IS $\mathrm{QCoh}(\mathrm{B}\,\mathbb{G}_\mathrm{a})$ DUALIZABLE?

Let k be a ring of characteristic p. All our sheaves(=stacks) will be assumed to be fppf sheaves of spaces types over finite type (underived) schemes over k. For $\mathscr{X}$ a stack/sheaf, we define the category of quasicoherent sheaves on $\mathscr{X}$ to be

$$\lim_{X \in SCH^{ft}_{/k}} \mathrm{QCoh(X)}$$

where the transition functors are quasicoherent pullback and the limit is taken in Pr^L_{St}.

From Cartier duality, we know that

$$\mathrm{QCoh}(\mathrm{B}\,\mathbb{G}_\mathrm{a}) \cong \mathrm{colim}_m(\mathrm{QCoh}(\alpha_\mathrm{p}^\mathrm{m}), *)$$

where the transition maps are pushforward along

$$(\mathrm{id} \times e_{\alpha_p}) : \alpha_p^m \times \operatorname{Spec} k \to \alpha_p^{m+1}$$

Here e_{α_p} is the inclusion of the unit of the group scheme α_p.

Let us consider the object Ω in (colimits are along pushforward functors and limits are along pullback functors)

$$\lim_n \mathrm{colim}_m \,\mathrm{QCoh}(\alpha_\mathrm{p}^\mathrm{m}) \otimes \mathrm{QCoh}(\alpha_\mathrm{p}^\mathrm{n})$$

whose projection to

$$\mathrm{colim}_m \,\mathrm{QCoh}(\alpha_\mathrm{p}^\mathrm{m}) \otimes \mathrm{QCoh}(\alpha_\mathrm{p}^\mathrm{n})$$

is given by the image of the diagonal sheaf on $\mathrm{QCoh}(\alpha_\mathrm{p}^\mathrm{n} \times \alpha_\mathrm{p}^\mathrm{n})$ under the inclusion into the colimit.

As

$$\lim_n \mathrm{colim}_m \,\mathrm{QCoh}(\alpha_\mathrm{p}^\mathrm{m}) \otimes \mathrm{QCoh}(\alpha_\mathrm{p}^\mathrm{n})$$

is isomorphic to

$$\lim_n (\lim_m (\mathrm{QCoh}(\alpha_\mathrm{p}^\mathrm{m+n}), \times), *)$$

(the notation indicates that the inner limit is along upper-cross transition functors), we can describe the above object as the infinite matrix

$$\begin{pmatrix} R & R & R & \dots \\ R & R \otimes \mathcal{O}_{\alpha_p} & R \otimes \mathcal{O}_{\alpha_p} & \dots \\ R & R \otimes \mathcal{O}_{\alpha_p} & R \otimes \mathcal{O}_{\alpha_p^2} & \dots \\ \dots & & & \end{pmatrix}$$

where m indexes the rows and n indexes the columns. Here R is a quasicoherent sheaf on α_p^i by pushforward along the inclusion of the unit. And $\mathcal{O}_{\alpha_p^i}$ is a quasicoherent sheaf on $\alpha_p^j \times \alpha_p^k$ by pushforward along the diagonal inclusion.

If $\mathrm{QCoh}(\mathrm{B}\,\mathbb{G}_\mathrm{a})$ were dualizable, then we would have isomorphisms

$$\mathrm{colim}_m \lim_n \mathrm{QCoh}(\alpha_\mathrm{p}^\mathrm{m+n}) \cong (\mathrm{colim}_m \,\mathrm{QCoh}(\alpha_\mathrm{p}^\mathrm{m})) \otimes (\lim_n \mathrm{QCoh}(\alpha_\mathrm{p}^\mathrm{n})) \cong \lim_n \mathrm{colim}_m \,\mathrm{QCoh}(\alpha_\mathrm{p}^\mathrm{m+n})$$

Therefore, the right adjoint of the natural functor from colim lim to lim colim should be an inverse. We simply compute the image of Ω under the composition of first the right adjoint then the natural functor.

First we consider how that the natural functor acts. Take an infinite matrix

$$\begin{pmatrix} M_{0,0} & M_{0,1} & M_{0,2} & \dots \\ M_{1,0} & M_{1,1} & M_{1,2} & \dots \\ M_{2,0} & M_{2,1} & M_{2,2} & \dots \\ \dots \end{pmatrix}$$

where the rows are compatible under pullback to the infinite matrix and they satisfy rules such as

$$M_{0,0} \cong \lim (e^i)^* e^\times M_{1,i}$$

By examining the functor

$$\lim_n \mathrm{QCoh}(\alpha_\mathrm{p}^{\mathrm{m+n}}) \lim_n \mathrm{colim}_m \mathrm{QCoh}(\alpha_\mathrm{p}^{\mathrm{m+n}})$$

and using the fact that any object $x \in \mathrm{colim}_j \mathcal{C}_j$ satisfies

$$x \cong \mathrm{colim}\, F_i G_i x$$

where $F_i : \mathcal{C}_j \to \mathrm{colim}\, \mathcal{C}_j$ is the inclusion functor and G_i is the right adjoint, we can identify that the natural functor takes the above element of colim lim to

$$\begin{pmatrix} \mathrm{colim}_m((e^m)^\times M_{m,0}) & \mathrm{colim}_m((e^m)^\times M_{m,1}) & \mathrm{colim}_m((e^m)^\times M_{m,2}) & \dots \\ \mathrm{colim}_m((e^m)^\times M_{m,0}) & \mathrm{colim}_m((e^m)^\times M_{m,1}) & \mathrm{colim}_m((e^m)^\times M_{m,2}) & \dots \\ \mathrm{colim}_m((e^m)^\times M_{m,0}) & \mathrm{colim}_m((e^m)^\times M_{m,1}) & \mathrm{colim}_m((e^m)^\times M_{m,2}) & \dots \\ \dots \end{pmatrix}$$

where now the compatibility is upwards (and also $M_{0,0} \cong \lim_i (e^i)^\times (\mathrm{id} \times e)^* M i, 1)$. Now we can also see using the above that an element

$$\begin{pmatrix} N_{0,0} & N_{0,1} & N_{0,2} & \dots \\ N_{1,0} & N_{1,1} & N_{1,2} & \dots \\ N_{2,0} & N_{2,1} & N_{2,2} & \dots \\ \dots \end{pmatrix}$$

of

$$\lim_n \mathrm{colim}_m \mathrm{QCoh}(\alpha_\mathrm{p}^{\mathrm{m+n}}) \cong \lim_n (\lim_m (\mathrm{QCoh}(\alpha_\mathrm{p}^{\mathrm{m+n}}), \times), *)$$

is taken to

$$\begin{pmatrix} \lim_n((e^n)^* M_{0,n}) & \lim_n((e^n)^* M_{1,n}) & \mathrm{colim}_n((e^n)^* M_{2,n}) & \dots \\ \lim_n((e^n)^* M_{0,n}) & \lim_n((e^n)^* M_{1,n}) & \mathrm{colim}_n((e^n)^* M_{2,n}) & \dots \\ \lim_n((e^n)^* M_{0,n}) & \lim_n((e^n)^* M_{1,n}) & \mathrm{colim}_n((e^n)^* M_{2,n}) & \dots \\ \dots \end{pmatrix}$$

Now we can compute where our element goes under the composition GF where F is the natural functor and G is its right adjoint. Looking at the upper left element, we see that it only depends on the first column of $F\Omega$.

The top element of this column is an infinite product of R's, where there is one R in degree 0, and countable many in every homological degree. The generators form an exterior algebra on infinitely many generators in homological degree 1. The next entry in the column loses one of the generators and so on. However, because it is a product, the colimit will not be able to kill everything in homological degrees. Therefore $\mathrm{QCoh}(\mathrm{B}\, \mathbb{G}_\mathrm{a})$ is not dualizable.