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Exposé I

AROUND THE LOCAL MONODROMY THEOREM

by Luc Illusie

0. Introduction.

This exposé contains no original results. We confine ourselves to describing some aspects of the local monodromy theorem, which, in various ways, have inspired the constructions of Hyodo-Kato (exp. V) and Fontaine (exp. VIII). We first briefly recall, in n° 1, the statement and the main corollaries of Grothendieck's l-adic local monodromy theorem, following closely Deligne's presentation in (SGA 7 I) and [11, 1.7]. In n° 2, we work over $\mathbf{C}$. We explain the geometric proof of the variant of the local monodromy theorem for complex analytic spaces, then we study in detail the semi-stable reduction case. For S an open disk centered at 0 in $\mathbf{C}$ and $f : X \to S$ a projective morphism of complex analytic spaces, smooth outside 0, and having semi-stable reduction at 0 (2.1.1), the unipotence of the monodromy T of $H^*(X_t, \mathbf{C})$ ($t \in S^* = S - \{0\}$)) has a classical interpretation in terms of the Gauss-Manin connection on the relative de Rham cohomology $R^*f_*\Omega^\bullet_{X^*/S^*}$ (where $X^* = f^{-1}(S^*)$). We recall more precisely how, according to Steenbrink, the complex of vanishing cycles $R\Psi(\mathbf{C})$ and the logarithm N of T can be calculated using relative de Rham complexes with logarithmic poles. Steenbrink's theory also provides a limit mixed Hodge structure H_0, in a certain sense, for $t \to 0$, of the pure Hodge structures of the $H^*(X_t)$, and for which N is a nilpotent endomorphism of type $(-1, -1)$. We explain in 2.3 the principle of the construction. On H_0 we then have, a priori, two filtrations: the weight filtration (from the mixed Hodge structure) and the monodromy filtration (derived from N). Their coincidence is the most profound result of the theory. We give some corollaries, due to M. Saito, generalizing in particular the local invariant cycle theorem (,). In n° 3, we explain some analogues of the preceding theory over a trait S of residual characteristic $p > 0$. For $f : X \to S$ proper and with semi-stable reduction (3.1.1), and l a prime number $\neq p$, we begin by summarizing the calculation, due to Grothendieck (SGA 7 I), completed by Rapoport-Zink, of the vanishing cycles $R\Psi^q(\mathbf{Z}_l)$, equipped with the action of monodromy: an important point, proved in, is that the inertia acts trivially. We then describe the analogue, constructed by Rapoport-Zink (loc. cit.), of the complex used by Steenbrink in, to which we alluded above. This construction gives rise to a "weight spectral sequence" ((3.6.9), (3.8.2)), leading to the cohomology of the geometric generic fiber. The coincidence between the limit filtration and the monodromy filtration is conjectural here. We review what is known. We also reformulate, in a slightly more precise form, certain questions of independence of l, already raised by Serre-Tate. The perversity of the complex $R\Psi(\mathbf{Q}_l)$, suitably shifted, is an underlying theme throughout this study. It follows from Artin's theorem on the cohomological dimension of affine schemes (SGA 4 XIV 3.1) and from the fact (well known, it seems) that the functor $R\Psi_*$ commutes with duality. We give a proof of this in n° 4, based on that of Deligne's finiteness theorems in (SGA 4 1/2 Th. finitude).

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Summary

2.4. Weights and monodromy.

3. Semi-stable reduction: positive or mixed characteristic case.
4. Appendix: vanishing cycles and duality in étale cohomology.

1. The l-adic local monodromy theorem.

1.1. In this section, we fix a Henselian discrete valuation ring R, with fraction field K and residue field k. We denote by p the characteristic exponent of k. We choose an algebraic closure $\bar{K}$ of K, we denote by $\bar{R}$ the normalization of R in $\bar{K}$, and by $\bar{k}$ the residue field of $\bar{R}$ (which is an algebraic closure of k). We denote by I the **inertia group**, given by the exact sequence

$$1 \to I \to \mathrm{Gal}(\bar{K}/K) \to \mathrm{Gal}(\bar{k}/k) \to 1.$$

It fits into a canonical exact sequence (and [8, §2])

$$1 \to P \to I \xrightarrow{t} \mathbf{Z}_{(p')}(1) \to 1,$$

where P is a pro-p-group and $\mathbf{Z}_{(p')}(1) = \prod_{l \neq p} \mathbf{Z}_l(1)$ is the **tame inertia group** ($\mathbf{Z}_l(1) = \varprojlim \mu_{l^n}(\bar{k})$). Let l be a prime number $\neq p$. We denote

$$t_l : I \to \mathbf{Z}_l(1)$$

the l-component of t; the kernel of t_l is a profinite group of order prime to l.

An **l-adic representation** of $\mathrm{Gal}(\bar{K}/K)$ (or more generally, of a profinite group G) is a homomorphism $\rho : G \to \mathrm{GL}(V)$, where V is a finite-dimensional $\mathbf{Q}_l$-vector space ($\bar{\mathbf{Q}}_l$ being an algebraic closure of $\mathbf{Q}_l$), such that there exists a finite extension E of $\mathbf{Q}_l$ contained in $\bar{\mathbf{Q}}_l$ and an E-structure V_E on V such that ρ factors through a continuous homomorphism $G \to \mathrm{GL}(V_E)$ ($\mathrm{GL}(V_E)$ being endowed with its natural topology as an l-adic Lie group).

Let $G = \mathrm{Gal}(\bar{K}/K)$. We say that an l-adic representation ρ of G is **quasi-unipotent** if there exists an open subgroup I_1 of I such that the restriction of ρ to I_1 is unipotent (i.e., such that $\rho(g)$ is unipotent for all $g \in I_1$). A fundamental result of Grothendieck asserts that this property is automatic as long as k is not too large.

THEOREM 1.2 (Grothendieck) [26, Appendix]. *Suppose that no finite extension of k contains all roots of unity of order a power of l. Then any l-adic representation of G is quasi-unipotent.*

1.3. Let X be a separated scheme of finite type over K. According to (SGA 4 XIV) (resp. (SGA 4 1/2 Th. finitude)), the cohomology groups $H^n(X_{\bar{K}}, \bar{\mathbf{Q}}_l)$ (resp. $H^n_c(X_{\bar{K}}, \bar{\mathbf{Q}}_l)$) are finite-dimensional over $\bar{\mathbf{Q}}_l$. The Galois group G acts by transport of structure, whence an l-adic representation

$$\rho : G \to \mathrm{GL}(H), \tag{1.3.1}$$

where H is one of the preceding groups.

THEOREM 1.4. *The representation ρ (1.3.1) is quasi-unipotent.*

As explained in (SGA 7 I 1), Grothendieck deduced this result from a variant of 1.2 by a passage to the limit using Néron's smoothing method (see [3, 3.1 theorem 3 + 3.6 lemma 5]).

It is plausible that the conclusion of 1.4 is still true if one replaces the constant sheaf $\bar{\mathbf{Q}}_l$ on X by a $\bar{\mathbf{Q}}_l$-sheaf "of geometric origin" (cf. [2, 6.2]).

1.5. Let $\rho : G \to \mathrm{GL}(V)$ be a quasi-unipotent representation. There exists a unique nilpotent morphism

$$N : V(1) \to V, \tag{1.5.1}$$

characterized by the fact that, if I_1 is an open subgroup of I such that the restriction of ρ to I_1 is unipotent, then

$$\rho(g) = \exp(N t_l(g)) \text{ for all } g \in I_1. \tag{1.5.2}$$

Uniqueness is indeed clear. For existence, it suffices to observe that, the kernel P_l of t_l being a profinite group of order prime to l, the image by ρ of P_l is finite (we can assume that ρ factors through $\mathrm{GL}(V_E)$ as above; if L is a G-stable lattice of V_E, the kernel K of the reduction map $\mathrm{GL}(L) \to \mathrm{GL}(L/m_E L)$ is a pro-l-group, so $K \cap \rho(P_l) = \{1\}$). As the elements of $\rho(P_l \cap I_1)$ are unipotent, we have $\rho(P_l \cap I_1) = \{1\}$. The restriction of ρ to I_1 thus factors through $t_l(I_1)$, and, by shrinking I_1, $\log \rho(t_l(g))$ is defined for $g \in I_1$.

The endomorphism N is called the **logarithm of the unipotent part of the local monodromy**. It follows from the characterization (1.5.2) that $N : \bar{\mathbf{Q}}_l(1) \to \mathrm{End}(V)$ is Galois invariant: for $z \in \bar{\mathbf{Q}}_l(1)$, $x \in V$, $g \in G$, we have

$$\rho(g)N(zx) = N(\chi(g)z \cdot \rho(g)x), \tag{1.5.3}$$

where $\chi : G \to \mathbf{Z}_l^*$ is the cyclotomic character. In particular, if k is the finite field $\mathbf{F}_q$ and $F \in G$ lifts the geometric Frobenius $(a \mapsto a^q)$ of $\mathrm{Gal}(\bar{k}/k)$, we have

$$N(zFx) = qFN(zx),$$

a relation which is sometimes written, by abuse of notation,

$$NF = qFN. \tag{1.5.4}$$

The pair (ρ, N) then determines a representation of the **Weil-Deligne group** $W'(\bar{K}/K)$ [8, §8].

The endomorphism N allows one to define the **local monodromy filtration** of V: it is the unique finite increasing filtration (separated and exhaustive) $\cdots \subset M_iV \subset M_{i+1}V \subset \dots$ such that $NM_iV(1) \subset M_{i-2}V$ and that N^k induces an isomorphism $\mathrm{gr}_k^M V(k) \xrightarrow{\sim} \mathrm{gr}_{-k}^M V$. If we denote by $K.V$ (resp. $I.V$) the **kernel (resp. image) filtration** defined by

$$K_kV = \mathrm{Ker}\ N^{k+1} \quad (\text{resp. } I^kV = \mathrm{Im}\ N^k),$$

the monodromy filtration is the "convolution product" of the filtrations K and I:

$$M_j = \sum_{k-i=j} I^i \cap K_k, \tag{1.5.5}$$

cf. [28, 2.3]. We will examine, in n° 3, some problems concerning these filtrations.

2. Semi-stable reduction and limit Hodge structure.

Throughout this section, we work exclusively over $\mathbf{C}$.

2.1. The geometric proof of the local monodromy theorem.

2.1.1. Let $f : X \to S$ be a proper morphism of complex analytic spaces. We assume that S is an open disk, and that f is smooth outside $0 \in S$. As the restriction of f to $S^* = S - \{0\}$ is a locally trivial fibration in the C^∞ sense, the positive generator of $\pi_1(S^*)$ induces, for $t \in S^*$, an automorphism of $H^*(X_t, \mathbf{Z})$, denoted T_t (or T if there is no risk of confusion), and called the **local monodromy automorphism** (see (SGA 7 XIV 1.1) for sign conventions). The local monodromy theorem states that this automorphism is quasi-unipotent. More precisely:

THEOREM 2.1.2. *Under the hypotheses of 2.1.1, there exists an integer $a \geq 1$ such that*

$$(T^a - 1)^{i+1} | H^i(X_t, \mathbf{Z}) = 0 \textit{ for all } i.$$

2.1.3. Several proofs have been given for this theorem. Let us briefly recall Grothendieck's (historically the first, to my knowledge) (cf. (SGA 7 I 3.3)). Using Hironaka's resolution of singularities, we can assume that X is smooth and that the special fiber X_0 is a normal crossing divisor (i.e., that f is given, in the neighborhood of a point of X_0, by $(z_1, \dots, z_n) \mapsto z_1^{e_1} \dots z_r^{e_r}$, where $(z_1, \dots, z_n)$ are local coordinates on X). As f is proper, we have the **vanishing cycles spectral sequence**

$$E_2^{pq} = H^p(X_0, R^q\Psi(\mathbf{Z})) \Rightarrow H^*(X_t, \mathbf{Z}), \tag{2.1.3.1}$$

which is T-equivariant (cf. SGA 7 XIV (1.3.3.2))) (recall that the vanishing cycle sheaves $R^q\Psi(\mathbf{Z})$ on X_0 are defined by

$$R^q\Psi(\mathbf{Z}) = i^*R^qj_*\mathbf{Z},$$

where $i : X_0 \to X$ is the inclusion, $\tilde{X}^*$ the space deduced from $X^* = X|_{S^*}$ by the base change via a universal covering $\tilde{S}^* \to S^*$, and $j : \tilde{X}^* \to X$ the canonical map). It is therefore sufficient to prove that there exists an integer $a \geq 1$ such that $T^a|R^q\Psi(\mathbf{Z}) = \mathrm{Id}$ for all q. It is sufficient to prove that such an integer exists locally on X_0. We can therefore drop the properness hypothesis on f, and assume that X is an open neighborhood of 0 in $\mathbf{C}^n$, and that f is given by $(z_1, \ldots, z_n) \mapsto z_1^{e_1} \ldots z_r^{e_r}$. Then $X_0 = \sum_{i=1}^r e_iD_i$, where D_i is the divisor $(z_i = 0)$. Let us show that $a = \mathrm{lcm}(e_i)$ works. Let $x \in X_0$, and let $I = \{i \in [1, r] | x \in D_i\}$. It is sufficient to show that, if $e = \gcd(e_i, i \in I)$, then

$$T^e|R^q\Psi(\mathbf{Z})_x = \mathrm{Id}$$

for all q. After replacing z_i, for $i \in I$, by u_iz_i, where u_i is a suitable unit, we can assume that f, in the neighborhood of x, is given by $z \mapsto \prod_{i\in I} z_i^{e_i}$. Changing notations, we can assume that $x = 0$ and $I = [1, r]$.

Applying the definition of vanishing cycles, we find that

$$R^q\Psi(\mathbf{Z})_0 = H^q(F, \mathbf{Z}),$$

where F is the analytic subspace of $\mathbf{C}^* \times \mathbf{C}$ ($\mathrm{Im}\ u > 0$) with equation

$$z_1^{e_1} \ldots z_r^{e_r} = \exp(2\pi iu)$$

the monodromy T acting by $u \mapsto u + 1$. This space is a disjoint union of the F_k $(0 \leq k < e - 1)$ with equations

$$z_1^{e'_1} \ldots z_r^{e'_r} = \zeta^k \exp(2\pi iu/e)$$

where $\zeta = \exp(2\pi i/e)$, $e'_i = e_i/e$. It is identified (analytically) with $(\mathbf{Z}/e\mathbf{Z}) \times F_0$, and (homotopically) with $(\mathbf{Z}/e\mathbf{Z}) \times V$, where V is the torus defined by the exact sequence

$$1 \to V \to (S^1)^r \to S^1 \to 1, \quad (z_i) \mapsto \prod z_i^{e'_i},$$

the monodromy acting by $n \mapsto n + 1$ on the factor $\mathbf{Z}/e\mathbf{Z}$. In particular, $T^e = \mathrm{Id}$ on $H^*(F, \mathbf{Z})$. More precisely, we find:

$$\begin{aligned} R^0\Psi(\mathbf{Z})_0 &= \mathbf{Z}[\mathbf{Z}/e\mathbf{Z}] \otimes H^0(V, \mathbf{Z}), \\ H^q(V, \mathbf{Z}) &= \Lambda^q H^1(V, \mathbf{Z}), \\ H^1(V, \mathbf{Z}) &= \mathrm{Coker}(\mathbf{Z} \to \mathbf{Z}^r, 1 \mapsto (e'_i)), \end{aligned} \tag{2.1.3.2}$$

with T acting by $n \mapsto n + 1$ on $\mathbf{Z}/e\mathbf{Z}$.

By the comparison theorems between étale cohomology and classical cohomology, as well as between the algebraic fundamental group and the classical fundamental group, we deduce from 2.1.2:

COROLLARY 2.1.4. *Let K be the fraction field of the henselization, at a closed point, of a smooth curve over $\mathbf{C}$ (or even a field of characteristic zero), and let X be a proper and smooth scheme over K. With the notations of 1.1, there then exists an open subgroup I_1 of the inertia group I ($\simeq \hat{\mathbf{Z}}(1)$) such that we have, for all $g \in I_1$ and all i,*

$$(\rho(g) - 1)^{i+1}|H^i(X_{\bar{K}}, \mathbf{Z}_l) = 0$$

(where ρ is the local monodromy representation, cf. 1.3).

This result is both less general and more precise than 1.4. We will see later that the exponent $i + 1$ can be improved. We will also examine, in n° 3, variants of this in mixed or positive characteristic.

2.1.5. Let S be as in 2.1.1, and let $f : X \to S$ be a morphism of analytic spaces, smooth outside 0, and having **semi-stable reduction** at 0 (i.e., given, in the neighborhood of any point of X_0, by $(z_1, \ldots, z_n) \mapsto z_1 \ldots z_m$, where $(z_1, \ldots, z_n)$ are local coordinates on X, with X being smooth). The special fiber $Y = X_0$ is then a reduced normal crossing divisor. Assume further that this divisor is globally a sum of smooth divisors Y_i $(1 \leq i \leq r)$ (intersecting transversally). We can then make the vanishing cycle sheaves $R^q\Psi(\mathbf{Z})$ explicit globally on Y (and no longer just punctually, as in (2.1.3.2)): we have canonical isomorphisms:

(i) $R^0\Psi(\mathbf{Z}) = \mathbf{Z}$

(ii) $R^q\Psi(\mathbf{Z}) = \Lambda^q R^1\Psi(\mathbf{Z}) \quad (q > 0)$

(iii) $R^1\Psi(\mathbf{Z}) = \mathrm{Coker}(\mathbf{Z}_Y \to \bigoplus_{1\leq i\leq r} \mathbf{Z}_{Y_i}, n \mapsto (n|_{Y_i}))(-1)$,

where $(-)(k)$ denotes the usual "Tate twist" in Hodge theory, $(-) \otimes (2\pi i)^k\mathbf{Z}$.

It follows immediately from (2.1.3.2) that the natural maps $\mathbf{Z} \to R^0\Psi(\mathbf{Z})$, $\Lambda^q R^1\Psi(\mathbf{Z}) \to R^q\Psi(\mathbf{Z})$ are isomorphisms. Let's give the definition of the isomorphism (iii). Let us denote $j_m : X - Y_m \to X$ the inclusion. We know that the class $\mathrm{cl}(Y_m) \in H^2_{Y_m}(X, \mathbf{Z})(1)$ of the divisor Y_m provides (by purity) an isomorphism

$$\mathbf{Z}_{Y_m} \xrightarrow{\sim} R^2 j_{m*}\mathbf{Z}(1). \tag{a}$$

If $j : X - Y = X^* \to X$ is the inclusion and $\tilde{j} : \tilde{X}^* \to X$ is the map defined in (2.1.3.1), we have natural maps

$$R^1 j_{m*}\mathbf{Z}(1) \to R^1 j_*\mathbf{Z}(1) \to R^1\tilde{j}_*\mathbf{Z}(1). \tag{b}$$

From (a) and (b) one deduces a map of sheaves on Y

$$\bigoplus_{1\leq m\leq r} \mathbf{Z}_{Y_m} \to R^1\Psi(\mathbf{Z})(1). \tag{c}$$

It follows from (2.1.3.2) that (c) identifies $(\bigoplus \mathbf{Z}_{Y_m})/\mathbf{Z}$ with $R^1\Psi(\mathbf{Z})(1)$: this is the isomorphism (iii).

For $q \geq 1$, we have $\Lambda^q(\bigoplus \mathbf{Z}_{Y_i}) = \bigoplus_{i_1<\cdots<i_q} \mathbf{Z}_{Y_{i_1}\cap\cdots\cap Y_{i_q}}$. The exterior product with $v = (1, \ldots, 1)$ provides an exact sequence

$$0 \to \mathbf{Z}_Y \xrightarrow{v} \bigoplus \mathbf{Z}_{Y_i} \xrightarrow{v} \bigoplus_{i<j} \mathbf{Z}_{Y_i\cap Y_j} \xrightarrow{v} \cdots \to \bigoplus_{i_1<\cdots<i_q} \mathbf{Z}_{Y_{i_1}\cap\cdots\cap Y_{i_q}} \to \ldots, \tag{2.1.5.2}$$

which is none other than the augmented Čech complex of the covering of Y by the Y_i. If we denote this complex by $C^\bullet$ $(C^0 = \bigoplus \mathbf{Z}_{Y_i})$, we have, according to (2.1.5.1)(ii):

$$R^q\Psi(\mathbf{Z})(q) = \mathrm{Coker}(C^{q-2} \to C^{q-1}) = \mathrm{Ker}(C^q \to C^{q+1}) \quad (q \geq 1). \tag{2.1.5.3}$$

One can also interpret these formulas in the following way. The maps (a) above give an isomorphism

$$\bigoplus_{1\leq i\leq r} \mathbf{Z}_{Y_i} \xrightarrow{\sim} R^1 j_*\mathbf{Z}(1)$$

(from which (c) is deduced by composition with $R^1 j_*\mathbf{Z}(1) \to R^1\Psi(\mathbf{Z})(1)$). From this one deduces, for $q \geq 1$, an isomorphism

$$C^{q-1} \xrightarrow{\sim} R^q j_*\mathbf{Z}(q) \tag{2.1.5.4}$$

(with the notation of (2.1.5.3)). To avoid confusion, let's denote G rather than $\mathbf{Z}$ the fundamental group of S^*. The complex $R\Psi(\mathbf{Z})$ underlies an object (still denoted $R\Psi(\mathbf{Z})$) of $D^+(Y, \mathbf{Z}[G])$, and we have

$$R\Gamma(G, R\Psi(\mathbf{Z})) = Rj_*\mathbf{Z}|Y. \tag{2.1.5.5}$$

In particular, we have a spectral sequence

$$E_2^{pq} = H^p(G, R^q\Psi(\mathbf{Z})) \Rightarrow R^{p+q}j_*(\mathbf{Z})|Y,$$

which provides short exact sequences

$$0 \to H^1(G, R^{q-1}\Psi(\mathbf{Z})(q)) \to R^q j_*\mathbf{Z}(q)|Y \to H^0(G, R^q\Psi(\mathbf{Z})) \to 0, \tag{2.1.5.6}$$

(G being of cohomological dimension 1). But the hypotheses on f imply, according to (2.1.3.2), that G acts trivially on the sheaves $R^q\Psi(\mathbf{Z})$. So we have

$$H^0(G, R^q\Psi(\mathbf{Z})) = R^q\Psi(\mathbf{Z}),$$

and a canonical isomorphism

$$H^1(G, R^{q-1}\Psi(\mathbf{Z}))(q) = R^{q-1}\Psi(\mathbf{Z})(q-1)$$

(deduced from $H^1(G, \mathbf{Z}) = H^1(S^*, \mathbf{Z}) = \mathbf{Z}(-1)$). The sequence (2.1.5.6) is thus rewritten as

$$0 \to R^{q-1}\Psi(\mathbf{Z})(q-1) \to R^q j_* \mathbf{Z}(q)|Y \to R^q\Psi(\mathbf{Z})(q) \to 0. \tag{2.1.5.7}$$

Taking into account (2.1.5.4), this is (up to a sign perhaps) the sequence deduced from (2.1.5.3). This interpretation is due to Rapoport-Zink, we will return to it in n° 3.6.

2.2. Local monodromy and Gauss-Manin connection.

2.2.1. Let $f : X \to S$ be as in 2.1.1. Assume that X is smooth over $\mathbf{C}$, and that the special fiber $Y = X_0$ is a normal crossing divisor (not necessarily reduced). For any q, the local system $R^q f_*(\mathbf{C}) = R^q f_*(\mathbf{Z}) \otimes \mathbf{C}$, of fiber $H^q(X_t, \mathbf{C})$ at $t \in S^*$, is the subsheaf of horizontal sections of the Gauss-Manin connection ∇ on $R^q f_* \Omega^\bullet_{X^*/S^*}$ (where $X^* = X - Y$). As Steenbrink shows in, the geometry of the situation allows one to exhibit a "canonical extension" (in the sense of Deligne-Manin, cf.) of this connection, and consequently to give another description of the monodromy T of $H^q(X_t, \mathbf{C})$.

Let us denote $\omega^\bullet_X = \Omega^\bullet_X(\log Y)$ the de Rham complex of X (over $\mathbf{C}$) with logarithmic poles along Y, and similarly $\omega^\bullet_S = \Omega^\bullet_S(\log 0)$ that of S with logarithmic poles along 0 (,). The $\mathcal{O}_X$-module

$$\omega^1_{X/S} := \omega^1_X / (\mathrm{Im}\ f^* : \omega^1_S \to \omega^1_X) \tag{2.2.1.1}$$

(denoted $\Omega^1_{X/S}(\log Y)$ in) is locally free of finite type (admitting locally as basis $dz_1/z_1, \ldots, dz_r/z_r, dz_{r+1}, \ldots, dz_n$ with the relation $\sum e_i dz_i/z_i = 0$, in a neighborhood of a point of Y where Y has for equation $z_1^{e_1} \ldots z_r^{e_r} = 0$ in a system of local coordinates $(z_1, \ldots, z_n)$). The differential of $\omega^\bullet_X$ gives, by passing to the quotient, a differential on $\omega^\bullet_{X/S} := \Lambda^\bullet \omega^1_{X/S}$, and we have an exact sequence of complexes

$$0 \to \omega^1_S \otimes \omega^{\bullet-1}_{X/S} \to \omega^\bullet_X \to \omega^\bullet_{X/S} \to 0, \tag{2.2.1.2}$$

(where the map on the left is $a \otimes b \mapsto f^*a \wedge b$). The boundary operator that is deduced from this,

$$\nabla : R^q f_* \omega^\bullet_{X/S} \to \omega^1_S \otimes R^q f_* \omega^\bullet_{X/S}, \tag{2.2.1.3}$$

extends the Gauss-Manin connection (cf. [N.M. Katz, The regularity theorem in algebraic geometry, Actes Congrès Int. Math. 1970, tome 1, 437-443, Gauthier-Villars, 1971]). Steenbrink proves the following result:

THEOREM 2.2.2 [27, 2.18, 2.20]. *(a) The sheaves $R^q f_* \omega^\bullet_{X/S}$ are locally free of finite type, of formation compatible with any base change; in particular,*

$$R^q f_* \omega^\bullet_{X/S} \otimes_{\mathcal{O}_S} \mathcal{O}_{\mathbf{C},\{0\}} \simeq H^q(Y, \omega^\bullet_Y),$$

where we have set

$$\omega^\bullet_Y := \omega^\bullet_{X/S} \otimes_{\mathcal{O}_S} \mathcal{O}_{\mathbf{C},\{0\}}. \tag{2.2.2.1}$$

(b) Let N be the residue at 0 of the connection (2.2.1.3). Then, if α is an eigenvalue of N, we have $\alpha \in \mathbf{Q}$ and $0 \le \alpha < 1$.

The vector bundle $R^q f_* \omega^\bullet_{X/S}$, equipped with ∇ (2.2.1.3), is therefore the canonical extension of $R^q f_* \Omega^\bullet_{X^*/S^*}$, equipped with the Gauss-Manin connection, in the sense of Deligne [6, II 5.4] (relative to the choice (II 5.3.1) of τ). According to [6, II 1.17] and [6, II 5.6], it follows:

COROLLARY 2.2.3. *(a) The monodromy automorphisms T_t of $H^q(X_t, \mathbf{C})$ ($t \in S^*$) are the fibers of an automorphism T of the bundle $R^q f_* \omega^\bullet_{X/S}$ whose fiber at 0 is given by*

$$T_0 = \exp(-2\pi i N)$$

(with N as in 2.2.2(b)).

(b) If we identify $H^q(X_t, \mathbf{C})$ and $R^q f_* \omega^\bullet_{X/S} \otimes_{\mathcal{O}_S} \mathbf{C}_{\{0\}}$ ($= H^q(Y, \omega^\bullet_Y)$ according to 2.2.2(a)) with the same vector space V, then T_t and T_0 are conjugate in $GL(V)$.

Taking into account 2.2.2(b), this statement establishes again the quasi-unipotence of T_t.

2.2.4. According to Deligne (or), the complex $\omega^\bullet_X$ "calculates" $Rj_*\mathbf{C}$, i.e. one has an isomorphism

$$\omega^\bullet_X \xrightarrow{\sim} Rj_*\mathbf{C} \quad (\text{in } D(X, \mathbf{C})). \tag{2.2.4.0}$$

Steenbrink deduces 2.2.2 from a finer result, according to which the complex $\omega^\bullet_Y$ (2.2.2.1) calculates $R\Psi(\mathbf{C})$ (in $D(Y, \mathbf{C})$). More precisely, let us choose a uniformizer $t : S \to \mathbf{C}$, a universal covering $p : \tilde{S}^* \to S^*$, and a logarithm, i.e. a function $\log t$ on $\tilde{S}^*$ such that $\exp(\log t) = p^* t$. Steenbrink constructs an isomorphism in $D(Y, \mathbf{C})$ (depending on these choices)

$$\alpha_t : \omega^\bullet_Y \xrightarrow{\sim} R\Psi(\mathbf{C}). \tag{2.2.4.1}$$

The homomorphism of degree 1 deduced from the exact sequence (2.2.1.2)

$$\omega^{\bullet-1}_{X/S} \to \omega^1_S \otimes \omega^\bullet_{X/S}$$

gives, by composition with the residue at 0, $\mathrm{Res}_0 : \omega^1_S \to \mathbf{C}_{\{0\}}$, a homomorphism

$$N : \omega^\bullet_Y \to \omega^\bullet_Y \tag{2.2.4.2}$$

of $D(Y, \mathbf{C})$. Steenbrink shows moreover (cf. 2.3.3) that the monodromy automorphism T of $R\Psi(\mathbf{C})$ corresponds, via (2.2.4.1), to $\exp(-2\pi i N)$. Assertion (a) of 2.2.2 follows from (2.2.4.1) and the isomorphism $H^*(Y, R\Psi(\mathbf{C})) = H^*(X_t, \mathbf{C})$, and, by an explicit calculation of N on $H^*\omega^\bullet_Y$, the formula $T = \exp(-2\pi i N)$ implies (b).

For the dependence of (2.2.4.1) with respect to the choices, see [27, 4.24] (at least in the case where Y is reduced).

Let us explain the definition of (2.2.4.1), placing ourselves, to simplify, in the case where Y is reduced. Let us denote i^{-1} the inverse image functor by $i : Y \to X$ for abelian sheaves. It results from the definition of $R\Psi(\mathbf{C})$ that one has

$$R\Psi(\mathbf{C}) = i^{-1} R\tilde{j}_*\mathbf{C} \quad (\text{in } D(Y, \mathbf{C})) \tag{a}$$

(with the notations of (2.1.3.1)). The complex $i^{-1}\omega^\bullet_X (= i^{-1}\Omega^\bullet_X(\log Y))$ is identified in a natural way with a subcomplex of $i^{-1} j_* \Omega^\bullet_{X^*}$. More generally, $i^{-1} R\tilde{j}_* \Omega^\bullet_{X^*}$ contains the subcomplex $i^{-1}\omega^\bullet_X[\log t]$ formed of sections writing locally $\sum_{0 \le k \le s} (\log t)^k \omega_k$, where ω_k is a section of $\omega^\bullet_X$. Steenbrink shows that the inclusion

$$i^{-1}\omega^\bullet_X[\log t] \to i^{-1} R\tilde{j}_*\mathbf{C} \tag{b}$$

and the homomorphism

$$i^{-1}\omega^\bullet_X[\log t] \to \omega^\bullet_Y \tag{c}$$

associating to $\sum_{0 \le k \le s} (\log t)^k \omega_k$ the class of ω_0 in $\omega^\bullet_Y$ are quasi-isomorphisms (for (c), see the remark following 2.3.2.5 below). The isomorphism (2.2.4.1) is defined by (b) and (c), taking (a) into account.

Let us mention the following interpretation of these isomorphisms, due to Navarro-Aznar (personal communication). The complex $i^{-1}\omega^\bullet_X[\log t]$ is a graded differential module over the graded differential algebra $\omega^\bullet_S$. It can be written as a tensor product

$$i^{-1}\omega^\bullet_X[\log t] = i^{-1}(\omega^\bullet_S[\log t] \otimes_{\omega^\bullet_S} \omega^\bullet_X).$$

Let us denote $i_0 : \{0\} \to S$ and $j_0 : S^* \to S$ the inclusions. The graded differential $\omega^\bullet_S$-module $i_0^{-1}\omega^\bullet_S[\log t]$ is a resolution of $\mathbf{C}_{\{0\}}$ (a special case of (c)), which one can easily check is acyclic for the functor $\otimes_{\omega^\bullet_S}$. We thus have

$$i^{-1}\omega^\bullet_X[\log t] \overset{L}{\otimes}_{\omega^\bullet_S} \omega^\bullet_X = \mathbf{C}_{\{0\}} \overset{L}{\otimes}_{\omega^\bullet_S} \omega^\bullet_X.$$

Taking into account (a) and (b), and as $\omega^\bullet_X$ (resp. $\omega^\bullet_S$) calculates $Rj_*\mathbf{C}$ (resp. $Rj_{0*}\mathbf{C}$), we can rewrite this formula (always in the case where Y is reduced) in the more striking form

$$R\Psi(\mathbf{C}) = \mathbf{C}_{\{0\}} \overset{L}{\otimes}_{Rj_{0*}\mathbf{C}} Rj_*\mathbf{C}, \tag{2.2.4.3}$$

which can be considered as a sort of inversion of (2.1.5.5).

2.3. Steenbrink's complex and limit Hodge structure.

2.3.1. Let S be as in 2.1.1. In this section, we assume that $f : X \to S$ is projective, with X smooth over $\mathbf{C}$, and that f is smooth outside 0, of relative dimension d, and has semi-stable reduction at 0, the special fiber $Y = X_0$ being a sum of smooth divisors Y_i. One of the main results of Steenbrink (, completed by) is that $R\Psi(\mathbf{Z})$ underlies a cohomological mixed Hodge complex on Y, $(R\Psi(\mathbf{Z}), (R\Psi(\mathbf{Q}), W), (R\Psi(\mathbf{C}), W, F))$ in the sense of Deligne. In particular, the cohomology groups $H^n(Y, R\Psi(\mathbf{Z}))$ are endowed with natural mixed Hodge structures. Moreover, the monodromy operator T is unipotent, and $N := (-1/2\pi i)\log T$ is a morphism of mixed Hodge structures $H^n(Y, R\Psi(\mathbf{Q})) \to H^n(Y, R\Psi(\mathbf{Q}))(-1)$. These results had been announced by Deligne in (P. Deligne, Théorie de Hodge, I, Actes, Cong. int. math., I, Gauthier-Villars (1971), 425-430). The theory of Morihiko Saito, offers a better formulation: $R\Psi(\mathbf{Z})[d]$ underlies a mixed Hodge module on Y (in particular, $R\Psi(\mathbf{Q})[d]$ is a perverse sheaf), for which N is (up to a twist) a nilpotent endomorphism.

2.3.2. We will confine ourselves to sketching Steenbrink's construction of a representative of $(R\Psi(\mathbf{C}), W, F)$. For that of $(R\Psi(\mathbf{Q}), W)$, which is not treated correctly in, we refer to (where the construction is inspired by Rapoport-Zink). We assume chosen, as in 2.2.4, a uniformizer t and a logarithm $\log t$. The basic observation is the following (cf. [27, 4.6] - which, literally, makes no sense...):

LEMMA 2.3.2.1. *Let $i : Y \to X$ be the inclusion, and $\theta = f^* dt/t$. The sequence of complexes*

$$i^{-1}\omega_X^\bullet[-1] \xrightarrow{\theta\wedge} i^{-1}\omega_X^\bullet \xrightarrow{\theta\wedge} i^{-1}\omega_X^\bullet$$

is exact, as is the sequence of sheaves deduced from it by applying H^q.

The verification of the first assertion is immediate (Koszul complex). The second results from the standard calculation of $H^q\omega_X^\bullet = R^q j_*\mathbf{C}$ (where $j : X - Y \to X$ is the inclusion), cf. (2.1.5.5) and [7, 3.1.8]. The exact sequence of 2.3.2.1 defines a bicomplex

$$M = (\cdots \to i^{-1}\omega_X^\bullet[-1] \xrightarrow{d''} i^{-1}\omega_X^\bullet \xrightarrow{d''} i^{-1}\omega_X^\bullet \to \dots),$$

with vertical differential d'' deduced from θ, whose columns are acyclic. This complex is concentrated in the oblique band $0 \le i + j \le d + 1 = \dim X$ (and poorly placed, because on a diagonal $i + j = n$, for $0 \le n \le d + 1$, all components are non-zero). Certain complexes deduced from M by truncation are particularly interesting. We will denote by $\sigma_{\le a}, \sigma_{\ge a}$ the naive truncations, $\tau_{\le a}, \tau_{\ge a}$ the canonical truncations (if L is a complex, $\sigma_{\le a}L = (\cdots \to L^a \to 0)$, $\sigma_{\ge a}L = (0 \to L^a \to \dots)$, $\tau_{\le a}L = (\cdots \to L^{a-1} \to Z^a \to 0)$, $\tau_{\ge a}L = (0 \to L^a/B^a \to L^{a+1} \to \dots)$). In the case of a bicomplex, if t is a truncation, we will denote t^{hor} (resp. t^{ver}) the truncation relative to the horizontal (resp. vertical) differential. Let us set

$$M' = \tau^{hor}_{\le 0}\sigma^{ver}_{\le 0}M, \quad M'' = \sigma^{hor}_{\ge 1}(\tau^{ver}_{\ge 1}M) \tag{2.3.2.2}$$

(where $[p, q]$ denotes the shift for bicomplexes). For example, if $d = 1$, M is the bicomplex

$$\begin{pmatrix} & \vdots & \vdots & \vdots & \\ 0 \to & \mathcal{O} & \to \omega_X^1 & \to \omega_X^2 & \to 0 \\ & \uparrow & \uparrow & \uparrow & \\ 0 \to & \mathcal{O} & \to \omega_X^1 & \to \omega_X^2 & \to 0 \\ & \uparrow & \uparrow & \uparrow & \\ 0 \to & \mathcal{O} & \to \omega_X^1 & \to \omega_X^2 & \to 0 \\ & \vdots & \vdots & \vdots & \end{pmatrix} \leftarrow \text{row of degree 0, O in degree 0}$$

M' is the bicomplex

$$\begin{pmatrix} 0 \to & \mathcal{O} & \to Z\omega_X^1 & \to 0 \\ & \uparrow & \uparrow & \\ 0 \to & 0 & \to Z^0 & \to 0 \\ & & (= \mathbf{C}_{\{0\}}) & \end{pmatrix} \leftarrow \text{row of degree 0, O in degree 0,}$$

and M'' is the bicomplex

$$\begin{pmatrix} 0 \to & \omega_X^1/B^2 & \to \omega_X^2 & \to 0 \\ & \uparrow & \uparrow & \\ 0 \to & \omega_X^0/B^1 & \to \omega_X^1 & \to 0 \end{pmatrix} \leftarrow \text{row of degree 0, } \omega_X^0/B^1 \text{ in degree 0}$$

(we have omitted i^{-1} for brevity). We have natural homomorphisms of bicomplexes

$$M' \to M'', \tag{2.3.2.3}$$

induced by $d'' : M^{\cdot,0} \to M^{\cdot,1}$, and

$$M' \to \omega_Y^\bullet, \tag{2.3.2.4}$$

induced by $\tau_{\leq d}\omega_X^\bullet \to \omega_Y^\bullet$.

LEMMA 2.3.2.5. *The morphisms (2.3.2.3) and (2.3.2.4) induce quasi-isomorphisms on the associated simple complexes.*

Indeed, the columns H^q_{ver} are acyclic according to (2.3.2.1) and the direct calculation of $H^q\omega_Y^\bullet$ (cf. [27, 1.14]).

One can also observe that, for the same reasons, the inclusion $M' \to \sigma^{ver}_{\leq 0}M$ induces a quasi-isomorphism on the associated simple complexes, and that the simple complex associated with $\sigma^{ver}_{\leq 0}M$ is none other than the complex $i^{-1}\omega_X^\bullet[\log t]$ considered in 2.2.4 (b) and (c): the fact that (c) is a quasi-isomorphism thus follows from 2.3.2.5.

The complex $\omega_X^\bullet$ is equipped with the weight filtration $0 \subset W_0\omega_X^\bullet \subset W_1\omega_X^\bullet \subset \dots$ [7, 3.1.5], and one has an inclusion $\tau_{\leq n}\omega_X^\bullet \subset W_n\omega_X^\bullet$. It is known [7, 3.1.8] that the identity of $\omega_X^\bullet$ gives a quasi-isomorphism of filtered complexes

$$(\omega_X^\bullet, \tau_{\leq \cdot}) \to (\omega_X^\bullet, W_\cdot). \tag{2.3.2.6}$$

We thus have, for any n, quasi-isomorphisms

$$\tau_{\geq n}\omega_X^\bullet \to \omega_X^\bullet/\tau_{\leq n-1}\omega_X^\bullet \to \omega_X^\bullet/W_{n-1}\omega_X^\bullet. \tag{2.3.2.6')}$$

Steenbrink considers the bicomplex $A = (A^{pq}, d', d'')$ defined by

$$A^{pq} = \omega_X^{p+q+1}/W_q\omega_X^{p+q+1}, \tag{2.3.2.7}$$

$$A : \begin{pmatrix} \vdots \\ \uparrow \\ (\omega_X^\bullet/W_1\omega_X^\bullet) \\ \uparrow \\ (\omega_X^\bullet/W_0\omega_X^\bullet) \end{pmatrix} \leftarrow \text{row of degree 0, } \omega_X^\bullet/W_0\omega_X^\bullet \text{ in degree 0,}$$

with differential d' (resp. d'') induced by the exterior differential (resp. $\theta\wedge$): for $x \in A^{pq}$, $d'x = (-1)^{p+1}dx$ (d = exterior differential), $d''x = (-1)^p\theta \wedge x$. Its rows are connected to those of M'' by the quasi-isomorphisms (2.3.2.6)': the q-th row of A corresponds to $\omega_X^\bullet[q]/\tau_{\leq 0}(\omega_X^\bullet[q]) \simeq \omega_X^\bullet[q]/(\tau_{\leq q}\omega_X^\bullet)[q] \simeq (\tau_{\geq q+1}\omega_X^\bullet)[q]$. By construction, A is supported on Y, and the simple complex sA associated with A is another "incarnation" of $\omega_Y^\bullet$ (and thus of $R\Psi(\mathbf{C})$) in $D(Y, \mathbf{C})$: we have canonical isomorphisms of $D(Y, \mathbf{C})$

$$R\Psi(\mathbf{C}) \simeq \omega_Y^\bullet \simeq sA. \tag{2.3.2.8}$$

Note that the bicomplex A is concentrated in the triangle ($p \geq 0, q \geq 0, p+q \leq d = \dim Y$); for example, for $d = 1$, A is the bicomplex

$$\begin{pmatrix} & \omega_X^2/W_1\omega_X^2 \\ & \uparrow \\ \omega_X^1/W_0\omega_X^1 \to & \omega_X^2/W_0\omega_X^2 \end{pmatrix} \leftarrow \text{row of degree 0.}$$

Steenbrink defines as follows the filtrations (W, F) on the bicomplex A. The (decreasing) filtration F is the filtration by the first degree: $F^pA = \sigma^{hor}_{\geq p}A(= A^{\geq p,\cdot})$. The (increasing) filtration W is given by

$$W_rA^{pq} = W_{2q+r+1}\omega_X^{p+q+1}/W_q\omega_X^{p+q+1}.$$

The filtration W is a "monodromy filtration" (see 2.3.3). Observe that, by (2.3.2.6), the q-th row of W_rA corresponds to $\tau_{\leq q+r}(\omega_X^\bullet[q])/\tau_{\leq 0}(\omega_X^\bullet[q]) = (\tau_{\leq 2q+r}\omega_X^\bullet)[q]/(\tau_{\leq q}\omega_X^\bullet)[q] \simeq \tau_{\leq q+r}((\tau_{\geq q+1}\omega_X^\bullet)[q])$.

The associated graded pieces are easily calculated. For W, noting that $\mathrm{gr}^W_r A$ has zero d'' differential, and taking into account (2.2.4.0) and the preceding remark, we find that $\mathrm{gr}^W_r A$ is cohomologically concentrated on the line $j - i = r$, sum of the $R^{r+1+2q}j_*\mathbf{C}$ placed in bidegree $(r+q, q)$, and thus

$$\mathrm{gr}^W_r sA = \bigoplus_{\substack{q\geq 0 \\ q+r\geq 0}} R^{r+1+2q}j_*\mathbf{C}[-r-2q]. \tag{2.3.2.9}$$

Recall that, according to (2.1.5.4), we also have, if $Y = \cup_{1 \le i \le h} Y_i$,

$$R^m j_* \mathbf{C} = \bigoplus \mathbf{C}_{Y_{i_1} \cap \cdots \cap Y_{i_m}},$$

sum extended over $1 \le i_1 < \cdots < i_m \le h$. As for F, Steenbrink shows that we have

$$\mathrm{gr}_p^F sA = \omega_Y^p[-p]. \tag{2.3.2.10}$$

He proves more precisely that $\theta\wedge$ induces a morphism of complexes $\omega_Y^\bullet \to A^{\cdot,0}$, and that the sequence of complexes

$$0 \to \omega_Y^\bullet \xrightarrow{\theta\wedge} A^{\cdot,0} \xrightarrow{d''} A^{\cdot,1} \xrightarrow{d''} \cdots \to A^{\cdot,q} \xrightarrow{d''} \ldots \tag{2.3.2.11}$$

is exact.

2.3.3. The endomorphism ν of A, of bidegree $(-1,1)$, equal to $(-1)^{p+q+1}$ (canonical projection) on A^{pq}, commutes with d' and d'', thus induces an endomorphism

$$\nu : sA \to sA. \tag{2.3.3.1}$$

Denoting by W (resp. F) the filtration of sA deduced from the filtration W (resp. F) of A, we have

$$\nu(W_r sA) \subset W_{r-2} sA, \quad \nu(F^n sA) \subset F^{n-1} sA.$$

One verifies moreover that, for $r \ge 0$, ν^r induces an isomorphism (of complexes)

$$\nu^r : \mathrm{gr}_r^W sA \xrightarrow{\sim} \mathrm{gr}_{-r}^W sA \tag{2.3.3.2}$$

so that W is the monodromy filtration of ν. Steenbrink finally proves that, via the isomorphism (2.3.2.8), ν corresponds to the endomorphism $N = \mathrm{Res}_0 \nabla$ (2.2.4.2). In fact, we have the following more precise results, which express the essence of the theory:

THEOREM 2.3.4. *Under the hypotheses of 2.3.1, with t and $\log t$ chosen as in 2.3.2:*

(a) *The complex $R\Psi(\mathbf{Z}) \in D(Y, \mathbf{Z})$ underlies a cohomological mixed Hodge complex $(R\Psi(\mathbf{Z}), (R\Psi(\mathbf{Q}), W), (R\Psi(\mathbf{C}), W, F)$ [10, 8.1.6], whose component $(R\Psi(\mathbf{C}), W, F)$ is isomorphic to (sA, W, F) in the bifiltered derived category $D^+F_2(Y, \mathbf{C})$.*

(b) *The endomorphism T of $R\Psi(\mathbf{C})$ is unipotent, and $-(1/2\pi i)\log T$ is the underlying morphism of a mixed Hodge complex morphism*

$$N : R\Psi(\mathbf{Q}) \to R\Psi(\mathbf{Q})(-1),$$

whose component on $(R\Psi(\mathbf{C}), W, F)$ is given by ν (2.3.3.1). For all $r \ge 0$, N^r induces an isomorphism

$$N^r : \mathrm{gr}_r^W R\Psi(\mathbf{Q}) \xrightarrow{\sim} \mathrm{gr}_{-r}^W R\Psi(\mathbf{Q})(-r).$$

(c) *One has a canonical isomorphism in $D(Y, \mathbf{C})$*

$$\mathrm{gr}_p^F R\Psi(\mathbf{C}) \simeq \omega_Y^p[-p].$$

Recall that, as f is proper and $f|S^*$ is smooth, we have (SGA 7 XIV (1.3.3.2))

$$H^*(Y, R\Psi\Lambda) = H^*(\tilde{X}^*, \Lambda) = H^*(X_t, \Lambda)$$

for any point $t \in S^*$ (these identifications being compatible with the action of the monodromy T). From 2.3.4, one therefore deduces:

COROLLARY 2.3.5. *Under the hypotheses of 2.3.4, for any n, $H^n(\tilde{X}^*, \mathbf{Z})$ is endowed with a mixed Hodge structure, and the (nilpotent) endomorphism $N = (-1/2\pi i)\log T$ of $H^n(\tilde{X}^*, \mathbf{C})$ is a morphism of mixed Hodge structures $H^n(\tilde{X}^*, \mathbf{Q}) \to H^n(\tilde{X}^*, \mathbf{Q})(-1)$.*

One can consider this mixed Hodge structure as a "limit" of the pure Hodge structures of the $H^n(X_s, \mathbf{Z})$, $s \in S^*$, "as s tends to 0". It depends on the choices of $(t, \log t)$ (the lattice $H^n(\tilde{X}^*, \mathbf{Z})$ and the weight filtration W are fixed, but the filtration F varies, cf. [27, 4.24] and).

From the theory of mixed Hodge structures, the **weight spectral sequence**

$$ {}_W E_1^{pq} = H^{p+q}(Y, \mathrm{gr}^W_{-p} R\Psi(\mathbf{Q})) \Rightarrow H^{p+q}(\tilde{X}^*, \mathbf{Q}) \tag{2.3.6} $$

degenerates at E_2, and the **Hodge spectral sequence**

$$ {}_F E_1^{pq} = H^{p+q}(Y, \mathrm{gr}^F_p R\Psi(\mathbf{C})) \Rightarrow H^{p+q}(\tilde{X}^*, \mathbf{C}) \tag{2.3.7} $$

degenerates at E_1. The limit filtrations of (2.3.6) and (2.3.7) are respectively the weight filtration W and the Hodge filtration F of $H^*(\tilde{X}^*)$.

One can make the initial terms of these spectral sequences explicit. First, the formula (2.3.2.9) refines to an isomorphism of Hodge complexes

$$ \mathrm{gr}^W_r R\Psi(\mathbf{Q}) = \bigoplus_{\substack{q \geq 0 \\ r+q \geq 0}} (a_{r+1+2q})_* \mathbf{Q}_{Y(r+1+2q)}(-r-q)[-r-2q] \tag{2.3.8} $$

where

$$ Y^{(m)} := \coprod_{1 \leq i_1 < \cdots < i_m \leq h} Y_{i_1} \cap \cdots \cap Y_{i_m}, $$

and

$$ a_m : Y^{(m)} \to Y $$

is the projection. In other words, $\mathrm{gr}^W_r R\Psi(\mathbf{Q})$ is the simple complex associated with the double complex, with zero differentials d' and d'',

$$ \begin{array}{ccccccccc}
a_{d+1*}\mathbf{Q} & & & & & & & & \\
 & \searrow N & & & & & & & \\
a_{d*}\mathbf{Q} & & a_{d+1*}\mathbf{Q}(-1) & & & & & & \\
 & & & & & & & & \\
\vdots & & \vdots & & \ddots & & & & \\
a_{2*}\mathbf{Q} & & a_{3*}\mathbf{Q}(-1) & & \dots & & a_{d+1*}\mathbf{Q}(-d+1) & & \\
 & \searrow N & & \searrow N & & \searrow N & & & \\
a_{1*}\mathbf{Q} & & a_{2*}\mathbf{Q}(-1) & & \dots & & a_{d*}\mathbf{Q}(-d+1) & & a_{d+1*}\mathbf{Q}(-d)
\end{array} \tag{2.3.8.1} $$

the gr_r is the sum of the terms on the diagonal $j - i = r$; $N : \mathrm{gr}_r \to \mathrm{gr}_{r-2}(-1)$ is given by the identical oblique arrows. Consequently, the initial term of (2.3.6) is rewritten

$$ {}_W E_1^{-r,n+r} = \bigoplus_{\substack{q \geq 0 \\ r+q \geq 0}} H^{n-r-2q}(Y^{(r+1+2q)}, \mathbf{Q})(-r-q). \tag{2.3.9} $$

It is pure of weight $n + r$. The weights of $H^n(\tilde{X}^*)$ are therefore in the interval $[n-d, n+d]$. One can also describe the differential d_1 of (2.3.6): we have $d_1 = d_1' + d_1''$, where d_1' (resp. d_1'') is an alternating sum of Gysin (resp. restriction) homomorphisms (cf. [19, 2.10] and). As for the initial term of (2.3.7), taking into account (2.3.2.10), it is given by

$$ {}_F E_1^{pq} = H^q(Y, \omega^p_Y). \tag{2.3.10} $$

By a counting argument, the degeneration at E_1 of (2.3.7) thus implies:

COROLLARY 2.3.11. *For any (p, q), the sheaf $R^q f_* \omega^p_{X/S}$ is locally free of finite type, and commutes with any base change. If h^{pq} denotes its rank, we have*

$$ h^{pq} = \dim \mathrm{gr}^p_F H^{p+q}(\tilde{X}^*, \mathbf{C}) = \dim H^q(X_t, \Omega^p_{X_t}) = \dim H^q(Y, \omega^p_Y) \quad (t \in S^*). $$

The fact that N is a morphism of mixed Hodge structures provides, on the other hand, a bound for its exponent of nilpotence, better than 2.1.4.

COROLLARY 2.3.12. *Let*

$$h_n = \sup\{b-a | \forall i \in [a,b], h^{i,n-i} \neq 0\},$$

where the h^{pq} are the Hodge numbers considered in 2.3.11. Then

$$N^{h_n+1} | H^n(\tilde{X}^*, \mathbf{Q}) = 0.$$

This results in fact from the existence of a bigraduation $H^n(\tilde{X}^*, \mathbf{C}) = \bigoplus H^{pq}$ such that $N(H^{pq}) \subset H^{p-1,q-1}$ and $\sum_p \dim H^{pq} = h^{p,n-p}$, cf. [27, 3.2].

2.4. Weights and monodromy.

2.4.1. We resume the hypotheses of 2.3.4. The fact that we assumed f not only proper, but projective, does not really serve in the construction of the limit Hodge structure: it would suffice to assume Y is algebraizable. On the other hand, the projectivity of f intervenes in an essential way[1](through polarizations) in the proof of the following result:

THEOREM 2.4.2. *For all $r \geq 0$ and all n, $N^r : H^n(\tilde{X}^*, \mathbf{Q}) \to H^n(\tilde{X}^*, \mathbf{Q})(-r)$ induces an isomorphism of (pure) Hodge structures*

$$N^r : \mathrm{gr}^W_{n+r} H^n(\tilde{X}^*, \mathbf{Q}) \xrightarrow{\sim} \mathrm{gr}^W_{n-r} H^n(\tilde{X}^*, \mathbf{Q})(-r).$$

In other words, the weight filtration on H^n coincides with the **monodromy filtration** (centered at n), characterized by $N(W_r) \subset W_{r-2}$ and $N^r : \mathrm{gr}^W_{n+r} \xrightarrow{\sim} \mathrm{gr}^W_{n-r}$.

An erroneous proof of 2.4.2 is given by Steenbrink in [27, 5.9]. A correction, also erroneous, is provided by El Zein in. To our knowledge, the only correct published proof is that of Morihiko Saito [20, 4.2.2]; up to a few writing details, it is reproduced by Guillen and Navarro Aznar. Morihiko Saito indicates (loc. cit.) that Deligne pointed out to him another proof of 2.4.2.

2.4.3. Let us only indicate where the difficulty lies. According to 2.3.4 (b), N induces an endomorphism of the weight spectral sequence (2.3.6), and, for $r \geq 0$, N^r is an isomorphism from ${}_WE_1^{-r,n+r}$ to ${}_WE_1^{r,n-r}$. As ${}_WE_2 = {}_WE_\infty$, it is therefore sufficient to prove that N^r still induces an isomorphism from ${}_WE_2^{-r,n+r}$ to ${}_WE_2^{r,n-r}$. But N^r is not an automorphism of the term E_1: in the morphism of complexes

$$\begin{array}{ccccc} E_1^{-r-1,n+r} & \xrightarrow{d_1} & E_1^{-r,n+r} & \xrightarrow{d_1} & E_1^{-r+1,n+r} \\ \downarrow N^r & & \downarrow N^r & & \downarrow N^r \\ E_1^{r-1,n+r} & \xrightarrow{d_1} & E_1^{r,n+r} & \xrightarrow{d_1} & E_1^{r+1,n+r}, \end{array}$$

only the middle vertical arrow is an isomorphism, the one on the right (resp. left) is only surjective (resp. injective). To analyze E_2, it is convenient (cf. (2.3.8.1)) to consider E_1 as the simple complex $\bigoplus C_r (= \bigoplus C^{-r})$ associated with the bicomplex $C^{\cdot,\cdot} (= C^{-i,j})$,

$$\begin{aligned} C_r &= C_{r+q}^{-r+q} \quad (q \geq 0, r+q \geq 0) \\ C_{r+q}^{-r+q} &= H^q(Y^{(r+1+2q)}, \mathbf{Q})(-r-q), \end{aligned} \tag{2.4.3.1}$$

with the notations of (2.3.9), the differential $d_1' : C_r \to C_{r-1}$ (resp. $d_1'' : C_r \to C_{r+1}$) being "of Gysin type" (resp. "Čech type"). As $N : C_i \to C_{i-2}(-1)$ $(i-1 \geq 0, j \geq 0)$, the decomposition (2.4.3.1) can also be interpreted as a primitive decomposition of the graded vector space C under the (nilpotent) operator N:

$$C_r^{r+q} = N^p PC_{r+2q}, \quad PC_{r+2q} = \mathrm{Ker}\ N^{r+2q+1} \subset C_{r+2q},$$

where PC_j is the primitive part Ker $N^{j+1} \subset C_j$. The projectivity hypothesis on f allows one to construct a Lefschetz operator $L : C_i \to C_{i+2}(1)$, commuting with N and d, and a pairing $<,>: C \times C \to \mathbf{Q}(-d)$ $(d = \dim Y)$, having certain positivity properties, and for which d, L, N are (up to sign) derivations. M. Saito formally deduces from this that the cohomology $H^r(C)$ of the complex $(C, d = d' + d'')$ admits a primitive decomposition analogous to (2.4.3.1), and thus in particular that $N^r : H^{-r}(C) \simeq H^r(C)$.

He also proves, from there, the following degeneration theorem:

[1]See however 2.4.7 (b)

THEOREM 2.4.4. *Under the hypotheses of 2.3.4, the vanishing cycles spectral sequence*

$$E_2^{pq} = H^p(Y, R^q\Psi(\mathbf{Q})) \Rightarrow H^{p+q}(\tilde{X}^*, \mathbf{Q}) \quad (= H^{p+q}(X_t, \mathbf{Q}), t \in S^*) \tag{2.4.4.1}$$

degenerates at E_3, and the limit filtration is defined by the kernels of the iterates of $N : F^{n-p}H^n(\tilde{X}^*, \mathbf{Q}) =$ Ker N^{p+1}.

Up to a renumbering, this spectral sequence coincides with the one defined by the canonical filtration $\tau_{\leq i}$ of $R\Psi(\mathbf{Q})$:

$$E_1^{-k,n+k} = H^n(Y, \mathrm{gr}_k^\tau R\Psi(\mathbf{Q})) = H^n(Y, R^k\Psi(\mathbf{Q})) \Rightarrow H^n(\tilde{X}^*, \mathbf{Q}). \tag{2.4.4.2}$$

It results easily from 2.3.2.1 and (2.3.2.6) that the quasi-isomorphism $\omega_Y^\bullet \to sA$ deduced from (2.3.2.11) defines filtered quasi-isomorphisms

$$(\omega_Y^\bullet, \tau_{\leq \cdot}) \to (sA, \tau_{\leq \cdot}^{hor}) \to (sA, K),$$

where $\tau_{\leq r}^{hor}$ (resp. K_r) is the filtration obtained by applying $\tau_{\leq r}$ (resp. W_{r+q+1}) to the q-th row of A. But we have

$$K_rA = \mathrm{Ker}\ N^{r+1} : A \to A, \tag{2.4.4.3}$$

which explains (but does not prove) the second assertion of 2.4.4. This had been verified, prior to, by Zucker.

A particular case of 2.4.4 is the **invariant cycle theorem** (cf. and [2, 6.2.9]):

COROLLARY 2.4.5. *Under the hypotheses of 2.4.4, the sequence*

$$H^n(Y, \mathbf{Q}) \xrightarrow{sp} H^n(\tilde{X}^*, \mathbf{Q}) \xrightarrow{T-1} H^n(\tilde{X}^*, \mathbf{Q}),$$

where sp is the specialization morphism, is exact for all n.

(The specialization map is the lateral homomorphism $E_2^{0,n} \to E_\infty^{0,n} \subset H^n$ of the spectral sequence (2.4.4.1).)

Example 2.4.6: If $d = 1$ (i.e. Y is a curve), the spectral sequence (2.4.4.1) reduces to the exact specialization sequence defined by the triangle $\mathbf{Q} \to R\Psi(\mathbf{Q}) \to R\Phi(\mathbf{Q})$ (SGA 7 XIV 1.3),

$$(*) \quad 0 \to H^1(Y, \mathbf{Q}) \to H^1(\tilde{X}^*, \mathbf{Q}) \to \bigoplus_{x \in Y^{(2)}} R^1\Psi_x(\mathbf{Q}) \to H^2(Y, \mathbf{Q}) \to H^2(\tilde{X}^*, \mathbf{Q}) \to 0 \tag{1}$$

($Y^{(2)}$ is the set of double points of Y, and $R^1\Psi_x(\mathbf{Q}) \simeq \mathbf{Q}$). In dual bases $(\delta_x), (\delta_x^\vee)$ ($x \in Y^{(2)}$) of $H^0(Y^{(2)}, \mathbf{Q}) \simeq \bigoplus R^1\Psi_x(\mathbf{Q})$ and $H^0(Y^{(2)}, \mathbf{Q})(-1) \simeq \bigoplus H_c^2(Y, R\Psi(\mathbf{Q}))(-1)$, $N = -(1/2\pi i)(T-1)$ is then given by $\delta_x \mapsto \delta_x^\vee$ (Picard-Lefschetz formula (SGA 7 XIV 3.2.11)). So, if $V = \bigoplus R^1\Psi_x(\mathbf{Q})$, $V^\vee = H_c^2(Y, R\Psi(\mathbf{Q}))(-1)$, $N : V \to V^\vee$ corresponds to a positive definite quadratic form on V. From the weight spectral sequence, we have

$$\mathrm{gr}_0^W H^1(\tilde{X}^*, \mathbf{Q}) = \mathrm{Ker}(V \to H^2(Y, \mathbf{Q}))$$

$$\mathrm{gr}_2^W H^1(\tilde{X}^*, \mathbf{Q})(-1) = \mathrm{Coker}(H^0(Y^{(1)}, \mathbf{Q})(-1) \to V^\vee),$$

and the fact that N induces an isomorphism from gr_2^W to $\mathrm{gr}_0^W(-1)$ (2.4.2) comes from the fact that the restriction of a positive definite form to a subspace is still positive definite. As for the invariant cycle theorem 2.4.5, it results from the exact sequence (*) and the factorization of N as

$$H^1(\tilde{X}^*, \mathbf{Q}) \to \bigoplus_x R^1\Psi_x(\mathbf{Q}) \xrightarrow{\delta_x'} \bigoplus_x \mathbf{Q}_x \to H_c^2(Y, \mathbf{Q})(-1) \xrightarrow{\delta_x} H^1(\tilde{X}^*, \mathbf{Q})(-1).$$

For variations on this, see Grothendieck's exposé (SGA 7 IX §12), and.

REMARKS 2.4.7. (a) By the theory of M. Saito (,), $R\Psi(\mathbf{Q})[d] = \mathcal{F}$ is a perverse, self-dual sheaf, and $N : \mathcal{F} \to \mathcal{F}(-1)$ is a nilpotent homomorphism. The perversity of $\mathcal{F}$ also results from that of the $\mathrm{gr}_r^W R\Psi(\mathbf{Q})[d]$, a consequence of (2.3.8). Moreover, the filtration W of sA is (up to $[d]$) a filtration of $\mathcal{F}$ in the category of perverse sheaves, and property 2.3.4 (b) shows that it is the monodromy filtration associated with N. On the other hand, the filtration K of sA defined by (2.4.4.3) is also (up to $[d]$) a filtration in the category of perverse sheaves, as is the filtration I of sA defined by the second degree, or, what amounts to the same thing, by the images of the iterates of N:

$$I^k sA = \bigoplus_{q \geq k} A^{p,q} = \mathrm{Im}\ N^k : sA \to sA \tag{2.4.7.1}$$

(this results from the perversity of $\mathrm{gr}_W \mathrm{gr}_K$ and $\mathrm{gr}_W \mathrm{gr}_I$, via (2.3.8)). We thus have

$$K_i sA[d] = \mathrm{Ker}\ N^{i+1} : \mathcal{F} \to \mathcal{F}, \quad I^k sA[d] = \mathrm{Im}\ N^k : \mathcal{F} \to \mathcal{F}$$

(kernels and images in the category of perverse sheaves), and

$$W = K * I,$$

with the notation of [28, 2.3], cf. (1.5.5). In parallel to the spectral sequence (2.4.4.2), which is associated with the filtration K, one can consider the spectral sequence

$${}_I E_1^{k,n-k} = H^n(Y, \mathrm{gr}_k^I R\Psi(\mathbf{Q})) \Rightarrow H^n(\tilde{X}^*, \mathbf{Q}), \tag{2.4.7.2}$$

associated with the filtration I, where

$$\mathrm{gr}_k^I R\Psi(\mathbf{Q}) = (\tau_{\geq k+1} Rj_* \mathbf{Q}(k+1)).$$

In parallel to 2.4.4, M. Saito shows that (2.4.7.2) degenerates at E_2 and that the limit filtration is the filtration $I^k = \mathrm{Im}\ N^k$. He observes moreover that, by the self-duality of $\mathcal{F}$, I and K correspond, which allows to put in duality the spectral sequences (2.4.4.2) and (2.4.7.2).

(b) The results stated in this section under the hypotheses of 2.3.4 are in fact valid under much more general hypotheses (it is sufficient to assume f proper and X bimeromorphically equivalent to a Kähler manifold).

3. Semi-stable reduction: positive or mixed characteristic case.

3.1. We resume the notations of 1.1. We set $S = \mathrm{Spec}\ R$, $s = \mathrm{Spec}\ k$, $\bar{s} = \mathrm{Spec}\ \bar{k}$, $\eta = \mathrm{Spec}\ K$, $\bar{\eta} = \mathrm{Spec}\ \bar{K}$, $G = \mathrm{Gal}(\bar{K}/K)$. We assume $p > 1$. We denote by Λ a torsion ring where p is invertible, or a finite extension of $\mathbf{Z}_l$ or $\mathbf{Q}_l$, or even $\bar{\mathbf{Q}}_l$. Let $f : X \to S$ be proper. The vanishing cycles spectral sequence (SGA 7 I 2.2.3 and XIII §3)

$$E_2^{ij} = H^i(X_{\bar{s}}, R^j\Psi(\Lambda)) \Rightarrow H^{i+j}(X_{\bar{\eta}}, \Lambda) \tag{3.1.1}$$

is G-equivariant. When one knows how to calculate the vanishing cycle sheaves $R^j\Psi(\Lambda)$, with the action of G with which they are endowed, it provides information on the local monodromy representation (cf. (1.3.1))

$$\rho : G \to \mathrm{GL}\ H^*(X_{\bar{\eta}}, \Lambda).$$

We will assume in what follows that f is **semi-stable**, by which we mean that, locally for the étale topology, X is S-isomorphic to $S[t_1, \ldots, t_n]/(t_1 \ldots t_r - \pi)$, where π is a uniformizer of R. The generic fiber X_η is then smooth, X is regular, and the special fiber $Y = X_s$ is a normal crossing divisor in X. We will further assume that Y is (globally) a sum of smooth divisors Y_i $(1 \leq i \leq h)$. One can then, as in 2.1.5, explicitly calculate the sheaves $R^j\Psi(\Lambda)$. Before stating the results, let us fix some notations. Let us consider the commutative diagram

$$\begin{array}{ccccccccccc}
\bar{Y} & \xrightarrow{\bar{i}} & \bar{X} & \xrightarrow{\bar{j}} & X_{\bar{\eta}} & & & \bar{s} & \to \bar{S} \to & \bar{\eta} \\
\downarrow & & \downarrow & & \downarrow & & & \downarrow & \downarrow & \downarrow \\
\tilde{Y} & \xrightarrow{\tilde{i}} & \tilde{X} & \xrightarrow{\tilde{j}} & X_{\tilde{\eta}} & \text{above} & & s & \to S \to & \tilde{\eta} \\
\downarrow & & \downarrow & & \downarrow & & & \downarrow & \downarrow & \downarrow \\
Y & \xrightarrow{i} & X & \xrightarrow{j} & X_\eta & & & s & \to S \to & \eta,
\end{array}$$

where $\tilde{\eta}$ is the spectrum of the maximal unramified extension of K, $\tilde{S}$ (resp. $\bar{S}$) the normalization of S in $\tilde{\eta}$ (resp. $\bar{\eta}$), i, j are the inclusions, and $\bar{i}$ (resp. $\tilde{i}$), $\bar{j}$ (resp. $\tilde{j}$) are deduced from them by extension of scalars to $\bar{S}$ (resp. $\tilde{S}$). The complex $R\Psi(\Lambda)$ is defined by

$$R\Psi(\Lambda) = \bar{i}^* R\bar{j}_* \Lambda. \tag{3.1.2}$$

It is an object of $D(Y \times_s \bar{\eta}, \Lambda)$, with the notation of (SGA 7 XIII) ("derived category of Λ-modules on Y endowed with a G-action compatible with that on $Y_{\bar{s}}$"; cf., for the case where Λ is not of torsion). By definition, we thus have

$$\tilde{i}^* R\tilde{j}_* \Lambda = R\Gamma(G, R\Psi(\Lambda)), \quad \bar{i}^* R\bar{j}_* \Lambda = R\Gamma(I, R\Psi(\Lambda)). \tag{3.1.3}$$

On the other hand, for $E \subset [1, h]$, we will set $Y_E = \cap_{i \in E} Y_i$, and we will denote, as in (2.3.8)

$$Y^{(m)} = \coprod_{\mathrm{card}(E) = m} Y_E,$$

and $a_m : Y^{(m)} \to Y$ the projection.

THEOREM 3.2. *(a) The purity conjecture of Grothendieck (SGA 5 I) is verified for the inclusions of the Y_i in X: one has*

$$R^i a_1^* \Lambda = 0 \text{ for } i \neq 2,$$

and a canonical isomorphism, given by the classes of the Y_i (SGA 4 1/2 Cycle)

$$R^2 a_1^! \Lambda = a_{1*}(-1).$$

(b) For all $q \geq 1$, one has canonical isomorphisms

$$\bar{i}^* \Lambda^q R^1 \bar{j}_* \Lambda = \bar{i}^* R^q \bar{j}_* \Lambda = a_{q*} R^{q+1} a_q^! \Lambda = a_{q*}(-q).$$

(c) (i) We have $R^0\Psi(\Lambda) = \Lambda$. (ii) The map $\bar{i}^* R^1 \bar{j}_* \Lambda \to R^1\Psi(\Lambda)$ (deduced from (3.1.3)) induces an isomorphism

$$(\bigoplus \Lambda_{Y_i} / \Lambda \text{ diagonal})(-1) \xrightarrow{\sim} R^1\Psi(\Lambda).$$

(iii) We have

$$\Lambda^q R^1 \Psi(\Lambda) \xrightarrow{\sim} R^q \Psi(\Lambda)$$

for all $q \geq 1$. If $C^\bullet$ denotes the (acyclic) augmented Čech complex defined by $\bar{a}_1 : Y_{\bar{s}}^{(1)} \to Y_{\bar{s}}$,

$$C^\bullet = (0 \to \Lambda_{Y_{\bar{s}}} \to \bar{a}_{1*}\Lambda \to \bar{a}_{2*}\Lambda \to \cdots \to \bar{a}_{q*}\Lambda \to \dots)$$

(where $\Lambda_{Y_{\bar{s}}}$ is in degree -1), we have (for $q \geq 1$)

$$R^q\Psi(\Lambda)(q) = \mathrm{Coker}(C^{q-2} \to C^{q-1}) = \mathrm{Ker}(C^q \to C^{q+1})$$

(compare with (2.1.5.3)). The isomorphisms of (i), (ii), (iii) are isomorphisms of G-Λ-sheaves on Y. In particular, the inertia I acts trivially on $R^q\Psi(\Lambda)$ for all q.

The last assertion of (c) implies, via (3.1.1):

COROLLARY 3.3. *For all $g \in I$, $(\rho(g) - 1)^{i+1} = 0$ on $H^i(X_{\bar{\eta}}, \Lambda)$.*

COROLLARY 3.4. *Let X_η be a proper and smooth scheme over η having potentially semi-stable reduction (i.e. such that there exists a finite extension η' of η such that $X_{\eta'}$ admits a proper and semi-stable model over the normalization S' of S in η'). Then, there exists an open subgroup I' of I such that $(\rho(g) - 1)^{i+1} = 0$ on $H^i(X_{\bar{\eta}}, \Lambda)$, for all $g \in I'$.*

According to the semi-stable reduction theorem („), the hypothesis of 3.4 is verified if $\dim X = 1$, and, if one is optimistic, one can conjecture that it always is.

3.5. Since I acts trivially on the $R^q\Psi(\Lambda)$, a fortiori the wild inertia subgroup P also does (cf. 1.1), thus

$$R^q\Psi_t(\Lambda) = R^q\Psi(\Lambda), \tag{3.5.1}$$

where $R\Psi_t(\Lambda)$ is the sheaf of tame vanishing cycles defined in (SGA 7 I 2.7), i.e. $R\Psi_t(\Lambda) = R\Psi(\Lambda)^P$. By means of 3.2 (a) and (3.5.1), the calculation of the geometric fibers of $R^q\Psi_t(\Lambda)$ is carried out in (SGA 7 I 3.3), and it is easy to deduce from it, as in 2.1.5, assertions (b) and (c). The difficulty is to prove (a) and (3.5.1). This is what is done by Rapoport-Zink, by a method inspired by that used by Deligne in (SGA 4 1/2 Th. Finitude). Assertion 3.2 (a) also follows, at least for $\Lambda = \mathbf{Q}_l$ and with some additional hypotheses on S, from the result of Thomason 4.18 in [R.W. Thomason, Algebraic K-theory and étale cohomology, Ann. Sci. ENS, 4th series, t. 13 (1985), 437-552, and Erratum, Ann. Sci. ENS, 4th series, t. 22 (1989), 675-677]; see also [R.W. Thomason, Absolute cohomological purity, Bull. SMF 112 (1984), 397-406].

3.6. Rapoport and Zink (loc. cit.) also construct, in this situation, an analogue of Steenbrink's complex (2.3.2.7) and the weight spectral sequence (2.3.6).

Let us briefly explain their construction. We will assume, to simplify, that k is algebraically closed (so $s = \bar{s} = \tilde{s}$ in the diagram of 3.1, $S = \tilde{S}$, $X = \tilde{X}$). Let P_l be defined by the exact sequence (cf. 1.1)

$$1 \to P_l \to I \xrightarrow{t_l} \mathbf{Z}_l(1) \to 1.$$

As P_l is of order prime to l and I, hence P_l, acts trivially on the $R^q\Psi(\Lambda)$, we have

$$R\Psi(\Lambda) = R\Gamma(P_l, R\Psi(\Lambda)). \tag{3.6.1}$$

This allows one to consider $R\Psi(\Lambda)$ as an object of $D^b_c(Y, \Lambda[\mathbf{Z}_l(1)])$ (with trivial action of $\mathbf{Z}_l(1)$ on its cohomology sheaves). Let K be a complex of $\Lambda[\mathbf{Z}_l(1)]$-modules on Y, bounded and with degrees ≥ 0, representing $R\Psi(\Lambda)$. According to (3.1.3) and (3.6.1), we have

$$\bar{i}^* R\bar{j}_*\Lambda = R\Gamma(\mathbf{Z}_l(1), R\Psi(\Lambda)). \tag{3.6.2}$$

Thus, if T denotes a topological generator of $\mathbf{Z}_l(1)$, $\bar{i}^* R\bar{j}_*\Lambda$ is represented by the bicomplex $K \xrightarrow{T-1} K$, with K being concentrated on lines of degree 0 and 1 (cf. [8, 10.7]). Let us denote

$$L = s(K \xrightarrow{T-1} K).$$

Let

$$\theta : \Lambda_\eta \to \Lambda_\eta(1)$$

be the opposite of the fundamental class of the closed point s in S (SGA 4 1/2 Cycle 2.1), considered as an element of

$$\mathrm{Hom}_{D(\eta,\Lambda)}(\Lambda_\eta, \Lambda_\eta(1)) = H^1(\eta, \Lambda(1)) = i^* R^1 j_*\Lambda(1).$$

(The last equality comes from the fact that k has been assumed algebraically closed). It is the class of torsors of the l^n-th roots of a uniformizer of R. As an element of $H^1(I, \Lambda(1))$ (where I acts trivially on $\Lambda(1)$), it is thus the composite of the projection $I \to \mathbf{Z}_l(1)$, and of the natural map $\mathbf{Z}_l(1) = \mathbf{Z}_l(1)(k) = \mathbf{Z}_l(1)(K) \to \Lambda(1) = \Lambda \otimes \mathbf{Z}_l(1)(K)$. It is also the class of the extension of sheaves on η corresponding to the trivial exact sequence of abelian groups

$$0 \to \Lambda(1) \to \Lambda(1) \oplus \Lambda \to \Lambda \to 0$$

endowed with the trivial action of $\mathbf{Z}_l(1)$ on the extremes, and the action of $\mathbf{Z}_l(1)$ on the central term given by $g.(x, y) = (x + gy, y)$. By functoriality, θ provides a map from $D(Y, \Lambda)$

$$\theta : \bar{i}^* R\bar{j}_*\Lambda \to \bar{i}^* R\bar{j}_*\Lambda(1) \tag{3.6.3}$$

(analogue to the one defined by $\theta\wedge$ in 2.3.2.1). One verifies, using the above description of θ, that (3.6.3) is represented, at the level of complexes, by the morphism of simple complexes associated to the morphism of double complexes

$$\begin{array}{ccc} K(1) & \xrightarrow{T-1} & K(1) \\ \uparrow 1 \otimes T & & \uparrow 1 \otimes T \\ K & \xrightarrow{T-1} & K \end{array}$$

(in the vertical arrow $1 \otimes T$, T is considered as the morphism $\mathbf{Z}_l \to \mathbf{Z}_l(1)$, $a \mapsto T^a$). Let us still denote

$$\theta : L \to L(1)$$

the morphism (of complexes of Λ-modules on Y) thus defined. We then obtain an exact sequence of complexes

$$\cdots \to L(-1)[-1] \xrightarrow{\theta} L \xrightarrow{\theta} L(1) \to \dots \tag{3.6.4}$$

As T acts trivially on the $H^i K$, we also see that, for all $i \in \mathbf{Z}$, the sequence deduced from (3.6.4) by application of H^i,

$$\cdots \to H^{i-1}L(-1) \xrightarrow{\theta} H^i L \xrightarrow{\theta} H^{i+1}L(1) \to \dots, \tag{3.6.5}$$

is exact. This is the analogue of 2.3.2.1. One can still observe that (3.6.5) is obtained by patching together the short exact sequences

$$0 \to H^1(\mathbf{Z}_l(1), H^{i-1}K) \to H^i L \to H^0(\mathbf{Z}_l(1), H^i K) \to 0, \tag{3.6.6}$$

analogues to (2.1.5.6), which are rewritten

$$0 \to R^{i-1}\Psi(\Lambda)(-1) \to \bar{i}^* R^i j_* \Lambda \to R^i \Psi(\Lambda) \to 0$$

(cf. (2.1.5.7)); these sequences are provided by the spectral sequence

$$E_2^{rs} = H^r(\mathbf{Z}_l(1), H^s K) \Rightarrow H^{r+s}L.$$

Denoting by M the bicomplex defined by (3.6.4), we define the bicomplex

$$A = (\sigma^{hor}_{\geq 1} \tau^{ver}_{\geq 0} M), \tag{3.6.7}$$

with the notations of (2.3.2.2) (this is the analogue of M''). In other words, A is the bicomplex

$$\begin{array}{ccccccc} (\tau_{\geq 1}L(1)) & \xrightarrow{0} & (\tau_{\geq 2}L(2)) & \xrightarrow{0} & \dots & \xrightarrow{0} & (\tau_{\geq j}L(j))[j] \quad \dots \\ 0 & & 1 & & & & j-1 \end{array}$$

(where $0, 1, \dots, j-1, \dots$ indicate the row number). We construct, as in (2.3.2.5), an isomorphism (in $D(Y, \Lambda[\mathbf{Z}_l(1)])$)

$$sA \simeq L (= R\Psi(\Lambda)). \tag{3.6.8}$$

From there, it is easy to paraphrase, in this context, the definitions given in 2.3.2 of a filtration by weight and of a lifting of the monodromy at the level of A. Let

$$0 \subset \cdots \subset W_r A \subset W_{r+1} A \subset \cdots \subset A \tag{3.6.9}$$

be the (finite, increasing) filtration of A such that $W_r A$ is obtained by applying $\tau_{\leq r+q}$ to the q-th row of A, and let us denote $W.sA$ the induced filtration on the associated simple complex. It follows from 3.2 (a) that one has, in $D(Y, \Lambda[\mathbf{Z}_l(1)])$

$$\mathrm{gr}^W_r sA = \mathrm{gr}^W_r sB, \tag{3.6.10}$$

where B is the double complex, with null differentials,

$$\begin{array}{ccccc} & & & & a_{d+1*}\Lambda \\ & & a_{d-1*}\Lambda & & a_{d*}\Lambda(-1) \\ & \cdot^{\cdot^{\cdot}} & & & \vdots \\ a_{1*}\Lambda & a_{2*}\Lambda(-1) & & \dots & a_{d+1*}\Lambda(-d) \end{array}$$

(d denoting the dimension of Y), endowed with the filtration W defined by the same formula as for A (compare with (2.3.8.1)). Let us call **weight spectral sequence** the spectral sequence

$${}_W E_1^{ij} = H^{i+j}(Y, \mathrm{gr}^W_j R\Psi(\Lambda)) \Rightarrow H^*(X_{\bar{\eta}}, \Lambda) \tag{3.6.11}$$

defined, thanks to (3.6.8), by the filtration W of sA. According to (3.6.10), its initial term is rewritten

$${}_W E_1^{-r,n+r} = \bigoplus_{\substack{q \geq 0 \\ r+q \geq 0}} H^{n-r-2q}(Y^{(r+1+2q)}, \Lambda)(-r-q), \tag{3.6.12}$$

and it is still possible to describe, as in (2.3.9), the differential d_1 as $d_1' + d_1''$, where d_1' (resp. d_1'') is an alternating sum of Gysin (resp. restriction) homomorphisms.

Rapoport and Zink show on the other hand that, if $\tilde{T}$ denotes the generator of $\mathbf{Z}_l(-1)$ dual to T (i.e. such that $T\tilde{T} = 1$), then the homomorphism of $D(Y, \Lambda[\mathbf{Z}_l(1)])$

$$(T-1) \otimes \tilde{T} : R\Psi(\Lambda) \to R\Psi(\Lambda)(-1)$$

is realized, after passing to the associated simple complexes, by the homomorphism of bicomplexes

$$\nu : A \to A(-1)[-1, 1]$$

given by $(-1)^{i+j+1}$ times the canonical projection of $A^{i,j}$ on $A^{i-1,j+1}$. In particular, $(T-1)\otimes\tilde{T}$ underlies a homomorphism of the filtered derived category sending W_r into $W_{r-2}(-1)$, and, for $r \geq 0$, $((T-1)\otimes\tilde{T})^r$ induces an isomorphism

$$((T-1) \otimes \tilde{T})^r : \mathrm{gr}_r^W R\Psi(\Lambda) \xrightarrow{\sim} \mathrm{gr}_{-r}^W R\Psi(\Lambda)(-r) \tag{3.6.13}$$

(corresponding, via (3.6.10), to the application T^r from $\mathrm{gr}_r^W B$ into $\mathrm{gr}_{-r}^W B$). Moreover, as one can choose K with degrees in $[0, d]$, hence L with degrees in $[0, d+1]$, one has

$$((T-1) \otimes \tilde{T})^{d+1} = 0 : R\Psi(\Lambda) \to R\Psi(\Lambda)(-d-1). \tag{3.6.14}$$

3.7. By the same considerations as in 2.4.7, the structure of filtered complex[2]on $R\Psi(\mathbf{Q}_l)$ defined in 3.6 from (sA, W) can be intrinsically reconstructed as follows. The formula (3.6.10) shows that $\mathrm{gr}_r^W sA[d]$ (for $\Lambda = \mathbf{Q}_l$) is perverse. The complex $R\Psi(\mathbf{Q}_l)[d]$ is thus perverse[3], and the filtration W of $R\Psi(\mathbf{Q}_l)[d]$ is a filtration by perverse sub-sheaves. The homomorphism $(T-1) \otimes \tilde{T} : R\Psi(\mathbf{Q}_l)[d] \to R\Psi(\mathbf{Q}_l)(-1)[d]$ sends W_r into $W_{r-2}(-1)$, and is nilpotent. The homomorphism

$$N = \log T \otimes \tilde{T} : R\Psi(\mathbf{Q}_l)[d] \to R\Psi(\mathbf{Q}_l)(-1)[d] \tag{3.7.1}$$

is thus defined, and does not depend on the choice of T. We have

$$N(W_r) \subset W_{r-2}(-1), \quad N^{d+1} = 0,$$

and, for $r \geq 0$,

$$N^r : \mathrm{gr}_r^W R\Psi(\mathbf{Q}_l)[d] \xrightarrow{\sim} \mathrm{gr}_{-r}^W R\Psi(\mathbf{Q}_l)(-r)[d]. \tag{3.7.2}$$

The filtration W is thus the monodromy filtration of the perverse sheaf $R\Psi(\mathbf{Q}_l)[d]$ associated with the operator N. The knowledge of this filtration (in the category of perverse sheaves) is sufficient to define the weight spectral sequence (3.6.11).

3.8. Let us resume the hypotheses of 3.1: S Henselian, f proper and semi-stable of relative dimension d, with special fiber a sum of smooth divisors (we no longer assume k is algebraically closed). The complex $R\Psi(\mathbf{Q}_l)[d]$ (on $Y_{\bar{s}}$) is a perverse sheaf, and the homomorphism $N = \log T \otimes \tilde{T}$,

$$N : R\Psi(\mathbf{Q}_l)[d] \to R\Psi(\mathbf{Q}_l)(-1)[d], \tag{3.8.1}$$

is defined, and commutes with the action of $G = \mathrm{Gal}(\bar{K}/K)$. It is nilpotent ($N^{d+1} = 0$), thus defines a monodromy filtration W, characterized by $N(W_r) \subset W_{r-2}(-1)$ and $N^r : \mathrm{gr}_r^W \xrightarrow{\sim} \mathrm{gr}_{-r}^W(-r)$ for $r \geq 0$. From this we deduce a "weight spectral sequence" (analogue to (3.6.11)),

$$_W E_1^{ij} = H^{i+j}(Y_{\bar{s}}, \mathrm{gr}_j^W R\Psi(\mathbf{Q}_l)) \Rightarrow H^*(X_{\bar{\eta}}, \mathbf{Q}_l), \tag{3.8.2}$$

equivariant under G. Moreover, the isomorphism $\mathrm{gr}_j^W R\Psi(\mathbf{Q}_l) = \mathrm{gr}_j^W B$ of (3.6.10) is functorial (with respect to isomorphisms of strictly local traits), thus the term E_1 of (3.8.2) is rewritten

$$_W E_1^{-r,n+r} = \bigoplus_{\substack{q \geq 0 \\ r+q \geq 0}} H^{n-r-2q}(Y_{\bar{s}}^{(r+1+2q)}, \mathbf{Q}_l)(-r-q), \tag{3.8.3}$$

the isomorphism being equivariant under G. Finally, N acts on the spectral sequence (3.8.2) (in a way compatible with G), and, at the E_1 level, induces, for $r \geq 0$, an isomorphism

$$N^r :_W E_1^{-r,n+r} \xrightarrow{\sim}_W E_1^{r,n-r}(-r). \tag{3.8.4}$$

The results of the complex case suggest the following conjecture:

[3]or, more exactly, quasi-filtered, in the sense of [20, 5.2.17]

[3](special case of 4.5)

CONJECTURE 3.9. *Under the hypotheses of 3.8:*

(a) *The spectral sequence (3.8.2) degenerates at E_2.*

(b) *For all $r \geq 0$ and all n, N^r induces an isomorphism (G-equivariant)*

$$N^r : \mathrm{gr}^W_{n+r} H^n(X_{\bar{\eta}}, \mathbf{Q}_l) \xrightarrow{\sim} \mathrm{gr}^W_{n-r} H^n(X_{\bar{\eta}}, \mathbf{Q}_l)(-r),$$

where the filtration W on $H^*(X_{\bar{\eta}}, \mathbf{Q}_l)$ is the limit filtration of (3.8.2).

This conjecture is sometimes called the **monodromy-weight conjecture**. Suppose k is finite. Then (a) is a consequence of the Weil conjectures: for $s \geq 2$, the source and target of d_s^{ij} are of different weights, so $d_s^{ij} = 0$. For all $s \geq 1$, ${}_W E_s^{-r,n+r}$ is pure of weight $n + r$, thus also $\mathrm{gr}^W_{n+r} H^n(X_{\bar{\eta}}, \mathbf{Q}_l) ={}_W E_\infty^{-r,n+r} ={}_W E_2^{-r,n+r}$. The filtration $W_{\cdot - n}$ of $H^n(X_{\bar{\eta}}, \mathbf{Q}_l)$ is thus the weight filtration, in the sense of [11, 1.7.5]. Let M be the monodromy filtration (centered at 0) of $H^n(X_{\bar{\eta}}, \mathbf{Q}_l)$, characterized by $NM_i(1) \subset M_{i-2}$ and $N^i : \mathrm{gr}^M_i \xrightarrow{\sim} \mathrm{gr}^M_{-i}(-i)$. Part (b) is thus equivalent to:

(b') $M_\cdot = W_{\cdot - n}$, i.e. (by the unicity of the weight filtration (loc. cit.)), gr^M_i is pure of weight i.

Taking (a) into account, (b) is also equivalent to:

(b") For all $r \geq 0$ and all n, N^r induces an isomorphism ${}_W E_2^{-r,n+r} \xrightarrow{\sim}{}_W E_2^{r,n-r}(-r)$.

If S is the henselization at a closed point of a smooth curve over a finite field, (b) is verified according to Deligne [11, 1.8.4]: $H^n(X_{\bar{\eta}}, \mathbf{Q}_l)$ is indeed the fiber at $\tilde{\eta}$ of a pure sheaf of weight n according to the Weil conjectures. In unequal characteristic (k always assumed finite), Rapoport and Zink verified (b) for $\dim Y \leq 2$ [19, 2.12, 2.13]; the case of relative dimension 1 is already treated by Grothendieck in (SGA 7 IX 12.5), as a consequence of the Picard-Lefschetz formula (modulo the verification of some compatibilities, cf.) (one can also proceed like Deligne in (SGA 7 I 6)). One can also consider (b") as a sort of generalization of the Picard-Lefschetz formula to the semi-stable case of higher dimension.

The proof of 2.4.4 given by M. Saito is formal, based on the fact that N^r induces an isomorphism from ${}_W E_1^{-r,n+r}$ to ${}_W E_1^{r,n-r}$ (and on the interpretation of the vanishing cycles spectral sequence as coming (up to a renumbering) from the filtration of $R\Psi$ by the kernels of the iterates of N). In the situation of 3.8, we have the same interpretation: this results from the realization of $(T-1) \otimes \tilde{T}$ at the level of the Rapoport-Zink-Steenbrink complex explained in 3.6. Consequently, Saito's arguments provide the following result:

PROPOSITION 3.10. *If X/S verifies 3.9 (a) and (b), the vanishing cycles spectral sequence (3.1.1) (for $\Lambda = \mathbf{Q}_l$),*

$$E_2^{ij} = H^i(X_{\bar{s}}, R^j\Psi(\mathbf{Q}_l)) \Rightarrow H^{i+j}(X_{\bar{\eta}}, \mathbf{Q}_l),$$

degenerates at E_3, and the limit filtration is defined by the kernels of the iterates of N : $F^{n-p}H^n(X_{\bar{\eta}}, \mathbf{Q}_l) =$ Ker N^{p+1}.

In particular ("local invariant cycle theorem"):

COROLLARY 3.11. *If X/S verifies 3.9 (a) and (b), the sequence*

$$H^n(X_{\bar{s}}, \mathbf{Q}_l) \xrightarrow{sp} H^n(X_{\bar{\eta}}, \mathbf{Q}_l) \xrightarrow{N} H^n(X_{\bar{\eta}}, \mathbf{Q}_l)$$

is exact for all n.

(As in 2.4.5, *sp* denotes the specialization morphism, defined as the lateral homomorphism of (3.1.1), or, more simply, as the homomorphism induced by $\mathbf{Q}_l = R^0\Psi(\mathbf{Q}_l) \to R\Psi(\mathbf{Q}_l)$).

In equal characteristic p, more precisely if S is the henselization of a smooth curve over a field of characteristic p, so that we have 3.9 (a) and (b), a direct proof of 3.11 is given by Deligne in [11, 3.6] (under more general hypotheses: X essentially smooth over k and X_s smooth).

REMARKS 3.12. (a) As in 2.4.7 (a), the filtration I of the perverse sheaf $R\Psi(\mathbf{Q}_l)[d]$ defined by $I^r = \mathrm{Im} N^r$ gives rise to a spectral sequence

$${}_I E_1^{r,n-r} = H^n(X_{\bar{s}}, \mathrm{gr}^I_r R\Psi(\mathbf{Q}_l)) \Rightarrow H^n(X_{\bar{\eta}}, \mathbf{Q}_l), \tag{3.12.1}$$

where

$$\mathrm{gr}^I_r R\Psi(\mathbf{Q}_l) = (\tau_{\geq r+1} Rj_* \mathbf{Q}_l(r+1)).$$

One proves similarly that (3.12.1) degenerates at E_2 and defines on the limit the filtration $I^r = \mathrm{Im} N^r$.

(b) The perversity of $R\Psi(\mathbf{Q}_l)[d]$ holds more generally as soon as X is flat, of relative dimension d, and generically smooth (cf. 4.5). One can probably prove, under these sole hypotheses, that the monodromy of $R\Psi(\mathbf{Q}_l)$ is quasi-unipotent, and define a logarithm N of its unipotent part. Whence a spectral sequence of type (3.8.2) and a conjecture generalizing 3.9.

When the residue field of S is finite, the quasi-unipotence in question reduces to Grothendieck's theorem 1.2. We have in fact, more generally, the following result:

LEMMA 3.12.2. *Let S be as in 3.1, with k finite, Y/s of finite type, and $L \in D^b_c(Y \times_s \bar{\eta}, \Lambda)$, with the notations of 3.1 and (SGA 7 XIII) (one can thus interpret L as a complex of Λ-sheaves on $Y_{\bar{s}}$, with constructible cohomology, endowed with a $G = Gal(\bar{K}/K)$ action compatible with that of G on $Y_{\bar{s}}$, through $Gal(\bar{k}/k)$). There then exists an open subgroup I_1 of the inertia group I and an integer N such that, for all $g \in I_1$, $(g-1)^N$ is zero on L.*

The details of the verification are left to the reader: one can reduce successively to assuming L concentrated in a single degree (truncation), then smooth (recursion on the dimension of Y), then Y is the spectrum of a finite extension of k (taking the "fiber" of $L|_{Y_{\bar{s}}}$ at a Frobenius orbit), then $Y = s$ (direct image), in which case Grothendieck's theorem applies.

When S is the henselization at a closed point of a smooth curve over a finite field, one can say more. For X separated of finite type over S, and K a pure perverse sheaf on X, $R\Psi(K)$ is perverse (cf. 4.5), and mixed according to the fundamental results of. According to 3.12.2, one can define the monodromy filtration $M.$ of $R\Psi(K)$ (in the category of perverse sheaves): it coincides, according to a theorem of Gabber, with the weight filtration (cf. [J.-L. Brylinski, Cohomologie d'intersection et faisceaux pervers, Séminaire Bourbaki, 34e année, 1981/82, n° 585, Astérisque 92-93, 1982, 129-157, th. 3.2.9]). One of the ingredients is the Künneth formula for $R\Psi$ (4.7). (NB. In (loc. cit.), p. 144, l. 9, 10, 13 and p. 148, l. -4, it is necessary to replace the inertia group I by the subgroup P_l kernel of $t_l : I \to \mathbf{Z}_l(1)$.)

3.13. Let X_η be a separated scheme of finite type over η, and H one of the cohomology groups $H^n(X_{\bar{\eta}}, \mathbf{Q}_l)$, $H^n_c(X_{\bar{\eta}}, \mathbf{Q}_l)$. We know (1.4) that the monodromy representation $\rho : G \to \mathrm{GL}(H)$ is quasi-unipotent. Assume that k is the finite field $\mathbf{F}_q$, and let $F_q \in \mathrm{Gal}(\bar{k}/k)$ be the "geometric Frobenius", $x \mapsto x^{1/q}$. Let us recall the question of Serre-Tate [26, Appendix, Problem 2]:

Question 3.13.1. Let g be an element of the Weil group $W(\bar{K}/K)$ (i.e. an element of G with image F_q^m in $\mathrm{Gal}(\bar{k}/k)$, with $m \in \mathbf{Z}$). Is it true that the characteristic polynomial $\det(1 - gt, H) \in \mathbf{Q}_l[t]$ has coefficients in $\mathbf{Q}$, independent of l? If so, and if moreover X_η is proper and smooth, and $H = H^n$, is it true that the inverses of the roots of $\det(1 - gt, H)$ are of weight mw, with $0 \le w \le 2n$?

Assume that X_η is the generic fiber of X over S satisfying the hypotheses of 3.8, and let M be the monodromy filtration of $H^n(X_{\bar{\eta}}, \mathbf{Q}_l)$, centered at n (i.e. such that $N^r : \mathrm{gr}^M_r \xrightarrow{\sim} \mathrm{gr}^M_{-r}(-r)$). One can refine question 3.13.1 as follows:

Question 3.13.2. Let $g \in W(\bar{K}/K)$ with image F_q in $\mathrm{Gal}(\bar{k}/k)$. Is it true that $\det(1-gt, \mathrm{gr}^M_{n+r} H^n(X_{\bar{\eta}}, \mathbf{Q}_l)) \in \mathbf{Q}_l[t]$ has coefficients in $\mathbf{Z}$, independent of l, and that any inverse α of a root is of weight $n + r$?

These two questions have positive answers if X is proper and smooth over S (thanks to Deligne), or if $\dim X \le 1$, according to the semi-stable reduction theorem (SGA 7 IX 4.3), or if X comes, by localization, from a proper scheme on a smooth curve over a finite field, according to Deligne [8, 9.8], and [11, 1.8.4]. Outside of these cases, they remain open. Assume that 3.9 (b) is verified. Then, for any embedding i of $\mathbf{Q}_l$ in $\mathbf{C}$, any g as in 3.13.2 is of i-weight $n + r$, but we don't know if $\det(1 - gt, \mathrm{gr}^M_{n+r})$ has rational coefficients, cf. the discussion of problem 3.2.2 p. 55 in Katz.

4. Appendix: vanishing cycles and duality in étale cohomology.

The main result of this section (4.2) is well known, but does not appear, it seems, in the literature. The proof, modeled on (SGA 4 1/2 Th. finitude §3), was communicated to the editor by O. Gabber. A variant in Hodge theory was developed by M. Saito. It refines results of Brylinski. We also give a complement of the same nature (4.7), which was suggested to us by M. Rapoport.

4.1. Let $(S, s, \eta, \bar{s}, \tilde{s}, \tilde{\eta}, G)$ be as in 3.1. We denote by Λ a commutative noetherian ring, annihilated by an integer invertible on S. For X/S, we have the functor

$$R\Psi_\eta : D^+(X_\eta, \Lambda) \to D^+(X_s \times_s \eta, \Lambda), \quad M \mapsto i^* R j_* M$$

(where $D(X_s \times_s \eta, \Lambda)$ denotes, as in (SGA 7 XIII) the derived category of sheaves of Λ-modules on X_s endowed with a (continuous) action of G compatible with that of G on $X_{\bar{s}}$). When X is of finite type over S, $R\Psi_\eta$ sends, according to (SGA 4 1/2 Th. finitude §3), D_{ctf} into D_{ctf}, where D_{ctf} denotes the full subcategory formed by complexes with bounded, constructible cohomology and which are of finite tor-dimension (cf. SGA 5 III).

THEOREM 4.2. *Let X be separated and of finite type over S and let $M \in D_{ctf}(X_\eta, \Lambda)$. The map from $D_{ctf}(X_s \times_s \eta, \Lambda)$ defined in (4.3.6),*

$$(*) \quad R\Psi_\eta(DM) \to D(R\Psi_\eta(M)), \tag{2}$$

is an isomorphism (in the left (resp. right) member, $D(-)$ denotes the functor $RHom(-, a^! \Lambda)$ where a is the projection on η (resp. $s \times_s \eta = \eta$)).

(The method of (SGA 4 XVIII 3.1) allows to define $a^! : D^+(\eta, \Lambda) \to D^+(X_s \times_s \eta, \Lambda)$, and more generally $f^! : D^+(Y_s \times_s \eta, \Lambda) \to D^+(X_s \times_s \eta, \Lambda)$ for $f : X \to Y$ with X and Y separated of finite type over S. According to (SGA 4 1/2, loc. cit.), D sends D_{ctf} into D_{ctf}.)

4.3. Let us define the map (*). (a) For L, M, N in $D_{ctf}(X_\eta, \Lambda)$ and $u : L \overset{L}{\otimes} M \to N$, we have an associated map

$$R\Psi(L) \overset{L}{\otimes} R\Psi(M) \to R\Psi(N), \tag{4.3.1}$$

deduced from the canonical map

$$Rj_* L \overset{L}{\otimes} Rj_* M \to Rj_*(L \overset{L}{\otimes} M).$$

(Here, and in the following, we abbreviate $R\Psi_\eta$ as $R\Psi$.)

In particular, if $u : M \overset{L}{\otimes} DM \to a^! \Lambda$ is the canonical pairing, (4.3.1) gives a pairing

$$R\Psi(M) \overset{L}{\otimes} R\Psi(DM) \to R\Psi(a^! \Lambda), \tag{4.3.2}$$

whence a map

$$R\Psi(DM) \to R\mathrm{Hom}(R\Psi M, R\Psi(a^! \Lambda)). \tag{4.3.3}$$

(b) Let X and Y be separated of finite type over S and $f : X \to Y$ an S-morphism. For $K \in D^+(Y_\eta, \Lambda)$ we have a map (cf. (SGA 7 XIII 1.3.9))

$$R\Psi(f^! K) \to f_s^! R\Psi(K) \tag{4.3.4}$$

defined in one of the following equivalent ways:

(i) We have a canonical morphism (where the R are omitted, for brevity)

$$f_{1!} j_*(f_\eta^! K_\eta) \to j_* f_{\eta !}(f_\eta^! K_\eta)$$

(coming, by adjunction, from $j^* f_{s!} = f_{\eta !} j^*$ and from $j^* j_* \to \mathrm{Id}$). As (by base change for f_1), $i^* f_{s!} = f_{\eta s!} i^*$, we deduce a map

$$f_{s!} R\Psi(f_\eta^! K) \to R\Psi(f_{s!} f_\eta^! K),$$

whence, by composing with $f_{s!} f_\eta^! \to \mathrm{Id}$,

$$f_{s!} R\Psi(f_\eta^! K) \to R\Psi(K),$$

whence finally (4.3.4) is deduced by adjunction.

(ii) According to (SGA 4 XVIII (3.1.12.3)), we have

$$j_* f_{\eta *}^! K_\eta = f_s^! j_* K_\eta;$$

composing with the "base change map" (SGA 4 XVIII (3.1.14.2))

$$f_s^! i^* \to i^* f^!,$$

we obtain (4.3.4). The verification of the equivalence of (i) and (ii) is left as an exercise: use the language of fibered (or cofibered) categories from (SGA 4 XVII 2) (cf. also (SGA 5 III).)

(c) We have (trivially)

$$R\Psi(\Lambda_\eta) = \Lambda_s. \tag{4.3.5}$$

Applying (4.3.4) and (4.3.5) for $a : X \to S$, we find a map

$$R\Psi(a^! \Lambda_\eta) \to a_s^! \Lambda_s,$$

whence, by composition with (4.3.3), a map (from $D_{ctf}(X_s \times_s \eta, \Lambda)$)

$$R\Psi(DM) \to D(R\Psi(M)) \tag{4.3.6}$$

for $M \in D_{ctf}(X_\eta, \Lambda)$, which is the announced map (*).

By construction, (4.3.6) is compatible with proper direct images: for $f : X \to Y$ proper, we have $f_{s*}\Psi = \Psi f_*$ (SGA 7 XIII 2.1.7) and $f_{s*}D = Df_*$ (global duality) SGA XVIII), and the square which is deduced from it

$$\begin{array}{ccc} f_{s*}\Psi(DM) & \xrightarrow{(4.3.6)} & f_{s*}D\Psi M \\ \wr\downarrow & & \wr\downarrow \\ \Psi D f_* M & \xrightarrow{(4.3.6)} & D\Psi f_* M \end{array} \tag{4.3.7}$$

is commutative (we have omitted the R). The verification is left to the reader: as for the equivalence of (i) and (ii) above, it is convenient to use the language of (SGA 4 XVII 2).

Let us prove that (4.3.6) is an isomorphism. The proof is entirely parallel to that of the invariance of vanishing cycles under change of traits in (SGA 4 1/2 Th. finitude 3.7). We can assume S is strictly local. We proceed by induction on $\dim X_\eta$. We assume that (4.3.6) is an isomorphism for $\dim X_\eta < n$. For $\dim X_\eta = n$, we reduce to assuming X is projective, and we denote C the cone of (4.3.6). We verify conditions (A) and (B) of (loc. cit.), namely: (A) the support of the local sections of the cohomology sheaves of C is finite (B) $R\Gamma(X_{\bar{s}}, C) = 0$. For (A), we proceed as in (loc. cit.): (4.3.6) commutes with passage to invariants by a pro-p-group. For (B), we apply (4.3.7).

VARIANT 4.4. If Λ is a finite extension of $\mathbf{Z}_l$ or $\mathbf{Q}_l$, the analogous map to (4.3.6) (for $M \in D_c^b(X_\eta, \Lambda)$), defined by passage to the limit from the maps (4.3.6) (cf.), is an isomorphism. In particular:

COROLLARY 4.5. Let Λ be a finite extension of $\mathbf{Q}_l$. If $M \in D_c^b(X_\eta, \Lambda)$ is perverse, so is $R\Psi(M)$ (as an object of $D_c^b(X_s, \Lambda)$), the perversities considered being the self-dual perversities [2, 2.1.16 and §4].

According to [2, 4.4.2], the functor $R\Psi_*$ is indeed right t-exact. The isomorphism (4.3.6) thus implies that it is t-exact.

COROLLARY 4.6. Let Λ be a finite extension of $\mathbf{Q}_l$. If $K \in D_c^b(X, \Lambda)$ is perverse, so is $R\Phi(K)[-1] \in D_c^b(X_s, \Lambda_l)$, where $R\Phi(K)$ is defined by the canonical distinguished triangle (SGA 7 XIII 2.1.2)

$$(i^*K)_s \to R\Psi(j^*K) \to R\Phi(K) \to \tag{4.6.1}$$

By definition, for $X \in D_c^b(X, \Lambda)$, "perverse" means relative to the t-structure obtained by gluing (in the sense of [2, 1.4.10]) the t-structure t associated with the self-dual perversity $p_{1/2}$ on X_s [2, 4.0] and the t-structure t' on X_η defined by $({}^pD^{\leq -1}, {}^pD^{\geq -1})$, where $p = p_{1/2}$. This t-structure on X is thus defined by

$$K \in {}^pD^{\leq 0}(X) \Leftrightarrow (j^*K \in {}^pD^{\leq -1}(X_\eta) \text{ and } i^*K \in {}^pD^{\leq 0}(X_s))$$

$$K \in {}^pD^{\geq 0}(X) \Leftrightarrow (j^*K \in {}^pD^{\geq -1}(X_\eta) \text{ and } i^*K \in {}^pD^{\geq 0}(X_s)).$$

Consequently, if δ is the rectified dimension function introduced by Artin in (SGA 4 XIV 2.2) relative to $X \to S$, i.e.

$$\delta(x) = \dim \overline{\{y\}} + \text{deg.tr.}k(x)/k(y)$$

for x in X with image y in S, or also (loc. cit.)

$$\delta(x) = \begin{cases} \dim \overline{\{x\}} & \text{if } \overline{\{x\}} \cap X_s \neq \emptyset \\ \dim \overline{\{x\}} + 1 & \text{if } \overline{\{x\}} \cap X_s = \emptyset, \end{cases}$$

$K \in D_c^b(X, \Lambda)$ is perverse if and only if, for all $x \in X$, we have

$$H^q i_x^! K = 0 \text{ for } q > -\delta(x) \text{ and } H^q i_x^* K = 0 \text{ for } q < -\delta(x).$$

If X/S is deduced from a separated scheme of finite type Z on a smooth curve C over a field by localization at a closed point of C, the perversity $p_{1/2}$ on Z induces the one considered here on X.

Corollary 4.6 is due to Gabber. It is not an immediate consequence of 4.5. Note that, if K is perverse, $i^*K[-1]$ is not in general, as the case where $X = S$ and $K = \Lambda_S$ already shows. Here is Gabber's argument (presented by Laumon).

(a) It is clear that the functor i^* is t-exact. So are the functors j_* and $j_!$ from $D_c^b(X_\eta, \Lambda)$, equipped with t', into $D_c^b(X, \Lambda)$. For j_*, as in [2, 4.1.1], this results easily from Artin's theorem (SGA 4 XIV 3.1) via the calculation of the fibers $(R^q j_* F)_x$ at $x \in X_s$ as an inductive limit of $H^q(V_\eta, F)$, where V runs through the étale neighborhoods of x, and from the Leray spectral sequence for $V_\eta \to V_{\bar{\eta}}$. The case of $j_!$ is deduced by duality.

(b) Thanks to (a), the distinguished triangle

$$j_! j^* K \to K \to i_* i^* K \to$$

shows that

$$i^* K[-1] \in {}^pD_c(X_s, \Lambda)$$

(i.e. ${}^pH^n(i^*K[-1]) = 0$ for $n \neq 0, 1$) and that we have an exact sequence of perverse sheaves on X_s

$$0 \to i_* {}^pH^0(i^*K[-1]) \to j_! j^* K \to K \to i_* {}^pH^1(i^*K[-1]) \to 0.$$

As $R\Psi(j^*K[-1])$ is perverse (4.5), we deduce from it, by the distinguished triangle (4.6.1), that

$$(i^*K[-1])_s \to R\Psi(j^*K[-1]) \to R\Phi(K)[-1] \to,$$

that

$$R\Phi(K)[-1] \in {}^pD_c^{[-1,0]}(X_s, \Lambda)$$

and that we have an exact sequence of perverse sheaves on X_s

$$0 \to {}^pH^{-1}(R\Phi(K)[-1]) \to {}^pH^0(i^*K[-1])_s \to R\Psi(j^*K[-1])$$
$$\to {}^pH^0(R\Phi(K)[-1]) \to {}^pH^1(i^*K[-1])_s \to 0.$$

It is a matter of proving that

$${}^pH^{-1}(R\Phi(K)[-1]) = 0.$$

We will establish (***) by devissage.

(c) For K perverse on X, there exists a filtration (in the category of perverse sheaves on X)

$$0 \subset K' \subset K'' \subset K$$

such that K' and K/K'' are supported on X_s (i.e. annihilated by j^*, or even of the form i_*M for $M \in D_c^b(X_s, \Lambda)$), and $K''/K' \simeq j_{!*} j^* K$. Consider indeed the commutative square in the abelian category of perverse sheaves on X:

$$\begin{array}{ccc} K & \to & j_* j^* K \\ \uparrow & & \uparrow \\ j_! j^* K & \to & j_{!*} j^* K \end{array}$$

where, by definition, $j_{!*} j^* K$ is the image of $j_! j^* K$ in $j_* j^* K$. It is sufficient to take for K'' the image of $j_! j^* K$ in K, and for K' the kernel of $K'' \to j_{!*} j^* K$. We are thus reduced to proving (***) in the two following cases: (i) K is supported on X_s, (ii) $K = j_{!*} j^* K$. Case (i). Then $i^*K = i^!K$ is perverse, in particular ${}^pH^0(i^*K[-1])_s = 0$, so ${}^pH^{-1}(R\Phi(K)[-1]) = 0$ thanks to (**). Case (ii). Let $K = j_! L$ with L perverse on X_η (for t'). We have an exact sequence of perverse sheaves on X

$$0 \to j_{!*} L \to j_* L \to j_* L / j_{!*} L \to 0,$$

so, from (*), we have an injection

$$ {}^pH^{-1}(R\Phi(j_{!*}L)[-1]) \to {}^pH^{-1}(R\Phi(j_*L)[-1]). $$

It is thus sufficient to prove that

$$ {}^pH^{-1}(R\Phi(j_*L)[-1]) = 0. $$

By definition, this amounts to proving that the map

$$ {}^pH^0(i^*j_*L[-1])_s \to {}^pH^0(R\Psi(L[-1])) \overset{4.5}{=} R\Psi(L[-1]) $$

is a monomorphism of perverse sheaves. We can assume S is strictly local. Let I be the inertia group. The map from $D^b_c(X_s \times_s \eta, \Lambda)(= D^b_c(X_s, \Lambda[I])$, "c" meaning with constructible cohomology as a complex of Λ-modules),

$$ i^*j_*(L[-1]) \to R\Psi(L[-1]) $$

is rewritten as the canonical map associated with $I \to \{1\}$

$$ R\Gamma(I, R\Psi(L[-1])) \to R\Psi(L[-1]). $$

It is thus sufficient to show that, for $M \in D^b_c(X_s, \Lambda[I])$, perverse as an object of $D^b_c(X_s, \Lambda)$,

$$ R\Gamma(I, M) \to M \tag{1} $$

induces an injection on ${}^pH^0$. Let P_l be as in 3.6. The functor $\Gamma(P_l, -)$ (invariants under P_l) being exact on the category of constructible $\Lambda[I]$-sheaves, for E in $D^b_c(X_s, \Lambda[I])$ and $x \in X_s$, we have $H^iR\Gamma(P_l, E)_x = R\Gamma(P_l, H^iE_x) = \Gamma(P_l, H^iE_x)$, and similarly with $i^!_x$; moreover, if C is the cone of $R\Gamma(P_l, E) \to E$, the sequences $0 \to H^iR\Gamma(P_l, E) \to H^ii^*_xE \to H^ii^*_xC \to 0$ (and similarly with $i^!_x$) are exact and split. Consequently, for M perverse as above, $R\Gamma(P_l, M)$ and the cone C of $R\Gamma(P_l, M) \to M$ are perverse, and we have an exact sequence of perverse sheaves

$$ 0 \to R\Gamma(P_l, M) \to M \to C \to 0. $$

As

$$ R\Gamma(I, M) = R\Gamma(\mathbf{Z}_l(1), R\Gamma(P_l, M)), $$

we are thus reduced to showing that for N in $D^b_c(X_s, \Lambda[\mathbf{Z}_l(1)])$, perverse (as a complex of Λ-modules),

$$ R\Gamma(\mathbf{Z}_l(1), N) \to N $$

induces an injection on ${}^pH^0$. If T, as in 3.6, is a topological generator of $\mathbf{Z}_l(1)$, $R\Gamma(\mathbf{Z}_l(1), N)$ is the simple complex associated with the double complex

$$ N \xrightarrow{T-1} N, $$

on lines of degree 0 and 1, and (2) is the projection figuring in the exact sequence

$$ 0 \to N[-1] \to s(N \xrightarrow{T-1} N) \to N \to 0. $$

The corresponding exact sequence of perverse cohomology provides the desired injection. This completes the proof of 4.6.

In the same vein as 4.2, we have the following result, which was pointed out to me by M. Rapoport:

THEOREM 4.7. *Let X and Y be of finite type over S. For $L \in D_{ctf}(X_\eta, \Lambda)$, $M \in D_{ctf}(Y_\eta, \Lambda)$, the map from $D_{ctf}((X \times_S Y)_s \times_s \eta, \Lambda)$ defined below,*

$$ R\Psi(L) \overset{L}{\otimes}_s R\Psi(M) \to R\Psi(L \overset{L}{\otimes}_\eta M), \tag{4.7.1} $$

is an isomorphism ($\overset{L}{\otimes}_s$ and $\overset{L}{\otimes}_\eta$ denote external tensor products over pr_1^*, pr_2^*).

This result, attributed to Gabber, appears in a preprint of A. Beilinson and J. Bernstein, A proof of Jantzen conjectures. See also [M. Schröter, Diplomarbeit, Bonn, 1990]. We define the map (4.7.1) as deduced, by application of i^*, from the Künneth map

$$ Rj_*L \overset{L}{\otimes}_S Rj_*M \to Rj_*(L \overset{L}{\otimes}_\eta M) $$

defined by the inclusions $\bar{j}$ of $X_\eta, Y_\eta, (X\times Y)_\eta$ in $X_{\bar{s}}, Y_{\bar{s}}, (X\times Y)_{\bar{s}}$ respectively. Just like (4.3.6), (4.7.1) is compatible with proper direct images: for $f: X\to X'$, $g: Y\to Y'$ proper, and $h = f\times_S g$, we have a commutative square

$$\begin{array}{ccc} f_{s*}(L)\overset{L}{\otimes}_s g_{s*}(M) & \xrightarrow{(1)} & h_{s*}(L\overset{L}{\otimes}_\eta M) \\ (3)\downarrow & & \downarrow(4) \\ \Psi(f_*L)\overset{L}{\otimes}_s \Psi(g_*M) & \xrightarrow{(2)} & \Psi h_*(L\overset{L}{\otimes}_\eta M) \end{array} \tag{4.7.2}$$

where (1) is composed of the Künneth isomorphism $f_{s*}\Psi(L)\overset{L}{\otimes}_s g_{s*}\Psi(M)\xrightarrow{\sim} h_{s*}(\Psi(L)\overset{L}{\otimes}_s \Psi(M))$ (SGA 4 XVII 5.4.3) and of (4.7.1), (2) is composed of (4.7.1) (for f_*L and g_*M) and the Künneth isomorphism, and (3) and (4) are defined by the canonical isomorphisms (SGA 7 XIII 2.1.2) (as in (4.3.7) we have abbreviated $R\Psi_\eta$ to $R\Psi$ and omitted the R). The verification is left as an exercise.

Let us prove that (4.7.1) is an isomorphism. The proof is analogous to that of 4.2. We can assume S is strictly local. We reason by induction on $\dim Z_\eta$, where $Z = X\times_S Y$. We assume that (4.7.1) is an isomorphism for $\dim Z_\eta < n$. For $\dim Z_\eta = n$, we reduce to assuming X and Y are projective, and we denote C the cone of (4.7.1). It is sufficient to verify conditions (A) and (B): (A) the support of the local sections of the cohomology sheaves of C is finite; (B) $R\Gamma(Z_{\bar{s}}, C) = 0$. The validity of (B) results from (4.7.2). For (A), we proceed again as in (SGA 4 1/2 Th. finitude 3.7). We can assume X_η (resp. Y_η) is dense in X (resp. Y). If $n = 0$, (A) is then automatically satisfied, so we can assume $n > 0$. After localizing, we can assume that we have quasi-finite morphisms $X\to \mathbf{A}^q_s, Y\to \mathbf{A}^r_s$, with $q+r = n$. Assume $q > 0$, and let us denote $f_1: X\to \mathbf{A}^1_s$ the composite of b and of $\mathrm{pr}_1: \mathbf{A}^q_s\to \mathbf{A}^1_s$. As in (loc. cit.), let us introduce the strict localization S' of $\mathbf{A}^1_s$ at a geometric generic point s' of $\mathbf{A}^1_s$, let us denote η' the generic point of S', $\eta'_1 = \mathrm{Spec}(k(\eta)\otimes_{k(\eta)} k(\eta'))$ the generic point of the strict localization of $\mathbf{A}^1_s$ at s', and $\tilde{\eta}'$ a geometric point above η'_1. We then have a commutative diagram with Cartesian squares, where $X'\to S'$ is deduced from $f_1: X\to \mathbf{A}^1_s$ by the base change $S'\to \mathbf{A}^1_s$:

$$\begin{array}{ccccccccc} X'_{s'} & \to & X' & \leftarrow & X'_{\eta'} & \leftarrow & X'_{\bar{\eta}'_1} & \leftarrow & X'_{\tilde{\eta}'} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ s' & \to & S' & \leftarrow & \eta' & \leftarrow & \bar{\eta}'_1 & \leftarrow & \tilde{\eta}' \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ s & \to & S & \leftarrow & \eta & \leftarrow & \bar{\eta} & & \end{array} \tag{*}$$

Let $I = \mathrm{Gal}(\bar{\eta}/\eta)$ and $I' = \mathrm{Gal}(\tilde{\eta}'/\eta')$ be the inertia groups of S and S'. We have $\mathrm{Gal}(\tilde{\eta}'/\eta'_1) = I$, and, as Deligne observes in (loc. cit.), in the exact sequence

$$1\to P\to I'\to I\to 1,$$

P is a pro-p-group, because $I'\to I$ induces an isomorphism on the tame quotients, a uniformizer of S being a uniformizer of S'. Let L' be the inverse image of L on $X'_{\eta'}$. The diagram (*) shows (cf. (loc. cit. 3.4)) that

$$R\Psi_\eta(L)_{s'} = R\Psi_{\eta'}(L')^P, \tag{1}$$

where $(-)^P = R\Gamma(P, -)$ denotes the "invariants under P" functor. Let us set on the other hand

$$Y' = Y\times_s S', \quad Z' = X'\times_{s'} Y',$$

and let us denote M' the inverse image of M on $Y'_{\eta'}$. By the compatibility of vanishing cycles with change of traits (loc. cit. 3.7), we have

$$R\Psi_\eta(M)_{s'} = R\Psi_{\eta'}(M'). \tag{2}$$

Finally, from the analogous diagram to (*) relative to Z', we deduce that

$$R\Psi_\eta(L\overset{L}{\otimes}_\eta M)_{s'} = R\Psi_{\eta'}(L'\overset{L}{\otimes}_{\eta'} M')^P. \tag{3}$$

As $\dim X'_{\eta'} < q$ and $\dim Y'_{\eta'} \le r$, the induction hypothesis applies to $(X'/S', Y'/S', L', M')$: the map (4.7.1)

$$R\Psi_{\eta'}(L')\overset{L}{\otimes}_{s'} R\Psi_{\eta'}(M')\to R\Psi_{\eta'}(L'\overset{L}{\otimes}_{\eta'} M') \tag{4}$$

is an isomorphism. According to (2), P acts trivially on $R\Psi_{\eta'}(M')$, so

$$R\Psi_{\eta'}(M') = R\Psi_{\eta'}(M')^P.$$

Thus, applying to (4) the functor $(-)^P$, we obtain an isomorphism

$$R\Psi_{\eta'}(L')^P \overset{L}{\otimes}_{s'} R\Psi_{\eta'}(M') \xrightarrow{\sim} R\Psi_{\eta'}(L' \overset{L}{\otimes}_{\eta'} M')^P.$$

Via (1), (2) and (3), this is identified with the fiber at s' of (4.7.1). Consequently, the cone C is cohomologically concentrated on the inverse image, by the composite

$$(X \times Y)_s \xrightarrow{b\times c} \mathbf{A}^q_s \times_s \mathbf{A}^r_s \to \mathbf{A}^n_s,$$

of the union of a finite number of hyperplanes in $\mathbf{A}^n_s$ parallel to $\mathrm{pr}_i^{-1}(0)$. As this holds for any projection $\mathrm{pr}_i : \mathbf{A}^n_s \to \mathbf{A}^1_s$ and as (X, L) and (Y, M) play symmetric roles, we conclude that C is concentrated on the inverse image by $b \times c$ of the union of a finite number of closed points of $\mathbf{A}^q_s \times_s \mathbf{A}^r_s$, i.e. that the support of $H^*(C)$ is finite, which proves (A) and completes the proof of 4.7.

COROLLARY 4.8. *Under the hypotheses of 4.7, assume that $R\Psi_\eta(L) = R\Psi_{\eta,t}(L)$ or $R\Psi_\eta(M) = R\Psi_{\eta,t}(M)$, where $R\Psi_{\eta,t}(-)$ denotes the "tame vanishing cycles" functor (= $R\Psi(-)^P$, where P is the wild inertia). Then the analogous map to (4.7.1)*

$$R\Psi_{\eta,t}(L) \overset{L}{\otimes}_s R\Psi_{\eta,t}(M) \to R\Psi_{\eta,t}(L \overset{L}{\otimes}_\eta M) \tag{4.8.1}$$

is an isomorphism. If moreover, $R\Psi_\eta = R\Psi_{\eta,t}$ for L and M, this is also the case for $L \overset{L}{\otimes}_\eta M$.

It suffices indeed to apply the functor $(-)^P$ to (4.7.1). In particular, taking into account (3.5.1):

COROLLARY 4.9. *If X or Y has semi-stable reduction over S, then the natural map*

$$R\Psi_{\eta,t}(\Lambda_{X_\eta}) \overset{L}{\otimes}_s R\Psi_{\eta,t}(\Lambda_{Y_\eta}) \to R\Psi_{\eta,t}(\Lambda_{(X\times_S Y)_\eta}) \tag{4.9.1}$$

is an isomorphism.

REMARK 4.10. If X and Y have semi-stable reduction, the $R^i\Psi_{\eta,t}(\Lambda_{(X\times_S Y)_\eta})$ are thus tame. In general, $X \times_S Y$ no longer has semi-stable reduction. Nevertheless, $X \times_S Y$ is log-smooth over S in the sense of Kato (S endowed with its canonical log-structure), cf. [K. Kato, Logarithmic structures of Fontaine-Illusie, Algebraic Analysis, Geometry and Number Theory, The Johns Hopkins University Press (1989), 191-224], and the exposés of Kato and Hyodo-Kato in this seminar. One may hope that the result of Rapoport-Zink (3.5.1) extends to the log-smooth case.

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