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FORMAL FIBERS OF A NOETHERIAN LOCAL RING

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Let A be an integral Noetherian local ring, $\mathfrak{m}$ its maximal ideal and A' the completion of A for the m-adic topology.

If A is universally Japanese (EGA, $\mathrm{O_{IV}}$, 23.1.1), or if A is a quotient of a Cohen-Macaulay ring, A' has no embedded associated prime ideals (EGA, IV, 7.6.4 and 6.3.8). We will construct an integral Noetherian local ring A, of dimension 2, such that A' possesses an embedded associated prime ideal.

If A is a quotient of a regular ring, the set of points of Spec(A) that satisfy property (S_n) is an open set (EGA, IV, 6.11.2) and, letting K be the field of fractions of A, $A' \otimes_A K$ is a Gorenstein ring ([4], V, ğ 10, p. 299 and prop. 10.1, p.300). We will construct an integral Noetherian local ring A of dimension 3 such that the set of points of Spec(A) where the local ring is Cohen-Macaulay (resp. is normal) is not open, and an integral Noetherian local ring A of dimension 1 such that $A' \otimes_A K$ is not a Gorenstein ring.

In these examples, the integral closure B of A is a regular local ring, the canonical morphism A' $\to$ B' is surjective, its kernel I is an ideal of square zero and the extension of B' by the B'-module I is trivial. Also, we had to precede the construction of the rings A above with some remarks on the integral closure of an integral Noetherian local ring and on the nilradical of its completion.

The terminology and notations used are those of EGA. Recall that if A is a ring, M an A-module and U an open set of S = Spec(A), we denote:

- $\widetilde{M}$ the quasi-coherent sheaf on S associated with M;
- $M(U)$ the module $\Gamma(U, \widetilde{M})$ of sections of the sheaf $\widetilde{M}$ over the open set U;

$D_A(M)$ the trivial extension of type (EGA, $\mathrm{O_{IV}}$, 18.2.3) of A by M, that is to say the ring with underlying additive group $A \times M$, whose multiplication is given by

$$(a, x) \cdot (a', x') = (aa', ax' + a'x).$$

If $t \in A$, the open set of invertibility of t is denoted by S_t. Finally, if x is a point of S, $\overline{x}$ denotes the topological space equal to the closure of x.

1 Preservation of the Noetherian property by the closure operation

In this section, A denotes a Noetherian local ring with maximal ideal $\mathfrak{m}$. We denote S = Spec(A), s the closed point of S, A' the completion of A for the m-adic topology, $\mathfrak{m}'$ the maximal ideal of A', S' = Spec(A'), and s' the closed point of S'. Let Z be a closed subset of S, Z' its inverse image in S', U = S - Z, U' = S' - Z', B = A(U), B' = A'(U'). Since A' is flat over A, the canonical morphism

$$B \otimes_A A' \to B'$$

is an isomorphism (EGA, IV, 5.9.4).

Proposition 1.1. *Suppose Z is non-empty and distinct from S.*

1. *For B to be integral over A, it is necessary and sufficient that for every maximal point x' of U', we have* $\operatorname{codim}(\overline{x'} \cap Z', \overline{x'}) \geq 2$.

2. *For B to be finite over A, it is necessary and sufficient that for every $x' \in \operatorname{Ass}(U')$, we have* $\operatorname{codim}(\overline{x'} \cap Z', \overline{x'}) \geq 2$.

3. *Let I' be the ideal of A' formed by the elements whose support is contained in the union of the closures $\overline{x'}$ of the points x' of* $\operatorname{Ass}(U')$ *such that* $\operatorname{codim}(\overline{x'} \cap Z', \overline{x'}) \leq 1$*. Assume that* $\operatorname{Supp}(I') \cap U'$ *is affine. Then, $B' = \Gamma(U', \widetilde{A'/I'})$ is finite over A' (hence Noetherian), it is a completion of B for the mB-adic topology and, for every integer $n \geq 0$, $B/\mathfrak{m}^n B$ is finite over $A/\mathfrak{m}^n$.*

4. *Under the hypotheses and with the notations of (3), if B is integral over A, then B is Noetherian if and only if B' is flat over B and in this case, for any finitely generated A-module M such that* $\operatorname{Ass}(M) \subset \operatorname{Ass}(A)$*, M(U) is a finitely generated B-module.*

Remark 1.1. Example 3.6 below shows that the hypotheses made in (3) do not imply that B is Noetherian, even if B is integral over A.

Proof. Since A' is faithfully flat over A, B is integral (resp. finite) over A if and only if B' is integral (resp. finite) over A'. Assertion (2) is therefore a consequence of EGA, IV, 5.11.1 since A' is a quotient of a regular ring.

Let us prove assertion (1). Let x' be a maximal point of U' such that $\operatorname{codim}(\overline{x'} \cap Z', \overline{x'}) = 1$ and let z' be a point of $\overline{x'} \cap Z'$ such that $\operatorname{codim}(\overline{\{z'\}}, \overline{x'}) = 1$. Then x' is an isolated point of $\operatorname{Spec}(\mathcal{O}_{S',z'}) \cap U'$ and, consequently, $B' \otimes_{A'} \mathcal{O}_{S',z'}$ contains a direct factor isomorphic to the localization $\mathcal{O}_{S',z'}$, hence it is not integral over the latter, so B' is not integral over A'; this shows that the stated condition is necessary. Conversely, if for every maximal point x' of U' we have $\operatorname{codim}(\overline{x'} \cap Z', \overline{x'}) \geq 2$, $A'_{red}(U')$ is finite over A' according to EGA, IV, 5.11.1; since B_{red} injects into $A'_{red}(U')$, we see that B' is integral over A'.

Let us prove (3). Let V' be the open set of S' complementary to the union of the closures $\overline{x'}$ of the points x' of $\operatorname{Ass}(U')$ such that $\operatorname{codim}(\overline{x'} \cap Z', \overline{x'}) \leq 1$. By definition, I' is the kernel of the restriction homomorphism A' → A'(V'); consequently $\operatorname{Ass}(A'/I')$ is contained in $\operatorname{Ass}(A') \cap V'$, so for all $x' \in \operatorname{Ass}(A'/I') \cap U'$, we have $\operatorname{codim}(\overline{x'} \cap Z', \overline{x'}) \geq 2$; the fact that B' is finite over A' thus results again from EGA, IV, 5.11.1. Let J' = I'(U'). Since the support of $\widetilde{I'}$ on U' is affine, we have $H^1(U', \widetilde{I'}) = 0$, whence an exact sequence

$$(*) \quad 0 \to J' \to B' \to \widetilde{B'} \to 0.$$

For any integer $n \geq 0$, $B/\mathfrak{m}^n B$ is isomorphic to $B'/\mathfrak{m}'^n B'$. Furthermore, since the support of $\widetilde{I'}$ in U' is affine and $s' \notin U'$, $J'/\mathfrak{m}'^n J' = 0$. It then follows from (*) that the canonical morphism $B/\mathfrak{m}^n B \to B'/\mathfrak{m}'^n B'$ is an isomorphism, whence the fact that $B/\mathfrak{m}^n B$ is finite over $A/\mathfrak{m}^n$ and that B' is the separated completion of B for the mB-adic topology.

Finally, let us prove (4). If B is Noetherian, its completion B' is obviously flat over B. Conversely, suppose that B' is flat over B. Since B is integral over A by hypothesis, it follows from (1) that the open set V' introduced previously contains the maximal points of S'; consequently, J' is a nilideal of B'. As B' is faithfully flat over B, we deduce that B' is even faithfully flat over B, hence B is Noetherian. Finally, let M be a finitely generated A-module such that $\operatorname{Ass}_A(M) \subset \operatorname{Ass}_A(A)$. Let $N = B \otimes_A M$ and identify U with its inverse image in $\operatorname{Spec}(B)$, so that $N(U) = M(U)$ and it suffices to show that $N(U)$ is a finitely generated B-module. Since B → B' is faithfully flat, it suffices to show that $N'(U')$ is a finitely generated B'-module, where $N' = B' \otimes_B N$. But, as $\operatorname{Ass}_A(M) \subset \operatorname{Ass}_A(A)$, we have $\operatorname{Ass}_B(N) \cap U \subset \operatorname{Ass}_B(B)$, and consequently, $\operatorname{Ass}_{B'}(N') \cap U' \subset \operatorname{Ass}_{B'}(B')$; it is therefore sufficient to apply EGA, IV, 5.11.1 once more. □

Corollary 1.2. *Suppose that A is reduced, that for every maximal point x' of U', we have* $\operatorname{codim}(\overline{x'} \cap Z', \overline{x'}) \geq 2$ *and that for every* $x' \in \operatorname{Ass}(U')$ *such that* $\operatorname{codim}(\overline{x'} \cap Z', \overline{x'}) = 1$*, we have* $\dim(\overline{x'}) = 1$*. Then B is Noetherian, with completion B'.*

Proof. Indeed, the hypotheses made imply that $\operatorname{Supp}(I') \cap U'$ is discrete, hence affine and that B is integral over A. It remains to show that B' is flat over B. Let R be the total ring of fractions of A, which is a finite product of fields since A is reduced. The Tor sequence, applied to the exact sequence (*), leads for any B-module N, to the following commutative diagram:

$$\begin{array}{ccc} \operatorname{Tor}_1^B(B', N) & \to & J' \otimes_B N \\ \downarrow & & \downarrow \\ \operatorname{Tor}_1^B(B', N) \otimes_B R & \to & J' \otimes_B N \otimes_B R. \end{array}$$

Now, by localization, we have

$$\mathrm{Tor}_1^B(B', N) \otimes_B R \simeq \mathrm{Tor}_1^{B\otimes_B R}(B' \otimes_B R, N \otimes_B R) = 0$$

since R is a product of fields. Furthermore, it follows from the exact sequence (*) and from the fact that B' is flat over B, that the upper horizontal arrow is injective. To see that $\mathrm{Tor}_1^B(B', N)$ is zero, it is therefore sufficient to show that the morphism $J' \otimes_B N \to J' \otimes_B N \otimes_B R$ is bijective, which will result from the fact that J' is in fact an R-module. Indeed, as $T' = \mathrm{Supp}(I') \cap U'$ is discrete, $J' = I'(U')$ is the product of the localizations of I' at the points of T'; for the same reason, T' is formed by maximal points of $\mathrm{Supp}(I')$, so $T' \subset \mathrm{Ass}(I') \subset \mathrm{Ass}(A')$; as $\mathrm{Ass}(A')$ is above $\mathrm{Ass}(A)$ (EGA, IV, 6.3.1), we finally see that J' is indeed an R-module. □

Corollary 1.3. *Let A be a reduced Noetherian local ring and B the ring of sections of $\widetilde{A}$ over the complement of the closed point of* $\mathrm{Spec}(A)$*. If B is integral over A, in particular if A is unibranch and if* $\dim(A) \geq 2$ *(EGA, IV, 18.9.7.5), then B is Noetherian.*

2 Nilradical of the completion of an integral Noetherian local ring

Let A be an integral Noetherian local ring, with maximal ideal $\mathfrak{m}$, field of fractions K and let A' be the completion of A for the m-adic topology. We set S = $\mathrm{Spec}(A)$, S' = $\mathrm{Spec}(A')$ and denote by U the complement of the closed point s of S and by U' its inverse image in S'. If A' has an embedded associated prime ideal $\mathfrak{p}'$, then $\mathrm{Ann}(\mathfrak{p}')$ is an ideal of square zero. On the other hand, if $\dim(A) = 1$, $K \otimes_A A'$ is an Artinian ring, so the property for $K \otimes_A A'$ to be Gorenstein essentially concerns its radical, that is to say the nilradical of A'. To introduce the examples of the following paragraph, we will therefore first study the nilradical of A' by making the additional hypotheses (i) to (iv) below:

(i) The integral closure B of A is a regular local ring.

In particular, B is thus Noetherian and $B/\mathfrak{m}B$ is Artinian; as the residue field of B is finite over that of A, $B/\mathfrak{m}B$ is finite over $A/\mathfrak{m}$; up to replacing A with a finite A-algebra contained in B, which does not modify the fiber $K \otimes_A A'$, we can make the following hypothesis:

(ii) The homomorphism $A/\mathfrak{m} \to B/\mathfrak{m}B$ is an isomorphism.

Let B' = $A' \otimes_A B$. Since A' is complete, $A'_{red} \otimes_{A_{red}} B_{red}$ is finite (EGA, $\mathrm{O_{IV}}$, 23.1.5) and hypothesis (ii) implies that this is an isomorphism. In particular, B'_{red} is Noetherian and therefore regular since its maximal ideal is generated by the image of that of B and B is regular. Since, moreover, B'_{red} is complete and its associated graded ring is isomorphic to that of B since they are both regular and of the same dimension, B'_{red} is the completion of B for the mB-adic topology. For simplicity, we also assume that:

(iii) A contains a field.

We then deduce from the Cohen structure theorem that A' contains a regular subring B" such that the composite homomorphism

$$B'' \to A' \to A'_{red}$$

is an isomorphism. We finally suppose that:

(iv) The nilradical I' of A' has square zero and its support has dimension 1.

We first deduce an isomorphism

$$D_{B'}(I') \simeq A'.$$

Let us then show that under these hypotheses, we have an isomorphism

$$D_{B'}(I'(U')) \simeq B'$$

making the diagram

$$\begin{array}{ccc} D_{B'}(I') & \to & D_{B'}(I'(U')) \\ \downarrow & & \downarrow \\ A' & \to & B' \end{array}$$

commutative, where the top arrow is deduced from the restriction application I' $\to$ I'(U'). Suppose first that A has dimension 1. Then B' is the integral closure of A' in $K \otimes_A A'$ (Appendix 4.3) and therefore contains the nilradical $K \otimes_A I' = I'(U')$ of $K \otimes_A A'$; it is an ideal of square zero and B'_{red} is isomorphic to B', whence the desired isomorphism.

If $\dim(A) \geq 2$, let us first show that U is regular. Let V' be the open set of S' complementary to $\mathrm{Supp}(I')$; since V' is regular, it is sufficient (EGA, $\mathrm{O_{IV}}$, 17.3.3) to show that the flat morphism V' $\to$ U is surjective. It is first clear that its image contains the generic point of U. On the other hand, U' $\to$ U is surjective and if $x' \in U' - V'$, x' is a generic point of $\mathrm{Supp}(I')$ since $\dim(I') = 1$, hence $x' \in \mathrm{Ass}(I')$, from which we deduce that the image of x' in U is the generic point of U (EGA, IV, 6.3.1) and, consequently, that V' $\to$ U is surjective. But then U being regular, is normal, and the morphism $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is an isomorphism over U, so we have A(U) = B(U). But B is a regular ring of dimension ≥ 2, so the restriction homomorphism B $\to$ B(U) is an isomorphism and, consequently, B = A(U).

Since A' is faithfully flat over A, B' = A'(U'); finally, since B' is a regular ring of dimension ≥ 2, B' $\to$ B'(U') is an isomorphism. In short, we indeed have an isomorphism

$$D_{B'}(I'(U')) \simeq B'.$$

The square

$$\begin{array}{ccc} A & \to & B \\ \downarrow & & \downarrow v \\ A' & \to & B' \end{array}$$

is Cartesian since v induces an isomorphism $B/A \simeq B'/A'$. The isomorphisms established previously show that $v = \phi \oplus D$, where ϕ is the canonical homomorphism from B into its completion B' and D is a derivation from B into I'(U'), so that A is the set of $x \in B$ such that $D(x) \in I'$. Finally, I'(U') is, in fact, a K-module and, in particular, a flat B-module. This tedious unwinding has allowed us to identify what is needed to construct the examples we have in mind. Indeed, we have the following converse:

Lemma 2.1. *Let B be a Noetherian local ring with maximal ideal $\mathfrak{n}$, B' its completion for the n-adic topology, $\phi : B \to B'$ the canonical homomorphism and U and U' the complements of the closed points in* $\mathrm{Spec}(B)$ *and* $\mathrm{Spec}(B')$*. Let, on the other hand, L be a finitely generated B'-module and $D : B \to L(U')$ a derivation. Let $A' = D_L(L)$, $B'' = D_{L(U')}(L(U'))$ and let $v : B \to B''$ be the homomorphism defined by*

$$v(x) = (\phi(x), D(x)).$$

Let A be the fiber product of A' and B over B", so that we have the Cartesian square

$$\begin{array}{ccc} A & \xrightarrow{j} & B \\ \downarrow u & & \downarrow v \\ A' & \xrightarrow{j'} & B'' \end{array}$$

where j' is deduced from the restriction application $L \to L(U')$. Assume that:

- *(i) The application $L \to L(U')$ is injective and $L(U')$ is a flat B-module.*
- *(ii) The maximal ideal $\mathfrak{n}$ of B is generated by elements annihilated by D and $B = \mathrm{Ker}(D) + \mathfrak{n}$.*

Then $j : A \to B$ is integral and injective, A is local and the morphism $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ *deduced from j is an isomorphism over the complements of the closed points. Suppose further that:*

- *(iii) The application $B/A \to B'/A'$ deduced from v is surjective (hence bijective).*
- *(iv) The B'-module L is generated by $L \cap D(B) = D(A)$.*
- *(v) A is Noetherian.*

Then the homomorphism $u : A \to A'$ makes A' a completion of A.

Proof. If, taking (i) into account, we identify L with its image in L(U'), A identifies with the subring of B formed by the b such that $D(b) \in L$ and u is then the restriction of v to A. Since A contains the kernel of D, to show that B is integral over A it suffices, according to (ii), to show that every element t of $\mathfrak{n}$ is integral over A. Now, the image of D(t) by the restriction application $L(U') \to L(U'_t) = L_t$ can be written in the form x/t^n with $x \in L$; according to (EGA, I, 9.3.1), there exists an integer $m \geq n$ such that $t^m D(t) \in L$, hence such that $t^{m+1} \in A$. This shows that B is integral over A, so, in particular, A is local. Let $\mathfrak{m}$ be its maximal ideal. According to (ii), there exists a finite family (t_i) of elements of $\mathfrak{n}$ annihilated by D and such that the union of the open sets U'_{t_i} is equal to U'. Let x be an element of B; invoking again (EGA, I, 9.3.1), we see that there exists an integer m such that, for all i, $t_i^m D(x) \in L$, hence such that $t_i^m x \in A$; this shows that the morphism $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is an isomorphism over the complements of the closed points. This proves the first part of the statement.

Let us show that u induces an isomorphism $A/\mathfrak{m} \to A'/\mathfrak{m}A'$. [We must be careful that $\mathfrak{m}A'$ denotes the ideal of A' generated by $u(\mathfrak{m})$; it is, as we will see, distinct from $\mathfrak{m}B' + \mathfrak{m}L$]. It is clear, taking (ii) into account, that the residue fields of A and B are isomorphic, hence also those of A and A'. It is therefore sufficient to show that $\mathfrak{m}A'$ is equal to the maximal ideal $\mathfrak{n}' + L$ of $A' = B' + L$. According to (ii) an element b' of $\mathfrak{n}'$ can be written in the form $b' = \sum b'_i a_i$ with $b'_i \in B'$, $a_i \in \mathfrak{n}$ and $D(a_i) = 0$. In A', we therefore have $(b', 0) = \sum u(a_i)(b'_i, 0) \in \mathfrak{m}A'$. It remains to show that for any $x \in L$, we have $(0, x) \in \mathfrak{m}A'$. Now, L is generated by $D(\mathfrak{m})$ [hypothesis (iv)]; we can therefore write $x = \sum D(c_j)c'_j$ with $c_j \in \mathfrak{m}$ and $c'_j \in B'$. We therefore have

$$(0, x) = \sum_j u(c_j)(0, c'_j) - \sum_j (c_j c'_j, 0).$$

Since $\sum c_j c'_j \in \mathfrak{n}'$, the preceding shows that $(0, x) \in \mathfrak{m}A'$.

We deduce that the canonical application $w' : B/A \to (B/A) \otimes_A A'$ is surjective. Indeed, since $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is an isomorphism on the complements of the closed points, B/A has for support the closed point of A; it is consequently, A being assumed Noetherian, a filtered union of submodules annihilated by a power of $\mathfrak{m}$; hence the result since, according to what precedes, for any integer $n \geq 0$, $A/\mathfrak{m}^n \to A'/\mathfrak{m}^n A'$ is surjective. Let us show that B' is canonically isomorphic to $A' \otimes_A B$. Consider the following commutative diagram where the rows are exact and where $v''v' = v$:

$$\begin{array}{ccccccccc}
0 & \to & A & \to & B & \to & B/A & \to & 0 \\
 & & \downarrow u & & \downarrow v' & & \downarrow w' & & \\
0 & \to & A' & \to & A' \otimes_A B & \to & A' \otimes_A B/A & \to & 0 \\
 & & \| & & \downarrow v'' & & \downarrow w'' & & \\
0 & \to & A' & \to & B' & \to & B'/A' & \to & 0.
\end{array}$$

Hypothesis (iii) states that $w''w'$ is an isomorphism. We have just seen that w' is surjective. We deduce that w'', and consequently, v'', are isomorphisms.

We can finally prove that u makes A' a completion of A for the m-adic topology. Indeed, as L is of finite type, A' is finite over B'; A' is therefore a complete Noetherian local ring. As $A/\mathfrak{m} \to A'/\mathfrak{m}A'$ is an isomorphism, it suffices to show that A' is a flat A-module to be able to conclude. Now, as $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is integral and birational, it suffices ([3], corollary) to show that $A' \otimes_A B = B'$ is flat over B, which is clear since $L(U')$ is flat over B by hypothesis (i). ☐

3 The examples

Proposition 3.1. *For any integer $r \geq 0$, there exists a 1-dimensional integral Noetherian local ring A whose completion A' possesses a minimal prime ideal $\mathfrak{p}'$ of square zero such that $\mathfrak{p}'$ is a free $A'/\mathfrak{p}'$-module of rank r.*

Remark 3.1. (i) An Artinian local ring is Gorenstein if the intersection of two non-zero ideals is non-zero. In particular, if L is a vector space of finite rank ≥ 2 over a field K, $D_K(L)$ is not a Gorenstein ring. We therefore see that in the example above, relative to $r = 2$, if we denote by K the field of fractions of A, $K \otimes_A A'$ is not a Gorenstein ring.

(ii) We will see that the maximal ideal $\mathfrak{m}$ of A is generated by $r + 1$ elements and that, denoting by B the integral closure of A, we have an isomorphism $A/\mathfrak{m} \simeq B/\mathfrak{m}B$. Taking r equal to 1, this therefore provides an example of a complete intersection ring (EGA, IV, 19.3.1) which is not Japanese.

On the other hand, an integral Noetherian local ring whose maximal ideal is generated by two elements is Gorenstein; we see by taking r equal to 2, that this is no longer necessarily true as soon as the maximal ideal is generated by three elements.

Proof. Let $B = \mathbb{C}\{X\}$ be the ring of convergent series with complex coefficients, $B' = \mathbb{C}[[X]]$ its completion and L a free B'-module of rank r, for which we choose a basis (e_i), $1 \leq i \leq r$. The field of fractions K of B is an extension of infinite transcendence degree over $\mathbb{C}$, as can be verified by adapting to convergent series the reasoning of remark 4, page 219 of [5]. Since $\mathbb{C}$ has characteristic zero, K admits a separating transcendence basis over $\mathbb{C}$ that can be chosen to consist of elements of XB. Let $d : K \to \Omega_{K/\mathbb{C}}$ be the exterior differential of K relative to $\mathbb{C}$. There therefore exists a family $(s_{i,n})$, $1 \leq i \leq r$, $n \geq 1$, of elements of XB such that dX and the $ds_{i,n}$ are part of a basis of $\Omega_{K/\mathbb{C}}$. We can therefore construct a $\mathbb{C}$-derivation

$$D : B \to K \otimes_B L$$

such that $D(X) = 0$ and $D(s_{i,n}) = X^{-n} e_i$. We take for A the subring of B formed by the $b \in B$ such that $D(b) \in L$. We must verify conditions (i) to (v) of Lemma 2.1. The verification of (i) and (ii) is immediate. Let's check (iii). We need to show that for any integer $m \geq 0$, for any element e_i of the basis of L and for any $b' \in B'$, there exist $z \in L$ and $b \in B$ such that $X^{-m} b' e_i = z + D(b)$. Now, if $b' = \sum c_n X^n$ with $c_n \in \mathbb{C}$, we can take

$$z = \left(\sum_{n=m}^{\infty} c_n X^{n-m} \right) e_i \quad \text{and} \quad b = \sum_{j=0}^{m-1} c_j s_{i,m-j}.$$

Condition (iv) is satisfied since $D(X^n s_{i,n}) = e_i$. It remains to see that A is Noetherian. Now, conditions (i) and (ii) imply that B is integral over A. As B is integral and of dimension 1, it suffices to show that the maximal ideal $\mathfrak{m}$ of A is of finite type. We will see that it is generated by the $r+1$ elements $X, Xs_{1,1}, \ldots, Xs_{r,1}$. Let a be an element of $\mathfrak{m}$. We have $D(a) = \sum b'_i e_i$ with $b'_i \in B'$. Writing $b'_i = c_i + X b''_i$, with $c_i \in \mathbb{C}$ and $b''_i \in B'$ and $z = \sum b'_i e_i$, we have $D(a - \sum c_i X s_{i,1}) = Xz$. Since $\mathfrak{m} \subset XB$, there exists $b \in B$ such that $a - \sum c_i X s_{i,1} = Xb$; but, in fact, b is in A because $XD(b) = Xz$, so $D(b) = z \in L$; hence the result. □

Proposition 3.2. *There exists an integral Noetherian local ring A of dimension 2 such that the nilradical I' of its completion A' is of square zero and is an A'-module isomorphic to a quotient A'/p' which is a discrete valuation ring. The maximal ideal $\mathfrak{m}$ of A is generated by three elements,* $\operatorname{Spec}(A)$ *is regular outside the closed point, the integral closure B of A is a regular local ring and $A/\mathfrak{m} \to B/\mathfrak{m}B$ is an isomorphism.*

Proof. Let $B = \mathbb{C}\{X, Y\}$ be the ring of convergent series in two variables, with complex coefficients and $B' = \mathbb{C}[[X, Y]]$ its completion. □

Lemma 3.3. *Let $s \in \mathbb{C}[[Y]]$ be a formal series not belonging to $\mathbb{C}\{Y\}$, and let $t = X + Y + Y^2 s$. Then B'/tB' is a discrete valuation ring and $B \cap tB' = 0$.*

Proof. The first assertion comes from the fact that t is not contained in the square of the maximal ideal of B'. It implies that tB' is a prime ideal. So, if $B \cap tB'$ is non-zero, it is a principal ideal since B is factorial and, consequently, there exists $f \in B$ and an invertible element u of B' such that $t = uf$. The order of the series reduced modulo Y of f is 1 from the very form of t. Using the Weierstrass preparation theorem for convergent series ([5], remark, p. 141), we see that there exist $v \in B$ and $g \in \mathbb{C}\{Y\}$ such that $X = fv + g$. In B', this relation is written $X = tu^{-1}v + g$. But, from the definition of t, we also have $X = t - (Y + Y^2 s)$. Using the uniqueness assertion of the Weierstrass theorem in B', we conclude that $Y + Y^2 s = g \in \mathbb{C}\{Y\}$, contrary to the choice of s.

Let's choose t as in 3.4. Then $R' = B'/tB'$ is a complete discrete valuation ring and the composite homomorphism $i : B \to B' \to R'$ is injective. Let π be a uniformizer of R', K' the field of fractions of R' and $|.|$ the normalized valuation on R'. We have $|i(X)| = |i(Y)| = 1$. Since the canonical morphism $\mathbb{C}\{X\} \hookrightarrow \mathbb{C}\{Y\} \to \mathbb{C}\{X, Y\}$ is injective, the transcendence degree of the field of fractions K of $\mathbb{C}\{X, Y\}$ over the field of fractions L of $\mathbb{C}\{X\}$ is at least equal to the transcendence degree of the field of fractions of $\mathbb{C}\{Y\}$ over $\mathbb{C}$, hence is infinite. We deduce (the field $\mathbb{C}$ being of characteristic o) that the module $\Omega_{K/L} = \Omega_{K/\mathbb{C}\{X\}} \otimes_B K$ of algebraic differentials of K relative to L, is a vector space over K of infinite dimension. It is generated by the image in $\Omega_{K/L}$ of $d(B)$, where $d : B \to \Omega_{B/\mathbb{C}\{X\}}$ is the exterior differential. As any element of B can be written in the form $f + Yg$ where $f \in \mathbb{C}\{X\}$ and $g \in B$, we can find a sequence $(b_n)_{n \geq 1}$ of elements of YB such that the images in $\Omega_{K/L}$ of $d(Y)$ and of the $d(b_n)$ form part of a basis of this vector space. Finally, as $i : B \to R'$ is injective, i induces a homomorphism $K \to K'$. There therefore exists a $\mathbb{C}\{X\}$-derivation

$$D : B \to K' \text{ such that } D(Y) = 0 \text{ and } D(b_n) = \pi^{-n} \text{ for } n \geq 1.$$

We will apply Lemma 2.1 to the derivation D and to the B'-module R', so that the sought-after ring A is here the subring of B formed by the $b \in B$ such that $D(b) \in R'$. We must verify that conditions (i) to (v) of the lemma are satisfied. Since the quotient R' of B' plays the role of the module L of statement 2.1, we have $L(U') = K'$ and condition (i) is satisfied. As D annihilates C, X and Y, condition (ii) is also satisfied. Condition (iii) is equivalent to the relation $K' = R' + D(B)$; now, R' is isomorphic to the ring of formal series in π, with coefficients in C; an element of K', of valuation $-r (r \geq 0)$ can thus be written in the form $c_{-r}\pi^{-r} + \cdots + c_{-1}\pi^{-1} + z$, with $c_j \in \mathbb{C}$ and $z \in R'$. As $\pi^{-i} = D(b_i)$, the relation is indeed verified. Let $Z = Xb_1$; since $i(X)$ has valuation 1, $D(Z) = i(X)\pi^{-1}$ is an invertible element of R', hence a generator of the B'-module R'; whence (iv). It remains to verify that A is Noetherian. Let's first show that the maximal ideal $\mathfrak{m}$ of A is generated by X, Y and Z. Let a be an element of $\mathfrak{m}$. As D(Z) is invertible in R', there exists $c \in \mathbb{C}$ such that $D(a - cZ)$ has valuation ≥ 1. We can therefore assume that $D(a)$ has valuation ≥ 1. In $B = \mathbb{C}\{X, Y\}$, we have $a = f + Yg$ where $f \in \mathbb{C}\{X\}$ and $g \in B$; since $D(a) = i(Y)D(g)$ is an element of valuation ≥ 1 and $i(Y)$ an element of valuation 1, $D(g)$ is in R', hence $g \in A$. As A contains $\mathbb{C}\{X\}$, this clearly shows that $\mathfrak{m}$ is generated by X, Y and Z. According to 2.1, conditions (i) and (ii) already imply that B is integral over A and that $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is an isomorphism over the complement of the closed point. We deduce first that A is integral and of dimension 2. To show that A is Noetherian, it suffices to show that every prime ideal of A is of finite type. We have just seen that the maximal ideal of A is of finite type. It is therefore sufficient to examine the prime ideals of height 1. Now, these are the inverse images of the prime ideals of height 1 of B, which are principal since B is factorial. In short, it suffices to show that for any element f of the maximal ideal $\mathfrak{n}$ of B, $\mathfrak{a} = A \cap fB$ is an ideal of finite type. Let N be the set of $b \in B$ such that $fb \in A$. From the relation $D(fb) = i(b)D(f) + i(f)D(b)$, we deduce that the values of $|D(b)|$ for b in N are bounded below. Let $e \in N$ be such that $|D(e)|$ is minimal. For any $b \in N$, we therefore have $|D(b)| = |D(e)| + r$, with $r \geq 0$. As $|i(X)| = 1$, we deduce that there exists a polynomial P in $\mathbb{C}\{X\}$ such that $b - Pe \in A$. To see that $\mathfrak{a}$ is of finite type, it is therefore sufficient to show that the ideal $\mathfrak{c}$ of A formed by the $c \in A$ such that $fc \in A$ is an ideal of finite type: indeed, if $\mathfrak{c}$ is generated by the family (c_i), $\mathfrak{a}$ is generated by fe and the fc_i. Now, as $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is an isomorphism over the complement of the closed point, the closed set $V(\mathfrak{c})$ of $\mathrm{Spec}(A)$ is reduced to the single point $\mathfrak{m}$; as $\mathfrak{m}$ is of finite type, $\mathfrak{c}$ contains a power $\mathfrak{m}^n$ of $\mathfrak{m}$; finally, $A/\mathfrak{m}^n$ is Artinian; whence the result. □

Proposition 3.4. *Let A be the ring constructed in 3.3. Then $C = A[[T]]$ is an integral Noetherian local ring of dimension 3, such that* $\mathrm{Spec}(C)$ *is normal in codimension* ≤ 1 *and which possesses an infinity of prime ideals* $\mathfrak{p}$ *of height 2, such that* $\mathrm{depth}(C_\mathfrak{p}) = 1$. *In particular, the set of points of* $\mathrm{Spec}(C)$ *where the local ring is Cohen-Macaulay (resp. is normal) is not open.*

Proof. Note first that C is an integral Noetherian local ring of dimension 3 and that its completion C' identifies with A'[[T]]. By the choice of A', A' possesses one and only one embedded associated prime ideal, say $\mathfrak{p}'$, hence C' possesses an embedded associated prime ideal $\mathfrak{p}'C'$ and we have $\dim(C'/\mathfrak{p}'C') = 2$. As C is integral and $\mathrm{Ass}(C')$ is over $\mathrm{Ass}(C)$, $\mathfrak{p}'C'$ is over the ideal 0 of C. Consider the prime ideals $\mathfrak{q}'$ of C' strictly containing $\mathfrak{p}'C'$ and not containing T. There exists an infinity of such ideals since $\dim(C'/\mathfrak{p}'C') = 2$, and we have $\dim(C'/\mathfrak{q}') = 1$. It is clear that $C'/\mathfrak{q}' + TC'$ is finite over C. But C is complete for the T-adic topology, so $C'/\mathfrak{q}'$ is finite over C and consequently, if $\mathfrak{q} = C \cap \mathfrak{q}'$, we have $C/\mathfrak{q} \simeq C'/\mathfrak{q}'$. In particular $\dim(C/\mathfrak{q}) = 1$, and $C/\mathfrak{q}$ is complete, so $\mathfrak{q}'$ is the unique prime ideal of C' over $\mathfrak{q}$ and we thus obtain an infinity of distinct prime ideals. We have moreover

$$\dim(C_\mathfrak{q}) = \dim(C'_{\mathfrak{q}'}) = 2 \quad \text{and} \quad \mathrm{depth}(C_\mathfrak{q}) = \mathrm{depth}(C'_{\mathfrak{q}'}) \quad \text{(EGA, IV, 6.1.2 and 6.3.1).}$$

Now $\mathfrak{p}'C'$ is an immediate generization of $\mathfrak{q}'$ which belongs to $\mathrm{Ass}(C')$, hence $\mathrm{depth}(C'_{\mathfrak{q}'}) \leq 1$ and finally $\mathrm{depth}(C_\mathfrak{q}) = 1$. If now r is a prime ideal of C distinct from 0, from the maximal ideal and from the ideals $\mathfrak{q}$ of the preceding type (for example if r is of height 1), and if r' is a prime ideal of C' over r, r' does not contain $\mathfrak{p}'C'$, hence $C'_{r'}$ is regular and a fortiori it is the same for C_r. Proposition 3.1 results immediately from these remarks. □

Proposition 3.5. *There exists a Henselian integral Noetherian local ring C, of dimension 3, and an open set U of Spec(C) such that C(U) is integral over C and such that condition (3) of 1.1 is satisfied, but nevertheless such that C(U) is not Noetherian.*

Proof. Let us take again the ring A of 3.3 and the ring C of 3.5. Let $\mathfrak{m}$ be the maximal ideal of A, S = Spec(C), Z the closed set of S defined by the ideal $\mathfrak{m}C$, S' = Spec(C'), Z' the inverse image of Z in S'. We know that the normalization B of A is a regular local ring and that Spec(A) is normal outside the closed point. By flatness we deduce that $B \otimes_A C = \Gamma(\mathrm{Spec}(A) - \{\mathfrak{m}\}, \mathcal{O}_{\mathrm{Spec}(A)})$. By flatness we deduce that $B \otimes_A C = \Gamma(S - Z, \mathcal{O}_S)$. We will see that $B \otimes_A C$ is not Noetherian. Note first that B is integral over A, so $B \otimes_A C$ is integral over C and condition (1) of 1.1 is satisfied. The support T' of the embedded part of S' is underlying the spectrum of a regular ring of dimension 2 and Z' is of dimension 1 and contained in T', so T' - Z' is affine and condition (3) of 1.1 is also satisfied. It then follows from loc. cit. that the separated completion of $B \otimes_A C$ is equal to C_{red}, and thus is regular. If $B \otimes_A C$ were Noetherian, it would therefore be regular; now if $\mathfrak{q}$ is a prime ideal of C, of height 2 and depth 1, other than $\mathfrak{m}C$ (cf. 3.5), $\mathfrak{q} \notin Z$, so $C_{\mathfrak{q}} \simeq (B \otimes_A C)_{\mathfrak{q}}$ is not regular. □

Remark 3.2. (i) All the examples of rings that we have constructed in paragraph 3 are algebras over the field of complex numbers $\mathbb{C}$ and their "pathology" is therefore not linked to radicial phenomena. The same constructions are valid in characteristic $p > 0$, provided one replaces $\mathbb{C}\{X\}$ by the ring $K\{X\}$ of convergent series with coefficients in a complete non-discrete valued field K of characteristic p; one must however ensure that if L is the field of fractions of $K\{X\}$ and M that of $K\{X, Y\}$, one has $\dim_L \Omega_{M/L} = +\infty$ and $\dim_M \Omega_{M/L} = +\infty$; these conditions are in particular satisfied if $K = k((T))$ with $[k : k^p]$ non-denumerable.

(ii) Proposition 3.3 does not provide a complete answer to the problem of associated prime ideals of the formal fibers of a Noetherian local ring. In particular, it is natural to wonder if A' can have embedded associated prime ideals in the case where A is normal, of dimension ≥ 3.

Appendix: A Descent Result

Proposition 3.6. *Let $X = \mathrm{Spec}(A)$ and $X' = \mathrm{Spec}(A')$ be two affine schemes, $f : X' \to X$ a faithfully flat morphism, Y a closed subscheme of X, $Y' = X' \times_X Y$ its inverse image in X'. We suppose that:*

(i) the morphism $Y' \to Y$ deduced from f is an isomorphism.

(ii) The open set $U = X - Y$ is quasi-compact.

We denote by $j : U \to X$ and $j' : U' = f^{-1}(U) \to X'$ the canonical immersions and by $g : U' \to U$ the restriction of f to U', so that we have the commutative diagram:

$$\begin{array}{ccc} U' & \xrightarrow{g} & U \\ \downarrow j' & & \downarrow j \\ X' & \xrightarrow{f} & X. \end{array}$$

Finally, for any scheme S, we denote by $\mathfrak{M}(S)$ the category of quasi-coherent $\mathcal{O}_S$-Modules.

Proposition 3.7. *Under the above hypotheses, the square of functors*

$$\begin{array}{ccc} \mathfrak{M}(X) & \xrightarrow{j^*} & \mathfrak{M}(U) \\ \downarrow f^* & & \downarrow g^* \\ \mathfrak{M}(X') & \xrightarrow{j'^*} & \mathfrak{M}(U') \end{array}$$

is Cartesian.

For a formal definition of what a Cartesian square of functors is, see [1], p. 358. In an intuitive way, the statement means that a quasi-coherent $\mathcal{O}_X$-Module "descends" by f as soon as its restriction to U' descends by g. We will demonstrate, in fact, the more precise statement as follows: Let (F', u, E) be a triplet where F' is a quasi-coherent $\mathcal{O}_{X'}$-Module, E is a quasi-coherent $\mathcal{O}_U$-Module and $u : g^*E \to F'|_{U'}$ is an isomorphism of $\mathcal{O}_{U'}$-Modules. Let $N' = F'(X')$, so that $F' = \widetilde{N'}$, $M' = F'(U')$ and $M = E(U)$. Consider M, M' and N' as A-modules. We have an A-linear application $m : M \to M'$ deduced from the canonical application $E \to g_* g^* E$ composed with $g_*(u)$. Let r': N' → M' be the restriction application. Let N be the fiber product of M and N' over M', so that we have the Cartesian square

$$\begin{array}{ccc} N & \xrightarrow{n} & N' \\ \downarrow r & & \downarrow r' \\ M & \xrightarrow{m} & M'. \end{array}$$

Then r induces an isomorphism $\widetilde{N}|_U \to E$ and n induces an isomorphism $f^*\widetilde{N} \to F'$. Let $P' = \mathrm{Ker}(r')$, $Q' = \mathrm{Coker}(r')$, $P = \mathrm{Ker}(r)$ and $Q = \mathrm{Coker}(r)$. Note that the supports of the modules $P' = H^0_Y(F')$ and $Q' = H^1_Y(F')$ are contained in Y'. On the other hand, as the square above is Cartesian, n induces an isomorphism $P \simeq P'$; we deduce that the support of P is contained in Y, therefore that the canonical application $P \to A' \otimes_A P$ is an isomorphism using hypothesis (i) and the flatness of f. For the same reason, m induces an injective application $Q \to Q'$, so the support of Q is contained in Y and $Q \to A' \otimes_A Q$ is an isomorphism. On the other hand, since U is quasi-compact and f is flat, we deduce from EGA, III, 1.4.15 an isomorphism $A' \otimes_A M \to (g^*E)(U')$, whence an isomorphism $\overline{m} : A' \otimes_A M \to M'$. Consider the following commutative diagram where the top row is exact since A' is flat over A and the bottom row is exact by definition of P' and Q':

$$\begin{array}{ccccccccc} 0 & \to & A' \otimes P & \to & A' \otimes N & \to & A' \otimes M & \to & A' \otimes Q \to 0 \\ & & \downarrow \overline{p} & & \downarrow \overline{n} & & \downarrow \overline{m} & & \downarrow \overline{q} \\ 0 & \to & P' & \to & N' & \to & M' & \to & Q' \to 0 \end{array}$$

To show that $\overline{n}$ is an isomorphism, it suffices to show that $\overline{p}$, $\overline{m}$ and $\overline{q}$ are isomorphisms. This has already been seen for $\overline{p}$ and $\overline{m}$; we have also seen that q is injective; now, the commutativity of the right-hand square implies that $\overline{q}$ is surjective. Since $\overline{n}$ is an isomorphism and F' is a quasi-coherent module, n induces an isomorphism $f^*\widetilde{N} \to F'$. We deduce that the application $\widetilde{N}|_U \to E$ becomes an isomorphism after the faithfully flat base change $U' \to U$; it is therefore an isomorphism; whence the result.

Corollary 3.8. *Let us keep the hypotheses and notations of 4.1. Let B (resp. B') be the integral closure of A in A(U) [resp. of A' in A'(U')]. Then the canonical homomorphism $A' \otimes_A B \to B'$ is an isomorphism.*

It suffices to apply the preceding result to the $\mathcal{O}_{X'}$-Module $\widetilde{B'}$ and to the $\mathcal{O}_U$-Module $\mathcal{O}_U$.

Corollary 3.9. *Let A be a Noetherian ring, I an ideal of A and A' the completion of A for the I-adic topology. Let $X = \mathrm{Spec}(A)$, $X_0 = V(I)$, $U = X - X_0$, $X' = \mathrm{Spec}(A')$, $X'_0 = V(IA')$ and $U' = X' - X'_0$. Finally, let $g : U' \to U$ be the restriction to U' of the canonical morphism $X' \to X$. Suppose that the pair (X, X_0) is Henselian (EGA, IV, 18.5.5). Then the application $O_f(U) \to O_f(U')$ which to an open and closed part Z of U associates $g^{-1}(Z)$, is bijective.*

Proof. Since the pair (X, X_0) is Henselian, I is contained in the radical of A (EGA, IV, 18.5.7) so A' is faithfully flat over A and, consequently, $g : U' \to U$ is surjective. This already shows that the application $O_f(U) \to O_f(U')$ is injective. Let B be the integral closure of A in A(U) and (B_i) the inductive system of finite sub-A-algebras of B, so that $Y = \mathrm{Spec}(B)$ is the projective limit of the system of $Y_i := \mathrm{Spec}(B_i)$. According to 4.3, $A' \otimes_A B$ is the integral closure of A' in A'(U'). Since the set of open and closed parts of a scheme is in one-to-one correspondence with the set of idempotents of its ring of global sections, we have a bijection

$$O_f(Y') \to O_f(U')$$

where $Y' = \mathrm{Spec}(A' \otimes_A B)$. By the same remark, we deduce the bijections

$$\varinjlim O_f(Y_i) \simeq O_f(Y) \quad \text{and} \quad \varinjlim O_f(Y'_i) \simeq O_f(Y'),$$

where $Y'_i = \mathrm{Spec}(A' \otimes_A B_i)$. As the pairs (X, X_0) and (X', X'_0) are Henselian, we have, taking into account the isomorphism $X_0 \to X'_0$, the bijections

$$O_f(Y_i) = O_f(Y_i \times_X X_0) \simeq O_f(Y'_i \times_{X'} X'_0) = O_f(Y'_i).$$

Hence the result. □

Remark 3.3. (i) The only interest of these results is that they do not call upon any hypothesis on the formal fibers; when the rings are supposed to be excellent and of characteristic zero, or when one can use Artin's approximation theorem, one obviously obtains much more profound results (cf. EGA, IV, 18.9).

(ii) If A is a unibranch integral Noetherian local ring of dimension 1, of field of fractions K, of completion A', one deduces from what precedes that $A' \otimes_A K$ is local. Recall that Nagata constructed an example of a normal Noetherian local ring of dimension 2 such that $\mathrm{Spec}(A' \otimes_A K)$ is not even connected.

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