subset \mathbb{A}^n_Y$ (usual devissages : cut Y into pieces, cut X into pieces, factorize f). Then, either $X = \mathbb{A}^n_Y$ and the conclusion results immediately from (3.4.1.1), or $X \neq \mathbb{A}^n_Y$. In this last case, one can suppose moreover Y normal, connected and flat over $\mathbb{Z}$, of generic point η

(noetherian induction on Y). Above η, which is the spectrum of a field of characteristic zero, $(X_{\mathrm{red}})_\eta$ is either $\mathbb{A}^n_\eta$ entirely, or a subscheme of $\mathbb{A}^n_\eta$ finite, étale over η, defined by a unitary polynomial with invertible discriminant. This situation propagates above a neighborhood of η in Y and one concludes by noetherian induction on Y. $\square$

(3.4.2) *Proof of theorem (3.1.2).* Thanks to lemma (3.4.1.2), it is sufficient to prove (3.1.2) for $f : X \to Y$ a very elementary morphism, with Y flat over $\mathbb{Z}$, and for the trivial stratification $\mathcal{X} = \{X\}$. The case $X = D \to Y$ is trivial (f is finite étale of degree $d \geq 0$) : one takes $N = 1$, $\mathcal{Y} = \{Y\}$, $C = d$ and, if $\varphi : X \to \mathbb{Z}$ is adapted to $\mathcal{X}$, i.e. constant of value a, one takes for $f_!\varphi : Y \to \mathbb{Z}$ the constant function of value ad (the flatness of Y over $\mathbb{Z}$ is not used in this case). It remains the case $X = \mathbb{A}^1_Y - D \to Y$ (D is finite étale of degree $d \geq 0$) : one takes then $N = 1$, $\mathcal{Y} = \{Y\}$, $C = d + 1$, and for $\varphi : X \to \mathbb{Z}$ adapted to $\mathcal{X}$, i.e. constant of value a, one takes for $f_!\varphi : Y \to \mathbb{Z}$ the constant function of value $(1 - d)a$; to see that $N, \mathcal{Y}, C, f_!$ work for (3.1.2) in this case, one remarks that, Y being flat over $\mathbb{Z}$, any object K of $D^b_c(X[1/\ell], \overline{\mathbb{Q}}_\ell)$ which is lisse in cohomology is automatically moderately ramified in cohomology along $D[1/\ell]$ and along the section at infinity of $\mathbb{A}^1_Y$ (cf. [11] (4.7.1)). Afterwards, any base change $Y' \to Y[1/\ell]$, any direct factor L of $K_{Y'}$ is still lisse in cohomology on $X_{Y'}$, and moderately ramified along $D_{Y'}$, and the section at infinity of $\mathbb{A}^1_{Y'}$. It results (cf. [11] (4.7.1)) that the cohomology sheaves of $Rf'_!L$ are smooth on Y' and that the function χ of $Rf'_!L$ is given by

$$\chi(Rf'_!L)(y') = (1 - d)\chi(L)(x')$$

for any $x' \in X'$ of image y' in Y'. To control $\|Rf'_!L\|$ one uses the following facts : for a smooth $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{F}$ on $(\mathbb{A}^1_{Y'} - D_{Y'})[1/\ell]$, only $R^1f'_!\mathcal{F}$ and $R^2f'_!\mathcal{F}$ can be non-zero and the rank of $R^2f'_!\mathcal{F}$ at $y' \in Y'[1/\ell]$ is bounded by the rank of $\mathcal{F}$ on $\mathbb{A}^1_{Y'} - D_{Y'}$. So let y' be a point of $Y'[1/\ell]$ and x' a point of $(\mathbb{A}^1_{Y'} - D_{Y'})[1/\ell]$ above y', i.e. a point of $\mathbb{A}^1_{Y'} - D_{Y'}$; we have then

$$\|Rf'_!(L)\|(y') = \sum_i \dim_{\overline{\mathbb{Q}}_\ell}[R^if'_!(L)]_{y'} \leq \sum_i \sum_j \dim_{\overline{\mathbb{Q}}_\ell}[R^if'_!(\mathcal{H}^j(L))]_{y'}$$

(spectral sequence of hypercohomology), thus

$$\|Rf'_!(L)\|(y') < \sum_j (\sum_i \dim_{\overline{\mathbb{Q}}_\ell}[R^if'_!(\mathcal{H}^j(L))]_{y'}).$$

But, for any one of the $\mathcal{H}^i(L)$, we have

$$\begin{aligned}\sum_{i=1}^{2} \dim_{\overline{\mathbb{Q}}_\ell}[(R^if'_!\mathcal{F})_{y'}] &= -\chi(Rf'_!\mathcal{F})(y') + 2\dim_{\overline{\mathbb{Q}}_\ell}[(R^2f'_!\mathcal{F})_{y'}] \\ &= (d-1)\dim_{\overline{\mathbb{Q}}_\ell}(\mathcal{F}_{\bar{x}'}) + 2\dim_{\overline{\mathbb{Q}}_\ell}[(R^2f'_!\mathcal{F})_{y'}] \\ &\leq (d-1)\dim_{\overline{\mathbb{Q}}_\ell}(\mathcal{F}_{\bar{x}'}) + 2\dim_{\overline{\mathbb{Q}}_\ell}(\mathcal{F}_{\bar{x}'}),\end{aligned}$$

whence

$$\|Rf'_!L\|(y') \leq (d+1)\|L\|(x'),$$

which completes the proof of (3.1.2).

(3.4.3) *Proof of theorem (3.2.1).* We can replace X, S by $X_{\mathrm{red}}, S_{\mathrm{red}}$ without changing anything. For any good reduced scheme, there exists an integer $N \geq 1$ and a dense open set U of $S[1/N]$ which is normal and flat over $\mathbb{Z}$, and thus a disjoint union of its irreducible components ; working separately on each irreducible component, we are thus reduced to proving theorem (3.2.1) in the case where $S = \mathrm{Spec}(R)$, with R an integral, normal ring, of fraction field k of characteristic 0. We now proceed by induction on the dimension of $X \otimes_R k$. If $\dim(X \otimes_R k) = 0$, $X \otimes_R k$ is finite étale over k, as well as $X_\alpha \otimes_R k$ for each stratum X_α of the given stratification $\mathcal{X}$ of X. This situation propagates over a dense open set U of S ; then $N = 1, U$ and $\tilde{\mathcal{X}} = \{(X_\alpha)_U\}$ work for (3.2.1). If $\dim(X \otimes_R k) > 0$, let $x_1, \ldots, x_r$ be the maximal points of X. Up to replacing S by a dense open set of S, we can suppose that, for each $i = 1, \ldots, r$, the local ring of X at x_i is a field of finite type over k (recall that X is reduced), so that there exists an open neighborhood V_i of x_i in X that is smooth over R and contained in the stratum of $\mathcal{X}$ passing through x_i. Let us then set

$$V = \bigcup_{i=1}^{r} V_i, \quad Y = (X - V)_{\mathrm{red}};$$

V is an open set of X and $\dim(Y \otimes_R k) < \dim(X \otimes_R k)$; moreover, any complex K adapted to $\mathcal{X}$ is also adapted to the stratification $\mathcal{X}'$ of X given by

$$X = V \coprod \coprod_{\alpha} (X_\alpha \cap Y).$$

By induction, theorem (3.2.1) is proved for Y endowed with the stratification $\mathcal{X}_Y = \{X_\alpha \cap Y\}$, so there exists $U_1 \subset S$ a dense open set and a stratification $\tilde{\mathcal{Y}}$ of Y working for (3.2.1) for $(Y_{U_1} \to U_1, \mathcal{X}_Y)$. The central point of the proof is then the following : by resolution of singularities applied to $V \otimes_R k \hookrightarrow X \otimes_R k$ (cf. [10]), we know that there exists a dense open set $U_2 \subset U_1 \subset S$ and a diagram where each square is Cartesian

$$\begin{tikzcd} V \arrow[r, hook, "j"] \arrow[d, equal] & \tilde{X} \arrow[d, "\pi"] & D \arrow[l, hook', "i"'] \arrow[d, "\pi_D"] \\ V_{U_2} \arrow[r, hook, "j_{U_2}"] & X_{U_2} & Y_{U_2} \end{tikzcd} \qquad (3.4.3.1)$$

with π proper, $\tilde{X}$ smooth over U_2 and D a divisor in $\tilde{X}$ with normal crossings relative to U_2. Let us denote by $\mathcal{D}$ the canonical stratification of D (locally for the étale topology, $D_{\mathrm{red}} = \cup D_i$, with D_i irreducible and smooth over U_2, and $\mathcal{D}$ is described as the family of successive differences in the decreasing sequence of closed sets

$$D_{\mathrm{red}} = \bigcup_i D_i \supset \bigcup_{i \neq j} (D_i \cap D_j) \supset \bigcup_{i \neq j \neq k} (D_i \cap D_j \cap D_k) \supset \ldots).$$

The theorem (3.1.2) applied to $(\pi_D : D \to Y_{U_2}, \mathcal{D})$ provides us an integer $N \geq 1$ and a stratification $\mathcal{C}$ of $Y_{U_2}[1/N]$ such that $(\mathcal{D}[1/N], \mathcal{C})$ is a universal !-stratification for π_D (we forget C and $\pi_{D!}$). We now claim that this $N \geq 1$, the dense open set $U = U_2[1/N]$ of $S[1/N]$ and the stratification

$$\tilde{\mathcal{X}} = \{V_{U_2}[1/N], ((\mathcal{Y}_1)_{U_2}[1/N]) \cap \mathcal{C}\}$$

work for (3.2.1). Let us first note that, for $X' \to X$ étale, the diagram deduced from (3.4.3.1) by the base change $X' \to X$ still satisfies the same conditions; this allows us in the following to limit ourselves to the case $X' = X$. So let K be an object of $D^b_c(X[1/\ell], \overline{\mathbb{Q}}_\ell)$ adapted to the stratification

$$\mathcal{X}' = \{V, \{X_\alpha \cap Y\}\}.$$

The "exact sequence", where $K_U = K|X_U[1/\ell]$,

$$0 \to j_! j^* K_U \to K_U \to i_* i^* K_U \to 0$$

gives a "triangle"

$$D_{X_U/U}(i_* i^* K_U) \to D_{X_U/U}(K_U) \to D_{X_U/U}(j_! j^* K_U)$$

(cf. [8] (1.1.2) for the notion of "triangle" or "exact sequence"), in which we have, by duality of Verdier (cf. (1.1.3)),

$$\begin{aligned} D_{X_U/U}(i_* i^* K_U) &= i_* D_{Y_U/U}(i^* K_U) \\ D_{X_U/U}(j_! j^* K_U) &= Rj_* D_{V_U/U}(j^* K_U). \end{aligned}$$

By induction hypothesis, $D_{Y_U/U}(i^* K_U)$ is adapted to the stratification $(\mathcal{Y}_1)_U[1/\ell]$ of $Y_U[1/\ell]$ and its formation is compatible with any base change $T \to U[1/\ell]$, with T good. It results trivially from this, since i is a closed immersion, that $D_{X_U/U}(i_* i^* K_U)$ is adapted to the stratification $\tilde{\mathcal{X}}$ of X_U and that its formation is compatible with any base change $T \to U$ with T good. It remains thus to prove that $Rj_* D_{V_U/U}(j^* K_U)$ is adapted to $\tilde{\mathcal{X}}$ and that its formation commutes with any base change $T \to U[1/\ell]$, with T good. On $V_U[1/\ell]$, $j^* K_U$ has smooth cohomology and, V_U/U being smooth, $D_{V_U/U}(j^* K_U)$ also has smooth cohomology and its formation commutes with any base change $T \to U[1/\ell]$, with T good. To control what happens on $X_U - V_U$, we use the factorization $j = \pi \circ \tilde{j}$ (cf. (3.4.3.1)), which gives

$$Rj_* D_{V_U/U}(j^* K_U) = R\pi_* R\tilde{j}_* D_{V_U/U}(j^* K_U)$$

then, as π is proper, the proper base change theorem (cf. [SGA 4] XII (5.1)) says that

$$i^* Rj_* D_{V_U/U}(j^* K_U) = R\pi_{D!}(i^* R\tilde{j}_* D_{V_U/U}(j^* K_U)).$$

By (3.1.2), applied to π_D, we are reduced, to complete the proof of (3.2.1), to verifying the following point : if $L \in \mathrm{ob}\, D^b_c(V_U[1/\ell], \overline{\mathbb{Q}}_\ell)$ has smooth cohomology (here $L = D_{V_U/U}(j^* K_U)$), the object $i^* R\tilde{j}_! L$ of $D^b_c(D_U[1/\ell], \overline{\mathbb{Q}}_\ell)$ is adapted to the stratification $\mathcal{D}_U$ of D_U and of formation compatible with any base change $T \to U[1/\ell]$ with T good. But, this is well known : U being irreducible with a generic point of characteristic 0, the cohomology sheaves of L on $(X_U - D_U)[1/\ell]$ are automatically moderately ramified along $D_U[1/\ell]$ (cf. [SGA 1] XIII) and one can apply [SGA $4\frac{1}{2}$], [Th. Finitude] (A.1.3.3).

4. A uniformity theorem for the Fourier transform

(4.0) Let S be a scheme of finite type over $\mathbb{Z}$, $E \to S$ a vector bundle of constant rank $r \geq 1$, $E^\vee \to S$ its dual vector bundle, and $\mu : E \times_S E^\vee \to \mathbb{A}^1_S$ the canonical pairing.

For any finite field $\mathbb{F}_q$, for any prime number l invertible in $\mathbb{F}_q$, and for any non-trivial additive character $\psi : \mathbb{F}_q \to \overline{\mathbb{Q}}_l^\times$, we have the Fourier transform functor (cf. (2.1.4))

$$F_{S\otimes\mathbb{F}_q,\psi} : D^b_c(E_{S\otimes\mathbb{F}_q}, \overline{\mathbb{Q}}_l) \to D^b_c(E^\vee_{S\otimes\mathbb{F}_q}, \overline{\mathbb{Q}}_l);$$

more generally, if $T \to S$ is a morphism with T a good $\mathbb{F}_q$-scheme, we have the Fourier transform functor

$$F_{T,\psi} : D^b_c(E_T, \overline{\mathbb{Q}}_l) \to D^b_c(E^\vee_T, \overline{\mathbb{Q}}_l).$$

Theorem (4.1). Let, moreover, $\mathcal{E} = \{E_\alpha\}$ be a stratification of E. Then there exists an integer $N \geq 1$, a stratification $\mathcal{E}^\vee = \{E^\vee_\beta\}$ of $E^\vee[1/N]$, with strata $E^\vee_\beta$ non-empty and flat over $\mathbb{Z}$, an integer $C \geq 1$, and an additive map

$$\left\{\begin{array}{c}\text{constructible functions}\\ E[1/N] \to \mathbb{Z} \text{ adapted}\\ \text{to } \mathcal{E}[1/N]\end{array}\right\} \xrightarrow{\varphi \mapsto \varphi^\vee} \left\{\begin{array}{c}\text{constructible functions}\\ E^\vee[1/N] \to \mathbb{Z} \text{ adapted}\\ \text{to } \mathcal{E}^\vee[1/N]\end{array}\right\}$$

having the following property : for any prime number l, for any étale morphism $S' \to S[1/Nl]$, for any finite field $\mathbb{F}_q$ such that $S' \otimes \mathbb{F}_q \neq \emptyset$, for any $\mathbb{F}_q$-morphism $T \to S' \otimes \mathbb{F}_q$ with T a good $\mathbb{F}_q$-scheme, for any non-trivial additive character $\psi : \mathbb{F}_q \to \overline{\mathbb{Q}}_l^\times$, for any object K of $D^b_c(E_{S'}, \overline{\mathbb{Q}}_l)$ adapted to $\mathcal{E}_{S'}$, and for any direct factor L of K_T in $D^b_c(E_T, \overline{\mathbb{Q}}_l)$, the Fourier transform

$$F_{T,\psi}(L) \in \text{ob } D^b_c(E^\vee_T, \overline{\mathbb{Q}}_l)$$

is adapted to $\mathcal{E}^\vee_T$ and, for any point $t \in T$, it satisfies the inequality

$$\sup_{e' \in E^\vee_t} \|F_{T,\psi}(L)\|(e') \leq C \cdot \sup_{e \in E_t} \|L\|(e);$$

furthermore, if L is χ-adapted to $\mathcal{E}_T$, $F_{T,\psi}(L)$ is χ-adapted to $\mathcal{E}^\vee_T$ and we have (in the sense of (3.1)(iii))

$$\chi(F_{T,\psi}(L)) = (\chi(L))^\vee.$$

Remark (4.1.1). The map $\varphi \mapsto \varphi^\vee$, if it exists, is uniquely determined by $E \to S$. Moreover, for $\mathcal{E}$, $\mathcal{E}^\vee$ as above, it is given by the following "recipe" : for each α, let x_α be a geometric point of $E_\alpha[1/N]$; for each β, let y_β be a geometric point of $E^\vee_\beta$, s_β the geometric point of S below y_β (i.e., $s_\beta = \pi^\vee(y_\beta)$), and u_β a generic geometric point of $\mathbb{A}^1_{s_\beta}$; let l be a prime number distinct from the residual characteristics of the s_β ; then

$$\varphi^\vee(y_\beta) = \sum_\alpha \varphi(x_\alpha) \cdot [\chi_{\alpha,\beta} - \chi'_{\alpha,\beta}]$$

with

$$\chi_{\alpha,\beta} = \sum_i (-1)^i \dim_{\overline{\mathbb{Q}}_l} [H^i_c((E_\alpha)_{s_\beta}, \overline{\mathbb{Q}}_l)]$$

and

$$\chi'_{\alpha,\beta} = \sum_i (-1)^i \dim_{\overline{\mathbb{Q}}_l} [H^i_c((E_\alpha)_{s_\beta} \cap H_{y_\beta,u_\beta}, \overline{\mathbb{Q}}_l)]$$

where H_{y_β,u_β} is the geometric hyperplane of E_{s_β} with equation $\mu(x, y_\beta) = u_\beta$.

Corollary (4.2). We further assume S to be integral, with a generic point of characteristic zero, and let again $\mathcal{E} = \{E_\alpha\}$ be a stratification of E. Then, there exists a dense open set $U^\vee \subset E^\vee$ of $E^\vee$ satisfying the following condition : for any prime number l, for any étale morphism $S' \to S[1/l]$, for any finite field $\mathbb{F}_q$ such that $S' \otimes \mathbb{F}_q \neq \emptyset$, for any $\mathbb{F}_q$-morphism $T \to S' \otimes \mathbb{F}_q$ with T a good $\mathbb{F}_q$-scheme and $U^\vee_T \neq \emptyset$, for any non-trivial additive character $\psi : \mathbb{F}_q \to \overline{\mathbb{Q}}_l^\times$, for any object K of $D^b_c(E_{S'}, \overline{\mathbb{Q}}_l)$ which is adapted to $\mathcal{E}_{S'}$, and for any direct factor L of K_T in $D^b_c(E_T, \overline{\mathbb{Q}}_l)$, we have :

- (o) the cohomology sheaves of $F_{T,\psi}(L)|U^\vee_T$ and $F_{T,\bar\psi}(D_{E_T/T}(L))|U^\vee_T$ are smooth,
- (i) if K is perverse relative to S, these cohomology sheaves on $U^\vee_T$ are all zero except possibly those in dimension $-r$; the $\mathcal{H}^{-r}$ are smooth on $U^\vee_T$ and in perfect duality with values in the constant sheaf $\overline{\mathbb{Q}}_l$ on $U^\vee_T$,
- (ii) if K is perverse relative to S and if L is χ-adapted to $\mathcal{E}_T$, then the common rank of the $\mathcal{H}^{-r}$ is constant on $U^\vee_T$,
- (iii) if K is perverse relative to S and if we assume that T is of finite type over $\mathbb{F}_q$ and that L is pure, of weight m, relative to T, then the $\mathcal{H}^{-r}$ are punctually pure of weight m and $-m$ respectively on $U^\vee_T$.

Remark (4.2.1). For L as in (4.2)(ii), the common rank of the $\mathcal{H}^{-r}$ is given by the following recipe : let $\tilde{L}$ be any object of $D^b_c(E[1/l], \overline{\mathbb{Q}}_l)$ that is χ-adapted to $\mathcal{E}$ and whose χ function induces that of L on E_T ; let η be a geometric generic point of S, u a geometric generic point of $\mathbb{A}^1_\eta$ and y a geometric generic point of $E^\vee_\eta$; let $H_{y,u}$ be the generic geometric hyperplane of E_η with equation $\mu(x, y) = u$, then

$$\operatorname{rank} \mathcal{H}^{-r} = (-1)^r [\chi_c((E_\eta)_{\mathrm{gen}}, \tilde{L}) - \chi_c(H_{y,u}, \tilde{L})].$$

(4.2.2) Proof of the corollary. We take a stratification $\mathcal{E}^\vee$ of $E^\vee[1/N]$, for a certain $N \geq 1$, which works for (4.1) and we take for $U^\vee$ an open set of $E^\vee$ which contains the generic point $\delta^\vee$ of $E^\vee$ and which is contained in the unique stratum $E^\vee_\beta$ which contains $\delta^\vee$. The statement is now a formal consequence of (4.1) and (2.3.1) ; similarly, remark (4.2.1) formally follows from remark (4.1.1).

Lemma (4.2.3). In (4.2), we can choose the dense open set $U \subset E^\vee$ to be homogeneous, i.e., we can assume that the complement $Z = E^\vee - U^\vee$ is defined in $E^\vee$ by a homogeneous ideal.

Proof of (4.2.3). Let $U^\vee \subset E^\vee$ be a dense open set that works for (4.2) and let $Z = V(\mathcal{I}) \subset E^\vee$ be a closed subscheme such that $U^\vee = E^\vee - Z$. After the base change $T \to S' \otimes \mathbb{F}_q$, the $F_{T,\psi}(L)$ are smooth on $U^\vee_T$ for any non-trivial additive character ψ of $\mathbb{F}_q$. For $a \in \mathbb{F}_q^\times$, we can therefore replace ψ by $\psi_a(x) = \psi(a^{-1}x)$: the $F_{T,\psi_a}(L)$ are still smooth on $U^\vee_T$; this amounts to saying that the $F_{T,\psi}(L)$ are smooth on $a \cdot U^\vee_T$, for all $a \in \mathbb{F}_q^\times$. Consequently, the $F_{T,\psi}(L)$ are smooth on the open set $\bigcup_{a\in\mathbb{F}_q^\times} a \cdot U^\vee_T$ of $E^\vee_T$. The complement of this open set is defined by the smallest ideal which contains $\mathcal{I}_T$ and which is "graded mod $q-1$", let's call it $\mathcal{I}_{T,q-1}$. Note that if $\mathcal{I}_T$ is generated by elements of degree $\leq \delta$, then $\mathcal{I}_{T,q-1}$ is generated by elements of degree $\leq q-2$, then $\mathcal{I}_{T,q-1}$ is none other than the smallest graded ideal $\mathcal{I}_{T,\mathrm{homog}}$ which contains $\mathcal{I}_T$. It follows that, if $\mathcal{I}$ is generated by elements of degree $\leq \delta$, then the homogeneous dense open set $E^\vee - V((\delta+1)!\mathcal{I}_{\mathrm{homog}})$, where $\mathcal{I}_{\mathrm{homog}}$ is the smallest graded ideal that contains $\mathcal{I}$, works for (4.2).

(4.3) Proof of theorem (4.1). We use the cohomological analogue of the classical formula (factorization of the Fourier transform by the Radon transform) :

$$\int_{\mathbb{R}^r} f(x)e^{2\pi i\langle x,y\rangle}dx = \int_{\mathbb{R}} \left[\int_{\langle x,y\rangle=u} f(x)dx\right] e^{2\pi iu}du.$$

Let us denote by $pr : E \times_S E^\vee \to E$, $pr^\vee : E \times_S E^\vee \to E^\vee$ the two projections, by $\nu : \mathbb{A}^1_S \to \mathbb{A}^1_S$, $\rho : \mathbb{A}^1_S \to E^\vee$ the canonical morphisms and by $\gamma : E \times_S E^\vee \to \mathbb{A}^1_S \times_S E^\vee = \mathbb{A}^1_{E^\vee}$ the morphism defined by $\gamma(x,y) = (\mu(x,y), y)$; then we have a commutative diagram of S-schemes

$$\begin{array}{ccccc} E & \xleftarrow{pr} & E \times_S E^\vee & & \\ \downarrow \mu & & \downarrow \gamma & & \\ \mathbb{A}^1_S & \xleftarrow{\rho} & \mathbb{A}^1_{E^\vee} & \xrightarrow{pr^\vee} & E^\vee \end{array}$$

and, by the projection formula, we have

$$F_{T,\psi}(L) = R\rho_{T,!}[(R\gamma_{T,!}pr_T^*L) \otimes_{\overline{\mathbb{Q}}_l} \nu_T^*\mathcal{L}_\psi]; \tag{4.3.1}$$

this is the desired cohomological analogue. To use (4.3.1) to prove (4.1), we first apply (3.1.2) to the morphism $\gamma : E \times_S E^\vee \to \mathbb{A}^1_{E^\vee}$ and to the stratification

$pr^*\mathcal{E}$ of $E \times_S E^\vee$; this gives us an integer $N_1 \geq 1$ and a stratification $\mathcal{A}$ of $\mathbb{A}^1_{E^\vee}[1/N_1]$ which controls $R\gamma_* pr^*$. To control $F_{T,\psi}(L)$ we then apply (3.4.1) to $Y = E^\vee[1/N_1]$ and to the stratification $\mathcal{A}$ of $\mathbb{A}^1_Y = \mathbb{A}^1_{E^\vee[1/N_1]}$; this gives us an integer $N_2 \geq 1$ and a stratification $\mathcal{E}^\vee$ of $Y[1/N_2] = E^\vee[1/N_1 N_2]$. That $N = N_1 N_2 \geq 1$ and $\mathcal{E}^\vee$ (stratification of $E^\vee$ with normal, connected, and flat strata over $\mathbb{Z}$) work for (4.1) then results from the following lemma :

Lemma (4.3.2). Let Y be a scheme of finite type and flat over $\mathbb{Z}$ and

$$\mathcal{X} = \{\mathbb{A}^1_Y - D, D_1, \ldots, D_n\}, \quad D = \coprod_{i=1}^n D_i$$

a stratification of $\mathbb{A}^1_Y$, with $D_1, \ldots, D_n$ being divisors in $\mathbb{A}^1_Y$, finite étale over Y, of constant ranks $d_1, \ldots, d_n$. For any finite field $\mathbb{F}_q$, for any $\mathbb{F}_q$-morphism $T \to Y \otimes \mathbb{F}_q$, with T a good $\mathbb{F}_q$-scheme, for any prime number l invertible in $\mathbb{F}_q$, for any non-trivial additive character $\psi : \mathbb{F}_q \to \overline{\mathbb{Q}}_l^\times$, for any object K of $D^b_c(\mathbb{A}^1_T[1/l], \overline{\mathbb{Q}}_l)$ adapted to $\mathcal{X}_T$ and for any direct factor L of K_T in $D^b_c(\mathbb{A}^1_T, \overline{\mathbb{Q}}_l)$, we then have :

(i) the object $R\rho_{T,!}(L \otimes \mathcal{L}_\psi)$ of $D^b_c(T, \overline{\mathbb{Q}}_l)$, where $\rho_T : \mathbb{A}^1_T \to T$ is the canonical projection and $\mathcal{L}_\psi$ is the Artin-Schreier sheaf associated to ψ on $\mathbb{A}^1_T$ (cf. (2.0)), is of smooth cohomology over T;

(ii) for all $t \in T$, we have the inequality $\|R\rho_{T,!}(L \otimes \mathcal{L}_\psi)\|(t) \leq 2d \sup_{x \in \mathbb{A}^1_T} \|L\|(x)$ where d is the constant degree of $D \to Y$;

(iii) if we assume that L is χ-adapted to $\mathcal{X}$, then the function χ of $R\rho_{T,!}(L \otimes \mathcal{L}_\psi)$ on T is constant of value (for all $t \in T$)

a) $\chi(R\rho_{T,!}(L \otimes \mathcal{L}_\psi))(t) = \chi(R\rho_{T,*}L)(y) - \chi(\tilde{L})(x)$ for an arbitrary choice of a $\tilde{L} \in \mathrm{ob}\, D^b_c(\mathbb{A}^1_Y[1/l], \overline{\mathbb{Q}}_l)$ χ-adapted to $\mathcal{X}$, whose χ function induces that of L, and of points $x \in \mathbb{A}^1_Y[1/l]$, $y \in Y[1/l]$ (in particular, one can take $x \in \mathbb{A}^1_Y(\mathbb{C}), y \in Y(\mathbb{C})$); we have

b) $\chi(R\rho_{T,!}(L \otimes \mathcal{L}_\psi))(t) = -da + \sum_{i=1}^n d_i a_i$ where a (resp. a_i) is the constant value of $\chi(L)$ on $\mathbb{A}^1_T - D_T$ (resp. $D_{i,T}$, for $i = 1, \ldots, n$) and where $d = \sum_{i=1}^n d_i$ is the constant degree of D over Y.

(4.3.3) Proof of (4.3.2). The flatness over $\mathbb{Z}$ of Y ensures that K is moderately ramified along $D[1/l]$ and the section at infinity of $\mathbb{A}^1_Y[1/l]$ (cf. [8] (4.7.1)); consequently, L on $\mathbb{A}^1_T$ is also moderately ramified along D_T and the section at infinity of $\mathbb{A}^1_T$. On the other hand, $\mathcal{L}_\psi$ is smooth of rank 1 on $\mathbb{A}^1_T$, with Swan conductor constant 1 along the section at infinity of $\mathbb{A}^1_T$ ($\mathrm{Swan}_\infty(\mathcal{L}_\psi | \mathbb{A}^1_t) = 1$, for any geometric point t of T). The smoothness of the cohomology sheaves of $R\rho_{T!}(L \otimes \mathcal{L}_\psi)$ on T is therefore a particular case of the semi-continuity theorem of the Swan conductor, demonstrated by Deligne (cf. [12] (2.1.2)). The formulas for $\chi(R\rho_{T,!}(L \otimes \mathcal{L}_\psi))$ result from the Grothendieck-Ogg-Shafarevich formula ([15] and [11] (4.8.2)) : for t a geometric point of T and $\bar{\eta}$ a generic geometric point of $\mathbb{A}^1_t$, we have by moderation of the $\mathcal{H}^i(L)$, $\chi(R\rho_{T,!}(\mathcal{H}^i(L) \otimes \mathcal{L}_\psi))(t) = \chi(R\rho_{T,!}\mathcal{H}^i(L))(t) - \mathrm{Swan}_\infty(\mathcal{L}_\psi) \cdot \chi(\mathcal{H}^i(L))(\bar{\eta}) = \chi(R\rho_{T,!}\mathcal{H}^i(L))(t) - \chi(\mathcal{H}^i(L))(\bar{\eta})$

whence $\chi(R\rho_{T,!}(L\otimes\mathcal{L}_\psi))(t)=\chi(R\rho_{T,!}L)(t)-\chi(L)(\bar\eta)$ which completes the demonstration of (iii).

To demonstrate (ii), we use the following facts (cf. [11] (4.8)) : for a $\overline{\mathbb{Q}}_l$-sheaf $\mathcal{F}$ on $\mathbb{A}^1_t$ which is moderately ramified and adapted to $(\mathbb{A}^1_t-D_t, D_{1,t},\dots,D_{n,t})$, we have

$$H^2_c(\mathbb{A}^1_t,\mathcal{F}\otimes\mathcal{L}_\psi)=0$$

and

$$\begin{aligned}\dim_{\overline{\mathbb{Q}}_l}(H^1_c(\mathbb{A}^1_t,\mathcal{F}\otimes\mathcal{L}_\psi)) &< \dim_{\overline{\mathbb{Q}}_l}(H^1_c(D_{i,t},\mathcal{F}\otimes\mathcal{L}_\psi))\\ &< \sum_{x\in D_t}\dim_{\overline{\mathbb{Q}}_l}(\mathcal{F}_x),\end{aligned}$$

and, consequently, we have

$$\sum_i \dim_{\overline{\mathbb{Q}}_l}(H^i_c(\mathbb{A}^1_t,\mathcal{F}\otimes\mathcal{L}_\psi))$$

$$\begin{aligned}&=-\chi_c(\mathbb{A}^1_t,\mathcal{F}\otimes\mathcal{L}_\psi)+2\dim_{\overline{\mathbb{Q}}_l}(H^1_c(\mathbb{A}^1_t,\mathcal{F}\otimes\mathcal{L}_\psi))\\ &=d\cdot\dim_{\overline{\mathbb{Q}}_l}(\mathcal{F}_{\bar\eta})-\sum_{x\in D_t}\dim_{\overline{\mathbb{Q}}_l}(\mathcal{F}_x)+2\dim_{\overline{\mathbb{Q}}_l}(H^1_c(\mathbb{A}^1_t,\mathcal{F}\otimes\mathcal{L}_\psi))\\ &< d\cdot\dim_{\overline{\mathbb{Q}}_l}(\mathcal{F}_{\bar\eta})+\sum_{x\in D_t}\dim_{\overline{\mathbb{Q}}_l}(\mathcal{F}_x).\end{aligned}$$

For $t\in T$, we thus have $\|R\rho_{T,!}(L\otimes\mathcal{L}_\psi)\|(t)\le\sum_i\dim_{\overline{\mathbb{Q}}_l}(H^i_c(\mathbb{A}^1_t,\mathcal{H}^i(L)\otimes\mathcal{L}_\psi))$

$$\le\sum_i[d\cdot\dim_{\overline{\mathbb{Q}}_l}(\mathcal{H}^i(L)_{\bar\eta})+\sum_{x\in D_t}\dim_{\overline{\mathbb{Q}}_l}(\mathcal{H}^i(L)_x)]$$

whence $\|R\rho_{T,!}(L\otimes\mathcal{L}_\psi)\|(t)\le\sum_{x\in D_t}[d\cdot\|L\|(\bar\eta)+\|L\|(x)]$ and the conclusion. This completes the demonstration of (4.3.2) and consequently that of (4.1) and of remark (4.1.1).

5. A first application of the uniformity theorem of the Fourier transform to trigonometric sums

(5.0) Let $R\subset C$ be a sub-$\mathbb{Z}$-algebra of $\mathbb{C}$ of finite type, r an integer >0, $\mathbb{A}^r_R$ the standard affine space of coordinates $x_1,\dots,x_r$ over R, X an affine R-scheme, smooth, purely of relative dimension m, and

$$f=(f_1,\dots,f_r):X\to\mathbb{A}^r_R$$

a finite R-morphism. Let us also fix an algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$. The data above are the "fixed" data ; to this fixed data, we will add "mobile" data :

— an invertible function g on X (possibly $g=1$) which will also be seen as an R-morphism $g:X\to\mathbb{G}_{m,R}$,

— a quintuple $(\mathbb{F}_q, \varphi, \mathbf{a}, \psi, \chi)$ composed of a finite field $\mathbb{F}_q$, a ring homomorphism $\varphi : R \to \mathbb{F}_q$ with $\varphi(1) = 1$, an element $\mathbf{a} = (a_1, \ldots, a_r)$ of $(\mathbb{F}_q)^r$, a non-trivial additive character $\psi : \mathbb{F}_q \to \overline{\mathbb{Q}}^\times$ and a multiplicative character (possibly trivial) $\chi : \mathbb{F}_q^\times \to \overline{\mathbb{Q}}^\times$.

We will denote by $f_\mathbb{C} : X_\mathbb{C} \to \mathbb{A}^r_\mathbb{C}$ and $g_\mathbb{C} : X_\mathbb{C} \to \mathbb{G}_{m,\mathbb{C}}$ the morphisms of $\mathbb{C}$-schemes $f \otimes_R \mathbb{C}$ and $g \otimes_R \mathbb{C}$; more generally, for any ring homomorphism $\varphi : R \to k$, with $\varphi(1) = 1$, we will denote by $f_\varphi : X_\varphi \to \mathbb{A}^r_k$ and $g_\varphi : X_\varphi \to \mathbb{G}_{m,k}$ the morphisms of k-schemes $f \otimes_R k$ and $g \otimes_R k$.

(5.1) The R-morphism $f : X \to \mathbb{A}^r_R$ being fixed as in (5.0), to any invertible function g on X and to any quintuple $(\mathbb{F}_q, \varphi, \mathbf{a}, \psi, \chi)$ as in (5.0), we associate the family of trigonometric sums $(S_n)_{n \geq 1}$ (n integer), with

$$S_n = S_n(g, \mathbb{F}_q, \varphi, \mathbf{a}, \psi, \chi) = \sum_{x \in X_\varphi(\mathbb{F}_{q^n})} \psi_n\left(\sum_{i=1}^r a_i \cdot f_{i,\varphi}(x)\right) \chi_n(g_\varphi(x)),$$

where

$$\psi_n = \psi \circ \mathrm{Tr}_{\mathbb{F}_{q^n}/\mathbb{F}_q} \quad \text{and} \quad \chi_n = \chi \circ N_{\mathbb{F}_{q^n}/\mathbb{F}_q},$$

and we associate the L function defined by

$$L(T) = L(g, \mathbb{F}_q, \varphi, \mathbf{a}, \psi, \chi; T) = \exp\left(\sum_{n \geq 1} S_n \frac{T^n}{n}\right).$$

Theorem (5.2). For any R-morphism $f : X \to \mathbb{A}^r_R$ as in (5.0), there exists a non-zero homogeneous polynomial $F(y_1, \ldots, y_r) \in R[y_1, \ldots, y_r]$ having the following property : for any invertible function g on X and for any quintuple $(\mathbb{F}_q, \varphi, \mathbf{a}, \psi, \chi)$, as in (5.0), such that

$$(\varphi F)(a_1, \ldots, a_r) \neq 0,$$

the function $L(T) = L(g, \mathbb{F}_q, \varphi, \mathbf{a}, \psi, \chi; T)$ is of the form

$$L(T) = P(T)^{(-1)^{m+1}}$$

where

$$P(T) = \prod_{\lambda=1}^d (1 - \alpha_\lambda T)$$

is a polynomial, with $P(0) = 1$, satisfying :

(i) the $(\alpha_\lambda)_{\lambda=1,\ldots,d}$ are algebraic integers such that

$$|\alpha_\lambda| = q^{m/2}$$

for all $\lambda = 1, \ldots, d$ and for any archimedean absolute value $||$ over $\overline{\mathbb{Q}}$,

(ii) the degree d of $P(T)$ is independent of the choices of g and of $(\mathbb{F}_q, \varphi, \mathbf{a}, \psi, \chi)$ and is given by the topological formula

$$d = (-1)^m [\chi(X_{\mathbb{C}}) - \chi(H_{\mathbb{C}})]$$

where $H_{\mathbb{C}}$ is the hypersurface of $X_{\mathbb{C}}$ with equation

$$\sum_{i=1}^{r} a_i f_{i,\mathbb{C}}(x) = b$$

for any $a_1, \ldots, a_r, b$ in $\mathbb{C}$ which are algebraically independent over R.

Furthermore, the function $L(T) = L(g, \mathbb{F}_q, \varphi, \mathbf{a}, \bar{\psi}, \bar{\chi}; T)$ for $\bar{\psi} = \psi^{-1}$ and $\bar{\chi} = \chi^{-1}$ is written

$$L(T) = \left[\prod_{\lambda=1}^{d} (1 - q^m \alpha_\lambda^{-1} T)\right]^{(-1)^{m-1}} .$$

Corollary (5.2.1). For any invertible function g on X, for any quintuple $(\mathbb{F}_q, \varphi, \mathbf{a}, \psi, \chi)$, as in (5.0), such that

$$(\varphi F)(a_1, \ldots, a_r) \neq 0,$$

for any integer $n \geq 1$ and for any archimedean absolute value $||$ over $\overline{\mathbb{Q}}$, we have the majoration

$$|S_n(g, \mathbb{F}_q, \varphi, \mathbf{a}, \psi, \chi)| \leq (-1)^m [\chi(X_{\mathbb{C}}) - \chi(H_{\mathbb{C}})] q^{mn/2},$$

with $H_{\mathbb{C}}$ as in (5.2).

(5.3) We will now deduce (5.2) from a cohomological statement. Given $g \in \Gamma(X, \mathcal{O}_X^\times)$ and a quintuple $(\mathbb{F}_q, \varphi, \mathbf{a}, \psi, \chi)$ as in (5.0), let us choose a prime number l invertible in $\mathbb{F}_q$ and an embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_l$; the homomorphism $\varphi : R \to \mathbb{F}_q$ extends to a homomorphism, still denoted φ, from $R[1/l(q-1)]$ into $\mathbb{F}_q$. The covering $\pi_\varphi : \tilde{X}_\varphi \to X_\varphi$, deduced by pullback by g_φ of the Lang torsor of $\mathbb{G}_{m,\mathbb{F}_q}$,

$$\begin{gathered} \tilde{X}_\varphi = X_\varphi \times_{\mathbb{G}_{m,\mathbb{F}_q}} [\sqrt[q-1]{g_\varphi}] \to \mathbb{G}_{m,\mathbb{F}_q} \\ \downarrow z \mapsto z^{q-1} \\ X_\varphi \xrightarrow{g_\varphi} \mathbb{G}_{m,\mathbb{F}_q} \end{gathered}$$

is finite étale and still an $\mathbb{F}_q^\times$-torsor ; we denote by $\mathcal{L}_\chi(g_\varphi)$ the rank 1 smooth $\overline{\mathbb{Q}}_l$-sheaf on X_φ, pullback by g_φ of the rank 1 smooth $\overline{\mathbb{Q}}_l$-sheaf on $\mathbb{G}_{m,\mathbb{F}_q}$, deduced from the $\mathbb{F}_q^\times$-torsor of Lang by extension of the structural group via

$$\mathbb{F}_q^\times \xrightarrow{\chi} \overline{\mathbb{Q}}^\times \hookrightarrow \overline{\mathbb{Q}}_l^\times$$

(cf. [SGA 4_1], [Sommes trig.] (1.7)). Then we have a decomposition

$$(\pi_\varphi)_*\overline{\mathbb{Q}}_l = \bigoplus_\chi \mathcal{L}_\chi(g_\varphi),$$

the sum being extended over the $q-1$ characters χ of $\mathbb{F}_q^\times$ with values in $\overline{\mathbb{Q}}_l^\times$. Similarly, we denote by $\mathcal{L}_\psi(\sum a_i f_i)$ the rank 1 smooth $\overline{\mathbb{Q}}_l$-sheaf on X_φ pullback by

$$\sum_{i=1}^r a_i f_{i,\varphi} : X_\varphi \to \mathbb{A}^1_{\mathbb{F}_q}$$

of the Artin-Schreier $\overline{\mathbb{Q}}_l$-sheaf $\mathcal{L}_\psi$ on $\mathbb{A}^1_{\mathbb{F}_q}$ (cf. (2.0)). The sheaves $\mathcal{L}_\chi(g_\varphi)$ and $\mathcal{L}_\psi(\sum a_i f_i)$ are punctually pure of weight 0 and smooth on an affine and smooth scheme X_φ over $\mathbb{F}_q$, purely of dimension m. Taking into account the cohomological interpretation of exponential sums (cf. [11] 3 or [SGA 4_1], [Sommes trig.] (1.9)) and Deligne's results (cf. [8] (3.3.1)), theorem (5.2) immediately results from the following theorem :

Theorem (5.4). Let R, X and $f : X \to \mathbb{A}^r_R$ be as in (5.0). Then there exists a non-zero homogeneous polynomial $F(y_1, \dots, y_r) \in R[y_1, \dots, y_r]$ such that, for any invertible function g on X, for any quintuple $(\mathbb{F}_q, \varphi, \mathbf{a}, \psi, \chi)$ such that

$$(\varphi F)(a_1, \dots, a_r) \neq 0$$

and for any prime number l invertible in $\mathbb{F}_q$, we have :

(i) the l-adic cohomology groups

$$H^j_c = H^j_c(X_\varphi \otimes_{\mathbb{F}_q} \bar{\mathbb{F}}_q, \mathcal{L}_\psi(\sum_{i=1}^r a_i f_{i,\varphi}) \otimes \mathcal{L}_\chi(g_\varphi))$$

are zero for all $j \neq m$, and the pairing defined by the cup-product and the trace

$$H^m_c \times H^m_c \xrightarrow{\cup} H^{2m}_c(X_\varphi \otimes \bar{\mathbb{F}}_q, \overline{\mathbb{Q}}_l) \xrightarrow{\mathrm{Tr}} \overline{\mathbb{Q}}_l(-m)$$

where $\bar{H}^m_c$ is defined as H^m_c but with $\bar{\psi}, \bar{\chi}$ in place of ψ, χ, is a perfect duality,

(ii) if we set

$$H^j = H^j(X_\varphi \otimes_{\mathbb{F}_q} \bar{\mathbb{F}}_q, \mathcal{L}_\psi(\sum_{i=1}^r a_i f_{i,\varphi}) \otimes \mathcal{L}_\chi(g_\varphi)),$$

the forgetful support map

$$H^j_c \to H^j$$

is an isomorphism for all j,

(iii) for all j, $H^j_c \simeq H^j$ is pure of weight j, and even zero if $j \neq m$,

(iv) the formula

$$\chi_c(X_\varphi \otimes \bar{\mathbb{F}}_q, \mathcal{L}_\psi(\sum a_i f_{i,\varphi}) \otimes \mathcal{L}_\chi(g_\varphi)) = \chi_c(X_\mathbb{C}, \overline{\mathbb{Q}}_l) - \chi_c(H_\mathbb{C}, \overline{\mathbb{Q}}_l).$$

Remark (5.4.1). The assertions (i) and (ii) of the statement above are equivalent and (iii) results from them (cf. [SGA 4_1], [Sommes trig.] (1.2.0)).

(5.4.2) Let's prove (5.4). Let's first construct a candidate for $F(y_1, \ldots, y_r)$. For this, we first apply (3.1.2) to the finite morphism $X \to \mathbb{A}^r_R$ and to the trivial stratification $\mathcal{X} = \{X\}$ of X ; this gives us an integer $N \geq 1$ and a stratification $\mathcal{A}$ of $\mathbb{A}^r_R[1/N]$. Then we apply (4.2) and (4.2.3) to $S = \mathrm{Spec}(R[1/N])$, $E = \mathbb{A}^r_S$, $\mathcal{E} = \mathcal{A}$; this gives us a homogeneous dense open set $U^\vee$ of $E^\vee = \mathbb{A}^r_S$ and therefore a dense open set U_V of $\mathbb{A}^r_{\mathrm{Spec}(R[y_1,\ldots,y_r])}$. We take for $F(y_1, \ldots, y_r)$ a non-zero homogeneous polynomial which vanishes on $\mathrm{Spec}(R[y_1, \ldots, y_r]) - U^\vee_V$, such that

$$\mathrm{Spec}(R[y_1, \ldots, y_r][1/F]) \subset U^\vee_V.$$

To see that F works for (5.4), we consider for each invertible function g and each finite field the covering

$$\pi : \tilde{X} \to X[1/(q-1)]$$

deduced by pullback by g of the μ_{q-1}-torsor of Kummer of $\mathbb{G}_{m,R[1/(q-1)]}$,

$$\tilde{X} \to \mathbb{G}_{m,R[1/(q-1)]}$$

$$\pi \downarrow z \mapsto z^{q-1}$$

$$X[1/(q-1)] \to \mathbb{G}_{m,R[1/(q-1)]}$$

it is a finite étale covering of degree $q-1$, so that, for any prime number l invertible in $\mathbb{F}_q$, the $\overline{\mathbb{Q}}_l$-sheaf $\pi_*\overline{\mathbb{Q}}_l$ on $X[1/l(q-1)]$ is smooth of rank $q-1$, punctually pure of weight 0. Because $X[1/l(q-1)]$ is smooth over $R[1/l(q-1)]$, purely of relative dimension m, the object

$$K_g = \pi_*\overline{\mathbb{Q}}_l[m]$$

of $D^b_c(X[1/l(q-1)], \overline{\mathbb{Q}}_l)$, which is adapted to the trivial stratification $\mathcal{X} = \{X\}$ of X, is perverse and pure, of weight m, relative to $\mathrm{Spec}(R[1/l(q-1)])$ (cf. (1.2.2)(iii) and (1.3.2)(iii)). Because $f : X \to \mathbb{A}^r_R$ is finite, the object

$$K = (f_*\pi_*\overline{\mathbb{Q}}_l[m])|\mathbb{A}^r_R[1/l(q-1)N]$$

of $D^b_c(\mathbb{A}^r_R[1/l(q-1)N], \overline{\mathbb{Q}}_l)$ is still perverse and pure, of weight m, relative to $\mathrm{Spec}(R[1/l(q-1)N])$; moreover, by choice of N, K is adapted to the stratification $\mathcal{A}[1/l(q-1)]$ of $E[1/l(q-1)]$. The hypothesis

$$(\varphi F)(a_1, \ldots, a_r) \neq 0$$

allows us to extend φ to $R[1/l(q-1)N]$ so that we have the object K_φ on $\mathbb{A}^r_{\mathbb{F}_q}$ deduced from K by the base change

$$\mathrm{Spec}(\mathbb{F}_q) \to \mathrm{Spec}(R[1/l(q-1)N]).$$

We have

$$K_\varphi = (f_{\varphi *}\pi_{\varphi *}\overline{\mathbb{Q}}_l)[m],$$

where π_φ coincides with the covering of the same name considered in (5.3) (identification of the Lang torsor and the μ_{q-1}-torsor of Kummer of $\mathbb{G}_{m,\mathbb{F}_q}$) ; consequently, K_φ decomposes into

$$K_\varphi = \bigoplus_\chi f_{\varphi *}\mathcal{L}_\chi(g_\varphi)[m]$$

where χ runs through the $q-1$ characters $\mathbb{F}_q^\times \to \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_l^\times$.

It only remains to apply (4.2) to $S' = \mathrm{Spec}(R[1/l(q-1)N])$, $T = \mathrm{Spec}(\mathbb{F}_q)$, $T \to S' \otimes \mathbb{F}_q$ induced by φ, ψ, K and

$$L = f_{\varphi *}\mathcal{L}_\chi(g_\varphi)[m]$$

which is a direct factor of $K_T = K_\varphi$ in $D^b_c(E_T, \overline{\mathbb{Q}}_l)$. The hypothesis

$$(\varphi F)(a_1, \dots, a_r) \neq 0$$

ensures that the point $\mathbf{a} = (a_1, \dots, a_r)$ is in $U_V^\vee$ and, by application at this point of $U_V^\vee$ of the statements (4.2)(i), (ii), (iii) and of the remark (4.2.1), we obtain (up to a shift) the assertions (5.4)(i), (iii) and (iv).

(5.5) The situation in equal characteristic $p > 0$. What become (5.2) and (5.4) if we allow the ring R to be an integral domain of finite type over a finite field $\mathbb{F}_p$?

Theorem (5.5.1). Let R be an integral domain, of finite type over a finite field $\mathbb{F}_p$, X an affine and smooth R-scheme, purely of relative dimension m, $r \geq 1$ an integer,

$$f = (f_1, \dots, f_r) : X \to \mathbb{A}^r_R$$

a finite R-morphism,

$$g : X \to \mathbb{G}_{m,R}$$

an invertible function on X, $\psi_0 : \mathbb{F}_p \to \overline{\mathbb{Q}}^\times$ a non-trivial additive character, $\chi_0 : \mathbb{F}_p^\times \to \overline{\mathbb{Q}}^\times$ a multiplicative character (possibly trivial), l a prime number invertible in $\mathbb{F}_p$ and $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_l$ a fixed embedding. Then, there exists a non-zero polynomial $F(y_1, \dots, y_r) \in R[y_1, \dots, y_r]$ and an integer $d \geq 0$ such that, for any triple $(\mathbb{F}_{q^\nu}, \rho, \mathbf{a})$ composed of a finite extension $\mathbb{F}_{q^\nu}$ of $\mathbb{F}_q$, an $\mathbb{F}_q$-homomorphism $\rho : R \to \mathbb{F}_{q^\nu}$ with $\rho(1) = 1$ and an element $\mathbf{a} = (a_1, \dots, a_r) \in (\mathbb{F}_{q^\nu})^r$ such that

$$(\rho F)(a_1, \dots, a_r) \neq 0,$$

we have :

(i) the L function associated to the exponential sums

$$S_n = \sum_{x \in X_\rho(\mathbb{F}_{q^{\nu n}})} \psi_{\nu n}\left(\sum_{i=1}^{r} a_i f_{i,\rho}(x)\right) \chi_{\nu n}(g_\rho(x))$$

$(n \geq 1)$ is of the form

$$L(T) = \left[\prod_{\lambda=1}^{d}(1-\alpha_\lambda T)\right]^{(-1)^{m+1}}$$

where the α_λ ($\lambda = 1, \ldots, d$) are algebraic integers whose all archimedean absolute values satisfy

$$|\alpha_\lambda| = q^{\nu m/2},$$

(ii) in l-adic cohomology, for any integer j, the forgetful support map

$$H_c^j(X_\rho \otimes \bar{\mathbb{F}}_{q^\nu}, \mathcal{L}_\psi(\sum a_i f_{i,\rho}) \otimes \mathcal{L}_{\chi_\nu}(g_\rho))$$

$$\to H^j(X_\rho \otimes \bar{\mathbb{F}}_{q^\nu}, \mathcal{L}_\psi(\sum a_i f_{i,\rho}) \otimes \mathcal{L}_{\chi_\nu}(g_\rho))$$

is an isomorphism (so that $H_c^j \simeq H^j$ is pure of weight j, for all j, and zero for $j \neq m$) and we have the formula

$$\chi_c(X_\rho \otimes \bar{\mathbb{F}}_{q^\nu}, \mathcal{L}_{\psi_\nu}(\sum a_i f_{i,\rho}) \otimes \mathcal{L}_{\chi_\nu}(g_\rho)) = (-1)^m d.$$

Indeed, this statement immediately results from (2.3.1) applied to $F_\psi(f_* \mathcal{L}_\chi(g)[m])$ and from (2.1.5)(i) (commutation of Fourier with base changes).

Remark (5.5.2). A major difference between (5.5.1) on one hand and (5.2), (5.4) on the other, is that in (5.5.1), the polynomial F and the degree d depend a priori on the choices of g, ψ, χ, l ; the interpretation of d as the degree of the L function shows however that d does not depend on l and its interpretation as an Euler-Poincaré characteristic of X_ρ with coefficients in $\mathcal{L}_{\psi_\nu}(\sum a_i f_{i,\rho}) \otimes \mathcal{L}_{\chi_\nu}(g_\rho)$ further shows that d does not depend on g and χ (cf. [11] (5.5.2), cor. 2) ; d can therefore only depend on the choice of ψ (for R, X and f fixed of course). Does d really depend on ψ ? We don't know. Is it possible, for a given ψ, to choose a polynomial F_ψ that "works" for all g, χ, l ? Does there exist a single ψ for which such an F_ψ exists ?

(5.6) Remarks on the concrete applications of (5.2) and (5.2.1).

(5.6.1) For $f : X \to \mathbb{A}_R^r$ given as in (5.0), the demonstrations (4.3) and (5.4.2) provide a sufficient condition, which can be made explicit, for the geometric purity of the trigonometric sums $S_n(\mathbb{F}_q, \varphi, \mathbf{a}, \psi, \chi)$, namely $(\varphi F)(a_1, \ldots, a_r) \neq 0$.

For example, for the trigonometric sums studied in [11] (5.1.1), we can easily verify that the condition $(\varphi F)(a_1, \ldots, a_r) \neq 0$ is expressed, with the notations of loc. cit., in the following way : the pencil of hypersurfaces of degree d of $\mathbb{P}$, generated by H and L^d, induces on $X \cap \mathbb{P}$ a Lefschetz pencil. One can also note on this example that the condition $(\varphi F)(a_1, \ldots, a_r) \neq 0$ is generally more restrictive than the purity criteria that can be derived from a precise analysis of the considered trigonometric sums (always in loc. cit., the purity criterion is simply : H is transverse to $X \cap L$, in other words the axis of the pencil generated by H and L^d is transverse to X, which is only one of the conditions required for the induced pencil on X to be of Lefschetz).

(5.6.2) The hypothesis that X is finite over $\mathbb{A}^r_R$ is less innocent than one might think. Of course, if one starts with a closed subscheme $X \subset \mathbb{A}^r_R$, which is smooth over R, there are many ways to find finite morphisms

$$f = (f_1, \ldots, f_r) : X \to \mathbb{A}^r_R.$$

The simplest way is to restrict to X polynomials $f_1, \ldots, f_r$ that define a finite endomorphism of $\mathbb{A}^n_R$; for example, this is automatic if we have

$$f_1 \in R[x_1], f_1 \text{ unitary of degree } d_1 \geq 1$$

$$\vdots$$

$$f_i \in R[x_1, \ldots, x_{i-1}][x_i], f_i \text{ unitary in } x_i \text{ of degree } d_i \geq 1$$

$$\vdots$$

On the other hand, if one starts from the open subscheme

$$X = \mathrm{Spec}(\mathbb{Z}[x, x^{-1}]) = \mathbb{G}_{m,\mathbb{Z}} \subset \mathbb{A}^1_{\mathbb{Z}} = \mathrm{Spec}(\mathbb{Z}[x]),$$

then "one does not have the right" to consider X equipped with the function $f = x$ and our results (5.2), (5.4), therefore say nothing about the Gauss sums

$$\sum_{x \in \mathbb{F}_q^\times} \psi(x)\chi(x).$$

We will partially fill this gap in the next chapter.

(5.6.3) Consider $X = \mathbb{G}_{m,\mathbb{Z}}$, equipped with the morphism $f : \mathbb{G}_{m,\mathbb{Z}} \to \mathbb{A}^2_{\mathbb{Z}}$ defined by a "true" Laurent polynomial,

$$f(x) = \sum_{i=-d_1}^{d_2} a_i x^i$$

with $d_1, d_2 \geq 1$ and $a_{d_2} = a_{-d_1} = 1$; then f is a finite morphism and our results therefore say something about the corresponding sums,

$$S_n = \sum_{x \in \mathbb{F}_{q^n}^\times} \psi_n(f(x))\chi_n(x) \quad (n \geq 1),$$

namely that the associated L function is, for $p \gg 0$, a polynomial of degree $d_1 + d_2$, pure of weight 1. But a detailed analysis of these sums shows that in any characteristic p prime to $d_1 d_2$, this is in fact the case : $L(T)$ is a polynomial of degree $d_1 + d_2$, pure of weight 1. This example illustrates the fundamental limitation of our results : they are generic results in p, i.e., for $p \gg 0$. Of course, our results are also generic in $(a_1, \ldots, a_r)$, but whereas our methods allow us to control the restriction to $\mathbb{A}^r_{R\otimes\mathbb{Q}}$ of the open set $\{F(y_1, \ldots, y_r) \neq 0\}$, they do not allow us to control the restriction of this same open set to $\mathbb{A}^r_{R\otimes\mathbb{F}_p}$, for a given prime number p.

(5.7) The situation when X is no longer assumed to be smooth over R. — The following variant of (5.2) was suggested to us by Deligne :

Theorem (5.7.0). Let $R \subset C$ be a sub-$\mathbb{Z}$-algebra of $\mathbb{C}$ of finite type, r an integer > 0, $\mathbb{A}^r_R$ the standard affine space of coordinates $x_1, \ldots, x_r$ over R, X an affine R-scheme, of relative dimension $\leq m$, and

$$f = (f_1, \ldots, f_r) : X \to \mathbb{A}^r_R$$

a finite R-morphism. Then there exists a non-zero homogeneous polynomial $F(y_1, \ldots, y_r) \in R[y_1, \ldots, y_r]$ and an integer $C \geq 1$ having the following property : for any invertible function g on X, for any quintuple $(\mathbb{F}_q, \varphi, \mathbf{a}, \psi, \chi)$ as in (5.0) such that

$$(\varphi F)(a_1, \ldots, a_r) \neq 0,$$

for any integer $n \geq 1$ and for any archimedean absolute value $||$ on $\overline{\mathbb{Q}}$, we have the majoration (with the notations of (5.1))

$$|S_n(g, \mathbb{F}_q, \varphi, \mathbf{a}, \psi, \chi)| \leq C \cdot q^{mn/2}.$$

Modulo the cohomological interpretation of exponential sums and Deligne's results on the weights of the H^j_c, theorem (5.7.0) immediately results from the following variant of (5.4).

Theorem (5.7.1). In the situation of (5.7.0), there exists a non-zero homogeneous polynomial $F(y_1, \ldots, y_r) \in R[y_1, \ldots, y_r]$ and an integer $C \geq 1$ such that, for any invertible function g on X, for any quintuple $(\mathbb{F}_q, \varphi, \mathbf{a}, \psi, \chi)$ such that

$$(\varphi F)(a_1, \ldots, a_r) \neq 0$$

and for any prime number l invertible in $\mathbb{F}_q$, we have :

(i) the l-adic cohomology groups

$$H^j_c = H^j_c(X_\varphi \otimes \bar{\mathbb{F}}_q, \mathcal{L}_\psi(\sum a_i f_{i,\varphi}) \otimes \mathcal{L}_\chi(g_\varphi))$$

are zero for $j > m$,

(ii) the inequality

$$\sum_j \dim_{\overline{\mathbb{Q}}_l}(H^j_c) \leq C.$$

Proof of (5.7.1). Just as in the demonstration (5.4.2) of (5.4), we first apply (3.1.2) to the finite morphism $f : X \to \mathbb{A}^r_R$ and to the trivial stratification of X ; this provides us with an integer $N_1 \geq 1$, a stratification $\mathcal{A}$ of $\mathbb{A}^r_R[1/N_1]$ and a constant $C_1 \geq 1$. Then we apply (4.1) to $S = \mathrm{Spec}(R[1/N_1])$, $E = \mathbb{A}^r_S$ and to $\mathcal{A}$; this provides us with an integer $N_2 \geq 1$, a stratification $\mathcal{A}^\vee$ of $\mathbb{A}^r_R[1/N_1N_2]$ and a constant $C_2 \geq 1$. We then take a homogeneous open set $U^\vee$ of $\mathbb{A}^r_R$ containing the generic point of $\mathbb{A}^r_R$ and contained in the unique stratum of $\mathcal{A}^\vee$ which contains this generic point (that we can take such a homogeneous $U^\vee$ is shown as in (4.2.3)), we take for $F(y_1, \ldots, y_r)$ any non-zero homogeneous polynomial with coefficients in R which vanishes on the complement of $U^\vee$ in $\mathbb{A}^r_R$ and we take for C the product C_1C_2. With these choices of F and C, the estimation (ii) immediately results from (3.1.2) and (4.1). It remains to prove (i). For any invertible function g on X and for any $(\mathbb{F}_q, \varphi, \psi, \chi)$ such that $(\varphi F)(y_1, \ldots, y_r) \neq 0$, the cohomology groups, for j fixed and $\mathbf{a}$ variable.

$$H^j_c(\mathbf{a}) = H^j_c(X_\varphi \otimes_{\mathbb{F}_q} \bar{\mathbb{F}}_q, \mathcal{L}_\psi(\sum a_i f_{i,\varphi}) \otimes \mathcal{L}_\chi(g_\varphi)),$$

are the fibers of a smooth $\overline{\mathbb{Q}}_l$-sheaf $\mathcal{H}^j$ on $U^\vee_V$ (cf. the demonstration (5.4.2) of (5.4)) ; we must show that $\mathcal{H}^j$ is zero on $U^\vee_V$ as soon as $j > m$. Now, by the "fundamental miracle" $F_{\psi,!} \simeq F_{\psi,*}$ (cf. (2.1.3)) and Deligne's generic base change theorem (cf. [SGA 4_2], [Th. Finitude] (1.9)), there exists a dense open set W of $U^\vee_V$ such that the forgetful support map

$$H^j_c(\mathbf{a}) \to H^j(\mathbf{a})$$

is an isomorphism for all $\mathbf{a} \in W$. From this and the Lefschetz affine theorem (cf. [SGA 4] XIV (3.2)), we deduce the nullity of the $H^j_c(\mathbf{a})$ for all $j > m$ and $\mathbf{a} \in W$, i.e., the nullity of the $\overline{\mathbb{Q}}_l$-sheaves $\mathcal{H}^j|W$ for all $j > m$ ($X_\varphi \otimes \bar{\mathbb{F}}_q$ is an affine $\bar{\mathbb{F}}_q$-scheme of finite type of dimension $\leq m$). As the $\overline{\mathbb{Q}}_l$-sheaves $\mathcal{H}^j$ are smooth on $U^\vee_V$, which is connected, it follows that $\mathcal{H}^j = 0$ on all of $U^\vee_V$, for all $j > m$, which was to be shown.

6. Trigonometric sums that can be made pure by adjoining a multiplicative character of sufficiently large order

(6.0) In section 5 we studied trigonometric sums where multiplicative characters intervene, but whose purity is in no way linked to their presence. In this section, on the contrary, we will study trigonometric sums which, a priori, are not (geometrically) pure but which become so after the adjunction of a supplementary multiplicative character of sufficiently large order ; the typical example is that of Gauss sums : the sums

$$\sum_{x \in \mathbb{F}^\times_{q^n}} \psi_n(x) = -1,$$

where $\psi : \mathbb{F}_q \to \overline{\mathbb{Q}}^\times$ is a non-trivial additive character, are not (geometrically) pure, but, if $\chi : \mathbb{F}_q^\times \to \overline{\mathbb{Q}}^\times$ is a non-trivial multiplicative character (i.e. of order > 1), the sums

$$\sum_{x \in \mathbb{F}_{q^n}^\times} \psi_n(x)\chi_n(x),$$

they are (geometrically) pure.

(6.1) Let $R \subset C$ be a sub-$\mathbb{Z}$-algebra of $\mathbb{C}$, of finite type over $\mathbb{Z}$, r an integer ≥ 1, $\mathbb{A}_R^r$ the standard affine space of coordinates $x_1, \dots, x_r$, X an affine R-scheme, smooth, purely of relative dimension m, and

$$(f; g) = (f_1, \dots, f_r; g) : X \to \mathbb{A}_R^r \times_R \mathbb{G}_{m,R}$$

a finite morphism (for example, $r = 1$, $X = \mathbb{G}_{m,R}$ of coordinate t, $f = g = t$). Let us also fix an algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$. To any quintuple $(\mathbb{F}_q, \varphi, \mathbf{a}, \psi, \chi)$ composed of a finite field $\mathbb{F}_q$, a ring homomorphism $\varphi : R \to \mathbb{F}_q$ with $\varphi(1) = 1$, an element $\mathbf{a} = (a_1, \dots, a_r)$ of $(\mathbb{F}_q)^r$, a non-trivial additive character $\psi : \mathbb{F}_q \to \overline{\mathbb{Q}}^\times$ and a multiplicative character $\chi : \mathbb{F}_q^\times \to \overline{\mathbb{Q}}^\times$, we attach the family of trigonometric sums $(S_n)_{n\geq 1}$ defined by

$$S_n = S_n(\mathbb{F}_q, \varphi, \mathbf{a}, \psi, \chi) = \sum_{x \in X_\varphi(\mathbb{F}_{q^n})} \psi_n\left(\sum_{i=1}^r a_i f_{i,\varphi}(x)\right) \chi_n(g(x))$$

and the corresponding L function

$$L(T) = L(\mathbb{F}_q, \varphi, \mathbf{a}, \psi, \chi; T) = \exp\left(\sum_{n \geq 1} S_n \frac{T^n}{n}\right).$$

Theorem (6.2). Let $R, X, (f; g) : X \to \mathbb{A}_R^r \times_R \mathbb{G}_{m,R}$ be as in (6.1). Then there exists a non-zero polynomial $F(y_1, \dots, y_r) \in R[y_1, \dots, y_r]$ and a finite set $\mathcal{S} \subset \mathbb{N}^*$ of strictly positive integers having the following property : for any quintuple $(\mathbb{F}_q, \varphi, \mathbf{a}, \psi, \chi)$, as in (6.1), such that

$$\begin{cases} (\varphi F)(a_1, \dots, a_r) \neq 0 \\ \operatorname{ord}(\chi) \notin \mathcal{S} \end{cases}$$

where $\operatorname{ord}(\chi)$ is the exact order of the character χ, we have :

(I) The function $L(T) = L(\mathbb{F}_q, \varphi, \mathbf{a}, \psi, \chi; T)$ is of the form

$$L(T) = P(T)^{(-1)^{m+1}},$$

where

$$P(T) = \prod_{\lambda=1}^d (1 - \alpha_\lambda T)$$

is a polynomial, with $P(0) = 1$, which satisfies :

(i) the α_λ are algebraic integers such that

$$|\alpha_\lambda| = q^{m/2}$$

for all $\lambda = 1, \ldots, d$ and any archimedean absolute value $||$ on $\overline{\mathbb{Q}}$,

(ii) the degree d of $P(T)$ is given by the topological formula

$$d = (-1)^m[\chi(X_{\mathbb{C}}) - \chi(H_{\mathbb{C}})]$$

where $X_{\mathbb{C}}$ is the complex variety $X \otimes_R \mathbb{C}$ and where $H_{\mathbb{C}}$ is the hypersurface of $X_{\mathbb{C}}$ with equation

$$\sum_{i=1}^{r} a_i f_{i,\mathbb{C}}(x) = b$$

for any $a_1, \ldots, a_r, b$ in $\mathbb{C}$ which are algebraically independent over R,

(iii) the function $L(T)$ associated to $(\mathbb{F}_q, \varphi, \mathbf{a}, \bar{\psi}, \bar{\chi})$ for $\bar{\psi} = \psi^{-1}, \bar{\chi} = \chi^{-1}$, is written

$$L(T) = \left[\prod_{\lambda=1}^{d}(1 - q^m\alpha_\lambda^{-1}T)\right]^{(-1)^{m+1}} \quad ;$$

(II) For any integer $n \geq 1$ and any archimedean absolute value on $\overline{\mathbb{Q}}$, we have the majoration

$$|S_n(\mathbb{F}_q, \varphi, \mathbf{a}, \psi, \chi)| \leq (-1)^m[\chi(X_{\mathbb{C}}) - \chi(H_{\mathbb{C}})]q^{mn/2}.$$

The same arguments as in (5.3) show that theorem (6.2) results from the following cohomological statement :

Theorem (6.3). Let $R, X, (f; g) : X \to \mathbb{A}_R^r \times_R \mathbb{G}_{m,R}$ be as in (6.1). Then, there exists a non-zero polynomial $F(y_1, \ldots, y_r) \in R[y_1, \ldots, y_r]$ and a finite set $\mathcal{S} \subset \mathbb{N}^*$ of strictly positive integers having the following property : for any quintuple $(\mathbb{F}_q, \varphi, \mathbf{a}, \psi, \chi)$, as in (6.1), such that

$$\begin{cases} (\varphi F)(a_1, \ldots, a_r) \neq 0 \\ \mathrm{ord}(\chi) \notin \mathcal{S} \end{cases}$$

and for any prime number l invertible in $\mathbb{F}_q$, we have the assertions (i), (ii), (iii) and (iv) of (5.4).

(6.4) Ideas for the demonstration of (6.3). Let's first fix a quintuple $(\mathbb{F}_q, \varphi, \mathbf{a}, \psi, \chi)$ and a prime number l invertible in $\mathbb{F}_q$. How to calculate the

$$H_c^j = H_c^j(X_\varphi \otimes \bar{\mathbb{F}}_q, \mathcal{L}_\psi(\sum a_i f_i) \otimes \mathcal{L}_\chi(g_\varphi))$$

(the notations being those of (5.3)) ? Let's consider the diagram

$$
\begin{array}{ccccc}
X_\varphi & \xrightarrow{(f_\varphi;g_\varphi)} & \mathbb{A}^r_{\mathbb{F}_q} \times_{\mathbb{F}_q} \mathbb{G}_{m,\mathbb{F}_q} & \xrightarrow{pr_2} & \mathbb{G}_{m,\mathbb{F}_q} \hookleftarrow \mathcal{L}_\chi \\
k_\varphi \downarrow & & pr_1 \downarrow & & \\
\mathbb{A}^r_{\mathbb{F}_q} & & & & \\
\downarrow \iota_a & & & & \\
\mathbb{A}^r_{\mathbb{F}_q} \ni \mathbf{a} & & & & \\
\downarrow f_{\varphi *} & & & & \\
\mathrm{Spec}(\mathbb{F}_q) & & & &
\end{array}
$$

where $k : X \to \mathrm{Spec}(R)$ and $\rho : \mathbb{A}^r_R \to \mathrm{Spec}(R)$ are the structural morphisms and where $\mathcal{L}_\psi$ and $\mathcal{L}_\chi$ are the rank 1 smooth $\overline{\mathbb{Q}}_l$-sheaves defined in (2.0) and (5.3) respectively. Then, the projection formula and the definition of the transformation $\mathcal{F}_\psi$ (in its version $F_{\psi,!}$, cf. (2.1.1)) show that

$$R(k_\varphi)_*(\mathcal{L}_\psi(\sum a_i f_{i,\varphi}) \otimes \mathcal{L}_\chi(g_\varphi))[m+r]$$

is the fiber at point $\mathbf{a} = (a_1, \ldots, a_r)$ of $\mathbb{A}^r_{\mathbb{F}_q}$ of

$$F_\psi(R(pr_1)_!(K_{l,\varphi} \otimes pr_2^* \mathcal{L}_\chi)) \tag{6.4.2}$$

with

$$K_{l,\varphi} = (f;g)_{\varphi,*}\overline{\mathbb{Q}}_l[m].$$

The important point is then that, apart from $\mathcal{L}_\psi$ which intervenes in F_ψ, all the other data in (6.4.2) "come from characteristic zero". Let us specify this. For any integer $v \geq 1$, let's consider the Kummer covering of degree v of $\mathbb{G}_{m,\mathbb{Z}[1/v]}$, i.e. the $\mu_{v,\mathbb{Z}[1/v]}$-torsor

$$\mathbb{G}_{m,\mathbb{Z}[1/v]} \xrightarrow{[v]:z\mapsto z^v} \mathbb{G}_{m,\mathbb{Z}[1/v]}$$

it is a finite étale covering of degree v which, by the base change

$$\mathbb{Z}[1/v] \to \mathbb{Z}[1/v, \zeta_v] = \mathbb{Z}[1/v, T]/\Phi_v(T)$$

where $\Phi_v(T)$ is the v-th cyclotomic polynomial, becomes a finite étale galoisian covering, of Galois group

$$G_v = \mu_v(\mathbb{Z}[1/v, \zeta_v]),$$

cyclic of order v. For any prime number l, we therefore have, over $\mathbb{G}_{m,\mathbb{Z}[1/vl,\zeta_v]}$, a decomposition

$$[v]_*\overline{\mathbb{Q}}_l = \bigoplus_\chi \mathcal{X}_\chi$$

extended to the v characters $\chi : G_v \to \overline{\mathbb{Q}}_l^\times$, where each $\mathcal{X}_\chi$ is a rank 1 smooth $\overline{\mathbb{Q}}_l$-sheaf (cf. [SGA 4_1], [Sommes trig.] (4.7)). But we see immediately that, for each divisor v_1 of v, the sum

$$\bigoplus_{\substack{\chi \\ \mathrm{ord}(\chi)=v_1}} \mathcal{X}_\chi \overset{\mathrm{dfn}}{=} \mathcal{K}_{[v_1,v]}$$

descends canonically to a rank $\varphi(v_1)$ smooth $\overline{\mathbb{Q}}_l$-sheaf on $\mathbb{G}_{m,\mathbb{Z}[1/v]}$, a $\overline{\mathbb{Q}}_l$-sheaf still denoted $\mathcal{K}_{[v_1,v]}$, and we see that, for v_1 fixed and a multiple v variable of v_1, we have the canonical identification

$$\mathcal{K}_{[v_1,v]} = \mathcal{K}_{[v_1,v_2]}|\mathbb{G}_{m,\mathbb{Z}[1/v_1]}.$$

We will simply denote by $\mathcal{K}_{[v_1]}$ the rank $\varphi(v_1)$ smooth $\overline{\mathbb{Q}}_l$-sheaf $\mathcal{K}_{[v_1,v_1]}$ on $\mathbb{G}_{m,\mathbb{Z}[1/v_1]}$. Let us now take the upper half of our diagram (6.4.1), but this time over R :

$$\begin{array}{c} X \xrightarrow{(f;g)} \mathbb{A}^r_R \times_R \mathbb{G}_{m,R} \xrightarrow{pr_2} \mathbb{G}_{m,R} \\ \downarrow pr_1 \\ \mathbb{A}^r_R \end{array}$$

and let us set, for any prime number l,

$$K_l = (f;g)_*\overline{\mathbb{Q}}_l[m]$$

it is an object of $D^b_c((\mathbb{A}^r_R \times_R \mathbb{G}_{m,R})[1/l], \overline{\mathbb{Q}}_l)$ which is perverse and pure, of weight m, relative to $\mathrm{Spec}(R[1/l])$ (cf. (1.2.2)(v) and (1.3.2)(v) : X is smooth over R purely of relative dimension m and $(f;g)$ is finite) ; moreover, up to a Tate twist, it is self-dual (for the duality relative to $\mathrm{Spec}(R[1/l])$). By (3.1.2) applied to $(f;g) : X \to \mathbb{A}^r_R \times_R \mathbb{G}_{m,R}$ and to the trivial stratification $\mathcal{X} = \{X\}$ of X, there exists an integer N_1 and a stratification $\mathcal{A}$ of $(\mathbb{A}^r_R \times_R \mathbb{G}_{m,R})[1/N_1]$ to which K_l is adapted for all l. By (3.3.3) applied to $pr_1 : (\mathbb{A}^r_R \times_R \mathbb{G}_{m,R})[1/N_1] \to \mathbb{A}^r_R[1/N_1]$ and to the stratification $\mathcal{A}$, there exists an element $b \in R - \{0\}$, a stratification $\mathcal{E}$ of $\mathbb{A}^r_R[1/N_1, b]$ and an integer $C \geq 1$ such that

(1) $(\mathcal{A}, \mathcal{E}, C, pr_{1!})$ is a !-stratification universal of pr_1 ;

(2) $(\mathcal{A}, \mathcal{E})$ is a $*$-stratification of pr_1.

Thanks to the uniformity statement (4.2) of the Fourier transform, our theorem (6.3) is now an immediate consequence of the following statement :

(6.4.3) There exists a finite set $\mathcal{S} \subset \mathbb{N}^*$ such that, for any integer $v \notin \mathcal{S}$ and for any prime number l, the object

$$R(pr_1)_!(K_l \otimes pr_2^*\mathcal{K}_{[v]})$$

of $D^b_c(\mathbb{A}^r_R[1/N_1bv], \overline{\mathbb{Q}}_l)$ is perverse and pure, of weight m, relative to $\mathrm{Spec}(R[1/N_1bv])$.

Remark. Of course, we use the fact that for $(\mathbb{F}_q, \varphi, \mathbf{a}, \psi, \chi)$ and l fixed, $R(pr_{1,\varphi})_!(K_{l,\varphi} \otimes pr_2^*\mathcal{L}_\chi)$ is a direct factor of $(R(pr_1)_!(K_l \otimes pr_2^*\mathcal{K}_{[v]}))_\varphi$, for $v = \mathrm{ord}(\chi)$: this results from the proper base change theorem (cf. [SGA 4] XII (5.1)) and from the identification of the Lang torsor of $\mathbb{G}_{m,\mathbb{F}_q}$ with the Kummer torsor of degree $q-1$ of $\mathbb{G}_{m,\mathbb{F}_q}$ (cf. [SGA 4_1], [Sommes trig.] (4.9)).

To prove (6.4.3), we will apply the criteria (1.2.2)(v) and (1.3.2)(v) : the object $K_l \otimes pr_2^*\mathcal{K}_{[v]}$ being perverse and pure, of weight m, relative to $R[1/N_1bv]$, and self-dual (up to a Tate twist), its $R(pr_1)_!$ is perverse and pure, of weight m, relative to $R[1/N_1bv]$ if, after any base change

$$T \to \mathrm{Spec}(R[1/N_1bv]),$$

with T good, the forgetful support map

$$R(pr_{1,T})_!(K_{l,T} \otimes pr_2^*\mathcal{K}_{[v],T}) \to R(pr_{1,T})_*(K_{l,T} \otimes pr_2^*\mathcal{K}_{[v],T}) \qquad (*)$$

is an isomorphism. Since $(\mathcal{A},\mathcal{E})$ is a $*$-stratification of pr_1, the formation of $R(pr_1)_*(K_l \otimes pr_2^*\mathcal{K}_{[v]})$ is compatible with the base changes $T \to \mathrm{Spec}(R[1/N_1bv])$ above; on the other hand, it is the same for the formation of $R(pr_1)_!(K_l \otimes pr_2^*\mathcal{K}_{[v]})$ according to [SGA 4] XII (5.1). Consequently, to show that (*) is an isomorphism for any T, it suffices to show that (*) is an isomorphism for $T = \mathrm{Spec}(R[1/N_1bv])$. Since $(\mathcal{A},\mathcal{E})$ is also a $!$-stratification of pr_1, the source and the target of (*) for $T = \mathrm{Spec}(R[1/N_1bv])$ are both adapted to the stratification $\mathcal{E}$; therefore, to verify that (*) for $T = \mathrm{Spec}(R[1/N_1bv])$ is an isomorphism, it suffices to do it at each maximal point of each stratum of $\mathcal{E}$. But provided we shrink $R[1/N_1bv]$, i.e., "enlarge" b, we can assume that each maximal point of each stratum of $\mathcal{E}$ is sent to the generic point of $\mathrm{Spec}(R)$; consequently, to verify that (*) is an isomorphism for $T = \mathrm{Spec}(R[1/N_1bv])$, it suffices to verify that (*) for $T = \mathrm{Spec}(\mathbb{C})$, $R[1/N_1bv] \to \mathbb{C}$, is an isomorphism. Finally, the comparison with transcendental cohomology and the fact that our starting data (the constant sheaf $\overline{\mathbb{Q}}_l$ and the $\mathcal{K}_{[v]}$ on $X \otimes_R \mathbb{C}$ and $\mathbb{G}_{m,\mathbb{C}}$ respectively) are deduced by $(-) \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}_l$ from local systems of $\mathbb{Q}$-vector spaces on $(X \otimes_R \mathbb{C})^{\mathrm{an}}$ and on $(\mathbb{G}_{m,\mathbb{C}})^{\mathrm{an}}$ show that, for a given v, the morphism (*) for $T = \mathrm{Spec}(\mathbb{C})$ is an isomorphism either for all l, or for no l. The discussion above clearly shows that (6.4.3) (and consequently our theorem (6.3)) is a consequence of the following statement :

Theorem (6.5). Let Y be a $\mathbb{C}$-scheme of finite type, l a prime number and $\mathcal{F}$ a $\overline{\mathbb{Q}}_l$-sheaf on $Y \times_{\mathbb{C}} \mathbb{G}_{m,\mathbb{C}}$. Let us denote

$$\begin{array}{c} Y \times_{\mathbb{C}} \mathbb{G}_{m,\mathbb{C}} \xrightarrow{pr_2} \mathbb{G}_{m,\mathbb{C}} \\ pr_1 \downarrow \\ Y \end{array}$$

the projections and, for all $v \in \mathbb{N}^*$, $\mathcal{X}_{[v]}$ the direct factor of the smooth $\overline{\mathbb{Q}}_l$-sheaf $[v]_*\overline{\mathbb{Q}}_l$ on $\mathbb{G}_{m,\mathbb{C}}$ defined in (6.4). Then, there exists a finite set $\mathcal{S} \subset \mathbb{N}^*$ such that, for all $v \notin \mathcal{S}$, the forgetful support map

$$R(pr_1)_!(\mathcal{F} \otimes pr_2^*\mathcal{X}_{[v]}) \to R(pr_1)_*(\mathcal{F} \otimes pr_2^*\mathcal{X}_{[v]})$$

is an isomorphism.

(6.5.1) Proof of (6.5). We have the following compactification of pr_1

$$\mathbb{G}_{m,Y} \xleftarrow{j} \mathbb{P}^1_Y \xrightarrow{i} \{0,\infty\} \times Y$$

$$pr_1 \downarrow$$

$$Y$$

and, to verify that the forgetful support map of the statement is an isomorphism, it is sufficient to verify that

$$i^* Rj_*(\mathcal{F} \otimes pr_2^* \mathcal{X}_{[v]}) = 0$$

in $D^b_c(\{0,\infty\} \times Y, \overline{\mathbb{Q}}_l)$ (we have $R(pr_1)_* = R(pr_{1*})j_*$, $R(pr_1)_! = R(pr_{1*})Rj_!$ and the triangle $Rj_! \to Rj_* \to i_*i^*Rj_*$). Let us now consider the "vanishing cycles situation"

$$\mathbb{G}_{m,Y} \xleftarrow{j} \mathbb{P}^1_Y \xrightarrow{i} \{0,\infty\} \times Y$$

$$pr_2 \downarrow$$

$$\mathbb{G}_{m,\mathbb{C}} \xleftarrow{j} \mathbb{P}^1_{\mathbb{C}} \xrightarrow{i} \{0,\infty\}$$

and, for $\alpha = 0, \infty$, the complex of vanishing cycles (cf. [SGA 7] XIII (2.1) and [SGA 4_1], [Th. Finitude] (3.2))

$$R\Psi_{\bar{\eta}_\alpha}(\mathcal{F} \otimes pr_2^* \mathcal{X}_{[v]}) \in \text{ob } D^b_c(\{\alpha\} \times Y, \overline{\mathbb{Q}}_l)$$

equipped with the action of the inertia group $I_\alpha = \text{Gal}(\bar{\eta}_\alpha/\eta_\alpha)$ of the strict local $(\mathbb{P}^1_{\mathbb{C}})_{(\alpha)}$ at α (η_α is the generic point of this strict local and $\bar{\eta}_\alpha$ a geometric point above η_α). If we denote by $i_\alpha : \{\alpha\} \times Y \hookrightarrow \mathbb{P}^1_Y$ the inclusion, we have (cf. [SGA 4_1], [Th. Finitude], (3.11))

$$i^*_\alpha Rj_*(\mathcal{F} \otimes pr_2^* \mathcal{X}_{[v]}) = R\Gamma(I_\alpha, R\Psi_{\bar{\eta}_\alpha}(\mathcal{F} \otimes pr_2^* \mathcal{X}_{[v]}));$$

or, according to [SGA 7] I (0.3), we have

$$I_\alpha \simeq \varprojlim_n \mu_n(\bar{\eta}_\alpha) \simeq \varprojlim_n \mu_n(\mathbb{C}),$$

so that I_α is (non-canonically) isomorphic to $\hat{\mathbb{Z}}$. Consequently, if we choose a topological generator T_α of I_α, the isomorphism above in $D^b_c(\{\alpha\} \times Y, \overline{\mathbb{Q}}_l)$ translates more concretely into the triangle of $D^b_c(\{\alpha\} \times Y, \overline{\mathbb{Q}}_l)$ below :

$$R\Psi_{\bar{\eta}_\alpha}(\mathcal{F} \otimes pr_2^* \mathcal{X}_{[v]}) \xrightarrow{T_\alpha - 1} R\Psi_{\bar{\eta}_\alpha}(\mathcal{F} \otimes pr_2^* \mathcal{X}_{[v]})$$

$$\downarrow +1$$

$$i^*_\alpha Rj_*(\mathcal{F} \otimes pr_2^* \mathcal{X}_{[v]})$$

(this is the "derived category" version of the calculation by Serre of the $H^i(\hat{\mathbb{Z}}, -)$, cf. [16] chap. I, p. 31). Finally, we have the projection formula

$$(R\Psi_{\bar{\eta}_\alpha}(\mathcal{F} \otimes pr_2^* \mathcal{X}_{[v]}), T_\alpha) = (R\Psi_{\bar{\eta}_\alpha}(\mathcal{F}), T_\alpha) \otimes ((\mathcal{X}_{[v]})_{\bar{\eta}_\alpha}, T_\alpha),$$

where $(\mathcal{X}_{[v]})_{\bar{\eta}_\alpha}$ is a $\overline{\mathbb{Q}}_l$-vector space of finite dimension $\varphi(v)$ on which T_α acts with, for eigenvalues, the primitive v-th roots of 1 in $\overline{\mathbb{Q}}_l$. Now, by "constructibility" of $R\Psi_{\bar{\eta}_\alpha}(\mathcal{F})$, there exists a unitary polynomial $P_\alpha(T_\alpha) \in \overline{\mathbb{Q}}_l[T_\alpha]$ such that

$$P_\alpha(T_\alpha)|R\Psi_{\bar{\eta}_\alpha}(\mathcal{F}) = 0 \quad (\alpha = 0, \infty).$$

Let then $\mathcal{S} \subset \mathbb{N}^*$ be the (finite) set of integers $v \geq 1$ such that P_0 or P_∞ admits at least one zero which is a primitive v-th root of 1 in $\overline{\mathbb{Q}}_l$; $\mathcal{S}$ works for (6.5) : indeed, for a $v \notin \mathcal{S}$, T_α, acting on any of the fibers of one of the cohomology sheaves of $R\Psi_{\bar{\eta}_\alpha}(\mathcal{F} \otimes pr_2^* \mathcal{X}_{[v]})$, never admits the eigenvalue 1, thus $T_\alpha - 1$ is an isomorphism of $R\Psi_{\bar{\eta}_\alpha}(\mathcal{F} \otimes pr_2^* \mathcal{X}_{[v]})$, q.e.d.

Remark (6.5.2). Essentially the same demonstration shows the same theorem where we have replaced $\mathbb{C}$ by a field, say algebraically closed, of characteristic $p > 0$ (with of course $l \neq p$) and where we only consider v prime to p.

7. Link with the Fourier transform for $\mathcal{D}$-Modules and (conjectural) application to the determination of a "large" open set of geometric purity of trigonometric sums

(7.0) Let us start with some reminders about $\mathcal{D}$-Modules in the algebraic setting (cf. [2], [3] and [4]). Throughout this section, k will denote a field of characteristic zero, a variety X will be a quasi-projective, smooth, purely of dimension d_X k-scheme and a morphism between varieties will be a k-morphism. For a variety X, we denote

$$\pi : T^*X = V((\Omega^1_X)^{\vee\vee}) \to X$$

the cotangent bundle, equipped with its natural symplectic structure, and $(\mathcal{D}_X, \mathcal{D}_{X,i})$ the Ring of differential operators on X (relative to k), equipped with the filtration by the order of the operators ; we have a canonical isomorphism of $\mathcal{O}_X$-Algebras

$$\pi_* \mathcal{O}_{T^*X} = \mathrm{gr}\ \mathcal{D}_X. \qquad (7.0.1)$$

If $\mathcal{M}$ is a coherent (left) $\mathcal{D}_X$-Module, its characteristic variety and its characteristic cycle are respectively the reduced closed set

$$|\mathrm{Car}(\mathcal{M})| = \bigcup_{i \in I} \Lambda_i \subset T^*X$$

and the cycle

$$\mathrm{Car}(\mathcal{M}) = \sum_{i \in I} m_i [\Lambda_i].$$

of T^*X defined as follows : we provide $\mathcal{M}$ with an exhaustive increasing filtration $(\mathcal{M}_i)_{i\in\mathbb{N}}$ by quasi-coherent sub-$\mathcal{O}_X$-Modules, making $(\mathcal{M}, \mathcal{M}_i)$ a filtered $(\mathcal{D}_X, \mathcal{D}_{X,i})$-Module and such that $\mathrm{gr}\mathcal{M}$ is a coherent $\mathrm{gr}\mathcal{D}_X$-Module (such a filtration always exists) ; then $\mathrm{gr}\mathcal{M}$ defines, via (7.0.1), a coherent $\mathcal{O}_{T^*X}$-Module whose reduced support

$$\bigcup_{i\in I}\Lambda_i \subset T^*X$$

and the lengths m_i at the generic point of the irreducible components Λ_i $(i \in I)$ of this support depend only on $\mathcal{M}$ and not on the chosen filtration $(\mathcal{M}_i)_{i\in\mathbb{N}}$. Each component Λ_i of $|\mathrm{Car}(\mathcal{M})|$ is conic and involutive (according to Kashiwara, cf. S-K-K, Lecture Notes, 287, and Gabber, cf. Amer. J. of Math., 103 (1981), 445-468) and, in particular, we have Bernstein's inequality :

$$\dim(\Lambda_i) \geq d_X \quad (\forall i \in I).$$

A (left) $\mathcal{D}_X$-Module $\mathcal{M}$ is said to be holonomic if it is coherent and if it satisfies the equivalent following conditions :

a) $\mathrm{Ext}^i_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X) = 0$ for all $i < d_X$,

b) $\dim|\mathrm{Car}(\mathcal{M})| \leq d_X$.

In particular, if $\mathcal{M}$ is a non-zero holonomic $\mathcal{D}_X$-Module, each component Λ_i of $|\mathrm{Car}(\mathcal{M})|$ is conic and Lagrangian and therefore coincides with $T^*_{Y_i}X$ where Y_i is the reduced closed set $\pi(\Lambda_i)$ of X (by abuse of notation, we still denote by $T^*_{Y_i}X$ the adherence in T^*X of the conormal bundle to the smooth part of Y_i in X). For $k = \mathbb{C}$ and $\mathcal{M}$ holonomic, we have a topological interpretation of $\mathrm{Car}(\mathcal{M})$ essentially due to Kashiwara (cf. [7] II) : let $\mathcal{M}$ be a holonomic $\mathcal{D}_X$-Module of characteristic cycle

$$\mathrm{Car}(\mathcal{M}) = \sum_{i\in I} m_i[T^*_{Y_i}X];$$

let's denote by $(-)^{\mathrm{an}}$ the "passage to analytic" functor, then the de Rham complex

$$DR(\mathcal{M}^{\mathrm{an}}) = [\mathcal{M}^{\mathrm{an}} \to \Omega^1_{X^{\mathrm{an}}} \otimes_{\mathcal{O}_{X^{\mathrm{an}}}} \mathcal{M}^{\mathrm{an}} \to \cdots \to \Omega^{d_X}_{X^{\mathrm{an}}} \otimes_{\mathcal{O}_{X^{\mathrm{an}}}} \mathcal{M}^{\mathrm{an}}]$$

(where $\mathcal{M}^{\mathrm{an}}$ is placed in degree $-d_X$) is a perverse object of the category $D^b_c(X^{\mathrm{an}}, \mathbb{C})$ of bounded complexes of sheaves of $\mathbb{C}$-vector spaces on X^{an} with algebraically constructible cohomology ; moreover, $DR(\mathcal{M}^{\mathrm{an}})$ is adapted to any Whitney stratification $X^{\mathrm{an}} = \{X^{\mathrm{an}}_\alpha\}$ of X^{an} for which each Y_i^{an} $(i \in I)$ is a union of strata X^{an}_α and the function $\chi(DR(\mathcal{M}^{\mathrm{an}}))$ is given by

$$\chi(DR(\mathcal{M}^{\mathrm{an}}))(x) = \sum_{i\in I}(-1)^{\dim Y_i}\mathrm{Eu}_{Y_i}(x) \cdot m_i$$

($\mathrm{Eu}_{Y_i}(x)$ designating Euler's obstruction of Y_i at x). We will denote by $D_{\mathrm{hol}}(\mathcal{D}_X)$ the derived category of bounded complexes of (left) quasi-coherent $\mathcal{D}_X$-Modules with holonomic cohomology. The definitions and statements that follow are borrowed from Bernstein and Kashiwara (cf. [3] for more details).

The category $D_{\rm hol}(\mathcal{D}_X)$ is equipped with three internal operations, $D, \tilde{\otimes}, \mathcal{H}om$ defined by

$$D(M) = R\mathcal{H}om_{\mathcal{D}_X}(M, \mathcal{D}_X) \otimes_{\mathcal{O}_X} \omega_X^{-1}[d_X]$$
$$M \tilde{\otimes} N = (M \otimes_k^{\mathbb{L}} N)[-d_X]$$
$$\mathcal{H}om(M, N) = D(M) \tilde{\otimes} N$$

for all $M, N \in \operatorname{ob} D_{\rm hol}(\mathcal{D}_X)$. Furthermore, for any morphism of varieties $f : X \to Y$, we have the functors

$$f_*, f_! : D_{\rm hol}(\mathcal{D}_X) \to D_{\rm hol}(\mathcal{D}_Y)$$

$$f!, f^* : D_{\rm hol}(\mathcal{D}_Y) \to D_{\rm hol}(\mathcal{D}_X)$$

where

$$f_*M = Rf_*(\mathcal{D}_{Y \leftarrow X} \otimes_{\mathcal{D}_X}^{\mathbb{L}} M),$$
$$f!N = (\mathcal{D}_{X \leftarrow Y} \otimes_{\mathcal{D}_Y}^{\mathbb{L}} f^{-1}N)[d_X - d_Y]$$

and

$$f_! = Df_*D, \quad f^* = Df!D$$

with

$$\mathcal{D}_{Y \leftarrow X} = f^*(\mathcal{D}_Y \otimes_{\mathcal{O}_Y} \omega_Y^{-1}) \otimes_{\mathcal{O}_X} \omega_X$$
$$\mathcal{D}_{X \leftarrow Y} = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}_Y.$$

These operations and functors satisfy the usual formalism, except that $\tilde{\otimes}$ and, therefore, $\mathcal{H}om$, do not play the roles of the usual $\otimes$ and $\mathcal{H}om$: in fact $\tilde{\otimes}$ plays the role of $D(D(-) \otimes D(-))$ (which justifies the "tilde") ; in particular, we have biduality and for any Cartesian square of morphisms of varieties

$$\begin{array}{ccc} X' & \xrightarrow{\alpha} & X \\ \downarrow \beta & & \downarrow f \\ Y' & \xrightarrow{\gamma} & Y \end{array}$$

we have the base change isomorphisms

$$\gamma^! f_* = (\beta_*)' \alpha^!, \tag{7.0.2}$$

$$\beta^* f_! = (f_!)' \alpha^*. \tag{7.0.3}$$

We will say, following Bernstein (cf. [3]), that a holonomic $\mathcal{D}_X$-Module $\mathcal{M}$ has regular singularities or is RS if it satisfies

— for $d_X = 1$, the following property : there exists a dense open set $U \subset X$ such that $\mathcal{M}|U$ is a locally free $\mathcal{O}_U$-Module of finite rank with (integrable) connection and that $\mathcal{M}|U$ has regular singularities in the classical sense at any point of $U - \tilde{U}$, $\tilde{U}$ being the non-singular completion of U,

— for $d_X > 1$, the following property : $i^*(\mathcal{M}[0])$ is of RS cohomology for any smooth locally closed curve $i : Y \hookrightarrow X$.

Remark (7.0.4). The RS condition above concerns both the singularities of $\mathcal{M}$ at finite distance and those "at infinity of X". We will denote by $D_{\mathrm{RS}}(\mathcal{D}_X)$ the full subcategory of $D_{\mathrm{hol}}(\mathcal{D}_X)$ formed by complexes with RS cohomology; $D_{\mathrm{RS}}(\mathcal{D}_X)$ is stable under the operations $D, \tilde{\otimes}, \mathcal{H}om$ and $f_*, f_!, f!, f^*$ preserve $D_{\mathrm{RS}} \subset D_{\mathrm{hol}}$. For $k = \mathbb{C}$, the functor $DR((-)^{\mathrm{an}})$ induces an equivalence of categories between $D_{\mathrm{RS}}(\mathcal{D}_X)$ and $D_c^b(X^{\mathrm{an}}, \mathbb{C})$ and sends a holonomic RS $\mathcal{D}_X$-Module placed in degree 0 to a perverse object of $D_c^b(X^{\mathrm{an}}, \mathbb{C})$. This equivalence of category transforms the operations $D, \tilde{\otimes}, \mathcal{H}om$ and the functors $f_*, f_!, f!, f^*$ on D_{RS} into the operations and functors of the same name on D_c^b (with, of course, $(-) \otimes (-) = D(D(-) \otimes D(-))$ on D_c^b and idem for $\mathcal{H}om$, cf. [20]).

Example (7.0.5). Let $E = \mathbb{A}_k^r = \mathrm{Spec}(k[x_1, \ldots, x_r])$ with $r \geq 1$, then

$$\Gamma(E, \mathcal{D}_E) = k[x_1, \ldots, x_r, \partial/\partial x_1, \ldots, \partial/\partial x_r]$$

(algebra of non-commutative polynomials); if $\mathcal{I}$ is a (left) $\mathcal{D}_E$-Ideal, $\mathcal{M} = \mathcal{D}_E/\mathcal{I}$ is a coherent (left) $\mathcal{D}_E$-Module that can be filtered by the

$$M_i = \mathcal{D}_{E,i}/(\mathcal{I} \cap \mathcal{D}_{E,i}) \quad (i \in \mathbb{N}),$$

so that $\mathrm{gr}\mathcal{I}$ is the Annihilator of $\mathrm{gr}\mathcal{M}$; then, if $P_1(x, \xi), \ldots, P_s(x, \xi)$ are homogeneous generators in the $\xi_1, \ldots, \xi_r$ (images of $\partial/\partial x_1, \ldots, \partial/\partial x_r$ in $\mathrm{gr}\Gamma(E, \mathcal{D}_E)$) of the graded ideal $\Gamma(E, \mathrm{gr}\mathcal{I})$, $|\mathrm{Car}(\mathcal{M})|$ is defined by the annihilation of the polynomials $P_1, \ldots, P_s$ in $T^*X = \mathrm{Spec}(k[x_1, \ldots, x_r, \xi_1, \ldots, \xi_r])$. In particular, let $f \in k[x_1, \ldots, x_r]$ be such that $X = f^{-1}(0) \subset E$ is a smooth hypersurface and let $\mathcal{I}$ be the (left) Ideal of $\mathcal{D}_E$ generated by f and the

$$f_i' \cdot \partial/\partial x_j - f_j' \cdot \partial/\partial x_i \quad (1 \leq i < j \leq r),$$

where f_i' is the partial derivative of f with respect to x_i, then $\mathcal{M} = \mathcal{D}_E/\mathcal{I}$ is none other than

$$\mathcal{M} = \mathcal{H}_X^r(\mathcal{O}_E) = i_*(\mathcal{O}_X[0]);$$

$\mathcal{M}$ is holonomic RS, we have

$$\mathrm{Car}(\mathcal{M}) = [T_X^*E]$$

and, if $k = \mathbb{C}$, we have

$$DR(\mathcal{M}^{\mathrm{an}}) = \mathbb{C}_{X^{\mathrm{an}}}[d_X].$$

(7.1) The Fourier transform (also called "Laplace transform") for differential equations is very classical, cf. Ince ([19] (8.2), p. 187). For a presentation in the language of $\mathcal{D}$-Modules, cf. [2] (3.3), and [5] (7.16).

Let us fix a field k of characteristic 0 and an integer $r \geq 1$; let

$$E = \mathrm{Spec}(k[x_1, \ldots, x_r])$$

be the standard vector space of dimension r over k and $E^\vee = \mathrm{Spec}(k[y_1, \ldots, y_r])$ the dual affine space. The Fourier transform for $\mathcal{D}$-Modules is the equivalence of categories, denoted $\mathcal{F}$, between the category of quasi-coherent (left) $\mathcal{D}_E$-Modules and that of quasi-coherent (left) $\mathcal{D}_{E^\vee}$-Modules, defined as follows : we have a k-algebra isomorphism

$$F : \Gamma(E^\vee, \mathcal{D}_{E^\vee}) \simeq \Gamma(E, \mathcal{D}_E) \tag{7.1.1}$$

defined by

$$F(y_i) = \partial/\partial x_i; \quad (i = 1, \ldots, r)$$
$$F(\partial/\partial y_i) = -x_i$$

and, if $\mathcal{M}$ is a quasi-coherent $\mathcal{D}_E$-Module, $\mathcal{F}(\mathcal{M})$ is the quasi-coherent $\mathcal{D}_{E^\vee}$-Module characterized by

$$\Gamma(E^\vee, \mathcal{F}(\mathcal{M})) = F^*(\Gamma(E, \mathcal{M}))$$

(the k-vector space $\Gamma(E, \mathcal{M})$ "seen" as a $\Gamma(E^\vee, \mathcal{D}_{E^\vee})$-Module via F). If $\mathcal{F}^\vee$ is defined like $\mathcal{F}$ but exchanging the roles of x and y, E and $E^\vee$, we have

$$\mathcal{F}^\vee \circ \mathcal{F} = (-1_E)^* \quad \mathcal{F} \circ \mathcal{F}^\vee = (-1_{E^\vee})^*.$$

On the other hand, it is clear from the definition of $\mathcal{F}$ that $\mathcal{F}$ respects coherence over $\mathcal{D}$ and holonomy (cf. the homological definition of holonomy). On the other hand, $\mathcal{F}$ does not respect in general the RS condition (for $r = 1$, $\mathcal{D}_E/\mathcal{D}_E(x-1)$ is holonomic RS, but $\mathcal{F}(\mathcal{D}_E/\mathcal{D}_E(x-1)) = \mathcal{D}_{E^\vee}/\mathcal{D}_{E^\vee}(\partial/\partial y + 1)$ is holonomic, irregular at infinity).

Example (7.1.2) (cf. [5] (8.2)). — Let $f \in k[x_1, \ldots, x_r]$ defining in E a smooth hypersurface $X = f^{-1}(0)$, then the $\mathcal{D}_E$-Module $\mathcal{H}^r_X(\mathcal{O}_E)$ (cf. (7.0.7)) admits for Fourier transform

$$\mathcal{F}(\mathcal{H}^r_X(\mathcal{O}_E)) = \mathcal{D}_{E^\vee}/\mathcal{J}^\vee$$

where $\mathcal{J}^\vee$ is the left $\mathcal{D}_{E^\vee}$-Ideal generated by

$$f(\partial/\partial y_1, \ldots, \partial/\partial y_r)$$

and the

$$f'_i(\partial/\partial y_1, \ldots, \partial/\partial y_r) \cdot y_j - f'_j(\partial/\partial y_1, \ldots, \partial/\partial y_r) \cdot y_i \quad (1 \leq i < j \leq r).$$

$\mathcal{F}$ is trivially derived and induces an equivalence of categories, still denoted $\mathcal{F}$, between $D_{\mathrm{hol}}(\mathcal{D}_E)$ and $D_{\mathrm{hol}}(\mathcal{D}_{E^\vee})$.

Lemma (7.1.3). Fourier "commutes" with duality ; more precisely, we have a canonical isomorphism of functors

$$D \circ \mathcal{F} \simeq (-1_{E^\vee})^* \circ \mathcal{F} \circ D$$

from $D_{\rm hol}(\mathcal{D}_E)$ to $D_{\rm hol}(\mathcal{D}_{E^\vee})$. Indeed, we can define a Fourier transform $\mathcal{F}_d$ for right $\mathcal{D}$-Modules in a manner analogous to $\mathcal{F}$ and so that we trivially have

$$R\mathcal{H}om_{\mathcal{D}_{E^\vee}}(\mathcal{F}(\mathcal{M}),\mathcal{D}_{E^\vee}) = \mathcal{F}_d(R\mathcal{H}om_{\mathcal{D}_E}(\mathcal{M},\mathcal{D}_E))$$

for any $M \in$ ob $D_{\rm hol}(\mathcal{D}_E)$; it then remains to verify that for any right $\mathcal{D}_E$-Module N, we canonically have

$$\mathcal{F}_d(N)\otimes_{\mathcal{O}_{E^\vee}}\omega_{E^\vee}\simeq(-1_{E^\vee})^*\mathcal{F}(N\otimes_{\mathcal{O}_E}\omega_E^{-1}),$$

which we leave to the reader. We will find in (7.5) a demonstration of the "well-known" lemma below :

Lemma (7.1.4). Let us denote $pr : E\times E^\vee \to E$, $pr^\vee : E\times E^\vee\to E^\vee$ the canonical projections and $\langle,\rangle : E\times E^\vee \to \mathbb{A}^1 = \mathrm{Spec}(k[t])$ the evaluation map ; let $\mathcal{L}$ be the $\mathcal{D}_{\mathbb{A}^1}$-Module defined by

$$\Gamma(\mathbb{A}^1,\mathcal{L}) = k[t]$$

and

$$(\partial/\partial t)\cdot P(t) = P'(t) - tP(t)$$

and let $L = \mathcal{L}[-1] \in$ ob $D_{\rm hol}(\mathcal{D}_{\mathbb{A}^1})$. Then, we have an isomorphism of functors

$$\mathcal{F}(-)\simeq pr^\vee_*(pr^*(-)\tilde{\otimes}\langle,\rangle^*L)[2-r]$$

from $D_{\rm hol}(\mathcal{D}_E)$ into $D_{\rm hol}(\mathcal{D}_{E^\vee})$.

Remark (7.1.5). This lemma highlights an analogy between the Fourier transform $\mathcal{F}$ above and the Fourier transforms F_ψ defined in Section 2. This analogy is reinforced by the following facts :

(1) We have, for F_ψ, the formula

$$F_\psi(-) = Rpr^\vee_*(pr^*(-)\otimes^{\mathbb{L}}\langle,\rangle^*\mathcal{L}_\psi)[2-r](1-r)$$

where $(-)\otimes^{\mathbb{L}}(-) = D(D(-)\otimes D(-))$ in $D(-,\overline{\mathbb{Q}}_l)$.

(2) For $k=\mathbb{C}$, $DR(L^{\rm an})$ is smooth of rank 1 concentrated in degree 0 (in fact equal to the constant sheaf $\mathbb{C}$ on $(\mathbb{A}^1)^{\rm an}$),

(3) We have the dictionary below between properties of L and $\mathcal{L}_\psi$

— L free of rank 1 over $\mathcal{O}_{\mathbb{A}^1} \leftrightarrow \mathcal{L}_\psi$ smooth of rank 1 over $\mathbb{A}^1_{\mathbb{F}_q}$

— L has irregularity i at infinity $\leftrightarrow \mathcal{L}_\psi$ has Swan conductor i at infinity

— $\Lambda(L) = \mathcal{O}_{\mathbb{A}^1_k} \leftrightarrow \Lambda(\mathcal{L}_\psi) = \overline{\mathbb{Q}}_l$

where $\Lambda(-) = s^*(-)\otimes pr_1^*(-)^{-1}\otimes pr_2^*(-)^{-1}$, with $pr_i : \mathbb{A}^1\times\mathbb{A}^1\to\mathbb{A}^1$ $(i=1,2)$ the two canonical projections and $s : \mathbb{A}^1\times\mathbb{A}^1\to\mathbb{A}^1$ the sum morphism.

More generally, if S is a variety over k (in the sense of (7.0)) and if $E\to S$ is a vector bundle of constant rank $r\geq 1$ and of dual bundle $E^\vee\to S$, we have an isomorphism

$$F : (\pi^\vee)_*\mathcal{D}_{E^\vee}\to\pi_*\mathcal{D}_E$$

generalizing (7.1.1) which allows to define an equivalence of categories

$$\mathcal{F} : D_{\mathrm{hol}}(\mathcal{D}_E) \to D_{\mathrm{hol}}(\mathcal{D}_{E^\vee}),$$

still called Fourier transform ; lemmas (7.1.3) and (7.1.4) extend as they are to this relative situation.

(7.2) Let X be a variety (in the sense of (7.0)) over a field k of characteristic 0 and $\mathcal{X} = \{X_\alpha\}$ a stratification of X (cf. (3.0)), with smooth strata over k ; for each α, we denote $i_\alpha : X_\alpha \hookrightarrow X$ the inclusion. We will say that an object M of $D_{\mathrm{hol}}(\mathcal{D}_X)$ is adapted to $\mathcal{X}$ if, for each α, all the cohomology sheaves of $i_\alpha^! M$ are coherent over $\mathcal{O}_{X_\alpha} \subset \mathcal{D}_{X_\alpha}$ and, therefore a fortiori, are $\mathcal{O}_{X_\alpha}$-locally free of finite rank. We will denote $\chi(M) : X \to \mathbb{Z}$ the constructible function defined by

$$\chi(M)(x) = \sum_j (-1)^j \dim_{k(x)} H^j(i_x^! M)$$

where $i_x : \{x\} \hookrightarrow X$ is the inclusion and $k(x)$ the residue field of x ; we will say that M is χ-adapted to $\mathcal{X}$ if M and $\chi(M)$ are adapted to $\mathcal{X}$ (cf. (3.0)).

Remark (7.2.1). By replacing the $i_\alpha^!, i_x^!$ by i_α^*, i_x^*, we obtain the notions !-adapted to $\mathcal{X}$, function $\chi^!$, and χ- !-adapted to $\mathcal{X}$; M is then adapted (resp. χ-adapted) to $\mathcal{X}$ if and only if $D(M)$ is !-adapted (resp. χ- !-adapted) to $\mathcal{X}$ and $\chi^!(D(M)) = \chi(M)$. Let $\mathcal{M}$ be a holonomic $\mathcal{D}_X$-Module and $M = \mathcal{M}[0] \in$ ob $D_{\mathrm{hol}}(\mathcal{D}_X)$. If M is χ-adapted to a stratification $\mathcal{X}$ as above and if $X_\alpha \in \mathcal{X}$ is an open stratum in X, $\mathcal{M}|X_\alpha$ is locally free of finite constant rank m_α over $\mathcal{O}_{X_\alpha}$ (m_α is the value of $\chi(M)$ on X_α) and

$$\mathrm{Car}(\mathcal{M})|\pi^{-1}(X_\alpha) = m_\alpha[T^*_{X_\alpha} X_\alpha].$$

Reciprocally, if $U \subset X$ is an open set such that

$$\mathrm{Car}(\mathcal{M})|\pi^{-1}(U) = m[T^*_U U]$$

for an integer $m \geq 0$, $\mathcal{M}|U$ is $\mathcal{O}_U$-locally free of constant rank m.

Remark (7.2.2). If $\mathcal{M}$ is a holonomic RS $\mathcal{D}_X$-Module, the data of the characteristic cycle $\mathrm{Car}(\mathcal{M})$ is essentially equivalent to the data of $\chi(\mathcal{M}[0])$ and a stratification $\mathcal{X}$ of X as above to which $\mathcal{M}[0]$ is χ-adapted ; moreover, for $k = \mathbb{C}$, $DR(\mathcal{M}^{\mathrm{an}})$ is adapted (resp. χ-adapted) to $\mathcal{X}^{\mathrm{an}}$ as soon as $\mathcal{M}[0]$ is adapted (resp. χ-adapted) to $\mathcal{X}$ and $\chi(DR(\mathcal{M}^{\mathrm{an}})) = \chi(\mathcal{M}[0])$. But all this falls into default as soon as $\mathcal{M}$ is not RS.

(7.3) Let S be a scheme of finite type over $\mathbb{Z}$, $E \to S$ a vector bundle of constant rank $r \geq 1$ and $E^\vee \to S$ its dual vector bundle. For any variety T over a field k of characteristic 0 (as in (7.0)) and for any morphism $T \to S$, we have the Fourier transform functor

$$\mathcal{F}_T : D_{\mathrm{hol}}(\mathcal{D}_{E_T}) \to D_{\mathrm{hol}}(\mathcal{D}_{E_T^\vee}).$$

Then, theorem (4.1) and its corollary (4.2) admit the following prolongations :

Theorem (7.3.1). Let, moreover, $\mathcal{E} = \{E_\alpha\}$ be a stratification of E with strata E_α smooth over S. Then, there exists an integer $N \geq 1$, a stratification $\mathcal{E}^\vee = \{E^\vee_\beta\}$ of $E^\vee[1/N]$, with strata $E^\vee_\beta$ non-empty, smooth over $S[1/N]$ and flat over $\mathbb{Z}$ and an additive map

$$\left\{\begin{array}{c}\text{constructible functions}\\ E[1/N] \to \mathbb{Z} \text{ adapted}\\ \text{to } \mathcal{E}[1/N]\end{array}\right\} \xrightarrow{\varphi \mapsto \varphi^\vee} \left\{\begin{array}{c}\text{constructible functions}\\ E^\vee[1/N] \to \mathbb{Z} \text{ adapted}\\ \text{to } \mathcal{E}^\vee[1/N]\end{array}\right\}$$

having the following properties :

a) there exists an integer $C \geq 1$ such that $N, \mathcal{E}^\vee, C$ and $\varphi \mapsto \varphi^\vee$ work for (4.1) ;

b) for any variety T (as in (7.0)) over a field k of characteristic 0, for any morphism $T \to S$ and for any object M of $D_{\mathrm{RS}}(\mathcal{D}_{E_T})$ adapted to $\mathcal{E}_T$, the transform of Fourier

$$\mathcal{F}_T(M) \in \mathrm{ob}\ D_{\mathrm{hol}}(\mathcal{D}_{E^\vee_T})$$

is adapted to $\mathcal{E}^\vee_T$ and if, moreover, M is χ-adapted to $\mathcal{E}_T$, $\mathcal{F}_T(M)$ is also χ-adapted to $\mathcal{E}^\vee_T$ and we have

$$\chi(\mathcal{F}_T(M)) = (\chi(M))^\vee.$$

Corollary (7.3.2). We further assume S to be integral, with a generic point of characteristic zero, and let again $\mathcal{E} = \{E_\alpha\}$ be a stratification of E with strata E_α smooth over S. Then, there exists a dense open set $U^\vee \subset E^\vee$ of $E^\vee$ satisfying the following conditions :

a) $U^\vee$ works for (4.2) ;

b) for any variety T (as in (7.0)) over a field k of characteristic 0, for any morphism $T \to S$ with $U^\vee_T \neq \emptyset$ and for any holonomic RS $\mathcal{D}_{E_T}$-Module $\mathcal{M}$, with $\mathcal{M}[0]$ adapted to $\mathcal{E}_T$,

$$\mathcal{F}_T(\mathcal{M})|U^\vee_T \quad \text{and} \quad (-1_{E^\vee_T})^*\mathcal{F}_T(\mathcal{M}^*)|U^\vee_T,$$

where $\mathcal{M}^* = \mathcal{H}^0(D(\mathcal{M}[0]))$, are $\mathcal{O}_{U^\vee_T}$-Modules locally free of finite rank with integrable connections in perfect duality ; moreover, if $\mathcal{M}[0]$ is χ-adapted to $\mathcal{E}_T$, the common rank of these two $\mathcal{O}_{U^\vee_T}$-Modules is constant on $U^\vee_T$.

Remarks (7.3.3.1). We have not had the courage to develop a theory of relative duality (cf. 1) for $\mathcal{D}$-Modules ; in (7.3.2 b), we use the absolute duality, but this amounts to the same thing because we assume T and therefore E_T to be smooth over k.

(7.3.3.2) The essential hypothesis is the RS hypothesis ; it corresponds to moderation in the l-adic framework.

(7.3.3.3) We would like to be able to formulate and demonstrate a statement analogous to (4.2)(iii) for $\mathcal{D}$-Modules, purity being understood in the sense of Hodge theory.

(7.3.4) Proof of (7.3.1) and (7.3.2). Taking into account the formalism recalled in (7.0), of (7.1.4), of (7.2.1) and of the base change theorem (7.0.2), by arguments at every point parallel to those of (4.3), we reduce these statements to lemma (4.3.2) and to its following analogue for $\mathcal{D}$-Modules :

Lemma (7.3.4.1). Let T be a variety (cf. (7.0)) over a field k of characteristic 0 and let

$$\mathcal{X} = \{\mathbb{A}^1_T - D, D_1, \ldots, D_n\}, \quad D = \coprod_{i=1}^{n} D_i,$$

a stratification of $\mathbb{A}^1_T$, with $D_1, \ldots, D_n$ being divisors in $\mathbb{A}^1_T$ finite étale over T, of constant ranks $d_1, \ldots, d_n$. For any object M of $D_{\mathrm{RS}}(\mathcal{D}_{\mathbb{A}^1_T})$!-adapted to $\mathcal{X}$ (cf. (7.2.1)), we then have :

(i) the object $pr_*(M \tilde{\otimes} L)$ of $D_{\mathrm{hol}}(\mathcal{D}_T)$, where $pr : \mathbb{A}^1_T \to T$ and $\alpha_T : \mathbb{A}^1_T \to \mathbb{A}^1_k$ are the projections and where $L = \mathcal{L}[-1]$ is defined in (7.1.4), is of $\mathcal{O}_T$-locally free cohomology of finite rank,

(ii) if we assume that M is χ- !-adapted to $\mathcal{X}$ (cf. (7.2.1)), then the function χ' of $pr_*(M \tilde{\otimes} L)$ is constant on T of value

$$-da + \sum_{i=1}^{n} d_i a_i$$

where a (resp. a_i) is the constant value of $\chi'(M)$ on $\mathbb{A}^1_T - D$ (resp. D_i, $i = 1, \ldots, n$) and where $d = \sum d_i$ is the constant degree of D on T.

(7.3.4.2) Let's prove (7.3.4.1). Let

$$\mathbb{A}^1_T \xleftarrow{j_T} \mathbb{P}^1_T \xrightarrow{i_T} \infty_T = \mathbb{P}^1_T - \mathbb{A}^1_T$$

be the natural compactification of pr_T, then

$$pr_{T*}(M \tilde{\otimes} L) = \bar{pr}_{T*} j_{T*}(M \tilde{\otimes} L)$$

and it suffices to show, for part (i), that the characteristic variety of each cohomology sheaf of $j_{T*}(M \tilde{\otimes} L) \in \mathrm{ob}\, D_{\mathrm{hol}}(\mathcal{D}_{\mathbb{P}^1_T})$ is contained in

$$T^*_T\mathbb{P}^1_T \cup T^*_{D_1}\mathbb{P}^1_T \cup \cdots \cup T^*_{D_n}\mathbb{P}^1_T \cup T^*_{\infty_T}\mathbb{P}^1_T \subset T^*\mathbb{P}^1_T$$

(indeed, this implies, since $\bar{pr}_T$ is proper, that the characteristic variety of each cohomology sheaf of $\bar{pr}_{T*} j_{T*}(M \tilde{\otimes} L)$ is contained in the zero section of the cotangent to T). Or, above $\mathbb{A}^1_T \subset \mathbb{P}^1_T$, this is clear since

$$|\mathrm{Car}(\mathcal{H}^i(M))| \subset T^*_{\mathbb{A}^1_T}\mathbb{A}^1_T \cup \bigcup_{i=1}^{n} T^*_{D_i}\mathbb{A}^1_T$$

for all $j \in \mathbb{Z}$ (M is RS and !-adapted to $\mathcal{X}$) ; on the other hand, locally for the étale topology along ∞_T, M is of the form $\alpha_{T,*}N$ with $N \in \mathrm{ob}\, D_{\mathrm{RS}}(\mathcal{D}_{\mathbb{A}^1_k})$ and if $j : \mathbb{A}^1_k \hookrightarrow \mathbb{P}^1_k$ is the inclusion and $\pi : \mathbb{P}^1_T \to \mathbb{P}^1_k$ the projection,

$$j_{T*}(\alpha_{T,*}N \tilde{\otimes} L) = \pi^*_{T*} j_*(N \tilde{\otimes} L),$$

which implies the conclusion. For part (ii), the proof is the same as that of (4.3.2)(iii) provided we replace the Grothendieck-Ogg-Shafarevich formula with Deligne's index formula (cf. P. Deligne, Equations différentielles à points singuliers réguliers, Lecture Notes in Mathematics, no 163, Springer-Verlag, formula (6.21.1)).

Remark. The proof of (7.3.4.1)(i) above was suggested to us by Mebkhout.

(7.4) As we have already remarked in (5.6.1), the demonstrations (4.3) and (5.4.2) provide sufficient conditions for the purity of the trigonometric sums $S_n(\mathbb{F}_q, \varphi, \mathbf{a}, \psi, \chi)$ defined in (5.0), conditions that can be made explicit but which are generally more restrictive than the purity criteria that can be derived from a precise geometric analysis. One can then wonder if there exists a general method for providing the optimal purity criteria for the sums $S_n(\mathbb{F}_q, \varphi, \mathbf{a}, \psi, \chi)$. We provide below a conjectural answer to this question. Let us resume the notations of (5.0). Furthermore, let us denote by K the field of fractions of R and $(-)_K$ the base change $(-) \otimes_R K$. Then

$$f_{K*}\mathcal{O}_{X_K}$$

is a holonomic RS $\mathcal{D}_{\mathbb{A}^r_K}$-Module (cf. (7.0) ; f_{K*} is the direct image in the sense of $\mathcal{D}$-Modules), and its Fourier transform (cf. (7.1))

$$\mathcal{F}(f_{K*}\mathcal{O}_{X_K})$$

is a holonomic $\mathcal{D}_{\mathbb{A}^r_K}$-Module. Consequently, there exists a largest dense open set $U^\vee_K$ of $\mathbb{A}^r_K$ such that

$$\mathrm{Car}(\mathcal{F}(f_{K*}\mathcal{O}_{X_K}))|\pi^{-1}(U^\vee_K) = d \cdot [T^*_{U^\vee_K} U^\vee_K] \tag{7.4.1}$$

where d is an integer ≥ 0 and $\pi : T^*\mathbb{A}^r_K \to \mathbb{A}^r_K$ is the cotangent to $\mathbb{A}^r_K$; this open set is homogeneous (cf. (4.2.3)). Let

$$F_1, \ldots, F_s \in R[y_1, \ldots, y_r]$$

be a system of homogeneous equations of the reduced closed set $\mathbb{A}^r_K - U^\vee_K$ of $\mathbb{A}^r_K$.

Conjecture (7.4.2). There exists $\rho \in R - \{0\}$ such that for any quintuple $(\mathbb{F}_q, \varphi, \mathbf{a}, \psi, \chi)$ satisfying

$$\varphi(\rho F_i)(a_1, \ldots, a_r) \neq 0$$

for at least one $i \in \{1, \ldots, s\}$, all the conclusions of (5.2) are satisfied with for d the integer defined by (7.4.1).

Comments (7.4.2.1). One will find in [5] (8.7) a demonstration of the equality of the integers d defined in (5.2) and (7.4.1); another demonstration is provided by (7.3.4.1)(ii).

(7.4.2.2) In the case where X is a smooth hypersurface of $\mathbb{A}^r_R$ defined by an equation $f \in R[x_1, \ldots, x_r]$, the considerations developed in (7.0.5) and (7.1.2) allow in principle an explicit determination of the open set $U_K^\vee$ and therefore of a system of polynomials $F_1, \ldots, F_s$.

(7.4.2.3) The conjectural purity criterion (7.4.2) is better than the criterion that can be made explicit from the demonstrations (4.3) and (5.4.2) (cf. (5.6.1)) and coincides for $\rho \in R - \{0\}$ sufficiently divisible with the purity criteria that can be derived from precise geometric analyses (cf. [5] (8.7), [11] and [14]); however, our conjecture says nothing about $\rho \in R - \{0\}$ (apart from its existence!).

Appendix (7.5) : proof of (7.1.4). We will give the demonstration for $r = 1$ (this is the essential case) and to simplify the notations we will set $\partial_x = \partial/\partial x$, $\partial_y = \partial/\partial y$. The functors $f_*, f!, \tilde{s}$ extend naturally to $D_{\mathrm{coh}} \supset D_{\mathrm{hol}}$, just like $\mathcal{F}$, and we will in fact show (7.1.4) for quasi-coherent $\mathcal{D}$-Modules. Let $\mathcal{M}$ be a quasi-coherent $\mathcal{D}_E$-Module and $M = \Gamma(E, \mathcal{M})$. We have then

$$\Gamma(E \times E^\vee, pr'^* \mathcal{M}[-1]) = M \otimes_k k[y]$$

as a $k[x, y]$-module and

$$\partial_x(m \otimes P(y)) = (\partial_x m) \otimes P(y)$$

$$\partial_y(m \otimes P(y)) = m \otimes P'(y);$$

we have similarly

$$\Gamma(E \times E^\vee, \langle , \rangle' L) = k[x, y]$$

as a $k[x, y]$-module and

$$\partial_x P(x, y) = P_x(x, y) - yP(x, y)$$

$$\partial_y P(x, y) = P_y(x, y) - xP(x, y).$$

By suite,

$$\Gamma(E \times E^\vee, pr'^* \mathcal{M} \tilde{\otimes} \langle , \rangle' L[1]) = M \otimes_k k[y]$$

as a $k[x, y]$-module and

$$\partial_x(m \otimes P(y)) = (\partial_x m) \otimes P(y) - m \otimes (yP(y))$$

$$\partial_y(m \otimes P(y)) = m \otimes P'(y) - (x \cdot m) \otimes P(y);$$

finally, we obtain that

$$\Gamma(E^\vee, pr_{V*}(pr'^* \mathcal{M} \tilde{\otimes} \langle , \rangle' L)[1]) = [M \otimes_k k[y] \xrightarrow{\partial_x} M \otimes_k k[y]]$$

in $D^b(k[y,\partial_y])$ (complex concentrated in degree -1 and 0). It only remains to remark that the sequence of k-vector spaces

$$0 \to M \otimes_k k[y] \xrightarrow{y} M \otimes_k k[y] \to M \to 0$$

$$\sum m_i y^i \mapsto \sum (\partial_x m_i) y^i$$

is exact and that the multiplication by y (resp. ∂_y) on $[M \otimes_k k[y] \xrightarrow{\partial_x} M \otimes_k k[y]]$ induces the multiplication by ∂_y (resp. $-x$) on M.

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Acronyms

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