Criteria for Flatness and Projectivity

Techniques for “Flatification” of a Module

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First Part: Flatness in Algebraic Geometry

Let A be a commutative ring, B an A-algebra with finite presentation, M a B-module with finite presentation. We are interested in the flatness of M as an A-module. This is a classical problem in relative algebraic geometry that has been studied in detail by Grothendieck (EGA IV 11, 12 ...). We propose here a different approach that seems better adapted to relative geometry: we prove that M is A-flat if and only if, locally for the étale topology on $\mathrm{Spec}(A)$ and $\mathrm{Spec}(B)$, the A-module M has a finite composition series whose successive quotients are A-submodules underlying free modules of finite type over smooth A-algebras B_i with geometrically integral fibers. This structure theorem largely allows us to reduce the local study of flat A-modules to that of free modules over smooth A-algebras.

It turns out that the preceding results tend to extend to the case where M is only assumed to be of finite type over B. The reason is that if $\mathfrak{q}$ is a prime ideal of B such that the $B_{\mathfrak{q}}$-module of finite type $M_{\mathfrak{q}}$ is A-flat, then $M_{\mathfrak{q}}$ is necessarily a $B_{\mathfrak{q}}$-module with finite presentation. When A is not "too bad", M is still A-flat and has finite presentation over B in an open neighborhood of $\mathfrak{q}$ in $\mathrm{Spec}(B)$. Thus, we obtain that if A is Noetherian, every flat A-algebra of finite type has finite presentation.

In §3 we characterize A-modules M with finite presentation over B that are projective A-modules (or equivalently, that are locally free on $\mathrm{Spec}(A)$). For example, if B is a smooth A-algebra with geometrically integral fibers of constant dimension, B is a free A-module.

In the last two paragraphs, we provide methods to make a B-module M A-flat. Thus in §4, we seek to make the B-module M A-flat by changing rings $A \to A'$. This leads us to study the functor of "universal flatification" and allows us to simplify and generalize the technical results of EGA IV 11 on the descent of flatness. Finally, in §5, we seek to make M A-flat by making a splitting on Spec(A) and replacing M by its strict transform under this splitting. The applications we give here of this last result are negligible, but in a subsequent article, we will show how the preceding techniques extend to formal and rigid geometry and provide a systematic procedure to reduce Tate, Kiehl... rigid geometry to algebraic geometry.

A particular effort has been made to eliminate Noetherian hypotheses and Noetherian methods; in particular, the techniques of passage to the limit, without being totally suppressed, are less frequent than in EGA IV. It seems to us that this is a necessary effort for a better understanding of the problems of relative algebraic geometry, even though the presentation is somewhat cumbersome.

§0. Some notations and conventions

In the first part, all rings and algebras considered are commutative.

Let S be a scheme. If s is a point of S, we denote by $k(s)$ the residual field at s. A pointed scheme (S, s) is a pair consisting of a scheme S and a point s of S.

An elementary étale neighborhood of the pointed scheme (S, s) is a pointed scheme (S', s'), where S' is an S-scheme étale and s' is a point of S' above s, with trivial residual extension ($k(s) \simeq k(s')$).

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All sheaves of modules on a scheme that we will consider are quasi-coherent.

Let X be an S-scheme, s a point of S. The fiber of X above s, that is to say the scheme $X \times_S \operatorname{Spec}(k(s))$, will also be denoted $X \otimes_S k(s)$.

Let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{O}_X$-modules, $u : \mathcal{M} \to \mathcal{N}$ a $\mathcal{O}_X$-linear morphism. We say that u is S-universally injective if u remains injective after any base change $S' \to S$. When $\mathcal{N}$ is S-flat, this condition is equivalent to requiring that $\mathcal{N}/\mathcal{M}$ is S-flat.

Suppose $X \to S$ locally of finite type and let $\mathcal{M}$ be a $\mathcal{O}_X$-module of finite type, x a point of X, s its image in S. We denote by $\dim_x(\mathcal{M}/S)$ the dimension at x of the module $\mathcal{M} \otimes_S k(s)$. The upper bound of the numbers $\dim_x(\mathcal{M}/S)$, as x ranges over the points of X, is denoted $\dim(\mathcal{M}/S)$ (relative dimension of $\mathcal{M}$ above S). When $\mathcal{M} = \mathcal{O}_X$, we also write $\dim_x(X/S)$ and $\dim(X/S)$.

§1. Relative devissage of a module

1.1. An application of Zariski's "Main theorem"

Theorem 1 (1.1.1)**.** *Let $f : (X, x) \to (S, s)$ be a morphism of pointed schemes. We assume X locally of finite type over S; we set $n = \dim_x(X/S)$; then there exists a commutative diagram of pointed schemes*

$$\begin{array}{ccccc} (Y, y) & \to & (T, t) & \to & (S', s') \\ & & \downarrow & & \downarrow \\ & & (X, x) & \longrightarrow & (S, s) \end{array}$$

satisfying the following conditions:

> *[a)]Y, T and S' are affine, $(Y, y) \to (X, x)$ and $(S', s') \to (S, s)$ are elementary étale neighborhoods; $T \to S'$ is smooth with geometrically integral fibers of dimension n; $Y \to T$ is finite and y is the only point of Y above t.*

2. *Proof.* The question being local on (S, s) and on (X, x), we can assume that S is affine with ring A, that X is affine with ring B and that all irreducible

components of $f^{-1}(s)$ contain x (so that $\dim(f^{-1}(s)) = n$). Let us proceed in several steps:

1) Let us choose a closed specialization ξ of x in $f^{-1}(s)$; we have $\dim(B_\xi \otimes k(s)) = n$ and $\dim(B_x \otimes k(s)) = r \leq n$. We can find a sequence $(b_1, \dots, b_n)$ of elements of B such that the image of $(b_1, \dots, b_n)$ (resp. $(b_1, \dots, b_r)$) in the local Noetherian ring $B_\xi \otimes k(s)$ (resp. $B_x \otimes k(s)$) is a system of parameters. Let Z be the S-scheme $S[T_1, \dots, T_n]$, $g : X \to Z$ the S-morphism defined by the sequence $(b_1, \dots, b_n)$, $z = g(x)$; then g is quasi-finite at x since, by construction, it is quasi-finite in the specialization ξ of x (EGA IV 13.1.3), $\square$

and the adherence of z in $Z \otimes k(s)$, which is identified with the closed subscheme defined by the annulation of functions $T_1, \dots, T_r$, is geometrically integral.

2) Let $(Z, \tilde{z})$ be a henselization of (Z, z); let $\tilde{X}$ be the scheme $X \times_Z \tilde{Z}$, $\tilde{x}$ the unique point of $\tilde{X}$ with respective projections x and $\tilde{z}$, $\tilde{Y}$ the spectrum of $O_{\tilde{X},\tilde{x}}$; since $(\tilde{X}, \tilde{x}) \to (Z, \tilde{z})$ is quasi-finite at $\tilde{x}$ and since $(Z, \tilde{z})$ is henselian, it follows from the Main theorem that $\tilde{Y}$ is finite over $\tilde{Z}$ and is an open and closed subscheme of $\tilde{X}$, defined by an idempotent $\tilde{e}$ of $\Gamma(\tilde{X}, O_{\tilde{X}})$ (EGA IV 18.5.11).

3) Since $(Z, \tilde{z})$ is the filtered projective limit of elementary étale affine neighborhoods of (Z, z), we can find an elementary étale affine neighborhood (Z_1, z_1) of (Z, z) and an idempotent e_1 of $\Gamma(X_1, O_{X_1})$ (where we set $X_1 = X \times_Z Z_1$) such that $\tilde{e}$ is the image of e_1 in $\Gamma(X, O_X)$. Let x_1 be the image of $\tilde{x}$ in X_1 and Y_1 the open and closed subscheme of X_1 defined by e_1; the diagram of pointed schemes

$$\begin{array}{ccc} (Y_1, x_1) & \to & (Z_1, z_1) \\ \downarrow & & \downarrow \\ (X, x) & \to & (S, s) \end{array}$$

is commutative, $(Y_1, x_1) \to (X, x)$ is an elementary étale affine neighborhood, $Z_1 \to Z$ is étale, thus $Z_1 \to S$ is smooth and the irreducible components of its fibers all have dimension n. Moreover, up to replacing (Z_1, z_1) by one of its elementary étale affine neighborhoods and (Y_1, x_1) by its reciprocal image on this neighborhood, we can assume that

$Y_1 \to Z_1$ is finite (because, at the limit, $\tilde{Y} \to \tilde{Z}$ is finite)

x_1 is the only point of Y_1 above z_1 (because $\tilde{x}$ is the only point of $\tilde{Y}$ above $\tilde{z}$)

$Z_1 \otimes k(s)$ is connected.

Let us show that $Z_1 \otimes k(s)$ is even geometrically integral. Since the adherence of z in $Z \otimes k(s)$ is geometrically integral, this follows from the following lemma:

Lemma 1 (1.1.2)**.** *Let k be a field, Z a k-scheme of finite type and geometrically normal, z a point of Z with geometrically irreducible adherence, (T, t) a connected elementary étale neighborhood of (Z, z); then T is geometrically irreducible.*

Proof. Let k' be a finite extension of k, Z' and T' the reciprocal images on $\mathrm{Spec}(k')$ of Z and T; by hypothesis $k(z)$ is a primary extension of k, thus the fiber of t in T', which is isomorphic to $\mathrm{Spec}(k' \otimes k(z))$, contains a single point t'. On the other hand, since Z is geometrically normal, Z' is normal, thus T', which is étale over Z', is normal, so the

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connected components of T' are irreducible. Let T'' be a connected component of T'. Since $T' \to T$ is finite and flat, $T'' \to T$ is finite and flat, thus surjective since T is connected. Therefore T'' contains t' i.e. $T'' = T'$.

4) Let us note the following lemma:

Lemma 2 (1.1.3)**.** *Let (S,s) be a pointed scheme, T a smooth S-scheme such that $T \otimes k(s)$ is geometrically integral of dimension n. There exists an elementary étale neighborhood (S',s') of (S,s) and an open U' of $T' = T \times_S S'$ such that U' contains $T' \otimes k(s')$ and such that the fibers of the morphism $U' \to S'$ are geometrically integral of dimension n.*

Proof. Let $(\tilde{S},\tilde{s})$ be a strict henselization of (S,s) and $\tilde{T} = T \times_S \tilde{S}$. Since $\tilde{T}$ is smooth over $\tilde{S}$ and $\tilde{T} \otimes k(\tilde{s})$ is non-empty, the morphism $\tilde{T} \to \tilde{S}$ has a section $\tilde{\sigma}$ (EGA IV 17.16.3). Consider $(\tilde{S},\tilde{s})$ as a projective limit of a filtering system of finite and étale schemas over the henselization of S at s. By passage to the limit, we find that there exists an elementary étale neighborhood (S',s') of (S,s) and a finite étale surjective morphism $S_1 \to S'$, having a single point s_1 above s', such that $T_1 = T \times_S S_1$ has a section σ_1 above S_1. Let $T' = T \times_S S'$. The union of the connected components of the fibers of $T_1 \to S_1$ that intersect $\sigma_1(S_1)$ is an open U_1 of T_1 (EGA IV 15.6.5). Since $T_1 \otimes k(s_1)$ is connected by hypothesis, U_1 contains $T_1 \otimes k(s_1)$. The smooth morphism $U_1 \to S_1$ admits a section and its fibers are connected, they are therefore geometrically integral (1.1.2) and up to restricting S', we can assume that they are of dimension n.

Let $p : T_1 \to T'$ be the canonical projection. Then p is finite, thus closed and consequently $p(T_1 - U_1)$ is closed in T' and does not intersect $T \otimes k(s')$. Thus $U' = T' - p(T_1 - U_1)$ is an open of T', containing $T' \otimes k(s')$ and such that $p^{-1}(U')$ is contained in U_1. A fortiori, the fibers of $U' \to S'$ are geometrically integral of dimension n.

We can apply (1.1.3) to the pointed scheme (S,s) and to the smooth S-scheme Z_1 introduced in 3); we can therefore find an elementary étale affine neighborhood (S',s') of (S,s) and an open U' of $Z_1' = Z_1 \times_S S'$ such that U' contains $Z_1' \otimes k(s')$ and that the fibers of $U' \to S'$ are geometrically integral of dimension n. Let t be the unique point of U' of respective projections z_1 and s' and let T be an open affine neighborhood of t in U'. Since $T \to S'$ is open, we can, up to restricting T and S', assume $T \to S'$ surjective. Let Y be the schema $Y_1 \times_{Z_1} T$ and y the unique point of Y of respective projections x_1 and t. It is clear that the diagram

$$\begin{array}{ccccc} (Y,y) & \to & (T,t) & \to & (S',s') \\ & & \downarrow & & \downarrow \\ & & (X,x) & \longrightarrow & (S,s) \end{array}$$

verifies the conditions a), b) and c) of the statement, cqfd.

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Remark 1 (1.1.4)**.** In statement (1.1.1), one can choose the smooth scheme T such that the fiber $T \otimes k(s)$ is isomorphic to an open subset of the affine space over $k(s)$, of dimension n. To see this, we use Henselian couples ([17] IX p. 120).

Corollary 1 (1.1.5)**.** *Let (S, s) be a pointed scheme, X an S-scheme of finite type. There exists an elementary étale affine neighborhood (S', s') of (S, s) and a $(X \times_S S')$ scheme Y satisfying the following conditions:*

> *[a)]Y is affine, étale over X; There exists a finite partition $(Y_i)_{i \in I}$ of Y into open and closed subschemes and, for all i, a factorization $Y_i \to T_i \to S'$, such that Y_i is finite over T_i and T_i is affine, smooth over S', with geometrically integral fibers of constant dimension n_i. For every point x of $X \otimes k(s)$, there exists $i \in I$ and a point y of Y_i, above x, such that $n_i = \dim_x(X \otimes k(s))$.*

3. *Proof.* For every point x of $X \otimes k(s)$, let $n(x) = \dim_x(X/S)$ and let us construct a commutative diagram of pointed schemes

$$\begin{array}{ccccc} (Y_x,y_x) & \to & (T_x,t_x) & \to & (S'_x,s'_x) \\ & & \downarrow & & \downarrow \\ & & (X,x) & \longrightarrow & (S,s) \end{array}$$

satisfying the conditions stated in 1.1.1 (with $n = n(x)$). Since $Y_x \to X$ is étale, its image is an open neighborhood V_x of x in X. But $X \otimes k(s)$ is noetherian and the function $x \mapsto n(x)$ is upper semi-continuous on $X \otimes k(s)$. It follows easily that there exists a finite family $(x_i)_{i \in I}$ of points of $X \otimes k(s)$ satisfying the following condition: for every point x of $X \otimes k(s)$, there exists $i \in I$ such that $x \in V_{x_i}$ and $n(x) = n(x_i)$. Let (S', s') be the fiber product over (S, s) of the family (S'_{x_i}, s'_{x_i}); it is an elementary étale neighborhood of (S, s). For all i, let (T_i, t_i) (resp. (Y_i, y_i)) be the product of (T_{x_i}, t_{x_i}) (resp. (Y_{x_i}, y_{x_i})) and of (S', s') over (S'_{x_i}, s'_{x_i}). Then the scheme Y, disjoint union of the family (Y_i), answers the question. □

0.1 Notion of relative devissage

Definition 1 (1.2.1)**.** Let $f : X \to S$ be a morphism of finite presentation of affine schemes, s a point of S, $\mathcal{M}$ an $\mathcal{O}_X$-module of finite type, n an integer ≥ 0.

An S-devissage (or devissage relative to S), in dimension n, of the $\mathcal{O}_X$-module $\mathcal{M}$ above s, consists of the following data:

[a)]A closed subscheme of finite presentation X' of X, containing the closed subscheme defined by the annihilator of $\mathcal{M}$ in $\mathcal{O}_X$ and such that

$\dim(X' \otimes k(s)) \leq n$. The module $\mathcal{M}$ is then the direct image on X of an $\mathcal{O}_{X'}$-module of finite type that we will still denote by $\mathcal{M}$.

b) A factorization $X' \to T \to S$ of the restriction of f to X', such that $X' \to T$ is finite and $T \to S$ is affine, smooth, with geometrically integral fibers of dimension n. We denote by τ the generic point of $T \otimes k(s)$.

c) A homomorphism α of a free $\mathcal{O}_T$-module $\mathcal{L}$ of finite type into the $\mathcal{O}_T$-module $\mathcal{N}$ direct image on T of the $\mathcal{O}_{X'}$-module $\mathcal{M}$, such that $\alpha \otimes k(\tau)$ is bijective.

We can make the following remarks:

(1.2.1.1) Condition a) implies $n \geq \dim(\mathcal{M} \otimes k(s))$. Conversely, if this condition is satisfied, there always exists a closed subscheme X' of X satisfying a). Indeed, let $\mathcal{I}$ be the annihilator of $\mathcal{M}$ in $\mathcal{O}_X$; there exists an ideal of finite type $\mathcal{I}'$ of $\mathcal{O}_X$, contained in $\mathcal{I}$ and having the same image as $\mathcal{I}$ in the noetherian ring of global sections of $X \otimes k(s)$, and it suffices to take $X' = V(\mathcal{I}')$.

(1.2.1.2) The morphism $X' \to T$ is finite and of finite presentation; thus $\mathcal{N}$ is of finite type, and even of finite presentation if $\mathcal{M}$ is of finite presentation. In particular, $\mathcal{N} \otimes k(\tau)$ is free of finite rank r over $k(\tau)$. Any sequence of r elements of $\Gamma(T, \mathcal{N})$, whose image in $\mathcal{N} \otimes k(\tau)$ is a basis over the field $k(\tau)$, defines a morphism $\alpha : \mathcal{O}_T^r \to \mathcal{N}$ that satisfies condition c). We have $\mathcal{L} = 0$ if and only if $\dim(\mathcal{M} \otimes k(s)) < n$.

(1.2.1.3) Let $\mathcal{P} = \operatorname{coker}(\alpha)$; then $\mathcal{P}$ is of finite type on T (and even of finite presentation if $\mathcal{M}$ is of finite presentation on X). Moreover, since $\alpha \otimes k(\tau)$ is surjective, α_s is surjective (Nakayama's lemma) and consequently $\dim(\mathcal{P} \otimes k(s)) < n$.

(1.2.1.4) Let us denote by $(X' \to T, \mathcal{L} \xrightarrow{\alpha} \mathcal{N} \to \mathcal{P}) = D$ the relative devissage described above.

(1.2.1.5) Let x be a point of $X \otimes k(s)$. If $x \in X'$ and is the only point of X' above its image t in T, we say that D is an S-devissage in dimension n of $\mathcal{M}$ at point x.

Definition (1.2.2). Let X, S, $\mathcal{M}$, s be as in 1.2.1.

Let r be an integer ≥ 0 and $n_1 > n_2 > \cdots > n_r \geq 0$ a strictly decreasing sequence of r integers ≥ 0. An S-devissage of $\mathcal{M}$ above s, in dimensions $n_1, \ldots, n_r$ is defined by recurrence on r using the following data.

a) An S-devissage $D_1 = (X_1 \to T_1, \mathcal{L}_1 \xrightarrow{\alpha_1} \mathcal{N}_1 \to \mathcal{P}_1)$, in dimension n_1, of $\mathcal{M}$ above s.

b) An S-devissage D of $\mathcal{P}_1$ above s, in dimensions $n_2, \ldots, n_r$.

Let x be a point of $X \otimes k(s)$. The devissage described above is an S-devissage of $\mathcal{M}$ at point x if D_i is a devissage of $\mathcal{M}$ at point x (1.2.1.5) and if D is a devissage of $\mathcal{P}_i$ at point t_1, the image of x in T_1.

We can specify the terminology as follows:

(1.2.2.0) Let $T_0 = X$ and $\mathcal{P}_0 = \mathcal{M}$. Then, the preceding devissage is equivalent to the data, for $i = 1, \ldots, r$ of an S-devissage

$$D_i = (X_i \to T_i, \mathcal{L}_i \xrightarrow{\alpha_i} \mathcal{N}_i \to \mathcal{P}_i)$$

in dimension n_i, of the module $\mathcal{P}_{i-1}$ on T_{i-1}.

Let t_0 be the point of T_0 corresponding to point x of X. Then, the devissage preceding is a devissage of $\mathcal{M}$ at x, if for $i = 1, \ldots, r$, the point t_{i-1} is a point of the closed subscheme X_i of T_{i-1} and is the only point of X_i above its image t_i in T_i.

(1.2.2.1) The integer r is the length of the devissage.

(1.2.2.2) Let n and n' be integers such that $n \geq n_1$ and $\dim(\mathcal{P}_n \otimes k(s)) < n' \leq n_r$. We then say that we have a devissage of $\mathcal{M}$ in dimensions between n and n'.

(1.2.2.3) If $\mathcal{P}_n = 0$, we say that we have a total devissage of $\mathcal{M}$. If n is as in (1.2.2.2), we also say that we have a devissage of $\mathcal{M}$ in dimensions $\leq n$.

Proposition (1.2.3). *Let $f : (X, x) \to (S, s)$ be a locally finite presentation morphism of pointed schemes, $\mathcal{M}$ a $\mathcal{O}_X$-module of finite type such that $\mathcal{M}_x \neq 0$, let $n = \dim_x(\mathcal{M} \otimes k(s))$ and $r = \operatorname{coprof}_{\mathcal{O}_{X,x} \otimes k(s)}(\mathcal{M}_x \otimes k(s))$ (EGA* O_{IV} *16.4.9). Then, there exists a commutative diagram of pointed schemes*

$$\begin{array}{ccc} (X', x') & \to & (S', s') \\ \downarrow & & \downarrow \\ (X, x) & \to & (S, s) \end{array}$$

where the columns are elementary étale neighborhoods and such that the reciprocal image $\mathcal{M}'$ of $\mathcal{M}$ on X' admits an S'-devissage total, at point x', in dimensions between n and $n - r$.

Demonstration. The question is local on S and on X; we can therefore assume S and X affine. There exists then a closed subscheme Y of X, of finite presentation on S, defined by an ideal contained in the annihilator of $\mathcal{M}$, and such that $\dim_Y(Y/S) = n$ (cf. 1.2.1.1). Moreover, it follows from the theory of étale morphisms (EGA IV 18.4.6) that if (Y', x') is an elementary étale neighborhood of (Y, x), there exists an open set (V', x') of (Y', x') which is the restriction above Y of an elementary étale neighborhood

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(X', x') of (X, x). We easily deduce from this that to establish (1.2.3), we can replace X by Y, thus assume that $\dim_x(X/S) = n$.

Let us consider a commutative diagram

$$\begin{array}{ccc} (X_1, x_1) & \rightarrow & (T, t) \rightarrow (S_1, s_1) \\ \downarrow & & \downarrow \\ (X, x) & \longrightarrow & (S, s) \end{array}$$

satisfying conditions a), b), c) of 1.1.1. To establish (1.2.3), it is permissible to replace (S, s) by (S_1, s_1), (X, x) by (X_1, x_1) and $\mathcal{M}$ by its reciprocal image on X_1. Henceforth we assume $X = X_1$.

Let $\mathcal{N}$ be the direct image of $\mathcal{M}$ on T. Since x is the only point of X above its image t in T, $\mathcal{N}_t$ is the $\mathcal{O}_{T,t}$-module underlying the $\mathcal{O}_{X,x}$-module $\mathcal{M}_x$ and therefore $r = \mathrm{coprof}_{\mathcal{O}_{T,t} \otimes k(s)}(\mathcal{N}_t \otimes k(s))$ (EGA $\mathrm{O_{IV}}$ 16.4.11).

Let us reason by recurrence on r. If $r = 0$, the $(\mathcal{O}_{T,t} \otimes k(s))$-module $\mathcal{N}_t \otimes k(s)$ is free (EGA $\mathrm{O_{IV}}$ 17.3.4); there exists therefore a homomorphism α of a free $\mathcal{O}_T$-module of finite type $\mathcal{L}$ into $\mathcal{N}$ such that $\alpha_t \otimes k(s)$ is bijective; let $\mathcal{P} = \mathrm{coker}(\alpha)$. Then $\mathcal{P}_t = 0$ (Nakayama's lemma), so there exists an open affine neighborhood U of t such that $\mathcal{P}|U = 0$. Since $T \to S$ is open, there exists an open affine neighborhood S' of s in S, contained in the image of U. By replacing S by S', T by the reciprocal image T' of S' in U and X by the reciprocal image of T' in X, we can assume that $\mathcal{P} = 0$. Then $(X \to T, \mathcal{L} \xrightarrow{\alpha} \mathcal{N} \to 0)$ is an S-devissage of length 1, in dimension n of the module $\mathcal{M}$ at point x.

Suppose $r > 0$. There exists an S-devissage $(X \to T, \mathcal{L} \xrightarrow{\alpha} \mathcal{N} \to \mathcal{P})$ in dimension n of the $\mathcal{O}_X$-module $\mathcal{M}$ at point x (1.2.1.2). Let τ be the generic point of $T \otimes k(s)$. Since $T \otimes k(s)$ is integral and $\alpha_\tau \otimes k(s)$ injective, $\alpha_t \otimes k(s)$ is injective. On the other hand, since $r > 0$, $\alpha_t \otimes k(s)$ is not bijective. It follows then from EGA $\mathrm{O_{IV}}$ 16.4.4 and the exact sequence of Ext that we have

$$\mathrm{prof}_{\mathcal{O}_{T,t} \otimes k(s)}(\mathcal{N}_t \otimes k(s)) = \mathrm{prof}_{\mathcal{O}_{T,t} \otimes k(s)}(\mathcal{P}_t \otimes k(s)).$$

Furthermore, we have $\dim(\mathcal{P} \otimes k(s)) < n$ (1.2.1.3), thus

$$\mathrm{coprof}_{\mathcal{O}_{T,t} \otimes k(s)}(\mathcal{P}_t \otimes k(s)) < r.$$

Let us apply the recurrence hypothesis: there exists a commutative diagram of pointed schemes

$$\begin{array}{ccc} (T', t') & \rightarrow & (S', s') \\ \downarrow & & \downarrow \\ (T, t) & \rightarrow & (S, s) \end{array}$$

where the columns are elementary étale neighborhoods, such that the reciprocal image $\mathcal{P}'$ of $\mathcal{P}$ on T' admits a total S'-devissage in dimensions between $n-1$ and $n-r$. Let $(X' \to T', \mathcal{L}' \xrightarrow{\alpha'} \mathcal{N}' \to \mathcal{P}')$ be the reciprocal image of the devissage $(X \to T, \mathcal{L} \xrightarrow{\alpha} \mathcal{N} \to \mathcal{P})$ of $\mathcal{M}$ by the étale morphism $T' \to T$. Let x' be the unique point of X' of projections x and t'. Then x' is the unique point of X' above t' and it is clear that the reciprocal image $\mathcal{M}'$ of $\mathcal{M}$ on X' admits a total relative devissage at point x', in dimensions between n and $n-r$.

Remark 2 (1.2.3.1)**.** It is immediate that the bounds n and $n-r$ introduced in (1.2.3) are the best possible.

Corollary 2 (1.2.4)**.** *Let (S,s) be a pointed scheme, X an S-scheme of finite presentation, $\mathcal{M}$ an $\mathcal{O}_X$-module of finite type, $n = \dim(\mathcal{M} \otimes k(s))$ and $r = \mathrm{coprof}_{X\otimes k(s)}(\mathcal{M} \otimes k(s))$. Then there exists an elementary étale neighborhood (S',s') of (S,s), a $(X \times_S S')$-scheme Y, affine, étale over X and whose image in X contains $\mathrm{Supp}(\mathcal{M}) \cap (X \otimes k(s))$ and a finite partition (Y_i) of Y in open and closed subschemes, such that, for all i, the reciprocal image $\mathcal{M}_i$ of $\mathcal{M}$ on Y_i admits an S'-devissage of length $\leq r+1$, in dimensions $\leq n$, above s'.*

The proof is identical to that of 1.1.6 taking into account 1.2.3.

Let us conclude with some elementary results on relative devissages.

First, note that if we have a relative devissage of $\mathcal{M}$ above S and if $(S',s') \to (S,s)$ is a morphism of pointed schemes, we define in a natural way an S'-devissage above s' of the reciprocal image $\mathcal{M}'$ of $\mathcal{M}$ on $X' = X \times_S S'$; this devissage will be called the relative devissage reciprocal image of the initial devissage by the morphism $(S',s') \to (S,s)$. This remark allows us to formulate the result of passage to the limit below which will be useful for reductions to the Noetherian case:

Proposition 1 (1.2.5)**.** *Let S_0 be an affine scheme, S a projective filtering limit of affine S_0-schemes S_i, X_0 an S_0-scheme of finite presentation, $\mathcal{M}_0$ an $\mathcal{O}_{X_0}$-module of finite type. For all i, let $X_i = X_0 \times_{S_0} S_i$, $\mathcal{M}_i = \mathcal{M} \times_{S_0} S_i$ and let $X = X_0 \times_{S_0} S$ and $\mathcal{M} = \mathcal{M}_0 \times_{S_0} S$. Let x be a point of X, s its image in S, x_i its image in X_i and s_i its image in S_i. Finally, let*

$$\begin{array}{ccc} (X',x') & \to & (S',s') \\ \downarrow & & \downarrow \\ (X,x) & \to & (S,s) \end{array} \qquad (*)$$

be a commutative diagram of pointed schemes whose columns are elementary étale neighborhoods, affines, and D' an S'-devissage at point x' of the reciprocal image $\mathcal{M}'$ of $\mathcal{M}$ on X'.

Then, there exists an index i, a commutative diagram of pointed schemes

$$\begin{array}{ccc} (X'_i,x'_i) & \to & (S'_i,s'_i) \\ \downarrow & & \downarrow \\ (X_i,x_i) & \to & (S_i,s_i) \end{array}_{\;i} \qquad (*)$$

where the columns are elementary étale affine neighborhoods and an S'_i-devissage D'_i at point x'_i of the reciprocal image $\mathcal{M}'_i$ of $\mathcal{M}_i$ on X'_i, such that the diagram $(*)$ and the devissage D' are isomorphic to the reciprocal image of the diagram $(*)_i$ and of the devissage D'_i by the morphism $(S,s) \to (S_i,s_i)$.

The proof is immediate from the results of EGA IV 8 and 9 (in particular EGA IV 9.7.7).

§2. A criterion for flatness

Let $f : (X,x) \to (S,s)$ be a morphism locally of finite presentation of pointed schemes, $\mathcal{M}$ an $\mathcal{O}_X$-module of finite presentation. According to 1.2.3, we can *find a commutative diagram of pointed schemes*

$$\begin{array}{ccc} (X', x') & \to & (S', s') \\ \downarrow & & \downarrow \\ (X, x) & \to & (S, s) \end{array}$$

where the columns are elementary étale affine neighborhoods, and an S'-devissage at point x', in dimension $n = \dim_s(\mathcal{M}/S)$, of the $\mathcal{O}_{X'}$-module $\mathcal{M}'$ reciprocal image of $\mathcal{M}$ on X'; let us denote this devissage by $(Y \to T, \mathcal{L} \xrightarrow{\alpha} \mathcal{N} \to \mathcal{P})$.

Let t be the image of x' in T and z the generic point of $T \otimes k(s')$. These data being fixed, we are able to state a criterion for S-flatness of $\mathcal{M}$ at x.

Theorem 2 (2.1)**.** *The following conditions are equivalent:*

- *(i) $\mathcal{M}_x$ is $\mathcal{O}_{S,s}$-flat;*
- *(i') $\mathcal{M}'_{x'}$ is $\mathcal{O}_{S',s'}$-flat;*
- *(ii) α_z is bijective and $\mathcal{P}_z$ is $\mathcal{O}_{S',s'}$-flat;*
- *(ii') α_z is injective and $\mathcal{P}_z$ is $\mathcal{O}_{S',s'}$-flat;*
- *(ii") there exists an open neighborhood U' of s' in S', such that α is S'-universally injective (§0) above U' and $\mathcal{P}$ is $\mathcal{O}_{S',s'}$-flat;*
- *(iii) $Tor_1^{\mathcal{O}_{S,s}}(\mathcal{M}_x, k(s)) = 0$.*

[2.1.1]

- *a) This result will be extended, to some extent, to the case where $\mathcal{M}$ is only assumed to be of finite type on X (3.4).*
- *b) Conditions i) and iii) are independent of the choice of the relative devissage of $\mathcal{M}$.*

Let us first prove a lemma:

Lemma 3 (2.2)**.** *Let $(T, t) \to (S, s)$ be a smooth morphism of pointed schemes, such that $T \otimes k(s)$ is integral with generic point τ, α a homomorphism of a free $\mathcal{O}_T$-module $\mathcal{L}$ to a $\mathcal{O}_T$-module $\mathcal{N}$ of finite type, such that $\alpha \otimes k(s)$ is surjective at point τ. The following conditions are equivalent:*

- *(i) α_τ is surjective;*
- *(ii) α_t is S-universally injective.*

Proof. Since $\alpha \otimes k(s)$ is surjective at τ, α_τ is surjective, thus (ii) $\Rightarrow$ (i). Suppose (i) is verified and consider the commutative diagram of $\mathcal{O}_{T,t}$-modules

$$\mathrm{L}_t[r, "\alpha_t"][d]\mathcal{N}_t[d]$$
$$\mathrm{L}_\tau[r, "\alpha_\tau"]\mathcal{N}_\tau$$

The second row is bijective; to prove (ii), it suffices to prove that the first column is S-universally injective; since $\mathcal{L}$ is free, it suffices to establish that the morphism $\mathcal{O}_{T,t} \to \mathcal{O}_{T,\tau}$ is S-universally injective. For this, we can assume S is local with closed point s. Consider (S, s) as a filtered projective limit of local noetherian schemes $(S_i, s_i)_{i \in I}$. We can assume T is affine, in which case T comes from a smooth T_i for i large enough (EGA IV 17.7.8). Let $T_j = T_i \otimes_{S_i} S_j$ for $j \geq i$ and let t_j and τ_j be the images of t and τ in T_j. Then the morphism $\mathcal{O}_{T,t} \to \mathcal{O}_{T,\tau}$ is the inductive limit of the morphisms $\mathcal{O}_{T_j,t_j} \to \mathcal{O}_{T_j,\tau_j}$. It suffices therefore to consider the case where S is noetherian.

Let F be the multiplicative part of $\mathcal{O}_{T,t}$ formed by the non-zero divisor elements in $\mathcal{O}_{T,t} \otimes k(s)$. Since $\mathcal{O}_{T,t} \otimes k(s)$ is integral, the ring $\mathcal{O}_{T,t}[F^{-1}]$ is none other than $\mathcal{O}_{T,\tau}$. Moreover, if $a \in F$, it follows from EGA $\mathrm{O_{III}}$ 10.2.4 that multiplication by a in $\mathcal{O}_{T,t}$ is injective and has an S-flat cokernel. By passage to the inductive limit, we deduce that we have an exact sequence

$$0 \to \mathcal{O}_{T,t} \to \mathcal{O}_{T,\tau} \to H \to 0$$

where H is S-flat, from which it follows that $\mathcal{O}_{T,t} \to \mathcal{O}_{T,\tau}$ is S-universally injective. $\square$

Proof of 2.1. Since $(X', x') \to (X, x)$ and $(S', s') \to (S, s)$ are étale, we have the equivalences

$$\mathcal{N}_x \otimes_{\mathcal{O}_{S,s}} \text{-flat} \Leftrightarrow \mathcal{N}'_{x'} \otimes_{\mathcal{O}_{S',s'}} \text{-flat}.$$

Moreover, since $Y \to T$ is finite and x' is the only point of Y above t, $\mathcal{N}_\tau$ and $\mathcal{N}'_{x'}$ are isomorphic as S-modules, hence (i) $\Leftrightarrow$ (i').

Flatness criteria

Since T is flat over S', the implication (ii') $\Rightarrow$ (i') follows from the exact sequence $0 \to \mathcal{L}'_i \to \mathcal{N}'_i \to \mathcal{P}'_i \to 0$.

We have (ii) $\Leftrightarrow$ (ii') according to 2.2.

To establish (i') $\Rightarrow$ (ii), let us consider the $\mathcal{O}_T$-module $\mathcal{R} = Ker(\alpha)$. Since $\mathcal{P}_i = 0$, we have an exact sequence of $\mathcal{O}_{T,\tau}$-modules

$$0 \to \mathcal{R}_\tau \to \mathcal{L}'_\tau \xrightarrow{\alpha_\tau} \mathcal{N}'_i \to 0.$$

Since $\mathcal{M}$ is of finite presentation over X, $\mathcal{N}'$ is of finite presentation over T (1.2.1.2), thus $\mathcal{R}_\tau$ is of finite type over $\mathcal{O}_{T,\tau}$. If $\mathcal{N}'_i$ is S'-flat, the preceding sequence remains exact after tensoring by $k(s)$

$$0 \to \mathcal{R}_\tau \otimes k(s) \to \mathcal{L}'_\tau \otimes k(s) \xrightarrow{\alpha_\tau \otimes k(s)} \mathcal{N}'_i \otimes k(s) \to 0.$$

But $\alpha_\tau \otimes k(s)$ is bijective, thus $\mathcal{R}_\tau \otimes k(s) = 0$ and consequently $\mathcal{R}_\tau = 0$ (Nakayama's lemma). According to 2.2, α_i is then S'-universally injective. Since $\mathcal{N}'_i$ is S'-flat, we conclude that $\mathcal{P}'_i$ is also S'-flat, thus (i') $\Rightarrow$ (ii).

The implication (ii') $\Rightarrow$ (ii) is clear; let us prove (ii) $\Rightarrow$ (ii'). Since $\mathcal{N}'$ is of finite presentation over T and $\mathcal{L}'$ of finite type over T, the set V' of points of T where α is bijective is open; it contains τ according to (ii). Moreover, a morphism that is smooth and open, thus the image U' of V' is an open subset of S' containing s'. Since the fibers of $T \to S'$ are integral, we deduce from 2.2 that α is S'-universally injective above U'.

The implication (i) $\Rightarrow$ (iii) is trivial; let us prove (iii) $\Rightarrow$ (i) by recurrence on $n = \dim_s(\mathcal{M}/S)$. If $n < 0$, there is nothing to prove. Suppose $n \geq 0$. The hypothesis implies $Tor_1^{\mathcal{O}_{S,s}}(\mathcal{M}'_{s'}, k(s)) = Tor_1^{\mathcal{O}_{S,s}}(\mathcal{N}'_i, k(s)) = 0$, hence by localization $Tor_1^{\mathcal{O}_{S,s}}(\mathcal{N}'_i, k(s))$. As in the demonstration of (i') $\Rightarrow$ (ii), we deduce that $\mathcal{R}_\tau = 0$. Applying lemma 2.2, we deduce an exact sequence

$$0 \to \mathcal{L}'_i \xrightarrow{\alpha_i} \mathcal{N}'_i \to \mathcal{P}'_i \to 0.$$

The exact sequence of tors then shows that $Tor_1^{\mathcal{O}_{S,s}}(\mathcal{P}'_i, k(s)) = 0$; thus $\mathcal{P}'_i$ is S-flat by the recurrence hypothesis; but then $\mathcal{N}'_i$ is S-flat as an extension of two flat modules. Q.E.D.

Suppose now given an S'-devissage of length r of the $\mathcal{O}_X$-module M' at point x', denoted $(Y_i \to T_i, \mathcal{L}'_i \xrightarrow{\alpha_i} \mathcal{N}'_i \to \mathcal{P}'_i)$ $1 \leq i \leq r$ and let τ_i be the generic point of $T_i \otimes k(s')$ and t_i the point of T_i deduced from x' by proximity. By recurrence on r, we immediately deduce from 2.1 the following flatness criterion:

Corollary (2.3). *The following conditions are equivalent:*

(i) $\mathcal{M}$ is S-flat at x;

(ii) α_i is bijective at τ_i for all i and $\mathcal{P}_r$ is S'-flat at t_r;

(iii) $\mathcal{N}'_i$ is free over a neighborhood of τ_i for all i and $\mathcal{P}_r$ is S'-flat at t_r;

(iv) α_i is injective for all i and $\mathcal{P}_r$ is S'-flat at t_r; (v) there exists an open neighborhood U' of s' in S', such that α_i is S'-universally injective above U' for all i and $\mathcal{P}_r$ is S'-flat at t_r.

Combining corollary 2.3 with 1.2.3, we obtain the following result:

Corollary (2.4). *Let $f : (X, x) \to (S, s)$ be a locally finitely presented morphism of pointed schemes, $\mathcal{M}$ a $\mathcal{O}_X$-module of finite presentation that is S-flat at point x, $n = \dim_x(\mathcal{M}/S)$ and $r = coprof_{e_x, \otimes \mathcal{O}_{k(s)}}(\mathcal{M}_x \otimes k(s))$. Then, there exists a commutative diagram of pointed schemes*

$$\begin{array}{ccc} (X', x') & \longrightarrow & (S', s') \\ \downarrow & & \downarrow \\ (X, x) & \longrightarrow & (S, s) \end{array}$$

where the columns are elementary étale affine neighborhoods and a total S'-devissage at point x' $(Y_i \to T_i, \mathcal{L}'_i \xrightarrow{\alpha_i} \mathcal{N}'_i \to \mathcal{P}'_i)_{1 \leq i \leq r}$ of the reciprocal image $\mathcal{M}'$ of $\mathcal{M}$ on X', in dimensions between n and $n - r$, such that α_i is injective for all i.

This being established, let us define

$$A = \Gamma(S, \mathcal{O}_S), \quad M' = \Gamma(X', \mathcal{M}'), \quad M'_{(i)} = Ker\ M' \to \Gamma(T_i, \mathcal{P}_i).$$

We thus define an increasing filtration on the A-module M' such that $M'_{(i)}/M'_{(i-1)} \simeq \Gamma(T_i, \mathcal{L}'_i)$. We have therefore proved the following result:

Corollary (2.5). *Let B be a finitely presented A-algebra, $\mathfrak{q}$ a prime ideal of B above a prime ideal $\mathfrak{p}$ of A and M a B-module of finite presentation flat over A at $\mathfrak{q}$. Then there exists an elementary étale neighborhood $(A', \mathfrak{p}')$ of $(A, \mathfrak{p})$ and an elementary étale neighborhood $(B', \mathfrak{q}')$ of $(B, \mathfrak{q})$ above $(A', \mathfrak{p}')$, such that the A'-module $M' = M \otimes_B B'$ has a finite composition series whose successive quotients are A'-modules underlying finite type free modules over smooth A'-algebras, with geometrically integral fibers.*

Corollary (2.6) *(cf. EGA IV 11.3.1). Let S be a scheme, X an S-scheme locally of finite presentation, $\mathcal{M}$ a $\mathcal{O}_X$-module of finite presentation; the set of points x of X such that $\mathcal{M}$ is S-flat at x is open.*

This follows from 2.4 taking into account the fact that the image of X' in X is open.

Corollary (2.7) *(cf. EGA IV 11.2.6). Under the general hypotheses of passage to the projective limit described in 1.2.5, if $\mathcal{M}_0$ is of finite presentation over X_0 and if $\mathcal{M}$ is S-flat at x, then $\mathcal{M}_i$ is S_i-flat at x_i for i large enough.*

Flatness criteria

Indeed, if we replace (X_0, x_0) and (S_0, s_0) with étale neighborhoods, we can assume that $\mathcal{M}_0$ has a total S_0-devissage at x_0 (1.2.3). It suffices then to note that the flatness criterion (2.3.iii)) passes to the projective limit (EGA IV 8.5.5).

Remark *(2.8). In this paragraph, we have used the properties for a smooth morphism to pass to the projective limit and to be open. In EGA IV, the demonstration of these properties uses the passage to the projective limit of flatness. But one can easily obtain direct demonstrations by proving them separately for étale morphisms ([17] V th. 3) and for polynomial algebras. One can thus avoid any vicious circle in the demonstration of the two corollaries above.*

Let us indicate, to conclude, a result similar to 2.3.

Corollary *(2.9). Let $f : (X, x) \to (S, s)$ be a morphism of finite presentation of pointed schemes, $\mathcal{M}$ an $\mathcal{O}_X$-module of finite presentation, Z the closed set of points of X where $\mathcal{M}$ is not S-flat, m an integer less than $n = \dim_x(\mathcal{M}/S)$. Suppose given a commutative diagram*

(X', x') [r] [d] (S', s') [d]
(X, x) [r] (S, s)

where the columns are elementary étale affine neighborhoods and an S'-devissage at point x' $(Y_i \to T_i, \mathcal{L}_i \xrightarrow{\alpha_i} \mathcal{N}_i \to \mathcal{R}_i)$ of the reciprocal image $\mathcal{M}'$ of $\mathcal{M}$ on X', in dimensions between n and m. Then the following conditions are equivalent:

> *[(i)]$\dim_x(Z/S) < m$; α_i is bijective at the generic point τ_i of $T_i \otimes k(s)$ for all i; α_i is injective at τ_i for all i; there exists a neighborhood of s' in S' above which α_i is S'-universally injective for all i.*

Proof. *Let Z' be the reciprocal image of Z in X' and Z_i the image of Z' in T_i. Then Z_i is the set of points of T_i where $\mathcal{N}_i$ is not S'-flat and since Y_i is finite over T_i and that x' is the only point of Y_i above t_i, we have* $\dim_x(Z/S) = \dim_{x'}(Z'/S') = \dim_{t_i}(Z_i/S')$.

Let us reason by recurrence on n, the case $n < 0$ being trivial. According to 2.1 (applied at point τ_i) and 2.2, each of the conditions (i) to (iv) implies $\tau_i \notin Z_i$. Moreover, if we restrict S' they imply that α_i is S-universally injective, in which case Z_i is the set of points of T_i where $\mathcal{R}_i$ is not S'-flat. Since $\dim_s(\mathcal{R}_i/S') < n$, *the recurrence hypothesis implies that the four conditions are equivalent.*

1 Flatness and projectivity

1.1 Lemmas on projective modules of countable type

We have gathered here some auxiliary results concerning projective modules; some of them will be studied in detail in the second part of this article.

We denote by A a commutative ring; for any A-module M, we note M^* the dual of M, that is the A-module $Hom_A(M, A)$.

(3.1.1) Every flat A-module is an inductive filtered limit of free A-modules of finite type.

(3.1.2) Let $u : M \to N$ be a universally injective A-linear application. If N is projective of countable type and if M is of countable type, then M is projective.

(3.1.3) Let (L_i, u_{ji}) be an inductive filtering system of free A-modules of finite type, indexed by a countable set I. Let $P = \lim_{\longrightarrow} L_i$. Then, for P to be projective, it is necessary and sufficient that the projective system (L_i^*) satisfies the Mittag-Leffler condition (EGA O_{III} 13.1.2).

(3.1.4) Let B be a faithfully flat commutative A-algebra, P an A-module of countable presentation. For P to be a projective A-module, it is necessary and sufficient that $B \otimes_A P$ is a projective B-module.

(3.1.5) Suppose A is Noetherian; let I be a set and P a submodule of countable type of A^I. If the inclusion of P in A^I is universally injective, P is projective.

(3.1.6) Suppose A is local with residue field k and let u be an A-linear application from a projective A-module P to a flat A-module M. For u to be universally injective, it is necessary and sufficient that $u \otimes_A k$ is injective.

Proof. *Assertions 3.1.1 and 3.1.2 are due to Daniel Lazard ([13] I.1.2 and I.3.2).*

Let us prove 3.1.3. The condition is sufficient: indeed, if it is verified, for any A-module M, the projective system $(Hom_A(L_i, M))$ also satisfies the Mittag-Leffler condition (because $Hom_A(L_i, M) \simeq L_i^* \otimes_A M$); taking into account the functorial isomorphism $Hom_A(P, M) \xrightarrow{\sim} \lim_{\longleftarrow} Hom_A(L_i, M)$, we deduce from this remark, from the hypothesis of countability of I and from EGA O_{III} 13.2.2 that the functor $Hom_A(P, \cdot)$ is exact, thus P is projective. The condition is necessary: to see this, we can assume P is free (up to replacing P by $P \oplus Q$ where Q is a projective module of countable type, taking into account 3.1.1). Let

$(e_s)_{s\in S}$ be a basis of P. For any finite part T of S, let P_T be the submodule of P generated by $(e_s)_{s\in T}$. If i is an element of I, there exists a finite part T of S such

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that P_T contains the image of L_i in P. Since $P = \lim_{\longrightarrow} L_i$, and since T is finite, there exists an index $j \geq i$ such that the inclusion of P_T in P factors through L_j. We have the inclusions

$$Im(P^* \to L_i^*) \subset Im(L_j^* \to L_i^*) \subset Im(P_T^* \to L_i^*) = Im(P^* \to L_i^*)$$

since P_T is a direct submodule of P. For all $k \geq j$ we thus have $Im(L_k^* \to L_i^*) = Im(L_j^* \to L_i^*)$, in other words, the projective system (L_i^*) satisfies the Mittag-Leffler condition.

Let us prove 3.1.4. Since P is a flat A-module of countable presentation, it is the inductive limit of a sequence (L_n) of free modules (cf. [13] I 3.2); applying 3.1.3 we are thus reduced to seeing that the Mittag-Leffler condition for projective systems of A-modules descends by faithful flatness; this is immediate.

Assertion 3.1.5 results from [8] 2.4. (See also Part 2.)

Let us prove 3.1.6. Up to replacing u by $u \oplus 1_Q$ where Q is a suitable projective module, we can assume P is free; up to replacing P by its various direct free factors of finite type, we can assume P is of finite type; taking into account 3.1.1 we can assume M is free of finite type; we are then reduced to seeing that u is left invertible if and only if $u \otimes_A k$ is injective, which is quite clear.

1.2 Lemmas on the relative assassin

Definition 2 (3.2.1)**.** Let S be a scheme, s a point of S, $\mathcal{M}$ an $\mathcal{O}_S$-module quasi-coherent. We say that s is associated to $\mathcal{M}$ if there exists an element f of $\mathcal{M}$ whose annihilator I in $\mathcal{O}_{S,s}$ has for its radical the maximal ideal of $\mathcal{O}_{S,s}$. We call assassin of $\mathcal{M}$ in S and denote by $\mathrm{Ass}_S(\mathcal{M})$, or simply $\mathrm{Ass}(\mathcal{M})$, the set of points of S associated to M.

Definition 3 (3.2.2)**.** Let S be a scheme, X an S-scheme, $\mathcal{M}$ an $\mathcal{O}_X$-module quasi-coherent. We call assassin of $\mathcal{M}$ in X relative to S and denote by $\mathrm{Ass}_{X/S}(\mathcal{M})$ (or simply $\mathrm{Ass}(\mathcal{M}/S)$) the set

$$\bigcup_{s\in S} \mathrm{Ass}_{X\otimes k(s)}(\mathcal{M} \otimes k(s)).$$

For the usual properties of the assassin, we refer to II 1. Let us note here some elementary properties of the relative assassin:

(3.2.3). Let

$$\begin{array}{ccc} X' & \longrightarrow & S' \\ \downarrow & & \downarrow \\ X & \longrightarrow & S \end{array}$$

a Cartesian diagram of schemes, where the lines are morphisms locally of finite type, $\mathcal{M}$ an $\mathcal{O}_X$-module of finite type, $\mathcal{M}'$ its reciprocal image on X', x' a point of X', with respective projections x, s, s' in X, S, S'. Then for $x' \in Ass(\mathcal{M}'/S')$, it is necessary and sufficient that $x \in Ass(\mathcal{M}/S)$ and that x' is a maximal point of $Spec(k(x) \otimes_{k(s)} k(s'))$ (EGA IV 4.2).

(3.2.4). Let S be a scheme, X an S-scheme locally of finite type, Y an X-scheme flat and locally of finite type, $\mathcal{M}$ an $\mathcal{O}_X$-module of finite type, $\mathcal{N}$ its reciprocal image on Y, y a point of Y with projection x in X. For $y \in Ass(\mathcal{N}/S)$, it is necessary and sufficient that $x \in Ass(\mathcal{M}/S)$ and $y \in Ass(Y/X)$ (EGA IV 3.3.1).

(3.2.5). Let $f : (X, x) \to (S, s)$ be a morphism locally of finite presentation of pointed schemes, $\mathcal{M}$ an $\mathcal{O}_X$-module of finite presentation and S-flat, $A = \mathcal{O}_{S,s}$, $B = \mathcal{O}_{X,x}$, Σ the multiplicative part of B formed by the non-zero-divisor elements in $\mathcal{M}_x \otimes k(s)$.

(i) Then $B[\Sigma^{-1}]$ is a semi-local ring whose spectrum is formed by the generalizations of $Ass_{B\otimes k(s)}(\mathcal{M}_x \otimes k(s))$ in $Spec(B)$.

(ii) Every element of Σ defines an A-universally injective homothety in $\mathcal{M}_x$.

(iii) The localization morphism $\mathcal{M}_x \to \mathcal{M}_x[\Sigma^{-1}]$ is A-universally injective.

Proof. Let $\mathfrak{p}_{i,i\in I}$ be the finite family of prime ideals of B that are associated to $\mathcal{M}_x \otimes k(s)$. Then $\Sigma = B - (\bigcup_{i\in I} \mathfrak{p}_i)$, which gives the first assertion. To establish (ii) and (iii), we reduce by passage to the inductive limit to the case where A is Noetherian (2.7). Let $a \in \Sigma$. According to EGA O_{III} 10.2.4, multiplication by a in $\mathcal{M}_x$ is injective and its cokernel is A-flat, thus multiplication by a is A-universally injective. Since the localization morphism $\mathcal{M}_x \to \mathcal{M}_x[\Sigma^{-1}]$ is the filtering inductive limit of the homotheties $\mathcal{M}_x \to \mathcal{M}_x$ defined by the elements of Σ, assertion (iii) follows from (ii).

***Corollary** (3.2.6). Let A be a ring, B an A-algebra of finite presentation, M a B-module of finite presentation and A-flat, B' a B-algebra flat such that the image of $Spec(B')$ in $Spec(B)$ contains $Ass_{B/A}(M)$. Then the canonical morphism $M \to M' = M \otimes_B B'$ is universally A-injective.*

Proof. It suffices to show that for every prime ideal $\mathfrak{q}$ of B, the morphism $M_{\mathfrak{q}} \to M'_{\mathfrak{q}}$ is A-universally injective. Let $\mathfrak{p}$ be the reciprocal image of $\mathfrak{q}$ in A. We can assume A local with maximal ideal $\mathfrak{p}$. Let Σ be the multiplicative part of $B_{\mathfrak{q}}$ formed by the A-universally

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where the columns are elementary étale neighborhoods, such that $\Gamma(X', \mathcal{M} \times_X X')$ *is a* $\Gamma(S', \mathcal{O}_{S'})$ *projective module.*

This follows from 2.5 and 3.3.1.

Definition 4 (3.3.3)**.** Let S be a scheme, X a locally finite type S-scheme, $\mathcal{M}$ a $\mathcal{O}_X$-module of finite type.

[(i)]Let s be a point of S; denote by $(\bar{S}, \bar{s})$ a henselization of (S, s), $\bar{X} = X \times_S \bar{S}$, $\bar{\mathcal{M}} = \mathcal{M} \times_S \bar{S}$. We say that $\mathcal{M}$ is *pure* along $X \otimes k(s)$ if $\mathrm{Ass}_{\bar{X}/\bar{S}}(\bar{\mathcal{M}})$ is contained in the generic fiber of $X \otimes k(s)$ (in other words, if for every point $\bar{x}$ of $\mathrm{Ass}(\bar{\mathcal{M}}/\bar{S})$, the adherence of $\bar{x}$ in $\bar{X}$ meets $X \otimes k(s)$). We say that $\mathcal{M}$ is S-pure (or pure relative to S), if $\mathcal{M}$ is pure along $X \otimes k(s)$ for every point s of S.

[3.3.4]

[(i)]If X is proper over S, any sheaf $\mathcal{M}$ of finite type on X is S-pure. Suppose X is of finite type and separated over S with finite fibers; then X is S-pure (i.e., $\mathcal{O}_X$ is S-pure) if and only if X is finite over S. Suppose X is flat over S with geometrically irreducible fibers, without immersed components; then X is S-pure.

The notion of purity will allow us to characterize projective modules among flat modules; indeed, we have the following result:

Theorem 3 (3.3.5)**.** *Let A be a ring, B an A-algebra of finite presentation, M a B-module of finite presentation, flat over A. Then M is a projective A-module if and only if M is A-pure (i.e., if the module $\tilde{M}$ on Spec(B) is pure relative to Spec(A)).*

Necessity of the condition. Suppose that M is a projective A-module and let us show that M is A-pure. We can assume A is local henselian, with maximal ideal $\mathfrak{p}$. Let Σ be the multiplicative part of B formed by the non-zero-divisors in $M \otimes_A k(\mathfrak{p})$. Then, it follows from 3.1.6 that the localization morphism $M \to M[\Sigma^{-1}]$ is universally A-injective; a fortiori, any element of $\mathrm{Ass}_{B/A}(M)$ is a generization of $\mathrm{Ass}(M \otimes_A k(\mathfrak{p}))$ and M is pure along $B \otimes_A k(\mathfrak{p})$.

Before addressing the sufficiency of the purity condition, we will give a characterization of S-pure modules.

Let (S, s) be a pointed scheme, X an S-scheme of finite presentation, $\mathcal{M}$ an $\mathcal{O}_X$-module of finite presentation, S-flat, Z the finite set $\mathrm{Ass}(\mathcal{M} \otimes k(s))$. We deduce from 3.3.2 that one can find an elementary étale neighborhood (S', s') of (S, s) and an étale morphism $Y' \to X' = X \times_S S'$ satisfying the following conditions:

[a)]Y' is affine and the image of Y' in X' contains $Z' = \mathrm{Ass}(\mathcal{M} \otimes k(s'))$; $\Gamma(Y', \mathcal{M} \times_X Y)$ is a $\Gamma(S', \mathcal{O}_{S'})$-projective module.

Criteria for purity

With these data fixed, we have the following result:

Proposition 2 (3.3.6)**.** *For $\mathcal{M}$ to be pure along $X \otimes k(s)$, it is necessary and sufficient that the image of Y' in X' contains $Ass_{X'/S'}(\mathcal{M} \times_X X')$ above $Spec(\mathcal{O}_{S',s})$.*

3. *Proof.* We can replace S by S', then assume S-local with closed point s.

Suppose $\mathcal{M}$ pure along $X \otimes k(s)$. Since the image of Y' in X is an open set, to establish that this open set contains $\mathrm{Ass}(\mathcal{M}/S)$, it suffices to show that any point x of $\mathrm{Ass}(\mathcal{M}/S)$ is a generization of Z. By hypothesis, the adherence $\bar{x}$ of x in X meets $X \otimes k(s)$. So there exists a point t of $X \otimes k(s)$ which is a specialization of x. Let Σ be the multiplicative part of $\mathcal{O}_{X,t}$ formed by the universally injective homothetic transformations of $\mathcal{M}_t$. According to 3.2.5, $\mathcal{O}_{X,t}[\Sigma^{-1}]$ is a semi-local ring whose maximal ideals are points of Z and the morphism

$$\mathcal{M}_t \to \mathcal{M}_t[\Sigma^{-1}]$$

is universally injective. It follows that the points associated with the fibers of $\mathcal{M}_t$ above S, in particular x, are generizations of Z.

Conversely, suppose that the image of Y' in X contains $\mathrm{Ass}(\mathcal{M}/S)$ and let us show that $\mathcal{M}$ is S-pure along $X \otimes k(s)$. We can assume (S, s) henselian (3.2.3). Let $x \in \mathrm{Ass}(\mathcal{M}/S)$. By hypothesis there exists a point y' of Y' above x. Since Y' is étale over X, $y' \in \mathrm{Ass}(\mathcal{M} \times_X Y'/S)$. According to the part already demonstrated in 3.3.5, $\mathcal{M} \times_X Y'$ is S-pure, so the adherence of y' in Y' meets $Y' \otimes k(s)$; a fortiori, the adherence of x in X meets $X \otimes k(s)$. Q.E.D. □

Corollary 3 (3.3.7)**.** *Let $(T, t) \to (S, s)$ be a morphism of pointed schemes, X an S-scheme of finite presentation, $\mathcal{M}$ an $\mathcal{O}_X$-module of finite presentation and S-flat. If $\mathcal{M}$ is pure along $X \otimes k(s)$, then $\mathcal{M} \times_S T$ is pure along $X \otimes k(t)$; the converse is true if T is S-flat.*

This follows immediately from 3.3.6 and 3.2.3.

Corollary 4 (3.3.8)**.** *Let us keep the notations of 3.3.6 and assume that $\mathcal{M}$ is pure along $X \otimes k(s)$. Then, there exists an open set V' of S', containing s', such that $Ass_{X'/S'}(\mathcal{M} \times_X X')$ is contained in the image of Y'. If V is the open set of S image of V', then $\mathcal{M}|X \times_S U$ is U-pure. In particular, the set of points $s \in S$ such that $\mathcal{M}$ is pure along $X \otimes k(s)$ is an open set.*

Indeed, the existence of the open set V' follows from 3.3.6 and the lemma below; $\mathcal{M} \times_S V'$ is then V'-pure according to 3.3.6, thus $\mathcal{M} \times_S V$ is V-pure (3.3.7).

Lemma 4 (3.3.9)**.** *Let $X \to S$ be a morphism of finite presentation, $\mathcal{M}$ an $\mathcal{O}_X$-module of finite presentation, U an open subset of X of finite type over S. Then, the set of points s of S such that $Ass(\mathcal{M} \otimes k(s))$ is contained in U is a locally constructible part of S.*

Indeed, by passing to the limit, we reduce to the case where S is Noetherian, in which case the property follows immediately from EGA IV 9.8.3.

Corollary 5 (3.3.10)**.** *Under the limit passage hypotheses of 1.2.5, suppose* $\mathcal{M}_0$ *of finite presentation over* X_0 *and* S_0*-flat. Then if* $\mathcal{M}$ *is pure along* $X \otimes k(s)$ *(resp. is* S*-pure), there exists an index* i *such that* $\mathcal{M}_i$ *is pure along* $X_i \otimes k(s_i)$ *(resp. is* S_i*-pure).*

Proof. The second assertion follows from the first, taking into account the last assertion of 3.3.8. To establish the first assertion, we choose a commutative diagram

$$\begin{array}{ccccc} X_0 & \leftarrow & X_0' & \leftarrow & Y_0' \\ \downarrow & & \downarrow & & \\ S_0 & \leftarrow & S_0' & & \end{array}$$

such that (S_0', s_0') is an elementary étale neighborhood of (S_0, s_0). $X_0' = X_0 \times_{S_0} S_0'$, Y_0' is affine and étale over X_0', the image of Y_0' in X_0' contains $\mathrm{Ass}(\mathcal{M}_0 \otimes k(s_0'))$ and the reciprocal image of $\mathcal{M}_0$ on Y_0' has global sections that form a projective module over $\Gamma(S_0')$ (cf. 3.3.6). The fact that the purity property along $X \otimes k(s)$ passes to the limit then follows from 3.3.6, 3.3.9 and EGA IV 8.3.11. $\square$

End of the proof of 3.3.5. Suppose that the A-module flat M is pure and let us show that M is then a projective A-module. Since M is of finite presentation over B, M is an A-module of countable presentation; the assertion to be demonstrated is therefore of a local nature for the étale topology on $\mathrm{Spec}(A)$ (3.1.4) and (3.3.7). Taking into account 3.3.8, we can then suppose that there exists an étale B-algebra B', such that $M \otimes_B B' = M'$ is a projective A-module and such that the image of $\mathrm{Spec}(B')$ in $\mathrm{Spec}(B)$ contains $\mathrm{Ass}_{B/A}(M)$. But then the morphism $M \to M'$ is universally A-injective (3.2.6) and consequently M is a projective A-module (3.1.2). This completes the proof of 3.3.5.

Corollary 6 (3.3.11)**.** *Under the hypotheses of 3.3.5, suppose that for every prime ideal* $\mathfrak{p}$ *of* A*, we have* $\dim_{B\otimes k(\mathfrak{p})}(M \otimes k(\mathfrak{p})) \geq 1$*. Then if* M *is an* A*-flat and* A*-pure* A*-module,* M *is a free* A*-module.*

Indeed, by passing to the limit (3.3.10), we can suppose A Noetherian. Without considering the connected components of $\mathrm{Spec}(A)$, we can suppose $\mathrm{Spec}(A)$ connected. Given the hypothesis made on the fibers of M, the A-module projective M is not of finite type, it is therefore free [6].

Criteria of flatness

Corollary 7 (3.3.12)**.** *Under the hypotheses of 3.3.5, if* M *is* A*-flat and* A*-pure, there exists a finite covering of* $\mathrm{Spec}(A)$ *by affine opens of rings* A_i *such that* $M \otimes_A A_i$ *is free over* A_i*.*

Corollary 8 (3.3.13)**.** *Let* $(X, x) \to (S, s)$ *be a morphism of finite presentation,* $\mathcal{M}$ *an* $\mathcal{O}_X$*-module of finite presentation,* S*-flat at* x*. Suppose* (S, s) *is local Henselian, then there exists an affine open neighborhood* U *of* x *in* X*, such that* $\Gamma(U, \mathcal{M})$ *is a free* $\Gamma(S, \mathcal{O}_S)$*-module.*

We can assume X affine, $\mathcal{M}$ S-flat and $\mathrm{Ass}(\mathcal{M} \otimes k(s))$ formed of generalizations of x. Since (S, s) is Henselian, there exists an étale affine neighborhood of (X, x), say X' such that $\Gamma(X', \mathcal{M} \times_X X')$ is a $\Gamma(S, \mathcal{O}_S)$-projective module (3.3.2). Let U be an affine open of X, containing x and contained in the image of X'; let U' be its reciprocal image in X'. Since $\mathcal{M}' = \mathcal{M} \times_X X'$ is S-pure, $\mathrm{Ass}(\mathcal{M}'/S)$ is formed of generalizations of $\mathrm{Ass}(\mathcal{M}' \otimes k(s))$ and consequently $\mathrm{Ass}((\mathcal{M}|U)/S)$ is formed of generalizations of $\mathrm{Ass}(\mathcal{M} \otimes k(s))$, thus $\mathcal{M}|U$ is pure along $\mathcal{M} \otimes k(s)$ and therefore S-pure (3.3.8). But then $\Gamma(U, \mathcal{M})$ is free over $\Gamma(S, \mathcal{O}_S)$ (3.3.12).

3.4. Application to modules of finite type

(3.4.0). Let $f : (X, x) \to (S, s)$ be a locally finite presentation morphism of pointed schemes, $\mathcal{M}$ an $\mathcal{O}_X$-module of finite type and S-flat at x.

We will see that the flatness hypothesis on $\mathcal{M}$ tends to automatically make the $\mathcal{O}_X$-module $\mathcal{M}$ of finite presentation in a neighborhood of x.

Let us begin with the case of a local Henselian base.

Theorem 4 (3.4.1)**.** *Under the conditions of 3.4.0, if (S, s) is local Henselian, there exists an open neighborhood U of x in X such that $\mathcal{M}|U$ is of finite presentation over $\mathcal{O}_U$ and S-flat.*

Proof. The question is local for the étale topology on (X, x). Since (S, s) is Henselian, we can therefore assume that X is affine and that $\mathcal{M}$ admits at x a total S-devissage $(Y_i \to T_i, \mathcal{L}_i \xrightarrow{\alpha_i} \mathcal{N}_i \to \mathcal{R}_i)_{1 \le i \le r}$ (1.2.3). We will prove that $\mathcal{M}$ is then of finite presentation over X and S-flat. It suffices to show that all α_i are injective; by recursion on i, it suffices to show that α_1 is S-universally injective. Let t_1 be the image of x in T_1 and τ_1 the generic point of $T_1 \otimes k(s)$. Now $T_1 \otimes k(s)$ is integral and by construction $(\mathcal{L}_1)_{t_1} \otimes k(s) \to (\mathcal{N}_1)_{t_1} \otimes k(s)$ is injective; it follows that the morphism

$$\Gamma(T_1, \mathcal{L}_1) \otimes k(s) \to (\mathcal{N}_1)_{t_1} \otimes k(s)$$

is also injective. Since $\Gamma(T_1, \mathcal{L}_1)$ is a projective $\Gamma(S, \mathcal{O}_S)$-module (3.3.1), it follows from 3.1.6 that α_1 is universally S-injective. □

Corollary 9 (3.4.2)**.** *Under the conditions of 3.4.0, $\mathcal{M}_x$ is an $\mathcal{O}_{X,x}$-module of finite presentation.*

Indeed, this follows from (3.4.1) in the case where (S, s) is Henselian; we deduce the general case by flat descent.

To study the passage from the punctual to the local, we need the following variant of EGA IV 3.3.1 (where the Noetherian hypothesis is replaced by a relative finiteness hypothesis):

Proposition 3 (3.4.3)**.** *Let $f : X \to S$ be a locally finitely presented morphism, $\mathcal{M}$ an $\mathcal{O}_X$-module of finite type S-flat, $\mathcal{N}$ an $\mathcal{O}_S$-module. We have* $\mathrm{Ass}_X(\mathcal{M} \otimes_{\mathcal{O}_X} f^*(\mathcal{N})) = \mathrm{Ass}_{X/S}(\mathcal{M}) \cap f^{-1}(\mathrm{Ass}_S(\mathcal{N}))$.

Let us begin by establishing two lemmas.

Lemma 5 (3.4.4)**.** *Let $f : X \to S$ be a finite morphism, $\mathcal{M}$ an $\mathcal{O}_X$-module. We have* $\mathrm{Ass}_S(f_*(\mathcal{M})) = f(\mathrm{Ass}_X(\mathcal{M}))$.

Proof. The inclusion $\mathrm{Ass}_S(f_*(\mathcal{M})) \subset f(\mathrm{Ass}_X(\mathcal{M}))$ follows from [13] II 3.1; it remains to show that if x is associated to $\mathcal{M}$ then $s = f(x)$ is associated to $f_*(\mathcal{M})$. We can assume S is local with closed point s. Let $A = \mathcal{O}_{S,s}$ and $B = \Gamma(X, \mathcal{O}_X)$. By hypothesis, there exists $m \in \Gamma(X, \mathcal{M})$ such that x is a maximal point of $V(\mathrm{Ann}_B(m))$. Since B is finite over A, the annihilator local A is semi-local; since x is above s, x is a maximal ideal of B, so there exists an element c of x such that x is the only maximal ideal of B containing c. On the other hand, there exists an element b of $B - x$ and an integer n such that $bc^n m = 0$. Then $\mathrm{Ann}_B(bm)$ contains c^n and $\mathrm{Ann}_B(m)$; since x is the only maximal ideal of B containing c^n, we deduce that $V(\mathrm{Ann}_B(bm)) = \{x\}$. Since $A/\mathrm{Ann}_A(bm) \to B/\mathrm{Ann}_B(bm)$ is injective and finite, we conclude that $V(\mathrm{Ann}_A(bm)) = \{s\}$ and thus $s \in \mathrm{Ass}_S(f_*(\mathcal{M}))$. □

Lemma 6 (3.4.5)**.** *(invariance of the assassin by étale localization). Let (S, s) be a pointed scheme, $(\tilde{S}, \tilde{s})$ a strict henselization of (S, s), M an $\mathcal{O}_S$-module. For s to be associated to M, it is necessary and sufficient that $\tilde{s}$ be associated to $M \times_S \tilde{S}$.*

Proof. We can assume S is local with closed point s; let $A = \mathcal{O}_{S,s}$ and $\tilde{A} = \mathcal{O}_{\tilde{S},\tilde{s}}$. If s is associated to M, there exists $m \in M$ such that $\{s\} = V(\mathrm{Ann}_A(m))$; let $\tilde{m}$ be the reciprocal image of m in $\tilde{M} = M \times_S \tilde{S}$; since $\tilde{A}$ is A-flat, we have $\mathrm{Ann}_{\tilde{A}}(\tilde{m}) = \tilde{A} \cdot \mathrm{Ann}_A(m)$; since $\tilde{s}$ is the only point of $\tilde{S}$ above s, we have $\{\tilde{s}\} = V(\mathrm{Ann}_{\tilde{A}}(\tilde{m}))$, thus $\tilde{s}$ is associated to $\tilde{M}$.

Suppose $\tilde{s}$ is associated to $\tilde{M}$; let $\tilde{m}$ be an element of $\tilde{M}_{\tilde{s}}$ such that $\{\tilde{s}\} = V(\mathrm{Ann}_{\tilde{A}}(\tilde{m}))$; there exists a local sub-A-algebra A' of $\tilde{A}$, localized from an étale A-algebra, such that m comes from an element m' of $M' = M \otimes_A A'$; since $\tilde{A}$ is A'-flat, the reasoning above shows that the projection s' of $\tilde{s}$ in $\mathrm{Spec}(A')$ is associated to M'. According to EGA IV 18.4.6, there exists a finite A-algebra B, which is a free A-module, and a prime ideal $\mathfrak{q}$ of B such that A' is A-isomorphic to $B_{\mathfrak{q}}$. Let $N = M \otimes_A B$, so

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that $q \in \mathrm{Ass}_B(N)$. According to 3.4.4, we have $s \in \mathrm{Ass}_4(N)$; but elsewhere $\mathrm{Ass}_4(N) = \mathrm{Ass}_4(M)$ since N is a free A-module q.e.d.

Proof of 3.4.3. Let x be a point of X, $s = f(x)$; we need to show that x is associated to $\mathcal{M} \otimes_{O_x} f^*(A)$ if and only if x is associated to $\mathcal{M} \otimes k(s)$ and s is associated to $\mathcal{N}$. We can restrict X to an open containing x, we can assume that $\mathrm{Ass}(\mathcal{M} \otimes k(s))$ is formed of generalizations of x. According to 3.4.5, 3.4.1, 3.3.2 and 3.3.12, we can assume (S, s) henselian with ring A, X affine with ring B and $\Gamma(X, \mathcal{M})$ free over A.

Suppose that $x \in \mathrm{Ass}_B(M \otimes_A N)$. The homothetic maps of $M \otimes_A N$, defined by the elements of x are therefore not injective, consequently the homothetic maps of M defined by the elements of x are not universally injective relative to A, thus $x \in \mathrm{Ass}(M \otimes k(s))$ by 3.2.5. Let us prove that s is in $\mathrm{Ass}_4(N)$. Since M is a free A-module, it suffices to show that $s \in \mathrm{Ass}_4(M \otimes_A N)$. Let u be an element of s. By hypothesis, there exists an element m of $M \otimes_A N$ such that x is a maximal point of $V(\mathrm{Ann}_B(m))$. There exists therefore an element b of $B - x$ and an integer $n > 0$, such that $a^n b \cdot m = 0$. Since $\mathrm{Ass}(M \otimes k(s))$ is contained in the generic of x, the homothetic map of M defined by b is A-universally injective (3.1.6), thus the homothetic map of $M \otimes_A N$ defined by b is injective and consequently $a^n \cdot m = 0$, thus $V(\mathrm{Ann}_A(m)) = \{s\}$ q.e.d.

Suppose $x \in \mathrm{Ass}(M \otimes k(s))$ and $s \in \mathrm{Ass}_S(N)$; let us show that $x \in \mathrm{Ass}_B(M \otimes_A N)$. Let n be an element of N such that $\{s\} = V(\mathrm{Ann}_A(n))$. We can replace N by the sub-A-module generated by n (which is permissible since M is A-flat) and S by the closed subscheme defined by $\mathrm{Ann}_A(n)$, we can assume that $N = A$ and that S has only one point s. The ring A is then self-associated ([13] II 4.1). According to the part already demonstrated of 3.4.3, $\mathrm{Ass}_B(M \otimes_A N)$ is contained in $\mathrm{Ass}(M \otimes k(s))$ thus is finite. By [13] II 1.3, we are thus led to show that the homothetic maps of M defined by the elements of x are not injective. But since M is a free A-module and since A is self-associated, any endomorphism of the A-module M, which is injective, is A-universally injective ([13] II 4.5). The assertion follows therefore from 3.2.5, since $x \in \mathrm{Ass}(M \otimes k(s))$. Q.e.d.

Theorem 5 (3.4.6)**.** *Let $f : X \to S$ be a locally finite presentation morphism, $\mathcal{M}$ an O_X-module of finite type. We assume that $\mathrm{Ass}(S)$ is locally finite. Then, the set U of points of X where $\mathcal{M}$ is S-flat is open and $\mathcal{M}|U$ is an O_U-module of finite presentation.*

Proof. Let x be a point of X where $\mathcal{M}$ is S-flat, $s = f(x)$; let us find an open neighborhood V of x in X such that $\mathcal{M}|V$ is S-flat and of finite presentation over O_V. According to 3.4.2 and 2.6, we can, after restricting X, construct a homomorphism of O_X-modules

$$u : \mathcal{N} \to \mathcal{M}$$

such that $\mathcal{N}$ is of finite presentation and S-flat and such that u_x is an isomorphism. Since $\mathrm{Ass}(S)$ is locally finite, the same is true for $\mathrm{Ass}(\mathcal{N})$ (3.4.3); therefore, after restricting X, we can assume that $\mathrm{Ass}(\mathcal{N})$ is contained in the generic point of x and that u is surjective. Let $\mathcal{R} = \mathrm{Ker}(u)$. Then $\mathrm{Ass}(\mathcal{R}) \subset \mathrm{Ass}(\mathcal{N})$, thus $\mathrm{Ass}(\mathcal{R}) = \emptyset$, since u_x is an isomorphism and consequently $\mathcal{R} = 0$ ([13] II 1.2), therefore u is bijective q.e.d.

Corollary 10 (3.4.7)**.** *Let A be an integral ring and B an A-algebra of finite type which is flat over A. Then B is of finite presentation.*

Remarks (3.4.7). (i) Assertion 3.4.2 generalizes the well-known fact that a flat module of finite type over a local ring is of finite presentation (and consequently free (Bourbaki – Alg. com. I §2 ex. 23)), while assertion 3.4.6 generalizes the well-known fact that a flat module of finite type over an integral ring is projective (Bourbaki Alg. com. II §5 ex. 6).

(ii) Taking into account 3.4.2, the equivalence of conditions (i), (i'), (ii) and (ii') of 2.1 remains valid if instead of assuming M of finite presentation, we assume only M of finite type; these conditions are also equivalent to (iii'), provided we replace U' by $\mathrm{Spec}(\mathcal{O}_{X,x})$. However, (i) is not equivalent to (iii) when M is of finite type: if A is a local ring whose maximal ideal $\mathfrak{m}$ is A-flat and such that $\mathfrak{m} = \mathfrak{m}^2$ (for example, a non-discrete valuation ring of height 1), then $\mathrm{Tor}_1^A(A/\mathfrak{m}, A/\mathfrak{m}) = 0$, although $A/\mathfrak{m}$ is not A-flat if $\mathfrak{m} \neq 0$.

(iii) In (3.4.6), the finiteness condition on $\mathrm{Ass}(S)$ cannot be removed. Thus, when A is an absolutely flat ring (i.e., the local rings of A are fields), for any A-module (flat) of finite type to be of finite presentation, it is necessary and sufficient that A be a finite product of fields.

(iv) Let A be the ring $\mathbb{Z} \oplus I$ where I is a square-zero ideal, considered as a $\mathbb{Z}$-module $\oplus\mathbb{Z}/p\mathbb{Z}$ (where p ranges over all prime numbers). Then, the set of platitude points of the A-module of finite type $\mathbb{Z}$ is the generic point of $\mathrm{Spec}(A)$ which is not open in $\mathrm{Spec}(A)$.

(v) Question: is the result of 3.4.1 valid when (S, s) is assumed to be local, not necessarily noetherian?

2 4. Universal Platificateur

2.1 4.1. Platificateur of a module of finite type (local henselian case)

The essential tool of this paragraph is the following result:

Theorem 6 (4.1.1)**.** *Let S be a scheme, X an S-scheme of finite presentation, $\mathcal{M}$ an $\mathcal{O}_X$-module of finite presentation, S-flat and S-pure (3.3.3), $\mathcal{N}$ an $\mathcal{O}_X$-module quotient of $\mathcal{M}$, $u : \mathcal{M} \to \mathcal{N}$ the canonical surjective homomorphism.*

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Let $F : (Sch/S)^0 \to (Ens)$ be the subfunctor of the final object which "makes u bijective" (more precisely, if T is an S-scheme, we have $F(T) = \{0\}$ if $u \times_S T$ is bijective and $F(T) = \emptyset$ otherwise); then, F is representable by a closed subscheme of S, which is of finite presentation if $\mathcal{N}$ is of finite presentation.

Proof. The question is local on S for the étale topology. We can therefore assume that S is affine with ring A and that there exists an étale morphism of an affine scheme Y into X, whose image contains $Ass_{X/S}(\mathcal{M})$ and such that $\Gamma(Y, \mathcal{M} \times_X Y)$ is a free A-module (3.3.6, 3.3.8, 3.3.12). Then $Y \to X$ is S-universally separating for $\mathcal{M}$ (cf. 3.2.6), thus, for any S-scheme T, $u \times_S T$

is invertible if and only if $(u \times_X Y) \times_S T$ is invertible. We are thus reduced to the case where $X = Y$ is affine and $M = \Gamma(X, \mathcal{M})$ is a free A-module. Let $N = \Gamma(X, \mathcal{N})$, $R = \Gamma(X, Ker(u))$. Then it is immediate that F is represented by $Spec(A/I)$, where I is the ideal of A generated by the coordinates of the elements of R with respect to a basis of M. If N is of finite presentation, we see that I is of finite type, by passage to the inductive limit and reduction to the case where A is noetherian (3.3.10).

Here is a first application of 4.1.1:

Theorem 7 (4.1.2)**.** *Let $f : (X, x) \to (S, s)$ be a morphism locally of finite presentation, $\mathcal{M}$ an $\mathcal{O}_X$-module of finite type. We assume (S, s) local henselian and we denote by C the category of local schemes pointed above (S, s).*

Let $F : C^0 \to (Ens)$ be the subfunctor of the final functor which "makes $\mathcal{M}$ flat over S at x" (more precisely, if $g : (S', s') \to (S, s)$ is an object of C, we have $F(g) = \{0\}$ if $\mathcal{M} \otimes_S S'$ is S'-flat at the points of $X \otimes_S k(s')$ which are above x and $F(g) = \emptyset$ otherwise). Then F is representable by a closed subscheme of S, which is of finite presentation if $\mathcal{M}$ is of finite presentation.

Proof. The last assertion follows from the first by reduction to the noetherian case. The question of the representability of F is a local problem for the étale topology on X. Since (S, s) is henselian, it follows from 1.2.3 that we can assume that X is affine and that $\mathcal{M}$ admits a total S-devissage at x, of length r $(Y_i \to T_i, \mathcal{L}_i \xrightarrow{\alpha_i} \mathcal{N}_i \to \mathcal{B}_i)_{1 \leq i \leq r}$. If $r = 0$, we have $\mathcal{M} = 0$ and F is the final functor. Suppose $r > 0$ and let us reason by recurrence on r. By the recurrence hypothesis, the functor G which makes $\mathcal{B}_1$ flat over S at the point t_1 image of x in T_1 is representable by a closed subscheme S_1 of S. According to 2.1 and 3.4.7 (ii), F is a subfunctor of G. We can therefore replace S by S_1, and assume that $\mathcal{B}_1$ is S-flat at t_1. Let τ_1 be the generic point of $X \otimes k(s)$ and U an affine open set of T_1, containing

τ_1, such that $\alpha_1|U$ is surjective. It follows again from 2.1 and 3.4.7 that the functor F coincides with the functor that makes $\alpha_1|U$ bijective. Since U is smooth over S with geometrically integral fibers, U is S-pure, thus F is representable by a closed subscheme according to 4.1.1, q.e.d.

2.2 Application to the improvement of previous results

Lemma 7 (4.2.1)**.** *Let A be a ring, I a nilpotent ideal of A, $u : M \to N$ an A-linear application. If $u \otimes_A (A/I)$ is A-universally injective, and if N is A-flat, then u is A-universally injective.*

Proof. We immediately reduce to the case where $I^2 = 0$. Suppose first u injective; we need to show that $P = \mathrm{coker}(u)$ is A-flat. Since P/IP is (A/I)-flat, it suffices to show that $I \otimes_A P \to P$ is injective (Bourbaki - Alg. com. III §5 Th.

1). By hypothesis, the diagram below has exact rows

$$\begin{array}{ccccccccc} 0 \longrightarrow & I \otimes_A M & \longrightarrow & I \otimes_A N & \longrightarrow & I \otimes_A P & \longrightarrow 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \longrightarrow & M & \longrightarrow & N & \longrightarrow & P & \longrightarrow 0 \end{array}$$

(the first row is exact because I is an (A/I)-module and $u \otimes_A (A/I)$ is universally A-injective). We conclude using the snake lemma. If we only assume that u is injective, we deduce from what precedes, by replacing M with $\mathrm{Im}(u)$, that $\mathrm{Im}(u)$ is A-flat, then by the exact sequence of tores, that $\mathrm{Ker}(u) = I \cdot \mathrm{Ker}(u)$, thus $\mathrm{Ker}(u) = 0$ q.e.d. □

Lemma 8 (4.2.2)**.** *Let $f : (X, x) \to (S, s)$ be a morphism of finite presentation, $\mathcal{M}$ an $\mathcal{O}_X$-module of finite type, N a flat $\mathcal{O}_{S,s}$-module, $u : \mathcal{M}_x \to N$ an $\mathcal{O}_{S,s}$-linear application. If $u \otimes k(s)$ is injective, u is S-universally injective and consequently $\mathcal{M}_x$ is S-flat.*

Proof. By flat descent, we can assume (S, s) henselian. We can restrict X and assume X affine. Since $X \otimes k(s)$ is noetherian and $\mathcal{M}$ of finite type, $\mathcal{M}$ is the filtered inductive limit of X-modules of finite presentation $\mathcal{M}_i$ such that the canonical morphism $\mathcal{M}_i \to \mathcal{M}$ induces an isomorphism $(\mathcal{M}_i)_x \otimes k(s) \xrightarrow{\sim} \mathcal{M}_x \otimes k(s)$ and we are reduced to the case where $\mathcal{M}$ is of finite presentation. Let I be the smallest ideal of $\mathcal{O}_{S,s}$ such that $\mathcal{M}_x / I\mathcal{M}_x$ is flat over $\mathcal{O}_{S,s}/I$ (4.1.2). It suffices to show that $I = 0$. Indeed, if this point is established, we can, after restricting X, assume that $\Gamma(X, \mathcal{M})$ is an $\mathcal{O}_{S,s}$-free module such that $\mathrm{Ass}[\mathcal{M} \otimes k(s)]$ is contained in the generic point of x (3.3.13); but then $\Gamma(X, \mathcal{M}) \to N$ is S-universally injective (3.1.6); by passage to the filtered inductive limit on the affine neighborhoods of x in X, we conclude that u is S-universally injective.

Criteria of flatness

This being the case, let us prove that $I = 0$. Since $\mathcal{M}$ is of finite presentation, I is an ideal of finite type (4.1.2) and it suffices to show that $I = I^2$. By replacing $\mathcal{O}_{S,s}$ with $\mathcal{O}_{S,s}/I^2$, we can assume $I^2 = 0$. But then, I is nilpotent, N is S-flat and $u \otimes (\mathcal{O}_{S,s}/I)$ is S-universally injective; it suffices therefore to apply 4.2.1.

Theorem 8 (4.2.3)**.** *Let $f : (X, x) \to (S, s)$ be a morphism of finite presentation, $\mathcal{M}$ an $\mathcal{O}_X$-module of finite type. For $\mathcal{M}$ to be S-flat at x, it is necessary and sufficient that $\mathcal{M}$ be S-flat at the points of Ass$(\mathcal{M} \otimes k(s))$ which are generalizations of x.*

Proof. The condition is evidently necessary; to see that it is sufficient, let Σ be the multiplicative part of $\mathcal{O}_{X,x}$ formed by the injective homothéties of $\mathcal{M}_x \otimes k(s)$; then $\mathrm{Spec}(\mathcal{O}_{X,x}[\Sigma^{-1}])$ is the semi-local generized scheme of $\mathrm{Ass}(\mathcal{M}_x \otimes k(s))$ and the condition of the statement expresses that $\mathcal{M}_x[\Sigma^{-1}]$ is S-flat. Lemma 4.2.2 then shows that the canonical application $\mathcal{M}_x \to \mathcal{M}_x[\Sigma^{-1}]$ is S-universally injective, thus $\mathcal{M}_x$ is S-flat. □

Corollary 11 (4.2.4)**.** *Let (S, s) be a local scheme, $X \to S$ a morphism of finite presentation, $\mathcal{M}$ an $\mathcal{O}_X$-module of finite type. We assume that $\mathcal{M}$ is pure along $X \otimes k(s)$ and is S-flat at the points of $X \otimes k(s)$. Then $\mathcal{M}$ is S-flat.*

Proof. Since $\mathcal{M}$ is S-pure, $\mathrm{Ass}(\mathcal{M}/S)$ is formed of generalizations of $X \otimes k(s)$, thus $\mathcal{M}$ is S-flat at the points of $\mathrm{Ass}(\mathcal{M}/S)$ and consequently $\mathcal{M}$ is S-flat (4.2.3). □

Under the conditions of 4.2.4, is $\mathcal{M}$ an $\mathcal{O}_X$-module of finite presentation?

Corollary 12 (4.2.5)**.** *Let $f : X \to S$ be a morphism of finite presentation, $\mathcal{M}$ an $\mathcal{O}_X$-module of finite type. We assume, either that $\mathcal{M}$ is of finite presentation, or that $Ass(S)$ is locally finite. Then, the set of points s such that $\mathcal{M}$ is S-flat at the points of $f^{-1}(s)$ and pure along $f^{-1}(s)$ is open.*

Proof. Let s be a point of the set in question. According to 4.2.4, $\mathcal{M}$ is flat above $\mathrm{Spec}(\mathcal{O}_{S,s})$. But then $\mathcal{M}$ is flat and of finite presentation above an open quasi-compact neighborhood U of $X \times_S \mathrm{Spec}(\mathcal{O}_{S,s})$ in X (2.6 and 3.4.6). Since X is of finite presentation over S, we see by passage to the limit that U contains the reciprocal image of an open V of S containing s. By restricting V, we can assume that $\mathcal{M}|X \times_S V$ is V-pure (3.3.8) *cqfd*. □

As an application of 4.2.4 and 3.4.6, we leave to the reader the task of specifying the theorem of "generic freedom" (EGA IV 6.9.2).

We will now give some simplifications and generalizations of certain technical results of EGA IV 11.

First, we will simplify the demonstration of EGA IV 11.2.9 and prove the local version of this result (which answers the question posed in EGA IV 11.2.10).

Theorem 9 (4.2.6)**.** *Let A_0 be a ring, $(A_\lambda)_{\lambda \in L}$ an inductive filtering system of A_0-algebras, B_0 an A_0-algebra of finite presentation, I_0 an ideal of finite type of B_0, M_0 a B_0-module of finite presentation.*

Let $B_\lambda = B_0 \otimes_{A_0} A_\lambda$, $I_\lambda = I_0 B_\lambda$, $M_\lambda = M_0 \otimes_{A_0} A_\lambda$, $A = \lim_{\to} A_\lambda$, $B = \lim_{\to} B_\lambda$, $I = \lim_{\to} I_\lambda$, $M = \lim_{\to} M_\lambda$. Let $\overline{\mathfrak{q}}$ be a prime ideal of $\overline{B} = B/I$, $\mathfrak{q}_\lambda$ its reciprocal image in $\overline{B}_\lambda = B_\lambda / I_\lambda$.

For the $\overline{B}$-module $\mathrm{gr}^*_I(M)$ to be A-flat at $\mathfrak{q}$, it is necessary and sufficient that the $\overline{B}_\lambda$-module $\mathrm{gr}^*_{I_\lambda}(M_\lambda)$ be A-flat at $\mathfrak{q}_\lambda$ for λ large enough; if λ is chosen this way, the canonical homomorphism

$$\mathrm{gr}^*_{I_\lambda}(M_\lambda) \otimes_{A_\lambda} A \to \mathrm{gr}^*_I(M)$$

is bijective at $\overline{\mathfrak{q}}$.

Proof. The sufficiency of the condition and the last assertion result from the local version of EGA IV 11.2.9.2; let us prove the necessity. Let $\mathfrak{p}$ (resp. $\mathfrak{p}_\lambda$) be the reciprocal image of $\overline{\mathfrak{q}}$ in A (resp. A_λ). By henselization and flat descent, we can assume A_λ local henselian with maximal ideal $\mathfrak{p}_\lambda$ for all λ, in which case A is henselian with maximal ideal $\mathfrak{p}$. By the artifice of EGA IV 11.2.9(iv), we can assume that B_0 and $\overline{B}_0$ are smooth over A_0, and that $C_0 = \mathrm{gr}^*_{I_0}(B_0)$ is a $\overline{B}_0$-algebra of polynomials, in which case the homomorphism $C_0 \otimes_{A_0} A \to C = \mathrm{gr}^*_I(B)$ is an isomorphism.

The essential point of the proof consists in showing that after replacing B by B_b, where b is a suitable element of $B - \mathfrak{q}$ (we denoted by $\mathfrak{q}$ the reciprocal image of $\overline{\mathfrak{q}}$ in B), we can assume that $\mathrm{gr}^*_I(M)$ is A-flat and of finite presentation over C.

Let $S = \mathrm{Spec}(A)$, s the closed point of S, $X = \mathrm{Spec}(C)$, $Y = \mathrm{Spec}(\overline{B})$, y the point of Y corresponding to $\overline{\mathfrak{q}}$, $f : X \to Y$ the canonical morphism, $U = X - Y$, $P = \mathrm{Proj}_Y(C)$, $\pi : U \to P$ the canonical surjective smooth morphism, $g : P \to Y$ the structural morphism; finally, the canonical augmentation of the $\overline{B}$-algebra C corresponds to a Y-section $\sigma : Y \to X$ of image $(X - U)$ (Y is the line of vertices of the cone projecting X from the Y-projective scheme P).

(X) at (0,2) X; (U) at (2,2) U; (P) at (2,0) P; (Y) at (0,0) Y; [->] (X) – node[above] $\sim$ (U); [->] (U) – node[right] π (P); [->] (X) – node[left] f (Y); [->] (Y) – node[below] g (P);

The C-module of finite type $\mathrm{gr}^*_I(M)$ defines an $\mathcal{O}_X$-module of finite type $\mathcal{N}$ and a $\mathcal{O}_P$-module of finite type $\mathcal{Q}$ such that $\mathcal{N}|U = \pi^*(\mathcal{Q})$.

Flatness criteria

By hypothesis, $\mathcal{N}$ is S-flat at the points of $f^{-1}(y)$, so $\mathcal{Q}$ is S-flat at the points of $f^{-1}(y)$. Since (S, s) is Henselian, and since y is above s, there exists an open neighborhood V of $f^{-1}(y)$ in X, such that $\mathcal{N}|V$ is S-flat and of finite presentation over $\mathcal{O}_V$ (3.4.1); similarly, there exists an open neighborhood V' of $g^{-1}(y)$ in P such that $\mathcal{Q}|V'$ is S-flat and of finite presentation over $\mathcal{O}_{V'}$. Since g is proper, we can take V' of the form $g^{-1}(W)$, where W is an open neighborhood of y in Y, contained in $\sigma^{-1}(V)$. It is clear that $\mathcal{N}$ is then S-flat and of finite presentation over $f^{-1}(W)$. It suffices therefore to choose an element b of $B - \mathfrak{q}$ such that Y_b is contained in W.

We now assume that $\mathrm{gr}^*_I(M)$ is a flat A-module of finite presentation over C. Proceeding as in EGA IV 11.2.9 I) and III) we reduce to the case where the A_i are Noetherian and the transition morphisms are injective; we then conclude as in EGA IV 11.2.9 VI).

Corollary 13 (4.2.7)**.** *Let A be a ring, B an A-algebra of finite presentation, I an ideal of finite type of B, M a B-module of finite presentation, $\overline{\mathfrak{q}}$ a prime ideal of $\overline{B} = B/I$, $\mathfrak{q}$ the reciprocal image of $\overline{\mathfrak{q}}$ in B. If $\mathrm{gr}^*_I(M)$ is A-flat at $\overline{\mathfrak{q}}$, it is A-flat in a neighborhood of $\overline{\mathfrak{q}}$ in* $\mathrm{Spec}(\overline{B})$ *and M is A-flat at $\mathfrak{q}$.*

Proof. By 4.2.6 we can assume A Noetherian. The first assertion follows from 2.6 and the second from EGA $\mathrm{O}_{\mathrm{III}}$ 10.2.6.

We now propose to deduce from 4.1.2 certain results of EGA IV 11 on the descent of flatness and to extend them to modules of finite type. $\square$

Theorem 10 (4.2.8)**.** *(cf. EGA IV 11.5.1) Let $(X, x) \to (S, s)$ be a morphism locally of finite presentation, $\mathcal{M}$ an $\mathcal{O}_X$-module of finite type, $u_i : (S_i, s_i) \to (S, s)_{i \in I}$ a family of morphisms. We assume that (S, s) is local Henselian of ring A, that (S_i, s_i) is local of ring A_i and that the homomorphism $A \to \prod_{i \in I} A_i$ is injective. Then M is S-flat at x in each of the following two cases:*

a) for all $i \in I$, $\mathcal{M}_i = \mathcal{M} \times_S S_i$ is S_i-flat at the points of $X_i \otimes k(s_i)$ above x.

b) $k(x)$ is a primary extension of $k(s)$ and for all $i \in I$, there exists a point x_i of $X_i \otimes k(s_i)$, above x, such that $\mathcal{M}_i$ is S_i-flat at x_i.

Proof. If a) is verified, $\mathcal{M}$ is S-flat at x according to 4.1.2. Suppose b) verified and let us show that b) $\Rightarrow$ a). We must verify that $\mathcal{M}_i$ is S_i-flat at all points of $X_i \otimes k(s_i)$ above x. According to 4.2.3, it suffices to show that $\mathcal{M}_i$ is S_i-flat at all points z_i of $\mathrm{Ass}(\mathcal{M}_i \otimes k(s_i))$ which is above a generization of x; finally it suffices to see that z_i is a generization of x_i. Let z be the image of z_i in $X \otimes k(s)$,

Z the adherence of z in $X \otimes k(s)$ (which thus contains the adherence Y of x), Z_i the adherence of z_i in $X_i \otimes k(s_i)$. Then, it follows from 3.2.3 that Z_i is an irreducible component of $Z \otimes_{k(s)} k(s_i)$; on the other hand, since $k(x)$ is a primary extension of $k(s)$, Y is geometrically irreducible. It follows then from the lemma below that Z_i contains the reciprocal image of Y and a fortiori x_i.

Lemma 9 (4.2.9)**.** *Let k be a field, Z an irreducible k-scheme of finite type, Y a closed geometrically irreducible subscheme of Z, k' an extension of k, $Z' = Z \otimes_k k'$, $Y' = Y \otimes_k k'$. Then, every irreducible component of Z' contains Y'.*

Proof. We may assume k' is algebraically closed, or k the integral closure of k in k'; since k is algebraically closed, every k-scheme irreducible is geometrically irreducible (EGA IV 4.4.4) and we can limit ourselves to the case where $k' = k$ is an algebraic extension of k. Let T' be an irreducible component of Z'. Since $Z' \to Z$ is faithfully flat, the image of T' in Z contains the generic point of Z; since the morphism $T' \to Z$ is entire, it is therefore surjective. Thus T' contains a point of Z' above the generic point of Y. Since Y is geometrically irreducible, this point is necessarily the generic point of Y', thus T' contains Y'.

We leave to the reader the task of deducing from 4.2.8 the statements of EGA IV 11.5.3 and 11.6.1 (extended to modules of finite type). The valuative criterion of flatness (EGA IV 11.8.1), can also be slightly improved: □

Corollary 14 (4.2.10)**.** *Let $(X, x) \to (S, s)$ be a morphism of finite presentation, $\mathcal{M}$ an $\mathcal{O}_X$-module of finite type. We assume (S, s) local reduced. For $\mathcal{M}$ to be S-flat at x, it suffices that the following condition is verified:*

For every morphism $(S', s') \to (S, s)$, where S' is the spectrum of a valuation ring with closed point s', $\mathcal{M} \times_S S'$ is S'-flat at the points of $X \otimes_{k(s)} k(s')$ above x.

Proof. Since the henselization of a reduced ring is reduced, we can assume (S, s) henselian of ring A. Then A embeds in the product of local integral rings $A/\mathfrak{p}_i$, where $\mathfrak{p}_i$ is a minimal prime ideal of A, and each of the $A/\mathfrak{p}_i$ is dominated by a valuation ring; it suffices therefore to apply 4.2.6.

The corollary below is to be compared with Bourbaki Alg. com. III § 5 th. 1. $\square$

Corollary 15 (4.2.11)**.** *Let $f : A \to B$ be a local morphism of local rings, I an ideal of A, M a B-module of finite type. We assume that A is henselian and separated for the I-adic topology and that B is a localization of an A-algebra of finite type. For M to be A-flat, it is necessary and sufficient*

that M/I^nM is A/I^n-flat for all n (or equivalently that $\mathrm{gr}_I^*(M)$ is $\mathrm{gr}_I^*(A)$-flat).

This is an immediate consequence of 4.1.2. (We do not know if 4.2.11 remains valid without A being henselian.)

Remark 3 (4.2.12)**.** The plan followed in 4.2 can be adapted to the following situation: Let $f : A \to B$ be a local homomorphism of local noetherian rings, M a B-module of finite type. We deduce from 4.2.1 and Krull's theorem the following analog of 4.2.2: Let N be an A-flat module, $u : M \to N$ an A-linear application, k the residue field of A; if $u \otimes_A k$ is injective, u is A-universally injective (cf. EGA $\mathrm{O}_{\mathrm{III}}$ 10.2.4). As in 4.2.3, we deduce that M is A-flat if and only if it is A-flat at the points of $\mathrm{Ass}_B(M \otimes_A k)$. When A is complete, we have an analog of 4.1.2: there exists a smallest ideal I of A such that M/IM is A/I-flat, and for any local homomorphism of local rings $g : A \to A'$, $M \otimes_A A'$ is A'-flat if and only if $g(I) = 0$. These results, combined with passage to the limit, are essentially the arguments used in the results of EGA IV 11 that have been cited above.

3 Flattening of an S-pure module (global case)

The global analog of 4.1.2 is the following result:

Theorem 11 (4.3.1)**.** *Let $f : X \to S$ be a morphism of finite presentation, $\mathcal{M}$ an $\mathcal{O}_X$-module of finite presentation S-pure, n an integer. Let*

$$F_n : (\mathrm{Sch}/S)^0 \to \mathrm{Ens}$$

be the subfunctor of the final functor that "makes $\mathcal{M}$ flat over S in dimensions $\geq n$". More precisely, if T is an S-scheme, we have $F_n(T) = \{\emptyset\}$ if $\dim(Z/T) < n$ (where Z denotes the closed subset of $X \times_S T$ consisting of points where $\mathcal{M} \times_S T$ is not T-flat) and $F_n(T) = \emptyset$ otherwise. Then $F_n \to S$ is representable by a finite presentation monomorphism.

Let us first prove a lemma:

Lemma 10 (4.3.2)**.** *Let $f : X \to S$ be a smooth morphism with geometrically integral fibers, $\mathcal{M}$ an $\mathcal{O}_X$-module of finite type, p an integer, $H_p : (\mathrm{Sch}/S)^0 \to$ (Ens) the subfunctor of the final functor that "makes $\mathcal{M}$ free of rank p at generic points of the fibers of f". Then H_p is representable by a finite presentation subscheme of S if $\mathcal{M}$ is of finite presentation.*

Proof. The second assertion follows from the first by reduction to the noetherian case; let us prove the first: we will first prove that if H_q is empty for all $q > p$ then H_p is representable by a closed subscheme of S. The question is local on S; let s be a point of S, let us find an open neighborhood of s in S above which the assertion is true. Let x be the generic point of $f^{-1}(s)$. By hypothesis we have □

$\mathrm{rg}_{k(s)}((\mathcal{M} \otimes k(s))_x) \leq p$; *there exists therefore an open affine neighborhood* V *of* x *in* X *and a surjective homomorphism* $u : \mathcal{O}_V^p \to \mathcal{M}|V$. *By restricting* S *we can assume* $f(V) = S$ *in which case* V *is a schematically dominant* S*-universally open subset of* X. *It is then clear that* H_p *is equal to the functor that makes* u *invertible, so that the assertion follows from 4.1.1.*

That said, since the assertion to be proven is local on S, *we can assume that* H_p *is empty for* p *large enough; let us reason by recursion on the largest integer* n *such that* $H_n \neq \emptyset$. *For* $n = 0$, H_n *is the final functor and all other* H_n *are empty. For* $n > 0$, H_n *is representable by a closed subscheme of* S *according to what precedes; the other* H_p *are disjoint from* H_n *thus are subfunctors of the open* $S - H_n = U$; *it suffices to apply the recursion hypothesis to* U.

Proof of 4.3.1. The question is local for the étale topology on S. *We can assume* $F_n = S$ *for* n *large enough and reason by decreasing recursion on* n; *let us suppose therefore* F_{n+1} *representable by a finite presentation monomorphism. Since* F_n *is a subfunctor of* F_{n+1} *we can replace* S *by* F_{n+1}: *indeed, locally for the étale topology on* F_{n+1} *and on* S, $F_{n+1} \to S$ *is a closed immersion, thus the purity hypothesis on* M *is preserved by the base change* $F_{n+1} \to S$. *We can therefore assume that* M *is* S*-flat in dimensions* $> n$.

Let s *be a point of* S, *let us find an elementary étale neighborhood* (S', s') *of* (S, s) *such that* $F_n \times_S S'$ *is representable. For this we can assume that* S *is affine and that there exists an étale morphism of an affine scheme* Y *in* X *with image containing* $X \otimes k(s)$, *such that* Y *is a sum of open and closed subschemes* Y_j, *and that, for all* j, *there exists an* S*-devissage* $(Z_{ji} \to T_{ji}, \mathcal{L}_{ji} \xrightarrow{\alpha_{ji}} \mathcal{N}_i \to \mathcal{P}_j \to 0)$ *of* $\mathcal{M}_j = \mathcal{M} \times_X Y_j$ *in dimensions* $\geq n$ *above* s, *such that, for all indices* i *such that* $\dim(T_{ji}/S) > n$, *the homomorphism* α_{ji} *is* S*-universally injective (cf. 2.9). Since the image of* Y *in* X *is open and contains* $X \otimes k(s)$ *it contains the points of* $\mathrm{Ass}_{X/S}(\mathcal{M})$ *above the generic point of* s *(because* $\mathcal{M}$ *is* S*-pure); by restricting* S *we can therefore assume that the image of* Y *in* X *contains* $\mathrm{Ass}_{X/S}(\mathcal{M})$. *It follows then from 4.2.3 that for any* S*-scheme* T, $\mathcal{M} \times_S T$ *is* T*-flat in dimensions* $\geq n$ *if and only if* $\mathcal{M}_j \times_S T$ *is* T*-flat in dimensions* $\geq n$ *for all* j; *thus* F_n *is the intersection of the* F_{nj} *where* F_{nj} *is the functor that makes* $\mathcal{M}_j$ *flat on* S *in dimensions* $\geq n$. *It suffices therefore to show that* F_{nj} *is representable by an* S*-scheme of finite presentation.*

Let k *be the index such that* $\dim(T_{jk}/S) = n$; *since the* α_{ji} *are* S*-universally injective for* $i < k$, *we see that for any* S*-scheme* T, $\mathcal{M}_j \times_S T$ *is* T*-flat in dimensions* $\geq n$ *if and only if* $\mathcal{N}_{jk} \times_S T$ *is, thus* F_n *is the functor that makes* $\mathcal{N}_{jk}$ *flat on* S *in dimension* n, *i.e. locally free at generic points of the fibers of* T_{jk} *over* S *(2.1). According to 4.2.3 this functor is representable by an* S*-scheme of finite presentation. q.e.d.*

Criteria of flatness

Remark 4 (4.3.3)**.** When S is Noetherian and $n = 0$, Theorem 4.3.1 is proven by a different method in [14] th. 2.

4 Flattening by blowups

4.1 Lemmas on blowups

The lemmas below are taken from a course by Hironaka at IHES (1968).

Definition 5 (5.1.1)**.** Let S be a scheme, Y a closed subscheme of S defined by an ideal I of $\mathcal{O}_S$, M an $\mathcal{O}_S$-module.

(i) We call the blowup of Y (or of I) in S the homogeneous spectrum of the graded $\mathcal{O}_S$-algebra $\bigoplus_{n\in\mathbb{N}} I^n$ (cf. EGA II 8.1).

(ii) Let $f : S' \to S$ be the blowup of Y in S. We call the strict transform of M by f the quotient of the $\mathcal{O}_{S'}$-module $f^*(M)$ by the submodule formed by sections with support in $f^{-1}(Y)$ (cf. EGA IV 5.9).

Remark 5 (5.1.2)**.** (i) With the notations of 5.1.1, it is well known that S' is a final object of the full subcategory of (Sch/S) formed by S-schemes T such that $I\mathcal{O}_T$ is an invertible ideal of $\mathcal{O}_T$.

In particular, if U is the open set $S - Y$ of S, we see that f is an isomorphism above U.

(ii) From now on, we consider only blowups of ideals of finite type. If I is of finite type, f is a projective morphism and $I\mathcal{O}_{S'}$ is a very ample $\mathcal{O}_{S'}$-module for f; moreover, the closed subscheme Y' of S' defined by the ideal $I\mathcal{O}_{S'}$ is canonically identified with the S-scheme $\mathrm{Proj}(\bigoplus_{n\in\mathbb{N}} I^n/I^{n+1})$ ("locus at infinity of the tangent cone to S along Y").

(iii) Since $I\mathcal{O}_{S'}$ is an invertible ideal of $\mathcal{O}_{S'}$, we see that $U' = f^{-1}(U)$ is a schematically dense open subset of S' and more generally that if M is S-flat, $f^*(M)$ is equal to the strict transform of M by f.

(iv) If T is an S-scheme, we see from (i) that the blowup of $I\mathcal{O}_T$ in T is S-isomorphic to the schematic adherence of $U' \times_S T$ in $S' \times_S T$.

(v) Let I and J be two ideals of finite type of $\mathcal{O}_S$; let $f : S' \to S$ be the blowup of I in S and $g : S'' \to S'$ the blowup of $J\mathcal{O}_{S'}$ in S'; we see from (i) that $f \circ g$ is S-isomorphic to the blowup of JI in S.

Definition 6 (5.1.3)**.** Let $f : S' \to S$ be a morphism of finite type, U an open subset of S. We say that f is a U-admissible blowup if there exists a closed subscheme of finite presentation Y of S, disjoint from U, such that f is isomorphic to the blowup of Y in S. (NB. The isomorphism is then unique.)

Lemma 11 (5.1.4)**.** *Let S be a quasi-compact and quasi-separated scheme, U an open subset of S, $f : S' \to S$ a U-admissible blowup, $g : S'' \to S'$ an $f^{-1}(U)$-admissible blowup; then $f \circ g$ is a U-admissible blowup.*

Proof. Suppose that f is the blowup in S of an ideal of finite type I defining a closed subscheme Y disjoint from U and that g is the blowup in S' of an ideal of finite type J of $\mathcal{O}_{S'}$ defining a closed subscheme Z disjoint from $f^{-1}(U)$. We can replace U by the larger open subset $S - Y - f(Z)$ which is quasi-compact.

Since S is quasi-compact and quasi-separated and since $I' = I\mathcal{O}_{S'}$ is S-ample, there exists an integer $n > 0$ such that $f^*f_*(I'J^n) \to I'J^n$ is surjective (EGA IV 1.7.15); replacing J by J^n we can assume $n = 1$.

Consider the canonical homomorphism $u : \mathcal{O}_S \to f_*(\mathcal{O}_{S'})$. It follows from EGA II 5.4.3 and IV 18.12.8 that $f_*(\mathcal{O}_{S'})$ is an entire $\mathcal{O}_S$-algebra. Moreover, u is an isomorphism above U. Consider the ideal $f_*(IJ)$ of $f_*(\mathcal{O}_{S'})$ which is equal to $f_*(\mathcal{O}_{S'})$ above U. According to EGA IV 1.7.7, there exists a sub-$\mathcal{O}_S$-module of finite type K of $f_*(IJ)$, equal to $\mathcal{O}_S$ above U and such that the canonical morphism $f^*(K) \to IJ$ is surjective. Since the sub-$\mathcal{O}_S$-algebra of $f_*(\mathcal{O}_{S'})$ generated by K is finite and isomorphic to $\mathcal{O}_S$ above U and since $f^{-1}(U) = S' - V(IJ)$, we easily see that there exists an integer $m > 0$, such that K^m is contained in $u(\mathcal{O}_S)$. According to EGA IV 1.7.7, there exists a sub-$\mathcal{O}_S$-module of finite type L of $u^{-1}(K^m)$ such that $K^m = u(L)$. By construction, L is then an ideal of finite type of $\mathcal{O}_S$ such that $L\mathcal{O}_{S'} = J^m I^m$; it follows immediately that $f \circ g$ is S-isomorphic to the blowup of the ideal LI in S *q.e.d.* □

Lemma 12 (5.1.5)**.** *Let S be a quasi-compact and quasi-separated scheme, U a quasi-compact open subset of S, (U_i) a partition of U into open sets. There exists a U-admissible blowup $f : S' \to S$ and a partition (S'_i) of S' into open sets such that $f^{-1}(U_i) = f^{-1}(U) \cap S'_i$ for all i.*

Proof. We can restrict to the case of a partition of U into two open sets U_1 and U_2. Let R_i $(i = 1, 2)$ be an ideal of finite type of $\mathcal{O}_S$ such that $U_i = S - V(R_i)$, which equals $\mathcal{O}_S$ on U_i and 0 on U_j $(j \neq i)$ (EGA IV 1.7.7). Let us blow up the ideal $R_1 + R_2$ in S; we are then reduced to the case where $R_1 + R_2$ is an invertible ideal and where U is schematically dense in S' (5.1.2(ii)). Since $R_1 \cap R_2 | U$ is null by construction, we have $R_1 \cap R_2 = 0$. Then $(R_1 + R_2)^{-1}R_i$ $(i = 1, 2)$ defines a partition of S into open sets that extends the given partition of U. □

4.2 Statement of the flattening theorem by blowups

To motivate the statement of the main result, we will first examine the global case of a projective morphism; we find that we then have a canonical "flattening" procedure.

5 Flatness criteria

Let $f : X \to S$ be a projective morphism of noetherian schemes, $\mathcal{M}$ an $\mathcal{O}_X$-module of finite type, $Q : (Sch/S)^0 \to Ens$ the functor whose set of points with values in an S-scheme T is the set of equivalence classes of quotient modules of $\mathcal{M} \times_S T$ with finite presentation over $X \times_S T$ and T-flat. According to (9) the functor Q is representable by a scheme which is a disjoint sum of projective S-schemes.

Let U be an open subset of S above which $\mathcal{M}$ is flat. Then the sheaf $\mathcal{M} \times_S U$ defines an element of $Q(U)$, thus an S-morphism $s : U \to Q$. Let S' be the closed image of s (EGA I.9.5). The immersion $S' \to Q$ defines an element of $Q(S')$, that is, a quotient module $\mathcal{M}'$ of $\mathcal{M} \times_S S'$ which is S'-flat. In fact, one verifies immediately from this construction that the structural morphism $g : S' \to S$ is projective and satisfies the following conditions:

a) g is an isomorphism above U and $g^{-1}(U)$ is a schematically dense open subset of S';

b) the quotient of $\mathcal{M} \times_S S'$ by the submodule of sections with support disjoint from $g^{-1}(U)$ (equal to $\mathcal{M}'$) is S'-flat.

Moreover, the morphism g is universal for properties a) and b). (N.B. It is not certain that g is necessarily a U-admissible blowup; cf. however EGA III 2.3.5.)

In the remainder of this paragraph, we will extend to morphisms of finite presentation, not necessarily projective, the assertion of existence of a projective morphism satisfying conditions a) and b) above, but we do not claim to find a morphism g universal for the properties in question.

Definition 7 (5.2.1)**.** Let $f : X \to S$ be a morphism of finite presentation, $\mathcal{M}$ an $\mathcal{O}_X$-module of finite type, n an integer. We say that $\mathcal{M}$ is S-flat in dimension $\geq n$ if there exists a retro-compact open subset V of X (EGA $\mathrm{O_{III}}$ 9.1.1) such that $\dim((X - V)/S) < n$ and such that $\mathcal{M}|V$ is an $\mathcal{O}_V$-module of finite presentation, S-flat.

Theorem 12 (5.2.2)**.** *Let S be a quasi-compact and quasi-separated scheme, U a quasi-compact open subset of S, $f : X \to S$ a morphism of finite presentation, $\mathcal{M}$ an $\mathcal{O}_X$-module of finite type, n an integer. Suppose that $\mathcal{M}|f^{-1}(U)$ is U-flat in dimension $\geq n$. Then there exists a U-admissible blowup $g : S' \to S$, such that the strict transform of $\mathcal{M}$ by g is S'-flat in dimension $\geq n$.*

5.1 Some elementary reductions

Let us first note the following lemma:

Lemma 13 (5.3.1)**.** *Let S be a quasi-compact and quasi-separated scheme, U and V two quasi-compact open subsets of S, $f : V' \to V$ a $U \cap V$-admissible of V. Then there exists a U-admissible blowup*

$$f : S' \to S$$

which extends f.

Proof. Suppose that f is the blowup of the ideal of finite type $\mathcal{I}$ of $\mathcal{O}_V$, such that $\mathcal{I}|U \cap V = \mathcal{O}_{U\cap V}$. According to EGA IV 1.7.7, there exists an ideal of finite type $\tilde{\mathcal{I}}$ of $\mathcal{O}_S$ such that $\tilde{\mathcal{I}}|U = \mathcal{O}_U$ and $\tilde{\mathcal{I}}|V = \mathcal{I}$ and it suffices to take for f the blowup of $\tilde{\mathcal{I}}$ in S.

Concerning the proof of (5.2.2), we can make the following remarks:

(5.3.2). We can assume U schematically dominant in S.

Indeed, it suffices to replace S by the blowup in S of an ideal of finite type $\mathcal{I}$ such that $U = S - V(\mathcal{I})$ (EGA IV 1.7.7 and 5.1.2(iii)).

Once this reduction is made, the strict transform of $\mathcal{M}$ by a U-admissible blowup $S' \to S$ is the quotient of $\mathcal{M}\times_S S'$ by the sub-module of sections supported on U.

(5.3.3). Theorem 5.2.2 is of local nature on S.

Indeed, let $(S_i)_{i\in I}$ be a finite open affine covering of S and $f_i : S_i' \to S_i$ a $U \cap S_i$-admissible blowup which is a solution of the restriction of the problem to S_i. Let $\mathcal{I}_i$ be an ideal of finite type of $\mathcal{O}_{S_i}$ such that the blowup of $\mathcal{I}_i$ is U-admissible and extends f_i (5.3.1). Then, it follows from 5.1.2(iii) and (v), that the blowup of $\mathcal{I} = \prod_i \mathcal{I}_i$ is a solution of the initial problem.

(5.3.4). Theorem 5.2.2 is of local nature on X for the étale topology.

This follows immediately from 5.1.2(v) and from the fact that the strict transform commutes with flat base changes $X' \to X$.

5.2 Proof of 5.2.2 in a particular case

We will prove 5.2.2 in the particular case where X is smooth over S with geometrically integral fibers of dimension n. Note that it follows from 2.1 that an $\mathcal{O}_X$-module $\mathcal{M}$ of finite presentation is S-flat in dimension $\geq n$ if and only if $\mathcal{M}$ is locally free above an open subset V of X which projects onto S. To "make $\mathcal{M}$ locally free", we will use the technique of Fitting ideals.

Reminders on Fitting ideals (5.4.1). Let S be a scheme, $\mathcal{M}$ an $\mathcal{O}_S$-module of finite type, r an integer; let us recall the definition of the r-th Fitting ideal of $\mathcal{M}$:

Flatness criteria

Let us first suppose S is an affine ring scheme A and consider a presentation of $\Gamma(S, \mathcal{M})$:

$$A^{(I)} \xrightarrow{u} A^n \to \Gamma(S, \mathcal{M}) \to 0. \qquad (*)$$

Then the ideal of A generated by the coefficients of the matrix of $A^{n-r}(u)$ does not depend on the choice of the exact sequence $(*)$; it is the r-th Fitting ideal of $\mathcal{M}$; we denote it by $F_r(\mathcal{M})$.

Let us move to the general case. When U runs through the set of affine open subsets of S, the ideals $F_r(\mathcal{M}|U)$ of $\mathcal{O}_U$ glue together and define the r-th Fitting ideal $F_r(\mathcal{M})$ of $\mathcal{M}$.

If $\mathcal{M}$ is of finite presentation, $F_r(\mathcal{M})$ is of finite type.

For any S-scheme T, we have $F_r(\mathcal{M} \times_S T) = F_r(\mathcal{M}) \cdot \mathcal{O}_T$.

A point s of S is in $V(F_r(\mathcal{M}))$ if and only if $\dim_{k(s)}(\mathcal{M} \otimes k(s))$ is $> r$ and $X - V(F_r(\mathcal{M}))$ is the largest open subset of X above which $\mathcal{M}$ can be locally generated by r elements.

Lemma 14 (5.4.2)**.** *If $F_r(\mathcal{M})$ is locally monogenic, $\mathcal{M}/Ann_{\mathcal{M}}(F_r(\mathcal{M}))$ is locally generated by r elements.*

Proof. We may assume S is a local ring scheme A. Consider an exact sequence

$$A^{(I)} \xrightarrow{u} A^n \to \Gamma(S, \mathcal{M}) \to 0 \tag{1}$$

and denote by $(u_{ji})_{1 \leq j \leq n, i \in I}$ the matrix of u. By hypothesis, the ideal $F_r(\mathcal{M})$, generated by the determinants of order $n-r$ extracted from this matrix, is monogenic. Since A is local, $F_r(\mathcal{M})$ is generated by one of these determinants. Up to replacing $\mathcal{M}$ by a module $\mathcal{M}'$ of which $\mathcal{M}$ is a quotient, and changing notation, we can assume that I is the interval $[1, n-r]$ of $\mathbb{N}$ and that $F_r(\mathcal{M})$ is generated by $D = \det((u_{ji})_{1 \leq i,j \leq n-r})$. Let $(m_j)_{1 \leq j \leq n}$ be the image in $\Gamma(S, \mathcal{M})$ of the canonical basis of A^n, so that the m_j are related by the system of equations

$$\sum_{1 \leq j \leq n} u_{ji} m_j = 0 \quad (1 \leq i \leq n-r). \tag{2}$$

For all i $(1 \leq i \leq n-r)$ and all j $(n-r < j \leq n)$, let D_{ji} be the determinant obtained by substituting in the expression of D the j-th column of the matrix of u for the i-th column. According to Cramer's rules, the m_j are related by the relations

$$Dm_i + \sum_{n-r < j \leq n} D_{ji} m_j = 0 \quad (1 \leq i \leq n-r); \tag{3}$$

since $D_{ji} \in F_r(\mathcal{M})$, there exists $b_{ji} \in A$ such that $D_{ji} = b_{ji} D$ and we have

$$D(m_i + \sum_{n-r < j \leq n} b_{ji} m_j) = 0 \quad (1 \leq i \leq n-r) \tag{4}$$

which proves that $\mathcal{M}/\mathrm{Ann}_{\mathcal{M}}(D)$ is generated by the image of $(m_j)_{n-r < j \leq n}$. □

Lemma 15 (5.4.3)**.** *If $F_r(\mathcal{M})$ is invertible and if $\mathcal{M}$ is free of rank r at the points of $Ass(S)$, then $\mathcal{M}/\mathcal{A}\backslash\backslash_{\mathcal{M}}(F_r(\mathcal{M}))$ is locally free of rank r.*

Proof. Let $\mathcal{N} = \mathcal{M}/\mathcal{A}\backslash\backslash_{\mathcal{M}}(F_r(\mathcal{M}))$. Since $F_r(\mathcal{M})$ is invertible, $\mathcal{N}$ is equal to $\mathcal{M}$ at the points of $\mathrm{Ass}(S)$ and consequently is locally free of rank r at these points. On the other hand, the question is local on S and according to (5.4.2) we can assume that we have an exact sequence

$$0 \to \mathcal{R} \to \mathcal{O}_S^n \to \mathcal{N} \to 0.$$

We then see that $\mathcal{R}$ is zero at the points of $\mathrm{Ass}_S(\mathcal{R}) \subset \mathrm{Ass}(S)$, thus is zero.

Let us now prove 5.2.2 in the case where X is smooth over S with geometrically integral fibers of dimension n. Since $\mathcal{M}$ is U-flat in dimension $\geq n$, it follows from 2.1 that there exists a quasi-compact open subset V of X, projecting onto U, such that $\mathcal{M}|V$ is locally free. Let $s \in U$, x the generic point of $X \otimes k(s)$; then $x \in V$; let $r(s)$ be the rank of the free $\mathcal{O}_{X,x}$-module $\mathcal{M}_x$. Since the morphism $V \to U$ is open, the function $s \mapsto r(s)$ is locally constant on U. According to 5.1.5, we can, after making a U-admissible blowup of S, and then taking a suitable open covering of S, assume that $r(s)$ is a constant function on U, with value r. Finally, taking into account 5.3 and 3.3.11, we can assume that $S = \mathrm{Spec}(A)$, $X = \mathrm{Spec}(B)$ and B is free over A with basis $e_{i,i\in I}$.

Since V is quasi-compact, there exists a finitely presented X-module $\mathcal{N}$ and a surjective morphism $u : \mathcal{N} \to \mathcal{M}$ such that $u|V$ is an isomorphism. Consider the r-th Fitting ideal $F_r(\mathcal{N})$ of $\mathcal{N}$; it is a finite type ideal and $F_r(\mathcal{N})|V = \mathcal{O}_V$.

We can form the ideal $\mathcal{I}$ of "coefficients" of $F_r(\mathcal{N})$ relative to S, that is the ideal of $\mathcal{O}_S$ defining the largest closed subscheme of S above which $\mathcal{O}_X$ is equal to $\mathcal{O}_X/F_r(\mathcal{N})$ (4.1.1). Recall (loc.cit.) that $\mathcal{I}$ is a finite type ideal generated by the coordinates with respect to the basis (e_i) of a family of generators of the $\mathcal{O}_X$-module $F_r(\mathcal{N})$. Since $F_r(\mathcal{N})|V = \mathcal{O}_V$, we have $\mathcal{I}|U = \mathcal{O}_U$. After making $\mathcal{I}$ blow up in S, we can assume that $\mathcal{I}$ is an invertible ideal. Given the definition of the coefficient ideal $\mathcal{I}$, it is clear that the finite type fractional ideal $\mathcal{J} = (\mathcal{I}\mathcal{O}_X)^{-1}F_r(\mathcal{M})$ of $\mathcal{O}_X$ is entire and has a coefficient ideal equal to $\mathcal{O}_S$. Consequently $W = X - V(\mathcal{J})$ is a quasi-compact open subset that projects onto S. After replacing X by W, we can assume that $F_r(\mathcal{N})$ is an invertible ideal of $\mathcal{O}_X$. Finally, we can assume U is schematically dominant in S (5.3.2), thus V is schematically dominant in X.

Let $\overline{\mathcal{N}}$ (resp. $\overline{\mathcal{M}}$) be the quotient of $\mathcal{N}$ (resp. $\mathcal{M}$) by the submodule of sections that vanish above U and let $\overline{u} : \overline{\mathcal{N}} \to \overline{\mathcal{M}}$ be the surjective morphism deduced from u by passage to the quotient. Since $F_r(\mathcal{N})$ is an invertible ideal and $\mathcal{N}$ is locally free of rank r on V thus at the points of $\mathrm{Ass}(X)$, it follows from 5.4.3 that $\mathcal{N}/\mathcal{A}\backslash\backslash_{\mathcal{N}}(F_r(\mathcal{N}))$ is locally free of

rank r. Since $F_r(\mathcal{N}) = \mathcal{O}_X$ above U, $\mathcal{A}\backslash\backslash_{\mathcal{N}}(F_r(\mathcal{N}))$ and null above U; it follows that $\mathcal{N}$ is a quotient of $\mathcal{N}/\mathcal{A}\backslash\backslash_{\mathcal{N}}(F_r(\mathcal{N}))$ so that we have an exact sequence

$$0 \to \mathcal{P} \to \mathcal{N}/\mathcal{A}\backslash\backslash_{\mathcal{N}}(F_r(\mathcal{N})) \to \overline{\mathcal{N}} \to 0.$$

Now $\mathrm{Ass}(\mathcal{P}) = \mathrm{Ass}(X) = V$ and $\mathcal{P}|V = 0$, thus $\mathcal{P} = 0$ and $\overline{\mathcal{N}}$ is locally free of rank r, the same reasoning proves that $\mathrm{Ker}(\overline{u}) = 0$, thus $\mathcal{M}$ is locally free of rank r, which completes the proof of 5.2.2 in this case.

5.3 Proof of 5.2.2 in the case where S is Noetherian

We will establish the following technical lemma:

Lemma 16 (5.5.1)**.** *Let S be a Noetherian scheme, X an S-scheme of finite presentation, n an integer ≥ 0, U an open subset of S, $\mathcal{M}$ an $\mathcal{O}_X$-module of finite presentation, $h' : S' \to S$ a blowup, $X' = X \times_S S'$, $\mathcal{M}' = \mathcal{M} \times_S S'$, $U' = U \times_S S'$, $\mathcal{M}'$ the strict transform of $\mathcal{M}$ by the blowup h'. We assume* *(i) $\mathcal{M}$ is S-flat in dimensions $\geq n+1$.*

(ii) $\mathcal{M}'$ is flat in dimensions $\geq n$ above U'.

(iii) $U \neq S$.

Then, there exists an open subset W of S, strictly larger than U and a U'-admissible blowup $S'' \to S'$ such that the strict transform of $\mathcal{M}'$ by this blowup is flat in dimensions $\geq n$, above the reciprocal image W'' of W in S''.

Let us first show how the lemma implies theorem 5.2.2 in the Noetherian case. For $n > \dim(X/S)$, $\mathcal{M}$ is evidently S-flat in dimensions $\geq n$. Proceeding by decreasing recurrence on n, we can thus assume $\mathcal{M}$ S-flat in dimensions $\geq n+1$. We can assume U schematically dense in S (5.3.2). Let E be the family of open sets V of S, containing U, such that theorem 5.2.2 is true for the restriction of the situation above V. Then E is non-empty because $U \in E$; since S is Noetherian, E has a maximal element V_0. The lemma above (applied with $U = V_0$) implies, taking into account 5.1.4 and 5.3.2 that $V_0 = S$ cqfd.

Let s be a maximal point of $S - U$. We can find a commutative diagram

$$\begin{array}{ccc} X & \leftarrow & X \times_S \tilde{S} \leftarrow Y \\ \downarrow & & \downarrow \\ (S,s) & \leftarrow & (\tilde{S},\tilde{s}) \end{array}$$

satisfying the following conditions:

a) $(\tilde{S},\tilde{s})$ is an elementary étale neighborhood affine of (S,s) such that $\tilde{s}$ is the only point of $\tilde{S}$ above s.

b) $Y \to X \times_S \tilde{S}$ is étale and its image contains $X \times_S k(\tilde{s})$.

c) Y is an affine sum disjoint of affine opens $Y_i, i \in I$.

d) For all $i \in I$, the reciprocal image $\mathcal{M}_i$ of $\mathcal{M}$ on Y_i possesses a $\tilde{S}$-devisage in dimensions $\geq n$ $(Z_{ij} \to T_{ij}, \mathcal{Z}_{ij} \xrightarrow{\alpha_{ij}} \mathcal{N}_{ij} \to \mathcal{P}_{ij})$ such that the morphisms α_{ij} are S-universally injective for the indices ij such that $\dim(T_{ij}/S) > n$ and are surjective above an open of T_{ij} that projects onto S, for all indices ij.

The existence of such data results essentially from 1.2.4 and 2.9; the auxiliary conditions stated in a) and b) can be realized after suitable restriction of the étale neighborhood $\tilde{S}$.

Let H be the reduced subscheme of S, with underlying space $S-U$, and s a maximal point of H. According to condition a), the morphism $\tilde{H} = H \times_S \tilde{S} \to H$ is an isomorphism above $\mathrm{Spec}(\mathcal{O}_{H,s})$, thus also above an open neighborhood of s in H. Up to restricting $\tilde{S}$, we can assume that the image of $\tilde{S}$ in S is an open S_1 such that $\tilde{S} \to S_1$ is an isomorphism above $H_1 = H \cap S_1$. We can also strengthen condition b) to condition b') as follows:

b') The morphism $Y \to X \times_S \tilde{S}$ is étale and its image contains $X \times_S H_1$.

We denote with index 1 (resp. with $\sim$) the objects deduced from objects above S by the open immersion $S_1 \to S$ (resp. the étale morphism $\tilde{S} \to S$).

Suppose we have solved lemma 5.5.1 after making the base change $\tilde{S} \to S$, with the open $\tilde{W} = \tilde{S}$ and a $\tilde{U}'$-admissible blowup $\tilde{S}'' \to \tilde{S}'$.

This blowup is that of an ideal $\tilde{\mathcal{J}}'$ of $\tilde{S}'$ such that $\tilde{\mathcal{J}}'|\tilde{U}' = \mathcal{O}_{\tilde{U}'}$. Consider the Cartesian square

$$\begin{array}{ccc} S_1' & \longleftarrow & \tilde{S}' \\ h_1' \downarrow & & \downarrow \tilde{h}' \\ S_1 & \longleftarrow & \tilde{S} \end{array}$$

Then $\tilde{S}' \to S_1'$ is an étale surjective morphism and is an isomorphism above $(h_1')^{-1}(H_1)$ (which is a closed subscheme of S_1', with underlying space $S_1' - U_1'$). It is clear under these conditions that the ideal $\tilde{\mathcal{J}}'$ is the reciprocal image of an ideal $\mathcal{J}_1'$ of S_1'; let $S_1'' \to S_1'$ be the blowup of $\mathcal{J}_1'$. By faithfully flat descent from $\tilde{S}''$ to S_1'', we see that we have proved lemma 5.5.1, for the restriction of the situation above S_1 and with the open $W_1 = S_1$. According to 5.3.1, we can extend the blowup $S_1'' \to S_1'$ to a blowup $S'' \to S'$; we have thus proved lemma 5.5.1 with open $W = U \cup S_1$.

Criteria of flatness

We are thus reduced to proving 5.5.1 in the case where $S = \tilde{S}$ with $W = S$. By flat descent (cf. condition b')), we can replace X by Y. Finally, to construct the U'-admissible blowup $S'' \to S'$, we can apply the reductions of 5.3.4 and thus reduce to the case where $Y = Y_i$; in short, it suffices to prove the following lemma:

Lemma 17 (5.5.2)**.** *Let S be a noetherian affine scheme, U an open subset of S, $X \to S$ an affine morphism of finite presentation, n an integer, $\mathcal{M}$ an $\mathcal{O}_X$-module of finite presentation, which possesses an S-filtration in dimensions $\geq n$ $(Z_j \to T_j, \mathcal{L}_j \xrightarrow{\alpha_j} \mathcal{N}_j \to \mathcal{P}_j)$ in which the α_j are S-universally injective for $j = \dim(T_j/S) \geq n+1$ and are surjective above an open subset of T_j that projects onto S, for $j \leq n$.*

Let $S' \to S$ be a blowup, $U' = U \times_S S'$, $X' = X \times_S S'$, $\mathcal{M}' = \mathcal{M} \times_S S'$ such that the strict transform $\mathcal{M}'$ of $\mathcal{M}$ is flat in dimensions $\geq n$ above U'. Then there exists a U'-admissible blowup $S'' \to S'$, such that the strict transform of $\mathcal{M}'$ by this blowup is S''-flat in dimensions $\geq n$.

We will need the following lemma:

Lemma 18 (5.5.3)**.** *Let S be a noetherian scheme, X a S-scheme with integral fibers of dimension r, $0 \to \mathcal{L} \xrightarrow{\alpha} \mathcal{N} \to \mathcal{P} \to 0$ an exact sequence of $\mathcal{O}_X$-modules of finite type such that $\mathcal{L}$ is locally free, U an open subset schematically dominant of S.*

i) Let $\mathcal{R}$ be a coherent submodule of $\mathcal{N}$ such that $\dim(\mathcal{R}/S) < r$. Then $\mathcal{R} \cap \mathcal{L} = 0$, thus the morphism $\mathcal{R} \hookrightarrow \mathcal{N} \to \mathcal{P}$ is injective and we have an exact sequence $0 \to \mathcal{L} \to \mathcal{N}/\mathcal{R} \to \mathcal{P}/\mathcal{R} \to 0$.

ii) Suppose $\dim(\mathcal{P}/S) < r$ and let $\mathcal{R}$ be a coherent submodule of $\mathcal{P}$, null above U, $\mathcal{R}'$ its reciprocal image in $\mathcal{N}$. Then the exact sequence $0 \to \mathcal{L} \to \mathcal{R}' \to \mathcal{R} \to 0$ is canonically split, a lifting of $\mathcal{R}$ in $\mathcal{R}'$ being given by the submodule $\mathcal{L}'$ of $\mathcal{R}'$

of null sections above U; in particular, we have an exact sequence

$$0 \to \mathcal{L} \to \mathcal{N}/\mathcal{L}' \to \mathcal{P}/\mathcal{R} \to 0.$$

Proof of 5.5.3. Let x be a point of $\mathrm{Ass}(\mathcal{R} \cap \mathcal{L})$; then the point x is in $\mathrm{Ass}(\mathcal{L})$, thus in $\mathrm{Ass}(X)$, since $\mathcal{L}$ is locally free on X. But then $x \in \mathrm{Ass}(X/S)$ (EGA IV 3.3.1), that is to say it is a generic point of a fiber of X above S; this contradicts the hypothesis $\dim(\mathcal{R}/S) < r$. We thus have $\mathrm{Ass}(\mathcal{R} \cap \mathcal{L}) = \emptyset$ and consequently $\mathcal{R} \cap \mathcal{L} = 0$ which proves (i).

Let us prove (ii). Let W be the complementary of the support of $\mathcal{R}$. Then $W = X \times_S U$ and maps surjectively onto S since $\dim(\mathcal{R}/S) < r$. Let us denote by j the open immersion $W \to X$ and by $\beta : \mathcal{L} \to \mathcal{R}'$ the canonical injection. Then the morphism $j^*(\beta) : \mathcal{L}|W \to \mathcal{R}'|W$ is an isomorphism. By

elsewhere let $x \in X - W$ and denote s the projection of x in S. Since U is schematically dense in S, we have $\mathrm{prof}(\mathcal{O}_{S,s}) \geq 1$ and since x is not the generic point of $X \otimes k(s)$, we have $\mathrm{prof}(\mathcal{O}_{X_s \otimes k(s)}) \geq 1$ thus $\mathrm{prof}(\mathcal{O}_{X,x}) \geq 2$ (EGA IV 6.3.1); but then $\mathcal{L}$ is $X - W$-closed (EGA IV 5.10.5) otherwise said the canonical morphism $\mathcal{L} \to i_* i^*(\mathcal{L})$ is an isomorphism. Let us consider then the following commutative diagram where the first line is exact

$$\begin{array}{ccc} 0 \to \mathcal{L}' \to \mathcal{R}' & \longrightarrow & i_* i^*(\mathcal{R}') \\ \uparrow \beta & & \uparrow i_* i^*(\beta) \\ \mathcal{L} & \xrightarrow{\sim} & i_* i^*(\mathcal{L}). \end{array}$$

This shows us that β admits a kernel inverse $\mathcal{L}'$. q.e.d.

Proof of 5.5.2. Let $\mathcal{R}'$ be the kernel of the canonical morphism $\mathcal{M}' \to \mathcal{M}''$. Given the required properties of the S-devissage of $\mathcal{M}$, $\mathcal{M}$ is S-flat in dimensions $\geq n+1$ and consequently we have $\dim(\mathcal{R}'/S) \leq n$. Let $(Z'_j \to T'_j, \mathcal{L}'_j \overset{\sigma'_j}{\to} \mathcal{N}'_j \to \mathcal{R}'_j)$ be the reciprocal image on X' of the S-devissage of $\mathcal{M}$. By decreasing recurrence on j, we deduce from lemma 5.5.3 i), that $\mathcal{R}'$ defines step by step a T'_j-module $\mathcal{R}'_j$ (isomorphic to $\mathcal{R}'$ as S'-module) which is a submodule of $\mathcal{N}'_j$ such that $\mathcal{R}'_j \cap \mathcal{L}'_j = 0$ for $j \geq n+1$. By passage to the quotient, we deduce an S' devissage of $\mathcal{M}' : (Z'_j \to T'_j, \mathcal{L}'_j \to \mathcal{N}'_j/\mathcal{R}'_j \to \mathcal{R}'_j/\mathcal{R}'_j)$. We can then replace S by S', $\mathcal{M}$ by $\mathcal{M}'$ thus assume $\mathcal{M}$ S-flat in dimensions $\geq n$, above U. Then $\mathcal{N}_n$ is U-flat in dimensions $\geq n$.

According to the smooth case treated in 5.4, there exists a U-admissible blowup $S'' \to S$, such that the strict transform $\mathcal{N}''_n$ of $\mathcal{N}_n$ is S''-flat in dimensions $\geq n$. Let $\mathcal{R}''_n$ be the kernel of the morphism $\mathcal{N}''_n \to \mathcal{N}''_n$; $\mathcal{R}''_n$ is thus the submodule of $\mathcal{N}''_n$ formed by the null sections above U. Consider the part of the S''-devissage of $\mathcal{M}''$ relative to dimension $n+1$; that is to say the exact sequence

$$0 \to \mathcal{L}''_{n+1} \overset{\sigma''_{n+1}}{\to} \mathcal{N}''_{n+1} \to \mathcal{R}''_{n+1} \to 0.$$

By definition of a relative devissage, Z''_n is a closed subscheme of T''_{n+1}, $\mathcal{R}''_{n+1}$ comes from a Z''_n-module (still denoted $\mathcal{R}''_{n+1}$) and $\mathcal{N}''_n$ is the direct image, by the finite morphism $Z''_n \to T''_n$, of the module $\mathcal{R}''_n$. It follows that $\mathcal{R}''_n$ comes in fact from the submodule of $\mathcal{R}''_{n+1}$ formed by the null sections above U. According to lemma 5.5.3(ii), $\mathcal{R}''_n$ canonically lifts to the submodule $\mathcal{R}''_{n+1}$ of $\mathcal{N}''_{n+1}$ formed by the null sections above U. Step by step, we lift $\mathcal{R}''_n$ to the submodule $\mathcal{R}''$ of $\mathcal{M}''$ formed by the null sections above U. It is clear under these conditions that the strict transform $\mathcal{M}'' = \mathcal{M}''/\mathcal{R}''$ of $\mathcal{M}$ is S''-flat in dimensions $\geq n$ q.e.d.

6 Flatness criteria

6.1 End of the demonstration of 5.2.2

We will need the following lemma:

Lemma 19 (5.6.1)**.** *Let S be a noetherian scheme, X an S-scheme of finite type, $\mathcal{N}$ an $\mathcal{O}_X$-module of finite type, S-flat, n an integer, U an open subset of S and V an open subset of $X \times_S U = X_U$, such that* $\dim(X_U - V/S) < n$. *Then, there exists a closed subset T of $X - V$, such that* $\dim(T/S) < n$ *which contains* $\mathrm{Ass}(\mathcal{M}/S) \cap (X_U - V)$.

Indeed, let $x \in \mathrm{Ass}(\mathcal{M}/S) \cap (X_U - V)$. Let s be the projection of x on S and $\bar{x}$ the adherence of x in X. Since $\dim(X_U - V/S) < n$, we also have $\dim(\bar{x} \otimes_k k(s)) < n$ and consequently $\dim(\bar{x}/S) < n$ since $\mathcal{N}$ is S-flat (EGA IV 12.1.1.5). We conclude with a constructibility argument (cf. EGA IV 9.8.3), which shows that $E = \mathrm{Ass}(\mathcal{M}/S) \cap (X_U - V)$ is contained in the adherence of a finite number of points of E.

Let us return to the demonstration of 5.2.2. It follows from 5.3 that we can assume X and S are affine and U schematically dense in S. Let V be a quasi-compact open subset of $X \times_S U$, such that $\mathcal{M}|V$ is S-flat and such that $\dim((X \times_S U - V)/U) < n$. We can find an $\mathcal{O}_X$-module of finite presentation $\mathcal{N}$ and a surjective morphism $u : \mathcal{N} \to \mathcal{M}$ which is an isomorphism above V.

Let us consider S as a filtered projective limit of affine noetherian schemes $S_i, i \in I$. For i large enough, we can assume that the objects $U, X, V, \mathcal{N}$ come from objects $U_i, X_i, V_i, \mathcal{N}_i$ above S_i, such that $\mathcal{N}_i|V_i$ is S_i-flat and $\dim((X_i \times_{S_i} U_i - V_i)/U_i) < n$. According to 5.5, there exists an ideal $\mathcal{I}_i$ of $\mathcal{O}_{S_i}$, with $\mathcal{I}_i|U_i = \mathcal{O}_{U_i}$, such that the strict transform $\mathcal{N}'_i$ of $\mathcal{N}_i$, by the blowup S'_i of S_i along $\mathcal{I}_i$, is S'_i-flat in dimensions $\geq n$. Let us define $X'_i = X_i \times_{S_i} S'_i$, $\mathcal{N}'_i = \mathcal{N}_i \times_{S_i} S'_i$. There exists then an open subset W'_i of X'_i, containing V_i, such that $\mathcal{N}'_i|W'_i$ is S'_i-flat and such that $\dim((X'_i - W'_i)/S'_i) < n$. According to 5.6.1, we can, by removing from W'_i a closed subset of $W'_i - V_i$ of relative dimension $< n$, assume that $\mathrm{Ass}(\mathcal{N}'_i/S'_i) \cap W'_i \times_{S'} U$ is contained in V_i.

Let $S' \to S$ be the U-admissible blowup defined by the ideal $\mathcal{I} = \mathcal{I}_i\mathcal{O}_S$, so that S' is a closed subscheme of $S \times_S S'_i$. Let us define $X' = X \times_S S'$, $\mathcal{N}' = \mathcal{N} \times_S S'$, $\mathcal{M}' = \mathcal{M} \times_S S'$, $W' = W'_i \times_{S_i} S'$. Then the strict transform $\mathcal{N}''$ of $\mathcal{N}$ by the blowup $S' \to S$ is isomorphic to $\mathcal{N}'_i \times_{S_i} S'$ and is therefore S'-flat in dimensions $\geq n$. More precisely, $\mathcal{N}'$ is S'-flat at the points of W'; we have $\dim((X' - W')/S') < n$ and $\mathrm{Ass}(\mathcal{N}'/S') \cap W' \times_{S'} U$ is contained in V. By replacing X' with W', S with S' we can assume that the following conditions are satisfied:

a) the quotient $\bar{\mathcal{N}}$ of $\mathcal{N}$ by the submodule of null sections above U is S-flat.

b) We have $\mathrm{Ass}(\mathcal{N}/S) \cap X \times_S U \subset V$.

Since $\mathcal{N} = \bar{\mathcal{N}}$ above U, $\mathcal{N}$ is U-flat and we have

$$\mathrm{Ass}(\mathcal{N}) \cap X \times_S U \subset V$$

(3.4.3). Let $\mathcal{R} = \mathrm{Ker}(u)$. By construction, $\mathcal{R}|V = 0$; this implies $\mathcal{R} = 0$ above U. But then, if $\overline{\mathcal{M}}$ is the quotient of $\mathcal{M}$ by the submodule of null sections on $U \times_S X$, it is clear that the canonical morphism $\overline{u} : \overline{\mathcal{N}} \to \overline{\mathcal{M}}$, deduced from u by passage to the quotient, is an isomorphism, thus $\overline{\mathcal{M}}$ is S-flat, which completes the proof of 5.2.2.

5.7. Applications to algebraic spaces

In this section, we provide some very simple applications of theorem 5.2.2. To accommodate the needs of the day, we formulate them in the more general framework of Artin's algebraic spaces [1]. Let us begin with some generalizations and variants of Knutson's results [12].

Let S be a scheme. We denote by (Sch/S) the category of S-schemes, (E/S) the category of sheaves on (Sch/S) for the étale topology, and we identify in the usual way (Sch/S) with a full subcategory of (E/S).

Definition 8 (5.7.1)**.** Let S be a scheme. An algebraic space Y over S (or an S-algebraic space) is a sheaf on (Sch/S) for the étale topology, quotient of an S-scheme X by an étale equivalence relation representable by R. More precisely, there exists an S-scheme X, an S-scheme R, an S-monomorphism $i : R \to X \times_S X$ and an S-morphism of étale sheaves $p : X \to Y$, such that the following conditions are satisfied:

(i) $i : R \to X \times_S X$ is the graph of an S-equivalence relation on X.

(ii) The canonical projections $\begin{matrix} q_1 \\ q_2 \end{matrix} : R \xrightarrow{i} X \times_S X \rightrightarrows X$ are étale morphisms.

(iii) (p, Y) is a coequalizer of the double arrow $\begin{matrix} q_1 \\ q_2 \end{matrix} : R \rightrightarrows X$ in the category (E/S).

An algebraic space (absolute) is an algebraic space over $\mathrm{Spec}(\mathbb{Z})$.

Lemma 20 (5.7.2)**.** *With the previous notations, the square*

$$\begin{array}{ccc} X & \xleftarrow{q_1} & R \\ p\downarrow & & \downarrow q_2 \\ Y & \xleftarrow{p} & X \end{array} \qquad (*)$$

is cartesian and the morphism $p : X \to Y$ is representable by étale surjective morphisms.

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Proof. The first assertion follows from general properties of categories of sheaves (SGA 3 IV 4.4.9); let us prove the last assertion. Let Z be a scheme, $u : Z \to Y$ a morphism in (E/S); we need to show that $F = X \times_Y Z$ is a scheme such that the projection $F \to Z$ is an étale surjective morphism. We may assume Z is affine. Since p is an epimorphism in (E/S), there exists a commutative diagram:

$$\begin{array}{ccc} X & \xleftarrow{u'} & Z' \\ p\downarrow & & \downarrow v \\ Y & \xleftarrow{u} & Z \end{array}$$

in which v is a morphism of schemes, affine, étale, surjective. Thus $F' = F \times_Y Z' \simeq R \times_X Z'$ is a scheme and $F' \to Z'$ is étale surjective since it is so for $q_2 : R \to X$. We are reduced to a problem of effectivity of the canonical descent data on F', relative to the morphism étale affine $Z' \to Z$. Note that the question of representability of $F = X \times_Y Z$ is local in nature on X. Let U be an affine open subset of X; then $U \times_Y Z'$ is an open subset of F', stable under the descent data and is a sub-object of the affine scheme $U \times_Z Z'$, thus is separated. Since $Z' \to Z$ is quasi-compact and universally open, we can cover $U \times_Y Z'$ by quasi-compact open subsets stable under the descent data; being separated on Z', they are then quasi-affine on Z' (EGA IV 18.12.12) and therefore descend to above Z (SGA 1 VIII 7.9).

Note that the following diagram in (E/S) (where δ is the diagonal map) is cartesian:

$$\begin{array}{ccc} R & \xrightarrow{pq_1} & Y \\ \downarrow i & & \downarrow \delta \\ X \times_S X & \xrightarrow{p \times p} & Y \times_S Y. \end{array}$$

This justifies the following definitions:

Definition 9 (5.7.3)**.** With the notations of 5.7.1, we say that Y is quasi-separated (resp. locally separated, resp. separated) over S if i is quasi-compact (resp. is an immersion, resp. is a closed immersion).

Taking into account 5.7.2, the quasi-separated algebraic spaces that we have just defined coincide with those of Knutson [12]. We refer to loc. cit. for the extension to algebraic spaces of the properties of schemes of local nature for the étale topology. Let us also recall that an algebraic space Y is quasi-compact if in 5.7.1, one can choose X quasi-compact.

Definition 10 (5.7.4)**.** Let Y be an algebraic space, Z an algebraic subspace of Y. An elementary étale neighborhood of Z in Y is the

given by a flat morphism $u : Y' \to Y$, where Y' is a scheme and u is an isomorphism above Z.

We note that Z is then representable and that, possibly after restricting Y', we can assume that $u^{-1}(Z)$ is a closed subscheme of Y'.

Lemma 21 (5.7.5)**.** *Let Z be an algebraic subspace of an algebraic space Y, X a scheme, $v : X \to Y$ a flat morphism which above Z is a finite morphism of rank $r \geq 1$. Suppose that any finite set of points of X is contained in an affine open set. Then there exists an elementary étale neighborhood Y' of Z in Y. If X is separated, we can take Y' separated.*

We revisit Knutson's idea of using symmetric products. By recursion on the integer $n \geq 0$, we prove that the product $X^n/Y = X \times_Y X \times_Y \cdots \times_Y X$ (n factors) is a scheme and, proceeding as in the proof of 5.7.2, we see that any finite set of points of X^n/Y is contained in an affine open set, as is the case for X. The symmetric group $\mathfrak{S}_r$ acts on X^r/Y by permutation of factors. Let U be the largest open subset of X^r/Y above which $\mathfrak{S}_r$ acts freely. Then the étale sheaf U', quotient of U by the action of $\mathfrak{S}_r$, is a scheme and $U \to U'$ is a finite étale Galois covering of degree $r!$ (SGA 1 V 1.8 and [17] chap. X). The canonical morphism $U \to Y$ factors through U' and we verify immediately (working locally for the étale topology on Y) that U' is an elementary étale neighborhood of Z in Y. If X is separated, so is U and therefore also U'.

Proposition 4 (5.7.6)**.** *Let Y be a quasi-compact and quasi-separated algebraic space. Then there exists a finite sequence $Z_i, i = 1, \ldots, r$ of algebraic subspaces of Y such that:*

> *[(i)]For all i, Z_i is reduced and quasi-compact. The Z_i are disjoint and cover Y. $Y_i = \bigcup_{j \leq i} Z_j$ is an open subset of the space Y. There exists a separated and quasi-compact elementary étale neighborhood Y_i' of Z_i in Y_i and the image of Y_i' in Y is Y_i.*

1. *Proof.* Consider Y as the quotient of an affine scheme X by an étale equivalence relation with graph R. We thus have a Cartesian square

$$\begin{array}{ccc} X & \xleftarrow{q_1} & R \\ \downarrow^{p} & & \downarrow^{q_2} \\ Y & \xleftarrow{p} & X \end{array}$$

and since X is separated and Y quasi-separated, the morphism q_2 is étale separated, of finite presentation. Then the function on X which to x associates

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the number of geometric points of $q_2^{-1}(x)$ is a constructible semi-continuous function from below (cf. EGA IV 18.10.17.1) and q_2 is finite if and only if this function is locally constant. This function is compatible with base changes on X, thus comes from a constructible function on Y. Let Z_i be the quasi-compact and reduced algebraic subspace above which it takes the value i. Then the restriction of p above Z_i is a finite étale covering of degree i and one can apply 5.7.5.

Proposition 5 (5.7.7)**.** *Let Y be a quasi-separated algebraic space. Then there exists an open subset of Y, containing the maximal points of Y (thus a fortiori dense) which is representable.*

Proof. The assertion is local on Y. We can thus assume that there exists an affine scheme X and an étale surjective morphism $p : X \to Y$. Then the largest open subset of Y above which p is finite answers the question. □

Proposition 6 (5.7.8)**.** *(cf. [12] chap. III th. 1.1). Let $\mathcal{M}$ be a quasi-coherent sheaf on a quasi-compact and quasi-separated algebraic space Y. Then $\mathcal{M}$ is the filtered inductive limit of its quasi-coherent subsheaves of finite type.*

Proof. Taking into account 5.7.6, we are led by recurrence on i to prove the following assertion: let U be a quasi-compact open subset of Y, Z the closed algebraic reduced subspace of space $Y - U$, $u : Y' \to Y$ an elementary étale separated quasi-compact neighborhood of Z in Y, such that u is surjective. Finally, let $\mathcal{M}$ be a quasi-coherent sheaf on Y. Assuming the proposition proved for $(U, \mathcal{M}|U)$, then it is true for $(Y, \mathcal{M})$. Let U' (resp. $\mathcal{M}'$) be the reciprocal image of U (resp. $\mathcal{M}$) by u. Then $\mathcal{M}'$ is the filtered inductive limit of its quasi-coherent subsheaves of finite type $\mathcal{M}'_i, i \in I$ (EGA I 9.4.9). Fix i in I. It suffices to construct a quasi-coherent subsheaf of $\mathcal{M}$, of finite type, whose reciprocal image on Y' contains $\mathcal{M}'_i$. By hypothesis one can find such a sheaf $\mathcal{N}$ above U. Let $\mathcal{N}'$ be its reciprocal image on U'. Then there exists a quasi-coherent subsheaf $\mathcal{N}''$ of $\mathcal{M}'$, of finite type, which extends $\mathcal{N}'$ (EGA I 9.4.7) and we can assume that $\mathcal{N}''$ majorizes $\mathcal{M}'_i$. But since Y' is an elementary étale neighborhood of Z in Y, it is clear that $\mathcal{N}''$ is automatically stable by the descent data on $\mathcal{M}'$ relative to the étale morphism $Y' \to Y$, thus comes from a subsheaf $\mathcal{F}$ of $\mathcal{M}$ which answers the question. □

We can now extend to algebraic spaces the theorem 5.2.2.

Theorem 13 (5.7.9)**.** *Let S be a quasi-compact and quasi-separated algebraic space, U a quasi-compact open subset of S, $f : X \to S$ an algebraic*

space of finite presentation, $\mathcal{M}$ an $\mathcal{O}_X$-module of finite type, n an integer. Suppose that $\mathcal{M}|f^{-1}(U)$ is U-flat in dimensions $\geq n$ (cf. 5.2.1). Then, there exists a U-admissible blowup $g : S' \to S$ (defined by a quasi-coherent sheaf of ideals of finite type) such that the strict transform of $\mathcal{M}$ by g is S'-flat in dimensions $\geq n$.

Let us first note that it follows from 5.7.8 that the lemmas on blow- ups proved in 5.1 and the reductions made in 5.3 are also valid when S is an algebraic space that is quasi-compact and quasi- separated.

Taking into account 5.7.6, we are reduced to proving that if V is an open quasi-compact subset of S such that the reduced closed complementary Z has an elementary étale neighborhood $\tilde{S}$, quasi-compact and separated, covering S and if (5.7.9) is true for the restriction of the situation above V, then (5.7.9)

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is true. So let $V' \to V$ be a $U \cap V$ admissible blowup such that the strict transform of $\mathcal{M}$ becomes V'-flat in dimensions $\geq n$. We extend this blowup of V to a U-admissible blowup $S' \to S$ (5.3.1). After replacing S by S' (5.1.4), we can assume that $U \supseteq V$. Let $\tilde{U}$ be the reciprocal image of U in $\tilde{S}$. Since any blow- up $\tilde{U}$-admissible of $\tilde{S}$ descends to a U-admissible blowup of S, we can replace S by $\tilde{S}$ and thus assume S representable. Taking into account 5.3.4, we can also assume X representable and we are reduced to the case treated in 5.2.2.

Corollary 16 (5.7.10)**.** *Let S be a quasi-compact and quasi-separated algebraic space, U a quasi-compact open subset of S, n an integer, X an S-space algebraic of finite type, which above U is of relative dimension $\leq n$. Then there exists a U-admissible blowup $S' \to S$, such that the strict transform of X by this blowup is of relative dimension $\leq n$ above S'.*

Proof. We apply 5.7.9 with $\mathcal{M} = \mathcal{O}_X$ and $n+1$ in place of n. □

Corollary 17 (5.7.11)**.** *Let S be a quasi-compact and quasi-separated algebraic space, U a quasi-compact open subset of S, $X \to S$ an S-algebraic space of finite type. We assume that $X \times_S U \to U$ is an open immersion. There exists a U-admissible blowup $S' \to S$, such that if $\overline{X}'$ is the strict transform of X by this blowup, then the morphism $\overline{X}' \to S'$ is flat, quasi- finite (and is an open immersion above U). If X is locally separated over S (resp. is separated over S), $\overline{X}' \to S'$ is étale (resp. an open immersion).*

Indeed, according to 5.7.9, there exists a U-admissible blowup $S' \to S$ which makes $\overline{X}' \to S'$-flat and we can assume U schematically dense in S' (5.3.2). It is then clear that $\overline{X}'$ is quasi-finite over S'. Suppose X locally separated over S, thus $\overline{X}'$ locally separated over S' and let us show that the diagonal immersion $\delta : \overline{X}' \to \overline{X}' \times_{S'} \overline{X}'$ is open which prov-

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will show that $\overline{X}'$ is étale over S'. Let V be an open subset of $\overline{X}' \times_{S'} \overline{X}'$ that contains $\delta(\overline{X}')$ as a closed algebraic subspace, defined by an ideal $\mathcal{I}$. Now $\overline{X}'$ is S'-flat, $\overline{X}' \to S'$ is an open immersion above U and U is schematically dense in S'; consequently δ is an isomorphism at the points of $\mathrm{Ass}(\overline{X}' \times_{S'} \overline{X}')$, thus $\mathcal{I} = 0$. If moreover X is separated over S, δ is a closed immersion and the monomorphism étale $\overline{X}' \to S'$ is an open immersion.

Corollary (5.7.12). *Let S be a quasi-compact and quasi-separated algebraic space, $X \to S$ an S-algebraic space proper, U an open quasi-compact of S above which $X \to S$ is an isomorphism. Then there exists a U-admissible blowup $S' \to S$, such that if $\overline{X}'$ is the strict transform of X by this blowup, the canonical morphism $\overline{X}' \to S'$ is an isomorphism (in other words, there exists a $X \times_S U$ admissible blowup of X, say X', such that the composite morphism $X' \to X \to S$ is a U-admissible blowup).*

Indeed according to 5.7.12, we can assume that $\overline{X}' \to S'$ is an open immersion; we can also assume U schematically dense in S'. Since $\overline{X}' \to S'$ is proper, it is an isomorphism.

Corollary (5.7.13) (Chow's lemma). *Let S be an algebraic space quasi-compact and quasi-separated, $f : X \to S$ an S-algebraic space quasi-separated of finite type, having only a finite number of irreducible components. Then, there exists a commutative diagram*

X X' [l, "g"']
S Y [l, "p"'] [uu, "h"]

such that:

(i) p is quasi-projective.

(ii) g is a U-admissible blowup, where U is a dense quasi-compact open subset of X.

(iii) h is a flat quasi-finite surjective morphism, which is an isomorphism above a quasi-compact open subset of Y, schematically dense in Y. If X is locally separated over S (resp. is separated over S), h is étale (resp. is an isomorphism).

(N.B. In the separated case 5.7.13 is a simple generalization of [12] chap. IV §3; the non-separated case was pointed out to us by Artin.)

Proof. Since X is quasi-separated and has only a finite number of irreducible components, we can find a dense open subset U of X, affine over S, retrocompact in X and representable (5.7.7). We easily deduce from 5.7.8 that V is realized as an open subset of a projective S-scheme P. Let Γ be the closed algebraic subspace of $X \times_S P$,

schematic adherence of the graph of the identity of U. Thus the morphism $\Gamma \to X$ is projective and is an isomorphism above U. According to 5.7.12, there exists a proper morphism $Z \to \Gamma$, such that the composite morphism $Z \to \Gamma \to X$ is a U-admissible blowup. The morphism $Z \to \Gamma \to P$ is a morphism of finite type and possesses above U a section $s : U \to Z$ which is an open immersion with schematically dense image in Z. Taking into account the fact that U has only a finite number of irreducible components, we immediately see that there exists an open quasi-compact dense V of U above which $Z \to P$ is an isomorphism. By replacing U with V we now assume that $Z \to P$ is an isomorphism above U. There exists then a U-admissible blowup $P' \to P$ such that the strict transform X' of Z is flat and quasi-finite over P' (5.7.11). Thus $X' \to Z \to X$ is a blowup U-admissible of X (5.1.4) and it suffices to take for Y the sub- algebraic open space of P', image of X'. The other assertions of 5.7.13 also result from 5.7.11.

We demonstrate similarly the following quantitative form of Chow's lemma:

Corollary (5.7.14). *Let S be a quasi-compact and quasi-separated algebraic space, $X \to S$ a separated S-algebraic space, of finite type, U an open subset of X quasi-projective over S. Then there exists a U-admissible blowup X' of X which is quasi-projective over S.*

Second part: Flatness and projectivity

Introduction

In this second part, we study the problem of the "descent" of flatness and projectivity properties in commutative algebra: let $f : A \to B$ be a homomorphism of commutative rings, M an A-module, possibly subject to certain preliminary conditions, and such that $B \otimes_A M$ is B-flat (resp. projective); under what conditions on f can we ensure that M is A-flat (resp. projective)?

The problem of the descent of flatness is well known when we make a preliminary hypothesis of "relative finiteness" on M: cf. EGA IV 11, as well as §4 of the first part. In §1 we study the problem without making preliminary hypotheses on M. One immediately perceives that it is then necessary to impose serious restrictions on f: in particular, if f is injective and descends flatness, it "descends nullity" (condition (O) of §1).

It is easy to prove that an injective and finite homomorphism verifies condition (O); in fact, D. Ferrand demonstrated in [7] that a homomorphism injective and finite of Noetherian rings descends flatness. We show here that this result remains valid without hypothesis

Noetherian. We also show that an injective homomorphism $f : A \to B$, which descends nullity, also descends flatness when A is Noetherian. For this, we study a condition on f, stricter than the descent of flatness and which, like condition (O), does not involve the multiplication of B. Furthermore, we provide an example of an injective homomorphism $f : A \to B$ which descends nullity and not flatness; in this example, A is a valuation ring (a monstrous one, in fact).

In the case where the base ring A is complete local Noetherian integral, condition (O) for an A-algebra B is equivalent to the existence of a non-null linear form on the A-module B; we show by an example that this condition does not always imply the existence of an A-algebra quotient of B that is finite and faithful, even when B is of finite type over A.

In order to study the descent of projectivity, we introduce in §2 a "Mittag-Leffler condition" on modules. Lazard has shown in [13] that every pure submodule of a free module is a filtered union of pure and projective submodules, and Goblot has remarked [8] that the same property holds for pure submodules of A^I when A is Noetherian. In our terminology, this property characterizes flat Mittag-Leffler modules. Another characterization of these modules is the following: the flat A-module M is Mittag- Leffler if and only if it is a filtered inductive limit of A-modules free of finite type L_i such that the projective system of duals verifies the condition of set-theoretic Mittag-Leffler (EGA $\mathrm{O_{III}}$ 13.1.2).

We state a certain number of criteria for the descent of the condition of Mittag-Leffler, under a preliminary hypothesis of flatness; these criteria resemble, at least in the Noetherian case, the criteria of EGA IV 11 for the descent of flatness under a hypothesis of relative finiteness.

In §3, we study the passage from the Mittag-Leffler property to the property of projectivity. For this purpose, we systematically use the method of transfinite recurrence introduced by Kaplansky [11]; this method allows in numerous questions to suppose satisfied a hypothesis of countability, and it turns out that every flat module, Mittag-Leffler and of countable type is projective (2.2.2).

In particular, the criteria for descent of the Mittag- Leffler property obtained in §2 provide analogous criteria for descent of projectivity. The same methods allow to calculate the "finitist dimension" of Noetherian rings [4]. Finally, the techniques of descent of projectivity can be used to study certain problems of homological dimension of flat modules; we provide an example (3.3).

Second part: Flatness and projectivity

Conventions and notations

Except in sections 1.1, 2.1, 2.2, 2.3, and 2.4, rings are assumed to be commutative.

For any set E, we denote by 1_E the identity map of E.

An ordinal is a set on which the relation ($x \in y$ or $x = y$) is a good order; a cardinal is an ordinal that is not equipotent to any of its elements. For any ordinal u, we denote by $\aleph_u$ the infinite cardinal of index u.

We denote by (Ens) the category of sets, (Ab) the category of abelian groups, $\mathrm{Mod}(A)$ the category of left modules over a ring A.

If A is a commutative ring, we denote by $\mathrm{Min}(A)$ the set of minimal prime ideals of A. For any ideal I of A, we denote by $D(I)$ the set of prime ideals of A not containing I.

The dual of a category C is denoted by C^0.

7 Descent of flatness

Let $f : A \to B$ be a ring homomorphism; we say that f *descends flatness* if it satisfies the condition

(P) any A-module P such that $B \otimes_A P$ is B-flat is A-flat.

This condition has been studied by Ferrand [7] and Olivier [15]. Let us recall the following results [15]:

Let $f : A \to B$ be a homomorphism satisfying (P). Any monogenic ideal contained in $\mathrm{Ker}(f)$ is generated by an idempotent, and the inclusion $f(A) \hookrightarrow B$ satisfies (P).

Let $f : A \to B$ be an injective homomorphism satisfying (P). The A-module underlying B satisfies the condition

(O) any A-module P such that $B \otimes_A P = 0$ is null.

Criteria for flatness

One may ask whether the converse of this last result is true; we will see that the answer is affirmative when A is Noetherian and negative in the general case.

1.1. Reminder on universally exact sequences

Let A be a not necessarily commutative ring; we denote by T the functor $\mathrm{Hom}_{\mathbb{Z}}(\cdot, \mathbb{Q}/\mathbb{Z})$ from $(\mathrm{Mod}(A))^0$ to $\mathrm{Mod}(A^0)$; recall that T is exact and faithful. The following proposition is an amalgamation of well-known results ([13, 18]):

Proposition 7 (1.1.1)**.** *Let $0 \to M \xrightarrow{u} N \xrightarrow{r} P \to 0$ be an exact sequence of Mod(A). The following conditions are equivalent:*

(i) u is A-universally injective, i.e., for any A^0-module Q, $1_Q \otimes_A u$ is injective;

(ii) $1_{TM} \otimes_A u$ is injective;

(iii) Tu is invertible on the right;

(iv) Tv is A^0-universally injective;

(v) for any A-module of finite presentation F, Hom$_A(F, v)$ is surjective;

(vi) there exists an inductive system filtering of split exact sequences of Mod(A), with limit isomorphic to $0 \to M \xrightarrow{u} N \xrightarrow{r} P \to 0$.

Let us quickly recall the demonstration: (i) $\Rightarrow$ (ii) is clear; (ii) $\Rightarrow$ (iii) is obtained by forming the commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_{A^0}(TM, TN) & \xrightarrow{\mathrm{Hom}_{A^0}(TM, Tu)} & \mathrm{Hom}_{A^0}(TM, TM) \\ \downarrow & & \downarrow \\ T(TM \otimes_A N) & \xrightarrow{T(1_{TM} \otimes_A u)} & T(TM \otimes_A M) \end{array}$$

(in which the columns are the isomorphisms deduced from the universal property of the tensor product) and by using the exactness of T; (iii) $\Rightarrow$ (iv) is clear; (iv) $\Rightarrow$ (v) is obtained by forming the commutative diagram

$$\begin{array}{ccc} TP \otimes_A F & \xrightarrow{Tv \otimes_A 1_F} & TN \otimes_A F \\ \downarrow & & \downarrow \\ T(\mathrm{Hom}_A(F,P)) & \xrightarrow{T(\mathrm{Hom}_A(F,v))} & T(\mathrm{Hom}_A(F,N)) \end{array}$$

(in which the columns are the isomorphisms described in Bourbaki, Alg. comm., chap. I, §2, ex. 14) and by using the faithfulness of T; (v) $\Rightarrow$ (vi) is obtained by proceeding by "pull-back" from a representation of P as an inductive limit filtering of A-modules of finite presentation ([13] I 2.3); (vi) $\Rightarrow$ (i) is clear.

We will say that the sequence $0 \to M \xrightarrow{u} N \xrightarrow{v} P \to 0$ is A-universally exact if it satisfies conditions (i) to (vi) of 1.1.1; in accordance with [13] we will say that a submodule M of an A-module N is pure if the sequence $0 \to M \to N \to N/M \to 0$ is A-universally exact.

We will say that an A-module P is relatively projective if the functor $\mathrm{Hom}_A(P,\cdot)$ transforms every A-universally exact sequence into an exact sequence; similarly we will say that an A-module I is relatively injective if the functor $\mathrm{Hom}_A(\cdot,I)$ transforms every A-universally exact sequence into an exact sequence. The following characterizations are well known [18]:

(1.1.2) An A-module P is relatively projective if and only if it is a direct factor of a direct sum of A-modules of finite presentation.

(1.1.3) For any A-module M, there exists an A-universally exact sequence $0 \to N \to P \to M \to 0$, such that P is relatively projective.

(To prove these assertions, we note that any A-module of finite presentation is relatively projective (1.1.1(v)), which proves the sufficiency of condition 1.1.2; on the other hand, if M is an A-module inductive filtering limit of A-modules of finite presentation F_i, we form the exact sequence $0 \to N \to \oplus F_i \to M \to 0$; this sequence is A-universally exact (indeed, if F is an A-module of finite presentation, the functor $\mathrm{Hom}_A(F,\cdot)$ commutes with inductive filtering limits) hence 1.1.3 and the necessity of condition 1.1.2.)

(1.1.4) An A-module I is relatively injective if and only if the canonical A-linear application $j_I : I \to T^2(I)$ is left invertible.

(1.1.5) For any A-module M, there exists an A-universally exact sequence $0 \to M \to I \to N \to 0$, such that I is relatively injective.

(To prove these assertions, we note that, for any A^0-module Q, the A-module TQ is relatively injective (given the isomorphism $T(Q \otimes_A \cdot) \to \mathrm{Hom}_A(\cdot, TQ)$); this proves the sufficiency of condition 1.1.4; moreover, for any A-module M, the canonical A-linear application $j_M : M \to T^2(M)$ is A-universally injective, since J_{TM} is a right inverse of T_{j_M}; this proves 1.1.5 and the necessity of condition 1.1.4.)

7.1 Generalization of a theorem of Ferrand

In [7], Ferrand proves that an injective and finite homomorphism of Noetherian rings descends flatness. We will show that this result is valid without the Noetherian hypothesis; on the other hand, if A is a Noetherian ring and if $f : A \to B$ is an injective homomorphism satisfying condition (C), we will show that f descends flatness.

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Let A be a (commutative) ring, M an A-module; let us introduce the following condition on M:

(Q) given an exact sequence $0 \to Q \xrightarrow{u} L \xrightarrow{v} P \to 0$ such that L is A-flat, if $\mathrm{Im}(1_M \otimes_A u)$ is a pure A-submodule of $M \otimes_A L$, then $\mathrm{Im}(u)$ is a pure A-submodule of L (i.e., P is A-flat).

It is clear that if $f : A \to B$ is a ring homomorphism, and if the A-module underlying B satisfies condition (Q), then f descends flatness.

Lemma 22 (1.2.1)**.** *(Olivier, cf. [15]). Let M be an A-module, N a pure submodule of M. If N satisfies (Q), then M satisfies (Q).*

(In particular, a universally injective homomorphism descends flatness.)

Proof. Let $0 \to Q \xrightarrow{u} L \xrightarrow{v} P \to 0$ be an exact sequence in $\mathrm{Mod}(A)$ such that L is A-flat. Let us use the functor T from 1.1. If R is an A-module, we deduce from 1.1.1 that $\mathrm{Im}(1_R \otimes_A u)$ is a pure A-submodule of $R \otimes_A L$ if and only if $\mathrm{Hom}_A(v, TR)$ is left invertible.

By hypothesis, TN is isomorphic to a direct factor of TM; thus, if $\mathrm{Hom}_A(v, TM)$ is left invertible, so is $\mathrm{Hom}_A(v, TN)$; if N satisfies (Q), then Tv is left invertible, thus M satisfies (Q). □

Let us now introduce the "dual" conditions of conditions (O) and (Q), relative to an A-module M:

(O') any A-module N, such that $\mathrm{Hom}_A(M, N) = 0$, is null;

(Q') given an A-linear injective application $u : J \to I$, such that I is an injective A-module, if the A-linear application $\mathrm{Hom}_A(M, u)$ is left invertible, then u is left invertible (i.e., J is injective).

Lemma 23 (1.2.2)**.** *(i) We have (O') $\Rightarrow$ (O); the converse is true if A is linearly compact for the discrete topology (e.g., local Noetherian complete).*

(ii) We have (Q') $\Rightarrow$ (Q).

(iii) If $\mathrm{Supp}(M) = \mathrm{Spec}(A)$*, we have (Q') $\Rightarrow$ (O').*

Proof. (i) If M, P and E are three A-modules, we have a functorial isomorphism

$$\mathrm{Hom}_A(M \otimes_A P, E) \to \mathrm{Hom}_A(M, \mathrm{Hom}_A(P, E)); \qquad (*)$$

taking $E = P$, we see that (O') $\Rightarrow$ (O). Suppose A linearly compact for the discrete topology; let R be the radical of A and E the injective envelope of A/R; the functor $\mathrm{Hom}_A(\cdot, E)$ defines a duality on $\mathrm{Mod}(A)$ for which any A-module of finite type is reflexive ([8] 5.17); thus,

to verify condition (O'), we can limit ourselves to taking N monogenic, the implication (O) $\Rightarrow$ (O') follows here from the isomorphism (*) where we take $P = \mathrm{Hom}_A(N, E)$.

(ii) Suppose that M satisfies (Q); let's use the functor T from 1.1. Let $0 \to Q \xrightarrow{u} L \xrightarrow{v} P \to 0$ be an exact sequence such that L is A-flat and $\mathrm{Im}(1_M \otimes_A u)$ is a pure A-submodule of $M \otimes_A L$. According to 1.1.1 and isomorphism (*) we see that $\mathrm{Hom}_A(M, Tv)$ is left invertible. Since TL is an injective A-module, condition (Q') shows that Tv is left invertible; thus M satisfies (Q).

(iii) Suppose that M satisfies (Q'). Let R be the intersection of ideals R' of A such that $\mathrm{Hom}_A(M, A/R') = 0$. Then $\mathrm{Hom}_A(M, A/R) = 0$, and any A-module N, such that $\mathrm{Hom}_A(M, N) = 0$, is annihilated by R and is injective. In particular, any A/R-module is injective, thus the ring A/R is semi-simple. Let $S = 1 + R$ and let's show that $A/R = A[S^{-1}]$. Without localizing by S, we can assume that R is contained in the radical of A. The functor $\mathrm{Hom}_A(\cdot, A/R)$ is then exact and faithful; thus $R = 0$, which proves the assertion. In particular, we have $M[S^{-1}] = 0$. If $\mathrm{Supp}(M) = \mathrm{Spec}(A)$, this implies $A[S^{-1}] = 0$, thus $R = A$; consequently M satisfies (O').

Let's now give examples of conditions that imply (Q'). The essential idea of these examples is the following: since the category $\mathrm{Mod}(A)$ has injective envelopes, it suffices, to verify (Q'), to take for u an injective envelope; we then try to show that $\mathrm{Hom}_A(M, u)$ is bijective, to be able to apply the following lemma:

Lemma 24 (1.2.3)**.** *Let M be an A-module, N a submodule of M, $u : J \to I$ an injective A-linear application such that the A-module I is injective. If* $\mathrm{Hom}_A(M, u)$ *is bijective,* $\mathrm{Hom}_A(N, u)$ *is bijective.*

Proof. We form the commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 \to \mathrm{Hom}_A(M/N, J) & \to & \mathrm{Hom}_A(M, J) & \to & \mathrm{Hom}_A(N, J) & & \\ \downarrow & & \downarrow & & \downarrow & & \\ 0 \to \mathrm{Hom}_A(M/N, I) & \to & \mathrm{Hom}_A(M, I) & \to & \mathrm{Hom}_A(N, I) & \to & 0 \end{array}$$

in which the columns are injective and the middle column is bijective; we deduce from the lemma of five that all columns are bijective. □

Theorem 14 (1.2.4)**.** *A faithful A-module of finite type satisfies (Q'). (In particular, an injective and finite homomorphism descends flatness.)*

Proof. Let $u : J \to I$ be an injective envelope, L a free A-module of finite type,

F a quotient A-module of L. The diagram

$$\begin{array}{ccc} \mathrm{Hom}_A(F,J) & \to & \mathrm{Hom}_A(F,I) \\ & \uparrow & \uparrow \\ \mathrm{Hom}_A(L,J) & \to & \mathrm{Hom}_A(L,I) \end{array}$$

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is Cartesian; since the second row is an injective envelope, the first row is an essential extension. Thus, if $\mathrm{Hom}_A(F,u)$ is invertible on the left, it is an isomorphism. If F is faithful, there exists a monomorphism of A into a finite power of F, and u is an iso- morphism (1.2.4) thus F satisfies condition (Q').

Corollary 18 (1.2.5)**.** *(Mollier). Let $f : A \to B$ be an injective and finite homomorphism. If B is Noetherian, A is Noetherian.*

Proof. According to Bass [5] it suffices to see that if $(I_r)_{r\in R}$ is a family of injective A-modules, then $\bigoplus_{r\in R} I_r$ is an injective A-module.

By hypothesis, $\mathrm{Hom}_A(B, \bigoplus_{r\in R} I_r) = \bigoplus_{r\in R} \mathrm{Hom}_A(B, I_r)$ is a B-module injective. It suffices therefore to apply 1.2.4 to the A-module B. Let us indicate in passing a result that slightly generalizes EGA IV 11.4.1. □

Lemma 25 (1.2.6)**.** *(Bass, cf. [4]). Let A be a ring, R an ideal of A; the following conditions are equivalent:*

(i) the A-module A/R satisfies (Q');

(i') every A-module M is an essential extension of $\mathrm{Ann}_M(R)$;

(ii) the A-module A/R satisfies (Q);

(iii) R is T-nilpotent, in other words, for any sequence $(a_n)_{n\in\mathbb{N}}$ of elements of R, there exists an integer N such that $\prod_{0\le n\le N} a_n = 0$.

□

Proof. (i) $\Leftrightarrow$ (i') is immediate; (i) $\Rightarrow$ (ii) by 1.2.2; if (ii) is verified, and if $(a_n)_{n\in\mathbb{N}}$ is a sequence of elements of R, we form the A-module P, inductive limit of the sequence of A-linear applications

$A \xrightarrow{a_0} A \xrightarrow{a_1} \cdots \to A \xrightarrow{a_n} A \xrightarrow{a_{n+1}} \cdots;$

thus $P/RP = 0$, therefore $P = 0$, therefore there exists an integer N such that $\prod_{0\le n\le N} a_n = 0$, hence (iii). If (i) is not verified, let M be an A-module non zero such that, for all non-zero element x of M, we have $Rx \ne 0$; we construct by recurrence a sequence $(a_n, x_n)_{n\in\mathbb{N}}$ of elements of $R \times M$ such that $x_{n+1} = a_n x_n$ and $x_n \ne 0$ for all integer n; thus $\prod_{0\le n\le N} a_n \ne 0$ for all integer N: this contradicts (iii). □

Proposition 8 (1.2.7)**.** . *Let A be a ring, R a T-nilpotent ideal of A, M an A-module admitting a faithful submodule of finite type; the A-module $M \oplus (A/R)$ satisfies (Q').*

Proof. Let $u : J \to I$ be an injective envelope. If $\mathrm{Hom}_A(A/R, u)$ is invertible on the left, it is an isomorphism, and the same is true for
$\mathrm{Hom}_A(N, u)$ for any A-module N annihilated by R. We form the commutative diagram

$$\mathrm{Hom}_A(M/RM, J)[r]\,\mathrm{Hom}_A(M/RM, I)$$
$$\mathrm{Hom}_A(M, J)[u][r]\,\mathrm{Hom}_A(M, I)[u]$$

in which the columns are essential (1.2.6 (i')). Since the first row is bijective, the second row is essential; if it is invertible on the left, it is an isomorphism, and u is an isomorphism by 1.2.3, q.e.d.

Corollary 19 (1.2.8)**.** *(cf. EGA IV 11.4.1). Let $f : A \to B$ be an injective homomorphism, R a T-nilpotent ideal of A, P an A-module such that $B \otimes_A P$ is B-flat and P/RP is (A/R)-flat. Then P is A-flat.*

Theorem 15 (1.2.9)**.** . *Let A be a reduced Noetherian ring, M an A-module satisfying (Q'). Then M satisfies (Q).*

Proof. By Noetherian induction, we can assume the statement is proven for all reduced quotient rings of A distinct from A. Let $u : J \to I$ be an injective envelope such that $\mathrm{Hom}_A(M, u)$ is invertible on the left.

If A is not integral, let $\mathfrak{p}$ be a minimal prime ideal of A; since the $(A/\mathfrak{p})$-module $M/\mathfrak{p}M$ satisfies (Q'), it follows from the induction hypothesis that $\mathrm{Hom}_A(A/\mathfrak{p}, J)$ is an $(A/\mathfrak{p})$-module injective. Since the homomorphism $A \to \prod_{\mathfrak{p} \in \mathrm{Min}(A)} A/\mathfrak{p}$ is injective and finite, it follows from 1.2.4 that J is injective.

If A is integral, we will first verify that the torsion submodule of J is injective, or in other words, that for any non-zero element s of A, the (A/sA)-module $\mathrm{Hom}_A(A/sA, J)$ is injective.

Since M satisfies (Q'), there exists a non-zero linear form f on M; let $F = f(M)$, choose a non-zero element t of F and set $R = \mathrm{Ann}_A(F/stF)$; then $R \subset sA$: indeed, if $r \in R$, there exists $a \in F$ such that $rt = sta$, from which $r = sa$ since t is A-regular.

Let R' be the radical of R. According to the induction hypothesis, the (A/R')-module $\mathrm{Hom}_A(A/R', J)$ is injective. Moreover F/RF is an (A/R)-module faithful of finite type; a fortiori M/RM has an (A/R)-submodule faithful of finite type. It follows from 1.2.7 that the (A/R)-module $\mathrm{Hom}_A(A/R, J)$ is injective. A fortiori $\mathrm{Hom}_A(A/sA, J)$ is an (A/sA)-module injective, hence the assertion.

We can now divide J by its torsion submodule, and assume that J is torsion-free; let K be the field of fractions of A; then I is a K-module, thus $\mathrm{Hom}_A(M, I)$ is a K-module and the same is true for its direct factor $\mathrm{Hom}_A(M, J)$.

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Let J' be the submodule $\sum_{u \in \mathrm{Hom}_A(M,J)} \mathrm{Im}(u)$ of the A-module J. Since M satisfies (O'), J' is an essential submodule of J. Since $\mathrm{Hom}_A(M,J)$ is a K-module, J' is divisible, thus is a K-module since J is without torsion. Consequently J' is injective, thus $J' = J$ and J is injective, q.e.d.

Corollary 20 (1.2.10)**.** *Let A be a Noetherian ring, $f : A \to B$ an injective homomorphism satisfying (O); then f descends flatness.*

Proof. By localization, completion and flat descent, we may assume A is a complete local ring; let A' be the quotient of A by its nilradical. By 1.2.2(i), B satisfies (O'). Let P be an A-module such that $B \otimes_A P$ is B-flat. By 1.2.9, $A' \otimes_A P$ is A'-flat. By EGA IV 11.4.1, P is A-flat, q.e.d. □

Remark 6 (1.2.11)**.** Let A be a complete local Noetherian integral ring, E the injective envelope of the residue field of A, M an A-module. Then, for M to satisfy (O), it is necessary and sufficient that $M \otimes_A E \neq 0$. Indeed, if this latter condition is verified, we have $\mathrm{Hom}_A(M \otimes_A E, E) \neq 0$, that is $\mathrm{Hom}_A(M, A) \neq 0$ since $A = \mathrm{End}_A(E)$; thus there exists a non-zero linear form on M, so M has a quotient module of finite type and faithful, which satisfies (O); a fortiori M satisfies (O).

8.1 Case of valuation rings

In this section, we consider a homomorphism of rings $f : A \to B$, where A is a valuation ring. We will give necessary and sufficient conditions for f to be universally injective (resp. satisfies (P), resp. satisfies (O)). The conditions found are not generally equivalent; in particular condition (O) is not always sufficient to descend flatness.

Lemma 26 (1.3.1)**.** *The following conditions are equivalent:*

> *[(i)]the homomorphism f is universally injective; the quotient of B by its torsion A-submodule is faithfully flat over A; there exists a specialization $\mathfrak{q} = \mathfrak{q}'$ in $\mathrm{Spec}(B)$, such that $f^{-1}(\mathfrak{q}) = 0$ and $f^{-1}(\mathfrak{q}')$ is the maximal ideal of A.*

3. *Proof.* Let K be the field of fractions of A. It is clear that (ii) implies (i) and (iii). If (iii) is verified, $B/\mathfrak{q}$ is faithfully flat over A, and a fortiori $\mathrm{Im}(B \to B \otimes_A K)$ is faithfully flat over A, thus (ii) is verified. If (i) is verified, the homomorphism $A \to \mathrm{Im}(B \to B \otimes_A K)$ is universally injective [15] and flat, thus faithfully flat; thus (ii) is verified. □

Lemma 27 (1.3.2)**.** *For any $\mathfrak{p} \in \mathrm{Spec}(A)$, the homomorphism $A \to A_{\mathfrak{p}} \times (A/\mathfrak{p})$ descends flatness.*

Proof. Let M be an A-module such that $M_{\mathfrak{p}}$ is A-flat and $M/\mathfrak{p}M$ is $(A/\mathfrak{p})$-flat; to show that M is A-flat, it suffices to show that the A-module $T = \mathrm{Ker}(M \to M_{\mathfrak{p}})$ is null. Let $x \in T$. If $M/\mathfrak{p}M$ is $(A/\mathfrak{p})$-flat, there exist $s \in \mathfrak{p}$ and $y \in M$ such that $x = sy$. If $M_{\mathfrak{p}}$ is A-flat, there exists $t \in A - \mathfrak{p}$ such that $ty = 0$. Since t divides s, we conclude that $x = 0$, q.e.d. □

Lemma 28 (1.3.3)**.** *For f to satisfy (O), it is necessary and sufficient that $B \neq 0$ and that for any non-closed point $\mathfrak{p}$ of* $\mathrm{Spec}(A)$*, there exists a specialization $\mathfrak{q} \subset \mathfrak{q}'$ in* $\mathrm{Spec}(B)$ *such that $\mathfrak{p} = f^{-1}(\mathfrak{q}) \neq f^{-1}(\mathfrak{q}')$.*

Proof. Necessity: since condition (O) is universal, we can assume $\mathfrak{p} = 0$. Let K be the field of fractions of A and

$$I = \mathrm{Ker}(B \to B \otimes_A K).$$

Since $(K/A) \otimes_A I = 0$, we have $(K/A) \otimes_A (B/I) \neq 0$ because f satisfies (O), thus the flat A-algebra B/I is not a K-algebra: it therefore contains prime ideals $\tilde{\mathfrak{q}} \subset \tilde{\mathfrak{q}}'$ such that $\tilde{\mathfrak{q}}$ is above 0 but not $\tilde{\mathfrak{q}}'$. It suffices then to take for $\mathfrak{q}$ and $\mathfrak{q}'$ the respective reciprocal images of $\tilde{\mathfrak{q}}$ and $\tilde{\mathfrak{q}}'$ in B.

Sufficiency: let M be a non-null A-module such that $B \otimes_A M = 0$; let us find a contradiction with the condition of the statement.

a) Let us show that $\mathrm{Supp}(M)$ admits a maximal point. Otherwise, we have $M_{\mathfrak{p}} = 0$ where $\mathfrak{p}$ denotes the intersection of the elements of $\mathrm{Supp}(M)$; let $\mathfrak{q} \subset \mathfrak{q}'$ be a specialization in $\mathrm{Spec}(B)$ such that $f^{-1}(\mathfrak{q}) = \mathfrak{p} \neq f^{-1}(\mathfrak{q}') = \mathfrak{p}'$. We have $M_{\mathfrak{p}'} \neq 0$ and $(M_{\mathfrak{p}})_{\mathfrak{p}} = 0$, thus $M_{\mathfrak{p}'}/\mathfrak{p}M_{\mathfrak{p}'} \neq 0$ (otherwise $M_{\mathfrak{p}'}$ would be flat (1.3.2) and non-null, so that $M_{\mathfrak{p}}$ would be non-null). But this contradicts the fact that the homomorphism $A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} \to B_{\mathfrak{q}}/\mathfrak{q}B_{\mathfrak{q}}$ is universally injective (1.3.1).

b) Let $\mathfrak{p}$ be the maximal point of $\mathrm{Supp}(M)$; let us show that $\mathfrak{p} = \bigcup_{\mathfrak{p}' \in D(\mathfrak{p})} \mathfrak{p}'$.

Otherwise, $D(\mathfrak{p})$ has a closed point $\mathfrak{p}'$; according to the condition of the statement, there exists a specialization $\mathfrak{q}' \subset \mathfrak{q}$ in $\mathrm{Spec}(B)$, above the specialization $\mathfrak{p}' \subset \mathfrak{p}$; since we have $M_{\mathfrak{p}'} = 0 \neq M_{\mathfrak{p}}$, the reasoning of a) provides a new contradiction.

c) Let us show that $\mathfrak{p}M_{\mathfrak{p}} = 0$. Let $x \in M_{\mathfrak{p}}$ and $s \in \mathfrak{p}$; let $\mathfrak{p}'$ be the maximal point of $V(sA)$; by b), we have $\mathfrak{p}' \neq \mathfrak{p}$, thus $M_{\mathfrak{p}'} = 0$; therefore there exists $t \in A - \mathfrak{p}'$ such that $tx = 0$; since t divides s, we have $sx = 0$, hence the assertion.

That said, we see that $M_{\mathfrak{p}}$ is a $k(\mathfrak{p})$-vector space that is non-null; since there exists a point of $\mathrm{Spec}(B)$ above $\mathfrak{p}$, we cannot have $M_{\mathfrak{p}} \otimes_A B = 0$. □

Lemma 29 (1.3.4)**.** *For f to satisfy (P) it is necessary and sufficient that it satisfies (O) and that, for any point $\mathfrak{p}'$ of* $\mathrm{Spec}(A)$ *that is not maximal, there exists a specialization $\mathfrak{q} \subset \mathfrak{q}'$ in* $\mathrm{Spec}(B)$ *such that $f^{-1}(\mathfrak{q}') = \mathfrak{p}' \neq f^{-1}(\mathfrak{q})$.*

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Proof. Necessity: suppose that f satisfies (P); it is clear that f satisfies (O); let us show that it satisfies the last condition. By localization, we can assume that

$\mathfrak{p}'$ is the closed point of $\operatorname{Spec}(A)$. If $D(\mathfrak{p}')$ admits a closed point, the existence of the sought specialization follows from condition (O) by 1.3.3; we can thus assume that $\mathfrak{p}' = \bigcup_{\mathfrak{p}\in D(\mathfrak{p}')} \mathfrak{p}$. If f satisfies (P), $B/\mathfrak{p}'B$ is not B-flat, so there exists $\mathfrak{q}' \in V(\mathfrak{p}'B)$ such that $\mathfrak{p}'B_{\mathfrak{q}'} \neq 0$. Therefore, there exists $\mathfrak{p} \in D(\mathfrak{p}')$ such that $\mathfrak{p}B_{\mathfrak{q}'} \neq 0$. Let $x \in \mathfrak{p}' - \mathfrak{p}$. For any integer n, we have $s^n A \subset \mathfrak{p}$, thus $s^n B_{\mathfrak{q}'} \neq 0$. The ideal $sB_{\mathfrak{q}'}$ of $B_{\mathfrak{q}'}$ is not nilpotent; since it is homogeneous, it is not a nilideal; there exists therefore a prime ideal $\mathfrak{q}$ of B, contained in $\mathfrak{q}'$, such that $sB_{\mathfrak{q}} \subset \mathfrak{q}B_{\mathfrak{q}}$; a fortiori $\mathfrak{q}$ is not above $\mathfrak{p}'$.

Sufficiency: it is a matter of proving that under the conditions of the statement, any A-module of torsion M such that $B\otimes_A M$ is B-flat is null. If not, since f satisfies (O), we see as in the proof of 1.3.3 a) that $\operatorname{Supp}(M)$ has a maximal point $\mathfrak{p}$. Since M is of torsion, we have $\mathfrak{p} \neq 0$; we can thus find a specialization $\mathfrak{q}' \subset \mathfrak{q}$ in $\operatorname{Spec}(B)$ such that $\mathfrak{p}' = f^{-1}(\mathfrak{q}') \neq f^{-1}(\mathfrak{q}) = \mathfrak{p}$. Since $M_{\mathfrak{q}'} = 0 \neq M_{\mathfrak{q}}$, we have $M_{\mathfrak{p}'}/\mathfrak{p}'M_{\mathfrak{p}'} \neq 0$ (1.3.2). We can therefore change rings $A \to A_{\mathfrak{p}'}/A_{\mathfrak{p}'}$, and we can thus reduce to the case where f is universally injective (1.3.1); but then f descends flatness, thus M is flat, thus $M = 0$. □

Proposition 9 (1.3.5)**.** *For there to exist a homomorphism $f : A \to B$ that satisfies (O) and does not satisfy (P), it is necessary and sufficient that* $\operatorname{Spec}(A)$ *is not well-ordered by the order opposite to inclusion (i.e., that there exists a prime ideal $\mathfrak{p}$ of A which is the union of prime ideals strictly smaller than $\mathfrak{p}$).*

Proof. The necessity is obtained by comparing the statements of 1.3.3 and 1.3.4. Conversely, suppose given a strictly increasing sequence $(\mathfrak{p}_n)_{n\in\mathbb{N}}$ of prime ideals of A, such that $\mathfrak{p}_0 = 0$; let $\mathfrak{p}$ be the union of the $\mathfrak{p}_n$, which is a prime ideal of A. The homomorphism product

$$f : A \to (A/\mathfrak{p}) \times \prod_{n\in\mathbb{N}} (A_{\mathfrak{p}_{n+1}}/\mathfrak{p}_n A_{\mathfrak{p}_{n+1}}) = B$$

satisfies (O) (1.3.3) and does not satisfy (P): indeed $B/\mathfrak{p}B$ is B-flat, because it is the inductive limit of the $B/\mathfrak{p}_n B$ which are direct factors of B. □

1.4. Problems and complements

(1.4.1) According to Olivier [15], certain fundamental properties of faithfully flat homomorphisms are also satisfied by universally injective homomorphisms (thus condition (P) and the "effective descent" of modules). It seems natural to seek to clarify this analogy; for example:

a) Ferrand raised the following question: let $f : A \to B$ be a universally injective homomorphism; is $\operatorname{Spec}(f)$ *covering for the fpqc topology, or in other words, does there exist a B-algebra C which is faithfully flat over A?*

b) Let $f : A \to B$ be a finite type homomorphism, universally injective, and suppose A is a complete local Noetherian integral domain; by analogy with EGA IV 14.5, one can ask if there exists a prime ideal $\mathfrak{p}$ of B such that the homomorphism $A \to B/\mathfrak{p}$ is injective and finite.

We will see that the answer to the questions posed in a) and b) is generally negative.

Let us take for A a complete local Noetherian normal ring of dimension 2. Let $S = \mathrm{Spec}(A)$, s the closed point of S, $U = S - \{s\}$, L an invertible O_U-module. For all $n \in \mathbb{Z}$, the A-module $A_n = \Gamma(U, L^{\otimes n})$ is of finite type (EGA IV 5.11.1). We equip $B = \bigoplus_{n\in\mathbb{Z}} A_n$ with its natural structure of A-algebra. Let $X = \mathrm{Spec}(B)$, $f : X \to S$ the structural morphism, $V = f^{-1}(U)$; then V is identified with the base torus U, under the multiplicative group G_m, defined by the invertible O_U-module L, and B is identified with $\Gamma(V, O_V)$.

Proposition 10 (1.4.1.1)**.** *(i) B is an A-algebra of finite type if and only if L is of finite order.*

(ii) Let C be a sub-A-algebra of B containing A_1 and A_{-1}; then

a) the homomorphism $A \to C$ is universally injective;

b) C admits a finite and faithful quotient ring over A if and only if L is of finite order;

c) the morphism $\mathrm{Spec}(C) \to \mathrm{Spec}(A)$ *is covering for the fpqc topology if and only if L is trivial.*

Proof. (i) If L is of finite order r, the A-algebra B is generated by $\bigoplus_{-r\leq n\leq r} A_n$. Conversely, suppose that the A-algebra B is of finite type. The homomorphism $A \to B$ has a left A-linear inverse; it is therefore universally injective, and f is surjective. Let us show that f is equidimensional. Indeed, the generic fiber of f is of dimension 1; on the other hand, x is a point of $X \otimes k(s)$; since the homomorphism $B \to \Gamma(V, O_V)$ is bijective, we have $\dim(O_{X,x}) \geq 2$ (EGA IV 5.10.5). It follows from EGA IV 5.6.5 that $\dim(X \otimes k(s)) \leq 1$. The assertion now follows from EGA IV 13.1.1. That said, since A is complete, f has a finite quasi-section (EGA IV 14.4.4 and 14.5.9): in other words, there exists an ideal I of B such that $\overline{B} = B/I$ is a finite and faithful A-module, which we can assume to be torsion-free. Let $\overline{S} = \mathrm{Spec}(\overline{B})$, $\overline{U}$ the reciprocal image of U in $\overline{S}$. Since U is regular of dimension 1, and connected, the $O_{\overline{U}}$-module defined by O_U is locally free of constant rank r. Since the reciprocal image

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of L on $\overline{U}$ is trivial, the norm operation shows that L is of finite dividing order r.

(ii) Since $A = A_0$ is a direct factor of the A-module B and a fortiori of the A-module C, assertion a) is clear. Since C contains A_1 and A_{-1}, the structural morphism $\mathrm{Spec}(B) \to \mathrm{Spec}(C)$ is an isomorphism above U, and the argument used at the end of the proof of (i) proves also c). Let now S' be an S-scheme faithfully flat quasi-compact such that $f \times_S S'$ has a section. Let U' be the reciprocal image of U in S', $i : U \to S$ and $i' : U' \to S'$ the canonical open immersions; since $f \times_S S'$ has a section, the reciprocal image L' of L'' on U' is trivial; since the canonical homomorphism $O_S \to i_*(O_U)$ is an isomorphism, we see by flatitude that the same holds for the canonical homomorphism $O_{S'} \to$

$i'_*(O_{U'})$. As a result $i'_*(L')$ is an $O_{S'}$-module inversible trivial. Since S is local, we see by faithful flatitude that $i_*(L)$ is an O_S-module inversible trivial; a fortiori L is trivial, *cqfd.*

Prop. 1.4.1.1 provides the sought counter-example: we can choose A and L such that L is of infinite order; we then take for C the sub-A-algebra of B generated by A_1 and A_{-1}; the homomorphism $A \to C$ is universally injective and of finite type, but the morphism $\mathrm{Spec}(C) \to \mathrm{Spec}(A)$ is not covering for the fpqc topology and does not admit a finite quasi-section.

In this counter-example, the generic fiber is of dimension 1; the remarks that follow show that such phenomena do not occur when the generic fiber is of dimension 0.

Lemma 30 (1.4.1.2)**.** *Consider a Cartesian square of rings*

A [r, "f"] [d, "g"'] B [d, "h"]
A' [r, "f'"'] B'

If A is Noetherian, and if f verifies (O), then $g(A) \to h(B)$ verifies (O).

Proof. We can assume A local complete and g injective. Up to replacing A by $A/\mathfrak{p}$, where $\mathfrak{p}$ runs through $\mathrm{Min}(A)$, we can assume A integral. By 1.2.11, there exists a non-zero A-linear form u on the A-module B. Then $u' = 1_A \otimes_A u$ induces a non-zero linear form on the $g(A)$-module $h(B)$. We conclude by 1.2.11. □

Lemma 31 (1.4.1.3)**.** *Let A be a complete local Noetherian integral ring, B an A-algebra verifying (O), $(R_i)_{i \in I}$ a finite family of ideals of B whose product is a nilpotent ideal. There exists $i \in I$ such that the A-algebra $B/R_i = B_i$ verifies (O).*

Proof. Since the B-algebra $\prod_{i \in I} B_i$ verifies (O), we can replace B by $\prod_{i \in I} B_i$. Let E be the injective envelope of the residue field of A. Since $B \otimes_A E \neq 0$, there exists $i \in I$ such that $B_i \otimes_A E \neq 0$. We conclude by 1.2.11. □

Proposition 11 (1.4.1.4)**.** *Let A be a complete local Noetherian integral ring with field of fractions K, B an A-algebra verifying (O) such that $B \otimes_A K$ is finite over K; there exists a prime ideal $\mathfrak{p}$ of B such that $B/\mathfrak{p}$ is finite and faithful over A (in particular, if B is integral, it is finite over A).*

Proof. By 1.4.1.2 we can assume that the A-module B is torsion-free; let us reason by induction on its rank n. For $n = 1$ the assertion follows from the existence of a non-zero linear form on the A-module B. For $n > 1$ we can assume that B is not integral (otherwise we would make the change of rings $A \to A'$ where A' is the integral closure of A in the field of fractions of B). Since $\mathrm{Min}(B) = \mathrm{Min}(B \otimes_A K)$ is finite, there exists $\mathfrak{p} \in \mathrm{Min}(B)$ such that the A-algebra $B/\mathfrak{p}$ verifies (O) (1.4.1.3); it is then finite by the induction hypothesis.
□

Example 1. Let A be a complete local Noetherian normal ring, I an ideal of A, $S = \mathrm{Spec}(A)$, $f : X \to S$ the blowup of I in S; for f to descend the platitude, it is necessary and sufficient that I be principal (i.e., that f be an isomorphism).

Indeed, if f descends the platitude, there exists by 1.4.1.3 an open affine subset of X, with ring B, such that the A-algebra B verifies (O); since A and B have the same field of fractions, and since A is normal, we have $B = A$ (1.4.1.4) thus I is principal (because IB is an invertible B-module).

(1.4.2). In relation with condition (P), Venken has proposed the following problem: let $f : A \to B$ be a ring homomorphism, and let P be an A-module such that $B \otimes_A P$ is B-flat; under what conditions does there exist an A-linear application $u : P \to Q$ such that Q is A-flat and that $1_B \otimes_A u$ is an isomorphism? Let us mention in this direction the following result:

Lemma 32 (1.4.2.1)**.** *Let A be a ring, S a multiplicative subset of A consisting of A-regular elements, P an A-module such that $P[S^{-1}]$ is A-flat and that P/sP is (A/sA)-flat for all $s \in S$, T the S-torsion submodule of P. Then T is S-divisible and P/T is A-flat.*

Proof. Let $j : A \to A[S^{-1}]$ be the canonical homomorphism; for any pair $(s, t) \in S \times S$ such that s divides t, let $u_{t,s} : A/sA \to A/tA$ be the A-linear application deduced by passage to the quotient of the homothety of A with ratio t/s. Since s is A-regular, $u_{t,s}$ is injective. The system

inductive filtering system $(A/sA, u_{ts})$ has a limit isomorphic to the A-module of S-torsion $\mathrm{Coker}(j)$.

To see that T is S-divisible, we form the commutative diagram

T/sT r$1_T \otimes u_{ts} dT/tTd$
P/sP r$1_P \otimes u_{ts} P/tP$

where the columns are injective (since T is the submodule of S-torsion of P). Since P/tP is (A/tA)-flat and since u_{ts} is injective, the second line is injective, thus the first one is also; on the other hand, the system inductive filtering $(T/sT, 1_T \otimes u_{ts})$ has a limit isomorphic to $\mathrm{Coker}(1_T \otimes j)$, which is null, since T is of S-torsion. Thus $T = sT$ for all $s \in S$.

Let us show that P/T is A-flat. Let $s \in S$; since s is A-regular and since P/sP is (A/sA)-flat, we have $\mathrm{Tor}_1^A(P/sP) \leq 1$; by varying s and passing to the inductive limit, we obtain $\mathrm{Tor}\cdot\dim_A(\mathrm{Coker}(1_P \otimes j)) \leq 1$; but we have an exact sequence

$0 \to P/T \to P[S^{-1}] \to \mathrm{Coker}(1_P \otimes j) \to 0$

and by hypothesis $P[S^{-1}]$ is A-flat; therefore P/T is A-flat.

Questions (1.4.3). 1) Let $f : A \to B$ be a ring homomorphism, such that the A-module B admits a faithful quotient of finite type. If A is Noetherian, does f descend flatness (1.2.10); is it the same in the general case?

Note that, according to 1.2.1 and 1.2.4, it is indeed the case when the A-module B admits a faithful direct factor of finite type.

2) Let $f : A \to B$ be an injective and integral ring homomorphism. When A is Noetherian, does f descend flatness? (When no finiteness hypothesis is made,

a counter-example of Lazard shows that f does not generally descend flatness, cf. [7].)

To study this problem, we can reduce by devissage to the case where A is a complete local normal ring and where B is the normalization of A in an extension algebraic of its field of fractions. If then A is a 0-algebra, the trace application shows that f is universally injective (since it has a left A-linear inverse). On the other hand, we can always find a local complete regular subring R of A such that A is finite over R; we show then, by introducing the A-module $\mathrm{Hom}_R(A, R)$ and by applying 1.2.10 and 1.2.11, that f descends flatness if and only if the homo- morphism composed $R \to B$ descends flatness; we can thus reduce to the case where A is regular. If $\dim(A) \leq 2$, then f is faithfully flat. We do not know what happens if $\dim(A) > 2$.

9 Modules of Mittag-Leffler

Let (E_i, u_{ij}) be a projective system of sets indexed by a filtered ordered set I. Following EGA $\mathrm{O_{III}}$ 13.1.2, we say that (E_i, u_{ij}) is a Mittag-Leffler projective system if it satisfies the condition (ML): for every $i \in I$, the decreasing filtered family $(u_{ij}(E_j))_{j \geq i}$ is stationary. Projective systems of Mittag-Leffler have useful cohomological properties (EGA $\mathrm{O_{III}}$ 13.2.2).

Let now C be a small category, (E_i, u_{ij}) a projective system indexed by a filtered ordered set I; we say that (E_i, u_{ij}) is a Mittag-Leffler projective system if for every functor $F : C \to (\mathrm{Ens})$, the projective system of sets $F(E_i)$ is Mittag-Leffler. One easily verifies that this condition depends only on the pro-object of C defined by (E_i, u_{ij}); we thus define the pro-objects of Mittag-Leffler; by duality we similarly define the ind-objects of Mittag-Leffler.

We are interested here in the case where C is the category of finitely presented modules over a ring A (not necessarily commutative); the category of ind-objects of C is then equivalent to the category of A-modules. We will express the Mittag-Leffler condition in this context and translate the corresponding cohomological results.

The essential ideas of this paragraph are implicit in Chapter I of D. Lazard's thesis [13].

9.1 Stabilizers

The notations are those of 1.1. Proposition 1.1.1 admits the following "relative" variant:

Proposition 12 (2.1.1)**.** *Let $u : M \to N$ and $v : M \to M'$ be two A-linear applications. The following conditions are equivalent:*

> *[(i)]for every A°-module Q, we have $\mathrm{Ker}(1_Q \otimes_A u) \subset \mathrm{Ker}(1_Q \otimes_A v)$; we have $\mathrm{Ker}(1_{T_{M'}} \otimes_A u) \subset \mathrm{Ker}(1_{T_{M'}} \otimes_A v)$; Tu factorizes Tv on the left;*

if $\mathrm{Coker}(u)$ is of finite presentation, these conditions are equivalent to

> *[(i)]u factorizes v on the right.*

1. *Proof.* Let us form the Cartesian diagram of A-modules

M [r, "u"] [d, "v"'] N [d, "w"]
M' [r, "u'"'] N'

and note that, for every A°-module Q, we have the exact sequence

$$0 \to \mathrm{Ker}(1_Q \otimes u) \cap \mathrm{Ker}(1_Q \otimes v) \to \mathrm{Ker}(1_Q \otimes u) \to \mathrm{Ker}(1_Q \otimes u') \to 0;$$

□

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in particular, we have $\mathrm{Ker}(1_Q \otimes u') = 0$ *if and only if* $\mathrm{Ker}(1_Q \otimes u) \subset \mathrm{Ker}(1_Q \otimes v)$. *Applying 1.1.1, we see that conditions (i) to (iii) each express that u' is A-universally injective. If* $\mathrm{Coker}(u)$ *is of finite presentation, the same is true for* $\mathrm{Coker}(u')$, *thus u' is A-universally injective if and only if u' is left invertible, i.e., if and only if u factorizes v on the right.* □

We will say that v dominates *u if conditions (i) to (iii) of 2.1.1 are satisfied.*

Let us recall that every A-module is a filtered inductive limit of A-modules of finite presentation, and that, for every A-module of finite presentation F, the functor $\mathrm{Hom}_A(F, -)$ *commutes with filtered inductive limits (these two assertions express the equivalence between the category* $\mathrm{Mod}(A)$ *and the category of "ind-A-modules of finite presentation").*

Lemma 33 (2.1.2)**.** *Let $(F_i, u_{ji})_{i\in I}$ be a filtered inductive system of A-modules of finite presentation, with limit (M, u_i); let i and j be two indices such that $i \leq j$; the following conditions are equivalent:*

[(i)]u_{ji} dominates u_i; for all $k \geq i$, u_{ki} factorizes u_{ji} on the right.

2. *Proof.* Since u_i dominates u_{ki}, u_{ji} dominates u_{ki} if (i) is satisfied, thus (ii) is satisfied (2.1.1); the converse is clear. □

Definition 11 (2.1.3)**.** Let $u : F \to M$ and $v : F \to G$ be two A-linear maps such that F and G are of finite presentation. We say that v stabilizes u (or that v is a stabilizer of u) if u and v dominate each other mutually.

We call *Mittag-Leffler A-module* an A-module M that satisfies

(ML) every A-linear map from an A-module of finite presentation to M admits a stabilizer.

Let us justify the terminology.

Proposition 13 (2.1.4)**.** *Let (F_i, u_{ji}) be a filtered inductive system of A-modules of finite presentation, with limit (M, u_i); the following conditions are equivalent:*

[(i)]M satisfies (ML); for all i there exists $j \geq i$ such that u_{ji} dominates u_i; for every A-module N, the projective system

$$(\mathrm{Hom}_A(F_i, N), \mathrm{Hom}_A(u_{ji}, N))$$

satisfies condition (ML) of EGA O_{III} 13.1.2; same statement as (iii) but restricted to the module $N = \prod_{i \in I} F_i$.

2. *Proof.* If (i) is satisfied, for all i there exists a stabilizer $v : F_i \to G$ of u_i; since $\mathrm{Hom}_A(G, \cdot)$ commutes with filtered inductive limits, and since v factorizes u_i on the right, there exists $j \geq i$ such that v factorizes u_{ji} on the right, and u_{ji} dominates u_i, thus (ii) is satisfied. If (ii) is satisfied, and if $u : F \to M$ is an A-linear map such that F is of finite presentation, there exist i and $v \in \mathrm{Hom}_A(F, F_i)$ such that $u = u_i v$; if u_{ji} dominates u_i, it is clear that $u_{ji}v$ stabilizes u. Thus (i) $\Leftrightarrow$ (ii).

If (ii) is satisfied, let i be an index; there exists $j \geq i$ such that u_{ji} dominates u_i; by 2.1.2, u_{ki} factorizes u_{ji} on the right for all $k \geq i$, thus, for all A-module N, $\mathrm{Im}(\mathrm{Hom}_A(u_{ji}, N)) = \mathrm{Im}(\mathrm{Hom}_A(u_{ki}, N))$; thus (iii) is satisfied. It is clear that (iii) $\Rightarrow$ (iv). If (iv) is satisfied, for all i there exists $j \geq i$ such that for all $k \geq j$ we have $\mathrm{Im}(\mathrm{Hom}_A(u_{ji}, F_j)) = \mathrm{Im}(\mathrm{Hom}_A(u_{ki}, F_j))$; thus u_{ki} factorizes u_{ji} on the right, thus (ii) is satisfied. $\square$

Proposition 14 (2.1.5)**.** *For an A-module M to satisfy (ML) it is necessary and sufficient that for every family $(Q_r)_{r \in R}$ of A^0-modules, the canonical Z-linear application*

$$\left(\prod_{r \in R} Q_r\right) \otimes_A M \to \prod_{r \in R} (Q_r \otimes_A M) \text{ is injective.}$$

Proof. We know that if F is an A-module of finite presentation, the functor $\cdot \otimes_A F$ commutes with direct products. Let now (F_i, u_{ji}) be a filtered inductive system of A-modules of finite presentation with limit (M, u_i).

Suppose that M satisfies (ML); let $(Q_r)_{r \in R}$ be a family of A^0-modules. Let x be an element of $\mathrm{Ker}\left(\left(\prod_{r \in R} Q_r\right) \otimes_A M \to \prod_{r \in R}(Q_r \otimes_A M)\right)$. For i large enough, x comes from an element x_i of $\left(\prod_{r \in R} Q_r\right) \otimes_A F_i$. Let $j \geq i$ such that u_{ji} stabilizes u_i; we form the commutative diagram

$$\begin{array}{ccc}
\left(\prod_{r \in R} Q_r\right) \otimes_A F_i & \longrightarrow & \prod_{r \in R}(Q_r \otimes_A F_i) \\
\downarrow & & \downarrow \\
\left(\prod_{r \in R} Q_r\right) \otimes_A F_j & \longrightarrow & \prod_{r \in R}(Q_r \otimes_A F_j)
\end{array}$$

where the horizontal lines are bijective. The image of x_i in $\prod_{r \in R}(Q_r \otimes_A F_j)$ is null since $\mathrm{Ker}(1_{Q_r} \otimes_A u_i) = \mathrm{Ker}(1_{Q_r} \otimes_A u_{ji})$ for all $r \in R$; thus $x = 0$ and M satisfies the condition of the statement.

Suppose that M satisfies the condition of the statement. Let i be an index, Q an A^0-module, $(x_r)_{r \in R}$ a family of elements of $\mathrm{Ker}(1_Q \otimes_A u_i)$. We form the commutative diagram

$$\begin{array}{ccc}
Q^R \otimes_A F_i & \longrightarrow & (Q \otimes_A F_i)^R \\
\downarrow & & \downarrow \\
Q^R \otimes_A F & \longrightarrow & (Q \otimes_A F)^R
\end{array}$$

in which the first line is bijective and the second line injective; consequently (x_r) comes from an element of $Q^R \otimes_A \mathrm{Ker}(u_i)$, thus also from an element of

$Q^R \otimes_A \mathrm{Ker}(u_{ji})$ for j large enough; thus $x_r \in \mathrm{Ker}(1_Q \otimes u_{ji})$ for all r. In particular, let $Q = T(M \oplus F_i)$ and let $(x_r)_{r \in R}$ be a family of generators of the abelian group $\mathrm{Ker}(1_Q \otimes_A u_i)$; we see then that u_{ji} stabilizes u_i, and M satisfies (ML).

Corollary 21 (2.1.6)**.** *Let $0 \to M \to N \to P \to 0$ be an A-universally exact sequence. If N satisfies* (ML)*, M satisfies* (ML)*. If M and P satisfy* (ML)*, N satisfies* (ML)*.*

This is clear by the criterion of 2.1.5.

Corollary 22 (2.1.7)**.** *Let P be a flat A-module satisfying* (ML)*. For any filtered projective system (Q_r, u_{rs}) of A^0-modules, the canonical application $(\lim\limits_{\leftarrow} Q_s) \otimes_A P \to \lim\limits_{\leftarrow}(Q_r \otimes_A P)$ is injective; it is bijective if the u_{rs} are injective.*

Proof. The first assertion follows from 2.1.5 and the flatness of P. The second assertion follows from the snake lemma applied to the commutative diagram with exact rows

$$\begin{array}{ccccccc}
0 \to (\lim\limits_{\leftarrow} Q_s) \otimes_A P & \longrightarrow & Q_r \otimes_A P & \longrightarrow & (\lim\limits_{\leftarrow}(Q_r/Q_s)) \otimes_A P \to 0 \\
\downarrow & & \downarrow & & \downarrow \\
0 \to \lim\limits_{\leftarrow}(Q_s \otimes_A P) & \longrightarrow & Q_r \otimes_A P & \longrightarrow & \lim\limits_{\leftarrow}((Q_r/Q_s) \otimes_A P)
\end{array}$$

where r is a fixed index. □

In particular, let P be a flat A-module of Mittag-Leffler type; then, for any A^0-module Q and any $x \in Q \otimes_A P$, the set of submodules Q' of Q such that $x \in Q' \otimes_A P$ has a smallest element, which we call the "module of coefficients of x in Q". In the inverse sense:

Proposition 15 (2.1.8)**.** *Let P be a flat A-module, such that, for any A^0-module L of finite type and any $x \in L \otimes_A P$, the set of submodules Q of L, such that $x \in Q \otimes_A P$, has a smallest element. Then P satisfies* (ML)*.*

Proof. For any A-module M let M^* be the A^0-module $\mathrm{Hom}_A(M, A)$. If L and M are two A-modules, L being free of finite type, we have a functorial isomorphism $L^* \otimes_A M \to \mathrm{Hom}_A(L, M)$; let us agree to identify the source and the target of this isomorphism. Then, for any $x \in L^* \otimes_A M$, we easily see that the module of coefficients of x in M is identified with the image of x, considered as an element of $\mathrm{Hom}_A(L, M)$.

That said, let us write P as a filtered inductive limit of A-modules free of finite type (L_i, u_{ji}) ([13] 11.2); for any i, let $u_i : L_i \to P$ be the limit of the u_{ji} and let us look for a stabilizer of u_i. Let Q_i be the module of

coefficients of u_i in L_i^*. Then $u_i \in Q_i \otimes_A P$, thus $u_j \in Q_i \otimes_A L_j$ for j sufficiently large. Therefore $Q_i = \mathrm{Im}(\mathrm{Hom}_A(u_{ji}, A))$ for j sufficiently large, and the projective system of L_i^* satisfies the Mittag-Leffler condition. By the functorial isomorphism above, we see that for any A-module M, the projective system $(\mathrm{Hom}_A(L_j, M))$ satisfies the Mittag- Leffler condition; according to 2.1.4, P satisfies (ML).

Let $u : F \to M$ be an A-linear application such that F has finite presentation. Suppose that $u(F)$ is contained in a pure submodule of finite presentation G of M; then it is clear that $F \to G$ stabilizes u. One may wonder if the converse is true; in this direction, we have:

Proposition (2.1.9). *Suppose that u admits a stabilizer.*

(i) If A is local and if M is flat, there exists a pure free submodule of finite type L of M containing $u(F)$.

(ii) If A is commutative local Henselian, there exists a pure submodule of finite presentation of M containing $u(F)$.

Proof. (i) We write M as a filtered inductive limit of A-modules free of finite type (L_i, u_{ji}); for all i we denote u_i the limit of u_{ji}; for i sufficiently large, u admits a stabilizer $v : F \to L_i$ such that $u = u_i v$. Let $R = \mathrm{Im}(\mathrm{Hom}_A(v, A)) \subset F^*$; then $R = \mathrm{Im}(\mathrm{Hom}_A(u_i, v, A))$ for all $j \geq i$. Since A is local, and since R is an A^0-module of finite type, we can find a minimal epimorphism $p : L' \to R$, where L' is a free A^0-module of finite type. We choose $q \in \mathrm{Hom}_A(L_i^*, L')$ such that $\mathrm{Hom}_A(v, A) = pq$. Then q is surjective, and the same is true for $q \circ (\mathrm{Hom}_A(u_{ji}, A))$ for all $j \geq i$. By transposition, p and q define A-linear applications $s : F \to L'^* = L$ and $t : L \to L_i$. We have $v = ts$, thus $u = u_i ts$, thus $u(F) \subset u_i(t(L))$. On the other hand $u_{ji}t$ is invertible on the left for all $j \geq i$, thus $u_i t$ is A-universally injective.

(ii) Let $v : F \to G$ be a stabilizer of u. Since A is local, we can, after replacing G by one of its direct factors containing $v(M)$, assume that any non-zero direct factor of G intersects $v(F)$ in a non-zero submodule. Let then $w : G \to M$ be an A-linear application such that $u = wv$. We will prove that w is A-universally injective. Let us write M as an inductive limit of A-modules of finite presentation F_i; after replacing M by the various F_i for i sufficiently large, we can assume M of finite presentation; there exists then $w' : M \to G$ such that $v = w'u$. It suffices to see that $w'w$ is an automorphism of G. Since $(1 - w'w)v = 0$, it suffices to verify the following lemma:

Lemma (2.1.10). *Let A be a commutative local Henselian ring, G an A-module of finite type, f an endomorphism of G, such that, for any non-zero direct factor G' of G, we have $G' \cap \mathrm{Ker}(f) \neq \emptyset$. Then $1 - f$ is bijective.*

Criteria for flatness

Demonstration. Since A is Henselian and G of finite type, the ring $A[f]$ is a direct product of local rings; by replacing A with the various local components of $A[f]$, we can assume that f is a scalar. But then f is not invertible, since the homothety of G that it defines is not injective; therefore $1 - f$ is invertible.

9.2 Structure of Mittag-Leffler modules

Let M be an A-module that is relatively projective (1.1). It is clear that M satisfies (ML) (1.1.2 and 2.1.6). The converse is false; however, we have the following result:

Theorem 16 (2.2.1)**.** *Let M be an A-module of Mittag-Leffler. Any sequence $(x_n)_{n\in\mathbb{N}}$ of elements of M is contained in a pure submodule, of countable type and relatively projective of M.*

Demonstration. We follow the demonstration of th. 3.1 and 3.2 of [13] I. We construct by recurrence on the integer n a sequence (F_n, u'_n, v_n) that satisfies the following conditions:

a) F_n is an A-module of finite presentation, $u'_n \in \mathrm{Hom}_A(F_n \oplus A, F_{n+1})$, $v_n \in \mathrm{Hom}_A(F_n, M)$;

b) $F_0 = 0$;

c) u'_n stabilizes the element v'_n of $\mathrm{Hom}_A(F_n \oplus A, M)$ of matrix (p_n, x_n) and we have $v_{n+1}u'_n = v'_n$.

Since M satisfies (ML), the construction is possible. Let u_n be the restriction of u'_n to F_n, F the inductive limit of (F_n, u_n), $v : F \to M$ the limit of the v_n. We will show that $v(F)$ satisfies the conditions of the statement; this results from the following remarks:

(i) Since $x_n \in \mathrm{Im}(v'_n) \subset \mathrm{Im}(v_{n+1}) \subset \mathrm{Im}(v)$, $\mathrm{Im}(v)$ contains the sequence (x_n).

(ii) v is universally injective: indeed u_n stabilizes v_n, thus we have

$$\mathrm{Ker}(1_{TF} \otimes u_n) = \mathrm{Ker}(1_{TF} \otimes v_n)$$

and

$$\mathrm{Ker}(1_{TF} \otimes v) = \varinjlim \mathrm{Ker}(1_{TF} \otimes v_n) = \varinjlim \mathrm{Ker}(1_{TF} \otimes u_n) = 0.$$

(iii) F is of countable type and relatively projective: the first assertion is clear since $F = \lim_{\rightarrow} F_n$; let us prove the second assertion. According to (ii) and 2.1.6, F satisfies (ML), thus, for any A-module N', the projective system $(\mathrm{Hom}_A(F_n, N'))$ is of Mittag-Leffler; its limit is $\mathrm{Hom}_A(F, N')$. Let

$$0 \to N' \to N \to N'' \to 0$$

be an A-universally exact sequence; for any integer n, the sequence

$$0 \to \mathrm{Hom}_A(F_n, N') \to \mathrm{Hom}_A(F_n, N) \to \mathrm{Hom}_A(F_n, N'') \to 0$$

is exact (1.1.1). By passing to the limit, we deduce from EGA $\mathrm{O_{III}}$ 13.2.2 that the sequence

$$0 \to \mathrm{Hom}_A(F, N') \to \mathrm{Hom}_A(F, N) \to \mathrm{Hom}_A(F, N'') \to 0$$

is exact, which proves that F is relatively projective. Q.E.D.

Corollary 23 (2.2.2)**.** *A module of countable type is relatively projective if and only if it satisfies* (ML)*; in particular, a flat module of countable type is projective if and only if it satisfies* (ML)*.*

The property stated in 2.2.1 characterizes Mittag-Leffler modules, according to the following lemma:

Lemma 34 (2.2.3)**.** *Let (M_i, u_{ji}) be a filtered inductive system of A-modules of Mittag-Leffler, with limit (M, u_i). If, for all i, there exists $j \geq i$ such that u_{ji} dominates u_i, then M satisfies* (ML)*; in particular, if the u_{ji} are A-universally injective, then M satisfies* (ML)*.*

Proof. Let $u : F \to M$ be an A-linear map such that F is of finite presentation. There exist i and $v \in \mathrm{Hom}_A(F, M_i)$ such that $u = u_i v$. Let $j \geq i$ such that u_{ji} dominates u_i. Then, for any A^0-module Q, we have $\mathrm{Ker}(1_Q \otimes u) = \mathrm{Ker}(1_Q \otimes u_{ji} v)$, thus any stabilizer of $u_{ji}v$ stabilizes u. □

9.3 Strictly Mittag-Leffler Modules

Let (F, u_{ji}) be a filtered inductive system of A-modules of finite presentation, with limit (M, u_i). If M is Mittag-Leffler, it follows from 2.1.4 that for all i, the family $\mathrm{Im}(\mathrm{Hom}_A(u_{ji}, N))_{j\geq i}$ is stationary. If M is of countable type, we have $\mathrm{Im}(\mathrm{Hom}_A(u_{ji}, N)) = \mathrm{Im}(\mathrm{Hom}_A(u_i, N))$ for j sufficiently large (cf. EGA $\mathrm{O}_{\mathrm{III}}$ 13.2.2); this equality is no longer valid in general.

Definition 12 (2.3.1)**.** Let $u : F \to M$ and $v : F \to G$ be two A-linear maps, such that F and G are of finite presentation. We say that v strictly stabilizes u if u and v factor mutually on the right.

We say that an A-module M is *strictly Mittag-Leffler* if it satisfies

(SML) any A-linear map from an A-module of finite presentation F to M admits a strict stabilizer.

Proposition 16 (2.3.2)**.** *Let (F_i, u_{ji}) be a filtered inductive system of A-modules of finite presentation, with limit (M, u_i). The following conditions are equivalent:*

(i) *M satisfies* (SML)*;*

(ii) *(resp.* (ii')*) For any submodule of finite type (resp. monogenic) F of M, there exists an endomorphism of M inducing the identity on F and which factors through an A-module of finite presentation;*

(iii) *for any A-module N and any index i, there exists $j \geq i$ such that $\mathrm{Im}(\mathrm{Hom}_A(u_i, N)) = \mathrm{Im}(\mathrm{Hom}_A(u_{ji}, N))$;*

(iv) *for any A-module N, the canonical application $TN \otimes_A M \to T(\mathrm{Hom}_A(M, N))$ is injective (where T is the functor $\mathrm{Hom}_{\mathbb{Z}}(\cdot, \mathbb{Q}/\mathbb{Z})$, cf. 1.1).*

Proof. It is clear that (i) ⇔ (ii) ⇒ (ii'); let us show that (ii') ⇒ (ii).

Let B be the ring $\mathrm{End}_A(M)$, I the set of elements of B that factor through an A-module of finite presentation; it is easy to see that I is a two-sided ideal of B; thus the set S of elements s of B such that $1 - s \in I$ is multiplicatively stable. Condition (ii') states that any element of M is annihilated by an element of S; suppose this condition is verified and let $(x_i)_{1\leq i\leq n}$ be a finite sequence of elements of M. We construct by recurrence a sequence $(s_i)_{1\leq i\leq n}$ of elements of S such that s_{i+1} annihilates $s_i s_{i-1}\dots s_1 x_{i+1}$ for any i. Then the product of the s_i annihilates the sub-A-module of M generated by the x_i, thus (ii) is verified. The equivalence (ii) $\Leftrightarrow$ (iii) is proven like the equivalence (ii) (iii) of 2.1.4. To show the equivalence (i) $\Leftrightarrow$ (iv) we proceed as in 2.1.5:

Suppose that M satisfies (SML), let N be an A-module and x an element of $\mathrm{Ker}(TN\otimes_A M\to T(\mathrm{Hom}_A(M,N)))$. For i large enough, x comes from an element x_i of $TN\otimes_A F_i$. Let $j\geq i$ such that u_{ji} strictly stabilizes u_i. We form the commutative diagram

$$\text{TN }\otimes_A F_i[r][d]T(\mathrm{Hom}_A(F_i,N))[d]$$
$$\text{TN }\otimes_A F_j[r]T(\mathrm{Hom}_A(F_j,N))$$

in which the lines are isomorphisms (Bourbaki, Alg. comm., chap. I, §2, ex. 14). Since u_{ji} strictly stabilizes u_i, we have

$$\mathrm{Coker}(\mathrm{Hom}_A(u_i,N))=\mathrm{Coker}(\mathrm{Hom}_A(u_{ji},N));$$

the diagram above then shows that the image of x_i in $TN\otimes_A F_j$ is zero; a fortiori x is zero, hence (iv).

Suppose that M satisfies (iv); since (iv) manifestly implies the condition of 2.1.5, M satisfies (ML); therefore, if u_{ji} stabilizes u_i, we have $\mathrm{Ker}(1_{TN}\otimes u_i)=\mathrm{Ker}(1_{TN}\otimes u_{ji})$ for any A-module N, thus

$$\mathrm{Coker}(\mathrm{Hom}_A(u_i,N))=\mathrm{Coker}(\mathrm{Hom}_A(u_{ji},N))$$

by (iv), so that u_{ji} strictly stabilizes u_i. Q.E.D.

[2.3.3]

1) Condition (SML) implies (ML) (2.3.2 and 2.1.4); it is clear that any relatively projective module satisfies (SML).

2) If M satisfies (SML), any pure submodule of M satisfies (SML) (use criterion 2.3.2(iv)).

3) If M is an inductive filtered limit of direct factors of M that satisfy (SML), it is clear that M satisfies (SML). In particular, any direct sum of modules satisfying (SML) satisfies (SML). If A is a local Henselian commutative ring, M satisfies (SML) if and only if it is an inductive filtered limit of direct factors with finite presentation (2.1.9).

4) If A is commutative and linearly compact for the discrete topology, the conditions (ML) and (SML) are equivalent. Indeed, A is a direct product of local rings, so we may assume A is local; it is then Henselian; by 2.1.9, any A-module of Mittag-Leffler type M is an inductive filtered limit of pure submodules of finite presentation F_i; since F_i is linearly compact for the discrete topology, it is necessarily a direct factor of M [8].

If A is a non-complete local Noetherian commutative ring, we can provide examples of flat A-modules P satisfying (SML) such that $\mathrm{Ext}^1_A(P,A)\neq 0$; if

$0 \to A \to Q \to P \to 0$ is a non-split exact sequence, the flat A-module Q satisfies (ML) but does not satisfy (SML).

Let us now study the condition (SML) for flat modules:

Proposition 17 (2.3.4). *Let M be an A-module. The following conditions are equivalent:*

(i) M is flat and satisfies (SML);

(ii) (resp. (ii′)) for any submodule of finite type (resp. monogenic) F of M, there exists an endomorphism of M inducing the identity on F and which factors through a free A-module of finite type;

(iii) for any $x \in M$, we have $x \in o_M(x) \cdot M$, where $o_M(x)$ denotes the right ideal of A, image of the linear form on $\mathrm{Hom}_A(M, A)$ defined by x.

Proof. The equivalence (i) ⇔ (ii) is immediate from [13] 11.3. The equivalence (ii) ⇔ (ii′) is obtained like the equivalence (ii) ⇔ (ii′) of 2.3.2 (where we take this time for I the set of endomorphisms of B that factor through a free A-module of finite type). If (ii′) is verified, let $x \in M$; there exists a sequence $M \xrightarrow{u} L \xrightarrow{v} M$ of A-linear maps, such that L is free of finite type and $v \circ u(x) = x$. Since L is free of finite type, we clearly have $u(x) \in o_L(u(x)) \cdot L$, thus $x = v \circ u(x) \in o_L(u(x)) \cdot M$; since $o_L(u(x)) = o_M(x)$, we have a fortiori (iii). If (iii) is verified, for any $x \in M$, we can find a sequence $(u_i, x_i)_{1 \leq i \leq n}$ of $\mathrm{Hom}_A(M, A) \times M$ such that $x = \sum_{1 \leq i \leq n} u_i(x) x_i$; let us denote $u : M \to A^n$ the product map of the u_i and $v : A^n \to M$ the linear map defined by the x_i; then $v \circ u(x) = x$, hence (ii′).

(Note that if (iii) is verified, for any $x \in M$, $o_M(x)$ is the ideal of coefficients of x: indeed, any right ideal I of A such that $x \in IM$ necessarily contains $o_M(x)$.) □

10 Criteria for flatness

10.1 Examples of flat Mittag-Leffler modules

(2.4.1) Let A be a not necessarily commutative noetherian ring on the left, C the (abelian) category of A-modules of finite type, M an A-module. We assume that the functor $\mathrm{Hom}_A(M, \cdot) : C \to (Ab)$ is exact. Let M^* be the A^0-module $\mathrm{Hom}_A(M, A)$. The functorial morphism $k_N : M^* \otimes_A N \to \mathrm{Hom}_A(M, N)$ is an isomorphism when N is free of finite type; since both functors in the presence are exact on the right, k_N is an isomorphism for any object N of C. If now N is any A-module, we see by writing N as an inductive limit of finite type submodules that k_N is injective. In particular, M^* is flat and satisfies condition (iii) of 2.3.4 (for any $x \in M^*$, taking $N = \mathrm{Coker}(x)$, we see that $x \in M^* \cdot \mathrm{Im}(x)$), thus it satisfies (SML).

(2.4.2) The hypotheses of 2.4.1 are verified when M is projective; we see in particular that if A is a noetherian ring on the right and I is a set, the A-module A^I is flat and satisfies (SML) [8].

Let now A be a commutative noetherian ring, I a set, J an ideal of A; for any A-module of finite type M, let $C_J(I, M)$ be the submodule of M^I formed by families that tend to zero for the J-adic topology. According to Krull's theorem, $C_J(I, M)$ is an exact functor of M. Consequently, the functorial morphism $C_J(I, A) \otimes_A M \to C_J(I, M)$ is bijective (because it is for M free of finite type), thus $C_J(I, A) = C_J(I)$ is a pure A-submodule of A^I; in particular it is flat and satisfies (SML).

(2.4.3) Let A be a complete local commutative noetherian ring with maximal ideal $\mathfrak{m}$. We know [10] that $\mathrm{Ext}^1_A(M, N) = 0$ when M is flat and N is of finite type. It follows from 2.4.1 that the dual of a flat A-module is a strictly flat Mittag-Leffler A-module.

Proposition 18 (2.4.3.1)**.** *Let P be a flat A-module, $\hat{P}$ the $\mathfrak{m}$-adic completion of P. There exists a set I and an isomorphism $C_{\mathfrak{m}}(I) \to \hat{P}$. Moreover, the following conditions are equivalent:*

(i) P satisfies (ML) (or (SML), it's the same thing by 2.3.3.4);

(ii) $P \to \hat{P}$ is A-universally injective;

(iii) $P \to P^{**}$ is A-universally injective.

Proof. Let $(e_i)_{i \in I}$ be a family of elements of P forming a basis of $P/\mathfrak{m}P$ over $A/\mathfrak{m}$. For any integer n, $P/\mathfrak{m}^n P$ is an $(A/\mathfrak{m}^n)$-module free with basis the image of $(e_i)_{i \in I}$; if $u_n : (A/\mathfrak{m}^n)^{(I)} \to P/\mathfrak{m}^n P$ is the corresponding isomorphism, $\lim u_n$ is an isomorphism of $C_{\mathfrak{m}}(I)$ onto $\hat{P}$, hence the first assertion. Moreover, (i) $\Rightarrow$ (iii) by 2.3.4, (iii) $\Rightarrow$ (ii) because P and $\hat{P}$ have the same dual, (ii) $\Rightarrow$ (i) by 2.3.3.2); qed. ☐

11 Descent of the condition (ML)

From now on, the rings are assumed to be commutative.

Proposition 19 (2.5.1)**.** *Let $f : A \to B$ be a universally injective homomorphism, $u : F \to M$ and $v : F \to G$ two A-linear applications such that F and G are of finite presentation. If $1_B \otimes_A v$ stabilizes $1_B \otimes_A u$, then v stabilizes u. In particular, if $B \otimes_A M$ is a B-module of Mittag-Leffler, M is an A-module of Mittag-Leffler.*

Proof. Let us form the cocartesian diagram of A-modules

F [r, "u"] [d, "v"'] M [d, "s"]
G [r, "t"'] N.

By hypothesis, $1_B \otimes_A s$ and $1_B \otimes_A t$ are B-universally injective and a fortiori A-universally injective (1.1.1 (vii)). Since f is universally injective, it follows that s and t are A-universally injective (because, for any A-module P, $u \otimes_A 1_P : P \to B \otimes_A P$ is A-universally injective). In other words, v stabilizes u.

Let now (F_i, u_{ji}) be an inductive filtering system of A-modules of finite presentation, with limit (M, u_i). Suppose that the B-module $B \otimes_A M$ satisfies

(ML); then, for any i, there exists $j \geq i$ such that $1_B \otimes u_{ji}$ stabilizes $1_B \otimes u_i$; therefore u_{ji} stabilizes u_i and M satisfies (ML). □

Proposition 20 (2.5.2)**.** *Let $f : A \to B$ be a homomorphism satisfying condition (O) of §1 (e.g., a finite homomorphism with T-nilpotent kernel), M a flat A-module such that the B-module $B \otimes_A M$ satisfies (ML); then M satisfies (ML).*

Proof. For any set I, let $t_I : \mathrm{Mod}(A) \to \mathrm{Mod}(A)$ be the functor defined by $t_I(N) = \mathrm{Ker}(N^I \otimes_A M \to (N \otimes_A M)^I)$. Since M is A-flat, t_I is exact on the left. If I and J are two sets, if we identify $N^{I\times J}$ and $(N^I)^J$ for any A-module N, we have the inclusion $t_I(N^J) \subset t_{I\times J}(N)$[1].

Let now C be the full subcategory of $\mathrm{Mod}(A)$ formed by the modules that annihilate all functors t_I; according to the properties above, C is stable under subobjects, products and extensions; on the other hand, since the B-module $B \otimes_A M$ satisfies (ML), it follows from 2.1.5 that $TB = \mathrm{Hom}_A(B, \mathbb{Q}/\mathbb{Z})$ is an object of C.

Let N be an A-module. There exists a smallest submodule N' of N such that N/N' is an object of C, and N' has no non-zero quotient in C: in particular, $\mathrm{Hom}_A(N', TB) = 0$, thus $B \otimes_A N' = 0$, thus $N' = 0$ according to

property (O); but then N is an object of C, thus $C = \mathrm{Mod}(A)$ and M satisfies (ML) by 2.1.5.

Corollary 24 (2.5.3)**.** *Let A be a Noetherian ring, I an ideal of A such that A is complete for the I-adic topology, M a flat A-module such that M/IM satisfies (ML). For M to satisfy (ML), it is necessary and sufficient that, for all $\mathfrak{p} \in D(I)$, $M_{\mathfrak{p}}$ is separated for the I-adic topology.*

Proof. Let us first note that if $u : F \to N$ and $v : F \to G$ are A-linear applications such that F and G are of finite type and v stabilizes u, the filtrations of F induced by the respective I-adic filtrations of N and G are identical, and that, if u is injective, v is injective; we deduce that any A-module of Mittag-Leffler type is separated for the I-adic topology. The condition of the statement is therefore necessary.

Conversely, let us assume it is satisfied.

Let us first show that, for any A-module of finite type F, $M \otimes_A F$ is separated for the I-adic topology. By devissage, it suffices according to the property of the statement to prove that if $0 \to F' \to F \to F'' \to 0$ is an exact sequence of A-modules of finite type, and if $M \otimes_A F'$ and $M \otimes_A F''$ are separated for the I-adic topology, $M \otimes_A F$ is separated for the I-adic topology. But it follows from the Artin-Rees lemma and the flatness of M that the filtration of $M \otimes_A F'$ induced by the I-adic filtration of $M \otimes_A F$ is I-good, thus separated. We easily deduce that the I-adic filtration of $M \otimes_A F$ is separated.

To prove that M satisfies (ML) it suffices according to 2.1.8 to show that for any A-module of finite type F, any element x of $M \otimes_A F$ has a module of coefficients in F. Given 2.5.2 and the hypothesis on M/IM, the image of x

[1] More precisely, we have the exact sequence $0 \to t_I(N^J) \to t_{I\times J}(N) \to (t_I(N))^J$.

in $M \otimes_A (F/I^nF)$ has a module of coefficients J_n in F/I^nF for any integer n. Let J be the submodule $\lim_{\longleftarrow} J_n$ of $F = \lim_{\longleftarrow} F/I^nF$. Since $M \otimes_A (F/J)$ is separated for the I-adic topology, we have $x \in M \otimes_A J$, thus J is the sought module of coefficients. Q.E.D.

We will now improve prop. 2.5.2 in the case where A is Noetherian. Let us recall the following definition:

Definition 13 (EGA IV 15.7.8)**.** Let $f : X \to S$ be a morphism of schemes. We say that f is submersive if it is surjective and if the topology of S is quotient of that of X. We say that f is universally submersive if $f \times_S S'$ is submersive for any S-scheme S'.

Lemma 35 (2.5.4)**.** *Let $f : X \to S$ be a morphism of schemes. We assume S Noetherian and f universally submersive. Let s be a point of S, s' a specialization of s in S. There exists a point x of X above s and a specialization x' of x above s'.*

This is immediate by reduction to the case where S is the spectrum of a discrete valuation ring.

Proposition 21 (2.5.5)**.** *Let $f : A \to B$ be a ring homomorphism, M a flat A-module; we assume that A is Noetherian, that (f) is universally submersive and that the B-module $B \otimes_A M$ satisfies (ML). Then M satisfies (ML).*

Proof. Let us first assume A is a complete local ring with maximal ideal $\mathfrak{m}$. Let F be the set of ideals I of A such that M/IM does not satisfy (ML); if F is non-empty, it has a maximal element I which is necessarily a prime ideal of A (since the set of ideals of A that do not belong to F is stable under product, according to 2.5.2). Let us find a contradiction with the hypotheses of the statement. We can assume $I = 0$, in which case A is integral and f is injective. By 2.5.3, it suffices to show that M is separated for the $\mathfrak{m}$-adic topology. Let V be a discrete valuation ring dominating A. By 2.5.4 and 1.3.1, the homomorphism $1_V \otimes_A f$ is universally injective. By 2.5.1, $V \otimes_A M$ is a Mittag-Leffler module. Since V is separated for the $\mathfrak{m}$-adic topology, $V \otimes_A M$ is also, thus M is as well, since it identifies with a sub-A-module of $V \otimes_A M$, hence the assertion.

The case where A is local follows from the preceding by flat descent (2.5.1).

To study the general case, we will use the following lemma: □

Lemma 36 (2.5.6)**.** *Let A be a ring, M a flat A-module such that, for every prime ideal $\mathfrak{p}$ of A, $M_\mathfrak{p}$ is an $A_\mathfrak{p}$-module of Mittag-Leffler. For M to be an A-module of Mittag-Leffler, it is necessary and sufficient that the following condition be satisfied: for any A-linear application $u : F \to M$ where F is of finite presentation, the set of prime ideals $\mathfrak{p}$ of A such that $u_\mathfrak{p}$ is A-universally injective is open in (A).*

Proof. The necessity is easily proved by introducing a stabilizer of u. Conversely, suppose the condition of the statement is satisfied. Let $v : G \to M$ be an A-linear application such that G is of finite presentation, and let $\mathfrak{p}$ be a prime ideal of A. By 2.1.9, $v_\mathfrak{p}(G_\mathfrak{p})$ is contained in a pure, free $A_\mathfrak{p}$-submodule of finite type $L_{\mathfrak{p}}$ of $M_{\mathfrak{p}}$. We deduce from the property of the statement that there exists an element f of $A - \mathfrak{p}$ and a pure, free A_f-submodule of finite type L_f of M_f that contains $\mathrm{Im}(v_f)$; a fortiori v_f has a stabilizer. Thus, locally for the Zariski topology on (A), v has a stabilizer. We deduce from 2.5.1 by a quasi-compactness argument that v has a stabilizer, Q.E.D.

Returning to the proof of 2.5.5, let us show that the A-module M satisfies the conditions of 2.5.6. We already know that M is pointwise Mittag-Leffler. Let $u : L \to M$ be an A-linear application such that L is free of finite type; since M is A-flat and since $B \otimes_A M$ is a B-module of Mittag-Leffler, there exists an A-linear application $v : L \to L'$ such that L' is free of finite type, that v factorizes u on the right and that $1_B \otimes_A v$

dominates $1_B \otimes_A u$. The set U of points of (A) where v is invertible on the left is open. It suffices therefore to see that u is A-universally injective above U. Let $\mathfrak{p} \in U$. Since $1_B \otimes_A v_{\mathfrak{p}}$ is invertible on the left and dominates $1_B \otimes_A u_{\mathfrak{p}}$, we see that $1_B \otimes_A u_{\mathfrak{p}}$ is B-universally injective, thus that $1_B \otimes_A u \otimes_A k(\mathfrak{p})$ is injective. Since (f) is surjective, it follows that $u \otimes_A k(\mathfrak{p})$ is injective. Since L is free and M is flat, $u_{\mathfrak{p}}$ is A-universally injective by 3.1.6, Q.E.D.

12 Applications

12.1 Descent of projectivity

In this paragraph, we will systematically use the technique of devissage introduced by Kaplansky [11]; let us recall what it is about.

Definition 14 (3.1.1)**.** Let A be a ring, M an A-module. We call *transfinite devissage* of M a family $(M_u)_{u \in U}$ of submodules of M, indexed by an ordinal U, increasing, such that $M_0 = 0$, that $M = \bigcup_{u \in U} M_u$ and that, for any limit ordinal $u' \in U$, we have $M_{u'} = \bigcup_{u \in u'} M_u$.

We say that $(M_u)_{u \in U}$ is a *Kaplansky devissage* if, for any ordinal u such that $u + 1 \in U$, M_u is a direct factor of M_{u+1} and M_{u+1}/M_u is of countable type.

Lemma 37 (3.1.2)**.** *Let $(M_u)_{u \in U}$ be a transfinite devissage of an A-module M. If, for any ordinal u such that $u + 1 \in U$, M_u is a direct factor of M_{u+1}, then M is isomorphic to the direct sum of the M_{u+1}/M_u.*

The proof is done easily, by transfinite recurrence, cf. [11].

We see in particular that an A-module is a direct sum of submodules of countable type if and only if it admits a Kaplansky devissage.

We will now deduce from the results of 2.5 criteria for the descent of projectivity. For brevity, we will say that a homomorphism $f : A \to B$ satisfies condition (C) (resp. (C')) if any flat A-module P such that the B-module $B \otimes_A P$ is of Mittag-Leffler (resp. projective), is necessarily an A-module of Mittag-Leffler (resp. projective).

Theorem 17 (3.1.3)**.** *(C) implies (C').*

Proof. Let $f : A \to B$ be a homomorphism satisfying condition (C). Let P be a flat A-module such that $B \otimes_A P$ is a projective B-module. By [11] we know that the B-module $B \otimes_A P$ is a direct sum of a family $(Q_i)_{i \in I}$ of submodules of countable type. We will say that a submodule P' of P is *adapted* to the decomposition $(Q_i)_{i \in I}$ if it is pure and if $\mathrm{Im}(B \otimes_A P' \to B \otimes_A P)$ is the sum of a subfamily of $(Q_i)_{i \in I}$.

According to property (C), P satisfies (ML). By 2.2.1, the set of submodules of countable type of P has a cofinal part formed of pure submodules. We will show that this set has a cofinal part formed of adapted submodules. Let P_0 be a pure submodule of countable type of P. We construct, by recurrence on n, a sequence (P_n, I_n) satisfying the following conditions:

a) P_n is a pure submodule of countable type of P, I_n is a countable part of I;

b) I_n is the set of $i \in I$ such that the projection of $B \otimes_A P_n$ in Q_i is non-zero;

c) the image of $B \otimes_A P_{n+1}$ in $B \otimes_A P$ contains the Q_i $(i \in I_n)$.

This construction is possible: it suffices to note that the Q_i are of countable type and that, for any sub-B-module of countable type Q of $B \otimes_A P$, there exists a sub-A-module of countable type R of P such that $Q = \mathrm{Im}(B \otimes_A R \to B \otimes_A P)$.

Let us set $P' = \bigcup_{n \in \mathbb{N}} P_n$ and $J = \bigcup_{n \in \mathbb{N}} I_n$; it is clear that P' is a pure submodule of countable type of P containing P_0, and that $\mathrm{Im}(B \otimes_A P' \to B \otimes_A P)$ is the sum of $(Q_i)_{i \in J}$; hence the assertion.

Let A be the set of adapted submodules of P distinct from P. For any $N \in A$, let us choose a non-zero submodule of countable type N' of P/N, adapted to the decomposition of $B \otimes_A (P/N)$ quotient of $(Q_i)_{i \in I}$. The reciprocal image of N' in P will be denoted $s(N)$; it is an adapted submodule of P strictly larger than N. According to the principle of transfinite recurrence, there exists a unique transfinite devissage $(P_u)_{u \in U}$ of P such that, for any ordinal u such that $u + 1 \in U$, we have $P_{u+1} = s(P_u)$. By 2.2.2, P_{u+1}/P_u is projective of countable type. Therefore P is projective (3.1.1) cqfd.

Examples (3.1.4). 1) Condition (O) of §1 implies (C) (2.5.2) and thus also (C'). In particular, a faithfully flat homomorphism and an injective and finite homomorphism satisfy (C').

2) Let $f : A \to B$ be a homomorphism. If A is Noetherian and if $\mathrm{Spec}(f)$ is universally submersive, f satisfies (C) (2.5.5) and thus also (C').

3) Let A be a ring, M an A-module; by 1), the condition "P is projective" is local for the Zariski topology on $\mathrm{Spec}(A)$.

Let now S be a scheme, $\mathcal{M}$ an $\mathcal{O}_S$-module; we will say that $\mathcal{M}$ is *locally projective* if for any affine open U of S, $\Gamma(U, \mathcal{M})$ is a projective $\Gamma(U, \mathcal{O}_S)$-module. This condition is local for the Zariski topology on S, and even for the fpqc topology, according to 1).

We will say that a morphism of schemes $f : X \to S$ satisfies condition (C') if every $\mathcal{O}_S$-module $\mathcal{M}$, such that $f^*(\mathcal{M})$ is a locally projective $\mathcal{O}_X$-module, is locally projective. According to 2), if f is quasi-compact, quasi-separated and universally submersive, and if S is Noetherian, f satisfies (C'); in particular, if f is proper surjective and if S is Noetherian, f satisfies (C').

12.2 Finite dimension of Noetherian rings

Definition 15 (3.2.1)**.** (see [4]). Let A be a ring.

1. We call *finitely generated dimension* of A and denote by $FPD(A)$ the smallest integer n such that every A-module of finite projective dimension has projective dimension $\leq n$, or $+\infty$ if no such integer exists.

2. We call *weak finitely generated dimension* of A and denote by $FWD(A)$ the smallest integer n such that every A-module of finite Tor-dimension has Tor-dimension $\leq n$, or $+\infty$ if no such integer exists.

Let us recall the following results:

(3.2.2) (Auslander-Buchsbaum, see [3]). If A is a Noetherian ring, we have $FWD(A) = \sup_{p \in \mathrm{Spec}(A)} \mathrm{prof}(A_p)$.

(3.2.3) (Bass, see [5]). If A is a Noetherian ring, we have $FPD(A) \geq \dim(A)$ (Krull dimension); equality holds if A is Gorenstein.

We will see that for any Noetherian ring A, we have $FPD(A) = \dim(A)$, which clarifies 3.2.3. We will use the following auxiliary result:

Proposition 22 (3.2.4)**.** *Let A be an $\aleph_0$-Noetherian ring, P an A-module projective equipped with a differential d. There exists a transfinite devissage $(P_u)_{u \in U}$ of the A-module P satisfying the following condition: for all $u \in U$, P_u is stable under d, and the linear map $H(P_u) \to H(P)$ induced by the inclusion $P_u \hookrightarrow P$ via passage to homology modules is injective.*

(Recall that a ring A is $\aleph_0$-Noetherian when every ideal of A is of countable type.)

Proof. Let $(Q_i)_{i \in I}$ be a decomposition of the projective A-module P as a direct sum of submodules of countable type. Using the exact homology sequence, we proceed by recurrence to define how to show that if P is non-zero, there exists a non-zero submodule P' of P, sum of a countable subfamily of $(Q_i)_{i \in I}$, such that $d(P') \subset P'$ and such that $H(P') \to H(P)$ is injective.

Let I_0 be any countable subset of I; we construct by recurrence on n an increasing sequence (I_n) of countable subsets of I

such that, if $P_n = \sum_{i \in I_n} Q_i$, we have $d(P_n) \subset P_{n+1}$ and $d(P) \cap P_n \subset d(P_{n+1})$. The possibility of this construction comes from the fact that, since A is $\aleph_0$- Noetherian, $d(P) \cap P_n$ is of countable type. Let us set $J = \bigcup_{n \in \mathbb{N}} J_n$

and $P' = \bigcup_{n \in \mathbb{N}} P_n$; we have $P' = \sum_{i \in J} Q_i$; P' is stable under d and contains P_0; since

$d(P) \cap P' = d(P')$, $H(P') \to H(P)$ is injective; this proves the assertion.

Corollary 25 (3.2.5)**.** *Let A be an $\aleph_0$-Noetherian ring, M an A-module of finite projective dimension q. There exists a transfinite devissage $(M_u)_{u \in U}$ of M such that, for any ordinal u such that $u+1 \in U$, M_{u+1}/M_u is of countable type and of projective dimension $\leq q$.*

We apply 3.2.4 to a projective resolution of M of length q. Recall [2] that conversely, if A is any ring and if M is an A-module equipped with a transfinite devissage whose successive quotients are of projective dimension $\leq q$, then M is of projective dimension $\leq q$.

It follows in particular that, if A is an $\aleph_0$-Noetherian ring, $FPD(A)$ is the upper bound of the projective dimensions of A-modules of countable type and of finite projective dimension.

Theorem 18 (3.2.6)**.** *For any Noetherian ring A, we have $FPD(A) = \dim(A)$.*

Proof. By 3.2.3 it suffices to show that $FPD(A) \leq \dim(A)$. For this we can assume $\dim(A) = n < +\infty$ and reason by induction on n. For $n = 0$, A is Artinian and the assertion is well known (see for example [4]); suppose $n > 0$. It suffices to show that if M is an A-module of countable type and of finite projective dimension, and if

$0 \to P_n \to P_{n-1} \to \cdots \to P_0 \to M \to 0$

is a resolution of M such that P is projective of countable type for $i < n$, then P_n is projective. By 3.2.2, P_n is flat; since it is of countable type, it suffices to show that it satisfies (ML).

Suppose first A is local with maximal ideal $\mathfrak{m}$. By completion and flat descent (2.5.1) we can assume A complete. If $\mathrm{prof}(A) < n$, P_{n-1}/P_n is A-flat by 3.2.2, thus P_n satisfies (ML); suppose therefore that $\mathrm{prof}(A) = n$, in other words, that A is Cohen-Macaulay. By 2.5.3, it suffices to show that $P_n/\mathfrak{p}P_n$ is separated for the $\mathfrak{m}$-adic topology, for any prime ideal $\mathfrak{p}$ of A. Since A is Cohen-Macaulay, we can by EGA O_{IV} 16.5.9 find an A-regular sequence of elements of $\mathfrak{p}$, generating an ideal I such that $\mathfrak{p} \in \mathrm{Ass}(A/I)$. We can also assume that $\mathfrak{p}$ is not maximal, in which case $\mathrm{Tor}_n^A(M, A/I) = 0$, so that the linear map $P_n/IP_n \to P_{n-1}/IP_{n-1}$ is injective; thus P_n/IP_n is separated for the $\mathfrak{m}$-adic topology. On the other hand, since $\mathfrak{p} \in \mathrm{Ass}(A/I)$, there exists a map

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A-linear injective $j : A/\mathfrak{p} \to A/I$; then $1_{P_n} \otimes_A j : P_n/\mathfrak{p}P_n \to P_n/IP_n$ is injective, since P_n is flat; thus $P_n/\mathfrak{p}P_n$ is separated for the $\mathfrak{m}$-adic topology, which proves the assertion.

If A is arbitrary, the local case shows that P is pointwise projective. By 2.5.6, it suffices to show that if $u : L \to P_n$ is an A-linear application such that L is free of finite type, the set of points of $\mathrm{Spec}(A)$ where u is A-universally injective is open.

Let $\mathfrak{p}$ be a point of $\mathrm{Spec}(A)$ such that $u_{\mathfrak{p}}$ is A-universally injective. Let $v : L \to P_{n-1}$ be the composite application of the sequence $L \to P_n \to P_{n-1}$. The set of points $\mathfrak{q}$ of $\mathrm{Spec}(A)$ where v is invertible on the left is an open U containing $\mathrm{Ass}(A_{\mathfrak{p}})$ ([13] II 4.4). Instead of replacing A by A_f where f is a suitable element of $A - \mathfrak{p}$, we can assume that U is schematically dense in $\mathrm{Spec}(A)$. Let s be a regular element of A such that $D(s) \subset U$. Since $\mathrm{Tor}_i^A(M, A/sA) = 0$ for $i > 1$, we have the exact sequence

$0 \to P_n/sP_n \to P_{n-1}/sP_{n-1} \to \cdots \to P_1/sP_1$

thus, according to the recurrence hypothesis, P_n/sP_n is an (A/sA)-projective module. In particular, the set of points of $\mathrm{Spec}(A)$ where $u \otimes_A 1_{A/sA}$ is invertible on the left is an open containing $\mathfrak{p}$; instead of localizing again, we can assume that $u \otimes_A 1_{A/sA}$ is invertible on the left. Let $Q = \mathrm{Coker}(u)$. Then Q_s is flat, Q/sQ is (A/sA)-flat, and $\mathrm{Tor}_1^A(Q, A/sA)$ is of finite type and zero at $\mathfrak{p}$. Instead of localizing again, we can assume that s is Q-regular. It follows from 1.4.2.1 that Q is flat i.e. that u is A-universally injective, q.e.d.

Corollary 26 (3.2.7)**.** *Let A be a Noetherian ring of finite dimension n. Any flat A-module is of projective dimension $\leq n$.*

Indeed $FPD(A) = n < +\infty$, thus any flat A-module is of finite projective dimension (necessarily $\leq n$) by [10], prop. 6.

In the "relative" case, we note the following result:

Theorem 19 (3.2.8)**.** *Let A be a ring, B a finite and free A-algebra, M a B-module of finite Tor-dimension. We have the equalities*

$$Tor \cdot \dim_B(M) = Tor \cdot \dim_A(M), \tag{1}$$

$$\dim \cdot proj_B(M) = \dim \cdot proj_A(M). \tag{2}$$

We will use the following lemma:

Lemma 38 (3.2.9)**.** *Let P be a flat B-module. If P is an A-module of Mittag-Leffler, it is a B-module of Mittag-Leffler.*

Proof. Let F be the B-module $\mathrm{Hom}_A(B, A)$: it is clear that F is of finite type and faithful. If M is a B-module and N is an A-module,

on a functorial isomorphism

$$\mathrm{Hom}_B(M, B \otimes_A N) \to \mathrm{Hom}_A(M \otimes_B F, N), \tag{$*$}$$

where B is a free A-module of finite type.

We will resume the demonstration of 2.5.2; as before, let us introduce the full subcategory C of $\mathrm{Mod}(B)$ formed by the B-modules M such that $M' \otimes_B P \to (M \otimes_B P)^I$ is injective for any set I. As in loc. cit. we see that C is stable under sub-objects, products, and extensions. We need to show that $C = \mathrm{Mod}(B)$, in other words, that for any non-zero B-module M there exists a non-zero B-linear application of M into an object of C.

Since the A-module P satisfies (ML) and since B is a finite and free A-algebra, it is clear that, for any A-module N, the B-module $B \otimes_A N$ is an object of C. Let M be a B-module, and let us take for N the A-module $M \otimes_B F$ and suppose $\mathrm{Hom}_B(M, B \otimes_A N) = 0$. By the isomorphism $(*)$ it follows that $\mathrm{End}_A(N) = 0$, thus $N = 0$. Since F is a faithful B-module of finite type, we also have $M = 0$, q.e.d.

Let us show equality (1) of 3.2.8. By localization, we reduce to the case where A and B are local with maximal ideals $\mathfrak{m}$ and $\mathfrak{n}$ respectively. It suffices to show that if M is A-flat and if $\mathrm{Tor} \cdot \dim_B(M) \leq 1$, then M is B-flat. We form an exact sequence of $\mathrm{Mod}(B)$

$$0 \to P \xrightarrow{u} L \to M \to 0 \tag{5}$$

where L is B-free; we need to show that u is B-universally injective. Since L is A-free and since u is A-universally injective, P is an A-module of Mittag-Leffler (2.1.6); since P is B-flat, it is a B-module of Mittag-Leffler (3.2.9); it is therefore an inductive filtered limit of pure, free B-submodules of finite type (2.1.9); we can therefore assume that P and L are free of finite type. Since u is A-universally injective, it defines by passage to the quotient a B-linear injective application of $P/\mathfrak{n}P$ into $L/\mathfrak{n}L$; since $B/\mathfrak{n}B$ is Artinian (thus of finite dimension), this linear application is B-universally injective, thus $P/\mathfrak{n}P \to L/\mathfrak{n}L$ is injective; since P and L are free of finite type, u is B-universally injective (I 3.1.6) q.e.d.

Let us show equality (2) of 3.2.8. Given equality (1), it suffices to show that if M is B-flat and A-projective, it is B-projective. We reason as in the demonstration of 3.1.3, using 3.2.9. Let $(Q_i)_{i \in I}$ be a decomposition of the A-module M as a direct sum of submodules of countable type. We say that a B-submodule of M is adapted to the decomposition $(Q_i)_{i \in I}$ if it is pure and the sum of a subfamily of $(Q_i)_{i \in I}$. By transfinite recurrence (see 3.1.3) it suffices to verify that the set of B-submodules of countable type of M has a cofinal

formed of adapted submodules. Let I_0 be a countable part of I. We construct by recurrence on n a sequence (P_n, I_n) such that P_n is a pure B-submodule of countable type of M, that I_n is a countable part of I and that, if P'_n designates the sum of Q_i $(i \in I_n)$, we have $P'_n \subset P_n \subset P'_{n+1}$. Since M satisfies (ML) (3.2.9) the construction is possible (2.2.1); the union of P_i is then an adapted submodule of M containing P'_0, which completes the demonstration.

12.3 Projective dimension of flat modules

Let A be a ring; we denote by $d(A)$ the upper bound of the projective dimensions of flat A-modules. If A is Noetherian, we have $d(A) \leq \dim(A)$ (3.2.7); one may ask if $d(A) = \dim(A)$. According to Jensen (unpublished) the answer is negative, in particular if A is a countable Noetherian ring of Gorenstein, we have $d(A) = 1$. We will connect this result to the work of Osofsky [16]:

Theorem 20 (3.3.1)**.** *Let A be a Noetherian ring. We have*

$$d(A) = \sup(\dim \cdot_A (A[S^{-1}]))$$

where S ranges over the set of multiplicative parts of A.

Proof. Let us denote d the second member of the equality to demonstrate; it is about seeing that $d(A) \leq d$. By Noetherian recurrence and by 3.1.4 1), we can assume A integral of field of fractions K, and the equality proven for all quotient rings of A distinct from A.

We can evidently assume A infinite; let $(A) = \aleph_n$ and let $v \mapsto s_v$ be a bijection of $\aleph_n$ onto $A - \{0\}$. For all $v \in \aleph_n$ let S_v be the multiplicative part of A generated by $(s_v)_{v' \in v}$. Let U be the ordinal of the product lexicographic of $\aleph_n$ by $\aleph_n$; for all $w \in U$, let w' and w'' be the respective projections of w on $\aleph_n$ and $\aleph_n$; then w'' is an integer, and we have $w_1 < w_2$ if and only if $w_1' < w_2'$ or $w_1' = w_2'$ and $w_1'' < w_2''$.

Let $w \in U$. We denote K_w the A-submodule $s_w^{-w''}.(A[S_{w'}^{-1}])$ of K. Then $(K_{w'}/A)_{w \in U}$ is a transfinite devissage of the A-module K/A. For all $w \in U$, K_{w+1}/K_w is isomorphic to $A[S_{w'}^{-1}]/s_{w'}.A[S_{w'}^{-1}]$.

Let P be a flat A-module, let us show that $\dim \cdot_A(P) \leq d$. We have $\dim \cdot_A(K) \leq d$; since $K \otimes_A P$ is a K-vector space, we have thus $\dim \cdot_A(K \otimes_A P) \leq d$; it suffices therefore to see that

$$\dim \cdot_A((K/A) \otimes_A P) \leq d + 1.$$

Since P is flat, it is clear that $((K_w/A) \otimes_A P)_{w \in U}$ is a devissage transfinite of $(K/A) \otimes_A P$. According to [2] it suffices to show that the quotients successifs are of projective dimension $\leq d + 1$. Let $w \in U$. Then

$(K_{w+1}/K_w) \otimes_A P$ is a flat $(A/s_w A)$-module, which is of projective dimension $\leq d$ by the noetherian recurrence hypothesis; it is therefore an A-module of projective dimension $\leq d + 1$, hence the assertion.

Corollary 27 (3.3.2)**.** *Let A be a noetherian ring of finite dimension n and of infinite cardinality $\aleph_a$. Then $d(A) \leq \inf(n, a + 1)$. If A is complete, we have $d(A) = \inf(n, a + 1)$.*

Proof. The first assertion follows from 3.3.1, and from the known fact that a flat A-module of type $\aleph_a$ is of projective dimension $\leq a + 1$ (see a simple proof of this last point in [8] appendix *I*). Let us prove the second assertion. We know ([16] 6.10) that if A is a complete local noetherian regular ring with field of fractions K, then $\dim \cdot_A(K) = \inf(n, a + 1)$. Suppose now A is a complete local noetherian integral ring with field of fractions K. There exists an injective and finite homomorphism $f : R \to A$, such that R is a complete local noetherian regular ring; moreover, if L is the field of fractions of R, we have $K = A \otimes_R L$. By 3.1.4 1), we have $\dim \cdot_A(K) = \dim \cdot_R(L) = \inf(n, a + 1)$, so the equality still holds in this case. The general case is easily reduced to the integral case. □

[3.3.3]

1. We do not know if, when A is an integral noetherian ring with field of fractions K, we always have $d(A) = \dim \cdot_A(K)$.

2. Let A be a noetherian ring. Given a flat A-module P, let us call the *relative injective dimension* of P the smallest integer n such that ${}^{n+1}_{A}(Q, P) = 0$

for all flat A-module Q. One can easily show that this integer is also the injective dimension of P in the sense of the relative homology theory defined on the category (A) by the class of universally exact sequences. One can show [10] that a local noetherian ring is complete if and only if it is of relative injective dimension zero; more generally, if A is a complete noetherian ring for an I-adic topology, one can show that the relative injective dimensions of the A-module A and of the (A/I)-module A/I are equal.

Let us note without proof the following result, in this order of ideas: Let k be an uncountable field, X and Y indeterminates, A the localization of the ring $k[X,Y]$ at the maximal ideal (X,Y), $K = k(X,Y)$ the field of fractions of A. We have ${}^1_A(K,A) = 0 \neq^2_A (K,A)$. In particular, A is of relative injective dimension 2.

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References

1. Artin, M.: The implicit function theorem in algebraic geometry. Interscience colloquium in algebraic geometry. Bombay 1968 (Oxford University Press).

2. Auslander, M.: On the dimension of modules and algebras. III. Nagoya Math. J. **9**, 67–77 (1955).

3. – Buchsbaum, D.: Homological dimension in Noetherian rings. II. Trans. Am. Math. Soc. **88**, 194–206 (1958).

4. Bass, H.: Finitistic dimension and a homological generalization of semiprimary rings. Trans. Am. Math. Soc. **95**, 466–488 (1960).

5. – Injective dimension in Noetherian rings. Trans. Am. Math. Soc. **102**, 18–29 (1962).

6. – Big projective modules are free. Illinois J. of Math. **7**, 23–31 (1963).

7. Ferrand, D.: Descent of flatness by a finite homomorphism. C.R.A.S. **246**, 946–949 (1969).

8. Goblot, R.: Sur deux classes de catégories de Grothendieck. Thèse (to appear).

9. EGA: Grothendieck, A., Dieudonné, J., Éléments de géométrie algébrique. Publ. Math. I.H.E.S. 4, 8, 11, 17, 20, 24, 28, 32.

10. SGA: Grothendieck, A., Séminaire de géométrie algébrique (I.H.E.S.).

11. Grothendieck, A.: Seminaire Bourbaki, exposé 221.

12. Jensen, C.U.: On the vanishing of $\lim^{(i)}$. J. of Algebra **15**, 151–166 (1970).

13. Kaplansky, I.: Projective modules. Annals of Math. **68**, 372–377 (1958).

14. Knutson, D.: Algebraic spaces. Thèse M.I.T. Lectures Notes N° 203. Berlin-Heidelberg-New York: Springer 1971.

15. Lazard, D.: Thèse (= Autour de la platitude). Bull. S.M.F. **97**, 81–128 (1969); et: Disconnexités des spectres d'anneaux et des préschémas. Bull. S.M.F. **95**, 95–108 (1967).

16. Murre, J.P.: Representation of unramified functors. Séminaire Bourbaki, exposé 294.

17. Olivier, J.-P.: Thèse (à paraître).

18. Osofsky, B.L.: Homological dimension and the continuum hypothesis. Trans. Am. Math. Soc. **132**, 217–230 (1968).

19. Raynaud, M.: Anneaux henséliens, Lecture notes, n° 169. Berlin-Heidelberg-New York: Springer 1970.

20. Warfield, R.B.: Purity and algebraic compactness for modules. Pacific J. of Math. **28**, 699–719 (1969).

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