Gabber's work on local uniformization and the étale cohomology of quasi-excellent schemes.

Séminaire à l'École polytechnique 2006–2008

dirigé par

Luc Illusie, Yves Laszlo et Fabrice Orgogozo

Avec la collaboration de Frédéric Déglise, Alban Moreau, Vincent Pilloni, Michel Raynaud, Joël Riou, Benoît Stroh, Michael Temkin et Weizhe Zheng.

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Résumé
Les travaux d'Ofer Gabber présentés dans ce volume comportent deux parties étroitement liées, l'une, géométrique, l'autre, cohomologique. La première est constituée de théorèmes d'uniformisation locale, affirmant que tout couple formé par un schéma noethérien quasi-excellent et un fermé rare devient, après localisation par des morphismes étales et des altérations convenables, isomorphe au couple formé par un schéma régulier et un diviseur à croisements normaux. Il s'agit de résultats de nature locale, mais leur démonstration fournit des corollaires globaux, raffinant des théorèmes d'altération de de Jong pour les schémas de type fini sur un corps ou un anneau de Dedekind. Des techniques de géométrie logarithmique, et, pour les résultats les plus fins, de désingularisation canonique en caractéristique nulle jouent un rôle clef dans les démonstrations. Dans la seconde partie, on donne des applications, accompagnées d'exemples et contre-exemples, à des théorèmes de finitude (abéliens), de dimension cohomologique, et de dualité en cohomologie étale sur les schémas quasi-excellents. On y démontre notamment la conjecture de dualité locale de Grothendieck, et, par une nouvelle méthode, sa conjecture de pureté cohomologique absolue. Des résultats de rigidité et finitude non abéliens sont également établis dans les derniers exposés.

Abstract
The work of Ofer Gabber presented in this book can be divided roughly into two closely related parts, a geometric one and a cohomological one. The first part contains local uniformization theorems which state that any pair consisting of a quasi-excellent Noetherian scheme and a nowhere dense closed subscheme becomes isomorphic, after localization by suitable étale morphisms and alterations, to a pair consisting of a regular scheme and a normal crossings divisor. These are local results, but their proofs have global theorems as corollaries, refining alteration theorems of de Jong for schemes of finite type over a field or a Dedekind ring. Techniques from logarithmic geometry and, as regards the finest results, canonical desingularization in characteristic zero, play a key role in the proofs. In the second part, we give applications, with examples and counter-examples, to Abelian finiteness theorems, as well as theorems on cohomological dimension and duality in étale cohomology over quasi-excellent schemes. In particular, Grothendieck's local duality conjecture is proved, and his absolute cohomological purity conjecture is proved by a new method. Non-Abelian rigidity and finiteness results are also established in the final exposés.

Introduction

The present volume collects the talks of a working group that was held at the École polytechnique from spring 2006 to spring 2008 on the work of Gabber presented in [**Gabber, 2005a**] and [**Gabber, 2005b**]. These deal with étale cohomology and the uniformization of quasi-excellent schemes.

Regarding étale cohomology, a central result is the following finiteness theorem:

Théorème 1. *Let Y be a quasi-excellent noetherian scheme (cf.* **I**-*2.10), $f : X \to Y$ a morphism of finite type, n an integer invertible on Y, and F a constructible sheaf of $\mathbf{Z}/n\mathbf{Z}$-modules on X. Then, for all q, $\mathrm{R}^q f_\star F$ is constructible, and there exists an integer N such that $\mathrm{R}^q f_\star F = 0$ for $q \geq N$.*

Recall that, without any hypothesis on Y or on n, but when one assumes f is proper and of finite presentation, the sheaves $\mathrm{R}^q f_\star F$ are constructible, and zero for $q > 2d$ if d bounds the dimension of the fibers of f [**SGA 4** xiv 1.1]. If f is not assumed to be proper, the hypothesis that n is invertible on Y is essential: if k is an algebraically closed field of characteristic $p > 0$ and X is the affine line over $Y = \mathrm{Spec}(k)$, $\mathrm{H}^1(X, \mathbf{Z}/p\mathbf{Z})$ is an infinite-dimensional $\mathbf{F}_p$-vector space (cf. [**SGA 4** xiv 1.3]). For Y excellent of characteristic zero, 1 is proven in Artin's talk [**SGA 4** xix]. If S is a regular noetherian scheme of dimension ≤ 1 (not necessarily quasi-excellent) and f is a morphism of S-schemes of finite type, the conclusion of 1 is still true, according to [**SGA 4½** [Th. finitude] 1.1] (this result can moreover be deduced from 1, cf. talk XIII).

Artin's proof in [**SGA 4** xix] uses Hironaka's resolution of singularities to reduce to the case where f is the inclusion of the complement of a regular divisor in a regular scheme and F is a constant sheaf, in which case the conclusion follows from the absolute cohomological purity theorem also established in *loc. cit.*

Gabber's proof of 1 follows the same method, but:

(a) one must appeal to the absolute cohomological purity theorem established in the general case by Gabber [**Fujiwara, 2002**],

(b) resolution of singularities in Hironaka's form is no longer available; it is replaced by a local uniformization theorem, due to Gabber ([**Gabber, 2005b**]), which is stated as follows (**IX**-1.1):

Théorème 2. *Let X be a quasi-excellent noetherian scheme, Z a rare closed subset of X and ℓ a prime number invertible on X. There exists a finite family of morphisms $(p_i : X_i \to X)_{i \in I}$, which is a covering for the topology of ℓ'-alterations and such that, for all $i \in I$:*

(i) X_i is regular and integral,

(ii) $p_i^{-1}(Z)$ is the support of a strict normal crossings divisor.

The topology of ℓ'-alterations is a topology of the type considered by Voevodsky (cf. [**Goodwillie & Lichtenbaum, 2001**]), for which ℓ'-alterations (i.e. proper, surjective, generically finite morphisms whose generic residual degree is prime to ℓ) and completely decomposed étale coverings (*i.e. Nisnevich*) are covering families; see (**II**-2.3) for a precise definition.

The first part of this volume is devoted, after some recollections in I on the notions of quasi-excellent or excellent schemes, to the proof of 2 and of some complements and variants.

There are three main steps.

(A) *Fibration into curves*. The question being local for the Nisnevich topology, and in particular for the Zariski topology, we can assume X has finite dimension. We argue by induction on the dimension d of X. We can assume X is a local henselian scheme. The quasi-excellence hypothesis on X allows us to apply Popescu's theorem ([**Popescu, 1986**], **I**-10.3) to the completion $\widehat{X}$ of X, and to reduce, via approximation techniques due to Gabber, explained in III, to the case where X is a complete local scheme of dimension d, and even integral, normal, with $d \geq 2$. We then have the classical Cohen structure theorems. As they are, however, they are insufficient. But a delicate refinement, due to Gabber, proven in IV, allows us to reduce, up to replacing X by a finite extension of generic degree prime to ℓ, to the case where X is the completion at a closed point of a scheme X' of finite type and relative dimension 1 over a complete regular noetherian local scheme of dimension $d-1$, and the closed set Z is the completion of a rare closed subset Z' of X'. This "fibration", or "partial algebrization", theorem is established in V. After a few more easy reductions, we are reduced to the case where X is a normal integral scheme, proper over an excellent, normal, integral affine scheme Y of dimension $d-1$, with geometrically integral, smooth generic fiber of dimension 1, and the closed set Z is a divisor with étale generic fiber.

(B) *de Jong and log regularity.* We can then apply de Jong's nodal curve theorem, in its equivariant form, [**de Jong, 1997**, 2.4]: there exists a finite group G and a G-equivariant projective alteration of the morphism $f : X \to Y$ into a morphism $f' : X' \to Y'$, which is a G-equivariant projective nodal curve, and the inverse image Z' of Z is a divisor with dominant étale components transverse to the smooth locus of f' (**IX**-1.2). Applying the induction hypothesis to the quotient of Y' by a Sylow ℓ-subgroup of G, and using Abhyankar's lemma, we finally reduce to the case where G is an ℓ-group, $X = X'/G$, $Y = Y'/G$, Y' contains a (G-equivariant) closed subset T' such that the pair (Y', T') is *log regular*, as is the pair $(X', f'^{-1}(T') \cup D)$, where D is a (G-equivariant) divisor which is étale over Y', with $Z' \subset f'^{-1}(T') \cup D$ (cf. **VI**-1.9, **VI**-2.1).

(C) *Equivariant modifications.* The quotient of a log regular log scheme by the action of a finite group is not in general log regular. In particular, $X = X'/G$, endowed with the closed set $(f'^{-1}(T') \cup D)/G$, is not necessarily log regular. If this pair were log regular, the Kato-Nizioł desingularization of log regular schemes ([**Kato, 1994**], [**Nizioł, 2006**]), which generalizes the classical desingularization of toric varieties [**Kempf et al., 1973**], would complete the proof of 2. Gabber has identified sufficient conditions ensuring that the quotient of a log regular log scheme by a finite group is still log regular. This is the notion of a *very tame action*, studied in VI. Gabber shows that there exists a G-equivariant projective modification $p : X'' \to X'$ and a G-equivariant closed subset D'' of X'' containing the inverse image of $f'^{-1}(T') \cup D$ such that the pair (X'', D'') is log regular, and the action of G on (X'', D'') is very tame. One then concludes by applying the Kato-Nizioł desingularization to the quotient of (X'', D'') by G. The proof of this modification theorem is given in VIII. It relies on the theory of canonical desingularizations in characteristic zero (Hironaka, Bierstone-Milman, Villamayor, Temkin).

We establish in **X**-1.1 a *relative variant* of this modification theorem, due to Gabber, where the considered G-equivariant log scheme is not only log regular, but log smooth over a log regular base S, with a trivial action of G: one can construct an equivariant modification respecting log smoothness over S. From this, we deduce in particular the following refinements (also due to Gabber) of theorems of de Jong (**X**-2.1, **X**-2.4, **X**-3.5):

Théorème 3. (1) *Let X be a separated scheme of finite type over a field k, $Z \subset X$ a rare closed subset, ℓ a prime number $\neq$ car(k). Then there exists a finite extension k' of k of degree prime to ℓ and an ℓ'-alteration $h : X' \to X$ over $\operatorname{Spec} k' \to \operatorname{Spec} k$, with X' smooth and quasi-projective over k', and $h^{-1}(Z)$ the support of a strict normal crossings divisor. If k is perfect, one can take $k' = k$ and choose h to be generically étale.*

(2) *Let S be a separated, integral, excellent, regular noetherian scheme of dimension 1, with generic point η, X a separated, flat scheme of finite type over S, ℓ a prime number invertible on S, $Z \subset X$ a rare closed subset. Then there exists a finite extension η' of η of degree prime to ℓ, and a projective ℓ'-alteration $h : \tilde{X} \to X$ over $S' \to S$, where S' is the normalization of S in η', with $\tilde{X}$ regular and quasi-projective over S', a strict normal crossings divisor $\tilde{T}$ on $\tilde{X}$, and a finite closed subset Σ of S' such that:*

(i) *outside of Σ, the morphism $\tilde{X} \to S'$ is smooth and $\tilde{T} \to S'$ is a relative normal crossings divisor;*

(ii) *locally for the étale topology around each geometric point x of $\tilde{X}_{s'}$, where s' belongs to Σ, the pair $(\tilde{X}, \tilde{T})$ is isomorphic to the pair formed by*

$$X' = S'[t_1, ..., t_n, u_1^{\pm 1}, ..., u_s^{\pm 1}]/(t_1^{a_1} \cdots t_r^{a_r} u_1^{b_1} \cdots u_s^{b_s} - \pi),$$

$$T' = V(t_{r+1} \cdots t_m) \subset X'$$

around the point $(u_i = 1)$, $(t_j = 0)$, *with* $1 \leq r \leq m \leq n$, *for strictly positive integers* $a_1, \cdots, a_r, b_1, \cdots, b_s$ *such that* $\gcd(p, a_1, \cdots, a_r, b_1, \cdots, b_s) = 1$, *where* p *denotes the characteristic exponent of* s', *and* π *is a local uniformizer at* s';

(*iii*) $h^{-1}(Z)_{\mathrm{réd}}$ *is a sub-divisor of* $\tilde{T} \cup \bigcup_{s' \in \Sigma} \tilde{X}_{s'}$.

Various generalizations, due to Temkin, of this theorem and of the Abramovich-Karu theorem on weak semi-stable reduction in characteristic zero (see [**Abramovich & Karu, 2000**]) are given in section 3 of talk X.

The uniformization theorem 2 has the following corollary, called the "weak" uniformization theorem, where the prime number ℓ no longer appears:

Corollaire 4. *Let* X *be a quasi-excellent noetherian scheme,* Z *a rare closed subset of* X. *There exists a finite family of morphisms* $(p_i : X_i \to X)_{i \in I}$, *which is a covering for the topology of alterations and such that, for all* $i \in I$:

(*i*) X_i *is regular and integral,*

(*ii*) $p_i^{-1}(Z)$ *is the support of a strict normal crossings divisor.*

The topology of alterations is defined analogously to that of ℓ'-alterations. It is finer than the étale topology, and no constraint is imposed on the degree of the generic residual extensions, cf. **II**-2.3.3. One can prove 4 independently of 2, by following only steps (A) and (B) described above, with the help, in (A), of a weak form of the Cohen-Gabber theorem. The modification theorem of (C) is unnecessary, there is no need to appeal to canonical resolution of singularities in characteristic zero. The Kato-Nizioł resolution of toric singularities is sufficient. This proof is presented in VII.

The proof of 1 is given in (**XIII**-3), as an application of 2 and the absolute cohomological purity theorem. The statement of 1 contains two assertions:

(i) the constructibility of $\mathrm{R}^q f_\star F$ for all q,

(ii) the existence of an integer N (depending on (f, F)) such that $\mathrm{R}^q f_\star F = 0$ for $q \geq N$, in other words, the fact that $\mathrm{R}f_\star$ sends $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X, \mathbf{Z}/n\mathbf{Z})$ to $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(Y, \mathbf{Z}/n\mathbf{Z})$, where the index c denotes the full subcategory of D^{b} formed by complexes with constructible cohomology.

In (**XIII**-3), these two assertions are proven simultaneously. One can however prove (i) by invoking only the weak uniformization theorem 4 (and the purity theorem). This is done in (**XIII**-2). The idea is as follows. If, in 4, the morphisms p_i were proper, one could, after reducing to the case where f is an open immersion and F a constant sheaf, reduce, by proper cohomological descent, to the case of the immersion of the complement of a strict normal crossings divisor in a regular scheme, which is amenable to the purity theorem. However, the p_i are not proper in general. Gabber bypasses this difficulty with the help of Deligne's generic constructibility theorem [**SGA 4½** [Th. finitude] 1.9 (i)] and a "hyper-base change" theorem (**XII$_\mathbf{A}$**-2.2.5). This theorem is deduced in *loc. cit.* from an "oriented" cohomological descent theorem (**XII$_\mathbf{A}$**-2.2.3), which uses the notion of *oriented product of topoi*, due to Deligne [**Laumon, 1983**]. The basic definitions and properties are recalled in talk XI. A key result is a base change theorem for "tubes" (**XI**-2.4). A proof of **XII$_\mathbf{A}$**-2.2.5 independent of the notion of oriented product, due to W. Zheng, is given in talk $\mathrm{XII_B}$.

According to classical examples by Nagata, a quasi-excellent noetherian scheme does not necessarily have finite dimension. If Y has finite dimension, then $\mathrm{R}f_\star$ has finite cohomological dimension, according to a theorem by Gabber, presented in talk $\mathrm{XVIII_A}$, and consequently (ii) follows from (i). Gabber proves, more precisely, that if X is a strictly local noetherian scheme of dimension $d > 0$, and ℓ is a prime number invertible on X, then, for any open subset U of X, we have $\mathrm{cd}_\ell(U) \leq 2d - 1$.

The quasi-excellence hypothesis in 1 is essential, as shown by the example given in XIX, of an open immersion $j : U \to X$ of noetherian schemes, with 2 invertible on X, such that $\mathrm{R}^1 j_\star(\mathbf{Z}/2\mathbf{Z})$ is not constructible.

Gabber has proven variants of 1 for sheaves of sets or non-commutative groups [**Gabber, 2005a**]:

Théorème 5. *Let* $f : X \to Y$ *be a morphism of finite type between noetherian schemes. Then:*

(1) *For any constructible sheaf of sets* F *on* X, $f_\star F$ *is constructible.*

(2) *If Y is quasi-excellent, and if L is a set of prime numbers invertible on Y, for any constructible sheaf of groups with L-torsion F on X, $\mathrm{R}^1 f_\star F$ is constructible.*

The proof is given in talk XXI. It does not appeal to the preceding uniformization theorems. It uses, for the key point, an ultraproduct technique, and a rigidity theorem for non-abelian coefficients, also due to Gabber, which is established in talk XX. This theorem is a variant of a theorem by Fujiwara-Gabber for abelian coefficients [**Fujiwara, 1995**, 6.6.4, 7.1.1] (mentioned in *loc. cit.*, 6.6.5). It is stated as follows:

Théorème 6. *Let (X, Y) be a henselian pair, where $X = \operatorname{Spec} A$, $Y = V(I)$, the ideal I being assumed to be of finite type, $\widehat{X}$ the I-adic completion of X, U an open subset of X containing $X - Y$, and $\widehat{U} = \widehat{X} \times_X U$. Then, for any ind-finite stack in groupoids C on U, the restriction map $\Gamma(U, C) \to \Gamma(\widehat{U}, C|\widehat{U})$ is an equivalence.*

The proof given in talk XX is independent of [**Fujiwara, 1995**].

The rest of the volume is devoted to three other applications of the uniformization theorems.

(a) *Affine Lefschetz*. These are generalizations of the theorems from [**SGA 4** xiv 3.1] (for affine morphisms between schemes of finite type over a field) and [**SGA 4** xix 6.1] (for affine morphisms of finite type of excellent schemes of characteristic zero), as well as Gabber's theorem for affine morphisms of schemes of finite type over the spectrum of a DVR [**Illusie, 2003**]. The main statement is as follows (**XV**-1.1.2):

Théorème 7. *Let $f : X \to Y$ be an affine morphism of finite type, where Y is a quasi-excellent noetherian scheme, endowed with a dimension function δ_Y, and let n be an integer invertible on Y. Then, for any constructible sheaf F of $\mathbf{Z}/n\mathbf{Z}$-modules on X, and any integer q, we have:*

$$\delta_Y(\mathrm{R}^q f_\star F) \leq \delta_X(F) - q \tag{7.1}$$

A *dimension function* δ on a scheme T is a function $\delta : T \to \mathbf{Z}$ such that $\delta(y) = \delta(x) - 1$ if y is an immediate étale specialization of x. This notion is due to Gabber. It generalizes that of *rectified dimension* introduced in [**SGA 4** xiv]. It is defined and studied in talk XIV. In (7.1), the function δ_X is related to δ_Y by $\delta_X(x) = \delta_Y(f(x)) + \mathrm{deg.\,tr.}(k(x)/k(f(x))$, and for a sheaf G on X (resp. Y) $\delta_X(G)$ (resp. $\delta_Y(G)$) denotes the upper bound of $\delta_X(x)$ (resp. $\delta_Y(x)$) for $G_x \neq 0$. The proof of 7 is done by reduction to Gabber's theorem cited above, for schemes of finite type over the spectrum of a DVR, with the help of the "weak" uniformization theorem 4 and the hyper-base change theorem ($\mathrm{XII}_{\mathrm{A,B}}$). From 7 follows:

Corollaire 8. *If X is a local noetherian, henselian, quasi-excellent scheme of dimension d, with residue field k, and if U is an affine open subset of X, then, for any prime number ℓ invertible on X, we have*

$$\mathrm{cd}_\ell(U) \leq d + \mathrm{cd}_\ell(k). \tag{8.1}$$

In particular, if X is *strictly* henselian, integral, with fraction field K, we deduce $\mathrm{cd}_\ell(K) = d$ when $\ell \neq 2$, a formula conjectured in [**SGA 4** x 3.1]. More generally, the virtual cohomological dimension $\mathrm{vcd}_\ell(K)$ is equal in the henselian case to $\mathrm{vcd}_\ell(k) + d$, for any ℓ invertible on X, and the same for the usual cohomological dimension cd_ℓ, when $\ell \neq 2$[(i)]. The possible values of $\mathrm{cd}_\ell(U)$, for U an open, not necessarily affine, subset of X are studied in $\mathrm{XVIII}_{\mathrm{A}}$ and $\mathrm{XVIII}_{\mathrm{B}}$. Gabber also gives counter-examples to (8.1) when the quasi-excellence hypothesis is omitted.

(b) *A new proof of the absolute purity conjecture*. The proof of this conjecture given in [**Fujiwara, 2002**] uses, in its last part, techniques from algebraic K-theory (results by Thomason). Gabber announced in [**Gabber, 2005b**] that any recourse to algebraic K-theory can be avoided, by using, instead, the refined form of de Jong's theorem 3(2). This new proof is presented in detail in talk XVI. This talk also contains a theory of generalized fundamental classes (due to Gabber), used to construct a theory of Gysin morphisms for lissifiable complete intersection morphisms, generalizing the constructions of [**SGA 4½** [Cycle]].

(i) This formula then fails for $\ell = 2$ if k is a formally real field of finite virtual cohomological 2-dimension and K is not formally real, for example for the henselization or the completion of the local ring at the origin of the real variety $\sum_{i=1}^n X_i^2 = 0$, $n \geq 2$.

(c) *Dualizing complexes*. The notion of a dualizing complex is due to Grothendieck. The uniqueness, existence, and general properties of dualizing complexes are studied in [**SGA 5** I]. However, in *loc. cit.* existence is only established in characteristic zero, or under hypotheses of the existence of resolution of singularities, and of the validity of the absolute purity theorem (conjectural at the time). In the case of schemes of finite type over a regular scheme of dimension ≤ 1, existence is proven unconditionally by Deligne in [**SGA 4½** [Dualité]]. In the general case, existence, and the local duality theory that results from it, were announced by Gabber in [**Gabber, 2005b**]. Talk XVII presents this theory in detail. If X is a noetherian scheme, and $\Lambda = \mathbf{Z}/n\mathbf{Z}$, where n is an integer invertible on X, a *dualizing complex* on X is an object of $\mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X,\Lambda)$ such that the functor $\mathrm{D}_K = \mathrm{R}\,\mathbf{Hom}(-,K)$ sends $\mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X,\Lambda)$ to $\mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X,\Lambda)$ and that, for any $L \in \mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X,\Lambda)$, the biduality map $L \to \mathrm{D}_K\,\mathrm{D}_K(L)$ is an isomorphism. This definition differs slightly from that of [**SGA 5** I 1.7], see (**XVII**-0.1). Uniqueness, up to a shift and twisting by an invertible Λ-module, is proven in [**SGA 5** I 2.1]. The main result of talk XVII is that, if X is excellent and admits a dimension function, in the sense specified above, then X admits a dualizing complex. Moreover, these dualizing complexes have the expected functoriality properties, and, if X is regular, the constant sheaf Λ_X is dualizing. This last assertion was a conjecture in [**SGA 5** I], proven in the characteristic zero case. We refer the reader to the introduction of talk XVII for more complete statements, and for indications on the proof method, whose essential ingredients are the finiteness theorem 1 and the partial algebrization theorem of talk V (see (A) *supra*).

The results presented in this volume and their proofs are due to Ofer Gabber, with a few exceptions, which are specified in the appropriate place in the corresponding talks.

Luc Illusie
Yves Laszlo
Fabrice Orgogozo

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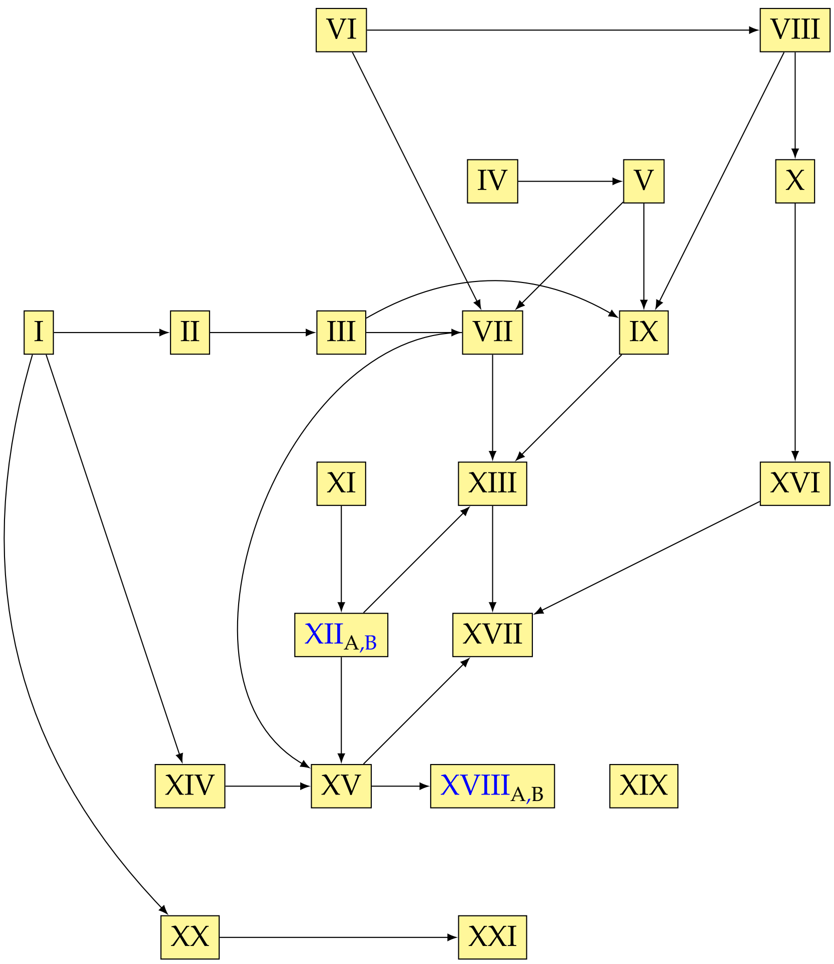

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TALK I

Excellent Rings

Michel Raynaud, notes by Yves Laszlo

This text is a slightly modified version of a talk by Michel Raynaud.

1. Introduction

The goal is to familiarize the reader with the notion of excellence and to provide a guide for navigating **ÉGA IV**, where the main properties of excellent rings can be found. Its ambition is certainly not to give a complete exposition of the theory, but an idea of the strategy which essentially reduces the proofs to statements, often difficult, in the complete case. In a second part, we show that all the properties defining excellent rings can fail, even in small dimensions. In particular, there exist non-excellent discrete valuation rings as well as one-dimensional noetherian integral domains whose regular locus is not open. This last example is a by-product of a construction proposed by Gabber (**XIX**-2.6). It shows that the constructibility theorem for direct images (**XIII**-1.1.1) is no longer true if one omits the quasi-excellence condition.

2. Definitions

Let A be a noetherian ring and $X = \mathrm{Spec}(A)$ its spectrum. We will be interested in two kinds of conditions on X.

• **Global conditions:**

2.1. Condition 1: Openness conditions. *Every integral scheme Y which is finite over X contains a dense open subset that is*

1.a) *regular.*
1.b) *normal.*

Remarque 2.2. Condition 1.a) implies, by Nagata's openness criterion, that the regular locus of any scheme finite over X is open ([**ÉGA** IV$_2$ 6.12.4]). Similarly, condition 1.b) implies that the normal locus of any scheme finite over X is open ([**ÉGA** IV$_2$ 6.13.7]) (i). These openness criteria also ensure that to test 1.a) or 1.b), one can restrict to integral schemes Y that are moreover finite *and* radicial over X.

• **Local conditions.**
They are of two types.

2.3. Condition 2: Conditions on the formal fibers. *For every* closed *point x of X, the completion morphism* (ii) $\mathrm{Spec}(\widehat{\mathscr{O}_x}) \to \mathrm{Spec}(\mathscr{O}_x)$ *is*

2.a) *regular.*
2.b) *normal.*
2.c) *reduced.*

A ring satisfying 2.a) is called a "G-ring" in English, in honor of Grothendieck who highlighted the importance of the notion and studied its properties.

(i) And in fact, 1.a) (resp. 1.b)) implies that the regular (resp. normal) locus of any integral scheme of finite type over X is open

(ii) Its fibers are called the formal fibers (of X or A) at x.

Remarque 2.4. Recall that a morphism of noetherian schemes is called **regular** (resp. **normal**, **reduced**) if it is flat and if the geometric fibers at every point of the base are regular (resp. normal, reduced). We say that the formal fibers of X at x are geometrically regular, geometrically normal, or geometrically reduced if the completion morphism $\mathrm{Spec}(\widehat{\mathcal{O}_x}) \to \mathrm{Spec}(\mathcal{O}_x)$ is regular, normal, or reduced. Of course, it is sufficient to test the regularity, normality, or reduction of the fibers after a finite radicial base change ([**ÉGA** IV$_2$ 6.7.7]). Note that the closed fiber of $\mathrm{Spec}(\widehat{\mathcal{O}_x}) \to \mathrm{Spec}(\mathcal{O}_x)$ is the spectrum of the residue field $k(x)$: it is always geometrically regular. The formal fiber at $y \in \mathrm{Spec}(\mathcal{O}_x)$ identifies with the *generic* formal fiber of the closed subscheme $\overline{\{y\}}$ (endowed with its reduced structure), the closure of y in $\mathrm{Spec}(\mathcal{O}_x)$; this explains the interest in the literature in the generic formal fibers of integral domains. In the case where A is local but not a field, they can have arbitrary dimensions between 0 and $\dim(A) - 1$ and contain closed points of different heights, even in the excellent (2.10) regular case ([**Rotthaus, 1991**]). In the case where A is a localization of an integral algebra of finite type over a field, the dimension of the generic formal fiber is indeed $\dim(A) - 1$ ([**Matsumura, 1988**]).

2.5. Condition 3: Formal catenarity condition. *For every closed point y of an irreducible closed subscheme Y of X, the completion*(iii) $\widehat{\mathcal{O}_{Y,y}}$ *is equidimensional.*

We then say that X is **formally catenary**. For example, if X is of dimension 1, X is formally catenary.

Exemple 2.6. Every *complete* noetherian local ring is formally catenary.

Lemme 2.7. *Let A be a noetherian local ring with an equidimensional completion. Then*

(i) *A is equidimensional and catenary.*
(ii) *For any ideal I of A, the quotient A/I is equidimensional if and only if its completion is; in particular, A/I is formally catenary.*
(iii) *In particular, a formally catenary noetherian affine scheme X is catenary and even universally catenary.*

Recall that X is said to be **catenary** if all saturated chains of irreducible closed subsets of X having the same endpoints have the same length, and **universally catenary**(iv) if every affine scheme of finite type over X is catenary. Catenarity is a local notion. The terminology of **formal catenarity** is then justified by the following elementary proposition ([**ÉGA** IV$_2$ 7.1.4]), a proposition which results from the faithful flatness of the completion morphism.

Note that (iii) follows immediately from (i) since X is catenary if and only if its irreducible components are. We will see later in section 5 that the property of formal catenarity is notably stable under finite extension, hence the announced universal catenarity (cf. the proof of proposition 7.1 and, for a converse, see (7.1.1)).

Exemple 2.8. Let $B \to A$ be a surjective local morphism of noetherian rings and assume B is Cohen-Macaulay (for example, regular). Since $\widehat{B}$ is Cohen-Macaulay, it is equidimensional, so A is formally catenary according to (2.7).

Let's see what happens in the complete case. For the record, let's recall the Cohen structure theorem for complete noetherian local rings ([**ÉGA** 0_{IV} 19.8.8]):

Théorème 2.9 (Cohen). *Let A be a complete noetherian local ring with residue field k.*

(iii) Of course, even if $\mathcal{O}_{Y,y}$ is an integral domain, its completion is generally not an integral domain: think of a nodal curve.

(iv) This last notion is useful in dimension theory: if A is a universally catenary integral domain contained in an integral domain B of finite type over A, we have for any $\mathfrak{p} \in \mathrm{Spec}(B)$ over $\mathfrak{q} \in \mathrm{Spec}(A)$ the formula $\dim B_{\mathfrak{p}} + \mathrm{deg.\,tr.}_{k(\mathfrak{q})}\, k(\mathfrak{p}) = \dim A_{\mathfrak{q}} + \mathrm{deg.\,tr.}_A\, B$. But, in practice, one rather tests for formal catenarity which, as we will see shortly, implies universal catenarity, and is even equivalent to it (see (7.1.1) below)!

(i) *A is isomorphic to a quotient of a formal power series ring over a Cohen ring* [(v)]. *If A contains a field, it is isomorphic to a quotient of a formal power series ring over k.*
(ii) *If A is also an integral domain, there exists a subring B isomorphic to a formal power series ring over a Cohen ring or a field* [(vi)] *such that the inclusion $B \to A$ is local, finite, and induces an isomorphism of residue fields.*

Every complete noetherian local ring is therefore a quotient of a regular ring.

Definition 2.10. Let X be a noetherian scheme (resp. $X = \mathrm{Spec}(A)$ an affine scheme). We say that X (resp. A) is

- **excellent** if X satisfies 1.a) + 2.a) + 3).
- **quasi-excellent** if X satisfies 1.a) + 2.a).
- **universally Japanese** [(vii)] if X satisfies 1.b) + 2.c).

2.11. The existence of a class of schemes stable under finite extension for which the desingularization theorem holds requires us to restrict to quasi-excellent schemes. Precisely, if all integral and finite schemes Y over X admit a desingularization (in the sense of the existence of $Y' \to Y$ proper and birational with Y' regular), then X is quasi-excellent ([**ÉGA** IV_2 7.9.5]) [(viii)]. Conversely, Hironaka's desingularization theorem generalizes to any reduced quasi-excellent scheme of characteristic zero ([**Temkin, 2008**, 3.4.3]) [(ix)]

We group together below (11) examples of "bad rings". Let's start with a more positive look.

3. Immediate examples.

Proposition 3.1. *A field, a Dedekind domain whose field of fractions has characteristic zero is excellent.*

Proof. Let's check that a field is excellent. Indeed, a finite integral algebra over a field is a field: properties 1.a), 2.a), and 3) are therefore satisfied, which proves that every field is excellent.

Let A be a Dedekind domain with field of fractions K of *characteristic zero* is excellent.

- Let's check 1.a). Let B be an integral domain finite over A. Either B is a field, in which case 1.a) is satisfied, or A embeds into B. Since K has characteristic zero, B is generically étale over A, proving that the regular locus of B contains a non-empty open set (the étale locus, for example).
- For 2.a), consider a closed point x in $\mathrm{Spec}(A)$. The non-closed formal fiber at x is the completion $\widehat{K_x}$ of K for the valuation defined by x. Since K has characteristic zero, the field $\widehat{K_x}$ is separable over K, hence 2.a).
- Property 3) is clear since the completion of A at x is an integral domain and therefore equidimensional.

□

We will see later (11.5) that there are many discrete valuation rings that are not quasi-excellent.

[(v)] Recall ([**ÉGA** 0_{IV} 19.8.5]) that Cohen rings C are fields of characteristic zero and complete discrete valuation rings of unequal characteristic $(0, p)$ that are unramified. When the residue field κ of C is perfect, C is none other than the ring of Witt vectors of κ.

[(vi)] See (4.2) for an improvement.

[(vii)] Or *Nagata* in English, or even *pseudo-geometric* (in Nagata's work, notably).

[(viii)] If, moreover, X can be locally embedded in a regular scheme, then X satisfies 3) and is therefore excellent.

[(ix)] This result was long considered "well-known to experts," whereas its proof, which is quite non-trivial, dates from 2008.

4. The basic example: complete noetherian local rings.

Let's explain with Nagata why complete noetherian local rings are excellent[(x)].

Property 2.a) is tautological. Formal catenarity has been seen (2.6). What remains is 1.a). A finite extension of a complete ring being complete, we must prove the following result (cf. [**ÉGA** IV_2 22.7.6]).

Théorème 4.1 (Nagata). *If X is a complete noetherian local integral scheme*[(xi)]*, then the regular locus is open.*

Proof. We will distinguish between the equal and unequal characteristic cases.

Cas I (Cf. [**ÉGA** 0_{IV} 22.7.6].) Assume that A contains a field and let k_0 be its prime field (which is perfect!) so that the residue field k of $A_{\mathfrak{p}}$ is separable over k_0 for all $\mathfrak{p} \in \operatorname{Spec}(A)$. The ring $A_{\mathfrak{p}}$ is regular if and only if $A_{\mathfrak{p}}$ is formally smooth over k_0 (see in this case [**ÉGA** 0_{IV} 19.6.4]). On the other hand, the Cohen structure theorem (2.9) ensures that A is isomorphic to $k[[T_1, \dots, T_n]]/I$ so that $\mathfrak{p}$ identifies with an ideal of $B = k[[T_1, \dots, T_n]]$ containing I. *Nagata's Jacobian criterion for formal smoothness* ([**ÉGA** 0_{IV} 22.7.3]) ensures that $A_{\mathfrak{p}}$ is regular if and only if there exist k_0-derivations D_i, $i = 1, \dots, m$, from B to B and f_i, $i = 1, \dots, m$, elements generating $I_{\mathfrak{p}}$ such that $\operatorname{dét}(D_i f_j) \notin \mathfrak{p}$. This condition being visibly open, the theorem follows.

Cas II Assume that A has unequal characteristic, and therefore its field of fractions K has characteristic zero. According to the Cohen structure theorem (2.9), A contains a regular (and complete) subring B such that A is a finite B-algebra. The field of fractions L of B has characteristic zero, like K. Up to replacing B by a localization, we can assume that A is a free B-module of finite rank with basis $y_1, \dots, y_m$. But $\operatorname{Spec}(A) \to \operatorname{Spec}(B)$ is étale outside the closed set defined by $d = \operatorname{dét}(\operatorname{Tr}(y_i y_j)) \neq 0$ in $\operatorname{Spec}(B)$. This set is a proper closed set since the extension $\operatorname{Frac}(A)/\operatorname{Frac}(B)$ is separable (being of characteristic zero)! Since B is regular, the theorem follows.

□

Remarque 4.2. Thus, a formal power series ring over a field is excellent.

Note that the proof simplifies if one knows Gabber's improvement of the Cohen structure theorem (**IV**-2.1.1 and **IV**-4.2.2): if A is a complete noetherian local integral domain, it contains a ring B isomorphic to a formal power series ring over a Cohen ring or a field such that $\operatorname{Spec}(A) \to \operatorname{Spec}(B)$ is finite and *generically étale*. One then does not need to distinguish the characteristics of the fields of fractions in the proof. But the proof of this improvement is difficult.

5. Permanence under localization and finite type extensions

The notion of (quasi-)excellence is remarkably stable. Precisely, we have

Théorème 5.1. *Any algebra of finite type, or more generally essentially of finite type, over an excellent (resp. quasi-excellent) ring is excellent (resp. quasi-excellent). In particular, any localization of an algebra of finite type over a field or over a Dedekind domain (**Z** for example) with field of fractions of characteristic zero is excellent.*

(Recall that, in this context, a morphism $\operatorname{Spec}(B) \to \operatorname{Spec}(A)$ is said to be **essentially of finite type** if B is a localization of an A-algebra of finite type by a multiplicative system.)

Let's outline the proof.

[(x)] This allows the construction of many examples of excellent discrete valuation rings of positive characteristic (by completion of regular schemes at points of height 1).

[(xi)] According to (2.2), this implies that the regular locus of a complete noetherian local scheme is open, whether it is integral or not.

5.2. Condition 1). Passing to localization poses no problem. Let B be of finite type over A. If A satisfies 1.a) or 1.b), Nagata's openness criterion ([**ÉGA** IV$_2$ 6.12.4 and 6.13.7]) implies that the same holds for B.

5.3. Condition 2). This is the most difficult part of the theory ([**ÉGA** IV$_2$ 7.4.4]), due entirely to Grothendieck. The most delicate point is localization:

THÉORÈME 5.3.1. *If A satisfies 2.a) (resp. 2.b) or 2.c)), then for any $\mathfrak{p} \in \operatorname{Spec}(A)$, the ring $A_{\mathfrak{p}}$ satisfies 2.a) (resp. 2.b) or 2.c()); in other words, the formal fibers at every point of $\operatorname{Spec}(A)$ are geometrically regular (resp. geometrically normal or geometrically reduced).*

Proof. The proof proceeds by reduction to the complete case. We restrict to property 2.a); the cases 2.b) and 2.c) are analogous. Let $\mathfrak{m}$ be a maximal ideal containing $\mathfrak{p} \in \operatorname{Spec}(A)$ and let B be the $\mathfrak{m}$-adic completion of A. Since $A_{\mathfrak{m}} \to B$ is faithfully flat, there exists $\mathfrak{q} \in \operatorname{Spec}(B)$ lying over $\mathfrak{p}$. By hypothesis, $A_{\mathfrak{m}} \to B$ is regular. As regular morphisms are stable under localization, $A_{\mathfrak{p}} \to B_{\mathfrak{q}}$ is regular. Consider the commutative diagram

$$\begin{tikzcd}
\widehat{A_{\mathfrak{p}}} \arrow[r, "\hat{f}"] & \widehat{B_{\mathfrak{q}}} \\
A_{\mathfrak{p}} \arrow[u, "\alpha"] \arrow[r, "f"] & B_{\mathfrak{q}} \arrow[u, "\beta"']
\end{tikzcd}$$

Assume that β is regular. Then $\hat{f} \circ \alpha$ is regular as a composition of regular morphisms. Since $\hat{f}$ is faithfully flat (being the completion of the flat local morphism f), we deduce that α is regular (exercise or [**ÉGA** IV$_2$ 6.6.1]), as desired. Thus we are reduced to β, i.e. to the complete case. The regularity of β follows from

THÉORÈME 5.3.2. *Let B be a complete noetherian local ring. Then the formal fibers of B at $\mathfrak{q} \in \operatorname{Spec}(B)$ are geometrically regular.*

This theorem is the hard core of the theory. We reduce (2.4) to studying the generic formal fibers. First, one shows ([**ÉGA** 0_{IV} 22.3.3]) that if $\mathfrak{p}$ is a prime ideal of a complete noetherian local domain A, the generic formal fiber $\widehat{A_{\mathfrak{p}}} \otimes_{A_{\mathfrak{p}}} \operatorname{Frac}(A_{\mathfrak{p}})$ of $A_{\mathfrak{p}}$ is formally smooth over $\operatorname{Frac}(A_{\mathfrak{p}}) = \operatorname{Frac}(A)$ at every point. Second, one shows ([**ÉGA** 0_{IV} 22.5.8]) that a noetherian local algebra over a field is formally smooth[xii] if and only if it is geometrically regular[xiii]. □

Once the permanence under localization is proven, one can show:

THÉORÈME 5.3.3. *Let B be an A-algebra of finite type. If A satisfies 2.a) (resp. 2.b) or 2.c)), then B satisfies 2.a) (resp. 2.b) or 2.c)).*

The proof is by induction on the number of generators of B. Thanks to the invariance under localization, one easily reduces to studying the formal fibers of B at a maximal ideal in the case where B is generated by *one* element and A is complete. The proof is not easy, but much simpler than those of ([**ÉGA** 0_{IV} 22.3.3 et 0.22.5.8]).

5.4. Condition 3). As for the conditions of type 2), stability under localization and finite extension follows as above ([**ÉGA** IV$_2$ 7.1.8]) from the complete case, the flatness of the localization morphism allowing one to descend from the completion to the ring — it is not immediate, however —. The complete case is easy as we have seen (2.6).

[xii] Recall that a local k-algebra B (endowed with the adic topology) is formally smooth over k if any continuous k-algebra morphism $B \to C/I$ with $I^2 = 0$ lifts continuously to the discrete C-algebra C.

[xiii] In fact, we only need the implication "formally smooth ⇒ geometrically regular," which is the easier direction. The equivalence was considerably simplified by Faltings ([**Faltings, 1978**]; see also [**Matsumura, 1989**, 28.7] for an English exposition).

5.5. Application to the local case. In the local case, the openness condition for the regular locus follows from 2.a). To be precise.

PROPOSITION 5.5.1.

(i) *The regular locus of a noetherian local ring satisfying 2.a) is open.*
(ii) *In particular, a noetherian local ring is quasi-excellent (resp. excellent) if and only if it satisfies 2.a) (resp. if it satisfies 2.a) and 3)).*

Proof. Let $f : X \to Y$ be a faithfully flat morphism of noetherian schemes with regular (resp. normal or reduced) fibers. Then, $\mathcal{O}_x$ is regular (resp. normal or reduced) if and only if $\mathcal{O}_{f(x)}$ is ([**ÉGA** IV_2 6.4.2, 6.5.1]). Let $U_R(X)$ be the set of $x \in X$ such that $R(\mathcal{O}_x)$ is regular (resp. normal or reduced). In other words, we have $f^{-1}(U_R(Y)) = U_R(X)$. Now, the regular or normal locus of a complete integral domain is open (4.1). Moreover, the completion morphism of a noetherian local ring A is regular if and only if A satisfies 2.a) according to (5.3.2). It follows that 2.a) implies 1.a) (resp. 2.b) implies 1.b)) in the local case. □

6. Comparison with ÉGA IV: the case of universally Japanese rings

Let us recall the usual definition of universally Japanese rings ([**ÉGA** 0_{IV} 23.1.1]).

DEFINITION 6.1. X is said to be

(i) **Japanese** if it is an integral domain and the integral closure of A in any finite extension (xiv) of its field of fractions is finite over A (xv);
(ii) **universally Japanese** if any integral domain which is a finite type extension of A is Japanese (xvi).

The definition of a Japanese ring is only technical in that it only serves to define the only truly useful (and verifiable, to be honest) notion: that of a universally Japanese ring. This definition is compatible with 2.10. Let's explain why. According to Nagata, X is universally Japanese in the sense of 6.1 if and only if X satisfies 1.b) and if all integral quotients of the localizations $\mathcal{O}_{X,x}$ at the closed points $x \in X$ are Japanese ([**ÉGA** IV_2 7.7.2]). Now, the Zariski-Nagata theorem ([**ÉGA** IV_2 7.6.4]) ensures that the integral quotients of $\mathcal{O}_{X,x}$ are Japanese if and only if the formal fibers of $\mathcal{O}_{X,x}$ are geometrically reduced (xvii). Hence the equivalence between the two definitions of universally Japanese rings.

If we strengthen condition 2.c) to 2.b) (geometrically normal formal fibers), passing to the integral closure commutes with completion. Precisely, we have ([**ÉGA** IV_2 7.6.1 et 7.6.3])

PROPOSITION 6.2. *Assume that A is a noetherian local ring satisfying 2.b) and is reduced. Then, the integral closure A' of A in its total ring of fractions is finite over A and its completion is isomorphic to the integral closure of $\widehat{A}$ (xviii) in its total ring of fractions.*

We deduce the important criterion for the integrity of the completion.

COROLLAIRE 6.3. *Let A be a noetherian local ring.*

(i) *Assume A is an integral domain satisfying 2.b). Then, $\widehat{A}$ is an integral domain if and only if A is unibranch* (i.e. *A' is local*).
(ii) *Assume A is henselian. Then A is excellent if and only if it satisfies 2.a). If A is also an integral domain, so is its completion.*

(xiv) One can restrict to finite radicial extensions if desired: exercise or [**ÉGA** IV_1 23.1.2].

(xv) As a module or as an algebra: it is the same thing because the integral closure is integral over A by construction.

(xvi) Or, equivalently ([**ÉGA** IV_2 7.7.2]), if any integral quotient is Japanese.

(xvii) Or, equivalently, that the completion of any finite and reduced $\mathcal{O}_{X,x}$-algebra is reduced. As usual, the proof is by reduction to the complete case, and even to the regular complete case thanks to the Cohen structure theorem. The Japanese character of such rings is guaranteed by Nagata's theorem ([**ÉGA** 0_{IV} 23.1.5]).

(xviii) Which is reduced since A is Japanese (cf. note (xvii)).

Proof. Let's prove (i). Since A is unibranch, the integral closure A' of A is local: so is its completion $\widehat{A'}$. According to (6.2), we have $\widehat{A'} = (\widehat{A})'$ and it is therefore normal. Now, a normal and local ring is an integral domain. Since $\widehat{A'}$ contains $\widehat{A}$, the result follows.

Let's prove (ii). According to (5.5.1), we only need to convince ourselves that a henselian local ring satisfying 2.a) also satisfies 3), *i.e.* is formally catenary. We can assume A is an integral domain and we must prove that $\widehat{A}$ is equidimensional. But since A is a henselian integral domain, it is unibranch [**ÉGA** IV$_4$ 18.8.16], so $\widehat{A}$ is an integral domain according to the first point, which ensures equidimensionality. □

7. Comparison with ÉGA IV: the case of excellent rings

Grothendieck's definition of excellent noetherian rings is *a priori* different from the one given here. Notably, it involves, somewhat strangely, *universal catenarity* instead of *formal catenarity*. Precisely, it involves three properties. In this part, A denotes a noetherian ring and $X = \operatorname{Spec}(A)$ the corresponding affine scheme.

1ÉGA) For any integral quotient B of A and any finite radicial extension K' of the field of fractions K of B, there exists a finite sub-B-algebra B' of K' containing B, with field of fractions K', such that the regular locus of $\operatorname{Spec}(B')$ is a dense open set.

2ÉGA) The formal fibers of X at any point (closed or not) are geometrically regular.

3ÉGA) A is universally catenary.

Excellent rings in the sense of **ÉGA** are the noetherian rings satisfying the 3 preceding properties ([**ÉGA** IV$_2$ 7.8.2]).

Let us note immediately, which is elementary, that the universal catenarity of A is equivalent to that of the local rings $\mathscr{O}_{X,x}$ at all its closed points — or all its points if one prefers — ([**ÉGA** IV$_2$ 5.6.3]).

For Grothendieck's definition of excellent rings ([**ÉGA** IV$_2$ 7.8.2]) to be the same as (2.10), we must prove the following proposition.

Proposition 7.1. *For any noetherian ring and $i = 1, 2, 3$, the properties i) and i*ÉGA) *are equivalent. In particular, the notions of quasi-excellence and excellence of the first part coincide with those of **ÉGA**.*

Proof. Condition 1ÉGA) is equivalent to 1) according to [**ÉGA** IV$_2$ 6.12.4] (only the part 1ÉGA) implies 1) is delicate, although it does not use Nagata's regularity criterion but only standard commutative algebra — essentially the regularity criterion via fibers and the non-degeneracy of the trace of finite separable field extensions —).

For the equivalence of 2) and 2ÉGA), one must be convinced that the geometric regularity of formal fibers at every closed point implies the geometric regularity of formal fibers at every point: this is a particular case of the permanence properties (5).

This proves the compatibility of the definitions of quasi-excellence.

If X satisfies 3), all its local rings are formally catenary (permanence under localization, cf. section 5) and therefore are catenary (2.7). Since any (affine) scheme of finite type over X satisfies 3) (permanence under finite type extension, cf. section 5), we deduce that X is universally catenary and thus X satisfies 3ÉGA).

The converse is due to Ratliff:

Proposition 7.1.1 (Ratliff). *A universally catenary noetherian ring is formally catenary.*

Precisely, Ratliff proves ([**Ratliff, 1971**, 3.12]) that if A is catenary, $A_{\mathfrak{p}}$ is formally catenary as soon as $\mathfrak{p}$ is not maximal[(xix)]. To prove the proposition, we can therefore assume $\mathfrak{p}$ is maximal and A is a local integral domain. Then, $\mathfrak{p}[X]$ is a non-maximal prime ideal in $A[X]$ so that the $\mathfrak{p}[X]$-adic completion $\widehat{A[X]_{\mathfrak{p}[X]}}$ is formally equidimensional. Since $\widehat{A} \to \widehat{A[X]_{\mathfrak{p}[X]}}$ is local and flat, the flatness argument ([**ÉGA** IV$_2$ 7.1.3]) used above ensures that $\widehat{A}$ is equidimensional. □

(xix) In the author's strange terminology, this is the condition $\operatorname{depth}(\mathfrak{p}) > 0$, which thus means that the *dimension* of $A/\mathfrak{p}$ is > 0.

8. Henselization and excellent rings

Recall that a morphism of noetherian rings $A \to B$ is said to be **absolutely flat** if it is reduced with discrete fibers and if the residue extensions are algebraic and separable. Or, equivalently, if it is flat as is the diagonal morphism $B \otimes_A B \to B$ (cf. [**Ferrand, 1972**, prop. 4.1], and [**Olivier, 1971**, 3.1]). When B is (locally) of finite type over A, this is equivalent to B being étale over A. In particular, the residue extensions are separable so that such a morphism is in fact regular. For example, any ind-étale morphism is absolutely flat. We then have the following result ([**Greco, 1976**]).

Théorème 8.1. *Let $f : A \to B$ be an absolutely flat morphism of noetherian rings. Then*

(i) *If A satisfies 2.a) (resp. 2.b) or 2.c)), B satisfies 2.a) (resp. 2.b) or 2.c))* (xx).
(ii) *If A is universally Japanese, B is universally Japanese.*
(iii) *If A is quasi-excellent, B is quasi-excellent.*
(iv) *If A is excellent, B is excellent.*
(v) *If f is faithfully flat, the converse of* (i), (ii) *and* (iii) *is true.*
(vi) *If f is faithfully flat and B is locally integral, the converse of* (iv) *is true.*

Since henselization and strict henselization morphisms are absolutely flat ([**ÉGA** iv$_4$ 18.6.9 et 18.8.12]) and faithfully flat (they are local), we find, in particular, that quasi-excellence and excellence are stable under henselization and strict henselization, and that the quasi-excellent or universally Japanese character of a local ring can be tested on the henselization or strict henselization. In the case of the henselization, these results were known to Grothendieck ([**ÉGA** iv$_4$ 18.7]), notably 18.7.6).

On the other hand, one cannot hope for a descent property for excellence as in (vi) without a local integrity condition (cf. § 11).

9. Formal completion and excellent rings

Let I be an ideal of a noetherian ring A contained in its Jacobson radical and $\widetilde{A}$ its I-adic completion. One may wonder if the properties of excellence pass to the completion. The answer is yes in general. Precisely, we have:

Proposition 9.1. *Let I be an ideal of a noetherian ring A contained in its Jacobson radical and $\widetilde{A}$ its I-adic completion.*

(i) *If A is (semi)-local quasi-excellent (resp. excellent), so is $\widetilde{A}$;*
(ii) *If A is an excellent* **Q***-algebra, so is $\widetilde{A}$.*

The permanence of quasi-excellence in the (semi)-local case, *i.e.* of the geometric regularity of formal fibers, is due to Rotthaus ([**Rotthaus, 1977**]), while that of universal catenarity is due to Seydi (xxi) (theorem 1.12 of [**Seydi, 1970**] proves that a formal power series ring $A[[t_1, \dots, t_n]]$ is universally catenary as soon as A is; it is then sufficient to consider generators $i_1, \dots, i_n$ of I defining a surjection $A[[t_1, \dots, t_n]] \twoheadrightarrow \widetilde{A}$). For (ii), it remains to study the openness of the regular locus. This is done in [**Brodmann & Rotthaus, 1980**], using Hironaka's desingularization theorem in a crucial way. The techniques of [**Brodmann & Rotthaus, 1980**] have moreover made it possible to show that if the desingularization theorem were true in the excellent local case, any I-adic completion of an excellent ring as above would be excellent ([**Nishimura & Nishimura, 1987**]).

In fact, the result is general. More precisely, Gabber ([**Gabber, 2007**]) can replace Hironaka's theorem with his uniformization theorem (**VII**-1.1) in the arguments of [**Nishimura & Nishimura, 1987**] to prove the following result

Théorème 9.2 (Gabber). *Let A be an I-adically complete noetherian ring. Then, if A/I is quasi-excellent, A is quasi-excellent.*

(xx) Or more generally, if A is a **P**-ring in the sense of Grothendieck ([**ÉGA** iv$_2$ 7.3]), B is a **P**-ring

(xxi) As Christel Rotthaus explained to me (private communication), if A, B are local such that $A \subset B \subset \widehat{A}$ and $\widehat{B} = \widehat{A}$, then the universal catenarity of A implies that of B.

One cannot replace quasi-excellent with excellent in the preceding theorem. Indeed, Greco ([**Greco, 1982**]) constructed an ideal I of an integral domain A of dimension 3, noetherian semi-local, I-adically complete and separated, which is quasi-excellent but not excellent, while A/I is excellent. One can even assume that A is a $\mathbf{Q}$-algebra. The construction is done by pinching maximal ideals of different heights (cf. 11.1). Nevertheless, as we have just seen, formal catenarity passes to partial completions ([**Seydi, 1970**]) so that the I-adic completion of an excellent ring A is excellent as long as I is contained in the Jacobson radical of A.

10. Artin approximation and excellent rings

Recall the following definition (cf. [**Artin, 1969**]).

Definition 10.1 (M. Artin). A noetherian local ring $(A, \mathfrak{m})$ has the **approximation property** (AP) if for any affine variety X of finite type over A, the set $X(A)$ is dense in $X(\widehat{A})$.

Of course, this is equivalent to saying that for any X as above, we have

$$X(\widehat{A}) \neq \varnothing \Rightarrow X(A) \neq \varnothing.$$

If A satisfies AP, A is certainly henselian. But example 11.4 proves that it is not sufficient for A to be henselian for it to have the approximation property. In fact, Rotthaus observed that excellence was a necessary condition for Artin approximation:

Lemme 10.2 ([**Rotthaus, 1990**]). *A noetherian local ring satisfying AP is henselian and excellent.*

Let k be a field of characteristic zero equipped with a non-trivial real-valued valuation. Artin proved ([**Artin, 1968**]) that rings of convergent series with coefficients in k (not necessarily complete) have the approximation property. They are therefore henselian and excellent[(xxii)].

The situation is now completely clarified thanks to the work of Popescu, culminating in the following result ([**Swan, 1998**]):

Théorème 10.3 (Popescu). (i) *Let $A \to B$ be a regular morphism of noetherian rings. Then, B is a filtered inductive limit of smooth A-algebras.*

(ii) *Any henselian and excellent noetherian local ring satisfies the approximation property* AP[(xxiii)].

The fact that (i) implies (ii) is a simple exercise. Indeed, if A is quasi-excellent, the completion morphism $A \to \widehat{A}$ is regular and thus we can write $\widehat{A} = \operatorname{colim} L$ where L is smooth over A. Let $X = \operatorname{Spec}(B)$ with B of finite type and $\hat{a} \in X(\widehat{A})$ with image $\hat{a}(0) \in X(k)$ where k is the residue field of A. There thus exists L smooth over A such that $\hat{a}$ comes from $l \in X(L)$. Since A is henselian, there exists $a \in X(A)$ (such that $a(0) = \hat{a}(0)$)[(xxiv)].

11. Examples of bad noetherian rings

Let A be a noetherian ring and $X = \operatorname{Spec}(A)$ the corresponding affine scheme. It will emerge from this inventory that the unpleasant properties of rings from the point of view of excellence are generally not only related to characteristic > 0 but can also occur for $\mathbf{Q}$-algebras.

11.1. Formal catenarity: condition 3). Let's first look at bad rings from the point of view of formal catenarity.

(xxii) In the same vein, the ring $k\{x_1, \dots, x_n\}$ of restricted formal series (series whose sequence of coefficients tends to 0) with coefficients in a non-archimedean complete valued field k is excellent as soon as k has characteristic zero or k is of finite degree over k^p, $p = \operatorname{car}(k)$ ([**Greco & Valabrega, 1974**]). The general case was obtained by Kiehl ([**Kiehl, 1969**] and also [**Conrad, 1999**] for a proof and developments). This partially answers a question of Grothendieck ([**ÉGA** IV$_2$ 7.4.8 B)]). On the other hand, if k is a non-archimedean non-complete valued field of positive characteristic such that the completion morphism $k \to \widehat{k}$ is not separable, Gabber knows how to prove that $k\{x_1\}$ is not excellent.

(xxiii) See [**Spivakovsky, 1999**, th. 11.3] for a slightly more general statement.

(xxiv) The argument only uses the geometric regularity of the fibers — and the henselian character — but not the formal catenarity. This is not paradoxical, because a henselian local ring is excellent if and only if its formal fibers are geometrically regular (5.5.1).

11.1.1. *Catenarity does not imply formal catenarity.* In [**ÉGA** IV$_2$ 5.6.11], Grothendieck constructs an example of a 2-dimensional noetherian local integral domain which is catenary but not universally catenary, hence not formally catenary according to 7.1.1 (*i.e.* not satisfying 3)).

Let's explain the construction, which consists of pinching a smooth surface over a field k, of characteristic zero if one wishes, along two points of different heights having residue fields isomorphic to k.

We start with a field k which is a pure transcendental extension of infinite degree over its prime field, for example, which we can even assume to be $\mathbf{Q}$. Let S be a smooth surface equipped with a projective morphism onto $T = \mathrm{Spec}(k[\tau])$ and $t \in S(T)$. We assume that there exists a point $s \in S(k)$ in the fiber S_0 of $S \to T$ over $0 \in T$ which is not in the image of t. For example, one can take $S = \mathrm{Spec}(k[\sigma, \tau])$ with t the section with image $\sigma = 0$ and $s = (1, 0)$. The fields $k(s) = k$ and $k(t) = \mathrm{Frac}(k[\tau])$ are pure transcendental extensions of $\mathbf{Q}$ of the same (infinite) degree, so we can choose a field isomorphism $k(s) \simeq k(t)$. This allows us to define the subring $\mathscr{O}_\Sigma$ of $\mathscr{O}_S$ of functions that coincide at s and t. We thus have a morphism $\pi : S \to \Sigma$ that sends s, t to $\sigma \in \Sigma$. Let $A = \mathscr{O}_{\Sigma,\sigma}$ and let B be the coordinate ring of $S \times_\Sigma \mathrm{Spec}(\mathscr{O}_{\Sigma,\sigma})$. By construction, $\dim(B_s) = 2$ and $\dim(B_t) = 1$. Then, A is noetherian, and B is the normalization of A and is finite over A. Since A is 2-dimensional and integral, it is obviously catenary. If A were universally catenary, the dimension formula (see note (iv)) would imply $\dim(A) = \dim B_s = \dim B_t$, a contradiction.

One can even find for any $n \geq 2$ noetherian local integral domains of dimension n satisfying 2.a), catenary but not universally catenary, thus not satisfying 3) ([**Heinzer et al., 2004**]).

11.1.2. *Formal catenarity cannot be tested on the henselization.* Using similar pinching techniques on surfaces over k, thus of characteristic zero if one wishes, as above, Grothendieck constructs an example of a local ring that is not universally catenary (thus not formally catenary) whose henselization is excellent ([**ÉGA** IV$_4$ 18.7.7]). Up to a base change by the separable closure, we see that formal catenarity can no more be tested on the strict henselization, unlike quasi-excellence (8.1).

11.1.3. *Formal catenarity certainly does not imply 2.a) (or even 2.c)).* For example, a non-excellent discrete valuation ring A (cf. part 11.5) has a non-geometrically regular generic formal fiber (indeed, it is formally catenary (2.8) and 2.a) implies 1.a) in the local case (5.5.1)). Now, this formal generic fiber is artinian in this case (it does not contain the maximal ideal of $\widehat{A}$) and so geometric regularity is equivalent here to geometric reduction.

11.1.4. *There exist* normal *integral domains that are not formally catenary.* Ogoma ([**Ogoma, 1980**]) constructed a local $\mathbf{Q}$-algebra A which is an integral *normal* domain of dimension 3 whose completion has a component of dimension 2 and a component of dimension 3 and thus is not equidimensional. Worse, this ring is not even catenary: it has infinitely many saturated chains of prime ideals of length 2 or 3.

11.2. Quasi-excellence: openness conditions 1.a) and 1.b). Here we are interested in rings with a non-open regular or normal locus.

As we will see later (**XIX**-2.6), Gabber constructed an example of a scheme, which we can even assume to be a $\mathbf{Q}$-scheme, which is an integral domain of dimension 1, whose regular locus (or normal locus, it's the same thing here) contains infinitely many points and in particular is not open. The construction ensures that the formal fibers are geometrically regular. Since we are in dimension 1, normality and regularity coincide, so we have an example satisfying 2.a) and 3) but not 1.b).

In [**Rotthaus, 1979**], Rotthaus constructs a noetherian local integral $\mathbf{Q}$-algebra of dimension 3 which is formally catenary, universally Japanese but whose regular locus is not open.

11.3. Quasi-excellence. Formal fibers: conditions 2a), 2b) and 2c). Here we are interested in rings with formal fibers that are not geometrically regular, or worse.

- Rotthaus constructs a noetherian local $\mathbf{Q}$-algebra A of dimension 3 which is regular (hence formally catenary), universally Japanese but not excellent ([**Rotthaus, 1979**]). Precisely, the formal fiber over a point of height 1 is not regular. Thus, it satisfies 2.c), 3) because A is regular, but not 2.a).

In Ogoma's preceding example, the generic formal fiber is connected but not an integral domain (it has a component of dimension 1 and one of dimension 2 which intersect), hence not normal. We thus have an example of a noetherian local **Q**-algebra (of dimension 3) (integral and normal) not satisfying 2.b).

We can go down one dimension: Nagata constructs ([**Nagata, 1962**, ex. 7 of appendix A1]) a local **Q**-algebra B which is a normal integral domain, formally catenary and of dimension 2 but whose completion is not an integral domain (xxv). According to 6.2, this proves that B does not satisfy 2.b) (but satisfies 3)).

- In characteristic > 0, Rotthaus also constructs ([**Rotthaus, 1979**]) a noetherian local algebra of dimension 2 which is regular, universally Japanese but not excellent. In this last case, since the formal fibers are of dimension < 2, they are not geometrically normal either. Thus, it satisfies 2.c), 3) but not 2.b).
- In characteristic zero, Ferrand and Raynaud constructed ([**Ferrand & Raynaud, 1970**, prop. 3.3 et 3.5]) a noetherian local **C**-algebra A which is an integral domain of dimension 2 such that

  • the normalization A' of A is the ring of convergent series, denoted $\mathbf{C}\{x, y\}$ (not to be confused with the henselization of $\mathbf{C}[x, y]$) and is therefore excellent.

  • A is not Japanese (in fact, A' is not finite over A).

  • $\widehat{A}$ has embedded components (so that — by flatness — the generic formal fiber has embedded components and thus does not satisfy 2.c)).

  • The normal locus of $A[[T]]$ is not open. According to [**ÉGA** IV$_2$ 6.13.5] its regular locus is therefore not open either.

  • The ring A is formally catenary (the spectrum of its completion is irreducible). The same is therefore true for $A[[T]]$ ([**Seydi, 1970**]).
- In [**Nagata, 1962**, ex. 5 of appendix A1], Nagata even constructs a noetherian local integral domain of dimension 3 (of characteristic > 0) whose integral closure is not even noetherian; in particular, this ring is not Japanese (xxvi).
- Worse, from rings constructed by Nagata, Seydi constructs ([**Seydi, 1972**]) a normal noetherian integral domain A of dimension 3 whose field of fractions has characteristic zero and whose completion is not reduced. In particular, it is Japanese but not universally Japanese. Ogoma constructs ([**Ogoma, 1980**]) a noetherian normal **Q**-algebra, hence Japanese, which is neither universally Japanese nor catenary.
- The formal fibers can be dreadful, even in dimension 1: Ferrand and Raynaud construct a 1-dimensional local integral **C**-scheme whose generic formal fiber is an artinian scheme which is not even Gorenstein — hence certainly not reduced — ([**Ferrand & Raynaud, 1970**], prop. 3.1): X does not satisfy 2.c). In particular, X is not universally Japanese (and thus not quasi-excellent). Of course, X satisfies 1.a) and 3) for dimension reasons.
- The examples of non-excellent discrete valuation rings (thus of positive characteristic) give examples of rings not satisfying 2.c) (2.a) and 2.c) are equivalent in dimension ≤ 1) but satisfying 1.a) and 3).

(xxv) The construction is as follows: let x, y be algebraically independent over **Q** and $w = \sum_{i>0} a_i x^i \in \mathbf{Q}[[x]]$ be transcendental over $K(x)$. We set $z_1 = (y + w)^2$ and $z_{i+1} = (z - (y + \sum_{j<i} a_j x^j)^2)/x^i$. Let A be the localization of $\mathbf{Q}[x, y, z_i, i \geq 1]$ at $(x, y, z_i, i \geq 1)$. Then, $B = A[X]/(X^2 - z_1)$ is the desired example. One easily checks that the completion of A is $\mathbf{Q}[[x, y]]$ so that $\widehat{B} = \mathbf{Q}[[x, y]][X]/(X^2 - (y + w)^2)$ is not an integral domain. As usual in these constructions, it is the noetherian character of A that poses a problem. Once this is established, A is regular of dimension 2 and B is normal since it is Cohen-Macaulay of dimension 2 and singular only at the origin. Note that B is formally catenary as a quotient of a regular ring.

(xxvi) The construction is of the same type as that of a discrete valuation ring described in note (xxvii), whose notation we adopt. We consider this time the ring $B = k^p[[X, Y, Z]][k]$ and $d = Y \sum_{i>0} X_i X^i + Z \sum_{i>0} X_{2i+1} X^i$. The ring $B[d]$ is suitable.

Remarque 11.4. Nagata constructed ([**Nagata, 1962**, (E3.3)]) a discrete valuation ring whose generic formal fiber is a non-trivial radicial extension of its field of fractions, hence not excellent [(xxvii)].

We will now see that such rings are very easily encountered.

11.5. Systematic method for constructing non-quasi-excellent rings. In fact, one can (Orgogozo) systematically construct many non-quasi-excellent discrete valuation rings. To be precise [(xxviii)].

Proposition 11.6. *Let $k((t))$ be the field of Laurent series with coefficients in a field k of characteristic $p > 0$, equipped with its t-adic valuation, and let L/k be a finitely generated sub-extension of $k((t))/k$ of transcendence degree > 1 over k. Then, the subring A of L of elements with valuation ≥ 0 is a non-excellent discrete valuation ring.*

Proof. Let $\bar{L}$ be a field of characteristic > 0. The p-rank is the dimension, finite or not, of $\Omega_{\bar{L}}$, the module of absolute differentials. It is also $\log_p([\bar{L} : \bar{L}^p])$ where $[\bar{L} : \bar{L}^p]$ is the dimension of $\bar{L}$ over $\bar{L}^p$ ([**ÉGA** IV$_1$ 21.3.5]). The key remark is that the p-rank increases under a *separable* field extension $\bar{K}/\bar{L}$, finite or not, since in this case we have an embedding $\bar{K} \otimes_{\bar{L}} \Omega_{\bar{L}} \hookrightarrow \Omega_{\bar{K}}$ ([**ÉGA** IV$_2$ 20.6.3])

$$[\bar{L} : \bar{L}^p] \leq [\bar{K} : \bar{K}^p]. \tag{11.6.1}$$

Furthermore, if $\bar{L}$ is of *finite type* over a field k of finite p-rank, we have ([**Bourbaki**, **A**, V, § 16, nº 6, cor. 3])

$$[\bar{L} : \bar{L}^p] = p^{\operatorname{deg.tr.}_k(\bar{L})}[k : k^p]. \tag{11.6.2}$$

Let's place ourselves in the situation of the lemma. The ring A is a discrete valuation ring by construction and its field of fractions is L. The henselization A^{h} is a regular local ring of dimension 1, hence an integral domain, and its field of fractions $K = \operatorname{Frac}(A^{\mathrm{h}})$ contains $L = \operatorname{Frac}(A)$. The completion $\widehat{A^{\mathrm{h}}}$ is a formal power series ring $\widehat{K} = k[[\varpi]]$, ϖ being a uniformizer of A^{h} (as the completion of a regular local k-algebra of dimension 1) and its field of fractions $\widehat{K}$ is the generic formal fiber of $\operatorname{Spec}(\widehat{A^{\mathrm{h}}}) \to \operatorname{Spec}(A^{\mathrm{h}})$.

Assume that A^{h} is quasi-excellent (precisely, satisfies 2.a)) so that the extension $\widehat{K}/K$ is *separable*.

We thus have in this case

$$[K : K^p] \leq [\widehat{K} : \widehat{K}^p].$$

Since $\widehat{K} = k((\varpi))$, we have

$$[\widehat{K} : \widehat{K}^p] = p[k : k^p].$$

We then have (11.6.1), the extension K/L being separable,

$$[K : K^p] \geq [L : L^p].$$

so that, thanks to (11.6.2), we have

$$p[k : k^p] = [\widehat{K} : \widehat{K}^p] \geq [L : L^p] > p[k : k^p],$$

a contradiction. This also prevents A from being quasi-excellent (8.1). □

(xxvii) Here is the construction: let k be the field of fractions of $\mathbf{F}_p[X_n, n > 0]$ and K that of $\widehat{A} = k[[Y]]$. Let L be the subfield of $K = k((Y))$, the field of fractions of $A = k^p[[Y]][k]$. The completion of A is $\widehat{A}$. One shows, and this is the delicate point, that A is noetherian. The tool is Cohen's criterion: a semi-local ring is noetherian if and only if the maximal ideals are finitely generated and the finitely generated ideals are closed ([**Nagata, 1962**, 31.8]). Its completion being regular, it is itself regular, hence a discrete valuation ring (by dimension). Let L be the field of fractions of A. One easily checks that $c = \sum_{n>0} X_n Y^n$, is not in L. Choose a p-basis $\{c_i\}$ of K over L containing c (which is possible because $c \notin L^p$, cf. [**ÉGA** IV$_1$ 21.4.3]). Let K_0 be the field generated over L by the c_i distinct from c. The extension K/K_0 is radicial of degree p by construction. The ring $A \cap K_0$ is a discrete valuation ring with completion $k[[Y]]$, so that the generic formal fiber is not geometrically reduced.

(xxviii) This construction in fact generalizes, independently, an example obtained by Rotthaus in [**Rotthaus, 1997**]

EXPOSÉ II

Topologies adapted to local uniformization

Fabrice Orgogozo

In this talk, ℓ is a prime number, the integer 1 or the symbol ∞, and we denote by ℓ' the set of natural numbers prime to ℓ where, by convention, $\infty' = \{1\}$.

1. Maximally dominant morphisms and the category alt / S

1.1. Maximally dominant morphisms.

1.1.1. Recall ([**ÉGA** iv 1.1.4]) that a point of a scheme is said to be **maximal** if it is the generic point of an irreducible component or, equivalently, if it is maximal for the order on the set of points of the scheme defined by the relation: $x \geq y$ if and only if y is a specialization of x (that is, if $y \in \overline{\{x\}}$). The maximal points of an affine scheme correspond to the minimal prime ideals. Any dense open set of a noetherian scheme contains all the maximal points; this is more generally true when the number of irreducible components is locally finite, or when the open set is assumed to be *retrocompact*.

Définition 1.1.2. A morphism of schemes is said to be **maximally dominant** if it sends every maximal point of the source to a maximal point of the target.

A morphism between irreducible schemes is maximally dominant if and only if it is dominant. It is clear that the composite of two maximally dominant morphisms is maximally dominant.

Exemple 1.1.3. According to [**ÉGA** iv□ 2.3.4], a flat morphism, or more generally a quasi-flat morphism (*op. cit.*, 2.3.3), is generizing ([**ÉGA** i′ 3.9.1]) and therefore maximally dominant (*op. cit.*, 3.9.5).

Proposition 1.1.4. *Let $f : X \to Y$ be a maximally dominant morphism. Any maximal point of Y belonging to the image of f is the image of a maximal point of X.*

Proof. This follows from the fact that f is increasing for the preorder above and from the fact that any point of X has a maximal generization. □

Proposition 1.1.5. *Let $f : X \to Y$ be a maximally dominant morphism and $Y' \to Y$ be a* flat *morphism. Then, the morphism $X' = X \times_Y Y' \to Y'$ is maximally dominant.*

Proof. See [**ÉGA** iv□ 2.3.7 (ii)]. □

Let us recall the following proposition.

Proposition 1.1.6 ([**ÉGA** iv 20.3.5]). *Let $f : X \to Y$ be a maximally dominant morphism. Assume that X is* reduced *and that Y has only a finite number of irreducible components. Then, for any open set U of Y and any open set V dense in U, the open set $f^{-1}(V)$ is dense in $f^{-1}(U)$.*

The hypothesis on Y ensures that any maximal point of U belongs to V; it is satisfied when Y is noetherian.

Proposition 1.1.7. *Let Y be a noetherian scheme and let $f : X \to Y$ be a morphism of finite type,* maximally dominant. *The following conditions are equivalent:*

(i) *There exists a dense open set of Y over which f is finite.*
(ii) *For any maximal point η of X, the extension $\kappa(\eta)/\kappa(f(\eta))$ is finite.*

A morphism of schemes satisfying condition (i) above is called **generically finite** (below).

Proof. As recalled above, any dense open set of the noetherian scheme Y contains its maximal points. We can assume Y is irreducible and X, Y are reduced: we use the fact that f is finite if and only if $f_{\text{réd}}$ is. By passing to the limit ([**ÉGA** IV 8.10.5.(x)]), we can also assume that Y is the spectrum of a field k. In this case, the conclusion follows from [**ÉGA** I 6.4.2 and 6.4.4]. □

Définition 1.1.8. A morphism $f : X \to Y$ is said to be **maximally ℓ'-finite** if for every maximal point η of X, the extension $\kappa(\eta)/\kappa(f(\eta))$ is finite and if for every maximal point μ of Y in the image of f, there exists a maximal point η of X over μ such that the extension $\kappa(\eta)/\kappa(\mu)$ has degree belonging to ℓ'.

1.1.9. When $\ell = 1$, the second condition is empty. It is useful to make the following language conventions: a maximally ℓ'-finite and maximally dominant morphism is called **maximally ℓ'-finite dominant**, and a maximally $1'$-finite morphism is called "**maximally finite**".

Proposition 1.1.10. *Let S be a scheme and $f : X \to Y$ be an S-morphism between maximally dominant S-schemes. If Y/S is maximally finite, f is maximally dominant. If we further assume X / S is maximally finite, the morphism f is maximally finite dominant.*

Proof. Let x be a maximal point of X and s (resp. y) its image in S (resp. Y). Let $y' \geq y$ be a maximal generization of y. Since the schemes X and Y are maximally dominant over S, the point s is maximal and y' has image s. Finally, if Y/S is maximally finite, the extension $\kappa(y')/\kappa(s)$ is finite so that y' is closed in the fiber Y_s. The point y, belonging to the closure of y' in Y_s, therefore coincides with y': the morphism f is maximally dominant. If we further assume X/S is maximally finite, the finiteness of the extension $\kappa(x)/\kappa(s)$ implies that of the intermediate extension $\kappa(x)/\kappa(y)$. □

1.2. The category alt / S.

1.2.1. Let S be a noetherian scheme. Let η_S be the (finite) coproduct scheme of its maximal points.

Définition 1.2.2. We denote by alt/S the category of S-schemes of finite type, maximally finite dominant, with source a *reduced* scheme. The morphisms in alt/S are the S-morphisms.

1.2.3. Note the following facts:

- the S-scheme $S_{\text{réd}}$ is final in the category alt/S;
- any morphism in alt/S is maximally finite dominant (1.1.10);
- inverse images of divisors exist for any morphism in alt/S ([**ÉGA** IV 21.4.5.(iii)]);
- if $X \in \operatorname{Ob} \text{alt}/S$ and $S' \to S$ is a *reduced* morphism ([**ÉGA** IV□ 6.8.1]) with S' noetherian, the usual fiber product $X' = X \times_S S'$ is naturally an object of alt/S'. This is more generally the case for $X'_{\text{réd}}$ as soon as $S' \to S$ is flat. Similarly, if $X' \to X$ is a quasi-finite reduced morphism (for example étale) and $X \in \operatorname{Ob} \text{alt}/S$, then X' is also an object of alt/S.

Remarques 1.2.4.

(i) The usual fiber product of two maximally dominant S-schemes is not necessarily maximally dominant, as can be seen when $S = \mathbf{A}^2$ and $X = Y$ are the blow-up at the origin.
(ii) The original definition of the category alt / S, due to O. Gabber, is less restrictive on the scheme S, which is assumed only to be coherent and to have a finite number of irreducible components. The objects of alt / S are then the reduced, quasi-separated, maximally dominant, generically finite schemes of finite type over S. The noetherian framework seems sufficient for our needs. Let us point out, however, that the "localizations" of a noetherian scheme for the topology of alterations (introduced below) are not necessarily noetherian (see 4.2.1).

1.2.5. Let X be an S-scheme of finite type. We denote by X_{md} — or $X_{\text{md}/S}$ in case of ambiguity about S — the closure of the (set-theoretic) image of X_{η_S} in X, endowed with the reduced structure. It is the union of the irreducible components of X that dominate an irreducible component of S, endowed with the reduced structure. (When $X \in \operatorname{Ob} \text{alt}/S$, $X_{\eta_S} = \eta_X$.) The functor $T \mapsto T_{\text{md}}$ is right adjoint to the inclusion functor from the category of reduced schemes of finite type that are maximally dominant over S into the category of S-schemes of finite type.

Proposition 1.2.6. *Fiber products exist in* alt$/S$.

Proof. Let $X \to S' \leftarrow Y$ be two arrows in alt$/S$; according to 1.1.10, the schemes X and Y are naturally objects of alt$/S'$. Since the composite of two maximally finite dominant morphisms of finite type is of the same nature, a product of X and Y, seen in alt$/S'$, is — if it exists — a fiber product in alt$/S$. We can therefore assume $S = S'$ and S is reduced. It formally follows from the existence of the product in the category of S-schemes of finite type and from the adjunction property of $X \mapsto X_{\mathrm{md}}$ that the scheme $(X \times_S Y)_{\mathrm{md}}$, endowed with the two obvious projections, is the product of X and Y in the category of reduced schemes of finite type that are maximally dominant over S. It belongs to $\mathrm{Ob}\,\mathrm{alt}/S$ because $((X \times_S Y)_{\mathrm{md}})_{\eta_S} = (X_{\eta_S} \times_{\eta_S} Y_{\eta_S})_{\mathrm{réd}}$ is finite over η_S. □

1.2.7. To avoid any ambiguity, we will sometimes denote by $X \times^S_{S'} Y$ the fiber product of X and Y over S' in alt$/S$, or more generally in the category of reduced, finite type, maximally dominant schemes over S. Let us also point out the following facts, whose proof is immediate (the first having been seen during the previous proof):

- if $X \to S' \leftarrow Y$ is a diagram in alt$/S$, the morphism $X \times^{S'}_{S'} Y \to X \times^S_{S'} Y$ deduced from the obvious functor alt$/S' \to$ alt$/S$ is an isomorphism;
- If $X \to S' \leftarrow Y$ is a diagram of S-schemes of finite type, the counits induce an isomorphism $X_{\mathrm{md}/S} \times^S_{S'_{\mathrm{md}/S}} Y_{\mathrm{md}/S} \xrightarrow{\sim} (X \times_{S'} Y)_{\mathrm{md}/S}$.

Proposition 1.2.8. *Let $f : X \to Y$ be an open immersion (resp. a proper and surjective morphism, resp. quasi-finite) between two S-schemes of finite type. The morphism $f_{\mathrm{md}} : X_{\mathrm{md}} \to Y_{\mathrm{md}}$ is an open immersion (resp. a proper and surjective morphism, resp. quasi-finite). Finally, if $(U_i)_{i \in I}$ is a covering by Zariski open sets of an S-scheme of finite type X, the open sets $(U_{i_{\mathrm{md}}})_{i \in I}$ cover the scheme X_{md}.*

Proof. The case where f is an open immersion is a consequence of the following well-known general fact about induced topologies: the trace on an *open set* of the closure of a subset coincides with the closure of the trace of that subset (see e.g. [**Bourbaki**, **TG**, I, § 3, nº 1, prop. 1] for a variant). If $f : X \to Y$ is proper and surjective, the morphisms $X_{\mathrm{md}} \to X$ and $Y_{\mathrm{md}} \to Y$ are closed immersions; the schemes X_{md} and Y_{md} are therefore proper over Y. The Y-morphism f_{md} is therefore proper and its image contains $f_{\eta_S}(X_{\eta_S}) = Y_{\eta_S}$ and therefore its closure Y_{md}. The case of a quasi-finite morphism is left to the reader. Let's check the last statement. That the $U_{i_{\mathrm{md}}}$ are open sets has already been seen; we need to check surjectivity. It results from the equalities $X_{\mathrm{md}} \cap U_i = U_{i_{\mathrm{md}}}$ and the fact that $X = \bigcup_i U_i$. □

2. Topologies: definitions

In this paragraph, we fix a noetherian scheme S.

2.1. Étale ℓ'-decomposed topology.

2.1.1. We will say that an étale covering $(X_i \to X)_{i \in I}$ of a scheme X is ℓ'-**decomposed** if any point of X can be lifted to a point x_i of some X_i such that the degree $[\kappa(x_i) : \kappa(x)]$ belongs to ℓ'. When $\ell = 1$ the imposed condition is empty and we simply say that the family constitutes an étale covering. When $\ell=\infty$, we recover the definition of [**Nisnevič, 1989**, § 1] and we then say, as in *op. cit.*, that the covering is **completely decomposed**.

The following stability result follows immediately from (i) of the lemma below (2.1.3).

Proposition 2.1.2. *The property of being an étale ℓ'-decomposed covering is stable under base change.*

According to 1.2.3, there is no need to specify whether this is the usual base change or, if applicable, in a category alt$/S$.

Lemme 2.1.3. *Let k be a field, K/k a finite extension, and k'/k any extension. Let K' be the finite k'-algebra $K \otimes_k k'$.*

(i) *Let ℓ be a prime number and assume the degree $[K : k]$ is prime to ℓ. Then, at least one residue field of K' has degree over k' prime to ℓ.*

(ii) *Assume K' is local and let κ' be its residue field. Then, the degree $[K : k]$ is the product of $[\kappa' : k']$ by a power of the characteristic exponent p of k.*

Proof. (i) Write $K' = \prod_i K'_i$ with K'_i local. There exists an index i such that $[K'_i : k']$ is prime to ℓ, since their sum is. Let $\mathfrak{m}_i$ be the maximal ideal of K'_i; the degree of the extension $[\kappa(\mathfrak{m}_i) : k']$ divides $[K'_i : k']$, since a field of representatives exists; this allows us to conclude. (ii) The result is trivial if the extension K/k is étale, since $[K : k] = [K' : k']$ and, in this case, $\kappa' = K'$. If K/k is finite radicial, $[K : k]$ is a power of p and, as we have seen, $[\kappa' : k']$ divides its degree. This allows us to conclude. □

2.1.4. We call **étale ℓ'-decomposed topology** the Grothendieck topology on alt / S, denoted $\text{ét}_{\ell'}$, defined by the pretopology consisting of étale ℓ'-decomposed coverings.

2.2. Sorites on the ℓ'-decomposed locus.

2.2.1. For each morphism of schemes $f : Y \to X$, let

$$\text{déc}_{\ell'}(f) = \{x \in X : \exists y \in Y \text{ such that } f(y) = x,\ [\kappa(x) : \kappa(y)] \text{ is finite and belongs to } \ell'\}.$$

When $\ell = \infty$, we recover the set $\text{cd}(f)$ introduced by Y. Nisnevič. We will also use this notation.

Proposition 2.2.2. *Let $f : Y \to X$ be an étale morphism, with X noetherian. The set* $\text{déc}_{\ell'}(f)$ *is* ind-constructible, *that is — since X is noetherian — a union of locally closed subsets.*

Proof. We can assume X and Y are integral, with generic points denoted η and μ respectively. By noetherian induction, it suffices to show that if η belongs to the set $\text{déc}_{\ell'}(f)$, then this set contains an open subset of X. We can further assume X and Y are affine with rings A and B respectively, and the morphism $A \to B$ is *finite*. The function $\mathfrak{p} \mapsto \dim_{\kappa(\mathfrak{p})} B/\mathfrak{p}B$, $X \to \mathbf{N}$, is locally constant for the Zariski topology because B is flat of finite presentation — hence locally free — over A. It takes the value $[\kappa(\mu) : \kappa(\eta)]$ — prime to ℓ — at η. This is therefore the case in a neighborhood of η; the conclusion follows immediately. □

Let's specify this result a bit when $\ell = \infty$.

Proposition 2.2.3. *Let $f : Y \to X$ be a locally of finite type morphism, with X noetherian. A point $x \in X$ belongs to* $\text{cd}(f)$ *if and only if there exists a subscheme Z of X containing x over which f has a section.*

Proof. The condition is of course sufficient. Consider $x \in \text{cd}(f)$; it is the generic point of the reduced closed subscheme $\overline{\{x\}}$. By hypothesis, there exists a section over this point. Since the morphism f is locally of finite presentation, this section extends by passing to the limit to an open set $Z = U \cap \overline{\{x\}}$ of $\overline{\{x\}}$, where U is an open set of X. □

Corollaire 2.2.4. *Let X be a noetherian scheme and $(U_i \xrightarrow{f_i} X)_{i \in I}$ be a covering of X for the étale ℓ'-decomposed topology. Then there exists a finite subset $I_0 \subset I$ such that the family $(U_i \to X)_{i \in I_0}$ is also a covering.*

Recall that an étale morphism is, by definition, locally of finite presentation.

Proof. According to [**ÉGA** IV 1.9.15], the topological space X^{cons}, whose underlying space is X and whose open sets are the ind-constructible subsets of X, is *compact* because X is coherent. The sets $\text{déc}_{\ell'}(f_i)$ constitute a covering by open sets. □

Proposition 2.2.5. *Let $(U_\alpha \xrightarrow{f_\alpha} X_\alpha)_{\alpha \in A}$ be a projective, filtering, cartesian system of étale morphisms between noetherian schemes, with affine transition morphisms. Let $f_\infty : U_\infty \to X_\infty$ be the morphism induced on the projective limit and, for each α, let π_α be the projection $X_\infty \to X_\alpha$. We have*

$$\text{cd}(f_\infty) = \bigcup_\alpha \pi_\alpha^{-1}(\text{cd}(f_\alpha)).$$

In particular, if $\text{cd}(f_\infty) = X_\infty$, *there exists α_0 such that* $\text{cd}(f_\alpha) = X_\alpha$ *for each $\alpha \geq \alpha_0$.*

Proof. The inclusion of the right-hand side in the left-hand side is obvious. Conversely, consider a point x_∞ in $\mathrm{cd}(f_\infty)$. The morphism f_∞ has a section on a subscheme of finite presentation Z_∞ containing x_∞. The morphism and the section can be descended by passing to the limit to a finite level α (see [**ÉGA** IV 8.6.3, 8.8.2]). If $\mathrm{cd}(f_\infty) = X_\infty$, it follows from the compactness of X_∞^{cons} — by coherence of X_∞, since the morphisms $X_\infty \to X_\alpha$ are affine — that $X_\infty = \pi_\alpha^{-1}(\mathrm{cd}(f_\alpha))$ for α sufficiently large. The equality $X_\alpha = \mathrm{cd}(f_\alpha)$ for large α then follows from [**ÉGA** IV 8.3.11]. □

The four preceding statements are valid, *mutatis mutandis*, when one makes assumptions of local finite presentation on the morphisms and coherence on their targets.

For future reference, let us mention the following result on the descent of a section.

Proposition 2.2.6. *Let k' / k be a finite field extension of degree prime to ℓ and K / k be a finite extension of degree a power of ℓ. Let $K' = k' \otimes_k K$. If the morphism* $\mathrm{Spec}(K') \to \mathrm{Spec}(K)$ *has a section, then the morphism* $\mathrm{Spec}(k') \to \mathrm{Spec}(k)$ *also has a section: it is an isomorphism.*

(Note that a morphism $X \to \mathrm{Spec}(k)$ has a section if and only if the morphism $X_{\mathrm{réd}} \to \mathrm{Spec}(k)$ deduced from it has a section.)

Proof. This follows from the fact that the image of k' in K by the composite morphism $k' \to K' \to K$, where the second arrow is the retraction whose existence is assumed, is both of degree prime to ℓ and of degree a power of ℓ over k. □

2.3. Topology of ℓ'-alterations.

2.3.1. The **topology of ℓ'-alterations** is the Grothendieck topology on alt / S, denoted $\mathrm{alt}_{\ell'}$, defined by the pretopology generated by

(i) *ℓ'-decomposed* étale coverings;
(ii) proper, surjective, *maximally ℓ'-finite* morphisms.

Be careful that the second condition ("maximally ℓ'-finite") concerns the maximal points while the first ("ℓ'-decomposed") concerns all points.

Remarques 2.3.2.

(i) The preceding families do not constitute a pretopology in the sense of [**SGA 4** II 1.3]: the condition of stability under composition is not satisfied. The other conditions are, in particular the existence of fiber products for morphisms (1.2.6).
(ii) A variant of the preceding definition can be found in [**Kelly, 2012**, § 3].

2.3.3. The topology $\mathrm{alt}_{1'}$ is called the **topology of alterations**, denoted simply alt.

3. Standard forms

In this paragraph, we fix a noetherian scheme S and X an object of alt / S.

3.1. Étale topology. The case $\ell = 1$ of the statement below is a well-known prototype of the results we are going to establish.

Proposition 3.1.1. *Any covering family $(U_i \to X)_{i \in I}$ for the étale ℓ'-decomposed topology is dominated by an $\mathrm{alt}_{\ell'}$-covering family of the type*

$$(V_i \to Y \to X)_{i \in I}$$

where $Y \to X$ is finite, maximally ℓ'-finite, surjective and $(V_i \to Y)_{i \in I}$ is a completely *decomposed étale covering. If $\ell = 1$, we can assume that $(V_i \to Y)_{i \in I}$ is a covering by Zariski open sets.*

Proof. We can assume the set I is finite (2.2.4) and X is integral. By passing to the limit, we can further assume X is normal, with a field of fractions having a pro-ℓ (absolute) Galois group. (The scheme X is therefore not necessarily noetherian.) Now, an ℓ'-decomposed étale morphism from such a scheme is necessarily completely decomposed because the Galois group of the residue fields is also pro-ℓ. For the complement when $\ell = 1$, see e.g. [**Orgogozo, 2006**, lemma 10.3]. □

3.2. Topology of alterations. In this subparagraph, we fix a prime number ℓ.

Théorème 3.2.1. *Assume the noetherian scheme X is irreducible. Any covering family for the topology of ℓ'-alterations of X is dominated by a covering family of the following type:*

$$(V_i \to Y \to X)_{i \in I},$$

where Y is an integral scheme, $Y \to X$ is proper and surjective of generic degree prime to ℓ, and $(V_i \to Y)_{i \in I}$ is a covering for the completely decomposed topology. If, moreover, $\ell = 1$, we can assume that $(V_i \to Y)_{i\in I}$ is a covering by Zariski open sets.

Let us begin with the proof of the particular case $\ell = 1$, although it follows from the general case (together with 3.1.1 for the complement).

Proof in the case where $\ell = 1$. We can assume the set I is finite. According to 3.1.1, it suffices to show that if $(U_i \to X)_{i\in I}$ is a covering of X by Zariski open sets and $(X_i \to U_i)_{i\in I}$ is a collection of proper and surjective morphisms, there exists a proper and surjective morphism $Y \to X$ in alt / S and a covering by Zariski open sets $(V_i \to Y)_{i\in I}$ such that each composite morphism $V_i \to X$ factors through $X_i \to X$. Let's choose for each i a compactification $\overline{X_i} \to X$ of $X_i \to X$; we have $\overline{X_i}_{|U_i} = X_i$. Let $Y = \overline{X_1} \times_X \cdots \times_X \overline{X_r}$, where $I = \{1, ..., r\}$, and $V_1 = X_1 \times_X \overline{X_2} \times_X \cdots \times_X \overline{X_r}$, $V_2 = \overline{X_1} \times_X X_2 \times_X \overline{X_3} \times_X \cdots \times_X \overline{X_r}$, etc. The open sets V_i cover the scheme $\overline{X}$, which is proper and surjective over X. By projection on the i-th factor, each V_i maps to X_i. Up to applying the functor $T \mapsto T_{\mathrm{md}}$ (1.2.5), which transforms a proper and surjective morphism (resp. a Zariski covering) into a proper and surjective morphism (resp. into a Zariski covering) (see 1.2.8), we obtain a covering of the desired type in alt / S. If, as we assumed, X is irreducible, the scheme Y can be assumed to be integral. □

Proof in the general case. It suffices to check that if $(U_i \to X)_{i\in I}$ is a completely decomposed étale covering of X and $(X_i \to U_i)_{i\in I}$ is a collection of proper, surjective, maximally ℓ'-finite morphisms, there exists a family as in the statement dominating it. We can assume the set I is finite and the scheme X is integral. Let's start by treating the case where X is universally Japanese (e.g., quasi-excellent); we can then also assume it is normal, which we do. We are free to assume the U_i are connected (normal) and the morphisms $X_i \to U_i$ are *finite*, surjective, flat, and of generic degree prime to ℓ: since each $X_i \to X$ is generically flat (X is reduced), up to replacing X by a modification $X' \to X$, we can platify them ([**Raynaud & Gruson, 1971**, I 5.2.2]), which makes them *finite*. Up to considering only one irreducible component of each X_i, of generic degree prime to ℓ, and normalizing it, we can assume the X_i are normal connected. Let's summarize:

— the schemes X, U_i, and X_i are normal integral domains (with fields of fractions denoted respectively K, K_i, and K_i');

— the morphisms $X_i \to U_i$ are finite of generic degrees prime to ℓ.

Let L be a quasi-Galois (=normal) closure over K of a composite extension of the K_i', and consider X_L the normalization of X in L. Similarly, consider for each i, the fiber product $U_{iL} = U_i \times_X X_L$ (resp. the reduced normalized fiber product $X_{iL} = (X_i \times_X X_L)^{\mathrm{nor}}_{\mathrm{réd}}$). Given the choice of L, the morphisms $X_{iL} \to U_{iL}$ admit a section over Spec(L). These morphisms being finite and the U_{iL} being *normal*, the sections extend to sections $\sigma_i : U_{iL} \to X_{iL}$. Let now S_ℓ be an ℓ-Sylow subgroup of $\mathrm{Aut}(L/K) = G$ and let p be the characteristic exponent of the field K. If $\ell = p$, the extensions K_i'/K are thus étale, so we can assume L/K is étale, hence *Galois*; the extension L^{S_ℓ}/K is then of degree $(G : S_\ell)$, prime to ℓ. If $\ell \neq p$, the extension L^{S_ℓ}/K is of degree $(G : S_\ell)$ multiplied by a power of p; it is therefore also an integer prime to ℓ. As above, let $X_{L^{S_\ell}}$ be the normalization of X in L^{S_ℓ}, and $U_{iL^{S_\ell}}$ (resp. $X_{iL^{S_\ell}}$) be the fiber product (resp. the reduced normalized fiber product) of U_i (resp. X_i) with $X_{L^{S_\ell}}$ over X. The morphism $X_{L^{S_\ell}} \to X$ is finite, of generic degree prime to ℓ. From the preceding, we have for each i a commutative diagram of normal schemes:

$$\begin{tikzcd}
X_{iL^{s_\ell}} \arrow[r] \arrow[d, "\text{prime to } \ell"'] & X_{iL} \arrow[d] \\
U_{iL^{s_\ell}} \arrow[r, "\text{power of } \ell"'] & U_{iL} \arrow[u, bend right, "\sigma_i"']
\end{tikzcd}$$

By considering the irreducible components of the normal scheme $U_{iL^{s_\ell}}$ in isolation and, for each of them, a maximal point of $X_{iL^{s_\ell}}$ of generic degree prime to ℓ above it, one immediately shows (cf. proposition 2.2.6) that there exists a section $U_{iL^{s_\ell}} \to X_{iL^{s_\ell}}$. This completes the proof of the proposition because the $U_{iL^{s_\ell}}$ form a covering for the completely decomposed topology of the scheme $X_{L^{s_\ell}}$, which is irreducible and of generic degree prime to ℓ over X.

If X is not universally Japanese, we write the normalizations above as limits of finite affine morphisms and use classical theorems of passage to the limit to descend the schemes constructed on the normalizations. Since we will only use this result in the excellent case, we leave the details to the reader. (One could also proceed by "absolute noetherian approximation," that is, write the coherent scheme X as a filtered limit of schemes of finite type over $\mathbf{Z}$, with the transition morphisms being affine.) □

Remarque 3.2.2. It follows from proposition 3.1.1 that in the definition of § 2.3, one can replace the condition of being ℓ'-decomposed by that of being *completely* decomposed (resp. Zariski, if $\ell = 1$). This also follows from the preceding theorem because by passing to the limit over the ℓ'-alterations Y of X, one obtains a normal scheme whose function field has a pro-ℓ absolute Galois group. This property passes to the residue fields, so that an ℓ'-decomposed étale covering of such a scheme is necessarily *completely* decomposed.

We will also make use of the following variant of the preceding theorem.

Théorème 3.2.3. *Assume the noetherian scheme X is irreducible. Any covering family for the topology of ℓ'-alterations of X is dominated by a covering family of the following type:*

$$(W_i \to V_i \to Y \to X)_{i \in I},$$

where all schemes are irreducible, $Y \to X$ is proper birational, $(V_i \to Y)_{i \in I}$ is a covering for the completely decomposed topology, and the morphisms $W_i \to V_i$ are finite, flat, of degree prime to ℓ.

We have a variant for a reduced X not necessarily irreducible: consider the coproduct of its irreducible components.

Proof sketch. By platification, it suffices to show the following exchange result: if $Y \to X$ is finite, flat, of generic degree prime to ℓ and $(V_i \to Y)_{i \in I}$ is a completely decomposed étale covering, there exists a completely decomposed étale covering $(U_j \to X)_{j \in J}$ and morphisms $Z_j \to U_j$, finite, flat, of degree prime to ℓ such that the family of composite morphisms $(Z_j \to X)$ dominates that of the $(V_i \to X)$. By passing to the limit, we can assume X is henselian. By hypothesis, there exists a connected component Y^0 of Y which is flat of degree prime to ℓ over X. The scheme Y^0 being local henselian, there exists an index i such that the restriction of the morphism $V_i \to Y$ to Y^0 has a section, so that the composite morphism $Y^0 \to X$ factors through V_i. This allows us to conclude. □

4. Applications

4.1. Sorites.

Proposition 4.1.1. *Let X be a noetherian scheme, x a point of X, $(X_i \to X)_{i=1,\dots,n}$ a covering for the topology of alterations and, for each index i, an open set $X_i^0 \hookrightarrow X_i$ containing the fiber $(X_i)_x$. There exists an open Zariski neighborhood U of x such that the family $(X_i^0{}_{|U} \to U)_i$ is* alt-*covering.*

A particularly useful case — and to which one could reduce according to the limit results below — is when X is local with closed point x, so that $U = X$.

Proof. According to theorem 3.2.1, the family $(X_i \to X)_i$ is dominated by a family $(V_j \hookrightarrow Y \to X)_j$ where $f : Y \to X$ is notably proper and surjective, and the $(V_j \to Y)_j$ form an open covering of Y. For each j, let Y_j^0 be the inverse image of $X_i^0 \hookrightarrow X_i$ by the morphism $V_j \to X_i$ whose existence is assumed. By hypothesis, over the point x of X, the open sets Y_j^0 and V_j of Y coincide, so that the open sets Y_j^0 cover the fiber Y_x. Their union $Y^0 \coloneqq \bigcup_j Y_j^0$ is an open set of Y, containing this fiber, and we readily check that the open set $U = X - f(Y - Y^0)$ of X is suitable. □

Proposition 4.1.2. *Let $S' \to S$ be a maximally dominant morphism between noetherian schemes. The base change functor* $\mathrm{alt}/S \to \mathrm{alt}/S'$, $X \mapsto X'' = (X \times_S S')_{\mathrm{md}/S'}$, *commutes with fiber products and transforms an* $\mathrm{alt}_{\ell'}$*-covering family into an* $\mathrm{alt}_{\ell'}$*-covering family.*

4.1.3. In other words, the preceding functor induces a *morphism of sites* ([**SGA 4** iv 4.9]) from alt/S' to alt/S, where both categories are equipped with the topology of ℓ'-alterations. (We use here the criterion [**SGA 4** iii 1.6].) When $S' \to S$ is flat (hence maximally dominant, see 1.1.3), it follows from 1.1.5 and the preceding proposition that if $(X_i \to X)$ is an $\mathrm{alt}_{\ell'}$-covering family in alt/S, the family $(X_i \times_S S' \to X \times_S S')$ of morphisms in $\mathrm{alt}\ /\ S'$, obtained by usual base change, is also $\mathrm{alt}_{\ell'}$-covering.

Proof of the proposition. Commutation with fiber products is formal (use 1.2.7). Let's check that a covering family of a scheme $X \in \mathrm{Ob}\,\mathrm{alt}/S$ is transformed by base change into a covering family of X''. We can assume X is irreducible and the covering family is of the type $(V_i \to Y \to X)$ as in 3.2.1. By base change, we obtain a family $(V_i'' \to Y'' \to X'')$, where $(V_i'' \to Y'')$ is also étale and completely decomposed, taking into account the fact that $V_i'' = V_i \times_Y Y''$. To conclude, it suffices to show that the proper and surjective morphism (1.2.8) $Y'' \to X''$ is maximally ℓ'-finite; this follows from 2.1.3 (i). □

4.1.4. Let's now consider a filtering projective system $(X_\alpha)_{\alpha\in A}$ of noetherian schemes, with affine and *dominant* transition morphisms. Let's fix a prime number ℓ invertible on these schemes. Let X_∞ be the limit of the X_α; we assume it is a noetherian scheme. (We leave it to the reader to consider the case of coherent schemes with a finite number of irreducible components.) It follows from [**ÉGA** iv$_3$ 8.4.2 a) i)] that, for sufficiently large α, the morphism $X_\infty \to X_\alpha$ induces a bijection on the maximal points. In particular, for sufficiently large α and $\infty \geq \beta \geq \alpha$, the morphisms $X_\beta \to X_\alpha$ are maximally dominant and induce bijections on the sets of maximal points. It formally follows from this observation and the general results of *op. cit.* (notably [**ÉGA** iv 8.7.1, 8.10.5 (vi,xii)]) that the category alt/X_∞ is the 2-colimit of the categories alt/X_α, the functors $\mathrm{alt}/X_\alpha \to \mathrm{alt}/X_\beta$ being those considered in the preceding proposition.

From the above, we are free to assume the transition morphisms are maximally dominant; we can therefore consider the morphisms of *sites* $(\mathrm{alt}/X_\beta, \mathrm{alt}_{\ell'}) \to (\mathrm{alt}/X_\alpha, \mathrm{alt}_{\ell'})$. One can show that the site $(\mathrm{alt}/X_\infty, \mathrm{alt}_{\ell'})$ is the 2-limit ([**SGA 4** vi 8.2-3]) $\lim_\alpha(\mathrm{alt}/X_\alpha, \mathrm{alt}_{\ell'})$. Let's content ourselves with checking the following fact:

Proposition 4.1.5. *Under the hypotheses of 4.1.4, any* $\mathrm{alt}_{\ell'}$*-covering family of X_∞ is dominated by the inverse image of an* $\mathrm{alt}_{\ell'}$*-covering family of one of the X_α.*

Note that *a priori*, the "inverse image" in the statement is defined using the functor $T \mapsto T_{\mathrm{md}/X_\infty}$. However, as we have seen, this base change and the naive base change (usual fiber product) coincide for sufficiently large α.

Proof. Given the preceding, we can assume X_∞ and the X_α are *irreducible*. According to theorem 3.2.1, it suffices to check that (1) a proper and surjective X_∞-scheme Y_∞ which is integral of degree prime to ℓ and (2) a finite covering family for the completely decomposed étale topology can be descended to schemes and families of the same type on X_α for sufficiently large α. In the first case, only the statement on the degrees remains to be checked; it follows from 2.1.3 (ii). In the second case, we use [**ÉGA** iv$_4$ 17.7.8 (ii)] to descend a completely decomposed étale covering $(U_{i\infty})_i$ of X_∞ to an étale covering $(U_{i\alpha} \to X_\alpha)$. According to 2.2.5, applied to the morphisms $f_\beta : X_\beta \times_{X_\alpha} \coprod_i U_{i\alpha} \to X_\beta$ for $\beta \geq \alpha$, the induced étale covering of X_β is completely decomposed when β is sufficiently large. □

4.2. Pointwise characterization. Let us end with a characterization of the topology of alterations similar to the characterization of the étale topology using strictly henselian local rings.

THÉORÈME 4.2.1 ([**Goodwillie & Lichtenbaum, 2001**], 3.5). *Let X be a noetherian scheme and Y be an object of* alt/X. *The arrow $Y \to X$ is a covering for the topology of alterations if and only if it is valuatively surjective in the following sense: any maximally dominant morphism* $\mathrm{Spec}(V) \to X$, *where V is a valuation ring with an algebraically closed field of fractions, lifts to a morphism* $\mathrm{Spec}(V) \to Y$.

Proof. Let's show that the condition is necessary. According to the preceding theorem, we can assume $Y \to X$ is proper and surjective. (The case where Y is associated with a covering by Zariski open sets is clear because $\mathrm{Spec}(V)$ is local.) Let η_V be the generic point of $\mathrm{Spec}(V)$. The morphism $\eta_V \to X$ lifts to a (non-unique) morphism $\eta_V \to Y$ by the Nullstellensatz, because $Y \to X$ is surjective and $\kappa(\eta_V)$ is algebraically closed. It follows from the valuative criterion of properness that the morphism $\eta_V \to Y$ extends to an X-morphism $\mathrm{Spec}(V) \to Y$. (Note that it is not necessary to assume $\mathrm{Spec}(V) \to X$ is maximally dominant.)

Let's show that the condition is sufficient. For simplicity, we can assume X is affine integral, with generic point denoted η. We can also assume Y is affine, so that there exists an open immersion $Y \hookrightarrow \overline{Y}$ into a scheme that is proper and surjective over X. Let's choose an algebraic closure of $\kappa(\eta)$ inducing a generic geometric point $\overline{\eta} \to X$, and consider the Zariski-Riemann space $\mathrm{ZR}_{\overline{\eta}}(X)$, the limit of the ringed spaces X', where X' is an integral X-scheme, proper and surjective, endowed with an X-morphism $\overline{\eta} \to X'$. One can show that it is quasi-compact (see [**Goodwillie & Lichtenbaum, 2001**, 3.5] or [**Zariski & Samuel, 1975**, chap. VI, th. 40] for a variant) and that if $X = \mathrm{Spec}(A)$, the map that to an intermediate valuation ring $A \subset V \subset \kappa(\overline{\eta})$ associates the point of $\mathrm{ZR}_{\overline{\eta}}(X)$ corresponding by the valuative criterion of properness is a bijection.

Any lift $r : \overline{\eta} \to Y$ of the geometric generic point of X induces an X-morphism $\overline{\eta} \to \overline{Y}$ and therefore a continuous morphism $\pi_r : \mathrm{ZR}_{\overline{\eta}}(X) \to \overline{Y}$, which factors through the irreducible component of $\overline{Y}$ reached by r. By hypothesis, the open sets $\pi_r^{-1}(Y)$, for varying r, cover $\mathrm{ZR}_{\overline{\eta}}(X)$. By quasi-compactness, there thus exists a finite number of lifts $r_1, \ldots, r_n : \overline{\eta} \to Y$ such that $\mathrm{ZR}_{\overline{\eta}}(X)$ is covered by the $\pi_{r_i}^{-1}(Y)$. Let $\overline{Z}$ be the irreducible component of the fiber product $\overline{Y} \times_X \cdots \times_X \overline{Y}$ (n factors) with generic point the image of $r_1 \times_X \cdots \times_X r_n$. By construction, the morphism $\mathrm{ZR}_{\overline{\eta}}(X) \to X$ factors through the X-scheme $\overline{Z}$, which is thus the *union* of the n open preimages of Y. Thus, $Y \to X$ can be refined into a covering by Zariski open sets of a proper and surjective scheme. QED. ☐

REMARQUE 4.2.2. There is an analogue of theorem 4.2.1 for the topology of ℓ'-alterations, the case already treated being that where $\ell = 1$. When $\ell = \infty$, there is no condition on the field of fractions of the rings V; if ℓ is a prime number, we restrict to valuation rings V that are:
— with field of fractions K whose absolute Galois group is pro-ℓ;
— perfect if $\ell \neq \mathrm{car.}(K)$.

4.3. Reduction of local uniformization theorems to the henselian case. Recall that one of the objectives of this book is to prove the local uniformization theorems **VII**-1.1 and **IX**-1.1 (also stated in **Intro.**-2) whose statements we reproduce.

THÉORÈME 4.3.1 ([**Gabber, 2005b**], 1.1). *Let X be a quasi-excellent noetherian scheme and Z be a rare closed subset of X. There exists a finite family of morphisms $(X_i \to X)_{i \in I}$, which is a covering for the topology of alterations and such that for all $i \in I$ we have:*

(i) *the scheme X_i is regular and integral;*
(ii) *the inverse image of Z in X_i is the support of a strict normal crossings divisor.*

By convention, the empty set is considered to be a strict normal crossings divisor: it is a sum indexed by the empty set.

THÉORÈME 4.3.2 (*op. cit.*, 1.3). *Let X be a quasi-excellent noetherian scheme, Z be a rare closed subset of X, and ℓ be a prime number invertible on X. There exists a finite family of morphisms $(X_i \to X)_{i \in I}$, which is a covering for the topology of ℓ'-alterations and such that for all $i \in I$ we have:*

(i) *the scheme X_i is regular and integral;*
(ii) *the inverse image of Z in X_i is the support of a strict normal crossings divisor.*

We end this talk with the following reduction.

PROPOSITION 4.3.3. *If one of the preceding uniformization theorems is true for every local noetherian henselian excellent normal scheme X (resp. for every local noetherian henselian excellent normal scheme X of dimension less than or equal to a fixed integer d), then it is true in general (resp. for every noetherian excellent scheme of finite dimension less than or equal to d).*

The reduction to the case where X is a *complete* noetherian local scheme is much more delicate, and is the subject of the next talk.

Proof. Assume theorem 4.3.1 (resp. 4.3.2) is proven in the local henselian excellent case. Let X be a quasi-excellent noetherian scheme and Z a rare closed subset. We can assume X is normal and integral because the normalization morphism is a covering for the topology of ℓ'-alterations and the inverse image of Z remains rare. Fix $x \in X$. According to [**ÉGA** IV 18.7.6] the henselization $X_{(x)}$ of X at x is excellent and $Z_{(x)}$ is a rare closed subset of $X_{(x)}$. There thus exists a finite family of diagrams

$$\begin{tikzcd} Y \arrow[d] & Y_i \arrow[l] \arrow[d] \\ X_{(x)} & X_i \arrow[l] \end{tikzcd}$$

in $\mathrm{alt}/X_{(x)}$, where Y is integral, proper and surjective (resp. and of generic degree prime to ℓ) over $X_{(x)}$, $(Y_i \to Y)$ is a covering family for the Zariski (resp. completely decomposed) topology, and the family $(X_i \to X_{(x)})$ satisfies conclusions (i) and (ii) above. It follows from the proof of proposition 4.1.5 that this family of diagrams extends to a family of the same type on a completely decomposed étale neighborhood U of x in X. It remains to check that properties (i) and (ii) are indeed preserved. If a morphism $T \to X_{(x)}$ of finite type, with T *regular*, is the base change of a morphism $V \to U$ of finite type where U is an étale neighborhood of x, the scheme V is regular at the points of the image of the morphism $T \to V$. (A local scheme is regular if and only if its henselization is.) In particular, V is regular at the points of the fiber V_x. The regular locus being open, we can assume according to 4.1.1 — up to shrinking the neighborhood U of x — that V is regular. Finally, it follows from [**ÉGA** IV 19.8.1 (ii)] that the property of being a strict normal crossings divisor is descended if it is satisfied at the limit. We conclude by quasi-compactness of X for the completely decomposed étale topology (which is coarser than the alt and $\mathrm{alt}_{\ell'}$ topologies). The "resp." case of proposition 4.3.3 is an immediate corollary of the proof of the non-"resp." case. □

EXPOSÉ III

Approximation

Luc Illusie and Yves Laszlo

1. Introduction

We show here how to reduce the proof of the uniformization theorem (6.1) to the complete local noetherian case (6.2). For this, we use Popescu's theorem (which implies that excellent henselian noetherian local rings satisfy Artin's approximation property, cf. **I**-10.3) and methods for approximating complexes of length 2 adapted from [**Conrad & de Jong, 2002**] (cf. section 4).

The oral presentation given by Alban Moreau used stronger results on the approximation of complexes (due to Ofer Gabber) than those used here (4.5). A written version of his presentation was very useful for the writing of this text: we thank him for it. We also thank Fabrice Orgogozo for pointing out that the statement [**Conrad & de Jong, 2002**, 3.1] was sufficient for the intended applications.

2. Models and Artin-Popescu style approximations

Let A be a noetherian local ring, $\mathfrak{m}$ its maximal ideal, $\hat{A}$ its completion. We assume A is excellent and henselian. Let $\pi : \hat{S} = \mathrm{Spec}(\hat{A}) \to S = \mathrm{Spec}(A)$ be the canonical morphism. For any $n \geq 0$, we denote by

$$i_n : S_n \hookrightarrow \hat{S}$$

the closed immersion defined by the ideal $\hat{\mathfrak{m}}^{n+1} = \mathfrak{m}^{n+1}\hat{A}$ of $\hat{A}$. The composition

$$\pi i_n : S_n \to \hat{S} \to S$$

is the closed immersion $S_n \hookrightarrow S$ defined by the ideal $\mathfrak{m}^{n+1}$.

Definition 2.1. Let $g : \hat{S} \to T$ and $f : S \to T$ be morphisms of schemes and $n \in \mathbf{N}$. We will say that f and g are $(n+1)$**-close** if their restrictions $f\pi i_n$ and $g i_n$ to S_n coincide.

If X is an $\hat{S}$-scheme, we denote by X_n the S_n-scheme $X \times_{\hat{S}} S_n \to S_n$.

Let us write $\hat{A}$ as an inductive limit over a filtered ordered set E of finitely generated A-algebras A_α. We have commutative diagrams

(2.1.1)
$$\begin{tikzcd} & S_\alpha = \mathrm{Spec}(A_\alpha) \arrow[d, "t_\alpha"] \\ \hat{S} \arrow[ur, "s_\alpha"] \arrow[r, "\pi"'] & S \end{tikzcd}$$

with t_α of finite type and an isomorphism $\hat{S} = \varprojlim S_\alpha$ [**ÉGA** IV$_3$ 8.2.3].

Definition 2.2. Let X be an $\hat{S}$-scheme of finite type and $h : X \to Y$ a morphism of $\hat{S}$-schemes of finite type.

(i) A **model** of X over S_α is a cartesian diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] \arrow[dr, phantom, "\square"] & X_\alpha \arrow[d, "f_\alpha"] \\ \hat{S} \arrow[r] & S_\alpha \end{tikzcd}$$

where X_α is of finite type over S_α.

(ii) A **model** of h over S_α is an S_α-morphism $h_\alpha : X_\alpha \to Y_\alpha$ of S_α-schemes of finite type, endowed with an isomorphism $h \xrightarrow{\sim} (h_\alpha)_{\hat{S}}$.

Models of X over S_α exist provided that α is large enough [**ÉGA** IV$_3$ 8.8.3]. Moreover, if X_α, X_β are models of X over S_α, S_β, there exists $\gamma \geq \alpha, \beta$ and an S_γ-isomorphism

$$X_\alpha \times_{S_\alpha} S_\gamma \xrightarrow{\sim} X_\beta \times_{S_\beta} S_\gamma$$

(*loc. cit.*). Similarly, models h_α of $h : X \to Y$ over S_α exist provided that α is large enough and the pullbacks of such models h_α, h_β over S_γ are S_γ-isomorphic for $\gamma \geq \alpha, \beta$ large enough.

If T is an S-scheme and B is an A-algebra, we denote by $T(B) = \mathrm{Hom}_S(\mathrm{Spec}(B), T)$ the set of S-points of T with values in B. According to Popescu's theorem [**Popescu, 1986**, 1.3], since A is excellent and henselian, it satisfies the Artin approximation property, cf. **I**-10.3. Thus, as $S_\alpha \to S$ is of finite type, $S_\alpha(A)$ is dense in $S_\alpha(\hat{A})$ (for the $\mathfrak{m}$-adic topology). Therefore, for any $n > 0$ there exists a section $u : S \to S_\alpha$ of t_α which is n-close to $s_\alpha : \hat{S} \to S_\alpha$. We then define X_u by the cartesian diagram

(2.2.1)
$$\begin{tikzcd} X_u \arrow[r] \arrow[d, "f_u"'] \arrow[dr, phantom, "\square"] & X_\alpha \arrow[d, "f_\alpha"] \\ S \arrow[r, "u"] & S_\alpha \end{tikzcd}$$

Since u is n-close to s_α, we have by definition the equality

$$u\pi i_n = s_\alpha i_n$$

so that the restriction of $X_u \to S$ to S_n is identified with $X_n \to S_n$, in other words we have a cartesian square

(2.2.2)
$$\begin{tikzcd} X_n \arrow[r] \arrow[d, "f_n"'] \arrow[dr, phantom, "\square"] & X_u \arrow[d, "f_u"] \\ S_n \arrow[r, "i_n"] & S \end{tikzcd}$$

Similarly, if X, Y are of finite type over $\hat{S}$ and h_α is a model of $h \in \mathrm{Hom}_{\hat{S}}(X, Y)$ over S_α, the pullback $h_u : X_u \to Y_u$ is an S-morphism inducing the restriction $h_n : X_n \to Y_n$ of h over S_n.

3. Approximations and the topology of alterations

Let us start with a reminder (cf. lecture II) on the topology of alterations. Let T be a noetherian scheme. The category alt$/T$ is the full subcategory of the category of T-schemes whose objects are the reduced T-schemes of finite type X, for which every maximal point is sent to a maximal point of T with a finite residual extension. Note that the morphisms in alt$/T$ send maximal points to maximal points. We define two topologies on alt$/T$.

(i) The *topology of alterations* is the coarsest for which the following families are covering
  (a) Zariski open coverings;
  (b) proper and surjective morphisms.
  A covering family for the topology of alterations will be called alt-covering.
(ii) Let ℓ be a prime number. The *topology of ℓ'-alterations* on alt$/T$ is the coarsest for which the following families are covering
  (a) Nisnevich étale coverings;
  (b) proper surjective morphisms $X' \to X$ such that for every maximal point η of X, there exists a maximal point η' of X' over η with ℓ not dividing $\deg(k(\eta')/k(\eta))$.
  A covering family for the topology of ℓ'-alterations will be called alt$_{\ell'}$-covering.

For any T-scheme X of finite type, we denote by X_{md} the reduced closed subscheme of X which is the union of the irreducible components that dominate an irreducible component of T.

Proposition 3.1. *We resume the notation of 2: let A be a noetherian local ring, $\mathfrak{m}$ its maximal ideal, $\hat{A}$ its completion. We assume A is excellent and henselian. Let $\pi : \hat{S} = \mathrm{Spec}(\hat{A}) \to S = \mathrm{Spec}(A)$ be the canonical morphism. Let $X \to \hat{S}$ be a non-empty object of* alt$/\hat{S}$. *We further assume S is integral. Let us choose a model X_β of X over S_β, for some index $\beta \in E$ (cf. 2.2), and for $\alpha \geq \beta$ let us denote by X_α the model deduced by base change.*

(i) *Then, there exist $\alpha_0 \in E, \alpha_0 \geq \beta$, and an integer $n_0 > 0$ such that for all $\alpha \geq \alpha_0$, all integers $n \geq n_0$, and any section $u : S \to S_\alpha$ of t_α that is n-close to $s_\alpha : \hat{S} \to S_\alpha$, X_u (2.2.1) has a finite generic fiber and the composite morphism $(X_u)_{\mathrm{md}} \to X_u \to S$ is a non-empty object of* alt$/S$.

(ii) *Assume $X \to \hat{S}$ is* alt*-covering. Then, there exist $\alpha_0 \in E, \alpha_0 \geq \beta$, and an integer $n_0 > 0$ such that for all $\alpha \geq \alpha_0$, all integers $n \geq n_0$, and any section $u : S \to S_\alpha$ of t_α that is n-close to $s_\alpha : \hat{S} \to S_\alpha$, the composite morphism $(X_u)_{\mathrm{md}} \to X_u \to S$ is* alt*-covering.*

(iii) *Assume $X \to \hat{S}$ is* $\mathrm{alt}_{\ell'}$*-covering. Then, there exist $\alpha_0 \in E, \alpha_0 \geq \beta$, and an integer $n_0 > 0$ such that for all $\alpha \geq \alpha_0$, all integers $n \geq n_0$, and any section $u : S \to S_\alpha$ of t_α that is n-close to $s_\alpha : \hat{S} \to S_\alpha$, the composite morphism $(X_u)_{\mathrm{md}} \to X_u \to S$ is* $\mathrm{alt}_{\ell'}$*-covering.*

Proof. First, let us observe that, since S is henselian and excellent, $\hat{S}$ is integral, cf. **I**-6.3.

Let us prove (i). Since $X \to \hat{S}$ is generically finite, there exists $a \in \hat{A} - \{0\}$ such that X is finite, surjective, and free of rank $d > 0$ over the non-empty open set $\hat{S} - V(a)$. We can choose α_0 large enough so that

- a comes from $a_\alpha \in A_\alpha - \{0\}$ for $\alpha \geq \alpha_0$;
- $X_\alpha \to S_\alpha$ is finite, surjective ([**ÉGA** IV$_3$ 8.10.5]) and free of rank d over $S_\alpha - V(a_\alpha)$ (use [**ÉGA** IV$_3$ 8.5.2]).

Let us then choose an integer n such that $a \notin \hat{\mathfrak{m}}^{n+1}$. For any $\alpha \geq \alpha_0, m \geq n$, and any section u that is m-close to t_α,, we have

$$u^\star(a_\alpha) \notin \mathfrak{m}^{n+1}$$

and thus $u^\star(a_\alpha)$ is non-zero. This ensures that X_u is finite, surjective and free of rank d over the non-empty open set $S - V(u^\star(a_\alpha))$, the pullback of $S_\alpha - V(a_\alpha)$ by u. The first point follows from this.

Let us prove (iii) [The proof of (ii) is completely similar]. We therefore assume that $X \to \hat{S}$ is $\mathrm{alt}_{\ell'}$-covering. We know (**II**-3.2.1) that $X \to \hat{S}$ is dominated in alt$/\hat{S}$ by a standard covering

$$Y \to X' \to \hat{S}$$

with

- $Y \to X'$ a Nisnevich covering
- $X' \to \hat{S}$ proper and surjective, whose restriction to each irreducible component is dominant and generically finite, with the generic degree of one of them being prime to ℓ.

Up to replacing the reduced scheme X' by a suitable component and Y by the induced Nisnevich $\mathrm{alt}_{\ell'}$-covering, we can assume that X' is *integral* of generic degree $\deg(X'/\hat{S}) = \delta$ prime to ℓ.

Let η be the generic point of S. The construction $X \mapsto X_{\mathrm{md}}$ is functorial for the full subcategory of S-schemes X with finite generic fiber. Now, according to (i), for $\alpha_0 \geq \beta$ large enough and choices of models $Y_{\alpha_0} \to X'_{\alpha_0} \to S_{\alpha_0}$, $Y_{\alpha_0} \to X_{\alpha_0} \to S_{\alpha_0}$ and of a suitable section u of t_α $(\alpha \geq \alpha_0)$, we know that Y_u, X'_u and X_u have a finite generic fiber. We thus have a factorization

$$\begin{tikzcd} (Y_u)_{\mathrm{md}} \arrow[r] \arrow[d] & (X_u)_{\mathrm{md}} \arrow[d] \\ (X'_u)_{\mathrm{md}} \arrow[r] & S \end{tikzcd}$$

Now, still according to (i), we can also assume that $(Y_u)_{\mathrm{md}}, (X'_u)_{\mathrm{md}}$ and $(X_u)_r$ are objects of alt$/S$. To conclude that $(X_u)_{\mathrm{md}} \to S$ is $\mathrm{alt}_{\ell'}$-covering, it suffices to prove that for a suitable u, $(Y_u)_{\mathrm{md}} \to S$ is $\mathrm{alt}_{\ell'}$-covering.

Taking into account the usual permanence properties of models [**ÉGA** IV$_3$ 8.8.3 and 8.10.5], the proof of (i) ensures that for suitable models and u, the morphism $X'_u \to S$ is proper and surjective and its generic fiber has degree prime to ℓ. This ensures that the restriction of $X'_u \to S$ to at least one of the reduced components of X'_u dominating S has degree prime to ℓ. Thus, $(X'_u)_{\mathrm{md}} \to S$ is indeed $\mathrm{alt}_{\ell'}$ *– covering*.

The property of being a Nisnevich covering (resp. proper and surjective) being stable under base change, it remains to prove the following lemma.

Lemma 3.1.1. *There exists $\alpha_0 \geq \beta$ such that for any $\alpha \geq \alpha_0$, the model $Y_\alpha \to X'_\alpha \to S_\alpha$ of $Y \to X' \to \hat{S}$ (deduced from $Y_{\alpha_0} \to X'_{\alpha_0} \to S_{\alpha_0}$) has the property that $Y_\alpha \to X'_\alpha$ is a Nisnevich covering.*

Proof. To say that the morphism $Y \to X'$ is a Nisnevich covering is to say that it is smooth, quasi-finite, and that we have a stratification

$$\varnothing = X'_0 \subset X'_1 \cdots \subset X'_n = X'$$

with X'_i closed in X' and Y/X' has a section over $X'_{i+1} - X'_i$. The conclusion follows immediately from this remark and the usual permanence properties of models [**ÉGA** IV$_3$ 8.8.3 and 8.10.5] and [**ÉGA** IV$_4$ 17.7.8]. □

□

The aim of what follows is to improve the topological results of proposition 3.1 by showing that suitable thickenings of the normal cones of the special fibers of X (resp. X_u) in X (resp. X_u) are isomorphic. This will allow us to prove statements about the stability of properties in the passage from X to X_u, in this case dimension and regularity (corollary 5.4).

4. Higher graded modules and approximations of complexes

Let I be an ideal of a ringed topos $(\mathscr{X}, \mathscr{O})$, $\mathscr{F}$ an $\mathscr{O}$-module of $\mathscr{X}$ and a an integer ≥ 1. We set $I^n = \mathscr{O}$ if $n \leq 0$. We define the **Z**-graded module

$$\mathrm{gr}_a(\mathscr{F}) = \bigoplus_{n \in \mathbf{Z}} I^n \mathscr{F} / I^{n+a} \mathscr{F}$$

which is thus the sum

$$\mathrm{gr}_a(\mathscr{F}) = \mathscr{F}/I\mathscr{F} \oplus \cdots \oplus \mathscr{F}/I^a F \oplus I\mathscr{F}/I^{a+1}\mathscr{F} \oplus I^2\mathscr{F}/I^{a+2}\mathscr{F} \oplus \cdots$$

concentrated in degrees $\geq -(a-1)$. It is an $\mathscr{O}/I^a$-module; moreover, the product

$$I^n \otimes I^m \to I^{n+m}$$

induces a structure of a **Z**-graded $\mathscr{O}/I^a$-algebra on $\mathrm{gr}_a(\mathscr{O})$ and $\mathrm{gr}_a(\mathscr{F})$ is a **Z**-graded $\mathrm{gr}_a(\mathscr{O})$-module.

We are interested here in the case where $\mathscr{X}$ is the Zariski topos of an S-scheme X ringed by its structure sheaf $\mathscr{O}$ and $I = \mathfrak{m}\mathscr{O}$.

Remarque 4.1. The tautological surjective morphism $\mathrm{gr}_a(\mathscr{O}) \to \mathrm{gr}_1(\mathscr{O})$ has for its kernel $J = I \cdot \mathrm{gr}_a(\mathscr{O})$. We thus have $J^a = 0$ (since J is an $\mathscr{O}_{X_{a-1}}$-module) so that $C_a(X) = \mathrm{Spec}(\mathrm{gr}_a(\mathscr{O}))$ is a thickening of order $a-1$ of the normal cone $\mathrm{Spec}(\mathrm{gr}_1(\mathscr{O}))$.

Definition 4.2. Let X, Y be S-schemes (resp. $\hat{S}$-schemes). An ***a*-isomorphism** $X \xrightarrow{\sim}_a Y$ is the data of an S-isomorphism $\phi : X_{a-1} \xrightarrow{\sim} Y_{a-1}$ and an isomorphism of graded $\mathrm{gr}_a(A)$-algebras $\phi^{-1}\mathrm{gr}_a(\mathscr{O}_Y) \xrightarrow{\sim} \mathrm{gr}_a(\mathscr{O}_X)$. In this case, we say that X, Y are ***a*-close**.

We will then identify their special fibers X_0, Y_0 thanks to the isomorphism $X_{a-1} \xrightarrow{\sim} Y_{a-1}$.

4.3. We adapt here theorem 3.2 of [**Conrad & de Jong, 2002**] (and the key lemma 3.1 of *loc. cit.*). Let us start with a definition. Let B be a noetherian ring and I an ideal of B.

Definition 4.4. Let $f : M \to N$ be a morphism of B-modules of finite type. An integer $c \geq 0$ is an **Artin-Rees constant** for f if for any $n \geq c$ we have

$$I^n N \cap \mathfrak{I}(f) \subset I^{n-c}\mathfrak{I}(f).$$

The Artin-Rees lemma ensures the existence of an Artin-Rees constant.

Proposition 4.5. *Let $(L^\bullet, d_L^\bullet), (M^\bullet, d_M^\bullet)$ be complexes of free B-modules of finite type concentrated in degrees $[-2, 0]$ with $L^i = M^i$ for all i. Let c be an Artin-Rees constant for d_L^{-2} and d_L^{-1} and n an integer $\geq c$. Suppose $\mathrm{H}^{-1}(L^\bullet) = 0$ and*

$$d_L^\bullet = d_M^\bullet \mod I^{n+1}.$$

Then:

(i) *c is an Artin-Rees constant for d_M^{-1};*
(ii) *if I is contained in the radical of A* [(i)], $\mathrm{H}^{-1}(M^\bullet) = 0$*;*
(iii) *The identity of $L^0 = M^0$ induces an isomorphism of $\mathrm{gr}_{n+1-c}(B)$-modules*

$$\mathrm{gr}_{n+1-c}(\mathrm{H}^0(L^\bullet)) \xrightarrow{\sim} \mathrm{gr}_{n+1-c}(\mathrm{H}^0(M^\bullet));$$

(iv) *Moreover, if $L^0 = M^0 = B$, the preceding isomorphism is an isomorphism of $\mathrm{gr}_{n+1-c}(B)$-algebras, in other words the algebras $\mathrm{H}^0(L^\bullet)$ and $\mathrm{H}^0(M^\bullet)$ are $(n+1-c)$-isomorphic.*

Proof. The first two points are proven in lemma 3.1 of *loc. cit.* The last one is trivial. Point (iii) remains.

For $n = c$, it is theorem 3.2 of *loc. cit.*, whose proof we only adapt to the case $n > c$. Let $m \in \mathbf{Z}$. We write d_L, d_M for d_L^{-1}, d_M^{-1}. For $\delta = d_L, d_M$, we have

$$\mathrm{gr}^m_{n+1-c}(\mathrm{Coker}(\delta)) = I^m L^0/(I^{m+n+1-c}L^0 + I^m L^0 \cap \mathfrak{I}(\delta))$$

so that we need to show the equality

$$I^{m+n+1-c}L^0 + I^m L^0 \cap \mathfrak{I}(d_L) = I^{m+n+1-c}L^0 + I^m L^0 \cap \mathfrak{I}(d_M)$$

for all $m \in \mathbf{Z}$. Let $x \in L^{-1}$ be such that $d_L(x) \in I^m L^0$.

Suppose $m \leq c$. Since

$$d_L(x) - d_M(x) \in I^{n+1}L^0 \text{ and } m \leq c \leq n,$$

we have $d_L(x) - d_M(x) \in I^m L^0$ so that

$$d_M(x) = d_L(x) + d_M(x) - d_L(x) \in I^m L^0 \cap \mathfrak{I}(d_M).$$

Since $n + 1 \geq m + n + 1 - c$, we also have

$$d_L(x) - d_M(x) \in I^{n+1}L^0 \subset I^{m+n+1-c}L^0$$

so that

$$d_L(x) = d_L(x) - d_M(x) + d_M(x) \in I^{m+n+1-c}L^0 + I^m L^0 \cap \mathfrak{I}(d_M)$$

and thus

$$I^{m+n+1-c}L^0 + I^m L^0 \cap \mathfrak{I}(d_L) \subset I^{m+n+1-c}L^0 + I^m L^0 \cap \mathfrak{I}(d_M).$$

By symmetry of the roles of d_L and d_M, we have the desired equality in this case.

If $m > c$, the calculation is analogous. We have (4.4)

$$I^m L^0 \cap \mathfrak{I}(d_L) \subset I^{m-c} d_L(L^{-1})$$

so that

$$d_L(x) = d_L(x') \text{ with } x' \in I^{m-c}L^{-1}.$$

Since $d_L - d_M = 0 \mod I^{n+1}$, the matrix of $d_L - d_M$ has coefficients in I^{n+1} so that

$$d_L - d_M \in I^{n+1}\mathrm{Hom}_B(L^{-2}, L^{-1}).$$

[(i)] This hypothesis is missing in lemma 3.1 of *loc. cit.*

We thus have

$$d_L(x') - d_M(x') \in I^{n+1} I^{m-c} L^0 = I^{n+1+m-c} L^0.$$

Since

$$d_M(x') = d_L(x') + d_M(x') - d_L(x') = d_L(x) + d_M(x') - d_L(x'),$$

we have on one hand

$$d_M(x') \in (I^m L^0 + I^{n+1+m-c} L_0) \cap \mathfrak{I}(d_M) \subset I^m L^0 \cap \mathfrak{I}(d_M),$$

because $n \geq c$, and, on the other hand,

$$d_L(x) = d_L(x') - d_M(x') + d_M(x') \in I^{m+n+1-c} L^0 + I^m L^0 \cap \mathfrak{I}(d_M).$$

We conclude as above by symmetry. □

5. Models and a-isomorphisms

Théorème 5.1 (Approximation). *Let A be a noetherian local ring, $\mathfrak{m}$ its maximal ideal, $\hat{A}$ its completion. We assume A is excellent and henselian. Let $\pi : \hat{S} = \operatorname{Spec}(\hat{A}) \to S = \operatorname{Spec}(A)$ be the canonical morphism. Let X be of finite type over $\hat{S}$. We are also given $\alpha_0 \in E$ and a model (cf. 2.2) X_{α_0} of X over S_{α_0}. For any $\alpha \geq \alpha_0$ we denote $X_\alpha = X_{\alpha_0} \times_{S_{\alpha_0}} S_\alpha$ the model of X over S_α deduced by base change. There exist $\alpha_1 \geq \alpha_0$ and integers $n_0 \geq c > 0$ such that for any $n \geq n_0, \alpha \geq \alpha_1$ and any section u of t_α which is $(n+1)$-close to s_α, there exists a unique $(n+1-c)$-isomorphism $X \tilde{\to}_{n+1-c} X_u$ over the isomorphism $X_{n-c} \to (X_u)_{n-c}$ deduced from (2.2.2).*

Definition 5.2. Under the preceding conditions, we say that (X_α, α, u) (or, if no confusion is to be feared, X_u) is an **approximation** of X over S (of order $n-c$).

The assertion "There exist α_0, an integer n_0 such that for any $n \geq n_0, \alpha \geq \alpha_0$ and any section u of t_α that is $(n+1)$-close to s_α, X_u satisfies property P" can sometimes be condensed to "Any sufficiently fine approximation X_u of X satisfies property P". We will use analogous terminology for approximations of $\hat{S}$-morphisms.

Proof. Two $(n+1-c)$-isomorphisms differ by an automorphism

$$\iota : \operatorname{gr}_{n+1-c}(\mathscr{O}_X) \tilde{\to} \operatorname{gr}_{n+1-c}(\mathscr{O}_X)$$

of graded $\mathscr{O}_{X_{n-c}}$-algebras. It is in particular $\mathscr{O}_S$-linear. Since $\operatorname{gr}_{n+1-c}(\mathscr{O}_X)$ is generated over $\operatorname{gr}_a(\mathscr{O}_S)$ by $\mathscr{O}_{X_{n-c}}$, the automorphism ι is the identity. Hence the uniqueness.

We can therefore assume X is affine. Since X is of finite type over $\hat{S}$, X embeds into the affine space

$$\mathbf{A}^m_{\hat{S}} = \operatorname{Spec}(\hat{A}[t])$$

with coordinates $t = (t_1, \cdots, t_m)$ as the closed subscheme with ideal

$$J = \langle \tilde{P}_1, \cdots, \tilde{P}_N \rangle$$

where $\tilde{P}_i \in B = \hat{A}[t]$. Let us choose a partial resolution of the B-module $C = B/J$ by free B-modules of finite type

$$B^a \xrightarrow{\tilde{R}} B^b \xrightarrow{\tilde{P}=(\tilde{P}_i)} B \to C \to 0 \tag{5.2.1}$$

where $\tilde{R}$ is a matrix with coefficients in B.

For α_0 large enough, $\tilde{P}$ and $\tilde{R}$ come from matrices $P_{\alpha_0}, R_{\alpha_0}$ with coefficients in

$$B_{\alpha_0} = A_{\alpha_0}[t] \text{ such that } PR = 0$$

so that the closed set F of $\mathbf{A}^m_{A_{\alpha_0}}$ with equations $P_{\alpha_0,1} = \cdots = P_{\alpha_0,N} = 0$ is a model of X over S_{α_0}. As recalled in section 2, up to changing α_0 to a larger index, we can assume that we have $F = X_{\alpha_0}$. For $\alpha \geq \alpha_0$, let P_α, R_α be the matrices with coefficients in B_α deduced from $P_{\alpha_0}, R_{\alpha_0}$ by the morphism

$$B_{\alpha_0} = A_{\alpha_0}[t] \to B_\alpha = A_\alpha[t].$$

For any $\alpha \geq \alpha_0$, the matrices with coefficients in B deduced from P_α, R_α by the morphism

$$B_{\alpha_0} = A_{\alpha_0}[t] \to B = \hat{A}[t]$$

are the same: we denote them by P, R.

We have reduced, for $\alpha \geq \alpha_0$, to the case where

$$X_\alpha = \operatorname{Spec}(C_\alpha) \text{ with } C_\alpha = B_\alpha/(P_\alpha).$$

We thus have on one hand a complex (in degrees [-2,0]) of free B_α-modules

$$L_\alpha = (B_\alpha^a \xrightarrow{R_\alpha} B_\alpha^b \xrightarrow{P_\alpha=(P_{i,\alpha})} B_\alpha)$$

with $\mathrm{H}^0(L_\alpha) = C_\alpha$. The complex of free B-modules of finite rank

$$L = B \otimes_{B_\alpha} L_\alpha = (B^a \xrightarrow{R} B^b \xrightarrow{P=1\otimes P_\alpha} B)$$

is acyclic in degree -1 by construction.

Remarque 5.3. *A priori*, there is no reason for L_α to be acyclic in degree -1, even for large α.

On the other hand, the section u of t_α is defined by a morphism of A-algebras

$$u^\star : A_\alpha \to A$$

such that

$$u^\star \mod \mathfrak{m}^{n+1} = s_\alpha^\star \mod \hat{\mathfrak{m}}^{n+1},$$

where $s_\alpha^\star : A_\alpha \to \hat{A}$ is defined by $s_\alpha : \hat{S} \to S_\alpha$ (2.1.1). By acting on the coefficients of the polynomials, we obtain a ring morphism

$$\bar{u} : B_\alpha = A_\alpha[t] \to A[t] \to \hat{A}[t] = B$$

hence a complex

$$M = (B^a \xrightarrow{\bar{u}(R)} B^b \xrightarrow{\bar{u}(P)} B)$$

By construction, we have

$$L/\mathfrak{m}^{n+1}L = M/\mathfrak{m}^{n+1}M.$$

We then choose an Artin-Rees constant c for $B^b \xrightarrow{P} B$ and invoke proposition 4.5 to conclude. □

Corollaire 5.4. *Let X, Y be S-noetherian schemes that are a-close. Let $x \in X_0 = Y_0$.*

(i) *If $a \geq 1$, the dimensions of X and Y at x are the same.*
(ii) *If $a \geq 2$ and X is regular at x, then Y is regular at x.*
(iii) *Assume $X \to \hat{S}$ is of finite type and X is regular. Let X_β be a model of X over S_β, and for $\alpha \geq \beta$, let us denote by X_α the model over S_α deduced by base change. Then, there exist $\alpha_0 \in E, \alpha_0 \geq \beta$, and an integer $n_0 > 0$ such that for any $\alpha \geq \alpha_0$, any integer $n \geq n_0$, and any section u of t_α that is n-close to $s_\alpha : \hat{S} \to S_\alpha$, the scheme X_u deduced from X_α is regular in an open neighborhood of the special fiber.*

Proof. By hypothesis, the normal cones of X_0, Y_0 in X, Y are S-isomorphic. Since the dimension of X at x is equal to that of its normal cone [**Matsumura, 1989**, 15.9], the first point follows.

Now assume that X, Y are 2-close. According to (i), we know that X and Y have the same dimension at x. Since X, Y are 2-close, X_1 and Y_1 are isomorphic. Since the Zariski tangent space to X at a point of X_0 only depends on X_1, the Zariski cotangent $k(x)$-vector spaces at x to X and Y are isomorphic, hence (ii).

For the last point, it suffices to invoke the first two and theorem 5.1 to conclude that a sufficiently fine approximation is regular in a neighborhood of the special fiber. Since X_u is excellent (as it is of finite type over excellent S), its regular locus R is open, so that R is a regular open neighborhood of the special fiber. □

Remarque 5.5. O. Gabber knows how to generalize proposition 4.5 to the case where the considered complexes are only of finite type over a noetherian ring to obtain closeness of cohomology also in degree -2 (and not only in degrees $0, -1$). He can more precisely show closeness statements for the images and kernels of the differentials[(ii)]. Gabber deduces from this numerous permanence statements by approximation analogous to corollary 5.4. In particular, if X, Y are a-close for a large enough, then X being reduced (resp. normal) along X_0 implies Y is reduced (resp. normal) along Y_0. However, several natural questions remain open, such as the permanence of the properties S_n, R_n.

6. Reduction to the complete local noetherian case

Let us recall the statement of the uniformization theorem (**Intro.**-2, **II**-4.3.2).

Théorème 6.1 (Uniformization). *Let T be a quasi-excellent noetherian scheme and Z a nowhere dense closed subset of T. Let ℓ be a prime number invertible on T. There exists a finite family of morphisms $(X_i \to T)_{i\in I}$ such that for each $i \in I$ we have*

(i) *The finite family of morphisms $(X_i \to T)_{i\in I}$ is* alt*-covering (resp.* $\mathrm{alt}_{\ell'}$*-covering);*
(ii) *X_i is regular and integral;*
(iii) *the inverse image of Z in X_i is empty or the support of a strict normal crossings divisor.*

We are going to show the following reduction statement.

Proposition 6.2. *If (6.1) is true for any complete, local, noetherian T, then (6.1) is true.*

Proof. We can first assume T is local, excellent, and henselian (let us recall (**I**-6.3) that a local, henselian, and quasi-excellent scheme is excellent). See **II**-4.3.3 for this reduction.

So, let us assume T is local, noetherian, henselian, excellent.

Up to replacing T by the disjoint sum of its reduced components, we reduce to the case where T is also integral.

We can moreover assume $T = \mathrm{Spec}(A)$ is normal and integral. Indeed, since A is excellent, the normalization morphism is finite with generic degree 1, so it is alt-covering (resp. $\mathrm{alt}_{\ell'}$-covering). Since A is local, integral, and henselian, A is unibranch, so that the normalization of A is local, thus integral, and is noetherian and henselian since it is finite over A.

Since A is excellent, normalization commutes with completion (**I**-6.2) so that $\hat{A}$ is thence normal like A, thus also integral since it is normal and local.

We can therefore assume T is local, integral, normal, henselian, and excellent.

Since $\hat{T}$ is flat over T, the inverse image $\hat{Z}$ of Z is still a nowhere dense closed subset of $\hat{T}$. Let us choose a uniformization

$$(\tilde{X}_i \to \hat{T})_{i\in I'}$$

of $(\hat{T}, \hat{Z})$ as in 6.1. According to 3.1, 5.1 and 5.4, we can find $\alpha_0 \in E$ and an integer $n_0 \geq 1$ such that, for any $\alpha \geq \alpha_0$, any $n \geq n_0$ and any u that is n-close to s_α, we have models $(\tilde{X}_i)_\alpha$ of the $\tilde{X}_i$ over T_α and n-isomorphisms $\tilde{X}_i \to_n (\tilde{X}_i)_u$ such that

a) each T-scheme $(\tilde{X}_i)_u$ is regular along its special fiber $(\tilde{X}_i)_0$, thus in a neighborhood (the regular locus being open since the considered schemes are excellent).

b) the family $((\tilde{X}_i)_u)_{\mathrm{md}}$ is alt-covering (resp. $\mathrm{alt}_{\ell'}$-covering).

According to a), $(\tilde{X}_i)_u$ is regular in a neighborhood of the special fiber and is there the disjoint union of its connected components which are integral. Note that, for given α and $n \geq n_0$, since the kernel of $A_\alpha/\mathfrak{m}^n A_\alpha \to A/\mathfrak{m}^n A$ is of finite type, there exists $\beta \geq \alpha$ such that $A_\alpha/\mathfrak{m}^n A_\alpha \to A_\beta/\mathfrak{m}^n A_\beta$ factors through $A/\mathfrak{m}^n A$, and thus any section of t_β gives a section of t_α which is n-close to s_α. Thus, up to increasing α_0 and n_0 (or just α_0), we can assume that $(\tilde{X}_i)_u = ((\tilde{X}_i)_u)_{\mathrm{md}}$ in a neighborhood of the special fiber. This is indeed a consequence of the preservation of dimension (5.4 (i)). To see this, let us choose, as in the proof of 3.1, a non-zero element a of $\widehat{A}$ such that $\tilde{X}_i - V(a)$ is finite and flat over $\widehat{T} - V(a)$. We can assume the same holds for $(\tilde{X}_i)_u - V(a') over T - V(a'), where a' = u^\star(a_\alpha)$. We can assume that at each point x of the special fiber the local rings of X_i and $(\tilde{X}_i)_u$ have the same dimension

(ii)The proof of this generalization was presented by A. Moreau during the oral seminar.

d, and similarly for $V(a)$ and $V(a')$. Since $X_i - V(a)$ is dense in X_i, the dimension of the local ring of $V(a)$ at x is $<$ d, so the same holds for $V(a') \subset (\tilde{X}_i)_u$. As $\mathscr{O}_{(\tilde{X}_i)_u,x}$ is regular, the irreducible component of $(\tilde{X}_i)_u$ passing through x is thus dominant.

Thus, in a neighborhood of the special fiber, $(\tilde{X}_i)_u$ is schematically the disjoint union of components dominating T. Since a sufficiently small open neighborhood of the special fiber $(\tilde{X}_i)_0$ in $(\tilde{X}_i)_u$ is alt-covering (resp. $\mathrm{alt}_{\ell'}$-covering) (**II**-4.1.1), the family $(X_i \to T)_{i\in I}$ of connected components of suitable neighborhoods of the $(\tilde{X}_i)_0$ in $(\tilde{X}_i)_u$, $i \in I'$ satisfies conditions (i) and (ii).

Let D' be the inverse image of $\hat{Z}$ in $\tilde{X} = \coprod_{i\in I'} \tilde{X}_i$, which we can assume is non-empty. By hypothesis, $D = D'_{\text{réd}}$ is a strict normal crossings divisor, that is $D = \sum_{j\in J} D_j$ with

$$D_K = \bigcap_{j\in K} D_j$$

regular of codimension card(K) for any subset $K \subset J$. Up to increasing α, we can assume that the D_j have models over T_α, these models inducing models of the D_K. Since u is a section of t_α, the scheme D_u, the schematic union of the $(D_i)_u$, is, topologically, the inverse image of Z in X_u. According to 5.4, we can assume that each $(D_u)_K$ is regular everywhere of codimension card(K) along the special fiber, so that D_u is a strict normal crossings divisor along the special fiber. The regular loci of $(D_u)_K$ and X_u being open, we can assume that D_u is a strict normal crossings divisor in a neighborhood of the special fiber (excellence of X_u). □

EXPOSÉ IV

The Cohen-Gabber Theorem

Fabrice Orgogozo

1. p-bases and differentials (reminders)

1.1. Definition and differential characterization.

1.1.1. For the convenience of the reader, and to set the notations, we recall here some well-known results which we will use hereafter. We advise the reader to refer to them only when necessary.

Définition 1.1.2. Let k be a field of characteristic $p > 0$, K an extension of k, and $(b_i)_{i\in I}$ a family of elements of K. We say that the (b_i) constitute a **p-base** of K over k (resp. are **p-linearly independent** over k) if the monomials $\prod_i b_i^{n(i)}$ ($0 \le n(i) < p$, $(n(i))_{i\in I}$ of finite support) form a base of the $k(K^p)$-vector space K (resp. are linearly independent over $k(K^p)$).

If $k = \mathbf{F}_p$, we then speak of an **absolute** p-base, or simply a p-base if there is no ambiguity. Finally, a product as above is sometimes called a **p-monomial**. A connection between this notion and the structure of complete local rings emerges from the following theorem.

Théorème 1.1.3 ([**Bourbaki**, **AC**, IX, §3, n°3, th. 1 b)]). *Let A be a complete separated local ring of characteristic $p > 0$ and $(\beta_i)_{i\in I}$ a family of elements of A whose classes modulo the maximal ideal $\mathfrak{m}_A$ form a p-base of the residue field $A/\mathfrak{m}_A$. There then exists a unique field of representatives of A containing the elements β_i.*

Remarque 1.1.4. The notion of p-base can be obviously extended to the case of any ring of characteristic $p > 0$; see [**ÉGA** 0_{IV} 21.1-4]. We will not need it.

1.1.5. It is immediately verified that the $(b_i)_{i\in I}$ form a p-base of K over k if and only if, for all $i \in I$, the element b_i does not belong to the subfield $k(K^p, (b_j)_{j\neq i})$ of K. (See e.g. [**ÉGA** 0_{IV} 21.4.3].)

1.1.6. For any field extension K/k, we denote by $d_{K/k}$ the differential $K \to \Omega^1_{K/k}$.

Proposition 1.1.7. *Let k be a field of characteristic $p > 0$, and K an extension of k. A family $(b_i)_{i\in I}$ of elements of K is a p-base of K over k if and only if the differentials $d_{K/k}(b_i)$ form a basis of the K-vector space $\Omega^1_{K/k}$.*

Démonstration. Let $B = (b_i)_{i\in I}$ be a p-base of K over k. Any (set-theoretic) morphism $\Delta : B \to K$ extends uniquely to a k-derivation D of K: it suffices to set $D(b_1^{n_1} \cdots b_r^{n_r}) = \sum_i n_i b_1^{n_1} \cdots b_i^{n_i-1} \cdots b_r^{n_r} \Delta(b_i)$ and to extend it by $k(K^p)$-linearity. This is equivalent to the fact that the $d_{K/k}(b_i)$ form a basis of $\Omega^1_{K/k}$. Conversely, if the $d_{K/k}(b_i)$ form a basis, we observe that the p-monomials are $k(K^p)$-linearly independent: otherwise, for a suitable index i, we would have $b_i \in K^p((b_j)_{j\neq i})$, which would translate into a linear relation between the differentials. Let B' be a p-base of K over k containing the b_i (*loc. cit.*, 21.4.2); according to the previous implication, we necessarily have $B' = (b_i)_{i\in I}$. □

Corollaire 1.1.8. *Let k be a field of characteristic $p > 0$, and K an extension of k. An element x of K belongs to $k(K^p)$ if and only if $d_{K/k}(x) = 0$.*

1.1.9. Recall that the **p-rank** of a field is the cardinality of an absolute p-base (well-defined by what precedes). It is immediately verified that this *cardinality* (finite or not) is invariant under *finite* field extension.

1.2. Stabilization.

Lemme 1.2.1 (see e.g. [**ÉGA** 0_{IV} 21.8.1]). *Let K be a field, k a subfield, $(k_\alpha)_{\alpha\in I}$ $(I \neq \varnothing)$ a family of subfields of K such that $\bigcap_\alpha k_\alpha = k$ and* decreasingly filtered, *i.e., such that for any pair of indices α, β, there exists an index γ such that $k_\gamma \subset k_\alpha \cap k_\beta$. Let V be a K-vector space, and (v_i) $(1 \leq i \leq n)$ a finite family of vectors of V. If the family (v_i) is free over k, there exists an index γ such that it is also free over k_γ.*

Lemme 1.2.2. *Let K be a field of characteristic $p > 0$, k a subfield and $(K_\alpha)_{\alpha\in I}$ a decreasingly filtered family of subfields containing k. The following conditions are equivalent :*

(i) $\bigcap_\alpha K_\alpha(K^p) = k(K^p)$*;*

(ii) *for any finite set $\{b_1, \dots, b_n\} \subset K$, p-linearly independent over k, there exists an index α such that it is p-linearly independent over K_α ;*

(iii) *there exists a p-base of K over k such that every finite subset is p-linearly independent over some K_α for suitable α ;*

(iv) *the canonical morphism $\Omega^1_{K/k} \to \lim_\alpha \Omega^1_{K/K_\alpha}$ is* injective.

Démonstration. (i)⇒(ii) is an immediate consequence of the preceding lemma. (ii)⇒(iii) is trivial (any p-base works). (iii)⇒(iv) trivial (use 1.1.7). Let us verify (iv)⇒(i). Let $x \notin k(K^p)$. According to 1.1.8, $d_{K/k}(x) \neq 0$ so that there exists α such that $d_{K/K_\alpha}(x)$ is also non-zero. According to *loc. cit.*, this implies that $x \notin K_\alpha(K^p)$. □

From this, we deduce the following lemma, which is a particular case of [**ÉGA** 0_{IV} 21.8.5].

Lemme 1.2.3. *Let K be a field of characteristic p, k a subfield and $(K_\alpha)_{\alpha\in I}$ a decreasingly filtered family of subfields of K containing k such that $\bigcap_\alpha K_\alpha(K^p) = k(K^p)$. For any finite extension L of K, we also have $\bigcap_\alpha K_\alpha(L^p) = k(L^p)$.*

Démonstration. We are immediately reduced to the case where L/K has no non-trivial subextension. If L/K is (algebraic) separable, the conclusion results immediately from the existence of canonical isomorphisms $\Omega^1_{L/k} \xrightarrow{\sim} \Omega^1_{K/k} \otimes_K L$, $\Omega^1_{L/K_\alpha} \xrightarrow{\sim} \Omega^1_{K/K_\alpha} \otimes_K L$ and criterion (iv) above. Otherwise, $L = K(a)$, where $b = a^p \in K - K^p$. We naturally distinguish two cases. First case : $d_{K/k}(b) = 0$, i.e., $b \in k(K^p)$. It follows that for any subextension M of L / k, we have the equality $M(L^p) = M(K^p, b) = M(K^p)$. Thus,

$$\bigcap_\alpha K_\alpha(L^p) = \bigcap_\alpha K_\alpha(K^p) = k(K^p) = k(L^p).$$

Second case : $d_{K/k}(b) \neq 0$. We can then complete $\{b\}$ into a p-base of K over k, which we denote $(b, (b_j)_{j\in J})$. The family $(a, (b_j)_{j\in J})$ is then a p-base of L over k and we immediately verify criterion (iii) above : if $(b, b_1, \dots, b_n)$ is p-linearly independent over K_α, the same holds for $(a, b_1, \dots, b_n)$. □

Proposition 1.2.4 ([**Matsumura, 1980b**], § 30, lemma 6). *Let K be a field of characteristic $p > 0$ and (K_α) a decreasingly filtered family of subfields* cofinite *— that is, such that the degrees $[K : K_\alpha]$ are finite — such that $\bigcap_\alpha K_\alpha = K^p$. Then, for any finite extension L/K, there exists an index β such that for any cofinite subfield $K' \subset K_\beta$ we have :*

$$\operatorname{rang}_L \Omega^1_{L/K'} = \operatorname{rang}_K \Omega^1_{K/K'}.$$

Démonstration. Let us first show that we can assume L/K has no non-trivial subextension. Suppose there exists a sub-K-extension $K \subsetneq M \subsetneq L$, otherwise there is nothing to prove. By induction on $[L : K]$, we can assume the lemma established for the extension M/K. The subfields $M_\alpha = K_\alpha(M^p)$ are cofinite in M and, for α, β and γ as in the statement, we have $M_\gamma \subset M_\alpha \cap M_\beta$. By virtue of the preceding lemma, applied in the particular case where $k = K^p$, we have the equality $\bigcap_\alpha M_\alpha = M^p$. On the other hand, the extensions M_α/K_α are *finite* and, for any subfield K' of K, we have the equalities $\Omega^1_{M/K'} = \Omega^1_{M/K'(M^p)}$ and $\Omega^1_{L/K'} = \Omega^1_{L/K'(M^p)}$. This reduces us to the particular case where $M = K$, then to the case where L/K has no non-trivial subextension.

If this is (algebraic) separable, the theorem is trivial : we have $\Omega^1_{L/K'} \xleftarrow{\sim} \Omega^1_{K/K'} \otimes_K L$ for all $K' \subset K$. Otherwise, $L = K(a)$, where $a^p = b \in K - K^p$, and, for each $K' \subset K$, the module of differentials $\Omega^1_{L/K'}$

is naturally isomorphic to

$$(\Omega^1_{K/K'}/Kd_{K/K'}(b))\otimes_K L \oplus Ld_{L/K'}(a).$$

Furthermore, $d_{K/K^p}(b)$ and $d_{L/L^p}(a)$ are non-zero because b (resp. a) does not belong to K^p (resp. L^p). Since $K^p = \bigcap_\alpha K_\alpha$ (resp. $L^p = \bigcap_\alpha K_\alpha(L^p)$), there exists a β such that $d_{K/K_\beta}(b) \neq 0$ (resp. $d_{L/K_\beta}(a) \neq 0$). It results from the above isomorphism that for each $K' \subset K_\beta$, we have the equality $\mathrm{rang}_L \Omega^1_{L/K'} = \mathrm{rang}_K \Omega^1_{K/K'}$. Q.E.D. □

1.2.5. Finally, recall that if A and B are two linearly topologized rings ([**ÉGA** 0_{I} 7.1.1]), and $A \to B$ a *continuous* morphism, the B-module $\Omega^1_{B/A}$ is a *topological* B-module, the topology being derived from that of $B \otimes_A B$ by restriction and quotienting ([**ÉGA** 0_{IV} 20.4.3]). The underlying B-module does not depend on the topologies of A and B. We denote by $\widehat{\Omega}^1_{B/A}$ its *separated completion*; it is isomorphic to a limit of Ω^1 of morphisms between discrete topological rings (*loc. cit.*, § 20.7.14). For any topological B-module L such that the ring $D_B(L) = B \oplus L$ is linearly topologized(i), the canonical morphism induced by the universal derivation is an *isomorphism* :

$$\mathrm{Hom.cont}_B(\Omega^1_{B/A}, L) \xrightarrow{\sim} \mathrm{Dér.cont}_A(B, L).^{\text{(ii)}}$$

If the topological B-module L is furthermore separated and complete, the previous isomorphism can be rewritten $\mathrm{Hom.cont}_{\widehat{B}}(\widehat{\Omega}^1_{B/A}, L) \xrightarrow{\sim} \mathrm{Dér.cont}_A(B, L)$. As observed in the very simple particular case where A is a field and B a ring of formal power series, the B-module $\widehat{\Omega}^1_{B/A}$ has much more remarkable finiteness properties than $\Omega^1_{B/A}$ (*loc. cit.*, example 20.7.16 and prop. 20.7.15).

2. The Cohen-Gabber theorems in characteristic > 0

2.1. The non-equivariant Cohen-Gabber theorem in characteristic > 0. The purpose of this paragraph is to demonstrate the following variant of the structure theorem of complete Noetherian local rings [**ÉGA** 0_{IV} 19.8.8 (ii)], due to Irving S. Cohen.

Théorème 2.1.1 (Cohen-Gabber theorem ; [**Gabber, 2005a**], lemma 8.1). *Let A be a complete Noetherian* reduced *local ring, of equal characteristic $p > 0$, equidimensional of dimension d and with residue field k. There exists a subring A_0 of A, isomorphic to $k[[t_1, \ldots, t_d]]$, such that A is finite over A_0, torsion-free and generically étale. Furthermore, the morphism $A_0 \to A$ induces an isomorphism on the residue fields.*

Remarques 2.1.2. This result appears explicitly as a hypothesis, for A integral, in [**ÉGA** 0_{IV} 21.9.5]. The expression "generically étale" here means that there exists a dense open subset of $\mathrm{Spec}(A_0)$ above which the morphism $\mathrm{Spec}(A) \to \mathrm{Spec}(A_0)$ is étale.

2.1.3. The proof of the theorem, which is an adaptation to the non-irreducible case of [**Gabber, 2005a**], occupies the remainder of this section. We will henceforth assume $d > 0$, otherwise the statement is obvious. In paragraphs 2.1.4 to 2.1.11, we will show that there exists a field of representatives κ of A such that the completed A-module of differential forms $\widehat{\Omega}^1_{A/\kappa}$ has generic rank equal to d on each irreducible component. In 2.1.12, we will see how to rapidly deduce the theorem from this.

2.1.4. Let $(b_i)_{i\in E}$ be a p-base of $k = A/\mathfrak{m}_A$. Choose arbitrary liftings β_i of the b_i in A. Recall that there exists a unique field of representatives $\kappa \subset A$ containing the β_i and surjecting onto k (1.1.3). Changing the field of representatives therefore amounts to changing the β_i. Also fix a system of parameters $\tau_1, \ldots, \tau_d$ of A; we will only change it at the end of the proof (2.1.12).

(i)This hypothesis implies that the topology of L is defined by a family of sub-B-modules L' such that L/L' is annihilated by an open ideal of B. Conversely, a B-module L equipped with such a topology is a topological B-module (the action $B \times L \to L$ is continuous) and $D_B(L)$ is linearly topologized (i.e., by a family of sub-B-modules).

(ii)As O. Gabber pointed out to us, the previous isomorphism — taken from [**ÉGA** 0_{IV} 20.4.8 (ii)] — can be flawed if we assume only that L is a topological B-module.

2.1.5. For any *finite* subset $e \subset E$, set $\kappa_e := \kappa^p(\beta_i,\ i \notin e) \subset \kappa$. The following three properties are obvious :

$$\text{for any finite subset } e \subset E, [\kappa : \kappa_e] < +\infty,$$

$$\text{for any finite subsets } e, e' \subset E, \kappa_{e \cup e'} \subset \kappa_e \cap \kappa_{e'},$$

$$\bigcap_{e \subset E} \kappa_e = \kappa^p.$$

2.1.6. Let $\mathrm{Spec}(\overline{A})$ be an irreducible component of $\mathrm{Spec}(A)$, endowed with the reduced structure, and $\overline{\tau_1}, \dots, \overline{\tau_d}$ the images of the τ_i in $\overline{A}$ by the canonical surjection $A \twoheadrightarrow \overline{A}$. Consider the diagram of rings :

$$\begin{tikzcd}
\kappa_e[[\overline{\tau_1}^p, \dots, \overline{\tau_d}^p]] \arrow[r] \arrow[d] & \kappa[[\overline{\tau_1}, \dots, \overline{\tau_d}]] \arrow[r] \arrow[d] & \overline{A} \arrow[d] \\
L_{\kappa,e} \arrow[r] & L_\kappa \arrow[r] & L
\end{tikzcd}$$

where the horizontal arrows are the canonical homomorphisms, and the vertical arrows are the inclusions into the respective fraction fields. The horizontal arrows are injective and correspond to finite morphisms. For the second, this follows from the fact that the module $\overline{A}$ is *quasi-finite* ([**ÉGA** 0_{I} 7.4.1]) over $\kappa[[\overline{\tau_1}, \dots, \overline{\tau_d}]]$ and thus of finite type because the ideal $(\overline{\tau_1}, \dots, \overline{\tau_d})\overline{A}$ is a defining ideal ([**ÉGA** 0_{I} 7.4.4]). Finally, the $\overline{\tau_i}$ are analytically independent over κ : the subring $\kappa[[\overline{\tau_1}, \dots, \overline{\tau_d}]]$ of $\overline{A}$ is indeed a formal power series ring ([**ÉGA** 0_{IV} 16.3.10]).

We observed above that the family of $\kappa_e \subset \kappa$, $e \subset E$, satisfies the hypotheses of proposition 1.2.4. It is immediately verified that the same holds for the family of subfields $L_{\kappa,e}$ of L_κ ; we thus have the equality

$$\mathrm{rang}_L \Omega^1_{L/L_{\kappa,e}} = \mathrm{rang}_{L_\kappa} \Omega^1_{L_\kappa/L_{\kappa,e}}, \tag{2.1.6.1}$$

as soon as the finite set e is sufficiently large.

Let $R_\kappa = \kappa[[\overline{\tau_1}, \dots, \overline{\tau_d}]]$ and $R_{\kappa,e} = \kappa_e[[\overline{\tau_1}^p, \dots, \overline{\tau_d}^p]]$. The left term of (2.1.6.1) is the generic rank of the $\overline{A}$-module $\Omega^1_{\overline{A}/R_{\kappa,e}}$, i.e., the rank of its tensor product with L. Note that according to [**ÉGA** 0_{IV} 21.9.4], $\Omega^1_{\overline{A}/R_{\kappa,e}}$ is identified with the completed $\overline{A}$-module $\widehat{\Omega}^1_{\overline{A}/\kappa_e}$ of differential forms. The right term is the rank of the free $R_{\kappa,e}$-module $\Omega^1_{R_\kappa/R_{\kappa,e}}$. The latter is equal to $d + \mathrm{rang}_\kappa \Omega^1_{\kappa/\kappa_e} = d + |e|$ (where $|-|$ denotes the cardinality of a set), so that formula 2.1.6.1 can be rewritten :

$$\mathrm{rang}_{\overline{A}} \widehat{\Omega}^1_{\overline{A}/\kappa_e} = d + |e|. \tag{2.1.6.2}$$

2.1.7. The following proposition will allow us to modify the field of representatives in order to assume e is empty (equivalently : $\kappa_e = \kappa$).

Proposition 2.1.8. *There exists a finite subset e of E and elements β'_i, for $i \in e$, lifting the b_i such that, for each irreducible integral component* $\mathrm{Spec}(\overline{A})$ *of* $\mathrm{Spec}(A)$, *the following conditions are satisfied :*

(i) $\mathrm{rang}_{\overline{A}} \widehat{\Omega}^1_{\overline{A}/\kappa_e} = d + |e|$,

(ii) *the images of the $d\beta'_i$ in $\widehat{\Omega}^1_{\overline{A}/\kappa_e} \otimes_{\overline{A}} L$, where $L = \mathrm{Frac}(\overline{A})$, are L-linearly independent.*

The equality 2.1.6.1 (and thus 2.1.6.2) being valid, for each irreducible component, as soon as e is sufficiently large, one can choose such a set that works for each of them. Property (i) follows from this.

To demonstrate property (ii), we will use the following elementary lemma.

Lemme 2.1.9. *Let $\overline{A}$ and L be as above. For any non-zero ideal I of $\overline{A}$, the set of $df \otimes_{\overline{A}} L$, for $f \in I$, is a generating family of the L-vector space $\widehat{\Omega}^1_{\overline{A}/\kappa_e} \otimes_{\overline{A}} L$.*

Démonstration. Let $f_0 \in I$ be non-zero, and $\omega_0 = df_0$. For any $b \in \overline{A}$, $d(bf_0) = b\omega_0 + f_0 db$. The family of $d(bf_0)\otimes 1$ contains $\omega_0 \otimes 1$; according to the preceding formula, the L-vector space it generates therefore contains the $db \otimes 1$ for each $b \in \overline{A}$. □

Let $\{\mathfrak{p}_1, \dots, \mathfrak{p}_c\}$ be the set of minimal prime ideals of A. For each $j \in \{1, \cdots, c\}$, let $A_j = A/\mathfrak{p}_j$ and $X_j = \mathrm{Spec}(A_j)$ be the corresponding integral irreducible component of $X = \mathrm{Spec}(A)$. For all $i \in e$ and $j \in \{1, \dots, c\}$, denote $\beta_{i,j}$ the image in A_j of $\beta_i \in A$. (Recall that the β_i are part of a p-base of $\kappa \subset A$.) We will prove by induction on j ($0 \leq j \leq c$) that there exist elements $\{m_{i,j}\}$ in $\mathfrak{m}_A$, for $i \in e$, such that the images of the elements $\beta_i + m_{i,j}$ in each of the rings $A_1, \dots, A_j$ have differentials linearly independent in each of the vector spaces $\Omega^1_{A_1/R_{\kappa,e}} \otimes_{A_1} \mathrm{Frac}\ A_1, \dots, \Omega^1_{A_j/R_{\kappa,e}} \otimes_{A_j} \mathrm{Frac}\ A_j$. For $j = 0$, this condition is empty. Assume the assertion demonstrated for some $j \leq c-1$ and let's show it for $j+1$. By replacing β_i with $\beta_i + m_{i,j}$ if necessary, we can assume that $m_{i,j} = 0$ for all $i \in e$. Since the ring A is reduced, the $\mathfrak{p}_\alpha$ form a *reduced* primary decomposition of (0), so that the ideal $\mathfrak{q}_j := \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_j$ ($= \mathrm{Ker}(A \to A_1 \times \cdots \times A_j)$) is not contained in $\mathfrak{p}_{j+1}$. If $j > 0$, denote by I_{j+1} its image in $\overline{A} = A_{j+1} (= A/\mathfrak{p}_{j+1})$; it is a non-zero ideal. If $j = 0$, we consider $\mathfrak{m}_{\overline{A}}$. According to (i), $\mathrm{rang}_{\overline{A}} \widehat{\Omega}^1_{\overline{A}/\kappa_e} = d + |e| \geq |e|$; on the other hand, the family $d(I_{j+1})$ is generating in $\widehat{\Omega}^1_{\overline{A}/\kappa_e} \otimes_{\overline{A}} L$ (where $L = \mathrm{Frac}\ \overline{A}$).

Lemme 2.1.10. *Let V be a vector space of dimension at least n, $b_1, \dots, b_n$ vectors of V and W a generating family. There exists a family $w_1, \dots, w_n$ of elements of $W \cup \{0\}$ such that the $b_i + w_i$ are linearly independent.*

Démonstration. By immediate induction on n. □

There therefore exist elements $m'_{i,j+1} \in I_{j+1}$, $i \in e$, such that the differentials of the elements $d((\beta_i \mod \mathfrak{p}_{j+1}) + m'_{i,j+1})$, $i \in e$, are linearly independent in $\widehat{\Omega}^1_{\overline{A}/\kappa_e} \otimes_{\overline{A}} L$.

Lift the $m'_{i,j+1}$ to elements $m_{i,j+1}$ of $\mathfrak{q}_j$ if $j > 0$, or of $\mathfrak{m}_A$ if $j = 0$. By construction, they satisfy the desired property at step $j+1$.

2.1.11. Consider the unique field of representatives (1.1.3) $\kappa' \subset A$ containing the β_i, for $i \notin e$, and the β'_i, for $i \in e$, the latter chosen as in 2.1.8 and set, as in 2.1.5, $\kappa'_e := \kappa'^p(\beta_i,\ i \notin e) \subset \kappa'$. For each irreducible integral component $\mathrm{Spec}(\overline{A})$ of X, we have the identifications

$$\Omega^1_{\overline{A}/\kappa_e} = \Omega^1_{\overline{A}} / \big(\sum_{i \notin e} \overline{A} d\beta_i\big) = \Omega^1_{\overline{A}/\kappa'_e},$$

and similarly for the completed modules. From equality 2.1.6.2 and property (ii) of 2.1.8, we derive :

$$\mathrm{rang}_{\overline{A}} \widehat{\Omega}^1_{\overline{A}/\kappa'} = d.$$

We now denote this new field of representatives κ'[(iii)].

2.1.12. Since the A-module $\widehat{\Omega}^1_{A/\kappa}$ has generic rank d on each irreducible component, we show by proceeding as before that there exist elements $f_1, \dots, f_d$ of A such that the $d(f_i \bmod \mathfrak{p}_\alpha) \otimes_{A_j} \mathrm{Frac}\ A_j$ form a basis of $\widehat{\Omega}^1_{A_j/\kappa} \otimes_{A_j} \mathrm{Frac}\ A_j$ for each irreducible component $\mathrm{Spec}(A_j)$ of X. We can assume them in $\mathfrak{m}_A$ by multiplying them individually by a p-th power of an element belonging to $\mathfrak{m}_A - \bigcup_j \mathfrak{p}_j$. Recall that we have chosen a system of parameters $\tau_1, \dots, \tau_d$ in A, so that the morphism $\mathrm{Spec}(A) \to \mathrm{Spec}(k[[\tau_1, \dots, \tau_d]])$ is finite.

For $i \in \{1, \dots, d\}$, set

$$t_i := \tau_i^p (1 + f_i).$$

Let A_0 be the subring $\kappa[[t_1, \dots, t_d]]$ of A, $X_0 = \mathrm{Spec}(A_0)$. The morphism $X \to X_0$ is finite : this follows from the fact that the elements $1 + f_i$ are units in A. Let's check that it is generically étale. Since the ring A is Noetherian complete, the A-module of finite type Ω^1_{A/A_0} is also complete and thus coincides

(iii) The author warmly thanks Kazuma Shimomoto [下元数馬] and Kazuhiko Kurano [藏野和彦] for pointing out an error in a previous version of this paragraph.

with the completed module of differential forms $\widehat{\Omega}^1_{A/A_0}$. Since the rings A_0 and A are metrizable, and any sub-A-module of $\widehat{\Omega}^1_{A/\kappa}$ is closed, the sequence

$$\widehat{\Omega}^1_{A_0/\kappa}\widehat{\otimes}_{A_0}A \to \widehat{\Omega}^1_{A/\kappa} \to \widehat{\Omega}^1_{A/A_0} = \Omega^1_{A/A_0} \to 0$$

is *exact* ([**ÉGA** 0_{IV} 20.7.17]). It follows from the hypothesis on the elements f_i and the formula

$$d(t_i) = \tau_i^p d f_i$$

that above each maximal point of $X = \mathrm{Spec}(A)$, the first arrow is surjective. From this we deduce that the A-module Ω^1_{A/A_0} is generically zero, Q.E.D.

2.2. The equivariant Cohen-Gabber theorem in characteristic > 0.

2.2.1. We will demonstrate here a generalization of theorem 2.1.1 in the case of a ring not necessarily equidimensional, endowed with an action of a finite group.

Théorème 2.2.2. *Let A be a complete reduced Noetherian local ring of equal characteristic, dimension d, residue field κ and G a finite group acting on A with $|G|$ invertible in κ. Then, there exists a finite generically étale, G-equivariant morphism, $\kappa[[t_1, \dots, t_d]] \to A$, where $\kappa \to A$ lifts the identity of κ and G acts trivially on the t_i.*

Let's begin with a proposition.

Proposition 2.2.3. *Let A be a ring endowed with an action of a finite group G whose order is invertible in A and let $B = \mathrm{Fix}_G A$ be the subring of invariants.*

(i) *The ring B is*
  (a) *Noetherian if A is;*
  (b) *reduced if A is;*
  (c) *local with maximal ideal $\mathfrak{m} \cap B$ if A is local with maximal ideal $\mathfrak{m}$, and its residue field is isomorphic to the subfield $\mathrm{Fix}_G A \,/\, \mathfrak{m}$ of $\kappa = A \,/\, \mathfrak{m}$.*

(ii) *The morphism $\mathrm{Spec}(A) \to \mathrm{Spec}(B) = \mathrm{Spec}(A) \,/\, G$ is*
  (a) *finite if A is Noetherian;*
  (b) *generically étale if A is furthermore reduced.*

Démonstration. (i) Let Tr be the B-linear morphism $\mathrm{Tr} : A \to B$, $x \mapsto \frac{1}{|G|}\sum_{g\in G} g(x)$, sometimes called the "Reynolds operator". For any ideal I of B, we have $IA \cap B = I$. Indeed, the inclusion $I \subset IA \cap B$ is trivial and the opposite inclusion results from the fact that if $x \in IA \cap B$, its "trace" $x = \mathrm{Tr}(x)$ belongs, by I-linearity, to $IB = I$. From this, statement (a) is immediately deduced. Statement (b) is trivial. If A is local, we have $A - \mathfrak{m} = A^\times$. It follows on the one hand that G globally stabilizes $\mathfrak{m}$ and on the other hand that $\mathrm{Fix}_G A - \mathrm{Fix}_G\, \mathfrak{m} = (\mathrm{Fix}_G A)^\times$. Thus, B is maximal with ideal $\mathfrak{n} = \mathrm{Fix}_G\, \mathfrak{m}$. Finally, the canonical morphism $B \,/\, \mathfrak{n} \to \mathrm{Fix}_G\, \kappa$ deduced from the canonical inclusion $B \,/\, \mathfrak{n} \to \kappa$ is an isomorphism. Indeed, if $a \in A$ is an arbitrary lifting of $\lambda \in \mathrm{Fix}_G\, \kappa$, the element $b = \mathrm{Tr}(a)$ is a G-equivariant lifting. This completes the proof of (c).

(ii.a) We will show that the integral morphism $\mathrm{Spec}(A) \to \mathrm{Spec}(B)$ is finite by reduction to the well-known case where A is a field.

□ Reduction to the reduced case. Let N be the nilradical of A and $M = N \cap B$ that of B. For each integer $i \in \mathbf{N}$, the A/N-module N^i/N^{i+1} is of finite type, because A is assumed Noetherian, and zero for $i \gg 0$. The module $\mathrm{gr}_N(A) = \bigoplus_{n\geq 0} N^i/N^{i+1}$ is therefore of finite type over $\mathrm{gr}^0_N(A) = A/N$. If the latter is of finite type over $B/M = \mathrm{gr}^0_M(B)$, the same holds for $\mathrm{gr}_N(A)$ over $\mathrm{gr}_M(B)$ and finally ([**Bourbaki**, **AC**, III, § 2, nº 9, cor. 1]) for A over B, by completeness of the Noetherian ring B for the M-adic topology.

□ Reduction to the case of a product of fields. Assume A is reduced and consider the finite set $\{\mathfrak{p}_i\}_{i\in I}$ of minimal prime ideals of A. For each i, $\mathfrak{q}_i = \mathfrak{p}_i \cap B$ is a *minimal* prime ideal of B. This results from the Cohen-Seidenberg theorem ([**Bourbaki**, **AC**, V, § 2, nº 1, th. 1 et cor. 2]) and from the transitivity of the action of G on the fibers of $\mathrm{Spec}(A) \to \mathrm{Spec}(B)$ (*op. cit.*, nº 2, th. 2). Let $K = \mathrm{Frac}\, A$ (resp. $L = \mathrm{Frac}\, B$) be the total ring of fractions of A (resp. B); it is a product of fields into which A (resp. B) injects, isomorphic to the semi-localization of A at the $\{\mathfrak{p}_i\}_{i\in I}$ (resp. $\{\mathfrak{q}_i\}_{i\in I}$). Let $S = A - \bigcup_i \mathfrak{p}_i$; we therefore have $K = S^{-1}A$. According to (*op. cit.*, § 1, nº 1, prop. 23), we have $\mathrm{Fix}_G(S^{-1}A) = (\mathrm{Fix}_G S)^{-1}B$, so that $\mathrm{Fix}_G K = L$ and $A \otimes_B L \cong K$. Assume K is finite over L — as will be demonstrated in the next paragraph —, so that according to the previous isomorphism, there exist a finite number n of elements $a_1, \dots, a_n$ of A that generate K over L. To conclude, it suffices to verify that the operator $\mathrm{Tr} : A \to B$ defines, by composition with the product, a pairing $A \otimes_B A \to B$ that is non-degenerate when passing to the rings of fractions, i.e., that if an element $x \in K$ satisfies $\mathrm{Tr}(K \cdot x) = \{0\}$, then $x = 0$. Indeed, if this is the case, the map $A \to B^n$, $a \mapsto \big(\mathrm{Tr}(a_i a)\big)$ is a B-linear *embedding* and we can conclude by Noetherianity of B. The fact that the pairing $K \otimes_L K \to L$ is non-degenerate results from the fact that if e is an idempotent corresponding to a factor $\mathrm{Frac}\, A/\mathfrak{p}$ of K, the element $\mathrm{Tr}(e)$ is equal to $\frac{|H|}{|G|} f$, where H is the stabilizer of e, and f is the idempotent of L corresponding to the factor $\mathrm{Frac}\, B/\mathfrak{q}$, with $\mathfrak{q} = \mathfrak{p} \cap A$.

▫ Reduction to the case of a field. So let $A = \prod_i K_i$ be a finite product of fields and set $X = \mathrm{Spec}(A) = \coprod_i \eta_i$. If $X = X_1 \amalg X_2$, where X_1 and X_2 are G-stable, $X/G = (X_1/G) \amalg (X_2/G)$ so that we are immediately reduced to the case where X/G is connected, i.e., where the action of G is *transitive*. For each i, let G_i be the corresponding decomposition group. According to the classical case (field case), $\eta_i \to \eta_i/G_i$ is finite étale. It follows that the morphism $X \to \coprod \eta_i/G_i$ is finite. Finally, since for all i, $\eta_i/G_i \xrightarrow{\sim} X/G$ (*loc. cit.*, § 2, nº 2, prop. 4), the result (ii.a) follows.

Statement (ii.b) is now obvious. □

2.2.4. Let A and G be as in the statement of theorem 2.2.2. It follows from the preceding proposition that we have the equality $\dim(B) = \dim(A) < +\infty$, where we denote $B = \mathrm{Fix}_G A$. We will denote d their common dimension. Let B/I be the maximal d-equidimensional quotient of B. According to the Cohen-Gabber theorem 2.1.1, there exists a field of representatives $\lambda \hookrightarrow B/I$ and a system of parameters $t_1, \dots, t_d$ of B/I such that $\lambda[[t_1, \dots, t_d]] \to B/I$ is finite, generically étale. We can lift the inclusion $\lambda \hookrightarrow B/I$ to an inclusion $\lambda \hookrightarrow B$: this results, for example, in positive residual characteristic (the only non-trivial case), from the correspondence between subfields of representatives and liftings of a given p-base of the residue field. Finally, we can lift the system of parameters of B/I to a system of parameters of B : this results, by unravelling, from the following lemma.

Lemme 2.2.5. *Let $A \twoheadrightarrow B$ be a surjection of Noetherian local rings and $b \in \mathfrak{m}_B$ be a* secant *element for B, i.e., such that* $\dim(B/b) = \dim(B) - 1$. *There exists a lifting of b in A that is secant for A.*

For generalities on secant sequences, see for example *op. cit.*, chap. viii, § 3, nº 2.

Démonstration. We are immediately reduced to the case where $B = A/(f)$, $f \in A$. Let $a \in A$ be an arbitrary lifting of b; by hypothesis, we have $\dim(A/(f, a)) = \dim(B) - 1$. If $\dim(B) = \dim(A) - 1$, we necessarily have $\dim(A/a) = \dim(A) - 1$ because the dimension drops by at most one per equation. Otherwise, f belongs to the union $\bigcup_{i=1}^n \mathfrak{p}_i$, where the $\mathfrak{p}_i$ are the prime ideals of A of coheight $\dim(A)$. Suppose that $f \in \mathfrak{p}_1, \dots, \mathfrak{p}_r$, and only these ideals. The conclusion can only be false if $a + (f) \subset \bigcup_{i=1}^n \mathfrak{p}_i$, i.e., if all liftings of b are non-secant. For each $i \le r$, we have $a \notin \mathfrak{p}_i$ because f belonging to $\mathfrak{p}_i$, we would have $\dim(A/(f, a)) = \dim(A)$. It follows in particular that $r \neq n$. It suffices therefore to show that the hypothesis $a + (f) \subset \bigcup_{i=r+1}^n \mathfrak{p}_i$ is absurd. Indeed, we would have $a + f^m = a + f \cdot f^{m-1} \in \bigcup_{i=r+1}^n \mathfrak{p}_i$ for all m and finally $f^m(1 - f^{m-m'}) \in \mathfrak{p}_i$ for two integers $m > m'$ and an index $r + 1 \le i \le n$. From this we immediately deduce $f \in \mathfrak{p}_i$, which contradicts the hypothesis. □

2.2.6. Since the extension κ/λ is étale, because $\lambda = \mathrm{Fix}_G \kappa$, the morphism $\kappa \to A/\mathfrak{m}$ lifts *uniquely* to a λ-homomorphism $k \to A$; this morphism is G-equivariant. Since the morphism A/B is finite, generically étale, this completes the proof of theorem 2.2.2.

3. Around Epp's theorem

3.1. Statement (reminder).

3.1.1. If X is a *reduced* scheme with only a finite number of irreducible components, we denote by X^{nor} its normalization ([**ÉGA** II 6.3.6–8]).

Théorème 3.1.2 (Helmut Epp, [**Epp, 1973**], Theorem 1.9). *Let $T \to S$ be a dominant local morphism of complete traits, of residual characteristic $p > 0$. Let κ_S and κ_T be their respective residue fields. Assume κ_S is* perfect *and the maximal perfect subfield of κ_T is* algebraic *over κ_S. There exists a finite extension of traits $S' \to S$ such that the reduced normalized fibered product*

$$T' := {(T \times_S S')_{\mathrm{réd}}}^{\mathrm{nor}}$$

has a reduced special fiber *over S'.*

Remarque 3.1.3. In mixed characteristic, the fibered product $T \times_S S'$ is reduced. Indeed, the morphism $T' \to S'$ (obtained by base change of a flat morphism) is flat, and S' is integral, so that the ring of functions of T' injects into the ring of functions of its generic fiber. It suffices therefore to prove that the latter is reduced. However, in characteristic zero, every field extension is separable. It is also easily verified that the conclusion of the theorem is still valid if we only assume S to be complete, but not necessarily T (cf. *loc. cit.*, § 2).

3.2. Sorites.

3.2.1. We will say that a field extension K/k of characteristic exponent $p \geq 1$ has the **Epp property** if every element of the maximal perfect subfield $K^{p^\infty} := \bigcap_{i\geq 0} K^{p^n}$ of K is separable algebraic over k. For k perfect, this is the hypothesis made on κ_T/κ_S in 3.1.2. In this short paragraph, we recall some elementary stability results for this notion. Let's start with a lemma.

Lemme 3.2.2. *For any field K of characteristic exponent $p > 1$, we have, in a separable closure $K^{\text{sép}}$ of K,*

$$(K^{p^\infty})^{\text{sép}} = (K^{\text{sép}})^{p^\infty}.$$

Démonstration. The inclusion $(K^{p^\infty})^{\text{sép}} \subset (K^{\text{sép}})^{p^\infty}$ is obvious : K^{p^∞} is perfect, so every algebraic extension, in particular its separable closure $(K^{p^\infty})^{\text{sép}}$, is also perfect. As the latter is contained in $K^{\text{sép}}$, it is also contained in its largest perfect subfield $(K^{\text{sép}})^{p^\infty}$.

Conversely, consider $x \in (K^{\text{sép}})^{p^\infty}$, and denote, for each integer $n \geq 0$, x_n its p^n-th root in $K^{\text{sép}}$ and f_n its minimal polynomial (monic). Taking into account on the one hand the expression of f_n in terms of symmetric polynomials in the Galois conjugates of x_n and on the other hand the injectivity and additivity of raising to the p^n-th power, we have the equality $f_0 = f_n^{(p^n)}$, where $f_n^{(p^n)}$ is the polynomial obtained from f_n by raising the coefficients to the p^n-th power. It follows that the coefficients of the minimal polynomial f_0 of x belong to K^{p^∞}. □

Proposition 3.2.3 (See [**Epp, 1973**], § 0.4). *Let k be a field of characteristic exponent p.*

(i) *Let L/K and K/k have the Epp property. Then L/k has the Epp property.*
(ii) *Any finite extension of k has the Epp property.*
(iii) *If $p > 1$, for any natural integer d, the extension* (Frac $k[[x_1, \dots, x_d]]$)$/k$ *has the Epp property.*
(iv) *If $p > 1$, for any inclusion $k \subset A$, where A is an integral complete Noetherian local ring, inducing an isomorphism on the residue fields, the extension* (Frac A)$/k$ *has the Epp property.*

Démonstration. We immediately assume $p > 1$, otherwise (i) and (ii) are trivial.

(i) By hypothesis we have in a separable closure of L the inclusion $L^{p^\infty} \subset K^{\text{sép}}$. Since the field L^{p^∞} is *perfect*, we deduce that $L^{p^\infty} \subset (K^{\text{sép}})^{p^\infty} = (K^{p^\infty})^{\text{sép}} \subset k^{\text{sép}}$, where the equality results from the previous lemma.

(ii) Any étale extension tautologically has the Epp property. According to (i), it remains to consider the case of a radical extension K/k. If it is of height $\leq r$, we have $K^{p^r} \subset k$ and in particular $K^{p^\infty} \subset k \subset k^{\text{sép}}$.

(iii) Let $A = k[[x_1, \dots, x_d]]$ and K its field of fractions. Let's show that $K^{p^\infty} = k^{p^\infty}$. Since K is contained in $k((x_1, \dots, x_{d-1}))((x_d))$, we reduce by induction to the case where $d = 1$. Any non-zero element of $k((t))^{p^\infty}$ has an infinitely p-divisible valuation, thus zero, so that $k((t))^{p^\infty} - \{0\}$ is contained in

$k[[t]]^\times$ and finally in k^{p^∞} by an immediate calculation. (iv) This results from the previous observations and Cohen's structure theorem. □

4. The Cohen-Gabber theorem in mixed characteristic

4.1. Cohen rings and formal smoothness (reminders).

4.1.1. For the convenience of the reader, we state some results, mainly due to Cohen. For the proofs, we refer to [**Bourbaki**, **AC**, IX, § 2] and [**ÉGA** 0_{IV} § 19].

Définition 4.1.2 ([**ÉGA** 0_{IV} 19.3.1]). Let A be a topological ring(iv). An A-topological algebra B is said to be **formally smooth** if for any *discrete* A-topological algebra C, and any *nilpotent* ideal I of C, any continuous A-morphism $u : B \to C/I$ factors into $B \xrightarrow{v} C \overset{\varphi}{\twoheadrightarrow} C/I$, where v is a continuous A-morphism and φ is the canonical homomorphism.

One also says that $A \to B$ is a **formally smooth morphism**. The following proposition states a slightly more general lifting property than that of the definition.

Proposition 4.1.3 (*loc. cit.*, 19.6.1). *Let A be a topological ring, and B a formally smooth A-algebra. Let C be a topological A-algebra, I an ideal of C satisfying the following conditions :*

(i) *C is metrizable and complete ;*
(ii) *I is closed and the sequence $(I^n)_{n\in\mathbf{N}}$ tends to zero.*

Then, any continuous A-morphism $u : B \to C/I$ factors into $B \xrightarrow{v} C \twoheadrightarrow C/I$, where v is a continuous A-morphism.

The following two theorems give two important criteria for formal smoothness.

Théorème 4.1.4 (*loc. cit.*, 19.6.1). *An extension of fields equipped with the discrete topology is formally smooth if and only if the extension is* separable.

Théorème 4.1.5 (*loc. cit.*, 19.7.1). *Let A, B be two Noetherian local rings, $\mathfrak{m}, \mathfrak{n}$ their respective maximal ideals and $k = A/\mathfrak{m}$ the residue field of A. Equip A and B respectively with the $\mathfrak{m}$-adic and $\mathfrak{n}$-adic topologies. Let $\varphi : A \to B$ be a* local *morphism, and let $B_0 = B \otimes_A k$. The following properties are equivalent :*

(i) *B is a formally smooth A-algebra ;*
(ii) *B is a flat A-module, and B_0 equipped with the quotient topology is a formally smooth k-algebra.*

The following theorem, together with the previous one, is the basis for the demonstration of the existence of Cohen rings defined below.

> *In the following statements, local rings are endowed with the topology of the maximal ideal.*

Théorème 4.1.6 (*loc. cit.*, 19.7.2). *Let A be a Noetherian local ring, I a strict ideal, $A_0 = A/I$, B_0 a complete Noetherian local ring, $A_0 \to B_0$ a formally smooth local morphism. Then there exists a complete Noetherian local ring B, a local morphism $A \to B$ making B a* flat *A-module, and an A_0-isomorphism $u : B \otimes_A A_0 \xrightarrow{\sim} B_0$.*

Définition 4.1.7 (*loc. cit.*, 19.8.4 et 5). A **Cohen ring** is a ring that is either a field of characteristic zero, or a complete discrete valuation ring with residue field of characteristic $p > 0$ and maximal ideal generated by p.

Théorème 4.1.8 (Cohen, *loc. cit.*, 19.8.6 et 21.5.3).

(i) *Let W be a Cohen ring with residue field K, C a complete Noetherian local ring, and I a strict ideal of C. Then, any local morphism $u : W \to C/I$ factors into $W \xrightarrow{v} C \to C/I$, where v is local. Moreover, the factorization is* unique *if and only if $\Omega^1_K = 0$ or $I = 0$.*
(ii) *Let K be a* field. *There exists a Cohen ring W with residue field isomorphic to K. If W' is a second Cohen ring, with residue field K', any isomorphism $u : K \xrightarrow{\sim} K'$ arises by quotienting from an isomorphism $v : W \xrightarrow{\sim} W'$.*

(iv)Here, and below, we follow the convention [**ÉGA** 0_{IV} 19.0.3] : rings are assumed to be *linearly* topologized.

4.1.9. Recall that the hypothesis $\Omega^1_K = 0$ is equivalent to the fact that K is *perfect* if it is of characteristic > 0 or else is an algebraic extension of $\mathbf{Q}$ if it is of characteristic zero.

Note that if K is *perfect* of characteristic $p > 0$, the morphism v of (ii) is *unique*. In this case, W is isomorphic to the ring of Witt vectors over K.

4.2. The Cohen-Gabber theorem in mixed characteristic.

4.2.1. Let A be a complete Noetherian local ring of residual characteristic $p > 0$. The scheme $X = \mathrm{Spec}(A)$ is uniquely a $\mathrm{Spec}(\mathbf{Z}_p)$-scheme. Let X_p be the closed subscheme of X, the fiber above the closed point of $\mathrm{Spec}(\mathbf{Z}_p)$. We will say that an open subset $U \subset X$ is p-**dense** if $U \cap X_p$ is dense in X_p.

Théorème 4.2.2. *Let $X = \mathrm{Spec}(A)$ be a normal complete Noetherian local scheme with residue field k, of dimension $d \geq 2$ and with generic point of characteristic zero. There exists a finite surjective morphism $X' \to X$, where X' is normal integral with residue field k', and a finite surjective morphism $X' \to \mathrm{Spec}(V[[t_1, \dots, t_{d-1}]])$, where V is a discrete valuation ring with residue field k',* étale *above a p-dense open subset of the target.*

The rest of this paragraph is devoted to the proof of the preceding theorem.

4.2.3. Let X be as in the statement. Consider the maximal perfect subfield $k_0 = k^{p^\infty}$ of the residue field k of $A = \Gamma(X, \mathcal{O}_X)$ and denote by $W_0 = W(k_0)$ the corresponding ring of Witt vectors. It follows from Cohen's theorem that there exists a unique morphism $X \to S_0 = \mathrm{Spec}(W_0)$ which extends the morphism $\mathrm{Spec}(k) \to \mathrm{Spec}(k_0)$ between the closed points (4.1.8, (i)).

For any maximal point $got p$ of the special fiber X_p of this morphism, the discrete valuation ring $A_\mathfrak{p}$ has as its residue field Frac $A/\mathfrak{p}$, where the ring $A/\mathfrak{p}$ is an integral complete Noetherian local ring with residue field k. According to 3.2.3 (i) & (iv), the extension $\mathrm{Frac}(A/\mathfrak{p})/k_0$ has the Epp property. Such ideals $got p$ being finite in number and the conclusion of Epp's theorem (3.1.2, 3.1.3) being stable by finite base change because it is a formal smoothness result, there therefore exists a finite base change $S'_0 = \mathrm{Spec}(W_0') \to S_0$ such that the special fiber of the normalized fibered product $X'_0 := (X \times_{S_0} S'_0)^{\mathrm{nor}} = \mathrm{Spec}(A'_0)$ is reduced at its maximal points. (We use the fact that the maximal points of the special fiber of $X'_0 \to S'_0$ lie above the maximal points of the special fiber of $X \to S_0$; cf. e.g. [**ÉGA** 0_{IV} 16.1.6].)

According to the following lemma, the special fiber of the morphism $X'_0 \to S'_0$ is then reduced.

Lemme 4.2.4. *Let X be a normal Noetherian scheme. Any effective Cartier divisor generically reduced is reduced.*

Démonstration. We can assume X is affine and the effective Cartier divisor is defined by a function $f \in A = \Gamma(X, \mathcal{O}_X)$. Let $a \in A$ and $n \geq 1$ be such that $a^n \in (f)$; we want to show that $a \in (f)$. Since the ring $A/(f)$ is generically reduced, the element a/f of Frac A belongs to $A_\mathfrak{p}$ for any prime ideal $\mathfrak{p}$ of height 1 containing f. The same is obviously true for $f \notin \mathfrak{p}$. Since the ring A is normal, $\bigcap_\mathfrak{p} A_\mathfrak{p} = A$ where $\mathfrak{p}$ runs through the ideals of height 1 (see [**Bourbaki**, **AC**, VII, § 1, n$^{\mathrm{o}}$ 6, th. 4]) so that $a/f \in A$. Q.E.D. □

4.2.5. Let k'_0 be the residue field of W_0', ϖ' a uniformizing element of W_0', and consider a connected component $X' = \mathrm{Spec}(A')$ of X'_0; this is a finite surjective scheme over X. Let k' be its residue field. The inclusion $k'_0 \hookrightarrow k'$ derived from the morphism $X' \to S'_0$ is formally smooth, because k'_0 is perfect, so it lifts according to 4.1.5 and 4.1.6 to a *formally smooth* morphism $W_0' \to V$ where V is a complete Noetherian local ring. This ring is a discrete valuation ring. Since the ring A'/ϖ' is *reduced*, equidimensional of dimension $d-1$, with residue field k', there exists, according to the Cohen-Gabber theorem (2.1.1), a k'_0-linear lifting $k' \hookrightarrow A'/\varpi'$ and elements $x_1, \dots, x_{d-1}$ in the maximal ideal of A'/ϖ' such that the induced morphism $k'[[t_1, \dots, t_{d-1}]] \to A'/\varpi'$, sending the indeterminate t_i to x_i, is finite, *generically étale* from top to bottom.

By formal smoothness of $W_0' \to V$, the composite morphism $V \to k' \to A'/\varpi'$ lifts to a W_0'-morphism $\psi : V \to A'$. By lifting the x_i into A', we obtain a morphism $V[[t_1, \dots, t_{d-1}]] \to A'$, finite injective (cf. e.g. [**ÉGA** 0_{IV} 19.8.8 (proof)]), *étale* above the generic point of the special fiber. Q.E.D.

4.3. The Cohen-Gabber theorem prime to ℓ in mixed characteristic.

Théorème 4.3.1. *Let $X = \operatorname{Spec}(A)$ be a normal complete Noetherian local scheme of dimension $d \geq 2$, with residue field k of characteristic $p > 0$ and with generic point of characteristic zero. Let ℓ be a prime number different from p. Then there exist :*

(i) *a normal integral Noetherian local scheme Y equipped with an action of a finite ℓ-group H and a finite surjective H-equivariant morphism $Y \to X$ such that the quotient $Y \,/\, H$ is of generic degree prime to ℓ over X;*
(ii) *a complete discrete valuation ring V with the same residue field k' as Y, of mixed characteristic, equipped with an action of H compatible with its action on k' ;*
(iii) *a local morphism $Y \to Y' = \operatorname{Spec}(V[[t_1, \dots, t_{d-1}]])$ which is finite, étale above a p-dense open subset of Y', and H-equivariant with trivial action of H on the t_i.*

These morphisms are represented in the diagram below, where all arrows are finite surjective morphisms.

$$\begin{tikzcd}
\operatorname{Spec}(V[[t_1, \dots, t_{d-1}]]) = Y' & Y \arrow[l, "p\text{-generically étale}"'] \arrow[r] \arrow[d] & Y/H \arrow[dl, "\text{degree prime to } \ell"] \\
& X &
\end{tikzcd}$$

Remarque 4.3.2. Note that conditions (i) — (iii) on the morphisms $Y \to X$ and $Y \to Y'$ do not imply that the scheme Y/H is étale above a p-dense open subset of $\operatorname{Spec}(\operatorname{Fix}_H(V)[[t_1, \dots, t_{d-1}]]) = Y'/H$. Here is an example, due to Takeshi Saitô. Let k be an algebraically closed field of characteristic $p > 0$, W the Witt vector ring over k, ℓ a prime number different from p, $A = W[[x, y]]/(x^\ell y - p)$. Let $W' = W[\pi]/(\pi^\ell - p)$ and B be the normalization of $A \otimes_W W'$, W'-isomorphic to $W'[[x, z]]/(xz - \pi)$. The group $H = \mu_\ell(k)$ acts on B, via its action on W' : $\zeta \cdot x = x$ and $\zeta \cdot z = \zeta z$. The morphism $Y = \operatorname{Spec}(B) \to X = \operatorname{Spec}(A)$ defined by $x \mapsto x$ and $y \mapsto z^\ell$ satisfies the properties of the theorem because $Y/H \xrightarrow{\sim} X$ and $Y \to Y' = \operatorname{Spec}(W'[[t]])$, $t \mapsto x + z^\ell$, is p-generically étale. However, Y/H has a special fiber isomorphic to the non-reduced scheme $\operatorname{Spec}\big(k[[x, y]]/(x^\ell \cdot y)\big)$. It is therefore not étale above a dense open subset of the special fiber of Y'.

The remainder of this paragraph is devoted to the proof of the preceding theorem. Note that if the special fiber X_p of X over $\operatorname{Spec}(\mathbf{Z}_p)$ is reduced, this theorem — like the previous one — simply results from theorem 2.2.2, in the particular case where the group G is trivial : we can take $Y = X$ and H trivial.

4.3.3. Consider again the maximal perfect subfield k_0 of the residue field k of A and $W_0 = W(k_0) \hookrightarrow A$ the unique morphism lifting the inclusion $k_0 \hookrightarrow k$. Let W_0^ν be the integral closure of W_0 in A.

Lemme 4.3.4. *The extension W_0^ν/W_0 is* finite, *totally ramified.*

Démonstration. Let W'/W_0 be a finite extension of traits, where W' is contained in A. The residue field of W' is a finite extension of k_0 ; it is therefore a perfect field, contained in k and containing k_0. It is therefore equal to k_0 : the extension is totally ramified. The degree of the extension W'/W_0 is consequently equal to its ramification index, which is *bounded* by the integer N such that p belongs to $\mathfrak{m}_A^N - \mathfrak{m}_A^{N+1}$. If W'' is such that the degree of the extension $\operatorname{Frac}(W'') \,/\, \operatorname{Frac}(W_0)$ is maximal, we necessarily have $W' \subset W''$, as is immediately seen by considering the composite subextension, in $\operatorname{Frac}(A)$, of the fraction fields. Thus, $W_0^\nu = W''$ is finite over W_0. □

4.3.5. According to Epp's theorem (3.1.2), there exists a finite extension of discrete valuation rings $W_0^\nu \to W_0'$, which can be assumed generically Galois with a finite group G, such that the special fiber over W_0' of the normalization A' of $A \otimes_{W_0^\nu} W_0'$ is *reduced*. Note that since the ring W_0^ν is integrally closed in A, the ring A' is local. Let k_0^ν (resp. k_0') be the residue field of W_0^ν (resp. W_0') and k' be the residue field of A'. Choose Cohen rings $I(k')$ and $I(\operatorname{Fix}_G k')$ relative to the fields k' and $\operatorname{Fix}_G k'$. There exists a morphism $I(\operatorname{Fix}_G k') \to I(\operatorname{Fix}_G k')$ lifting the inclusion. Since this morphism is finite étale between complete local (hence Henselian) rings, the action of the quotient $\operatorname{Gal}(k'/\operatorname{Fix}_G k')$ of G

on k' lifts to an $I(\mathrm{Fix}_G k')$-linear action on $I(k')$ (cf. e.g. [**Serre, 1968**, III, § 5, th. 3]). Since the field k'_0 is perfect, there exists, according to theorem 4.1.8, a *G-equivariant* morphism $W(k'_0) \to I(k')$. Finally, let $V = W_0{}' \otimes_{W(k'_0)} I(k')$, ϖ a uniformizing element of $W_0{}'$, $\overline{A'} = A'/\varpi A'$ and H an ℓ-Sylow subgroup of G. According to theorem 2.2.2, there exists a *finite, generically étale, H-equivariant* morphism $\varphi : k'[[t_1, \dots, t_{d-1}]] \to \overline{A'}$, where the t_i map into $\mathrm{Fix}_H \overline{A'}$. Since the morphism $\mathrm{Fix}_H A' \to \mathrm{Fix}_H \overline{A'}$ is surjective — as can be seen by using the trace — we can lift the images of the t_i to elements x'_i in $\mathrm{Fix}_H A'$. Moreover, by formal smoothness of $V/W_0{}'$ (for the p-adic topologies), we can lift $k' \to \overline{A'}$ to a $W_0{}'$-morphism $\psi : V \to A'$: this results for example from [**ÉGA** 0_{IV} 19.3.10]. By proceeding step by step, and considering isobarycenters in the affine spaces defined by the well-known lemma below, we find that there even exists such a lifting which is *H-invariant*.

Lemme 4.3.6. *Let $A \to B$ be a ring morphism and $C \twoheadrightarrow C'$ a surjection of A-algebras, with kernel $\mathcal{N}$ of square zero. Then, the set of A-linear liftings of an A-linear morphism $B \to C'$ to C is either empty or a torsor under* $\mathrm{Dér}_A(B, \mathcal{N})$*. The first case does not occur if $A \to B$ is* formally smooth*.*

4.3.7. The X-scheme $Y = \mathrm{Spec}(A')$ is indeed finite p-generically étale over $Y' = \mathrm{Spec}(V[[t_1, \dots, t_{d-1}]])$ if we map V into A' by ψ as above and the variables t_i to the x'_i. By construction, Y is, generically over X, Galois with group G; its quotient Y/H is therefore generically of degree prime to ℓ over X. This completes the proof of the theorem.

LECTURE V

Partial Algebraization

Fabrice Orgogozo

1. Preliminaries (Recap)

1.1. Weierstrass Preparation Theorem. A proof of the following theorem can be found in [**Bourbaki**, **AC**, VII, § 3, n° 7-8].

THÉORÈME 1.1.1. *Let A be a complete separated local ring with maximal ideal $\mathfrak{m}$, $d \geq 0$ an integer and $f \in A[[\underline{X}, T]]$ a formal power series, where we set $\underline{X} = (X_1, \dots, X_d)$.*

(i) *Let ρ be a non-negative integer such that f is **ρ-regular** with respect to T, i.e., congruent to $(u \in A[[T]]^\times) \cdot T^\rho$ modulo $(\mathfrak{m}, \underline{X})$. Then, for any $g \in A[[\underline{X}, T]]$, there exists a unique pair $(q, r) \in A[[\underline{X}, T]] \times A[[\underline{X}]][T]$ such that $g = qf + r$ and $\deg_T(r) < \rho$. Furthermore, there exists a unique polynomial $P = T^\rho + \sum_{i<\rho} p_i T^i$, where the coefficients p_i belong to $(\mathfrak{m}, \underline{X})A[[\underline{X}]]$, and a unit $u \in A[[\underline{X}, T]]^\times$ such that $f = uP$.*

(ii) *If f is* non-zero modulo $\mathfrak{m}$*, there exists a non-negative integer ρ and an $A[[T]]$-linear automorphism c of $A[[\underline{X}, T]]$, such that $c(X_i) = X_i + T^{N_i}$ ($N_i > 0$) and the power series $c(f)$ is ρ-regular.*

1.1.2. Note that condition (ii) can be satisfied simultaneously for a *finite* number of elements: cf. *loc. cit.*, n° 7, lemma 2 where a (finite) product of formal power series will be considered.
We will make use of the following property of polynomials as in (i) above.

LEMME 1.1.3. *Let B be a complete noetherian local ring and $P \in B[X]$ a polynomial of the form $X^\rho + \sum_{i<\rho} b_i X^i$, where $b_i \in \mathfrak{m}_B$ and $\rho > 0$. Then, the (P)-adic completion of $B\{X\}$ can be identified with $B[[X]]$.*

Recall that $B\{X\}$ denotes the henselization at the origin of the ring $B[X]$. A polynomial P as above is sometimes called a **Weierstrass polynomial**.

Proof. Let N be a non-negative integer and $Q = P^N$. It follows from 1.1.1 (i) that the quotient ring $B[[X]]/(Q)$ is isomorphic as a B-module to $B[X]/(X^{\deg(Q)})$ and in particular is finite over B. By faithfully flatness of the morphism $B\{X\} \to B[[X]]$, we have $QB[[X]] \cap B\{X\} = QB\{X\}$ so that the B-morphism $B\{X\}/Q \to B[[X]]/Q$ is *injective*. The ring $B\{X\}/Q$ is therefore also finite over the complete ring B; it is thus isomorphic to its completion $B[[X]]/Q$. By letting N tend to infinity, we deduce that the (P)-adic separated completion of $B\{X\}$ is isomorphic to that of $B[[X]]$; the latter is isomorphic to $B[[X]]$ since $\deg(P) > 0$. □

1.2. Elkik's Algebraization Theorem.

DÉFINITION 1.2.1. A pair (C, J), where J is an ideal of a ring C, is said to be **henselian** if for any polynomial $f \in C[T]$, any root β of f in C/J such that $f'(\beta)$ is a unit in C/J lifts to a root in C.

REMARQUES 1.2.2. The notion of a *simple root* introduced in the definition is stronger than that of [**Bourbaki**, **A**, IV, § 2, n° 1, déf. 1] and the lift above is necessarily unique. Furthermore, the definition above depends only on the closed set $F = V(J)$. Indeed, if $I \subset \sqrt{J}$ and (C, J) is henselian, then (C, I) is also henselian; see [**Kurke et al., 1975**, 2.2.1] and the lemma below for a particular case. This allows us to say that a pair (X, F), where $X = \mathrm{Spec}(C)$ and $F = \mathrm{Spec}(J)$, is henselian when the pair (C, J) is.

LEMME 1.2.3. *Let C be a* local *henselian ring with maximal ideal $\mathfrak{m}$, and $J \subset \mathfrak{m}$ an ideal. The pair $(\mathrm{Spec}(C), V(J))$ is henselian.*

In particular, for B and P as in lemma 1.1.3, the pair $(\mathrm{Spec}(B\{X\}), V(P))$ is henselian.

Proof. Let f and β be as above. Since C is a local henselian ring, the image γ of β in the residue field $C/\mathfrak{m}$ lifts to a root α of P. Let us denote β' its image in C/J and verify that $\beta = \beta'$. First, note that since $P'(\alpha)$ is a unit in C, $P'(\beta')$ is a unit in C/J. Furthermore, the equality $P(\beta) = P(\beta') + (\beta-\beta')P'(\beta') + (\beta-\beta')^2 b$ where $b \in B/J$ reduces to $\beta - \beta' = (\beta-\beta')^2 \frac{-b}{P'(\beta')}$; if we set $x = \beta - \beta'$, we therefore have $x(1-ax) = 0$ for some $a \in C/J$. Since x belongs to $\mathfrak{m}$ (because β and β' have γ as their image in $C/\mathfrak{m}$), we have $x = 0$. □

1.2.4. The definition given above — taken from *op. cit.* § 2.2 and [**Gabber, 1992**, p. 59] — is equivalent to the usual definitions: a pair (X, F) is henselian in the preceding sense if and only if it satisfies the idempotent lifting property of [**ÉGA** IV 18.5.5] or if it satisfies the implicit function theorem over F (see, e.g., [**Gruson, 1972**, definition]). For the proof of these equivalences, see, for example, [**Kurke et al., 1975**, 2.6.1], [**Crépeaux, 1967**, prop. 2] and [**Raynaud, 1970**, chap. XI, § 2, prop. 1].

1.2.5. Note that in these last two references, it is assumed that J is contained in the Jacobson radical of A; this is automatic here, as can be seen by considering adequate polynomials of degree 1 (cf. [**Kurke et al., 1975**, 2.2.1]). Also note that in [**Crépeaux, 1967**] and [**Raynaud, 1970**], only *monic* polynomials are considered. The equivalence between the two viewpoints can be verified as follows. Let $f = a_0 + a_1 T + \cdots + a_n T^n \in C[T]$ with $a_0 \in J$ and $a_1 \in C^\times$; this polynomial has, modulo J, a simple root at 0. Let us look for a root of f in C of the form a_0/u, with $u \in C^\times$. By substitution, it suffices to show that the equation $g = 0$, where $g = U^n + a_1 U^{n-1} + a_2 a_0 U^{n-2} + \cdots + a_0^{n-1} a_n$ is a *monic* polynomial, has an invertible root. Now, this polynomial has, modulo J, the class of $-a_1$ as a simple root.

Let us conclude these recollections with the statement of Renée Elkik's algebraization theorem ([**Elkik, 1973**, théorème 5]).

Théorème 1.2.6. *Let $(X = \mathrm{Spec}(A), F)$ be a henselian pair with A noetherian, and U the open subscheme complementary to F in X. Let $X_{\widehat{F}}$ denote the completion of X along F, $\widehat{F}$ the closed set corresponding to F and $\widehat{U}$ its complement in $X_{\widehat{F}}$. The functor $X' \mapsto X' \times_X X_{\widehat{F}}$ induces an equivalence of categories between the category of X-schemes finite, étale over U, and the category of $X_{\widehat{F}}$-schemes finite, étale over $\widehat{U}$.*

2. Partial Algebraization in Equal Characteristic

2.1. Statement.

2.1.1. Let A be a complete noetherian local ring and $\{I_e\}_{e \in E}$ a collection of ideals of A. We say that the pair $(A, \{I_e\}_{e\in E})$ is **partially algebraizable** if there exists a complete noetherian local ring B of dimension strictly lower than that of A, a finitely generated B-algebra C, a maximal ideal $\mathfrak{n}$ over the maximal ideal of B, and an isomorphism $A \simeq \widehat{C_\mathfrak{n}}$ such that the ideals I_e $(e \in E)$ come from ideals of $C_\mathfrak{n}$.

Théorème 2.1.2. *Let A be a complete reduced noetherian local ring of equal characteristic which is not a field. Then, A endowed with any* finite *set of ideals is partially algebraizable.*

2.2. Proof.

2.2.1. Let $X = \mathrm{Spec}(A)$ and $I_1, \dots, I_n \subset A$ as in the statement. It follows from definition 2.1.1 that if an ideal I of A is of the form $J_1 \cap \cdots \cap J_r$ and that the J_i are simultaneously partially algebraizable (i.e., the pair $(A, \{J_i\}_{1,\dots,r})$ is partially algebraizable in the sense of the preceding definition), the ideal I is also. According to the primary decomposition theorem for ideals, we can assume the I_i are *primary*.

2.2.2. Let k be the residue field of A and $d > 0$ its dimension. According to **IV**-2.1.1 if X is equidimensional or **IV**-2.2.2 (with $G = \{1\}$) in the general case, there exists a finite generically étale morphism $\pi : X \to X_0$, where $X_0 = \mathrm{Spec}(k[[t_1, \dots, t_d]])$.

2.2.3. Let I be one of the I_i. Two cases arise.

(i) $\dim(A/I) = d$. The ideal I is therefore a minimal prime ideal of A.

(ii) $\dim(A/I) < d$. The image of $V(I)$ in X_0 is therefore of dimension at most $d-1$ and thus contained in a closed set $V(g_I)$ where $g_I \in A_0 = \Gamma(X_0, \mathcal{O}_{X_0}) - \{0\}$.

Let $g = \prod_{I_i} g_{I_i}$ where $I_i \in \{I_1, \dots, I_n\}$ ranges over the subset of ideals of the second type, and $f \in A_0 - \{0\}$ such that the ramification locus of π is contained in $V(f)$. Let $h = gf$. According to 1.1.1 (ii) and (i), up to changing the base by an automorphism (i.e., changing the coordinates), we can assume that h is a Weierstrass polynomial in t_d. (If h is a unit, we replace it by t_d.) Consider the subring $\widetilde{A_0} = k[[t_1, \dots, t_{d-1}]]\{t_d\}$ of A_0. It is henselian and contains h. According to lemmas 1.1.3 and 1.2.3, the pair $(\widetilde{X_0} = \mathrm{Spec}(\widetilde{A_0}), V(h))$ is henselian. We are therefore able to apply theorem 1.2.6 and deduce that there exists a cartesian diagram:

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \widetilde{X} \arrow[d] \\ X_0 \arrow[r] & \widetilde{X_0} \end{tikzcd}$$

where the left vertical arrow is, by hypothesis, étale outside $V(h)$ and the horizontal arrows are completion morphisms (both for the h-adic topology and that defined by their respective maximal ideals).

The ideals I of the first type (i.e., minimal prime) descend to $\widetilde{X}$ according to the following lemma.

Lemme 2.2.4. *Let B be a quasi-excellent henselian local ring with completion denoted $\widehat{B}$. Any minimal prime ideal of $\widehat{B}$ comes by extension from a minimal prime ideal of B.*

Proof. By restricting to the closure of the generic point of the closed set, it suffices to show that the completion of an integral henselian quasi-excellent ring is integral. This fact is well known and results immediately from Popescu's approximation theorem, applied to the equation $xy = 0$. □

2.2.5. As for the ideals I of the second type, it suffices to observe that each $V(I)$ is finite over $\mathrm{Spec}(k[[t_1, \dots, t_{d-1}]])$, thus over $\widetilde{X} = \mathrm{Spec}(\widetilde{A})$, and to apply the

Lemme 2.2.6. *Let B be a noetherian local ring, $J \subset \mathfrak{m}_B$ an ideal, and $\widehat{B}$ the J-adic completion of B. Any quotient of $\widehat{B}$ finite over B descends to B.*

2.2.7. Let us momentarily admit this lemma and finish the proof of 2.1.2. As we have seen, the ideals $I_1, \dots, I_n$ come from ideals of $\widetilde{A}$. This ring is finite — and thus *a fortiori* of finite presentation by noetherianity — over the ring $k[[t_1, \dots, t_{d-1}]]\{t_d\}$. The latter is the henselization at the origin of $k[[t_1, \dots, t_{d-1}]][t_d]$; it is isomorphic to the filtered colimit of rings of finite type over $k[[t_1, \dots, t_{d-1}]][t_d]$ hence over the ring $B = k[[t_1, \dots, t_{d-1}]]$. The conclusion results from [**ÉGA** IV 8.8.2] which ensures the existence of a B-algebra C as in 2.1.1 from which the ideals I_i originate.

2.2.8. Let us return to the proof of lemma 2.2.6 above. Let $I \subset \widehat{B}$ such that $\widehat{B}/I$ is finite over B. Up to replacing B by $B/\mathrm{Ker}(B \to \widehat{B}/I)$, i.e., $\mathrm{Spec}(B)$ by the schematic image of $V(I)$, we can assume $B \to \widehat{B}/I$ is injective, i.e., $V(I) \to \mathrm{Spec}(B)$ is schematically dominant. Since the B-module $\widehat{B}/I$ is finite, the J-adic topology on $\widehat{B}/I$ induces the J-adic topology on B. Since the map $B \to \widehat{B}/I$ is injective, with dense image, and continuous, it follows that $\widehat{B}/I$ is the *separated completion* of B for the J-adic topology. We therefore have $I = (0)$; it descends tautologically to B.

3. Partial Algebraization Prime to ℓ in Mixed Characteristic

3.1. Statement.

Théorème 3.1.1. *Let A be a complete normal noetherian local ring of mixed characteristic* $(0, p)$ *of dimension $d \geq 2$ and $\ell \neq p$ a prime number. There exists a finite injective morphism $A \to A'$ of generic degree prime to ℓ, where A' is an integral normal ring whose any* finite *family of ideals is partially algebraizable.*

Remarque 3.1.2. Note that to prove **XIII**-1.1.1, it suffices to establish the weaker variant of the preceding statement according to which — resuming the notations of 2.1.1 — any nowhere dense closed subset of $\mathrm{Spec}(A') \simeq \mathrm{Spec}(\widehat{C_\mathfrak{n}})$ is set-theoretically contained in the inverse image of a *divisor* of $\mathrm{Spec}(C_\mathfrak{n})$.

3.1.3. *Geometric Reformulation.* Let X be a complete noetherian local scheme of mixed characteristic, of dimension $d \geq 2$ and ℓ a prime number invertible on X. There exists a normal local scheme X' and a finite morphism $\pi : X' \to X$ of generic degree prime to ℓ such that for each finite family $\{Z'_i\}_{i\in I}$ of closed sets of X', there exists a diagram

$$\begin{tikzcd} X & X' \arrow[l, "\pi"'] \arrow[r, "a"] & Y \arrow[d, "f"] \\ & & S \end{tikzcd}$$

where:

- S is a complete regular noetherian scheme of mixed characteristic and of dimension $d-1$;
- f is a finite type morphism;
- a induces an isomorphism between X' and the completion of Y at a closed point of the special fiber of f,

and closed sets F_i of Y such that $Z'_i = a^{-1}(F_i)$ for all $i \in I$.

Remarque 3.1.4. It follows from 2.1.2 that the preceding result is also true in equal characteristic, and that one can then assume $X = X'$.

3.2. Proof.

3.2.1. Let $X = \operatorname{Spec}(A)$ of dimension $d \geq 2$ and ℓ as in the statement. According to theorem **IV**-4.3.1, there exists a commutative diagram

$$\begin{tikzcd} V[[t_1, \dots, t_{d-1}]] = B_0 \arrow[r] & B & A' = \operatorname{Fix}_H(B) \arrow[l] \\ & A \arrow[u] \arrow[ur] & \end{tikzcd}$$

where V is the spectrum of a complete discrete valuation ring with maximal ideal $\mathfrak{m}_V$, and H is an ℓ-group acting on the normal ring B, its subring V, and trivially on the variables t_i $(1 \leq i \leq d-1)$. Furthermore, $A \to A'$ is a finite injection of generic rank prime to ℓ and $\pi : \operatorname{Spec}(B) \to \operatorname{Spec}(B_0)$ is finite, p-generically étale.

3.2.2. We will show that any finite family of ideals of A' is partially algebraizable. Let $I'_1, \cdots, I'_n$ be such a family, which can be assumed to consist of primary ideals (cf. § 2.2.1).

3.2.3. Let I' be one of the I'_i. Two cases arise.

(i) $\dim(A'/I' + (p)) = d-1$. Suppose $I' \neq (0)$ and let $\mathfrak{p}$ be the prime ideal, necessarily of height one, for which I' is primary. By hypothesis, the prime ideal $\mathfrak{p}$ contains p; it is a minimal prime ideal of $A'/(p)$. On the other hand, A' is normal because B is. It results for example from [**Serre, 1965**, chap. iii, C, § 1] that I' is a *symbolic power* of $\mathfrak{p}$, i.e., the inverse image in A' of a power of the (principal) ideal $\mathfrak{p}A'_{\mathfrak{p}}$.

(ii) $\dim(A'/(I' + (p))) < d-1$. The image of $V(I')$ in $\operatorname{Spec}(\operatorname{Fix}_H(V)[[t_1, \dots, t_{d-1}]])$ is therefore contained in a closed set $V(g_{I'})$ where $g_{I'} \in \operatorname{Fix}_H(V)[[t_1, \dots, t_{d-1}]]$ is *non-zero modulo* $\mathfrak{m}_{\operatorname{Fix}_H(V)}$.

Let $g = \prod_{I'_i} g_{I'_i}$ where $I'_i \in \{I'_1, \dots, I'_n\}$ ranges over the subset of ideals of the second type, and an equation $f \in V[[t_1, \dots, t_{d-1}]]$ non-zero modulo $\mathfrak{m}_V$ such that the ramification locus of π is contained in $V(f)$. We can assume that f is not a unit. On the other hand, up to multiplying it by its H-conjugates, we can assume that the equation f is H-invariant. Let $h = gf$. According to the preparation theorem (1.1.1), we can assume that h is a Weierstrass polynomial (non-invertible) in t_{d-1}, invariant under the action of H. (Recall that H acts trivially on the variables). As in § 2, the morphism $B_0 \to B$ therefore descends according to 1.2.6 to a morphism $\widetilde{B_0} \to \widetilde{B}$, where $\widetilde{B_0} = V[[t_1, \dots, t_{d-2}]]\{t_{d-1}\}$. The group H preserving the open set $D(h)$ of $\operatorname{Spec}(B_0)$, its action descends. The diagram above is therefore completed into a diagram

$$\begin{tikzcd} V[[t_1,\dots,t_{d-2}]]\{t_{d-1}\} = \widetilde{B}_0 \arrow[r] \arrow[d] & \widetilde{B} \arrow[r] \arrow[d] & \widetilde{A}' = \mathrm{Fix}_H(\widetilde{B}) \arrow[d] \\ B_0 \arrow[r] & B \arrow[r] & A' \end{tikzcd}$$

where the vertical arrows are the completion morphisms and the horizontal arrows are finite.

3.2.4. The ideals I' of the second type descend from A' to $\widetilde{A}'$ because A'/I' is finite over $\mathrm{Fix}_H(V)[[t_1,\dots,t_{d-2}]]$ and thus *a fortiori* over $\widetilde{A}'$ (cf. 2.2.6). As for the ideals of the first type (symbolic powers), it suffices to apply lemma 2.2.4 to the pair consisting of $\widetilde{A}'/(p)$ and its completion $A'/(p)$. As in § 2, we use the fact that $\widetilde{A}'$ is finite — finite type would suffice — over $\mathrm{Fix}_H(V)[[t_1,\dots,t_{d-2}]]\{t_{d-2}\}$ to descend, by passing to the limit, the ideals to a ring of finite type over $\mathrm{Fix}_H(V)[[t_1,\dots,t_{d-2}]]$.

LECTURE VI

Log Regularity, Very Tame Actions

Luc Illusie

1. Log Regularity

1.1. For the language of log schemes we refer the reader to [**Kato, 1988**], [**Nizioł, 2006**], [**Gabber & Ramero, 2013**]. Unless otherwise stated, the log structures considered will be in the sense of the étale topology. A **fine** (resp. fs, *i.e.* **fine and saturated**) log scheme [**Kato, 1988**] is a scheme equipped with a log structure locally admitting (for the étale topology) a chart over a fine (resp. fine and saturated) monoid. We generally denote M_X the sheaf of monoids of a log scheme X, $\alpha : M_X \to \mathscr{O}_X$ the structural morphism, and $\overline{M}_X = M_X/\mathscr{O}_X^*$, $\overline{M}_X^{\mathrm{gp}} = M_X^{\mathrm{gp}}/\mathscr{O}_X^*$. Unless otherwise stated, the log schemes considered are assumed to be locally Noetherian.

In what follows, X denotes an fs log scheme.

1.2. Let $\overline{x}$ be a geometric point of X, with image $x \in X$, $\mathscr{O}_{X,\overline{x}}$ the strict localization of X at $\overline{x}$. Let $I_{X,\overline{x}}$ (or $I_{\overline{x}}$ if no confusion is to be feared) be the ideal of $\mathscr{O}_{X,\overline{x}}$ generated by $\alpha(M_{X,\overline{x}} - \mathscr{O}_{X,\overline{x}}^*)$, $C_{X,\overline{x}}$ the closed subscheme of $X_{(\overline{x})} = \operatorname{Spec} \mathscr{O}_{X,\overline{x}}$ defined by $I_{\overline{x}}$. The closed set underlying $C_{X,\overline{x}}$ is the trace on $X_{(\overline{x})}$ of the stratum of X where the rank of $\overline{M}_X^{\mathrm{gp}}$ is equal to $r(x) = \mathrm{rg}(\overline{M}_{X,\overline{x}}^{\mathrm{gp}})$.

We say that X is **log regular** at x (or $\overline{x}$) if $C_{X,\overline{x}}$ is regular and we have

$$\dim(X_{(\overline{x})}) = \dim(C_{X,\overline{x}}) + \mathrm{rang}_{\mathbf{Z}}(\overline{M}_{X,\overline{x}}^{\mathrm{gp}}), \tag{1.2.1}$$

(this condition depends only on x). We say that X is **log regular** if X is log regular at every point. The analogous definition for Zariski log schemes is due to Kato [**Kato, 1994**]. The variant in the étale setting has been treated by Nizioł [**Nizioł, 2006**]. See also [**Mochizuki, 1999**] and [**Gabber & Ramero, 2013**, 9.5]. We recall below some properties of this notion.

1.3. Assume X is endowed with a chart $P_X \to M_X$, with P fine and saturated. Then, for every x, X is log regular at x if and only if X, equipped with the Zariski log structure $M_X^{\mathrm{Zar}} := \varepsilon_\star M_X$, where $\varepsilon : X_{\text{ét}} \to X_{\mathrm{Zar}}$, is log regular at x in Kato's sense [**Kato, 1994**], [**Tsuji, 1997**, II 4.6], [**Nizioł, 2006**, 2.4]. In particular, if X is log regular at x, X is log regular at any generalization y of X (the stability of log regularity (Zariski) under generalization is stated in [**Kato, 1994**, 7.1], but, as Gabber observed, the proof given is insufficient; see [**Gabber & Ramero, 2013**, 9.5.47] for a correct argument).

If the log structure of X is trivial, X is log regular if and only if X is regular in the usual sense.

1.4. Assume X is log regular. Let $j : U \hookrightarrow X$ be the inclusion of the open set of triviality of its log structure. Then U is a dense open set of X and we have

$$M_X = \mathscr{O}_X \cap j_\star \mathscr{O}_U^*$$

([**Nizioł, 2006**, 2.6]).

We say that a pair (X, Z) consisting of a scheme X and a closed set Z is a **log regular pair** if, for the log structure on X defined by $M_X = \mathscr{O}_X \cap j_\star \mathscr{O}_U^*$, where $j : U \hookrightarrow X$ is the open complement of Z, X is log regular and Z is the complement of the open set of triviality of its log structure. The preceding log structure on X will be called **associated** with the pair (X, Z).

1.5. Assume X is log regular. It follows from 1.6 below (cf. [**Kato, 1994**, 4.1]) that the underlying scheme of X is Cohen-Macaulay and normal. In particular, in (1.2.1), we have

$$\operatorname{codim}_{\overline{x}}(C_{X,\overline{x}}, X_{(\overline{x})}) = \operatorname{rang}_{\mathbf{Z}}(\overline{M}^{\mathrm{gp}}_{X,\overline{x}}). \tag{1.5.1}$$

For $i \in \mathbf{N}$, let $X^{(i)}$ be the set of points x of X such that $r(x) = i$, with the notation of 1.2. This is a locally closed set, underlying a regular subscheme of X, of codimension i, whose trace on $X_{(\overline{x})}$, at each geometric point $\overline{x}$ localized at $x \in X^{(i)}$, is $C_{X,\overline{x}}$. We say that $X^{(i)}$ is the **codimension stratum** i defined by the rank of $\overline{M}^{\mathrm{gp}}$. The stratification by the $X^{(i)}$ is called **stratification by the rank of** $\overline{M}^{\mathrm{gp}}$, or **canonical stratification**. Here are two examples.

(i) If X is a regular Noetherian scheme, endowed with the log structure defined by a divisor with normal crossings D, $X^{(i)}$ is the set of points through which exactly i branches of D pass, *i.e.* such that the normalization of D has i points above x.

(ii) If X is a toric variety over a field k, with torus T, endowed with its canonical log structure, X is a log regular log scheme, the open set of triviality of the log structure is T, and $X^{(i)}$ is the union of the orbits of T of codimension i.

1.6. Assume X is log regular at x, image of the geometric point $\overline{x}$. Let $P = \overline{M}_{X,\overline{x}}$, $k = k(\overline{x})$. Note that P is a fs **sharp** monoid (*i.e.* such that $P^* = 0$). Let $\widehat{X}_{\overline{x}}$ be the completion of $X_{(\overline{x})}$ at the closed point. Then, according to [**Kato, 1994**, 3.2], X admits a chart modeled on P at $\overline{x}$, which gives rise to isomorphisms

(i)

$$\widehat{X}_{\overline{x}} \xrightarrow{\sim} \operatorname{Spec} k[[P]][[t_1, \cdots, t_n]]$$

if $\mathscr{O}_{X,x}$ is of equal characteristic,

(ii)

$$\widehat{X}_{\overline{x}} \xrightarrow{\sim} \operatorname{Spec} C(k)[[P]][[t_1, \cdots, t_n]]/(f)$$

if $\mathscr{O}_{X,x}$ is of unequal characteristic $(0, p)$, where $C(k)$ is a Cohen ring for k, and f is congruent to p modulo the ideal generated by $P - \{0\}$ and the t_i.

1.7. Assume X is log regular. Then X is regular at x, image of $\overline{x}$, if and only if $\overline{M}_{\overline{x}} \simeq \mathbf{N}^r$ ([**Nizioł, 2006**, 5.2], see also [**Vidal, 2001b**, 1.8]). It follows that the set of regular points of X coincides with the set of regular points of the monoid $\overline{M}_{\overline{x}}$, and in particular is open in X ([**Nizioł, 2006**, 5.3]). If X is log regular and regular, the open set of triviality of the log structure is then the complement of a divisor with normal crossings.

1.8. Let $f : X \to Y$ be a morphism of fs log schemes. If Y is log regular and f is log smooth, X is log regular.

The analogue for Zariski log structures is [**Kato, 1994**, 8.2]. The proof of *loc. cit.* applies, *mutatis mutandis*, in the étale setting.

The following corollary will play a key role in the application of de Jong's results to the proof of Gabber's uniformization theorem. A similar statement is given in [**Mochizuki, 1999**, 4.2]. Recall that, if S is a scheme, a **nodal curve** $f : C \to S$ is a flat morphism, locally of finite presentation, purely of relative dimension 1, whose geometric fibers have only normal crossing singularities [(i)].

PROPOSITION 1.9. *Let (Y, T) be a log regular pair (1.4). Let $f : X \to Y$ be a nodal curve, smooth over $Y - T$. Let D be an effective divisor on X, whose support is contained in the locus of smoothness of f, and which is étale over Y. Then the pair $(X, f^{-1}(T) \cup D)$ is log regular, and for the associated log structures, f is a morphism of log schemes and is log smooth.*

(i) It is often required that geometric fibers be connected. This additional condition is unnecessary for the following statement.

The question is local for the étale topology on X and the assertion is trivial on the open set of smoothness of f and on $f^{-1}(Y-T)$. We can therefore assume $D = \emptyset$. According to [**SGA 7** xv 1.3.2] (see also [**de Jong, 1996**, 2.23]), we can assume that Y is affine, $Y = \operatorname{Spec} R$, and that X is defined by

$$X = \operatorname{Spec} R[u,v]/(uv-h), \tag{1.9.1}$$

where h is a section of $\mathscr{O}_Y$ invertible over $Y-T$. We can further assume that we have a chart $c : P \to M_Y$ with $h = \varepsilon c(a)$, for $\varepsilon \in R^*$ and $a \in P$. Let Q be the fs monoid defined by the cocartesian square

$$\begin{tikzcd} \mathbf{N}^2 \arrow[r] & Q \\ \mathbf{N} \arrow[u] \arrow[r] & \mathbf{Z} \times P \arrow[u, "g"'] \end{tikzcd},$$

where the arrow $\mathbf{N} \to \mathbf{N}^2$ (resp. $\mathbf{N} \to \mathbf{Z} \times P$) is given by $1 \mapsto (1,1)$ (resp. $1 \mapsto (1,a)$). The square

$$\begin{tikzcd} \operatorname{Spec} R[u,v]/(uv-h) \arrow[r, "d"] \arrow[d] & \operatorname{Spec} \mathbf{Z}[Q] \arrow[d, "\operatorname{Spec} \mathbf{Z}[g]"] \\ \operatorname{Spec} R \arrow[r, "c'"] & \operatorname{Spec} \mathbf{Z}[\mathbf{Z} \times P] \end{tikzcd},$$

where the arrow c' is given by c and $\mathbf{Z} \to R$, $1 \mapsto \varepsilon$, and d by $(1,0) \mapsto u$, $(0,1) \mapsto v$ on $\mathbf{N}^2$ and c' on $\mathbf{Z} \times P$, is cartesian. Then d defines a log structure on X whose open set of triviality is $f^{-1}(Y-T)$, and (c', d, g) is a chart for f. Since g is injective and $\operatorname{Coker} g^{\mathrm{gp}} \simeq \mathbf{Z}$, f is log smooth. Since Y is log regular, X is therefore log regular, and since the open set of triviality of the log structure of X is $f^{-1}(Y-T)$, the pair $(X, f^{-1}(T))$ is log regular.

2. Kummer Étale Coverings

In this section and the next, we will consider group actions on schemes or log schemes: unless otherwise specified, these will be *right actions*.

2.1. Let us recall some definitions (cf. [**Illusie, 2002**, 3], [**Stix, 2002**, 3.1], [**Vidal, 2001a**]). A homomorphism $h : P \to Q$ of integral monoids is said to be **Kummer** if h is injective and, for every $q \in Q$, there exists $n \geq 1$ and $p \in P$ such that $nq = h(p)$. A morphism $f : X \to Y$ of fs log schemes is said to be **Kummer** if, for every geometric point $\overline{x}$ of X with image $\overline{y} = f(\overline{x})$ in Y, the induced homomorphism $\overline{M}_{\overline{y}} \to \overline{M}_{\overline{x}}$ is Kummer. A morphism $f : X \to Y$ is said to be **Kummer étale** if f is Kummer and log étale. We say that f is a **Kummer étale covering** if f is Kummer étale and the underlying scheme morphism is finite.

The **Kummer étale site** of an fs log scheme X is the category of fs log schemes Kummer étale over X endowed with the topology defined by surjective families of X-morphisms (which are automatically Kummer étale). For X connected non-empty, the Kummer étale coverings of X form a Galois category, equivalent to the category of representations of a profinite group, the *logarithmic fundamental group* of X, $\pi_1^{\log}(X, x)$, where x is a *logarithmic geometric point* of X, cf. *loc. cit.*

2.2. A morphism $f : X \to Y$ of fs log schemes which is deduced by base change by a strict morphism $g : Y \to \operatorname{Spec} \mathbf{Z}[P]$ from a morphism $\operatorname{Spec} \mathbf{Z}[h] : \operatorname{Spec} \mathbf{Z}[Q] \to \operatorname{Spec} \mathbf{Z}[P]$, where $h : P \to Q$ is a Kummer homomorphism between fs monoids such that the cokernel of h^{gp} is annihilated by an integer n invertible on Y, is a Kummer étale covering. Such a covering is called a **standard** Kummer étale covering.

A standard Kummer étale covering $f : X \to Y$ as above is **Galois** in the following sense: the diagonalizable (étale) group $G = \operatorname{Hom}(Q^{\mathrm{gp}}/P^{\mathrm{gp}}, \mathbf{G}_{mY})$ acts on X by automorphisms of Y-log schemes, and the canonical morphism

$$X \times_Y G \to X \times_Y X, (x,g) \mapsto (x, xg), \tag{2.2.1}$$

where the product in the second member is taken in the category of fs log schemes, is an isomorphism; the log scheme Y is a sheaf quotient of X by G in the category of sheaves on the Kummer étale site of Y, and as a scheme, a geometric quotient in the usual sense of X by G: we have

$$\mathscr{O}_Y = (f_\star \mathscr{O}_X)^G \ ; \ M_Y = (f_\star M_X)^G. \tag{2.2.2}$$

Moreover, the morphism f is finite, open and surjective, and remains so after any fs base change $Y' \to Y$. These assertions are a particular case of Kato's descent theorems [**Kato, 1991**, 3.1, 3.4.1, 3.5]. For the fact that 2.2.1 is an isomorphism, see [**Illusie, 2002**, 3.2]. For the first formula of 2.2.2, see [**Kato, 1991**, 3.4.1] (and [**Illusie et al., 2013**, 2.1] for the fact, used in the proof of [**Kato, 1991**, 3.4.1], that $\mathbf{Z}[P]$ is a direct factor of $\mathbf{Z}[Q]$ as a $\mathbf{Z}[P]$-module). One could also invoke the structure of representations of diagonalizable groups and the fact that $Q \cap P^{\mathrm{gp}} = P$. The verification of the second formula is more delicate. We provide a direct proof in 2.4.

Any Kummer étale morphism is, locally for the étale topology, isomorphic to a standard Kummer étale covering (cf. [**Stix, 2002**, 3.1.4]). More precisely, if $f : X \to Y$ is a surjective Kummer étale morphism, with X (resp. Y) strictly local with closed point x (resp. y), we have a cartesian square

$$\begin{tikzcd} X \arrow[r, "a"] \arrow[d, "f"'] & \operatorname{Spec} \mathbf{Z}[Q] \arrow[d, "\operatorname{Spec} \mathbf{Z}[h]"] \\ Y \arrow[r, "b"] & \operatorname{Spec} \mathbf{Z}[P] \end{tikzcd} ,$$

where a (resp. b) is a chart of X (resp. Y), and $h : P \to Q$ is a Kummer morphism such that $\operatorname{Coker} h^{\mathrm{gp}}$ is annihilated by an integer n invertible on Y. Then f is a Galois Kummer étale covering with group $G = \operatorname{Hom}(\operatorname{Coker} h^{\mathrm{gp}}, \mu_n(k(y)))$.

2.3. In this case, the action of G on X is simply described. More generally, consider a standard Kummer étale covering $f : X \to Y$, given by a cartesian square of schemes

$$\begin{tikzcd} X \arrow[r, "a"] \arrow[d, "f"'] & \operatorname{Spec} \Lambda[Q] \arrow[d, "\operatorname{Spec} \Lambda[h]"] \\ Y \arrow[r, "b"] & \operatorname{Spec} \Lambda[P] \end{tikzcd} , \tag{2.3.1}$$

where $\Lambda = \mathbf{Z}[\mu_n][1/n]$, $\mu_n = \mu_n(\mathbf{C})$ (n integer ≥ 1), $h : P \to Q$ is a Kummer morphism such that $\operatorname{Coker} h^{\mathrm{gp}}$ is annihilated by n, and X and Y are endowed with the log structures defined by the horizontal arrows. Let $C = \operatorname{Coker} h^{\mathrm{gp}}$. The exact sequence

$$0 \to P^{\mathrm{gp}} \to Q^{\mathrm{gp}} \to C \to 0$$

gives, by applying the Cartier dual $D = \operatorname{Hom}(-, \mathbf{G}_m)$, an exact sequence of diagonalizable groups over $\operatorname{Spec} \Lambda$,

$$0 \to G \to T_Q \to T_P \to 0,$$

where $G = D(C)$. Let $Z_P = \operatorname{Spec} \Lambda[P]$, $Z_Q = \operatorname{Spec} \Lambda[Q]$. The canonical pairing

$$T_P \otimes P^{\mathrm{gp}} \to \mathbf{G}_m \tag{2.3.2}$$

defines a family of characters

$$(\chi_p : T_P \to \mathbf{G}_m)_{p \in P^{\mathrm{gp}}},$$

which determines the action of T_P on Z_P by

$$g.p = \chi_p(g)p, \tag{2.3.3}$$

for a point g of T_P (with values in a scheme S), and a point p of Z_P with values in S. The action of T_Q on Z_Q is described similarly, and the chart $a : X \to Z_Q$ is equivariant relative to the actions of G and T_Q on X and Z_Q respectively: for $g \in G$ and $q \in Q$,

$$g.a^\star(q) = \chi_q(g)a^\star(q), \tag{2.3.4}$$

where $a^\star(q)$ is the inverse image section of M_X of q by a.

2.4. From this description, we deduce the second formula of (2.2.2). The following argument is due to Gabber. We must verify that $M_Y \to (f_\star M_X)^G$ induces an isomorphism on the stalks at every geometric point of Y, so we can assume Y is strictly local with closed point y. Hereafter, we will sometimes omit $h : P \to Q$ from the notation, and still denote a (resp. b) the homomorphism $Q \to \mathscr{O}(X)$ (resp. $P \to \mathscr{O}(Y)$) deduced from a (resp. b). By replacing P by its localization $P_{(\mathfrak{p})}$ at the prime ideal $\mathfrak{p}$ complementary to the face $b^{-1}(\mathscr{O}^*_{Y,y})$, and Q by $Q \oplus_P P_{(\mathfrak{p})}$, we can assume that $P^* = b^{-1}(\mathscr{O}^*_{Y,y})$. And, for $q \in Q$, q is invertible if and only if $a(q)$ is invertible at a point above y (because if this is the case, and $n \geq 1$ is such that $q^n \in P$, we have $a(q^n) = b(q^n) \in \mathscr{O}^*_{Y,y}$ and $q^n \in P^*$). If x is a point of X above y, thus (the extension $k(x)/k(y)$ being separable, hence trivial) corresponding to a lift to Q of $P \to k(y)$, it follows, by formula (2.3.4) (applied to the action of T_Q on X), that $g \in G$ fixes this point if and only if g is orthogonal to $Q^*/P^* \subset Q^{\mathrm{gp}}/P^{\mathrm{gp}}$ for the pairing $G \otimes Q^{\mathrm{gp}}/P^{\mathrm{gp}} \to \mu_n$ deduced from (2.3.2). The homomorphism $P \to Q$ factors canonically as

$$P \hookrightarrow P' \hookrightarrow Q,$$

where $P' := Q^* \oplus_{P^*} P$. This factorization defines a factorization of f into

$$X \xrightarrow{u} Y' \xrightarrow{v} Y,$$

where v is a usual finite étale covering, Galois with group H equal to the Cartier dual of Q^*/P^*, and strict as a morphism of log schemes. In particular, $\mathscr{O}^*_Y \to (v_\star \mathscr{O}^*_{Y'})^H$ and $M_Y \to (v_\star M_{Y'})^H$ are isomorphisms. Replacing Y by Y' then by its strict localization at a point above y, we can therefore assume that $P^* = Q^*$. It follows from the remark made earlier that X then has a unique point x above y, so it is strictly local, and the inertia group at x is G itself. Consider the following commutative diagram, with exact rows, where the vertical arrows are the obvious canonical arrows:

(2.4.1)
$$\begin{tikzcd}
0 \arrow[r] & \mathscr{O}^*_{Y,y} \arrow[r] \arrow[d] & M^{\mathrm{gp}}_{Y,y} \arrow[r] \arrow[d] & \overline{M}^{\mathrm{gp}}_{Y,y} \arrow[r] \arrow[d, "i"] & 0 \\
0 \arrow[r] & (\mathscr{O}^*_{X,x})^G \arrow[r] & (M^{\mathrm{gp}}_{X,x})^G \arrow[r] & (\overline{M}^{\mathrm{gp}}_{X,x})^G \arrow[r, "d"] & \mathrm{H}^1(G, \mathscr{O}^*_{X,x})
\end{tikzcd}.$$

where d is the boundary operator of the exact cohomology sequence of G relative to the exact sequence of G-modules

$$0 \to \mathscr{O}^*_{X,x} \to M^{\mathrm{gp}}_{X,x} \to \overline{M}^{\mathrm{gp}}_{X,x} \to 0.$$

According to the first formula of (2.2.2), the left vertical arrow is an isomorphism. According to (2.3.4), G acts trivially on $\overline{M}^{\mathrm{gp}}_{X,x}$, so $(\overline{M}^{\mathrm{gp}}_{X,x})^G = \overline{M}^{\mathrm{gp}}_{X,x}$. A computation that will be done in 3.5 shows that the natural map $\mathrm{H}^1(G, \mathscr{O}^*_{X,x}) \to \mathrm{H}^1(G, k(x)^*) = \mathrm{Hom}(G, \mu_n)$ is an isomorphism, and that, via these identifications, the boundary operator

$$d : \overline{M}^{\mathrm{gp}}_{X,x} (= Q^{\mathrm{gp}}/P^*) \to \mathrm{Hom}(G, \mu_n)$$

is deduced from the canonical pairing $G \otimes Q^{\mathrm{gp}}/P^{\mathrm{gp}} \to \mu_n$ (cf. (2.3.2)), which is a perfect pairing. The kernel of d is therefore $P^{\mathrm{gp}}/P^* = \overline{M}^{\mathrm{gp}}_{Y,y}$. It follows that i induces an isomorphism on the kernel of d, and therefore the middle vertical arrow of (2.4.1) is an isomorphism. Since $\overline{M}_{Y,y} = \overline{M}^{\mathrm{gp}}_{Y,y} \cap \overline{M}_{X,x} \subset M^{\mathrm{gp}}_{X,x}$ (f being Kummer), if $a \in (M_{X,x})^G$, there exists $b \in M_{Y,y}$ and $u \in \mathscr{O}^*_{X,x}$ such that $a = ub$, so $u \in (\mathscr{O}^*_{X,x})^G = \mathscr{O}^*_{Y,y}$, and $a \in M_{Y,y}$, and the map $M_{Y,y} \to (f_\star M_{X,x})^G$ is also an isomorphism.

Proposition 2.5. *Consider a standard Kummer étale covering (2.3.1). For every geometric point $\overline{x}$ of X with image $\overline{y}$ in Y, the $G_{\overline{x}}$-equivariant morphism*

(2.5.1)
$$C_{X,\overline{x}} \to C_{Y,\overline{y}},$$

where $G_{\overline{x}}$ is the inertia subgroup at $\overline{x}$, is an isomorphism. In particular, $G_{\overline{x}}$ acts trivially on $C_{X,\overline{x}}$.

Proceeding as in 2.4, we can assume X (resp. Y) is strictly local with closed point x (resp. y), with $P^* = b^{-1}(\mathcal{O}^*_{Y,y})$, (resp. $Q^* = a^{-1}(\mathcal{O}^*_{X,x})$), and $P^* \xrightarrow{\sim} Q^*$. It follows that the composite map

$$\mathbf{Z}[P^*] \to \mathbf{Z}[P] \to \mathbf{Z}[P]/b^{-1}(\mathcal{O}_{Y,y} - \mathcal{O}^*_{Y,y})$$

(resp.

$$\mathbf{Z}[Q^*] \to \mathbf{Z}[Q] \to \mathbf{Z}[Q]/a^{-1}(\mathcal{O}_{X,x} - \mathcal{O}^*_{X,x}))$$

is an isomorphism, and consequently the map

$$\varphi : \mathbf{Z}[P]/b^{-1}(\mathcal{O}_{Y,y} - \mathcal{O}^*_{Y,y}) \to \mathbf{Z}[Q]/a^{-1}(\mathcal{O}_{X,x} - \mathcal{O}^*_{X,x})$$

is an isomorphism. Since the map $\mathcal{O}_{Y,y}/I_{Y,y} \to \mathcal{O}_{X,x}/I_{X,x}$ is given by $\mathcal{O}_{Y,y} \otimes_{Z[P]} \varphi$, the projection $C_{X,x} \to C_{Y,y}$ (2.5.1) is therefore an isomorphism.

3. Very Tame Actions

3.1. Let X be an fs log scheme, equipped with an action of a finite group G. We propose to identify sufficient conditions on the action of G so that, when X is log regular, the quotient of X by G exists as a log scheme and is log regular.

We say that G **acts tamely on** X **at a geometric point** $\overline{x}$ of X localized at x, if the stabilizer $G_{\overline{x}}$ of $\overline{x}$ (**inertia group at** x) has order coprime to the characteristic of $k(x)$. We say that G **acts tamely on** X if G acts tamely at $\overline{x}$ for every $\overline{x}$. These definitions do not involve the log structure of X.

The definition and the following results are due to Gabber. We say that G **acts very tamely on** X ***at*** $\overline{x}$ if the following three conditions are satisfied:

(i) G acts tamely at $\overline{x}$;

(ii) G acts trivially on $\overline{M}_{X,\overline{x}}$;

(iii) $G_{\overline{x}}$ acts trivially on the stratum $C_{X,\overline{x}}$ (1.2).

We say that G **acts very tamely on** X if G acts very tamely on X at every geometric point.

In the situation of (2.3.1), it follows from (2.3.4) and 2.5 that, for every geometric point $\overline{x}$ of X, $G_{\overline{x}}$ acts trivially on $\overline{M}_{\overline{x}}$ and on the stratum $C_{X,\overline{x}}$. Therefore G acts very tamely on X. We will see that in a suitable sense, any very tame action generically free on a log regular log scheme is locally of this type. More precisely, the main result is the following:

Théorème 3.2. *Let X be an fs log regular log scheme, endowed with an action of a finite group G. Assume that G acts admissibly on the underlying scheme of X, freely on a dense open set, and very tamely. Let $Y = X/G$ be the quotient scheme, $f : X \to Y$ the projection. Then:*

(i) Y is locally Noetherian and the morphism f is finite;

(ii) the homomorphism $f_\star\alpha : (f_\star M_X)^G \to (f_\star \mathcal{O}_X)^G = \mathcal{O}_Y$ is an fs log structure on Y, which makes Y a log regular log scheme, and $f : X \to Y$ is a Kummer étale covering of group G. Furthermore, at every geometric point $\overline{x}$ of X, the inertia group $G_{\overline{x}}$ is abelian.

Recall ([**SGA 1** v 1]) that saying G acts admissibly on a scheme (not necessarily locally Noetherian) X means that X is a union of affine open sets stable under G, so that the quotient X/G exists as a scheme.

3.3. The proof of 3.2 will be given in 3.6. We will first establish a more precise result than 3.2, namely 3.4, of a local nature on X. For this we will need the following notion. Let n be an integer ≥ 1, G a finite group, and Q an fs monoid. Suppose given a homomorphism

$$\chi : G^{\mathrm{ab}} \otimes Q^{\mathrm{gp}} \to \mu_n := \mu_n(\mathbf{C}), \;\; g \otimes q \mapsto \chi(g,q).$$

For $g \in G$ and $q \in Q$, we will still write, by abuse, $\chi(g,q)$ (or $\chi_q(g)$) for $\chi(\overline{g},q)$, where $\overline{g}$ is the image of g in G^{ab}. Let $\Lambda = \mathbf{Z}[\mu_n][1/n]$. From χ, we deduce an action of G on the log scheme $\operatorname{Spec}\Lambda[Q]$, characterized by

$$g.q = \chi_q(g)q$$

for $g \in G$, $q \in Q$.

Let X be an fs log scheme equipped with an action of G. By a **G-equivariant chart of X modeled on** (χ, Q), we mean a strict, G-equivariant morphism

$$c : X \to \operatorname{Spec} \Lambda[Q].$$

Such a morphism is given by a homomorphism

$$c^\star : Q \to \Gamma(X, M_X)$$

such that, for every $q \in Q$ and every $g \in G$, we have $g^\star(c^\star(q)) = \chi(g,q)c^\star(q)$, where $g^\star$ denotes the endomorphism of $\Gamma(X, M_X)$ defined by g.

Proposition 3.4. *Let X be an fs log scheme, equipped with an action of a finite group G, and let $\overline{x}$ be a geometric point of X localized at x. We denote $H = G_{\overline{x}}$ the inertia group at x.*

(*a*) *Assume conditions* (*i*) *and* (*ii*) *of 3.1 are satisfied at $\overline{x}$. Let n be the order of H. There exists an affine étale H-equivariant neighborhood U of $\overline{x}$ and an H-equivariant chart of U modeled on (χ, Q), where $Q = \overline{M}_{\overline{x}}$ and $\chi : H^{\mathrm{ab}} \otimes Q^{\mathrm{gp}} \to \mu_n$.*

(*b*) *Assume furthermore that G acts very tamely at $\overline{x}$,* i.e. *that H acts trivially on the stratum $C_{X,\overline{x}}$. Then, there exists an affine étale H-equivariant neighborhood U of $\overline{x}$ such that the quotient scheme $V = U/H$, equipped with the log structure $(\pi_\star M_U)^H \to (\pi_\star \mathscr{O}_U)^H = \mathscr{O}_V$, where $\pi : U \to V$ is the projection, is an fs log scheme, and that we have an equivariant strict closed V-immersion, with nilpotent ideal, $i : U \hookrightarrow U'$, where $\pi' : U' \to V$ is a standard Kummer étale covering of V.*

(*c*) *If, under the hypotheses of* (*b*), *X is additionally assumed log regular at x, then there exists $\pi : U \to V$ as in* (*b*) *such that π is a standard Kummer étale covering of V, and V is log regular at $y = \pi(x)$. If moreover H acts freely on a dense open set of U, H is abelian, and equal to the group of the covering π.*

3.5. *Proof of 3.4.*

(a) First observe that the affine étale H-stable neighborhoods of $\overline{x}$ form a cofinal family of étale neighborhoods of $\overline{x}$. Indeed, let U_1 be an étale, affine, neighborhood of $\overline{x}$. Let $U_2 = \prod_{h \in H} U_1 h$, where $\prod$ denotes a fibered product over X. The scheme U_2 is an étale quasi-affine H-stable neighborhood of $\overline{x}$. If U_3 is an affine open neighborhood of the image x of $\overline{x}$ in U_3, $U_4 = \bigcap_{h \in H} U_3 h$ is an affine étale, H-stable, neighborhood of $\overline{x}$, above U. Since $k(\overline{x})$ contains μ_n, we can assume, by replacing X with a suitable H-stable étale neighborhood of $\overline{x}$, that X is above $\operatorname{Spec} \Lambda$, where $\Lambda = \mathbf{Z}[\mu_n][1/n]$, with trivial action of H on Λ.

Consider the exact sequence of abelian groups

$$0 \to \mathscr{O}^*_{X,\overline{x}} \to M^{\mathrm{gp}}_{X,\overline{x}} \to \overline{M}^{\mathrm{gp}}_{X,\overline{x}} \to 0. \tag{3.5.1}$$

It is H-equivariant, and if $Q = \overline{M}_{\overline{x}}$, Q^{gp} is finitely generated and torsion-free. Choose a splitting $s : Q^{\mathrm{gp}} \to M^{\mathrm{gp}}_{X,\overline{x}}$ of (3.5.1) (as an exact sequence of groups). Since H acts trivially on Q^{gp}, we have, for $q \in Q^{\mathrm{gp}}$ and $g \in H$

$$g.s(q) = z(g,q)s(q) \tag{3.5.2}$$

with $z(g,q) \in \mathscr{O}^*_{X,\overline{x}}$. For g_1, g_2 in H, we have

$$z(g_1 g_2, q) = (g_1 z(g_2, q)).z(g_2, q),$$

in other words, $g \mapsto (q \mapsto z(g,q))$ is a 1-cocycle

$$z \in Z^1(H, \operatorname{Hom}(Q^{\mathrm{gp}}, \mathscr{O}^*_{X,\overline{x}})).$$

The image $[z]$ of z in $\mathrm{H}^1(H, \operatorname{Hom}(Q^{\mathrm{gp}}, \mathscr{O}^*_{X,\overline{x}}))$ is the cohomology class of (3.5.1). In the commutative square of canonical arrows

$$\begin{tikzcd}
Z^1(H, \operatorname{Hom}(Q^{\mathrm{gp}}, \mathscr{O}^*_{X,\overline{x}})) \arrow[r] \arrow[d] & Z^1(H, \operatorname{Hom}(Q^{\mathrm{gp}}, k(\overline{x})^*)) \arrow[d] \\
\mathrm{H}^1(H, \operatorname{Hom}(Q^{\mathrm{gp}}, \mathscr{O}^*_{X,\overline{x}})) \arrow[r] & \mathrm{H}^1(H, \operatorname{Hom}(Q^{\mathrm{gp}}, k(\overline{x})^*))
\end{tikzcd},$$

the right vertical arrow is (trivially) an isomorphism. On the other hand, the bottom horizontal arrow is an isomorphism, because $(1 + \mathfrak{m}_{X,\overline{x}})^*$ is n-divisible. The image of z in $Z^1(H, \mathrm{Hom}(Q^{\mathrm{gp}}, k(\overline{x})^*))$ is a homomorphism

$$\chi : H \to \mathrm{Hom}(Q^{\mathrm{gp}}, \mu_n),\ g \mapsto (q \mapsto \chi(g, q)).$$

This homomorphism lifts to an element of $Z^1(H, \mathrm{Hom}(Q^{\mathrm{gp}}, \mathscr{O}^*_{X,\overline{x}}))$, still denoted χ, which has the same cohomology class as z in $\mathrm{H}^1(H, \mathrm{Hom}(Q^{\mathrm{gp}}, \mathscr{O}^*_{X,\overline{x}}))$. There therefore exists $u \in \mathrm{Hom}(Q^{\mathrm{gp}}, \mathscr{O}^*_{X,\overline{x}})$ such that $z(g, q)/\chi(g, q) = gu(q)/u(q)$. Then $g.(u(q)^{-1}s(q)) = \chi(g, q)u(q)^{-1}s(q)$. So by replacing s by $(q \mapsto u(q)^{-1}s(q))$, we can assume that $z = \chi$, in other words z is defined by a homomorphism (still denoted χ)

$$\chi : H^{\mathrm{ab}} \otimes Q^{\mathrm{gp}} \to \mu_n,\ (h, q) \mapsto \chi(h, q), \tag{3.5.3}$$

i.e. $z(h, q) = \chi(h, q)$ for all $h \in H$ and all $q \in Q$ (with the notation of 3.3). According to [**Kato, 1988**, 2.10], the homomorphism $Q \to M_{X,\overline{x}}$ induced by s extends to a chart $a : U \to \mathrm{Spec}\,\Lambda[Q]$ (corresponding to a homomorphism $a^\star : Q \to \Gamma(U, M_U)$) of an étale neighborhood U of $\overline{x}$. From what we have seen above, we can assume U to be H-stable. According to (3.5.2), the homomorphism $s : Q \to M_{X,\overline{x}}$ is H-equivariant, *i.e.* for $q \in Q$ and $h \in H$, we have (in $M_{X,\overline{x}}$)

$$h^\star a^\star(q)_{\overline{x}} = \chi(h, q)a^\star(q)_{\overline{x}}.$$

For fixed h and q, we therefore have $h^\star a^\star(q) = \chi(h, q)a^\star(q)$ on an H-stable étale neighborhood of $\overline{x}$ above U, and thus *a fortiori* on an H-stable open neighborhood U' of the image x of $\overline{x}$ in U. Since H is finite and Q is finitely generated, there exists an H-stable affine open neighborhood U'' of x contained in U' such that the preceding identity is satisfied above U''. So, by replacing U with a smaller H-stable affine open neighborhood, a is an H-equivariant chart of U modeled on (χ, Q).

(b) In view of (a), we can assume that we have an affine étale, H-stable neighborhood U of $\overline{x}$, and an H-equivariant chart

$$a : U \to \mathrm{Spec}\,\Lambda[Q]$$

modeled on (χ, Q). Since $n = |H|$ is invertible at x, by shrinking U, we can further assume that n is invertible on U. Since U is affine, the quotient $V = U/H$ exists and is affine. Moreover, according to **IV**-2.2.3, V is Noetherian, and $f : U \to V$ is finite. Let P' be the subgroup of Q^{gp} defined by

$$P' = \{q \in Q^{\mathrm{gp}} | \chi(h, q) = 1 \text{ for all } h \in H\}.$$

In other words, P' is defined by the exact sequence

$$0 \to P' \to Q^{\mathrm{gp}} \to \mathrm{Hom}(H^{\mathrm{ab}}, \mu_n),$$

where the second map is $q \mapsto (h \to \chi(h, q))$. It is therefore of finite index in Q^{gp}, coprime to the characteristic of $k(\overline{x})$, and the inclusion $Q^{\mathrm{gp}}/P' \hookrightarrow D(H^{\mathrm{ab}})$ (where $D = \mathrm{Hom}(-, \mu_n)$) defines by duality an epimorphism

$$H^{\mathrm{ab}} \twoheadrightarrow D(Q^{\mathrm{gp}}/P'). \tag{3.5.4}$$

Let

$$P = P' \cap Q.$$

This is an fs submonoid of Q, such that $P^{\mathrm{gp}} = P'$, and $P \to Q$ is Kummer. The morphism $a : U \to \mathrm{Spec}\,\Lambda[Q]$ from (a) (where $\Lambda = \mathbf{Z}[\mu_n][1/n]$) is equivariant relative to the epimorphism $H \twoheadrightarrow H^{\mathrm{ab}} \twoheadrightarrow D(Q^{\mathrm{gp}}/P')$, so it defines, by passing to the quotient, a morphism $b : V \to \mathrm{Spec}\,\Lambda[P]$ giving rise to a commutative square

$$\begin{tikzcd} U \arrow[r, "a"] \arrow[d, "\pi"] & \mathrm{Spec}\,\Lambda[Q] \arrow[d] \\ V \arrow[r, "b"] & \mathrm{Spec}\,\Lambda[P] \end{tikzcd}, \tag{3.5.5}$$

from which we get a commutative triangle

(3.5.6)
$$\begin{tikzcd} U \arrow[r, "i"] \arrow[d, "\pi"'] & U' \arrow[dl, "\pi'"] \\ V & \end{tikzcd} ,$$

where $U' = V \times_{\operatorname{Spec}\Lambda[P]} \operatorname{Spec}\Lambda[Q]$. If we equip V with the (fs) log structure defined by b, π is a morphism of log schemes, and (3.5.5) is a chart for it. The triangle (3.5.6) is a triangle of log schemes. Furthermore, π' is a standard Kummer étale covering with group $D(Q^{\mathrm{gp}}/P^{\mathrm{gp}})$, and i is equivariant relative to the epimorphism $H \twoheadrightarrow H^{\mathrm{ab}} \twoheadrightarrow D(Q^{\mathrm{gp}}/P^{\mathrm{gp}})$ (3.5.4). Since H acts trivially on the stratum $C_{X,\overline{x}} = C_{U,\overline{x}} = \operatorname{Spec}\mathscr{O}_{U,\overline{x}}/I_{\overline{x}}$, and H has order coprime to the characteristic of $k(\overline{x})$, the homomorphism

$$\mathscr{O}_{V,\overline{y}}\,(= \mathscr{O}^{H}_{U,\overline{x}}) \to \mathscr{O}_{U,\overline{x}}/I_{\overline{x}}\,(= (\mathscr{O}_{U,\overline{x}}/I_{\overline{x}})^{H})$$

is surjective, where $\overline{y} = f(\overline{x})$, so the same holds for the homomorphism

$$k(\overline{y})[Q] \;=\; k(\overline{x})[Q] \to \mathscr{O}_{U,\overline{x}}/\mathfrak{m}_{V,\overline{y}}\mathscr{O}_{U,\overline{x}}.$$

By Nakayama, it follows that the morphism i of (3.5.6) induces a closed immersion at $\overline{x}$, *i.e.* on the strict localizations at $\overline{x}$. By restricting (3.5.6) to a suitable affine étale neighborhood of $\overline{y}$ above V, we can therefore assume that i is a closed immersion. Since H acts transitively on the fibers of π, and thus, in view of (3.5.4), also on those of π', above every point of V, these fibers coincide, and therefore U and U' have the same underlying spaces. The closed immersion i is therefore defined by a nilpotent ideal I of $\mathscr{O}_{U'}$, and is strict for the log structures considered. Consider the homomorphisms

(3.5.7)
$$M_V \to (\pi'_\star M_{U'})^H \to (\pi_\star M_U)^H$$

defined by (3.5.6). Since $H \to D(Q^{\mathrm{gp}}/P^{\mathrm{gp}})$ is surjective, it follows from (2.4) that the first map of (3.5.7) is an isomorphism. Let us show that the same holds for the second. Since the immersion i is strict, we have an equivariant commutative diagram with exact rows,

(3.5.8)
$$\begin{tikzcd} 0 \arrow[r] & (1+I)^* \arrow[r] \arrow[d, "\mathrm{Id}"'] & M_{U'} \arrow[r] \arrow[d] & M_U \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & (1+I)^* \arrow[r] & M^{\mathrm{gp}}_{U'} \arrow[r] & M^{\mathrm{gp}}_U \arrow[r] & 0 \end{tikzcd} ,$$

where $(1+I)^* = \operatorname{Ker}\mathscr{O}^*_{U'} \to \mathscr{O}^*_U$. More precisely, $M_{U'}$ is a $(1+I)^*$-torsor over M_U induced by the $(1+I)^*$-torsor $M^{\mathrm{gp}}_{U'}$ over M^{gp}_U, in particular, the right square is cartesian. We deduce from this a commutative diagram where the square is cartesian and the bottom row is exact:

(3.5.9)
$$\begin{tikzcd} & & (\pi'_\star M_{U'})^H \arrow[r] \arrow[d] & (\pi_\star M_U)^H \arrow[d] \\ 0 \arrow[r] & (\pi'_\star(1+I)^*)^H \arrow[r] & (\pi'_\star M^{\mathrm{gp}}_{U'})^H \arrow[r] & (\pi_\star M^{\mathrm{gp}}_U)^H \end{tikzcd} .$$

Since π' is finite, at every geometric point $\overline{z}$ of V, we have $\mathrm{H}^q(H, \pi'_\star(1+I)^*_{\overline{z}}) = \prod_{\overline{t}\in U_{\overline{z}}} \mathrm{H}^q(H, (1+I)^*_{\overline{t}})$, and since the fibers of $(1+I)^*$ are n-divisible, we have $\mathrm{H}^q(H, \pi'_\star(1+I)^*) = 0$ for all $q > 0$. Moreover, $(\pi'_\star(1+I)^*)^H = 0$ (since $(\pi_\star\mathscr{O}_U)^H = (\pi'_\star\mathscr{O}_{U'})^H$ and thus $(\pi'_\star I)^H = 0$). The map $(\pi'_\star M^{\mathrm{gp}}_{U'})^H \to (\pi_\star M^{\mathrm{gp}}_U)^H$ of (3.5.9) is therefore an isomorphism, and the right square being cartesian, the same holds for the map $(\pi'_\star M_{U'})^H \to (\pi_\star M_U)^H$, as announced, which completes the proof of (b).

(c) In this case, with the notations of the proof of (b), the stratum $C_{U,\overline{x}}$ is regular, and projects isomorphically onto $C_{V,\overline{y}}$. Since $\mathrm{rg}(\overline{M}^{\mathrm{gp}}_{\overline{x}}) = \mathrm{rg}(\overline{M}^{\mathrm{gp}}_{\overline{y}}) = \mathrm{codim}(C_{V,\overline{y}}, V_{(\overline{y})})$, V is log regular at $y = \pi(x)$, hence at any generalization of y (cf. 1.3), so $V_{(\overline{y})}$ is log regular. Since π' is log étale, $U' \times_V V_{(\overline{y})}$ is therefore log regular, in particular, reduced, and the immersion i induces an isomorphism above $V_{(\overline{y})}$, and thus already above $V_{(y)} = \operatorname{Spec}\mathscr{O}_{V,y}$. By replacing V with a suitable open neighborhood of y, i is therefore an isomorphism, and $\pi : U \to V$ is a Kummer étale covering (with group $D(P^{\mathrm{gp}}/Q^{\mathrm{gp}})$, a quotient of H). Finally, if H acts freely on a dense open set, taking a geometric point t of V such that H acts freely on the fiber $V_{\pi(t)}$, we have $t.H = t.D(P^{\mathrm{gp}}/Q^{\mathrm{gp}})$, so $H = D(P^{\mathrm{gp}}/Q^{\mathrm{gp}})$.

3.6. *Proof of 3.2.*
The assertions of 3.2 follow from the following corollary of 3.4:

Corollaire 3.7. *We are under the hypotheses of* (*3.4* (*c*)), i.e. *X is log regular at x, the action of G on X is very tame at $\overline{x}$. We further assume that the action of G on X is admissible, and free on a dense open set. We denote $f : X \to Y = X/G$ the projection, and x* (*resp. y*) *the image of $\overline{x}$* (*resp. $\overline{y} = f(\overline{x})$*) *in X* (*resp. Y*). *Then $G_{\overline{x}}$ is abelian, and there exists an affine open neighborhood V of y in Y such that:*
(*i*) *the scheme V is Noetherian, and, equipped with the log structure $f_\star\alpha : (f_\star M_X)^G|V \to (f_\star \mathscr{O}_X)^G|V = \mathscr{O}_V$, is an fs log scheme, log regular at y;*
(*ii*) *$f_V : X \times_Y V \to V$ is a Kummer étale covering of V with group G.*

Proof of 3.7. The question is local on Y in the neighborhood of $\overline{y}$. By replacing Y with an étale neighborhood Y' of $\overline{y}$, and X with $X\times_Y Y'$, we can assume that x is a rational point of X_y, with stabilizer H, so that X_y is identified with $H\backslash G$ by $x \mapsto xg$. The morphism $f : X \to Y$ being integral, hence closed, by replacing Y with a neighborhood of y, we can find an open and closed neighborhood U of x, stable by H, whose translates by a system of representatives of $H\backslash G$ in G are pairwise disjoint, so that X is identified with the contracted product $U\wedge^H G$. Then $U/H \to X/G$ is an isomorphism, and in particular, according to **IV**-2.2.3, Y is Noetherian, and $f : X \to Y$ is finite. By the preceding argument, we reduce to the case where $G = H$, $X_y = \{x\}$. Since f is finite, the étale neighborhoods of $\overline{x}$ inverse images by f of étale neighborhoods of $\overline{y}$ form a cofinal system of étale neighborhoods of $\overline{x}$, and these are H-stable. Moreover, for any H-stable étale neighborhood Z of $\overline{x}$ there exists an étale neighborhood V of $\overline{y}$ and a morphism $h : Z' \to Z$ between neighborhoods of $\overline{x}$, where $Z' = f^{-1}(V)$, and, by shrinking V, we can require h to be H-equivariant. The proof of (3.4 (c)) shows that we can choose U of *loc. cit.* of this form, *i.e.* $U = f^{-1}(V)$, so that $f = \pi : U \to V = U/H$ is a standard Kummer étale covering of V with group H, which completes the proof.

Corollaire 3.8. *Under the hypotheses of 3.4* (*b*), *the subgroup* $\mathrm{Ker}\,(G_{\overline{x}} \to \mathrm{Hom}(\overline{M}_{X,\overline{x}}, k(\overline{x})))$ *acts trivially on the log scheme X in an open $G_{\overline{x}}$-stable neighborhood of x.*

Indeed, with the notations of 3.5 (b), we have

$$H = G_{\overline{x}} \twoheadrightarrow H^{\mathrm{ab}} \twoheadrightarrow D(Q^{\mathrm{gp}}/P^{\mathrm{gp}}) = \mathrm{Hom}(Q^{\mathrm{gp}}/P^{\mathrm{gp}}, k(\overline{x})) \hookrightarrow \mathrm{Hom}(\overline{M}_{X,\overline{x}}, k(\overline{x})),$$

thus 3.4 (b) implies that there exists an H-stable étale neighborhood $u : U \to X$ of $\overline{x}$ such that $\mathrm{Ker}\,H \to \mathrm{Hom}(\overline{M}_{X,\overline{x}}, k(\overline{x}))$ acts trivially on the log scheme U, and thus *a fortiori* on the open (H-stable) neighborhood $u(U)$ of x.

Let us also point out the following consequence of 3.4 (b):

Corollaire 3.9. *Let us place ourselves under the hypotheses of 3.4* (*b*), *and assume X is separated. Then G acts very tamely on X in an open G-stable neighborhood of x.*

Consider the H-stable étale neighborhood $u : U \to X$ of $\overline{x}$ constructed in 3.5 (b). As observed in 3.1, H acts very tamely on U', so on U, and *a fortiori* on $u(U)$. Let us provisionally admit the following lemma:

Lemme 3.10. *By replacing U with a smaller H-stable open neighborhood of the image x' of $\overline{x}$, we can assume that the following property is satisfied:*
($*$) *for every geometric point $\overline{z}$ of U, the homomorphism $H_{\overline{z}} \to G_{u(\overline{z})}$ is an isomorphism.*

Assume that U possesses property ($*$). Then, since H acts very tamely on $u(U)$, G acts very tamely on $u(U).G$. It remains to prove 3.10. Property ($*$) is equivalent to the conjunction of
(i) $u^{-1}(X^g) = \varnothing$ for $g \in G - H$,
(ii) $u^{-1}(X^g) = U^g$ for $g \in H$,

where $(-)^g$ denotes a fixed-point scheme of g. Since X is separated, X^g is a closed subscheme of X. If g is not in H, X^g does not contain x, so, by removing from U the union of $u^{-1}(X^g)$ for $g \in G - H$, condition (i) is satisfied. For $g \in H$, U^g is defined by the cartesian square

$$\begin{tikzcd} U^g \arrow[r] \arrow[d] & U \arrow[d, "\Delta"] \\ u^{-1}(X^g) \arrow[r, "{(1,g)}"] & U \times_X U \end{tikzcd},$$

where Δ is the diagonal. Since Δ is open and closed, U^g is open and closed in $u^{-1}(X^g)$. But U^g and $u^{-1}(X^g)$ are closed subschemes of U containing x'. They therefore coincide in an open neighborhood of x'. So by replacing U with a smaller H-stable open neighborhood of x', we can satisfy condition (ii) for all $g \in H$.

4. Fixed Points

4.1. Let X be a Noetherian scheme on which a finite group G acts. For each subgroup H of G, we denote X^H the fixed point scheme of H. This is a subscheme of X, closed if X is separated, representing the functor $S \mapsto X(S)^H$, intersection of the graphs of translations $h : X \to X, x \mapsto xh$ for $h \in H$. Assume X is separated. At each geometric point $\overline{x}$ of X^H with image x, the inertia subgroup $G_{\overline{x}}$ of G contains H, and is equal to H if and only if x belongs to the (locally closed) subscheme

$$X_H = X^H - \bigcup_{H' \supset H, H' \neq H} X^{H'}.$$

For $g \in G$, we have

$$(X^H)g = X^{g^{-1}Hg},$$

and similarly

$$(X_H)g = X_{g^{-1}Hg}$$

so that the union X^C (resp. X_C) of X^H (resp. X_H) for H in a conjugacy class C of subgroups of G is G-stable. The X_C, for C ranging over the conjugacy classes of subgroups of G, form a stratification of X by sub-G-schemes, with the property that for every geometric point $\overline{x}$ localized in X_C, the inertia group $G_{\overline{x}}$ belongs to C. We will call this stratification the **inertia stratification**.

The goal of this section is to give examples of very tame actions of finite groups G on log regular and regular log schemes X, where a refinement of the inertia stratification is deduced from the canonical stratification of a G-stable divisor with normal crossings.

The following result is classical, we provide a proof for lack of a reference.

Proposition 4.2. *Let X be a regular Noetherian scheme on which a finite group G acts tamely 3.1. Then the fixed point scheme X^G is regular.*

If y is a point of a scheme Y, let $Y_{(y)} = \operatorname{Spec} \mathscr{O}_{Y,y}$. Assume $X^G \neq \varnothing$. Let x be a point of X^G. We need to show that $(X^G)_{(x)}$ is regular. The group G is the inertia group at x: it fixes x and $k(x)$. There exists an affine open neighborhood of x stable by G: if U is an affine open neighborhood of x, the intersection V of Ug for $g \in G$ is a quasi-affine open neighborhood of x stable by G, and if W is an affine open neighborhood of x contained in V, the intersection of Vg is an affine open neighborhood of x stable by G. So G acts on $X_{(x)}$, trivially on $k(x)$, and, if $A = \mathscr{O}_{X,x}$, we have

$$(X^G)_{(x)} = (X_{(x)})^G = \operatorname{Spec} A_G,$$

where A_G is the algebra of co-invariants of G in A, *i.e.*

$$A_G = A/I,$$

where I is the ideal generated by $ga - a$, for g in G and a in A. We are therefore reduced to assuming $X = X_{(x)}$. If $Y = X/G = \operatorname{Spec} A^G$, G acts by Y-automorphisms of X. Since G is of invertible order at x, according to **IV**-2.2.3, X is finite over Y, and Y is local, Noetherian, with closed point y, and its residue field $k(y) = k(x)^G = k(x)$. The morphisms $X^G \hookrightarrow X \twoheadrightarrow Y$ correspond to the local homomorphisms

$A^G \hookrightarrow A \twoheadrightarrow A_G$, which induce isomorphisms on the residue fields. Since $\widehat{X}^G = \operatorname{Spec} \hat{A}/I\hat{A} = (\hat{X})^G$, we can assume X is local and complete. The same holds for Y. Indeed, as observed in the proof of **IV**-2.2.3, for any ideal J of B, we have $JA \cap B = J$. In particular, if $\mathfrak{m}_A$ (resp. $\mathfrak{m}_B$) denotes the maximal ideal of A (resp. B), we have $\mathfrak{m}_B^n A \cap B = \mathfrak{m}_B^n$, so $\hat{B} = \varprojlim B/\mathfrak{m}_B^n \to \varprojlim A/\mathfrak{m}_B^n A = A$ is injective, with image contained in $A^G = B$, so $B = \hat{B}$. Let $k = k(x) = k(y)$, and $\mathfrak{m}_A = \mathfrak{m}$.

We will prove the proposition by *linearizing* the action of G.

First assume that A is of equal characteristic. Choose a basis $t = (t_i)_{1 \le i \le r}$ over k of the cotangent space $T = \mathfrak{m}/\mathfrak{m}^2$ and elements x_i in $\mathfrak{m}$ lifting the t_i. Since B is a complete Noetherian local ring, also choose a field of representatives, still denoted k, of k in B. Denote

$$\varphi : k[[T]] \to A$$

the homomorphism sending t_i to x_i. The homomorphism

$$f : k[[T]] \to A$$

sending t to the system

$$y = \frac{1}{\operatorname{card}(G)} \sum_{g \in G} g\varphi g^{-1} t$$

is G-equivariant. Furthermore, y is congruent to x modulo $\mathfrak{m}^2$, so it is a regular system of parameters for A, and therefore f is an isomorphism. Consequently f induces an isomorphism

$$k[[T]]_G = k[[T_G]] \to A_G,$$

where T_G is the space of co-invariants of G on T, which proves 4.2 in this case.

Now assume that A is of mixed characteristic $(0, p)$. Let C be a Cohen ring for B, with residue field k, so that B is a quotient of a formal power series ring over C. Since G is of order coprime to p, there exists an (essentially) unique $C[G]$-module V, free of finite type over C, lifting T. Let $t = (t_i)_{1 \le i \le r}$ be a basis of T, $v = (v_i)$ a basis of V lifting t. As above, choose liftings x_i in $\mathfrak{m}$ of the t_i, which therefore form a regular sequence of parameters in A. Extend the homomorphism $C \to A$ to

$$\varphi : C[[V]] \to A$$

by sending v_i to x_i. The homomorphism

$$f : C[[V]] \to A$$

sending $v = (v_i)$ to

$$y = \frac{1}{\operatorname{card}(G)} \sum_{g \in G} g\varphi g^{-1} v$$

is G-equivariant, and y is congruent to x modulo $\mathfrak{m}^2$, so it is a regular system of parameters for A. The homomorphism f is therefore surjective, and its kernel is defined by an element F congruent to p modulo the ideal generated by the v_i. Thus X is a regular G-equivariant divisor in $X' = \operatorname{Spec} C[[V]]$, with equation $F = 0$. Observe that the module of coinvariants V_G is free over C. Indeed, since the order of G is invertible in C, V is projective of finite type over $C[G]$, and therefore, if I_G is the augmentation ideal of $\mathbf{Z}[G]$, $V_G = V/I_G V = V \otimes_{C[G]} C$ is projective of finite type, hence free of finite type, over C (the same holds incidentally for V^G and the composite homomorphism $V^G \hookrightarrow V \twoheadrightarrow V_G$ is an isomorphism). The image of F in $C[[V_G]]$ is a regular parameter, so, since $X'^G = \operatorname{Spec} C[[V_G]]$, $X^G = X \times_{X'} X'^G$ is regular, which completes the proof of 4.2.

Corollaire 4.3. *Let X be a separated, regular Noetherian scheme, equipped with a tame action of a finite group G. Then the stratification of X by inertia consists of regular schemes.*

Remarque 4.4. For future reference, let us note the following complementary result: under the hypotheses of 4.2, for any $x \in X^G$, the canonical homomorphism

$$T_x(X^G) \to T_x(X)^G \tag{4.4.1}$$

is an isomorphism, where for a locally Noetherian scheme Y we denote $T_y(Y)$ the Zariski tangent space at a point y of Y ($= \operatorname{Hom}_{k(y)}(\mathfrak{m}_y/\mathfrak{m}_y^2, k(y))$). More generally, as Gabber observes, (4.4.1) is an isomorphism for a Noetherian local scheme $X = \operatorname{Spec} A$, with closed point x, equipped with an action of a finite group G such that the homomorphism $A^G \to A/\mathfrak{m} = k(x)$ is surjective. Indeed, in this case, if I is the ideal of A generated by $ga - a$ for $g \in G$ and $a \in A$, so that $X^G = \operatorname{Spec} A_G$, where $A_G = A/I$, I is also the ideal of A generated by $ga - a$ for $g \in G$ and $a \in \mathfrak{m}$, in other words, $I = I_G\mathfrak{m}$. Denoting $T_x^*(-)$ a cotangent space, dual (with values in $k(x)$) of $T_x(-)$, the homomorphism

$$\operatorname{Hom}(T_x(X)^G, k(x)) = T_x^*(X)_G \to T_x^*(X^G)$$

is identified with the homomorphism

$$(\mathfrak{m}/\mathfrak{m}^2)_G = \mathfrak{m}/(I_G\mathfrak{m} + \mathfrak{m}^2) \to \mathfrak{m}/(I + \mathfrak{m}^2)$$

which is an isomorphism, since $I = I_G\mathfrak{m}$.

4.5. Under the hypotheses of 4.3, let Y be a strict G-stable divisor with normal crossings, a union of irreducible components Y_i, $1 \le i \le m$. We equip X with the log structure defined by Y. Recall (cf. [**de Jong, 1996**, 7.1]) that we say that Y is ***G*-strict** if the following condition is satisfied: for all i and for all $g \in G$, if $Y_i g \cap Y_i \neq \emptyset$, then $Y_i g = Y_i$. If Y is G-strict, then condition (ii) of 3.1 is satisfied at each geometric point $\overline{x}$ of X. Indeed, if $(D_i)_{1 \le i \le r}$ is the set of branches of Y passing through $\overline{x}$, then $D_i g = D_i$ for all i. Since $\overline{M}_{\overline{x}}^{\mathrm{gp}} = \bigoplus_{1 \le i \le r} \mathbf{Z} e_i$, e_i corresponding to D_i, the inertia group $G_{\overline{x}}$ acts trivially on $\overline{M}_{\overline{x}}^{\mathrm{gp}}$. Recall also ([**de Jong, 1996**, 7.2]) that there exists a canonical G-equivariant modification $f : \tilde{X} \to X$ such that $f^{-1}(Y)_{\mathrm{red}}$ is G-strict.

Corollaire 4.6. *Assume that Y is G-strict, that G acts tamely, admissibly and generically freely, and that the stratification of X by the irreducible components of the strata of the canonical stratification* (*1.5*) *is finer than the inertia stratification* (*4.1*), i.e. *that each of these irreducible components is contained in a stratum of the inertia stratification. Then G acts very tamely on X* (*and thus the conclusion of 3.2 applies*). *The inertia group is constant along each irreducible component c of the stratum $X^{(i)}$* (*1.5* (*i*)), *with value G_c* (*the minimum number of generators of G_c being, according to 3.8, at most equal to i*). *In particular, G acts freely on the stratum $X^{(0)} = X - Y$.*

4.7. *Examples.*

(a) Let k be an algebraically closed field, n an integer ≥ 2 coprime to the characteristic of k, G the group $\mu_n = \mu_n(k)$. We let G act on $X = \mathbf{A}_k^2$ by homotheties ($(\lambda, x) \mapsto \lambda x$ for $\lambda \in G$, $x \in X(k)$). The inertia stratification has two strata, $X - \{0\}$, where G acts freely, and $X^G = \{0\}$. The data of two lines Y_1, Y_2 such that $Y_1 \cap Y_2 = \{0\}$ defines a G-strict divisor with normal crossings $Y = Y_1 \cup Y_2$, and the pair (X, Y) satisfies the conditions of 4.6. The choice of parameters t_1, t_2 such that $Y_i = V(t_i)$ allows to define an equivariant chart 3.3 $c : \operatorname{Spec} \mathbf{Z}[\mathbf{N}^2] \leftarrow X$, $e_i \mapsto t_i$, associated to the homomorphism $\chi : G \otimes \mathbf{Z}^2 \to \mu_n$ such that $\chi(\lambda \otimes e_i) = \lambda$. The quotient X/G is the toric scheme $\operatorname{Spec} k[x^n, x^{n-1}y, \cdots, xy^{n-1}, y^n]$.

More generally, let n be an integer ≥ 1, G an abelian group of order n, S a separated, regular Noetherian scheme, over $\operatorname{Spec} \mathbf{Z}[1/n, \mu_n]$, E an $\mathscr{O}_S$-locally free module of finite rank, equipped with a linear action of G, X the vector bundle $V(E) = \operatorname{Spec} \operatorname{Sym}(E)$. For each character $\chi : G \to \mu_n$, let L_χ be the corresponding G-$\mathscr{O}_S$-module. The canonical G-equivariant homomorphism

$$\bigoplus_\chi L_\chi \otimes E_\chi \to E,$$

where $E_\chi = \mathscr{H}om_G(L_\chi, E)$ and χ ranges over the group of characters of G, is an isomorphism. It defines a G-equivariant decomposition

$$X \xrightarrow{\sim} \bigoplus_\chi X_\chi,$$

where $X_\chi = V(E_\chi)$, equipped with the action of G via χ. In particular, $X^G = X_1$, where $1 : G \to \mu_n$ is the trivial character. Assume S is local, $S = \operatorname{Spec} A$. For each $\chi \in \operatorname{Hom}(G, \mu_n)$, choose a basis $(t_i)_{i \in I_\chi}$ of E_χ, so that $X_\chi = \operatorname{Spec} A[(t_i)_{i \in I_\chi}]$, with $gt_i = \chi(g)t_i$ for $g \in G$, $i \in I_\chi$. The pair formed by X and the divisor with normal crossings (relative to S) $Y = \sum_{\chi, i \in I_\chi} Y_i$, where $Y_i = (t_i = 0)$ for $i \in I_\chi$, satisfies the conditions of 4.6 (and indeed satisfies them fiber by fiber).

(b) Let k be an algebraically closed field of characteristic exponent p, n an integer ≥ 2 such that $(2n, p) = 1$, G the dihedral group $D_n = \langle s, r : s^2 = 1, r^n = 1, srs = r^{-1} \rangle$. Let $\zeta \in k$ be a primitive n-th root of unity. Let $\rho : G \to \mathrm{GL}(E)$ be the representation of degree 2 induced by the character χ of $\mu_n \subset G$ such that $\chi(r) = \zeta$: $\rho(s) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\rho(r) = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}$. Let X be the G-scheme $V(E) = \operatorname{Spec} k[u, v]$, $s(u) = v, r(u) = \zeta u$. For $0 \leq i \leq n-1$, let $Z_i \subset X$ be the line $v = \zeta^i u$, and $Z = \bigcup_{0 \leq i \leq n-1} Z_i$. The inertia stratification has $n + 2$ strata: $X - Z$, where G acts freely, $Z_i - \{0\}$ $(0 \leq i \leq n-1)$, where the inertia group is of order 2 (generated by $r^i s$), and $\{0\} = X^G$.

For $n = 2$, $G = (\mathbf{Z}/2\mathbf{Z})^2$, Z is a G-strict divisor with normal crossings, and the pair (X, Y) satisfies the conditions of 4.6. For $n > 2$, Z is no longer a divisor with normal crossings, and the inertia at $\{0\}$ is no longer abelian. Let $f : X' \to X$ be the blow-up of $\{0\}$ in X, $E = f^{-1}(0)$ the exceptional divisor, $Z' = \bigcup_{0 \leq i \leq n-1} Z'_i$ the strict transform of Z. Then G acts naturally on X', the projection f is G-equivariant, and $Y' = f^{-1}(Z) = E \cup Z'$ is a G-strict divisor with normal crossings. The pair (X', Y') satisfies the conditions of 4.6. The stratification of X' by inertia is composed of the strata Z'_i, where the inertia is a group of two elements, and of $X' - Z'$, where G acts freely. The canonical stratification associated with Y' refines it: $X'^{(0)} = X' - Y'$, $X'^{(1)} = Y' - \bigcup_{0 \leq i \leq n-1} (E \cap Z'_i)$, $X'^{(2)} = \bigcup_{0 \leq i \leq n-1} (E \cap Z'_i)$.

4.8. The preceding construction, which makes inertias abelian, generalizes. Let X be a regular Noetherian scheme, separated, equipped with a tame action of a finite group G, and let Y be a G-strict divisor with normal crossings. If H is a subgroup of G, X^H is regular (and separated), so the same holds for the blow-up $X' = \mathrm{\acute{E}cl}_{X^H}(X)$ of X along X^H. The normalizer $N = N_G(H)$ of H in G stabilizes X^H, so it acts on X', and the morphism $f : X' \to X$ is equivariant relative to $N \to G$. Moreover, $f^{-1}(X^H)$ is a regular divisor in X'. If D is a component of Y, since D is H-stable, $D \times_X X^H = D^H$ is regular, and the strict transform $\tilde{D} = \mathrm{\acute{E}cl}_{D^H}(D)$ is a regular divisor crossing $f^{-1}(X^H)$ transversally. It follows that the total reduced transform $Y' = f^{-1}(Y)_{\mathrm{red}}$ is an N-strict divisor with normal crossings in X'.

Proposition 4.9. *Under the hypotheses of 4.8, let $\overline{x}$ be a geometric point of X at which the inertia group $G_{\overline{x}}$ is not abelian, and let H be the commutator subgroup $(G_{\overline{x}}, G_{\overline{x}})$. Then $G_{\overline{x}} = N_{G_{\overline{x}}}(H)$ acts on $X' = \mathrm{\acute{E}cl}_{X^H}(X)$, and at each geometric point $\overline{y}$ of X' above $\overline{x}$, the inertia group $(G_{\overline{x}})_{\overline{y}}$ is strictly smaller than $G_{\overline{x}}$.*

Indeed, the point $\overline{y}$ corresponds to a line L in $(T_{\overline{x}}/T_{\overline{x}}^H) \otimes_{k(\overline{x})} k(\overline{y})$, where $T_{\overline{x}} = T_{\overline{x}}(X)$. Suppose that $\overline{y}$ is fixed under $G_{\overline{x}}$. Then $G_{\overline{x}}$ acts on L by a character, so H acts trivially on L. But $(T_{\overline{x}}/T_{\overline{x}}^H)^H = 0$, a contradiction. (Note that this argument shows in particular that, if $H \neq \{1\}$ and the action of G on X is generically free, X^H has codimension ≥ 2 in X at x.)

LECTURE VII

Proof of the Local Uniformization Theorem (Weak Form)

Fabrice Orgogozo

1. Statement

The purpose of this lecture is to prove Theorem **II**-4.3.1 (see also **Intro.**-4), whose statement we recall below:

Théorème 1.1. *Let X be a quasi-excellent noetherian scheme and Z a nowhere dense closed subset of X. There exists a finite family of morphisms $(X_i \to X)_{i \in I}$, covering for the topology of alterations and such that for every $i \in I$ we have:*

(i) *the scheme X_i is regular and integral;*

(ii) *the inverse image of Z in X_i is the support of a strict normal crossings divisor.*

2. Reductions: Recall of Previous Results

2.1. Reduction to the local, normal, finite-dimensional case. We saw in **II**-4.3.3 that it suffices to prove the theorem when the scheme X is local noetherian normal henselian excellent. Let us make this additional hypothesis. Such a scheme is necessarily finite-dimensional, which we will denote here by d. Furthermore, we saw in *loc. cit.* that if the theorem is established for every local noetherian henselian excellent scheme of dimension at most d, it also holds for quasi-excellent noetherian schemes of dimension at most d.

2.2. Reduction to the complete case. It follows from Proposition **III**-6.2 that it suffices to prove the theorem for the complete local noetherian scheme $\widehat{X}$, the latter having the same dimension as X and also being normal.

2.3. Recurrence. It follows from the above that we can assume the scheme X is a complete normal noetherian local scheme of dimension d and the theorem is known for every quasi-excellent noetherian scheme of dimension at most $d-1$. When $d = 1$, the theorem is well known; we will assume henceforth that $d \geq 2$.

3. Fibration in Curves and Application of a Theorem by A. J. de Jong

3.1. Let $X = \mathrm{Spec}(A)$ be a complete normal noetherian local scheme as in 2.3 and Z a nowhere dense closed subset. Up to replacing X (resp. Z) by a finite X-scheme which is also local noetherian normal excellent of dimension d (resp. by its inverse image), we can assume according to **V**-3.1.3, that there exists a regular noetherian local scheme S of dimension $d-1$, a finite type dominating S-scheme X' integral and affine, a closed point x' of the special fiber of $f : X' \to S$, a nowhere dense closed subset Z' of X', and finally a morphism $c : X \to X'$ satisfying the following conditions:

— the morphism c induces an isomorphism $X \xrightarrow{\sim} \mathrm{Spec}(\widehat{\mathcal{O}_{X',x'}})$;

— the inverse image $c^{-1}(Z')$ of Z' coincides with Z.

$$\begin{tikzcd} X \arrow[r, "c"] & X' \arrow[d, "f"] \\ & S \end{tikzcd}$$

3.2. Suppose the existence of a family $(X'_i \to X')$ covering for the topology of alterations (**II**-2.3) such that each X'_i is regular and each inverse image Z'_i of Z' in X'_i is the support of a strict normal crossings divisor. It follows from **II**-4.1.2 that the family $(X_i \to X)$ obtained by base change (flat) $X \to X'$ is also alt-covering. Furthermore, the excellence hypothesis on the schemes guarantees that the completion morphism c is *regular* (**I**-2.10). Since the regularity of a morphism is stable under locally finite type base change ([**ÉGA** IV□ 6.8.3]), and preserves the regularity of schemes ([**ÉGA** IV□ 6.5.2 (ii)]) it follows that each X_i is regular. Similarly, the inverse image Z_i of Z'_i in X_i — which coincides with the inverse image of Z in X_i via the obvious morphism — is the support of a strict normal crossings divisor for each index i.

3.3. Up to replacing X by X', which is permissible according to the above, we can assume the scheme X is integral of dimension d, equipped with a dominant finite type morphism $f : X \to S$, with generic fiber of dimension 1, where S is a regular noetherian local scheme of dimension $d-1$. Up to compactifying f, we can assume it is *proper*; up to blowing up, we can assume that the closed set Z is a *divisor* (i.e., the support of an effective Cartier divisor).

3.4. We are in the conditions for applying Theorem [**de Jong, 1997**, 2.4], according to which, up to altering S and X, we can assume the following facts:
— the morphism f is a nodal curve;
— the divisor Z is contained in the union of a divisor D étale over S, contained in the smooth locus of f, and the inverse image $f^{-1}(T)$ of a nowhere dense closed subset T of S.

4. Resolution of Singularities

4.1. Resolution of singularities of the base. The preceding alterations lead to a situation where the schemes X and S are not necessarily local (nor even affine) and S is no longer necessarily regular. It is however excellent of dimension $d-1$ thus falls under the recurrence hypothesis 2.3. Thus, we can assume that the pair (S, T) is regular, i.e., that the scheme S is regular and that T is a normal crossings divisor. This is indeed the case locally for the topology of alterations.

4.2. According to **VI**-1.9, the pair $(X, D \cup f^{-1}(T))$ is log regular in the sense of **VI**-1.2. That a divisor contained in a strict normal crossings divisor is also a strict normal crossings divisor allows us to assume that $Z = D \cup f^{-1}(T)$. The conclusion then results from the following theorem by Katô K. ([**Kato, 1994**, 10.3, 10.4]), complemented by W. Nizioł ([**Nizioł, 2006**, 5.7]). (See also [**Gabber & Ramero, 2013**, 9.6.32 & 53] and **VIII**-3.4.)

THÉORÈME 4.3. *Let (X, Z) be a log regular pair, where X is a noetherian scheme. There exists a regular noetherian scheme Y and a projective birational morphism $\pi : Y \to X$ such that the set-theoretic inverse image $\pi^{-1}(Z)$ is the support of a strict normal crossings divisor.*

(We use the procedure [**de Jong, 1996**, 7.2] allowing a normal crossings divisor to be made strict.)

EXPOSÉ VIII

Gabber's modification theorem (absolute case)

Luc Illusie and Michael Temkin [(i)]

In this exposé we state and prove Gabber's modification theorem mentioned in the introduction (see step (C)). Its main application is to Gabber's refined — i.e. prime to ℓ — local uniformization theorem. This is treated in exposé IX. A relative variant of the modification theorem, also due to Gabber, has applications to prime to ℓ refinements of theorems of de Jong on alterations of schemes of finite type over a field or a trait. This is discussed in exposé X. In §1, we state Gabber's modification theorem in its absolute form (Theorem 1.1). The proof of this theorem occupies §§4—5. A key ingredient is the existence of functorial (with respect to regular morphisms) resolutions in characteristic zero; the relevant material is collected in §2. We apply it in §3 to get resolutions of log regular log schemes, using the language of Kato's fans and Ogus's monoschemes. The main results, on which the proof of 1.1 is based, are Theorems 3.3.16 and 3.4.15. §§2 and 3 can be read independently of §§1, 4, 5.

Though we basically follow the lines of Gabber's original proof, our approach differs from it at several places, especially in our use of associated points and saturated desingularization towers, whose idea is due to the second author. In 2.3.13 and 2.4 we discuss material from Gabber's original proof.

We wish to thank Sophie Morel for sharing with us her notes on resolution of log regular log schemes and Gabber's magic box. They were quite useful.

1. Statement of the main theorem

Theorem 1.1. *Let X be a noetherian, qe, separated, log regular fs log scheme (**VI**-1.2), endowed with an action of a finite group G. We assume that G acts tamely (**VI**-3.1) and generically freely on X (i.e. there exists a G-stable, dense open subset of X where the inertia groups $G_{\overline{x}}$ are trivial). Let Z be the complement of the open subset of triviality of the log structure of X, and let T be the complement of the largest G-stable open subset of X on which G acts freely. Then there exists an fs log scheme X' and a G-equivariant morphism $f = f_{(G,X,Z)} : X' \to X$ of log schemes having the following properties:*

(*i*) *As a morphism of schemes, f is a projective modification, i.e. f is projective and induces an isomorphism of dense open subsets.*

(*ii*) *X' is log regular and $Z' = f^{-1}(Z \cup T)$ is the complement of the open subset of triviality of the log structure of X'.*

(*iii*) *The action of G on X' is very tame (**VI**-3.1).*

When proving the theorem we will construct $f_{(G,X,Z)}$ that satisfies a few more nice properties that will be listed in Theorem 5.6.1. We remark that Gabber also proves the theorem, more generally, when X is not assumed to be qe. However, the quasi-excellence assumption simplifies the proof so we impose it here. Most of the proof works for a general noetherian X, so we will assume that X is qe only when this will be needed in §5.

1.2. (a) Note that we do not demand that f is log smooth. In general, it is not. Here is an example. Let k be an algebraically closed field of characteristic $\neq 2$. Let $G = \{\pm 1\}$ act on the affine plane $X = \mathbf{A}^2_k$, endowed with the trivial log structure, by $x \mapsto \pm x$. Then X is regular and log regular, and $T = \{0\}$. The action of G on X is tame, but not very tame, as $G_{\{0\}}$ $(= G)$ does not act trivially on the (only) stratum X

(i) The research of M.T. was partially supported by the European Union Seventh Framework Programme (FP7/2007-2013) under grant agreement 268182.

of the stratification by the rank of $M^{\mathrm{gp}}/\mathscr{O}^*$. Let $f\colon X' \to X$ be the blow up of T, with its natural action of G. Then the pair $(X', Z' = f^{-1}(T))$ is log regular, f is a G-equivariant morphism of log schemes, $X' - Z'$ is at the same time the open subset of triviality of the log structure and the largest G-stable open subset of X' where G acts freely, and G acts very tamely on X'. However, f is not log smooth (the fiber of f at $\{0\}$ is the line Z' with the log structure associated to $\mathbf{N} \to \mathscr{O}, 1 \mapsto 0$, which is not log smooth over Spec k with the trivial log structure).

(b) In the above example, let D_1, D_2 be distinct lines in X crossing at $\{0\}$, and put the log structure $M(D)$ on X defined by the divisor with normal crossings $D = D_1 \cup D_2$. Then $\tilde{X} = (X, M(D))$ is log smooth over Spec k endowed with the trivial log structure, and G acts very tamely on $(X, M(D))$ (**VI**-4.6). Moreover, the modification f considered above underlies the log blow up $\tilde{f}\colon \tilde{X}' \to \tilde{X}$ of X at (the ideal in $M(D)$ of) $\{0\}$. While f depends only on X, the log étale morphism $\tilde{f}$ is not canonical, as it depends on the choice of D. However, one can recover f from the *canonical* resolutions of toric singularities (discussed in the next section). Namely, as G acts very tamely on $\tilde{X}$, the quotient $\tilde{Y} = \tilde{X}/G$ is log regular (**VI**-3.2): $\tilde{Y} = \operatorname{Spec} k[P]$, where P is the submonoid of $\mathbf{Z}^2$ generated by $(2,0)$, $(1,1)$ and $(0,2)$, and the projection $p\colon \tilde{X} \to \tilde{Y}$ is a Kummer étale cover of group G, in particular, U is a G-étale cover of $V = p(U)$, where $U = X - D$. Let $g\colon \tilde{Y}' \to \tilde{Y}$ be the log blow up of $\{0\} = p(\{0\})$ in $\tilde{Y}$. We then have a cartesian diagram of log schemes

$$\begin{tikzcd} \tilde{X}' \arrow[r] \arrow[d] & \tilde{Y}' \arrow[d, "g"] \\ \tilde{X} \arrow[r, "p"] & \tilde{Y} \end{tikzcd},$$

where the horizontal maps are Kummer étale covers of group G. Now, as a morphism of schemes, $\tilde{Y}' \to \tilde{Y}$ is the *canonical resolution* of $\tilde{Y}$, and the underlying scheme X' of $\tilde{X}'$ is the *normalization* of $\tilde{Y}'$ in the G-étale cover $p\colon U \to V$. This observation, suitably generalized, plays a key role in the proof of 1.1.

2. Functorial resolutions

For simplicity, all schemes considered in this section are quasi-compact and quasi-separated. In particular, all morphisms are quasi-compact and quasi-separated.

2.1. Towers of blow ups. In this section we review various known results on the following related topics: blow ups and their towers, various operations on towers, such as strict transforms and pushforwards, associated points of schemes and schematic closure.

2.1.1. *Blow ups.* We start with recalling basic properties of blow ups; a good reference is [**Conrad, 2007**, §1]. Let X be a scheme. By a **blow up** of X we mean a triple consisting of a morphism $f\colon Y \to X$, a finitely presented closed subscheme V of X (the **center**), and an X-isomorphism $\alpha\colon Y \xrightarrow{\sim} \operatorname{Proj}(\bigoplus_{n\in\mathbf{N}} \mathscr{I}^n)$, where $\mathscr{I} = \mathscr{I}(V)$ is the ideal of V. We will write $Y = \mathrm{Bl}_V(X)$. When there is no risk of confusion we will omit V and α from the notation. A blow up (f, V, α) is said to be **empty** if $V = \varnothing$. In this case, $Y = X$ and f is the identity. The only X-automorphism of Y is the identity. Also, it is well known that Y is the universal X-scheme such that $V \times_X Y$ is a Cartier divisor (i.e. the ideal $\mathscr{I}\mathscr{O}_Y$ is invertible).

2.1.2. *Total and strict transforms.* Given a blow up $f\colon Y = \mathrm{Bl}_V(X) \to X$, there are two natural ways to pullback closed subschemes $i\colon Z \hookrightarrow X$. The **total transform** of Z under f is the scheme-theoretic preimage $f^{\mathrm{tot}}(Z) = Z \times_X Y$. The **strict transform** $f^{\mathrm{st}}(Z)$ is defined as the schematic closure of $f^{-1}(Z - V) \xrightarrow{\sim} Z - V$.

Remark 2.1.3. (i) The strict transform depends on the centers and not only on Z and the morphism $Y \to X$. For example, if $D \hookrightarrow X$ is a Cartier divisor then the morphism $\mathrm{Bl}_D(X) \to X$ is an isomorphism but the strict transform of D is empty.

(ii) While $f^{\mathrm{tot}}(Z) \to Z$ is just a proper morphism, the morphism $f^{\mathrm{st}}(Z) \to Z$ can be provided with the blow up structure because $f^{\mathrm{st}}(Z) \xrightarrow{\sim} \mathrm{Bl}_{V \times_X Z}(Z)$ (e.g., if $Z \hookrightarrow V$ then $f^{\mathrm{st}}(Z) = \varnothing = \mathrm{Bl}_Z(Z)$). Thus, the strict transform can be viewed as a genuine blow up pullback of f with respect to i.

2.1.4. *Towers of blow ups.* Next we introduce blow up towers and study various operations with them (see also [**Temkin, 2012**, §2.2]). By a **tower of blow ups** of X we mean a finite sequence of length $n \geq 0$

$$X_\bullet = (\, X_n \xrightarrow{f_{n-1}} X_{n-1} \to \cdots \xrightarrow{f_0} X_0 = X\,)$$

of blow ups. In particular, this data includes the centers $V_i \hookrightarrow X_i$ for $0 \leq i \leq n-1$. Usually, we will denote the tower as $X_\bullet$ or $(X_\bullet, V_\bullet)$. Also, we will often use notation $X_n \dashrightarrow X_0$ to denote a sequence of morphisms.

If $n = 0$ then we say that the tower is **trivial** . Note that the morphism $X_n \to X$ is projective, and it is a modification if and only if the centers V_i are nowhere dense. If $X_\bullet$ is a tower of blow ups, we denote by $(X_\bullet)_c$ the **contracted** tower deduced from $X_\bullet$ by omitting the empty blow ups.

2.1.5. *Strict transform of a tower.* Assume that $\mathscr{X} = (X_\bullet, V_\bullet)$ is a blow up tower of X and $h\colon Y \to X$ is a morphism. We claim that there exists a unique blow up tower $\mathscr{Y} = (Y_\bullet, W_\bullet)$ of Y such that $Y_i \to Y \to X$ factors through X_i and $W_i = V_i \times_{X_i} Y_i$. Indeed, this defines Y_0, W_0 and Y_1 uniquely. Since $V_0 \times_X Y_1 = W_0 \times_{Y_0} Y_1$ is a Cartier divisor, $Y_1 \to X$ factors uniquely through X_1. The morphism $Y_1 \to X_1$ uniquely defines W_1 and Y_2, etc. We call $\mathscr{Y}$ the **strict transform** of $\mathscr{X}$ with respect to h and denote it $h^{\mathrm{st}}(\mathscr{X})$.

Remark 2.1.6. The following observation motivates our terminology: if $h\colon Y \hookrightarrow X$ is a closed immersion then $Y_i \hookrightarrow X_i$ is a closed immersion and Y_{i+1} is the strict transform of Y_i under the blow up $X_{i+1} \to X_i$.

2.1.7. *Pullbacks.* One can also define a naive base change of $\mathscr{X}$ with respect to h simply as $(Y_\bullet, W_\bullet) = \mathscr{X} \times_X Y$. This produces a sequence of proper morphisms $Y_n \dashrightarrow Y_0$ and closed subschemes $W_i \hookrightarrow Y_i$ for $0 \leq i \leq n-1$. If this datum is a blow up sequence, i.e. $Y_{i+1} \xrightarrow{\sim} \mathrm{Bl}_{W_i}(Y_i)$, then we say that $\mathscr{X} \times_X Y$ is the **pullback** of $\mathscr{X}$ and use the notation $h^\star(\mathscr{X}) = \mathscr{X} \times_X Y$.

Remark 2.1.8. The pullback exists if and only if $h^{\mathrm{st}}(\mathscr{X}) \xrightarrow{\sim} \mathscr{X} \times_X Y$. Indeed, this is obvious for towers of length one, and the general case follows by induction on the length.

2.1.9. *Flat pullbacks.* Blow ups are compatible with flat base changes $h\colon Y \to X$ in the sense that $\mathrm{Bl}_{V \times_X Y}(Y) \xrightarrow{\sim} \mathrm{Bl}_V(X) \times_X Y$ (e.g. just compute these blow ups in the terms of Proj). By induction on length of blow up towers it follows that pullbacks of blow up towers with respect to flat morphisms always exist. One can slightly strengthen this fact as follows.

Remark 2.1.10. Assume that $X_\bullet$ is a blow up tower of X and $h\colon Y \to X$ is a morphism. If there exists a flat morphism $f\colon X \to S$ such that the composition $g\colon Y \to S$ is flat and the blow up tower $X_\bullet$ is the pullback of a blow up tower $S_\bullet$ then the pullback $h^\star(X_\bullet)$ exists and equals to $g^\star(S_\bullet)$.

2.1.11. *Equivariant blow ups.* Assume that X is an S-scheme acted on by a flat S-group scheme G. We will denote by $p, m\colon X_0 = G \times_S X \to X$ the projection and the action morphisms. Assume that $V \hookrightarrow X$ is a G-equivariant closed subscheme (i.e., $V \times_X (X_0; m)$ coincides with $V_0 = V \times_X (X_0; p)$) then the action of G lifts to the blow up $Y = \mathrm{Bl}_V(X)$. Indeed, the blow up $Y_0 = \mathrm{Bl}_{V_0}(X_0) \to X_0$ is the pullback of $Y \to X$ with respect to both m and p, i.e. there is a pair of cartesian squares

$$\begin{tikzcd} Y_0 \arrow[r] \arrow[d, shift right, "p'"'] \arrow[d, shift left, "m'"] & X_0 \arrow[d, shift right, "p"'] \arrow[d, shift left, "m"] \\ Y \arrow[r] & X \end{tikzcd}$$

So, we obtain an isomorphism $Y_0 \xrightarrow{\sim} G \times_S Y$ (giving rise to the projection $p'\colon Y_0 \to Y$) and a group action morphism $m'\colon Y_0 \to Y$ compatible with m. Furthermore, the unit map $e\colon X \to X_0$ satisfies the condition of Remark 2.1.10 (with $X = S$), hence we obtain the base change $e'\colon Y \to Y_0$ of e.

It is now straightforward to check that m' and e' satisfy the group action axioms, but let us briefly spell this out using simplicial nerves. The action of G on V defines a cartesian sub-simplicial scheme $\mathrm{Ner}(G, V) \hookrightarrow \mathrm{Ner}(G, X)$. By the flatness of G over S and Remark 2.1.10, $\mathrm{Bl}_{\mathrm{Ner}(G,V)}(\mathrm{Ner}(G, X))$ is cartesian over $\mathrm{Ner}(G)$, hence corresponds to an action of G on $\mathrm{Bl}_V(X)$.

2.1.12. *Flat monomorphism.* Flat monomorphisms are studied in [**Raynaud, 1967**]. In particular, it is proved in [**Raynaud, 1967**, Prop. 1.1] that $i : Y \hookrightarrow X$ is a flat monomorphism if and only if i is injective and for any $y \in Y$ the homomorphism $\mathcal{O}_{X,y} \to \mathcal{O}_{X,x}$ is an isomorphism. Moreover, it is proved in [**Raynaud, 1967**, Prop. 1.2] that in this case i is a topological embedding and $\mathcal{O}_Y = \mathcal{O}_X|_Y$. In addition to open immersions, the main source of flat monomorphisms for us will be morphisms of the form $\mathrm{Spec}(\mathcal{O}_{X,x}) \hookrightarrow X$ and their base changes.

2.1.13. *Pushforwards of ideals.* Let $i : Y \to X$ be a flat monomorphism, e.g. $\mathrm{Spec}(\mathcal{O}_{X,x}) \hookrightarrow X$. By the **pushforward** $U = i_\star(V)$ of a closed subscheme $V \hookrightarrow Y$ we mean its schematic image in X, i.e. U is the minimal closed subscheme such that $V \hookrightarrow X$ factors through U. It exists by [**ÉGA** I 9.5.1].

Lemma 2.1.14. *The pushforward $U = i_\star(V)$ extends V in the sense that $U \times_X Y = V$.*

Proof. See [**Raynaud, 1967**, Proof of Proposition 1.2]. □

2.1.15. *Pushforwards of blow up towers.* Given a blow up $f : \mathrm{Bl}_V(Y) \to Y$ and a flat monomorphism $i : Y \hookrightarrow X$ we define the pushforward $i_\star(f)$ as the blow up along $U = i_\star(V)$, assuming that U is finitely presented over X. Using Lemma 2.1.14 and flat pullbacks we see that $i^\star i_\star(f) = f$ and $\mathrm{Bl}_V(Y) \hookrightarrow \mathrm{Bl}_U(X)$ is a flat monomorphism. So, we can iterate this procedure to construct pushforward with respect to i of any blow up tower $Y_\bullet$ of Y. It will be denoted $(X_\bullet, U_\bullet) = i_\star(Y_\bullet, V_\bullet)$.

Remark 2.1.16. (i) Clearly, $i^\star i_\star(Y_\bullet) = Y_\bullet$.

(ii) In the opposite direction, a blow up tower $(X_\bullet, U_\bullet)$ of X satisfies $i_\star i^\star(X_\bullet) = X_\bullet$ if and only if the preimage of Y in each center U_i of the tower is schematically dense.

2.1.17. *Associated points of a scheme.* Assume that X is a noetherian scheme. Recall that a point $x \in X$ is called associated if m_x is an associated prime of $\mathcal{O}_{X,x}$, i.e. $\mathcal{O}_{X,x}$ contains an element whose annihilator is m_x, see [**ÉGA** IV$_2$ 3.1.1]. The set of all such points will be denoted $\mathrm{Ass}(X)$. The following result is well known but difficult to find in the literature.

Lemma 2.1.18. *Let $i : Y \hookrightarrow X$ be a flat monomorphism. Then the schematic image of i coincides with X if and only if* $\mathrm{Ass}(X) \subset i(Y)$.

Proof. Note that the schematic image of i can be described as $\mathrm{Spec}(\mathcal{F})$, where $\mathcal{F}$ is the image of the homomorphism $\phi : \mathcal{O}_X \to i_\star(\mathcal{O}_Y)$. Thus, the schematic image coincides with X if and only if $\mathrm{Ker}(\phi) = 0$.

If $i(Y)$ omits a point x then any m_x-torsion element $s \in \mathcal{O}_{X,x}$ is in the kernel of $\phi_x : \mathcal{O}_{X,x} \to i_\star(\mathcal{O}_Y)_x$ (we use that $\mathcal{O}_Y = \mathcal{O}_X|_Y$ by 2.1.12 and since the closure of x is the support of an extension of s to a sufficiently small neighborhood, the restriction $s|_Y$ vanishes). So, if there exists $x \in \mathrm{Ass}(X)$ with $x \notin i(Y)$ then $\mathrm{Ker}(\phi) \neq 0$.

Conversely, if the kernel is non-zero then we take x to be any maximal point of its support and choose any non-zero $s \in \mathrm{Ker}(\phi_x)$. In particular, $s|_{Y \cap X_x} = 0$ and hence $x \notin i(Y)$. For any non-trivial generization y of x the image of s in $\mathcal{O}_{X,y}$ vanishes because $\mathrm{Ker}(\mathcal{O}_{X,y} \to i_\star(\mathcal{O}_Y)_y) = 0$ by maximality of x. Thus, the closure of x is the support of s, and hence s is annihilated by a power of m_x. Since X is noetherian, we can find a multiple of s whose annihilator is m_x, thereby obtaining that $x \in \mathrm{Ass}(X)$. □

2.1.19. *Associated points of blow up towers.* If $(X_\bullet, V_\bullet)$ is a blow up tower and all X_i's are noetherian then by the set $\mathrm{Ass}(X_\bullet)$ of its associated points we mean the union of the images of $\mathrm{Ass}(V_i)$ in X. Combining Remark 2.1.16(i) and Lemma 2.1.18 we obtain the following:

Lemma 2.1.20. *Let $i : Y \hookrightarrow X$ be a flat monomorphism and let $X_\bullet$ be a blow up tower of X. Then $i_\star i^\star(X_\bullet) = X_\bullet$ if and only if* $\mathrm{Ass}(X_\bullet) \subset i(Y)$.

2.2. Normalized blow up towers. For reduced schemes most of the notions, constructions and results of §2.1 have normalized analogs. We develop such a "normalized" theory in this section.

2.2.1. *Normalization.* The normalization of a reduced scheme X with finitely many irreducible components, as defined in [**ÉGA** II 6.3.8], will be denoted X^{nor}. Recall that normalization is compatible with open immersions and for an affine $X = \mathrm{Spec}(A)$ its normalization is $X^{\mathrm{nor}} = \mathrm{Spec}(B)$ where B is the integral closure of A in its total ring of fractions (which is a finite product of fields). The normalization morphism $X^{\mathrm{nor}} \to X$ is integral but not necessarily finite.

2.2.2. *Functoriality.* Recall (**II**-1.1.2) that a morphism $f\colon Y \to X$ is called **maximally dominating** if it takes generic points of Y to generic points of X. Normalization is a functor on the category whose objects are reduced schemes having finitely many irreducible components and whose morphisms are the maximally dominating ones. Furthermore, it possesses the following universal property: any maximally dominating morphism $Y \to X$ with normal Y factors uniquely through X^{nor}. (By definition, Y is normal if its local rings are normal domains. Both claims are local on Y and X and are obvious for affine schemes.)

2.2.3. *Normalized blow ups.* Assume that X is a reduced scheme with finitely many irreducible components. By the **normalized blow up** of X along a closed subscheme V of finite presentation we mean the morphism $f\colon \mathrm{Bl}_V(X)^{\mathrm{nor}} \to X$. The normalization is well defined since $\mathrm{Bl}_V(X)$ is reduced and has finitely many irreducible components. Note that f is universally closed but does not have to be of finite type. As in the case of usual blow ups, V is a part of the structure. In particular, $\mathrm{Bl}_V(X)^{\mathrm{nor}}$ has no X-automorphisms and we can talk about equality of normalized blow ups (as opposed to an isomorphism).

Proposition 2.2.4. (*i*) *Keep the above notation. Then* $\mathrm{Bl}_V(X)^{\mathrm{nor}} \to X$ *is the universal maximally dominating morphism* $Y \to X$ *such that* Y *is normal and* $V \times_X Y$ *is a Cartier divisor.*

(*ii*) *For any blow up* $f\colon Y = \mathrm{Bl}_V(X) \to X$ *its normalization* $f^{\mathrm{nor}}\colon Y^{\mathrm{nor}} \to X^{\mathrm{nor}}$ *is the normalized blow up along* $V \times_X X^{\mathrm{nor}}$.

Proof. Combining the universal properties of blow ups and normalizations we obtain (i), and (ii) is its immediate corollary. □

Towers of normalized blow ups and their transforms can now be defined similarly to their non-normalized analogs.

2.2.5. *Towers of normalized blow ups.* A **tower of normalized blow ups** is a finite sequence $X_n \dashrightarrow X_{-1}$ with $n \geq 0$ of normalized blow ups with centers $V_i \hookrightarrow X_i$ for $-1 \leq i \leq n-1$ and $V_{-1} = \emptyset$. The centers are part of the datum. Note that the map $X_0 \to X_{-1}$ is just the normalization map. The **contraction** of a normalized blow up tower removes the normalized blow ups with empty centers for $i \geq 0$. It follows from [**ÉGA** IV$_3$ 8.6.3] that each X_i with $i \geq 0$ is a normalization of a reduced projective X_{-1}-scheme. A tower is called **noetherian** if all X_i are noetherian.

2.2.6. *Normalization of a blow up tower.* Using induction on length and Proposition 2.2.4(ii), we can associate to a blow up tower $\mathcal{X} = (X_\bullet, V_\bullet)$ of a reduced scheme X with finitely many irreducible components a normalized blow up tower $\mathcal{X}^{\mathrm{nor}} = (Y_\bullet, W_\bullet)$, where $Y_{-1} = X$ and $Y_i = X_i^{\mathrm{nor}}$, $W_i = V_i \times_{X_i} X_i^{\mathrm{nor}}$ for $i \geq 0$. We call $\mathcal{X}^{\mathrm{nor}}$ the **normalization** of $\mathcal{X}$.

2.2.7. *Strict transforms.* If $\mathcal{X} = (X_\bullet, V_\bullet)$ is a normalized blow up tower of $X = X_{-1}$ and $f\colon Y \to X$ is a morphism between reduced schemes with finitely many irreducible components then we define the strict transform $f^{\mathrm{st}}(\mathcal{X})$ as the normalized blow up tower $(Y_\bullet, W_\bullet)$ such that $Y_{-1} = Y$ and $W_i = V_i \times_{X_i} Y_i$. Using induction on the length and the universal property of normalized blow ups, see 2.2.4 (i), one shows that such a tower exists and is the universal normalized blow up tower of Y such that $f = f_{-1}$ extends to a compatible sequence of morphisms $f_i\colon Y_i \to X_i$.

2.2.8. *Pullbacks.* The strict transform $f^{\mathrm{st}}(\mathcal{X})$ as above will be called the **pullback** and denoted $f^\star(\mathcal{X})$ if $Y_i \xrightarrow{\sim} X_i \times_X Y$ for any $-1 \leq i \leq n$. Recall that a morphism is **regular** if it is flat and has geometrically regular fibers, see [**ÉGA** IV$_2$ 6.8.1].

Lemma 2.2.9. *If* $f\colon Y \to X$ *is a regular morphism between reduced noetherian schemes then any normalized blow up tower* $\mathcal{X}$ *of* X *admits a pullback* $f^\star(\mathcal{X})$.

Proof. Blow ups are compatible with flat morphisms hence we should only show that normalizations in our tower are compatible with regular morphisms: if $f\colon Y \to X$ is a regular morphism of reduced noetherian schemes then the morphism $h\colon Y^{\mathrm{nor}} \to X^{\mathrm{nor}} \times_X Y$ is an isomorphism. Since h is an integral morphism which is generically an isomorphism, it suffices to show that $X^{\mathrm{nor}} \times_X Y$ is normal.

To prove the latter we can assume that X and Y are affine. Then f is a filtered limit of smooth morphisms $h_i\colon Y_i \to X$ by Popescu's theorem. If the claim holds for h_i then it holds for h, so we can assume that h is smooth. We claim that, more generally, if A is a normal domain and $\phi\colon A \to B$ is a smooth homomorphism then B is normal. Indeed, A is a filtered colimit of noetherian normal subdomains A_i and by [**ÉGA** IV$_3$ 8.8.2] and [**ÉGA** IV$_4$ 17.7.8] ϕ is the base change of a smooth homomorphism $\phi_i\colon A_i \to B_i$ for large enough i. For each $j \geq i$ let $\phi_j\colon A_j \to B_j$ be the base change of ϕ_i. Each B_j is normal by [**Matsumura, 1980a**, 21.E (iii)] and B is the colimit of B_j, hence B is normal. □

2.2.10. *Fpqc descent of blow up towers.* The classical fpqc descent of ideals (and modules) implies that there is also an fpqc descent for blow up towers. Namely, if $Y \to X$ is an fpqc covering and $Y_\bullet$ is a blow up tower of Y whose both pullbacks to $Y \times_X Y$ are equal then $Y_\bullet$ canonically descends to a blow up tower of X because the centers descend. In the same way, normalized blow up towers descend with respect to quasi-compact surjective regular morphisms.

2.2.11. *Associated points.* The material of 2.1.15–2.1.19 extends to noetherian normalized blow up towers almost verbatim. In particular, if $\mathscr{X} = (X_\bullet, V_\bullet)$ is such a tower then $\mathrm{Ass}(\mathscr{X})$ is the union of the images of $\mathrm{Ass}(V_j)$ and for any flat monomorphism $i\colon Y \hookrightarrow X$ (which is a regular morphism by 2.1.12) with a blow up tower $\mathscr{Y}$ of Y we always have that $i^\star i_\star \mathscr{Y} = \mathscr{Y}$, and we have that $i_\star i^\star \mathscr{X} = \mathscr{X}$ if and only if $\mathrm{Ass}(\mathscr{X}) \subset i(Y)$.

2.3. Functorial desingularization. In this section we will formulate the desingularization result about toric varieties that will be used later in the proof of Theorem 1.1. Then we will show how it is obtained from known desingularization results.

2.3.1. *Desingularization of a scheme.* By a **resolution** (or **desingularization**) **tower** of a noetherian scheme X we mean a tower of blow ups with nowhere dense centers $X_\bullet$ such that $X = X_0$, X_n is regular and no f_i is an empty blow up. For example, the trivial tower is a desingularization if and only if X itself is regular.

2.3.2. *Normalized desingularization.* We will also consider normalized blow up towers such that each center is non-empty and nowhere dense, $X = X_{-1}$ and X_n is regular. Such a tower will be called a **normalized desingularization tower** of X.

Remark 2.3.3. (i) For any desingularization tower $\mathscr{X}$ of X its normalization $\mathscr{X}^{\mathrm{nor}}$ is a normalized desingularization tower of X.

(ii) Usually one works with non-normalized towers; they are subtler objects that possess more good properties. All known constructions of functorial desingularization (see below) produce blow up towers by an inductive procedure, and one cannot work with normalized towers instead. However, it will be easier for us to deal with normalized towers in log geometry because in this case one may work only with fs log schemes.

2.3.4. *Functoriality of desingularization.* For concreteness, we consider desingularizations in the current section, but all what we say holds for normalized desingularizations too. Assume that a class $\mathscr{S}^0$ of noetherian S-schemes is provided with desingularizations $\mathscr{F}(X) = X_\bullet$ for any $X \in \mathscr{S}^0$. We say that the desingularization (family) $\mathscr{F}$ is **functorial** with respect to a class $\mathscr{S}^1$ of S-morphisms between the elements of $\mathscr{S}^0$ if for any $f\colon Y \to X$ from $\mathscr{S}^1$ the desingularization of X induces that of Y in the sense that $f^\star\mathscr{F}(X)$ is defined and its contraction coincides with $\mathscr{F}(Y)$ (so, $\mathscr{F}(Y) = (Y \times_X \mathscr{F}(X))_c$). Note that we put the = sign instead of an isomorphism sign, which causes no ambiguity by the fact that any automorphism of a blow up is the identity as we observed above.

Remark 2.3.5. (i) Contractions in the pulled back tower appear when some centers of $\mathscr{F}(X)$ are mapped to the complement of $f(Y)$ in X. In particular, if $f \in \mathscr{S}^1$ is surjective then the precise equality $\mathscr{F}(Y) = Y \times_X \mathscr{F}(X)$ holds.

(ii) Assume that $X = \bigcup_{i=1}^n X_i$ is a Zariski covering and the morphisms $X_i \hookrightarrow X$ and $\coprod_{i=1}^n X_i \to X$ are in $\mathscr{S}^1$. In general, one cannot reconstruct $\mathscr{F}(X)$ from the $\mathscr{F}(X_i)$'s because the latter are contracted pullbacks and it is not clear how to glue them with correct synchronization. However, all information about $\mathscr{F}(X)$ is kept in $\mathscr{F}(\coprod_{i=1}^n X_i)$. The latter is the pullback of $\mathscr{F}(X)$ hence we can reconstruct $\mathscr{F}(X)$ by gluing the restricted blow up towers $\mathscr{F}(\coprod_{i=1}^n X_i)|_{X_i}$. Note that $\mathscr{F}(\coprod_{i=1}^n X_i)|_{X_i}$ can be obtained from $\mathscr{F}(X_i)$ by inserting empty blow ups, and these empty blow ups make the gluing possible. This trick with synchronization of the towers $\mathscr{F}(X_i)$ by desingularizing disjoint unions is often used in the modern desingularization theory, and one can formally show (see [**Temkin, 2012**, Rem. 2.3.4(iv)]) that such approach is equivalent to the classical synchronization of the algorithm with an invariant.

(iii) Assume that $\mathscr{S}^1$ contains all identities Id_X with $X \in \mathscr{S}^0$ and for any $Y, Z \in \mathscr{S}^0$ there exists $T = Y \coprod Z$ in $\mathscr{S}^0$ such that for any pair of morphisms $a\colon Y \to X$, $b\colon Z \to X$ in $\mathscr{S}^1$ the morphism $(a, b)\colon T \to X$ is in $\mathscr{S}^1$. As an illustration of the above trick, let us show that even if $f, g\colon Y \to X$ are in $\mathscr{S}^1$ but not surjective, we have an equality $\mathscr{F}(X) \times_X (Y, f) = \mathscr{F}(X) \times_X (Y, g)$ of non-contracted towers. Indeed, set $Y' = Y \coprod X$ and consider the morphisms $f', g'\colon Y' \to X$ that agree with f and g and map X by the identity. Then $\mathscr{F}(X) \times_X (Y', f')$ and $\mathscr{F}(X) \times_X (Y', g')$ are equal because f' and g' are surjective, hence their restrictions onto Y are also equal, but these are precisely $\mathscr{F}(X) \times_X (Y, f)$ and $\mathscr{F}(X) \times_X (Y, g)$.

2.3.6. *Gabber's magic box.* Now we have tools to formulate the aforementioned desingularization result.

Theorem 2.3.7. *Let $\mathscr{S}^0$ denote the class of finite disjoint unions of affine toric varieties over* $\mathbf{Q}$, *i.e.* $\mathscr{S}^0 = \{\coprod_{i=1}^n \mathrm{Spec}(\mathbf{Q}[P_i])\}$, *where $P_1, \dots, P_n$ are fs torsion free monoids. Let $\mathscr{S}^1$ denote the class of smooth morphisms*

$$f\colon \coprod_{j=1}^m \mathrm{Spec}(\mathbf{Q}[Q_j]) \to \coprod_{i=1}^n \mathrm{Spec}(\mathbf{Q}[P_i])$$

such that for each $1 \le j \le m$ there exists $1 \le i = i(j) \le n$ and a homomorphism of monoids $\phi_j\colon P_i \to Q_j$ so that the restriction of f onto $\mathrm{Spec}(\mathbf{Q}[Q_j])$ *factors through the toric morphism* $\mathrm{Spec}(\mathbf{Q}[\phi_j])$. *Then there exists a desingularization $\mathscr{F}$ on $\mathscr{S}^0$ which is functorial with respect to $\mathscr{S}^1$ and, in addition, satisfies the following compatibility condition: if $\mathscr{O}_1, \dots, \mathscr{O}_l$ are complete noetherian local rings containing* $\mathbf{Q}$, $Z = \coprod_{i=1}^l \mathrm{Spec}(\mathscr{O}_i)$, *and $g, h\colon Z \to X$ are two regular morphisms with $X \in \mathscr{S}^0$ then*

$$(Z, g) \times_X \mathscr{F}(X) = (Z, h) \times_X \mathscr{F}(X). \tag{2.3.7.1}$$

In the above theorem, we use the convention that different tuples $P_1, \dots, P_n$ give rise to different schemes $\coprod_{i=1}^n \mathrm{Spec}(\mathbf{Q}[P_i])$. Before showing how this theorem follows from known desingularization results, let us make a few comments.

Remark 2.3.8. (i) Gabber's original magic box also requires that the centers are smooth schemes. This (and much more) can also be achieved as will be explained later, but we prefer to emphasize the minimal list of properties that will be used in the proof of Theorem 1.1.

(ii) It is very important to allow disjoint unions in the theorem in order to deal with synchronization issues, as explained in Remark 2.3.5(ii). This theme will show up repeatedly throughout the exposé.

2.3.9. *Desingularization of qe schemes over* $\mathbf{Q}$. In practice, all known functorial desingularization families are constructed in an explicit algorithmic way, so one often says a **desingularization algorithm** instead of a desingularization family. We adopt this terminology below.

Gabber's magic box 2.3.7 is a particular case of the following theorem, see [**Temkin, 2012**, Th. 1.2.1]. Indeed, due to Remark 2.3.5(iii), functoriality with respect to regular morphisms implies (2.3.7.1).

Theorem 2.3.10. *There exists a desingularization algorithm $\mathscr{F}$ defined for all reduced noetherian quasi-excellent schemes over* $\mathbf{Q}$ *and functorial with respect to all* regular *morphisms. In addition, $\mathscr{F}$ blows up only regular centers.*

Remark 2.3.11. Although this is not stated in [**Temkin, 2012**], one can strengthen Theorem 2.3.10 by requiring that $\mathscr{F}$ blows up only regular centers contained in the singular locus. An algorithm $\mathscr{F}$ is constructed in [**Temkin, 2012**] from an algorithm $\mathscr{F}_{\mathrm{Var}}$ that desingularizes varieties of characteristic zero, and one can check that if the centers of $\mathscr{F}_{\mathrm{Var}}$ lie in the singular loci (of the intermediate varieties) then the same is true for $\mathscr{F}$. Let us explain how one can choose an appropriate $\mathscr{F}_{\mathrm{Var}}$. In [**Temkin, 2012**], one uses the algorithm of Bierstone-Milman to construct $\mathscr{F}$, see Theorem 6.1 and its

Addendum in [**Bierstone et al., 2011**] for a description of this algorithm and its properties. It follows from the Addendum that the algorithm blows up centers lying in the singular loci until X becomes smooth, and then it performs some additional blow ups to make the exceptional divisor snc. Eliminating the latter blow ups we obtain a desingularization algorithm $\mathscr{F}_{\mathrm{Var}}$ which only blows up regular centers lying in the singular locus.

It will be convenient for us to use the algorithm $\mathscr{F}$ from Theorem 2.3.10 in the sequel. Also, to simplify the exposition we will freely use all properties of $\mathscr{F}$ but the careful reader will notice that only the properties of Gabber's magic box will be crucial in the end. Also, instead of working with $\mathscr{F}$ itself we will work with its normalization $\mathscr{F}^{\mathrm{nor}}$ which assigns to a reduced qe scheme over $\mathbf{Q}$ the normalized blow up tower $\mathscr{F}(X)^{\mathrm{nor}}$. It will be convenient to use the notation $\widetilde{\mathscr{F}} = \mathscr{F}^{\mathrm{nor}}$ in the sequel.

Remark 2.3.12. (i) Since normalized blow ups are compatible with regular morphisms, it follows from Theorem 2.3.10 that the normalized desingularization $\widetilde{\mathscr{F}}$ is functorial with respect to all regular morphisms.

(ii) The feature which is lost under normalization (and which is not needed for our purposes) is some control on the centers. The centers $\widetilde{V}_i$ of $\widetilde{\mathscr{F}}(X)$ are preimages of the centers $V_i \hookrightarrow X_i$ of $\mathscr{F}(X)$ under the normalization morphisms $X_i^{\mathrm{nor}} \to X_i$, so they do not have to be even reduced. It will only be important that $\widetilde{V}_i$'s are equivariant when a smooth group acts on X. In Gabber's original argument it was important to blow up only regular centers because they were not part of the blow up data, and one used that a regular center without codimension one components intersecting the regular locus is determined already by the underlying morphism of the blow up.

2.3.13. *Alternative desingularization inputs.* For the sake of completeness, we discuss how other algorithms could be used instead of $\mathscr{F}$. Some desingularization algorithms for reduced varieties over $\mathbf{Q}$ are constructed in [**Bierstone & Milman, 1997**], [**Włodarczyk, 2005**], [**Bravo et al., 2005**], and [**Kollár, 2007**]. They all are functorial with respect to equidimensional smooth morphisms (though usually one "forgets" to mention the equidimensionality restriction). It is shown in [**Bierstone et al., 2011**, §6.3] how to make the algorithm of [**Bierstone & Milman, 1997**] fully functorial by a slight adjusting of the synchronization of its blow ups. All these algorithms can be used to produce a desingularization of log regular schemes (see §3), so the only difficulty is in establishing the compatibility (2.3.7.1).

For the algorithm of [**Bierstone et al., 2011**] it was shown by Bierstone-Milman (unpublished, see [**Bierstone et al., 2011**, Rem. 7.1(2)]) that the induced desingularization of a formal completion at a point depends only on the formal completion as a scheme. This is precisely what we need in (2.3.7.1).

Finally, there is a much more general result by Gabber, see Theorem 2.4.1, whose proof uses Popescu's theorem and the cotangent complex. It implies that, actually, any desingularization of reduced varieties over $\mathbf{Q}$ which is functorial with respect to smooth morphisms automatically satisfies (2.3.7.1). So, in principle, any functorial desingularization of varieties over $\mathbf{Q}$ could be used for our purposes. Since Gabber's result and its proof are powerful and novel for the desingularization theory (and were missed in [**Bierstone et al., 2011**], mainly due to a not so trivial involvement of the cotangent complex), we include them in §2.4.

2.3.14. *Invariance of the regular locus.* Until the end of §2.3 we consider only qe schemes of characteristic zero, and our aim is to establish a few useful properties of $\mathscr{F}$ (and $\widetilde{\mathscr{F}}$) that are consequences of the functoriality property $\mathscr{F}$ satisfies. First, we claim that $\mathscr{F}$ does not modify the regular locus of X, and even slightly more than that:

Corollary 2.3.15. *All centers of $\mathscr{F}(X)$ and $\widetilde{\mathscr{F}}(X)$ sit over the singular locus of X. In particular, X is regular if and only if $\mathscr{F}(X)$ is the trivial tower.*

Proof. It suffices to study $\mathscr{F}$. The claim is obvious for $S = \mathrm{Spec}(\mathbf{Q})$ because S does not contain non-dense non-empty subschemes. By functoriality, $\mathscr{F}(T)$ is trivial for any regular T of characteristic zero, because it is regular over S. Finally, if T is the regular locus of X then $\mathscr{F}(T) = (\mathscr{F}(X) \times_X T)_c$ and hence any center $V_i \hookrightarrow X_i$ of $\mathscr{F}(X)$ does not intersect the preimage of T. □

2.3.16. *Equivariance of the desingularization.* It is well known that functorial desingularization is equivariant with respect to any smooth group action (and, moreover, extends to functorial desingularization of stacks, see [**Temkin, 2012**, Th. 5.1.1]). For the reader's convenience we provide an elementary argument.

Corollary 2.3.17. *Let S be a qe scheme over $\mathbf{Q}$, G be a smooth S-group and X be a reduced S-scheme of finite type acted on by G. Then there exists a unique action of G on $\mathscr{F}(X)$ and $\widetilde{\mathscr{F}}(X)$ that agrees with the given action on X.*

Proof. Again, it suffices to study $\mathscr{F}$. Let $\mathscr{F}(X)$ be given by $X_n \dashrightarrow X_0 = X$ and $V_i \hookrightarrow X_i$ for $0 \leq i \leq n-1$. By $p, m\colon Y = G \times_S X \to X$ we denote the projection and the action morphisms. Note that m is smooth (e.g. m is the composition of the automorphism $(g,x) \to (g,gx)$ of $G \times_S X$ and p). Therefore, $\mathscr{F}(X) \times_X (Y;m) = \mathscr{F}(Y) = \mathscr{F}(X) \times_S G$ by Theorem 2.3.10 and Remark 2.3.5(i). In particular, $V_0 \times_X (Y;p) = V_0 \times_X (Y;m)$, i.e. V_0 is G-equivariant. By 2.1.11, X_1 inherits a G-action. Then the same argument implies that V_1 is G-equivariant and X_2 inherits a G-action, etc. □

2.4. Complements on functorial desingularizations. This section is devoted to Gabber's result on a certain non-trivial compatibility property that any functorial desingularization satisfies. It will not be used in the sequel, so an uninterested reader may safely skip it.

Theorem 2.4.1. *Assume that S is a noetherian scheme, $\mathscr{S}^0$ is a class of reduced S-schemes of finite type and $\mathscr{S}^1$ is a class of morphisms between elements of $\mathscr{S}^0$ such that if $f\colon Y \to X$ is smooth and $X \in \mathscr{S}^0$ then $Y \in \mathscr{S}^0$ and $f \in \mathscr{S}^1$. Let $\mathscr{F}$ be a desingularization on $\mathscr{S}^0$ which is functorial with respect to all morphisms of $\mathscr{S}^1$. Then any pair of regular morphisms $g\colon Z \to X$ and $h\colon Z \to Y$ with targets in $\mathscr{S}^0$ induces the same desingularization of Z; namely, $\mathscr{F}(X) \times_X Z = \mathscr{F}(Y) \times_Y Z$.*

Note that the theorem has no restrictions on the characteristic (because no such restriction appears in Popescu's theorem). Before proving the theorem let us formulate its important corollary, whose main case is when $S = \mathrm{Spec}(k)$ for a field k and $\mathscr{S}^0$ is the class of all reduced k-schemes of finite type.

Corollary 2.4.2. *Keep the notation of Theorem 2.4.1. Then $\mathscr{F}$ canonically extends to the class $\widehat{\mathscr{S}^0}$ of all schemes that admit a regular morphism to a scheme from $\mathscr{S}^0$ and the extension is functorial with respect to all regular morphisms between schemes of $\widehat{\mathscr{S}^0}$.*

The main ingredient of the proof will be the following result that we are going to establish first.

Proposition 2.4.3. *Consider a commutative diagram of noetherian schemes*

$$\begin{tikzcd}
& Z \arrow[dl, "g"'] \arrow[d, "f"] \arrow[dr, "h"] & \\
X \arrow[dr, "a"'] & Z' \arrow[l, "g'"'] \arrow[r, "h'"] & Y \arrow[dl, "b"] \\
& S &
\end{tikzcd}$$

such that a and b are of finite type, g and h are regular and g' is smooth. Then h' is smooth around the image of f.

For the proof we will need the following three lemmas. In the first one we recall the Jacobian criterion of smoothness, rephrased in terms of the cotangent complex.

Lemma 2.4.4. *Let $f\colon X \to S$ be a morphism which is locally of finite presentation, and let $x \in X$. Then the following conditions are equivalent:*
(i) f is smooth at x;
(ii) $\mathrm{H}_1(L_{X/S} \overset{\mathrm{L}}{\otimes} k(x)) = 0$.

In the lemma we use the convention $\mathrm{H}_i = \mathrm{H}^{-i}$, and $L_{X/S}$ denotes the cotangent complex of X/S.

Proof. (i) ⇒ (ii) is trivial: as f is smooth at x, up to shrinking X we may assume f smooth, then $L_{X/S}$ is cohomologically concentrated in degree zero and locally free [**Illusie, 1972**, III 3.1.2]. Let us prove (ii) ⇒ (i). We may assume that we have a factorization

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"'] & Z \arrow[dl, "g"] \\ S & \end{tikzcd},$$

where i is a closed immersion of ideal I and g is smooth. Consider the standard exact sequence

$$(*)\qquad I/I^2 \to i^\star\Omega^1_{Z/S} \to \Omega^1_{X/S} \to 0.$$

By the Jacobian criterion [**ÉGA** IV$_4$ 17.12.1] and [**ÉGA** 0_{IV} 19.1.12], the smoothness of f at x is equivalent to the fact that the morphism

$$(**)\qquad (I/I^2)\otimes k(x) \to \Omega^1_{Z/S}\otimes k(x)$$

deduced from the left one in $(*)$ is injective. Now, $(I/I^2)\otimes k(x) = \mathrm{H}_1(L_{X/Z}\overset{\mathrm{L}}{\otimes} k(x))$ [**Illusie, 1972**, III 3.1.3], and $(**)$ is a morphism in the exact sequence associated with the triangle deduced from the transitivity triangle $Li^\star L_{Z/S} \to L_{X/S} \to L_{X/Z} \to Li^\star L_{Z/S}[1]$ by applying $\overset{\mathrm{L}}{\otimes} k(x)$:

$$\mathrm{H}_1(L_{X/S}\overset{\mathrm{L}}{\otimes} k(x)) \to \mathrm{H}_1(L_{X/Z}\overset{\mathrm{L}}{\otimes} k(x))(= (I/I^2)\otimes k(x)) \to \Omega^1_{Z/S}\otimes k(x).$$

By (ii), $\mathrm{H}_1(L_{X/S}\overset{\mathrm{L}}{\otimes} k(x)) = 0$, hence $(**)$ is injective, which completes the proof. □

Lemma 2.4.5. *Consider morphisms $f: X \to Y$, $g: Y \to S$, $h = gf: X \to S$, and let $x \in X$, $y = f(x) \in Y$. Assume that*

(i) $\mathrm{H}_1(L_{X/S}\overset{\mathrm{L}}{\otimes} k(x)) = 0$

(ii) $\mathrm{H}_2(L_{X/Y}\overset{\mathrm{L}}{\otimes} k(x)) = 0.$

Then $\mathrm{H}_1(L_{Y/S}\overset{\mathrm{L}}{\otimes} k(y)) = 0$. *In particular, if g is locally of finite presentation then g is smooth at y.*

Proof. It is equivalent to show that $\mathrm{H}_1(L_{Y/S}\overset{\mathrm{L}}{\otimes} k(x)) = 0$, and this follows trivially from the exact sequence

$$\mathrm{H}_2(L_{X/Y}\overset{\mathrm{L}}{\otimes} k(x)) \to \mathrm{H}_1(L_{Y/S}\overset{\mathrm{L}}{\otimes} k(x)) \to \mathrm{H}_1(L_{X/S}\overset{\mathrm{L}}{\otimes} k(x)).$$

□

Lemma 2.4.6. *Let $f: X \to S$ be a regular morphism between noetherian schemes. Then $L_{X/S}$ is cohomologically concentrated in degree zero and $\mathrm{H}_0(L_{X/S}) = \Omega^1_{X/S}$ is flat.*

Proof. We may assume $X = \operatorname{Spec} B$ and $S = \operatorname{Spec} A$ affine. Then, by Popescu's theorem [**Swan, 1998**, 1.1], X is a filtering projective limit of smooth affine S-schemes $X_\alpha = \operatorname{Spec} B_\alpha$. By [**Illusie, 1972**, II (1.2.3.4)], we have

$$L_{B/A} = \operatorname{colim}_\alpha L_{B_\alpha/A}.$$

By [**Illusie, 1972**, III 3.1.2 and II 2.3.6.3], $L_{B_\alpha/A}$ is cohomologically concentrated in degree zero and $\mathrm{H}_0(L_{B_\alpha/A}) = \Omega^1_{B_\alpha/A}$ is projective of finite type over B_α, so the conclusion follows. □

Proof of Proposition 2.4.3. The composition $ag': Z' \to S$ is locally of finite type. Since S is noetherian, ag' is locally of finite presentation, and so h' is locally of finite presentation too.

Next, we note that the question is local around a point $y = f(x) \in Z'$, $x \in Z$. In view of Lemma 2.4.4, by Lemma 2.4.5 applied to $Z \to Z' \to Y$ it suffices to show that $\mathrm{H}_1(L_{Z/Y}\overset{\mathrm{L}}{\otimes} k(x)) = 0$ and

$\mathrm{H}_2(L_{Z/Z'} \overset{\mathrm{L}}{\otimes} k(x)) = 0$. As Z is regular over Y, the first vanishing follows from Lemma 2.4.6. For the second one, consider the exact sequence

$$\mathrm{H}_2(L_{Z/X} \overset{\mathrm{L}}{\otimes} k(x)) \to \mathrm{H}_2(L_{Z/Z'} \overset{\mathrm{L}}{\otimes} k(x)) \to \mathrm{H}_1(L_{Z'/X} \overset{\mathrm{L}}{\otimes} k(x)).$$

By the regularity of Z/X and Lemma 2.4.6, $\mathrm{H}_2(L_{Z/X} \overset{\mathrm{L}}{\otimes} k(x)) = 0$. As Z' is smooth over X, $\mathrm{H}_1(L_{Z'/X} \overset{\mathrm{L}}{\otimes} k(x)) = 0$ by Lemma 2.4.4, which proves the desired vanishing and finishes the proof. □

Proof of Theorem 2.4.1. Find finite affine coverings $X = \bigcup_i X_i$, $Y = \bigcup_i Y_i$ and $Z = \bigcup_i Z_i$ such that $g(Z_i) \subset X_i$ and $h(Z_i) \subset Y_i$. Set $X' = \coprod_i X_i$, $Y' = \coprod_i Y_i$ and $Z' = \coprod_i Z_i$ and let $Z' \to X'$ and $Z' \to Y'$ be the induced morphisms. It suffices to check that $\mathscr{F}(X) \times_X Z$ and $\mathscr{F}(Y) \times_Y Z$ become equal after pulling them back to Z'. So, we should check that $(\mathscr{F}(X) \times_X X') \times_{X'} Z'$ coincides with $(\mathscr{F}(Y) \times_Y Y') \times_{Y'} Z'$. The morphisms $X' \to X$ and $Y' \to Y$ are smooth and hence contained in $\mathscr{S}^1$. So, $\mathscr{F}(X) \times_X X' = \mathscr{F}(X')$ and similarly for Y. In particular, it suffices to prove that $\mathscr{F}(X') \times_{X'} Z' = \mathscr{F}(Y') \times_{Y'} Z'$. This reduces the problem to the case when all schemes are affine, so in the sequel we assume that X, Y and Z are affine.

Next, note that it suffices to find factorizations $g = g_0 f$ and $h = h_0 f$, where $f : Z \to Z_0$ is a morphism with target in $\mathscr{S}^0$ and $g_0 : Z_0 \to X$, $h_0 : Z_0 \to X$ are smooth. By Popescu's theorem, one can write $g : Z \to X$ as a filtering projective limit of affine smooth morphisms $g_\alpha : Z_\alpha \to X$, $\alpha \in A$. As Y is of finite type over S, h will factor through one of the Z_α's ([**ÉGA** IV$_3$ 8.8.2.3]): there exists $\alpha \in A$, $f_\alpha : Z \to Z_\alpha$, $h_\alpha : Z_\alpha \to Y$ such that $g = g_\alpha f_\alpha$, $h = h_\alpha f_\alpha$. By Proposition 2.4.3, h_α is smooth around the image of f_α, so we can take Z_0 to be a sufficiently small neighborhood of the image of f_α. □

3. Resolution of log regular log schemes

All schemes considered from now on will be assumed to be noetherian. Unless said to the contrary, by log structure we mean a log structure with respect to the étale topology. We will say that a log structure M_X on a scheme X is **Zariski** if $\varepsilon^\star \varepsilon_\star M_X \xrightarrow{\sim} M_X$, where $\varepsilon : X_{\text{ét}} \to X$ is the morphism between the étale and Zariski sites. In this case, we can safely view the log structure as a Zariski log structure. A similar convention will hold also for log schemes.

3.1. Fans. Many definitions/constructions on log schemes are of "combinatorial nature". Roughly speaking, these constructions use only multiplication and ignore addition. Naturally, there exists a category of geometric spaces whose structure sheaves are monoids, and most of combinatorial constructions can be described as "pullbacks" of analogous "monoidal" operations. The first definition of such a category was done by Kato in [**Kato, 1994**]. Kato called his spaces **fans** to stress their relation to the classical combinatorial fans obtained by gluing polyhedral cones. For example, to any combinatorial fan C one can naturally associate a fan $F(C)$ whose set of points is the set of faces of C. The main motivation for the definition is that fans can be naturally associated to various log schemes.

It took some time to discover that fans are sort of "piecewise schemes" rather than a monoidal version of schemes. A more geometric version of combinatorial schemes was introduced by Deitmar in [**Deitmar, 2005**]. He called them $\mathbf{F}_1$-schemes, but we prefer the terminology of monoschemes introduced by Ogus in his book project [**Ogus, 2013**]. Note that when working with a log scheme X, we use the sheaf M_X in some constructions and we use its sharpening $\overline{M}_X$ (see 3.1.1) in other constructions. Roughly speaking, monoschemes naturally arise when we work with M_X while fans naturally arise when we work with $\overline{M}_X$.

In §3, we will show that: (a) a functorial desingularization of toric varieties over $\mathbf{Q}$ descends to a desingularization of monoschemes, (b) to give the latter is more or less equivalent to give a desingularization of fans, (c) a desingularization of fans can be used to induce a monoidal desingularization of log schemes, (d) the latter induces a desingularization of log regular schemes, which (at least in some cases) depends only on the underlying scheme.

In principle, we could work locally, using desingularization of disjoint unions of all charts for synchronization. In this case, we could almost ignore the intermediate categories by working only with fine monoids and blow up towers of their spectra. However, we decided to emphasize the actual geometric objects beyond the constructions, and, especially, stress the difference between fans and monoschemes.

3.1.1. *Sharpening.* For a monoid M, by M^* we denote the group of its invertible elements, and its **sharpening** $\overline{M}$ is defined as M/M^*.

3.1.2. *Localization.* By **localization** of a monoid M along a subset S we mean the universal M-monoid M_S such that the image of S in M_S is contained in M_S^*. If M is integral then M_S is simply the submonoid $M[S^{-1}] \subset M^{\mathrm{gp}}$ generated by M and S^{-1}. If M is a fine then any localization is isomorphic to a localization at a single element f, and will be denoted M_f.

3.1.3. *Spectra of fine monoids.* All our combinatorial objects will be glued from finitely many spectra of fine monoids. Recall that with any fine monoid P one can associate the set $\mathrm{Spec}(P)$ of prime ideals (with the convention that $\varnothing$ is also a prime ideal) equipped with the Zariski topology whose basis is formed by the sets $D(f) = \{p \in \mathrm{Spec}(P) | \ f \notin p\}$ for $f \in P$, see, for example, [**Kato, 1994**, §9]. The structure sheaf M_P is defined by $M_P(D(f)) = P_f$, and the sharp structure sheaf $\overline{M}_P = M_P/M_P^*$ is the sharpening of M_P (we will see in Remark 3.1.4(iii) that actually $\overline{M}_P(D(f)) = \overline{P_f} = P_f/(P_f^*)$, i.e. no sheafification is needed).

Remark 3.1.4. (i) Since $P \setminus P^*$ and $\varnothing$ are the maximal and the minimal prime ideals of P, $\mathrm{Spec}(P)$ possesses unique closed and generic points s and η. The latter is the only point whose stalk $M_{P,\eta} = P^{\mathrm{gp}}$ is a group.

(ii) The set $\mathrm{Spec}(P)$ is finite and its topology is the specialization topology, i.e. U is open if and only if it is closed under generizations. (More generally, this is true for any finite sober topological space, such as a scheme that has finitely many points.)

(iii) A subset $U \subset \mathrm{Spec}(P)$ is affine (and even of the form $D(f)$) if and only if it is the localization of $\mathrm{Spec}(P)$ at a point x (i.e. the set of all generizations of x). Any open covering $U = \bigcup_i U_i$ of an affine set is trivial (i.e. U is equal to some U_i), therefore any functor $\mathscr{F}(U)$ on affine sets uniquely extends to a sheaf on $\mathrm{Spec}(P)$. In particular, this explains why no sheafification is needed when defining $\overline{M}_P$. Furthermore, we see that, roughly speaking, any notion/construction that is "defined in terms of" localizations X_x and stalks M_x or $\overline{M}_x$ is Zariski local. This is very different from the situation with schemes.

3.1.5. *Local homomorphisms of monoids.* Any monoid M is local because $M \setminus M^*$ is its unique maximal ideal. A homomorphism $f : M \to N$ of monoids is **local** if it takes the maximal ideal of M to the maximal ideal of N. This happens if and only if $f^{-1}(N^*) = M^*$.

3.1.6. *Monoidal spaces.* A **monoidal space** is a topological space X provided with a sheaf of monoids M_X. A morphism of monoidal spaces $(f, f^\#) : (Y, M_Y) \to (X, M_X)$ is a continuous map $f : Y \to X$ and a homomorphism $f^\# : f^{-1}(M_X) \to M_Y$ such that for any $y \in Y$ the homomorphism of monoids $f_y^\# : M_{X,f(y)} \to M_{Y,y}$ is local.

Remark 3.1.7. Strictly speaking one should have called the above category the category of locally monoidal spaces and allow non-local homomorphisms in the general category of monoidal spaces. However, we will not use the larger category, so we prefer to abuse the terminology slightly.

Spectra of monoids possess the usual universal property, namely:

Lemma 3.1.8. *Let (X, M_X) be a monoidal space and let P be a monoid.*

(*i*) *The global sections functor Γ induces a bijection between morphisms of monoidal spaces $(f, f^\#) : (X, M_X) \to (\mathrm{Spec}(P), M_P)$ and homomorphisms $\phi : P \to \Gamma(M_X)$.*

(*ii*) *If M_X has sharp stalks then Γ induces a bijection between morphisms of monoidal spaces $(f, f^\#) : (X, M_X) \to (\mathrm{Spec}(P), \overline{M}_P)$ and homomorphisms $\phi : P \to \Gamma(M_X)$.*

Proof. (i) Let us construct the opposite map. Given a homomorphism ϕ, for any $x \in X$ we obtain a homomorphism $\phi_x : P \to M_{X,x}$. Clearly, $m = P \setminus \phi_x^{-1}(M_{X,x}^*)$ is a prime ideal and hence ϕ_x factors through a uniquely defined local homomorphism $P_m \to M_{X,x}$. Setting $f(x) = m$ we obtain a map $f : X \to \mathrm{Spec}(P)$, and the rest of the proof of (i) is straightforward.

If the stalks of M_X are sharp then any morphism $(X, M_X) \to (\mathrm{Spec}(P), M_P)$ factors uniquely through $(\mathrm{Spec}(P), \overline{M}_P)$. Also, $\Gamma(M_X)$ is sharp, hence any homomorphism to it from P factors uniquely through $\overline{P}$. Therefore, (ii) follows from (i). $\square$

3.1.9. *Fine fans and monoschemes.* A fine **monoscheme** (resp. a fine **fan**) is a monoidal space (X, M_X) that is locally isomorphic to $A_P = (\mathrm{Spec}(P), M_P)$ (resp. $\overline{A}_P = (\mathrm{Spec}(P), \overline{M}_P)$), where P is a fine monoid. We say that (X, M_X) is **affine** if it is isomorphic to A_P (resp. $\overline{A}_P$). A morphism of monoschemes (resp. fans) is a morphism of monoidal spaces. A monoscheme (resp. a fan) is called **torsion free** if it is covered by spectra of P's with torsion free P^{gp}'s. It follows from Remark 3.1.4(iii) that this happens if and only if all groups $M^{\mathrm{gp}}_{X,x}$ are torsion free.

Remark 3.1.10. (i) Any fs fan is torsion free because if an fs monoid is torsion free then any of its localization is so. In particular, if P is fs and sharp then P^{gp} is torsion free. This is not true for general fine fans. For example, if $\mu_2 = \{\pm 1\}$ then $P = \mathbf{N} \oplus \mu_2 \setminus \{(0, -1)\}$ is a sharp monoid with $P^{\mathrm{gp}} = \mathbf{Z} \oplus \mu_2$.

(ii) For any point x of a fine monoscheme (resp. fan) X the localization X_x that consists of all generizations of x is affine. In particular, by Remark 3.1.4(i) there exists a unique maximal point generizing x, and hence X is a disjoint union of irreducible components.

3.1.11. *Comparison of monoschemes and fans.* There is an obvious sharpening functor $(X, M_X) \mapsto (X, \overline{M}_X)$ from monoschemes to fans, and there is a natural morphism of monoidal spaces $(X, \overline{M}_X) \to (X, M_X)$. The sharpening functor loses information, and one needs to know M^{gp}_X to reconstruct M_X from $\overline{M}_X$ as a fibred product (see [**Ogus, 2013**]). Actually, there are much more fans than monoschemes. For example, the generic point $\eta \in \mathrm{Spec}(P)$ is open and $\overline{M}_{P,\eta}$ is trivial hence for any pair of fine monoids P and Q we can glue their sharpened spectra along the generic points. What one gets is sort of "piecewise scheme" and, in general, it does not correspond to standard geometric objects, such as schemes or monoschemes. We conclude that, in general, fans can be lifted to monoschemes only locally.

Remark 3.1.12. As a side remark we note that sharpened monoids naturally appear as structure sheaves of piecewise linear spaces (a work in progress of the second author on skeletons of Berkovich spaces). In particular, PL functions can be naturally interpreted as sections of the sharpened sheaf of linear functions on polytopes.

3.1.13. *Local smoothness.* A local homomorphism of fine monoids $\phi : P \to Q$ is called **smooth** if it can be extended to an isomorphism $P \oplus \mathbf{N}^r \oplus \mathbf{Z}^s \xrightarrow{\sim} Q$. The following lemma checks that this property is stable under localizations.

Lemma 3.1.14. *Assume that $\phi : P \to Q$ is smooth and P', Q' are localizations of P, Q such that ϕ extends to a local homomorphism $\phi' : P' \to Q'$. Then ϕ' is smooth.*

Proof. Recall that $P' = P_a$ for $a \in P$ (notation of 3.1.2), and ϕ' factors through the homomorphism $\phi_a : P' \to Q_a$, which is obviously smooth. Therefore, replacing ϕ with ϕ_a we can assume that $P = P'$. Let $b = (p, n, z) \in Q$ be such that $Q' = Q_b$. Then $p \in P^*$ because $P \to Q'$ is local, and hence Q' is isomorphic to $P \oplus (\mathbf{N}^r)_n \oplus \mathbf{Z}^s$. It remains to note that any localization of $\mathbf{N}^r$ is of the form $\mathbf{N}^{r-t} \oplus \mathbf{Z}^t$. $\square$

3.1.15. *Smoothness.* The lemma enables us to globalize the notion of smoothness: a morphism $f : Y \to X$ of monoschemes is called **smooth** if the homomorphisms of stalks $M_{X,f(y)} \to M_{Y,y}$ are smooth. In particular, X is smooth if its morphism to $\mathrm{Spec}(1)$ is smooth, that is, the stalks $M_{X,x}$ are of the form $\mathbf{N}^r \oplus \mathbf{Z}^s$. In particular, a smooth monoscheme is torsion free.

Analogous smoothness definitions are given for fans. Moreover, in this case we can consider only sharp monoids, and then the group component $\mathbf{Z}^s$ is automatically trivial. It follows that we can rewrite the above paragraph almost *verbatim* but with $s = 0$. Obviously, a morphism of torsion free fs monoschemes is smooth if and only if its sharpening is a smooth morphism of fans.

Remark 3.1.16. (i) Recall that any fine monoscheme X admits an open affine covering $X = \bigcup_{x \in X} \mathrm{Spec}(M_{X,x})$. It follows that a morphism of fine monoschemes $f : Y \to X$ is smooth if and only if it is covered by open affine charts of the form $\mathrm{Spec}(P \oplus \mathbf{N}^r \oplus \mathbf{Z}^s) \to \mathrm{Spec}(P)$.

(ii) Smooth morphisms of fine fans admit a similar local description, and we leave the details to the reader.

3.1.17. *Saturation.* As usually, for a fine monoid P we denote its saturation by P^{sat} (it consists of all $x \in P^{\mathrm{gp}}$ with $x^n \in P$ for some $n > 0$). Saturation is compatible with localizations and sharpening and hence extends to a saturation functor $F \mapsto F^{\mathrm{sat}}$ on the categories of fine monoschemes (resp. fine fans). We also have a natural morphism $F^{\mathrm{sat}} \to F$, which is easily seen to be bijective. So, actually, $(F, M_F)^{\mathrm{sat}} = (F, M_F^{\mathrm{sat}})$.

3.1.18. *Ideals.* A subsheaf of ideals $\mathscr{I} \subset M_X$ on a monoscheme (X, M_X) is called a **coherent ideal** if for any point $x \in X$ the restriction of $\mathscr{I}$ on X_x coincides with $\mathscr{I}_x M_{X_x}$. (Due to Remark 3.1.4(iii) this means that $\mathscr{I}$ is coherent in the usual sense, i.e. its restriction on an open affine submonoscheme U is generated by the global sections over U.) We will consider only coherent ideals, so we will omit the word "coherent" as a rule. An ideal $\mathscr{I} \subset M_X$ is **invertible** if it is locally generated by a single element.

3.1.19. *Blow ups.* Similarly to schemes, for any ideal $\mathscr{I} \subset \mathscr{O}_X$ there exists a universal morphism of monoschemes $h : X' \to X$ such that the *pullback ideal* $h^{-1}\mathscr{I} = \mathscr{I} M_{X'}$ is invertible. We call $\mathscr{I}$ the **center** of the blow up. (One does not have an adequate notion of closed submonoscheme, so unlike blow ups of the scheme it would not make sense to say that "$V(\mathscr{I})$" is the center.) An explicit construction of X' copies its scheme analog: it is local on the base and for an affine $X = \mathrm{Spec}(P)$ with an ideal $I \subset P$ corresponding to $\mathscr{I}$ one glues X' from the charts $\mathrm{Spec}(P[a^{-1}I])$, where $a \in I$ and $P[a^{-1}I]$ is the submonoid of P^{gp} generated by the fractions b/a for $b \in I$ (see [**Ogus, 2013**] for details).

Remark 3.1.20. Blow ups induce isomorphisms on the stalks of M^{gp}; this is an analog of the fact that blow ups of schemes along nowhere dense subschemes are birational morphisms.

3.1.21. *Saturated blow ups.* Analogously to normalized blow ups, one defines **saturated blow up** of a monoscheme X along an ideal $\mathscr{I} \subset M_X$ as the saturation of $\mathrm{Bl}_{\mathscr{I}}(X)$. The same argument as for schemes shows that $\mathrm{Bl}_{\mathscr{I}}(X)^{\mathrm{sat}}$ is the universal saturated X-monoscheme such that the pullback of $\mathscr{I}$ is invertible.

3.1.22. *Towers and pullbacks.* Towers of (saturated) blow ups of a monoscheme X are defined analogously to towers of (normalized) blow ups. In particular, saturated towers start with the saturation morphism $X_0 \to X_{-1}$. Given such a tower $X_\bullet$ with $X = X_0$ (resp. $X_{-1} = X$) and a morphism $f : Y \to X$ we define the **pullback tower** $Y_\bullet = f^\star(X_\bullet)$ as follows: $Y_0 = Y$ (resp. $Y_{-1} = Y$) and Y_{i+1} is the (saturated) blow up of Y_i along the pullback of the center $\mathscr{I}_i$ of $X_{i+1} \to X_i$. Due to the universal property of (saturated) blow ups this definition makes sense and $Y_\bullet$ is the universal (saturated) blow up tower of Y that admits a morphism to $X_\bullet$ extending f.

Remark 3.1.23. Unlike pullbacks of (normalized) blow up towers of schemes, see 2.1.7 and 2.2.8, we do not distinguish strict transforms and pullbacks. The above definition of pullback covers our needs, and we do not have to study the base change of monoschemes. For the sake of completeness, we note that fibred products of monoschemes exist and in the affine case are defined by amalgamated sums of monoids, see [**Deitmar, 2005**]. Also, it is easy to check that for a smooth f (which is the only case we will use) one indeed has that $f^\star(X_\bullet) = X_\bullet \times_X Y$ for any (saturated) blow up tower $X_\bullet$. For blow ups one checks this with charts and in the saturated case one also uses that saturation is compatible with a smooth morphism $f : Y \to X$, i.e. $Y^{\mathrm{sat}} \xrightarrow{\sim} X^{\mathrm{sat}} \times_X Y$.

3.1.24. *Compatibility with sharpening.* Ideals and blow ups of fans are defined in the same way, but with $\overline{M}_X$ used instead of M_X (Kato defines their saturated version in [**Kato, 1994**, 9.7]). Towers of blow ups of fans are defined in the obvious way.

Lemma 3.1.25. *Let $X = (X, M_X)$ be a monoscheme, let $(F, M_F) = (X, \overline{M}_X)$ be the corresponding fan and let $\lambda : M_X \to M_F$ denote the sharpening homomorphism.*

(*i*) *$\mathscr{I} \mapsto \lambda(\mathscr{I})$ induces a natural bijection between the ideals on X and on F.*

(*ii*) *Blow ups are compatible with sharpening, that is, the sharpening of $\mathrm{Bl}_{\mathscr{I}}(X))$ is naturally isomorphic to $\mathrm{Bl}_{\lambda(\mathscr{I})}(F)$. The same statement holds for saturated blow ups.*

(*iii*) *Sharpening induces a natural bijection between the (saturated) blow up towers of X and F.*

Proof. (i) is obvious. (ii) is shown by comparing the blow up charts. Combining (i) and (ii), we obtain (iii). □

3.1.26. *Desingularization.* Using the above notions of smoothness and blow ups, one can copy other definitions of the desingularization theory. By a **desingularization** (resp. **saturated desingularization**) of a fine monoscheme X we mean a blow up tower (resp. saturated blow up tower) $X_n \dashrightarrow X_0 = X$ (resp. $X_n \dashrightarrow X_{-1} = X$) with smooth X_n. By Remark 3.1.20, if X admits a desingularization then it is torsion free, and we will later see that the converse is also true.

For concreteness, we consider below non-saturated desingularizations, but everything extends to the saturated case verbatim. A family $\mathscr{F}^{\mathrm{mono}}(X)$ of desingularizations of torsion free monoschemes is called **functorial** (with respect to smooth morphisms) if for any smooth $f\colon Y \to X$ the desingularization $\mathscr{F}^{\mathrm{mono}}(Y)$ is the contracted pullback of $\mathscr{F}^{\mathrm{mono}}(X)$. The same argument as for schemes (see Remark 2.3.5(ii)) shows that $\mathscr{F}^{\mathrm{mono}}$ is already determined by its restriction to the family of finite disjoint unions of affine monoschemes.

The definition of a functorial desingularization $\mathscr{F}^{\mathrm{fan}}$ of fine torsion free fans is similar. Since blow up towers and smoothness are compatible with the sharpening functor, it follows that a desingularization of a monoscheme X induces a desingularization of its sharpening. Moreover, any affine fan can be lifted to an affine monoscheme and $\mathscr{F}^{\mathrm{fan}}$ is determined by its restriction onto disjoint unions of affine fans, hence we obtain the following result.

Theorem 3.1.27. *The sharpening functor induces a natural bijection between functorial desingularizations of quasi-compact fine torsion free monoschemes and functorial desingularizations of quasi-compact fine torsion free fans. A similar statement holds for saturated desingularizations.*

Remark 3.1.28. Similarly to the normalization of a desingularization tower, to any desingularization $\mathscr{F}$ of monoschemes or fans one can associate a saturated desingularization $\mathscr{F}^{\mathrm{sat}}$: one replaces all levels of the towers, except the zero level, with their saturations. In this case blow ups are replaced with saturated blow ups along the pulled back ideals. If $\mathscr{F}$ is functorial with respect to all smooth morphisms then the same is true for $\mathscr{F}^{\mathrm{sat}}$. Indeed, for any smooth $Y \to X$ the centers of $\mathscr{F}(Y)$, $\mathscr{F}^{\mathrm{sat}}(X)$ and $\mathscr{F}^{\mathrm{sat}}(Y)$ are the pullbacks of those of $\mathscr{F}(X)$. In addition, the saturation construction is compatible with the bijections from Theorem 3.1.27 in the obvious way.

Remark 3.1.29. In principle, (saturated) desingularization of fans or monoschemes can be described in purely combinatorial terms of fans and their subdivisions (e.g. see [**Kato, 1994**, 9.6] or [**Nizioł, 2006**, §4]). However, it is not easy to construct a functorial one directly. We will instead use a relation between monoschemes and toric varieties to descend desingularization of toric varieties to monoschemes and fans.

3.2. Monoschemes and toric varieties.

3.2.1. *Base change from monoschemes to schemes.* Let S be a scheme. The following proposition introduces a functor from monoschemes to S-schemes that can be intuitively viewed as a base change with respect to a "morphism" $S \to \mathrm{Spec}(1)$.

Proposition 3.2.2. *Let S be a scheme and F be a monoscheme. Then there exists an S-scheme $X = S[F]$ with a morphism of monoidal spaces $f\colon (X, \mathscr{O}_X) \to (F, M_F)$ such that any morphism $(Y, \mathscr{O}_Y) \to (F, M_F)$, where Y is an S-scheme, factors uniquely through f.*

Proof. Assume, first, that $F = \mathrm{Spec}(P)$ is affine. By Lemma 3.1.8(i), to give a morphism $(Y, \mathscr{O}_Y) \to (F, M_F)$ is equivalent to give a homomorphism of monoids $\phi\colon P \to \Gamma(\mathscr{O}_Y)$, and the latter factors uniquely through a homomorphism of sheaves of rings $\mathscr{O}_S[P] \to \mathscr{O}_Y$. It follows that $S[F] = \mathrm{Spec}(\mathscr{O}_S[P])$ in this case. Since the above formula is compatible with localizations by elements $a \in P$, i.e. $S[F_a] \xrightarrow{\sim} S[F]_{\phi(a)}$, it globalizes to the case of an arbitrary monoscheme. Thus, for a general monoscheme F covered by $F_i = \mathrm{Spec}(P_i)$, the scheme $S[F]$ is glued from $S[F_i]$. □

Remark 3.2.3. Note that if $S = \mathrm{Spec}(R)$ and $F = \mathrm{Spec}(P)$ then $S[F] = \mathrm{Spec}(R[P])$. However, we will often consider an "intermediate" situation where $S = \mathrm{Spec}(R)$ is affine and F is a general monoscheme. To simplify notation, we will abuse them by writing $R[F]$ instead of $\mathrm{Spec}(R[F])$. Such "mixed" notation will always refer to a scheme.

3.2.4. *Toric schemes.* If F is torsion free and connected then we call $S[F]$ a **toric scheme** over S. Recall that by Remark 3.1.10(ii), F possesses a unique maximal point $\eta = \mathrm{Spec}(P^{\mathrm{gp}})$, where $\mathrm{Spec}(P)$ is any affine open submonoscheme. Hence $X = S[F]$ possesses a dense open subscheme $T = S[\eta]$, which is a split torus over S, and the action of T on itself naturally extends to the action of T on X.

Remark 3.2.5. Assume that $S = \mathrm{Spec}(k)$ where k is a field. Classically, a toric k-variety is defined as a normal finite type separated k-scheme X that contains a split torus T as a dense open subscheme such that the action of T on itself extends to the whole X. If F is saturated then our definition above is equivalent to the classical one. However, we also consider non-normal toric varieties corresponding to non-saturated monoids.

3.2.6. *Canonical log structures.* For any monoscheme F, the S-scheme $X = S[F]$ possesses a natural log structure induced by the universal morphism $f\colon (X, \mathscr{O}_X) \to (F, M_F)$. Namely, M_X is the log structure associated with the pre-log structure $g^{-1}M_F \to \mathscr{O}_{X_{\text{ét}}}$, where $g\colon (X_{\text{ét}}, \mathscr{O}_{X_{\text{ét}}}) \to (X, \mathscr{O}_X) \to (F, M_F)$ is the composition. We call M_X the **canonical** log structure of $X = S[F]$.

Remark 3.2.7. (i) The canonical log structure is Zariski, as $\varepsilon_\star M_X$ coincides with the Zariski log structure associated with the pre-log structure $f^{-1}M_F \to \mathscr{O}_X$.

(ii) The log scheme $(X = S[F], M_X)$ is log smooth over the scheme S provided with the trivial log structure. In particular, if S is regular and F is saturated then (X, M_X) is fs and log regular. Without the saturation assumption we still have that $X^{\mathrm{sat}} \xrightarrow{\sim} S[F^{\mathrm{sat}}]$ is log regular, hence X is log regular in the sense of Gabber (see §3.5).

(iii) If η is the set of generic points of F then $T = S[\eta]$ is the open subset of X which is the triviality set of its log structure. However, the map $M_X \to \mathscr{O}_X \cap j_\star\mathscr{O}_T^*$ is not an isomorphism in general, as the case where $T = \mathrm{Spec}\,\mathbf{C}[t, t^{-1}] \subset X = \mathrm{Spec}\,\mathbf{C}[t^2, t^3]$ already shows: the image of $t^2 + t^3$ in $\mathscr{O}_{X,\{0\}}$ belongs to $(j_\star\mathscr{O}_T^*)_{\{0\}}$, but does not belong to $M_{\{0\}} = t^P\mathscr{O}_{X,\{0\}}^*$, where P is the (fine, but not saturated) submonoid of $\mathbf{N}$ generated by 2 and 3. (We use that $\mathscr{O}_{X,\{0\}}$ is strictly smaller than its normalization $\mathbf{C}[t]_{(t)} = \mathscr{O}_{X,\{0\}}[t]$ and hence $\frac{t^2+t^3}{t^2} = 1 + t$ is not contained in $\mathscr{O}_{X,\{0\}}$.)

3.2.8. *Toric saturation.* Saturation of monoschemes corresponds to normalization of schemes. This will play an essential role later, since we get a combinatorial description of the normalization.

Lemma 3.2.9. *If S is a normal scheme and F is a fine torsion free monoscheme then there is a natural isomorphism $S[F]^{\mathrm{nor}} \xrightarrow{\sim} S[F^{\mathrm{sat}}]$.*

Proof. Note that $f\colon \mathbf{Z}[F^{\mathrm{sat}}] \to \mathrm{Spec}(\mathbf{Z})$ is a flat morphism and its fibers are normal because they are classical toric varieties $\mathbf{F}_p[F^{\mathrm{sat}}]$. So, $f \times S\colon S[F^{\mathrm{sat}}] \to S$ is a flat morphism with normal fibers and normal target, and we obtain that its source is normal by [**Matsumura, 1980a**, 21.E(iii)]. It remains to note that $S[F^{\mathrm{sat}}] \to S[F]$ is a finite morphism inducing isomorphism of dense open subschemes $S[F^{\mathrm{gp}}]$, hence $S[F^{\mathrm{sat}}]$ is the normalization of $S[F]$. □

Remark 3.2.10. The same argument shows that if S is Cohen-Macaulay then so is $S[F^{\mathrm{sat}}]$.

3.2.11. *Toric smoothness.* Next, let us compare smoothness of morphisms of monoschemes as defined in 3.1.15 and classical smoothness of toric morphisms. The following lemma slightly extends the classical result (e.g. see [**Fulton, 1993**, §2.1]) that if P is fs and $\mathbf{C}[P]$ is regular then $P \xrightarrow{\sim} \mathbf{N}^r \oplus \mathbf{Z}^s$.

Lemma 3.2.12. *Let $f\colon F \to F'$ be a morphism of fine monoschemes and let S be a non-empty scheme.*
(i) If f is smooth then $S[f]$ is smooth (as a morphism of schemes).
(ii) If F is torsion free and the morphism $S[F] \to S$ is smooth then F is smooth.

Proof. Part (i) is obvious, so let us check (ii). We can also assume that $F = \mathrm{Spec}(P)$ is affine. Also, we can replace S with any of its points achieving that $S = \mathrm{Spec}(k)$. Then P is a fine torsion free monoid and $k[P] \subset k[P^{\mathrm{gp}}] \xrightarrow{\sim} k[\mathbf{Z}^n]$. It follows that $X = \mathrm{Spec}(k[P])$ is an integral smooth k-variety of dimension n. Note that $\mathrm{Spec}(k[P^{\mathrm{sat}}])$ is a finite modification of X which is generically an isomorphism. Since X is normal we have that $\mathrm{Spec}(k[P^{\mathrm{sat}}]) \xrightarrow{\sim} X$, and it follows that P is saturated. Now, $P \xrightarrow{\sim} \overline{P} \oplus \mathbf{Z}^l$

and hence $X \xrightarrow{\sim} \mathrm{Spec}(k[\overline{P}]) \times_k \mathbf{G}_m^l$. Obviously, $\mathrm{Spec}(k[\overline{P}])$ is smooth of dimension $r = n - l$ and our task reduces to showing that $\overline{P} \xrightarrow{\sim} \mathbf{N}^r$.

Let $m = \overline{P} \setminus \{1\}$ be the maximal ideal of $\overline{P}$. Then $I = k[m]$ is a maximal ideal of $k[\overline{P}]$ with residue field k. In particular, by k-smoothness of $k[\overline{P}]$ we have that $\dim_k(I/I^2) = r$. On the other hand, $I = \bigoplus_{x \in m} xk$ and $I^2 = \bigoplus_{x \in m^2} xk$, hence $I/I^2 \xrightarrow{\sim} \bigoplus_{x \in m \setminus m^2} xk$ and we obtain that $m \setminus m^2$ consists of r elements $t_1, \dots, t_r$. Note that these elements generate $\overline{P}$ as a monoid and hence they generate $\overline{P}^{\mathrm{gp}}$ as a group. Since $\overline{P}^{\mathrm{gp}} \xrightarrow{\sim} \mathbf{Z}^r$, the elements $t_1, \dots, t_r$ are linearly independent in $\overline{P}^{\mathrm{gp}}$, and we obtain that the surjection $\bigoplus_{i=1}^r t_i^{\mathbf{N}} \to \overline{P}$ is an isomorphism. $\square$

Remark 3.2.13. It seems very probable that, much more generally, f is smooth whenever $S[f]$ is smooth as a morphism of schemes and one of the following conditions holds: (a) S has points in all characteristics, (b) the homomorphisms $M^{\mathrm{gp}}_{F',x'} \to M^{\mathrm{gp}}_{F,x}$ induced by f have torsion free kernels and cokernels. We could prove this either in the saturated case or under some milder but unnatural restrictions. The main ideas are similar but the proof becomes more technical. We do not develop this direction here since the lemma covers our needs.

3.2.14. *Toric ideals.* Let F be a torsion free monoscheme, k be a field and $X = k[F]$. For any ideal $\mathscr{I}$ on F one naturally defines an ideal $k[\mathscr{I}]$ on X: in local charts, an ideal $I \subset P$ goes to the ideal $Ik[P] = k[I] = \bigoplus_{a \in I} ak$ in $k[P]$. We say that $\mathscr{J} = k[\mathscr{I}]$ is a **monoidal ideal** on $k[F]$. Note that $\mathscr{J}$ determines $\mathscr{I}$ uniquely via the following equality, which is obvious in local charts: $f^{\star}\mathscr{I} = \mathscr{J} \cap M_X$, where $f\colon X \to F$ is the natural map.

Lemma 3.2.15. *Assume that F is connected, η is its maximal point, $X = k[F]$, and $T = k[\eta]$ is the torus of the toric scheme X. A coherent ideal $\mathscr{J} \subset \mathscr{O}_X$ is T-equivariant if and only if it is monoidal.*

Proof. Any monoidal ideal is obviously T-equivariant, so let us prove the inverse implication. The claim is local on F, so we should prove that any T-equivariant ideal $J \subset A = k[P]$ is of the form $k[I]$ for a unique ideal I of P. Consider the coaction homomorphism $\mu\colon A \to A \otimes_k k[P^{\mathrm{gp}}] = B$. The equivariance of J means that JB (with respect to the embedding $A \hookrightarrow A \otimes_k k[P^{\mathrm{gp}}]$) is equal to $\mu(J)B$. In particular, $\mu|_J\colon J \to JB = J \otimes_k k[P^{\mathrm{gp}}]$ induces a P^{gp}-grading on J compatible with the P^{gp}-grading $A = \bigoplus_{\gamma \in P} A_\gamma$. Thus, J is a homogeneous ideal in A and, since $A_1 = k$ is a field and each k-module A_γ is of rank one, we obtain that $J = \bigoplus_{\gamma \in I} A_\gamma$ for a subset $I \subset P$. Thus, $J = k[I]$, and clearly I is an ideal. $\square$

3.2.16. *Toric blow ups.* We will also need the well known fact that toric blow ups are of combinatorial origin, i.e. they are induced from blow ups of monoschemes.

Lemma 3.2.17. *Assume that F is a monoscheme, $\mathscr{I} \subset M_F$ is an ideal, $X = k[F]$, and $\mathscr{J} = k[\mathscr{I}]$. Then there is a canonical isomorphism $\mathrm{Bl}_{\mathscr{J}}(X) \xrightarrow{\sim} k[\mathrm{Bl}_{\mathscr{I}}(F)]$.*

Proof. Assume first that $F = \mathrm{Spec}(P)$. Then $\mathscr{I}$ corresponds to an ideal $I \subset P$ and we can simply compare charts: $\mathrm{Bl}_{\mathscr{I}}(F)$ is covered by the charts $\mathrm{Spec}(P[a^{-1}I])$ for $a \in I$, and, since I generates J, the charts $k[a^{-1}J] = k[P[a^{-1}I]]$ cover $\mathrm{Bl}_J(X)$. This construction is compatible with localizations $(P, I) \mapsto (P_b, b^{-1}I)$ hence it globalizes to the case of a general fine monoscheme with an ideal. $\square$

Using Lemma 3.2.9 we obtain a similar relation between saturated blow ups and normalized toric blow ups.

Corollary 3.2.18. *Keep notation of Lemma 3.2.17 and assume that F is torsion free. Then* $\mathrm{Bl}_{\mathscr{J}}(X)^{\mathrm{nor}} \xrightarrow{\sim} k[\mathrm{Bl}_{\mathscr{I}}(F)^{\mathrm{sat}}]$.

3.2.19. *Desingularization of monoschemes.* Let F be a torsion free monoscheme and $X = k[F]$ for a field k of characteristic zero (e.g. $k = \mathbf{Q}$). Recall that the normalized desingularization functor $\widetilde{\mathscr{F}}$ from 2.3.9 is compatible with the action of any smooth k-group, hence the centers of $\widetilde{\mathscr{F}}(X)\colon X_n \dashrightarrow X_{-1} = X$ are T-equivariant ideals. By Lemma 3.2.15, the blown up ideal of $X_0 = k[F^{\mathrm{sat}}]$ is of the form $k[\mathscr{I}]$ for an ideal $\mathscr{I} \subset M_{F^{\mathrm{sat}}}$, hence $X_1 = k[F_1]$ for $F_1 = \mathrm{Bl}_{\mathscr{I}}(F^{\mathrm{sat}})^{\mathrm{sat}}$ by Lemma 3.2.18. Applying this argument inductively we obtain that the entire normalized blow up tower $\widetilde{\mathscr{F}}(X)$ descends to a saturated blow up tower of F, which we denote as $\widetilde{\mathscr{F}}^{\mathrm{mono}}(F)$ (in other words, $\widetilde{\mathscr{F}}(X) = k[\widetilde{\mathscr{F}}^{\mathrm{mono}}(F)]$). Since F_n is smooth by Lemma 3.2.12(ii), the tower $\widetilde{\mathscr{F}}^{\mathrm{mono}}(F)$ is a desingularization of F. Moreover, part (i) of the same lemma implies that $\widetilde{\mathscr{F}}^{\mathrm{mono}}$ is functorial with respect to smooth morphisms of monoschemes. Namely, for any smooth morphism $F' \to F$, $\widetilde{\mathscr{F}}^{\mathrm{mono}}(F')$ is the contracted pullback of $\widetilde{\mathscr{F}}^{\mathrm{mono}}(F)$ (see 3.1.22). Inspecting what is needed for 2.3.17, one obtains:

THEOREM 3.2.20. *Let k be a field and let $\widetilde{\mathscr{F}}$ be a normalized desingularization of finite disjoint unions of toric k-varieties which is functorial with respect to smooth toric morphisms. Then each normalized blow up tower $\widetilde{\mathscr{F}}(k[F])$ is the pullback of a uniquely defined saturated blow up tower of monoschemes $\widetilde{\mathscr{F}}^{\mathrm{mono}}(F)$. This construction produces a saturated desingularization of quasi-compact fine torsion free monoschemes which is functorial with respect to smooth morphisms.*

Combining theorems 3.1.27 and 3.2.20 we obtain a functorial saturated desingularization of fans that will be denoted $\widetilde{\mathscr{F}}^{\mathrm{fan}}$.

REMARK 3.2.21. We will work with normalized and saturated desingularizations, so we formulated the theorem for $\widetilde{\mathscr{F}}$. The same argument shows that $\mathscr{F}$ induces desingularizations $\mathscr{F}^{\mathrm{mono}}$ and $\mathscr{F}^{\mathrm{fan}}$ that are functorial with respect to all smooth morphisms. Moreover, the descent from toric desingularization is compatible with normalization/saturation, i.e. $\widetilde{\mathscr{F}}^{\mathrm{mono}} = (\mathscr{F}^{\mathrm{mono}})^{\mathrm{sat}}$ and similarly for fans.

3.3. Monoidal desingularization. In this section we will establish, what we call, monoidal desingularization of fine log schemes (X, M_X). This operation "resolves" the sheaf $\overline{M}_X$ but does not "improve" the log strata of X.

3.3.1. *Log stratification.* Using charts one immediately checks that for any fine log scheme (X, M_X) the monoidal rank function $x \mapsto \mathrm{rank}(\overline{M}^{\mathrm{gp}}_{\overline{x}})$ is upper semicontinuous. The corresponding stratification of X, whose strata $X^{(i)}$ are the locally closed subsets on which the monoidal rank function equals to i, will be called the **log stratification**. We remark that the analogous stratification in **VI**-1.5 was called canonical or stratification by rank.

REMARK 3.3.2. Sometimes it is more convenient to work with the **local log stratification** of (X, M_X) whose strata are the maximal connected non-empty locally closed subsets of X on which the monoidal rank function is constant. This stratification is obtained from the log stratification by replacing each stratum with the set of its connected components, in particular, all empty strata are discarded. For example, this stratification showed up in **VI**-3.9.

3.3.3. *Monoscheme charts of log schemes.* A (global) **monoscheme chart** of a Zariski log scheme (X, M_X) is a morphism of monoidal spaces $c\colon (X, \varepsilon_\star M_X) \to (F, M_F)$ such that the target is a monoscheme and $\varepsilon_\star M_X$ is isomorphic to the Zariski log structure associated with the pre-log structure $c^{-1}M_F \to \mathscr{O}_X$ (obtained as $c^{-1}M_F \to M_X \to \mathscr{O}_X$). In particular, M_X is the log structure associated with $(c \circ \varepsilon)^{-1}M_F \to \mathscr{O}_{X_{\text{ét}}}$. We say that the chart is **fine** if (F, M_F) is so. For example, any toric scheme $R[F]$, where F is a monoscheme, possesses a canonical chart $R[F] \to F$.

LEMMA 3.3.4. *Let (X, M_X) be a log scheme and let (F, M_F) be a monoscheme. Then any morphism of monoidal spaces $f\colon (X, M_X) \to (F, M_F)$ factors uniquely into the composition of a morphism of log schemes $(X, M_X) \to \mathbf{Z}[F]$ and the canonical chart $\mathbf{Z}[F] \to F$.*

Proof. Note that $(Id_X, \alpha)\colon (X, \mathscr{O}_X) \to (X, M_X)$ is a morphism of monoidal spaces, hence so is the composition $h\colon (X, \mathscr{O}_X) \to (F, M_F)$. If $F = \mathrm{Spec}(P)$ then h is determined by the homomorphism $P \to \Gamma(\mathscr{O}_X)$ by Lemma 3.1.8. Since the latter factors uniquely into the composition of the homomorphism of monoids $P \to \mathbf{Z}[P]$ and the homomorphism of rings $\mathbf{Z}[P] \to \Gamma(\mathscr{O}_X)$, we obtain a canonical factoring $X \to \mathrm{Spec}(\mathbf{Z}[P]) \to \mathrm{Spec}(P)$. Furthermore, this affine construction is compatible with localizations of P, hence it globalizes to the case when the monoscheme F is arbitrary. □

Remark 3.3.5. (i) Usually, one works with log schemes using local charts $(X, M_X) \to \mathrm{Spec}(\mathbf{Z}[P])$. By Lemma 3.3.4 this is equivalent to working with affine monoscheme charts.

(ii) In particular, any fine log scheme (X, M_X) admits a fine monoscheme chart étale locally, i.e., there exists a strict (in the log sense) étale covering $(Y, M_Y) \to (X, M_X)$ whose source possesses a fine monoscheme chart. Similarly, any Zariski fine log scheme admits a fine monoscheme chart Zariski locally.

3.3.6. *Chart base change.* Given a fine monoscheme chart $(X, M_X) \to F$ and a morphism of monoschemes $F' \to F$ we will write $(X, M_X) \times_F F'$ instead of $(X, M_X) \times_{\mathbf{Z}[F]} \mathbf{Z}[F']$, where the second product is taken in the category of fine log schemes. This notation is partially justified by the following result.

Lemma 3.3.7. *Keep the above notation and let* $(X', M_{X'}) = (X, M_X) \times_F F'$.

(*i*) *The morphism* $c' : (X', M_{X'}) \to F'$ *is a monoscheme chart.*

(*ii*) *If* (Y, M_Y) *is a log scheme over* (X, M_X) *and* $d : (Y, M_Y) \to F$ *is the induced morphism of monoidal spaces, then any lifting of* d *to a morphism* $(Y, M_Y) \to F'$ *factors uniquely through* c'.

Proof. Strictness is stable under base changes, hence $(X', M_{X'}) \to \mathbf{Z}[F']$ is strict and we obtain (i). The assertion of (ii) is a consequence of Lemma 3.3.4. □

3.3.8. *Log ideals.* By a **log ideal** on a fine log scheme (X, M_X) we mean any ideal $\mathscr{I} \subset M_X$ that étale locally on X admits a coherent chart as follows: there exists a strict étale covering $f : (Y, M_Y) \to (X, M_X)$, a fine monoscheme chart $c : (Y, M_Y) \to F$ and a coherent ideal $\mathscr{I}_F \subset M_F$ such that $f^{-1}(\mathscr{I})M_Y = c^{-1}(\mathscr{I}_F)M_Y$.

3.3.9. *Log blow ups.* It is proved in [**Nizioł, 2006**, 4.2] that there exists a universal morphism $f : (X', M_{X'}) \to (X, M_X)$ such that the ideal $f^{-1}(\mathscr{I})M_{X'}$ is invertible, i.e. locally (in the étale topology) generated by one element. (We use here that, unlike rings, any principal ideal aM of an integral monoid M is invertible in the usual sense, i.e. $M \xrightarrow{\sim} aM$ as M-sets.) Actually, the formulation in [**Nizioł, 2006**] refers only to saturated blow ups, but the proof deals also with the non-saturated ones.

The construction of log blow ups is standard and it also shows that they are compatible with arbitrary strict morphisms. If (X, M_X) and $\mathscr{I}$ admit a chart F, $\mathscr{J} \subset M_F$ then $(X', M_{X'}) = (X, M_X) \times_F \mathrm{Bl}_{\mathscr{J}}(F')$ is as required. If $(Y, M_Y) \to (X, M_X)$ is strict then F is also a chart of (Y, M_Y), hence the local construction is compatible with strict morphisms. The general case now follows by descent because any fine log scheme admits a chart étale locally. We call f the **log blow up** of (X, M_X) along $\mathscr{I}$ and denote it $\mathrm{LogBl}_{\mathscr{I}}(X, M_X)$ (it is called unsaturated log blow up in [**Nizioł, 2006**]). Log blow up towers are defined in the obvious way. As usual, contraction of such a tower is obtained by removing all empty log blow ups (i.e. blow ups along $\mathscr{I} = M_X$).

3.3.10. *Saturated log blow ups.* Saturated log blow up along a log ideal $\mathscr{I}$ is defined as $(\mathrm{LogBl}_{\mathscr{I}}(X, M_X))^{\mathrm{sat}}$. It satisfies an obvious universal property too. (It is called log blow up in [**Nizioł, 2006**]). Towers of saturated log blow ups, their pullbacks, and saturation of a tower of log blow ups are defined in the obvious way.

3.3.11. *Pullbacks.* Let $f : (Y, M_Y) \to (X, M_X)$ be a morphism of log schemes. By **pullback** of the log blow up $\mathrm{LogBl}_{\mathscr{I}}(X, M_X)$ along a log ideal $\mathscr{I} \subset M_X$ we mean the log blow up $\mathrm{LogBl}_{\mathscr{J}}(Y, M_Y)$, where $\mathscr{J} = f^{-1}(\mathscr{I})M_Y$. This is the universal log scheme over (Y, M_Y) whose morphism to (X, M_X) factors through $\mathrm{LogBl}_{\mathscr{I}}(X, M_X)$. The pullback of saturated blow ups is defined similarly, and these definitions extend inductively to pullbacks of towers of (saturated) log blow ups.

3.3.12. *Basic properties.* Despite the similarity with usual blow ups of schemes, log blow ups (resp. saturated log blow ups) have nice properties that are not satisfied by usual blow ups. First, it is proved in [**Nizioł, 2006**, 4.8] that log blow ups are compatible with any log base change $f : Y \to X$, i.e. $\mathrm{LogBl}_{f^{-1}\mathscr{I}}(Y) \xrightarrow{\sim} \mathrm{LogBl}_{\mathscr{I}}(X) \times_X Y$ for a monoidal ideal $\mathscr{I}$ on X. In particular, saturated blow ups are compatible with saturated base changes. Second, log blow ups (resp. saturated log blow ups) are log étale morphisms because so are both saturation morphisms and charts of the form $\mathbf{Z}[\mathrm{Bl}_{\mathscr{I}}(F)] \to \mathbf{Z}[F]$.

3.3.13. *Fan charts.* A **fan chart** of a Zariski log scheme (X, M_X) is a morphism $d\colon (X, \varepsilon_\star \overline{M}_X) \to (F, M_F)$ of monoidal spaces such that the target is a fan and $d^{-1}(M_F) \xrightarrow{\sim} \varepsilon_\star \overline{M}_X$. For example, for any monoscheme chart $c\colon (X, \varepsilon_\star M_X) \to (F, M_F)$, its sharpening $\overline{c}\colon (X, \varepsilon_\star \overline{M}_X) \to (F, \overline{M}_F)$ is a fan chart. Fan charts were considered by Kato (e.g., in [**Kato, 1994**, 9.9]). They contain less information than monoscheme charts, but "remember everything about ideals and blow ups" because there is a one-to-one correspondence between ideals and blow up towers of M_F and $M_{\overline{F}}$. Let us make this observation rigorous. For concreteness, we discuss only non-saturated (log) blow ups, but everything easily extends to the saturated case.

Remark 3.3.14. (i) Assume that $\overline{c}\colon (X, \overline{M}_X) \to (F, M_F)$ is a fan chart. Any ideal $\mathscr{I}_F \subset M_F$ induces a log ideal $\mathscr{I} \subset M_X$, which is the preimage of $\overline{c}^{-1}(\mathscr{I}_F)\overline{M}_X$ under $M_X \to \overline{M}_X$. We say that the blow up $F' = \mathrm{Bl}(\mathscr{I}_F)$ induces the log blow up $(X', \overline{M}_{X'}) = \mathrm{LogBl}_{\mathscr{I}}(X, \overline{M}_X)$ or that the latter log blow up is the **pullback** of $\mathrm{Bl}(\mathscr{I}_F)$. Furthermore, $(X', \overline{M}_{X'}) \to F'$ is also a fan chart (see [**Kato, 1994**, 9.9], where the fs case is treated), hence this definition iterates to a tower $F_\bullet$ of blow ups of F. We will denote the pullback tower of log blow ups as $\overline{c}^\star(F_\bullet)$.

(ii) By a slight abuse of notation, Kato and Nizioł denote $\overline{c}^\star(F_\bullet)$ as $(X, M_X) \times_F F_\bullet$. One should be very careful with this notation because, in general, there is no morphism $(X, M_X) \to F$ that lifts $\overline{c}$. Also, one cannot define analogous "base change" for an arbitrary morphism of fans $F' \to F$. The reason is that there are many "unnatural" gluings in the category of fans (e.g. along generic points), and such gluings cannot be lifted to log schemes (and even to monoschemes).

(iii) For blow up towers, however, the base change notation is safe and agrees with the base change from the monoschemes. Namely, if $\overline{c}$ is the sharpening of a monoscheme chart $c\colon (X, M_X) \to (F, M_F)$ then there exists a one-to-one correspondence between blow up towers of the monoscheme $F = (F, M_F)$ and the fan $\overline{F} = (F, \overline{M}_F)$, see Lemma 3.1.25. Clearly, the matching towers induce the same log blow up tower of (X, M_X). In particular, $\mathscr{F}^{\mathrm{mono}}(F, M_F)$ and $\mathscr{F}^{\mathrm{fan}}(F, \overline{M}_F)$ (see Theorem 3.1.27) induce the same log blow up tower of (X, M_X).

3.3.15. *Monoidal desingularization of log schemes.* Let (X, M) be a fine log scheme and assume that (X, M) is **monoidally torsion free** in the sense that the groups $\overline{M}^{\mathrm{gp}}_{\overline{x}}$ are torsion free. By a **monoidal desingularization** (resp. a **saturated monoidal desingularization**) of a fine log scheme (X, M) we mean a tower of log blow ups (resp. a tower of saturated log blow ups) $(X_n, M_n) \dashrightarrow (X_0, M_0) = (X, M)$ (resp. $(X_n, M_n) \dashrightarrow (X_{-1}, M_{-1}) = (X, M)$) such that for any geometric point $\overline{x} \to X_n$ the stalk of $\overline{M}_n$ at $\overline{x}$ is a free monoid. A morphism $(Y, N) \to (X, M)$ is called **monoidally smooth** if each induced homomorphism of stalks of monoids $\overline{M}_{\overline{x}} \to \overline{N}_{\overline{y}}$ can be extended to an isomorphism $\overline{M}_{\overline{x}} \oplus \mathbf{N}^r \xrightarrow{\sim} \overline{N}_{\overline{y}}$.

Theorem 3.3.16. *Let $\widetilde{\mathscr{F}}^{\mathrm{fan}}$ be a saturated desingularization of quasi-compact fine torsion free fans which is functorial with respect to smooth morphisms. Then there exists unique saturated monoidal desingularization $\widetilde{\mathscr{F}}^{\log}(X, M)$ of monoidally torsion free fine log schemes (X, M), such that $\widetilde{\mathscr{F}}^{\log}$ is functorial with respect to all monoidally smooth morphisms and $\widetilde{\mathscr{F}}^{\log}(X, M)$ is the contraction of $c^\star(\widetilde{\mathscr{F}}^{\mathrm{fan}}(F))$ for any log scheme (X, M) that admits a fan chart $c\colon (X, \overline{M}_X) \to F$. In the same way, a functorial desingularization $\mathscr{F}^{\mathrm{fan}}$ induces a monoidal desingularization $\mathscr{F}^{\log}$.*

Remark 3.3.17. Since any monoscheme chart induces a fan chart, it then follows from Remark 3.3.14(iii) that $\widetilde{\mathscr{F}}^{\log}(X, M)$ is the contraction of $d^\star(\widetilde{\mathscr{F}}^{\mathrm{mono}}(F))$ for any log scheme (X, M) that admits a monoscheme chart $d\colon (X, \overline{M}_X) \to F$.

Proof of Theorem 3.3.16. Both cases are established similarly, so we prefer to deal with $\mathscr{F}^{\mathrm{fan}}$ (to avoid mentioning saturations at any step of the proof). By descent, it suffices to show that the pullback from fans induces a functorial monoidal desingularization of those fine log schemes that admit a global fan chart. Thus, if $\mathscr{F}^{\log}$ exists then it is unique, and our aim is to establish existence and functoriality. Both are consequences of the following claim: assume that $f\colon (Y, M_Y) \to (X, M_X)$ is a monoidally smooth morphism whose source and target admit fan charts $d\colon (Y, \overline{M}_Y) \to G$ and

$d'\colon (X,\overline{M}_X) \to F$, then the contractions of $d^\star(\mathscr{F}^{\mathrm{fan}}(G))$ and $c^\star(\mathscr{F}^{\mathrm{fan}}(F))$ are equal, where $c = d' \circ \overline{f}\colon (Y,\overline{M}_Y) \to (X,\overline{M}_X) \to F$. Note that $d^{-1}(\overline{M}_G) \xrightarrow{\sim} \overline{M}_Y$ and the homomorphism $c^{-1}(\overline{M}_F) \to \overline{M}_Y$ is smooth.

Choose a point $y \in Y$ and consider the localizations $Y' = \mathrm{Spec}(\mathscr{O}_{Y,y})$, $F' = \mathrm{Spec}(M_{F,c(y)})$ and $G' = \mathrm{Spec}(M_{G,d(y)})$ at y and its images in the fans. Since $\phi\colon M_{G,d(y)} \to \overline{M}_{Y,y}$ is an isomorphism and the homomorphism $\psi\colon M_{F,c(y)} \to \overline{M}_{Y,y}$ is smooth, we obtain a factorization of ψ into a composition of a homomorphism $\lambda\colon M_{F,c(y)} \to M_{G,d(y)}$ and ϕ, where λ is smooth. Set $U = c^{-1}(F') \cap d^{-1}(G')$. Then U is a neighborhood of y, and c and d induce homomorphisms $\phi_U\colon M_{G,d(y)} \to \overline{M}_Y(U)$ and $\psi_U\colon M_{F,c(y)} \to \overline{M}_Y(U)$. Since the monoids are fine, we can shrink U so that the equality $\psi_U = \phi_U \circ \lambda$ holds. It then follows from Lemma 3.1.8(ii) that $c|_U$ factors into a composition of $d|_U$ and the smooth morphism $\mathrm{Spec}(\lambda)\colon G' \to F'$.

By quasi-compactness of Y we can now find finite coverings $Y = \bigcup_{i=1}^n Y_i$, $F = \bigcup_{i=1}^n F_i$ and $G = \bigcup_{i=1}^n G_i$, and smooth morphisms $\lambda_i\colon G_i \to F_i$ such that Y_i is mapped to F_i and G_i by c and d, respectively, and the induced maps of monoidal spaces $c_i\colon Y_i \to F_i$ and $d_i\colon Y_i \to G_i$ satisfy $c_i = \lambda_i \circ d_i$. Set $Y' = \coprod_{i=1}^n Y_i$, $F' = \coprod_{i=1}^n F_i$, $G' = \coprod_{i=1}^n G_i$, $c'\colon Y' \to F$ and $d'\colon Y' \to G$. By descent, it suffices to check that contractions of $c'^\star(\mathscr{F}^{\mathrm{fan}}(F))$ and $d'^\star(\mathscr{F}^{\mathrm{fan}}(G))$ are equal. Since, the morphism $Y' \to F$ factors through the surjective smooth morphism $F' \to F$, and similarly for G, these two pullbacks are equal to the contracted pullbacks of $\mathscr{F}^{\mathrm{fan}}(F')$ and $\mathscr{F}^{\mathrm{fan}}(G')$, respectively. It remains to note that $Y' \to F'$ factors through the smooth morphism $\coprod_{i=1}^n \lambda_i\colon G' \to F'$. Hence $\mathscr{F}^{\mathrm{fan}}(G')$ is the contracted pullback of $\mathscr{F}^{\mathrm{fan}}(F')$, and their contracted pullbacks to Y' coincide. □

3.4. Desingularization of log regular log schemes. In this section we will see how saturated monoidal desingularization leads to normalized desingularization of log regular log schemes. Up to now we freely considered saturated and unsaturated cases simultaneously, and did not feel any essential difference. This will not be the case in the present section because the notion of log regularity was developed by Kato and Nizioł in the saturated case. Gabber generalized the definition to the non-saturated case and extended to that case all main results about log regular log schemes. This was necessary for his original approach, but can be bypassed by use of saturated monoidal desingularization. So, we prefer to stick to the saturated case and simply refer to all foundational results about log regular fs log schemes to [**Kato, 1994**] and [**Nizioł, 2006**]. For the sake of completeness, we will outline Gabber's results about the general case in §3.5.

3.4.1. *Conventions.* Recall that Kato's notion of log regular fs log schemes was already used in **VI**-1.2. Throughout §3.4 we assume that (X, M_X) is a log regular fs log scheme. Note that the homomorphism $\alpha_X\colon M_X \to \mathscr{O}_X$ of $X_{\text{ét}}$-sheaves is injective by [**Nizioł, 2006**, 2.6], and actually $M_X = \mathscr{O}_U^* \cap \mathscr{O}_X$, where $U \subset X$ is the triviality locus of M_X. So, we will freely identify M_X with a multiplicative submonoid of $\mathscr{O}_X$.

3.4.2. *Monoidal ideals.* For any log ideal $\mathscr{I} \subset M_X$ consider the ideal $\mathscr{J} = \alpha(\mathscr{I})\mathscr{O}_X$ it generates. We call $\mathscr{J}$ a **monoidal ideal** and by a slight abuse of notation, we will write $\mathscr{J} = \mathscr{I}\mathscr{O}_X$.

Lemma 3.4.3. *Let (X, M_X) be as in 3.4.1. The rules $\mathscr{I} \mapsto \mathscr{I}\mathscr{O}_X$ and $\mathscr{K} \mapsto \mathscr{K} \cap M_X$ give rise to a one-to-one correspondence between log ideals $\mathscr{I} \subset M_X$ and monoidal ideals $\mathscr{K} \subset \mathscr{O}_X$.*

Proof. It suffices to show that any log ideal $\mathscr{I}$ coincides with $\mathscr{J} = \mathscr{I}\mathscr{O}_X \cap M_X$. Furthermore, it suffices to check the equality at the strict localizations of X, hence we can assume that $X = \mathrm{Spec}(A)$ for a strictly local ring A. Then the log structure admits a chart $X \to \mathrm{Spec}(\mathbf{Z}[P])$ and $\mathscr{I} = IM_X$ for an ideal $I \subset P$, and we should prove that $J := \mathscr{J}(X) = IA \cap P$ coincides with I.

Assume on the contrary that $I \subsetneq J$, and consider the exact sequence

$$I\mathbf{Z}[P] \otimes_{\mathbf{Z}[P]} A \to J\mathbf{Z}[P] \otimes_{\mathbf{Z}[P]} A \to J\mathbf{Z}[P]/I\mathbf{Z}[P] \otimes_{\mathbf{Z}[P]} A \to 0. \tag{3.4.3.1}$$

Since $\mathrm{Tor}_1^{\mathbf{Z}[P]}(\mathbf{Z}[P]/I\mathbf{Z}[P], A) = 0$ by [**Kato, 1994**, 6.1(ii)], $I\mathbf{Z}[P] \otimes_{\mathbf{Z}[P]} A = IA$ and similarly for J, and we obtain that the first morphism in the sequence (3.4.3.1) is $IA \to JA$. To obtain a contradiction, it suffices to show that $J\mathbf{Z}[P]/I\mathbf{Z}[P] \otimes_{\mathbf{Z}[P]} A \neq 0$. Note that $\mathbf{Z}[P]/\mathbf{Z}[m_P]$ is a quotient of $J\mathbf{Z}[P]/I\mathbf{Z}[P] = \mathbf{Z}[J]/\mathbf{Z}[I]$, so it remains to note that $m_P A \neq A$ and hence $\mathbf{Z}[P]/m_p\mathbf{Z}[P] \otimes_{\mathbf{Z}[P]} A \neq 0$. $\square$

3.4.4. *Interpretation of monoidal smoothness.* Note that by **VI**-1.7 (X, M_X) is monoidally smooth if and only if X is regular and in this case the non-triviality locus of M_X is a normal crossings divisor D.

3.4.5. *Saturated log blow ups of log regular log schemes.* Using Kato's Tor-independence result [**Kato, 1994**, 6.1(ii)] Nizioł proved in [**Nizioł, 2006**, 4.3] that saturated log blow ups of (X, M_X) are compatible with normalized blow ups along monoidal ideals. Namely, if $(Y, M_Y) = \mathrm{LogBl}_{\mathscr{I}}(X, M_X)^{\mathrm{sat}}$ then $Y \xrightarrow{\sim} \mathrm{Bl}_{\mathscr{I}\mathscr{O}_X}(X)^{\mathrm{nor}}$. We will also need more specific results that showed up in the proof of loc.cit., so we collect them altogether in the following lemma.

Lemma 3.4.6. *Let $f : (X, M_X) \to (Y, M_Y)$ be a strict morphism of fs log regular log schemes, let $\mathscr{I} \subset M_Y$ be a log ideal and $\mathscr{J} = f^{-1}\mathscr{I}M_X$. Set $(X', M_{X'}) = \mathrm{LogBl}_{\mathscr{J}}(X, M_X)$, $(X'', M_{X''}) = (X', M_{X'})^{\mathrm{sat}}$, $(Y', M_{Y'}) = \mathrm{LogBl}_{\mathscr{I}}(Y, M_Y)$ and $(Y'', M_{Y''}) = (Y', M_{Y'})^{\mathrm{sat}}$. Then*

(i) The (saturated) log blow up of (X, M_X) is compatible with (normalized) blow up of X: $X' \xrightarrow{\sim} \mathrm{Bl}_{\mathscr{J}\mathscr{O}_X}(X)$ and $X'' \xrightarrow{\sim} \mathrm{Bl}_{\mathscr{J}\mathscr{O}_X}(X)^{\mathrm{nor}}$.

(ii) The (normalized) blow up of X along $\mathscr{J}\mathscr{O}_X$ is the pullback of the (normalized) blow up of Y along $\mathscr{I}\mathscr{O}_Y$. In particular, $X' \xrightarrow{\sim} X \times_Y Y'$ and $X'' \xrightarrow{\sim} X \times_Y Y''$.

Proof. All claims can be checked étale locally, hence we can assume that there exists a chart $g : (Y, M_Y) \to (\mathbf{Z}[P], P)$ and $\mathscr{I} = g^{-1}(I_0)M_Y$ for an ideal $I_0 \subset P$. Then it suffices to prove (ii) for g and the induced chart $g \circ f$ of (X, M_X). In particular, this reduces the lemma to the particular case when $X = \mathrm{Spec}(A)$ and f is a chart $(X, M_X) \to (\mathbf{Z}[P], P)$. It is shown in the first part of the proof of [**Nizioł, 2006**, 4.3] that

$$X' \xrightarrow{\sim} \mathrm{Proj}(A \otimes_{\mathbf{Z}[P]} (\bigoplus_{n=0}^{\infty} I_0^n)) \xrightarrow{\sim} \mathrm{Proj}(\bigoplus_{n=0}^{\infty} \mathscr{J}^n)$$

The first isomorphism implies that $X' \to X$ is the base change of $\mathrm{Proj}(\bigoplus_{n=0}^{\infty} I_0^n) = Y' \to Y$, and the second isomorphism means that $X' \xrightarrow{\sim} \mathrm{Bl}_{\mathscr{J}\mathscr{O}_X}(X)$. This establishes the unsaturated and unnormalized part of the Lemma, and the second part follows in the same way from the second part of the proof of [**Nizioł, 2006**, 4.3]. $\square$

Remark 3.4.7. It follows from the lemma that the unsaturated log blow up $(X', M_{X'})$ is log regular in the sense of Gabber, see §3.5. Thus, once log regularity is correctly defined in full generality, it becomes a property preserved by log blow ups (as it should be, since log blow ups are log smooth).

3.4.8. *Desingularization of log regular log schemes.* By Lemma 3.4.6, any saturated log blow up tower $f : (X_n, M_n) \dashrightarrow (X, M_X)$ induces a normalized blow up tower $g : X_n \dashrightarrow X$ of the underlying schemes. Furthermore, g completely determines f as follows: if $X_{i+1} \to X_i$ is the normalized blow up along $\mathscr{I} \subset \mathscr{O}_{X_i}$ then $(X_{i+1}, M_{X_{i+1}}) \to (X_i, M_{X_i})$ is the saturated log blow up along $\mathscr{I} \cap M_{X_i}$ by Lemma 3.4.3. The convention that centers of blow ups are part of the data is used here essentially. Furthermore, by 3.4.4, f is a saturated monoidal desingularization if and only if g is a normalized desingularization. In particular, the saturated monoidal desingularization $\widetilde{\mathscr{F}}^{\log}(X, M_X)$ induces a normalized desingularization of the scheme X that depends on (X, M_X) and will be denoted $\widetilde{\mathscr{F}}(X, M_X)$.

Theorem 3.4.9. *The saturated monoidal desingularization $\widetilde{\mathscr{F}}^{\log}$ (see Theorem 3.3.16) gives rise to desingularization $\widetilde{\mathscr{F}}$ of log regular log schemes that possesses the same functoriality properties: if $\phi : (Y, M_Y) \to (X, M_X)$ is a monoidally smooth morphism of log regular log schemes then $\widetilde{\mathscr{F}}(Y, M_Y)$ is the contraction of $\phi^{\mathrm{st}}(\widetilde{\mathscr{F}}(X, M_X))$. Furthermore, if ϕ is strict then $\phi^{\mathrm{st}}(\widetilde{\mathscr{F}}(X, M_X)) = \widetilde{\mathscr{F}}(X, M_X) \times_X Y$.*

Strictness of ϕ in the last claim is not needed. To remove it one should work out the assertion of Remark 3.1.23.

Proof. We only need to establish functoriality. Let $(X', M_{X'}) = \mathrm{LogBl}_{\mathscr{I}}(X, M_X)^{\mathrm{sat}}$ be the first saturated blow up of $\widetilde{\mathscr{F}}^{\log}(X, M_X)$. Set $\mathscr{J} = \phi^{-1}\mathscr{I}M_Y$, then $(Y', M_{Y'}) = \mathrm{LogBl}_{\mathscr{J}}(Y, M_Y)^{\mathrm{sat}}$ is the first saturated blow up of $\widetilde{\mathscr{F}}^{\log}(Y, M_Y)$ by functoriality of $\widetilde{\mathscr{F}}^{\log}$. By Lemma 3.4.6, $X' = \mathrm{Bl}_{\mathscr{I}\mathscr{O}_X}(X)^{\mathrm{nor}}$ and $Y' = \mathrm{Bl}_{\mathscr{J}\mathscr{O}_Y}(Y)^{\mathrm{nor}}$, and using that $\phi^{-1}(\mathscr{I}\mathscr{O}_X)\mathscr{O}_Y = \mathscr{J}\mathscr{O}_Y$ we obtain that Y' is the strict transform of X'. It remains to inductively apply the same argument to the other levels of the towers. The last claim follows from Lemma 3.4.6(ii). $\square$

Remark 3.4.10. The same results hold for (non-saturated) monoidal desingularization, which induces a (usual) desingularization of log regular log schemes. For non-saturated log regular log schemes (see §3.5) one should first establish analogs of Lemmas 3.4.3 and 3.4.6 (where the input in the second one does not have to be saturated). Then the same proof as above applies.

3.4.11. *Canonical fans and associated points.* By the **canonical fan** $\mathrm{Fan}(X)$ of (X, M_X) we mean the set of maximal points of the log stratification (i.e. the maximal points of the log strata). Alternatively, $\mathrm{Fan}(X)$ can be described as in [**Nizioł, 2006**, §2.2] as the set of points $x \in X$ such that $m_{\overline{x}}$ coincides with the ideal $I_{\overline{x}} \subset \mathscr{O}_{\overline{x}}$ generated by $\alpha(M_{\overline{x}} - M^*_{\overline{x}})$.

We provide $F = \mathrm{Fan}(X)$ with the induced topology and define M_F to be the restriction of $\overline{M}_X$ onto F. For example, for a toric k-variety $X = \mathrm{Spec}(k[Z])$, where Z is a monoscheme, (F, M_F) is isomorphic to the sharpening of Z. More generally, if a log scheme (X, M_X) is Zariski then (F, M_F) is a fan and the map $c\colon X \to F$ sending any point to the maximal point of its log stratum is a fan chart of X. This follows easily from the fact that such (X, M_X) admits monoscheme charts Zariski locally.

Remark 3.4.12. In general, (F, M_F) does not have to be a fan, but it seems probable that it can be extended to a meaningful object playing the role of algebraic spaces in the category of fans. We will not investigate this direction here.

Lemma 3.4.13. *Let $X = (X, M_X)$ be an fs log regular log scheme with a monoidal ideal $\mathscr{I} \subset \mathscr{O}_X$. Then:*
(i) The set of associated points of $\mathscr{O}_X/\mathscr{I}$ is contained in $\mathrm{Fan}(X)$.
(ii) The fans of $\mathrm{Bl}_{\mathscr{I}}(X)$ and $\mathrm{Bl}_{\mathscr{I}}(X)^{\mathrm{nor}}$ are contained in the preimage of $\mathrm{Fan}(X)$.
(iii) For any tower of monoidal blow ups (resp. normalized monoidal blow ups) $X_n \dashrightarrow X$, its set of associated points is contained in $\mathrm{Fan}(X)$.

Proof. (i) Fix a point $x \in X - \mathrm{Fan}(X)$ and let us prove that it is not an associated point of $\mathscr{O}_X/\mathscr{I}$. Since associated points are compatible with flat morphisms, we can pass to the formal completion $\widehat{X}_x = \mathrm{Spec}(\widehat{\mathscr{O}}_{X,x})$. Let us consider first the more difficult case when $A = \widehat{\mathscr{O}}_{X,x}$ is of mixed characteristic $(0, p)$. By **VI**-1.6, $A \xrightarrow{\sim} B/(f)$ where $B = C(k)[[Q]][[\underline{t}]]$, Q is a sharp monoid defining the log structure, $\underline{t} = (t_1, \dots, t_n)$, and $f \in B$ reduces to p modulo $(Q \setminus \{1\}, \underline{t})$. Note that $n \geq 1$ as otherwise $Q \setminus \{1\}$ would generate the maximal ideal of A. In this case, x is the log stratum of $\widehat{X}_x$ and hence x is the maximal point of its log stratum in X_x, which contradicts that $x \notin \mathrm{Fan}(X)$. The completion $\widehat{\mathscr{I}}$ of $\mathscr{I}$ is of the form IA for an ideal $I \subset Q$. We should prove that $x \in \widehat{X}_x$ is not an associated point of $A/\widehat{\mathscr{I}}$, or, equivalently, $\mathrm{depth}(A/\widehat{\mathscr{I}}) \geq 1$. Since $B/(fB + \widehat{\mathscr{I}}B) = A/\widehat{\mathscr{I}}$, it suffices to show that $\mathrm{depth}(B/\widehat{\mathscr{I}}B) \geq 2$ and f is a regular element of $B/\widehat{\mathscr{I}}B$.

Regularity of f follows from the following easy claim by taking $C = C(k)$, $J = I$ and $R = Qt_1^{\mathbf{N}} \dots t_n^{\mathbf{N}}$: if C is a domain, R is a sharp fine monoid with an ideal J, f is an element of $C[[R]]$ with a non-zero free term (i.e. $c_1 \neq 0$, where $f = \sum_{r \in R} c_r r$), and $g \in C[[R]]$ satisfies $fg \in JC[[R]]$ then $g \in JC[[R]]$. Next, let us bound the depth of $B/\widehat{\mathscr{I}}B$. In view of [**Matsumura, 1980a**, 21.C], depth is preserved by completions of local rings hence it suffices to show that $\mathrm{depth}(D_r/ID_r) \geq 2$, where $D = C(k)[Q][\underline{t}]$ and $r = (m_Q, p, \underline{t})$ is the ideal generated by $m_Q = Q \setminus \{1\}$, p and $\underline{t}$. Note that $D/ID \xrightarrow{\sim} C(k)[\underline{t}][Q \setminus I]$ as a $C(k)[\underline{t}]$-module, hence it is a flat $C(k)[\underline{t}]$-module and the local homomorphism $C(k)[\underline{t}]_{(p,\underline{t})} \to D_r/ID_r$ is flat. By [**Matsumura, 1980a**, 21.C], $\mathrm{depth}(D_r/ID_r) \geq \mathrm{depth}(C(k)[\underline{t}]_{(p,\underline{t})}) = n + 1 \geq 2$, so we have established the mixed characteristic case. In the equal characteristic case we have that $A = k[[Q]][[\underline{t}]]$, and the same argument shows that $\mathrm{depth}(A/\widehat{\mathscr{I}}) \geq \mathrm{depth}(k[\underline{t}]_{(\underline{t})}) = n \geq 1$.

To prove (ii) we should check that if $x \in X - \mathrm{Fan}(X)$ then no point of $\mathrm{Fan}(\mathrm{Bl}_{\mathscr{I}}(X))$ sits over x. We will only check the mixed characteristic case since it is more difficult. As earlier, $\widehat{X}_x = \mathrm{Spec}(A)$ where $A = B/(f)$ and $B = C(k)[[Q]][[\underline{t}]]$ with $n \geq 1$. Note that $\psi\colon \widehat{X}_x \to X$ is a flat strict morphism of log schemes. Hence ψ is compatible with blow ups and it maps the fans of $\widehat{X}_x$ and $\mathrm{Bl}_{\widehat{\mathscr{I}}}(\widehat{X}_x)$ to the fans of X_x and $\mathrm{Bl}_{\mathscr{I}}(X)$, respectively. Therefore, we should only check that $\mathrm{Fan}(\mathrm{Bl}_{\widehat{\mathscr{I}}}(\widehat{X}_x))$ is disjoint from the preimage of x. The latter blow up is covered by the charts $V_a = \mathrm{Spec}(A[a^{-1}\widehat{\mathscr{I}}])$ with $a \in I$. Set $P' = Q[a^{-1}I]$, $B' = C(k)[[P']][[t_1,\dots,t_n]]/(f)$ and $A' = B'/(f)$ (where f is as above). Then the m_x-adic completion of V_a is $\widehat{V}_a = \mathrm{Spec}(A')$. The ideal defining the closed point of $\mathrm{Fan}(\widehat{V}_a)$ is generated by the maximal ideal m' of P'. This ideal does not contain any t_i. Indeed, $t_i \notin m'B' + fB'$ because $t_i \notin fC(k)[[t_1,\dots,t_n]]$ as $f-p \in (t_1,\dots,t_n)$. Thus, $\mathrm{Fan}(\widehat{V}_a)$ is disjoint from the preimage of x, and hence the same is true for $\mathrm{Fan}(V_a)$. This proves (ii) in the non-saturated case, and the saturated case is dealt with similarly but with P' replaced by its saturation. Finally, (iii) follows immediately from (i) and (ii). □

3.4.14. *Independence of the log structure.* Dependence of $\widetilde{\mathscr{F}}(X,M_X)$ on M_X is a subtle question. In this section we will use functoriality of $\widetilde{\mathscr{F}}$ to prove that $\widetilde{\mathscr{F}}(X,M_X)$ is independent of M_X in characteristic zero. This will cover our needs, but the restriction on the characteristic will complicate our arguments later. Conjecturally, $\widetilde{\mathscr{F}}(X,M_X)$ does not depend on M_X at all and the following result of Gabber supports this conjecture: if P and Q are two fine sharp monoids and $k[[P]] \simeq k[[Q]]$ for a field k (of any characteristic!) then $P \simeq Q$. For completeness, we will give a proof of this in §3.6.

Theorem 3.4.15. *Let $\widetilde{\mathscr{F}}$ be a functorial normalized desingularization of reduced qe schemes of characteristic zero, and let $\widetilde{\mathscr{F}}(X,M_X)$ be the normalized desingularization of log regular log schemes it induces* (*see Theorem 3.4.9*). *Assume that (X,M_X) and (Y,M_Y) are saturated log regular log schemes such that there exists an isomorphism $\phi\colon Y \xrightarrow{\sim} X$ of the underlying schemes. Assume also that the maximal points of the strata of the stratifications of X and Y by the rank of $\overline{M}^{\mathrm{gp}}$ are of characteristic zero. Then $\widetilde{\mathscr{F}}(X,M_X)$ and $\widetilde{\mathscr{F}}(Y,M_Y)$ are compatible with ϕ, that is, $\widetilde{\mathscr{F}}(Y,M_Y) = \phi^\star(\widetilde{\mathscr{F}}(X,M_X))$.*

Proof. We can check the assertion of the theorem étale locally. Namely, we can replace X with a strict étale covering X' and replace Y with $Y' = Y \times_X X'$ with the log structure induced from Y. In particular, we can assume that the log structures are Zariski, and so the canonical fans $\mathrm{Fan}(X)$ and $\mathrm{Fan}(Y)$ are defined. Our assumption on the maximal points actually means that $\mathrm{Fan}(X)$ is contained in $X_{\mathbf{Q}} = X \otimes_{\mathbf{Z}} \mathbf{Q}$. By Lemma 3.4.13(iii) and 2.2.11, $\widetilde{\mathscr{F}}(X,M_X)$ is the pushforward of its restriction onto $X_{\mathbf{Q}}$, and similarly for Y. So, it suffices to prove that the normalized desingularizations of $X_{\mathbf{Q}}$ and $Y_{\mathbf{Q}}$ are compatible with respect to $\phi \otimes_{\mathbf{Z}} \mathbf{Q}\colon X_{\mathbf{Q}} \xrightarrow{\sim} Y_{\mathbf{Q}}$. Thus, it suffices to prove the theorem for X and Y of characteristic zero, and, in the sequel, we assume that this is the case.

To simplify notation we identify X and Y, and set $N_X = M_Y$. It suffices to check that the blow up towers $\widetilde{\mathscr{F}}(X,M_X)$ and $\widetilde{\mathscr{F}}(X,N_X)$ coincide after the base change to each completion $\widehat{X}_{\overline{x}} = \mathrm{Spec}(\widehat{\mathscr{O}}_{X,\overline{x}})$ at a geometric point $\overline{x}$. By **VI**-1.6, we have that $\widehat{X}_{\overline{x}} \xrightarrow{\sim} \mathrm{Spec}(k[[P]][[t_1,\dots,t_n]])$, where $P \xrightarrow{\sim} \overline{M}_{X,\overline{x}}$, and the morphism of fs log schemes $(\widehat{X}_{\overline{x}},P) \to (X,M_X)$ is strict. By Theorem 3.3.16 the contracted pullback of $\widetilde{\mathscr{F}}(X,M_X)$ to $\widehat{X}_{\overline{x}}$ coincides with $\widetilde{\mathscr{F}}^{\log}(\widehat{X}_{\overline{x}},P)$. In the same way, the contracted pullback of $\widetilde{\mathscr{F}}(X,N_X)$ coincides with $\widetilde{\mathscr{F}}^{\log}(\widehat{X}_{\overline{x}},Q)$, where $\widehat{\mathscr{O}}_{X,\overline{x}} \xrightarrow{\sim} \mathrm{Spec}(k[[Q]][[t_1,\dots,t_m]])$ (we use that k is isomorphic to the residue field of this ring and hence depends only on the ring $\widehat{\mathscr{O}}_{X,\overline{x}}$).

Let us now recall how $\widetilde{\mathscr{F}}^{\log}(\widehat{X}_{\overline{x}},P)$ is constructed (Theorems 3.1.27, 3.2.20 and 3.3.16). We have the obvious strict morphism $\widehat{X}_{\overline{x}} \to Z := \mathrm{Spec}(\mathbf{Q}[P][t_1,\dots,t_n])$, hence $\widetilde{\mathscr{F}}^{\log}(\widehat{X}_{\overline{x}},P)$ is the pullback of $\widetilde{\mathscr{F}}^{\log}(Z,P)$. The latter is the pullback of $\widetilde{\mathscr{F}}^{\mathrm{mono}}(P) = \widetilde{\mathscr{F}}^{\mathrm{fan}}(P)$, which, in its turn, is induced from $\widetilde{\mathscr{F}}(Z)$. Therefore, $\widetilde{\mathscr{F}}(Z,P) = \widetilde{\mathscr{F}}(Z)$ and, by the functoriality of $\widetilde{\mathscr{F}}$, its pullback to $\widehat{X}_{\overline{x}}$ is $\widetilde{\mathscr{F}}(\widehat{X}_{\overline{x}})$. The same argument shows that $\widetilde{\mathscr{F}}(\widehat{X}_{\overline{x}})$ is the contracted pullback of $\widetilde{\mathscr{F}}(X,N_X)$.

In order to prove that $\widetilde{\mathscr{F}}(X,M_X) = \widetilde{\mathscr{F}}(X,N_X)$ it only remains to resolve the synchronization issues, i.e. to prove equality without contractions. For this one should take the union S of the fans of (X,M_X) and (X,N_X), and consider the morphism $\widehat{X}_S := \coprod_{x\in S}\widehat{X}_{\overline{x}}$ rather than the individual completions. The pullbacks of $\widetilde{\mathscr{F}}(X,M_X)$ and $\widetilde{\mathscr{F}}(X,N_X)$ to $\widehat{X}_S$ have no empty blow ups because the fans contain all associated points of the towers by Lemma 3.4.13(iii). Hence the same argument as above shows that they both coincide with $\widetilde{\mathscr{F}}(\widehat{X}_S)$. □

REMARK 3.4.16. (i) Without taking the completion, $\widetilde{\mathscr{F}}(X)$ does not even have to be defined as X may be non-qe. In order to pass to the completion we used functoriality of the monoidal desingularization with respect to strict morphisms, which may be very bad (e.g. non-flat) on the level of usual morphisms of schemes. Even when (X, M_X) is log regular, the formal completion morphism $\widehat{X}_x \to X$ can be non-regular in the non-qe case. However, one can show that it is regular over the fan, and this is enough to relate the (log) desingularization of X and $\widehat{X}_x$.

(ii) We used the very strong desingularization $\widetilde{\mathscr{F}}$ from Theorem 2.3.10. However, it is easy to see that only the properties listed in Theorem 2.3.7 were essential.

3.5. Complements on non-saturated log regular log schemes. For the sake of completeness, we mention Gabber's results on non-saturated log regular log schemes that will not be used. We only formulate results but do not give proofs. Gabber defines a fine log scheme (X, M_X) to be **log regular** if its saturation is log regular in the usual sense. Assume now that (X, M_X) is fine and log regular. The key result that lifts the theory off the ground is that Kato's Tor independence extends to non-saturated log regular log schemes. Namely, if (X, M_X) admits a chart $\mathbf{Z}[P]$ and $I \subset P$ is an ideal then $\mathrm{Tor}_1^{\mathbf{Z}[P]}(\mathscr{O}_X, \mathbf{Z}[P]/I) = 0$. For fs log schemes this is due to Kato, and Gabber deduces the general case using a non-flat descent. It then follows similarly to the saturated case that if $(Y, M_Y) = \mathrm{LogBl}_{\mathscr{I}}(X, M_X)$ then $Y \xrightarrow{\sim} \mathrm{Bl}_{\mathscr{I}\mathscr{O}_X}(X)$ and (Y, M_Y) is log regular. In addition, one shows that (X, M_X) is saturated if and only if X is normal, and if $(Y, M_Y) = (X, M_X)^{\mathrm{sat}}$ then $Y \xrightarrow{\sim} X^{\mathrm{nor}}$. Using these foundational results on log regular fine log schemes one can imitate the method of §3.4 to extend Lemma 3.4.13 to the non-saturated case. As a corollary, one obtains an analog of Theorem 3.4.15 for $\mathscr{F}$ and $\mathscr{F}^{\log}$.

3.6. Reconstruction of the monoid. This section will not be used in the sequel. Its aim is to prove that a fine torsion free monoid P can be reconstructed from a ring $A = k[[P]]$ where k is a field. The main idea of the proof is that an isomorphism $k[[P]] \xrightarrow{\sim} A$ defines an action of the torus $\mathrm{Spec}(k[P^{\mathrm{gp}}])$ on A, and any two maximal tori in $Aut_{k-\mathrm{aug}}(A)$ are conjugate.

3.6.1. *Automorphism groups of complete rings.* Let k be a field and A be a complete local noetherian k-algebra with residue field k. Let m denote the maximal ideal and set $A_n = A/m^{n+1}$. Let $G_n = Aut_{k-\mathrm{aug}}(A_n)$ be the automorphism group scheme of A_n viewed as an augmented k-algebra, i.e.

$$G_n(B) = \{\sigma \in \mathrm{Aut}_B(A_n \otimes_k B) |\ \sigma(mA_n \otimes_k B) = mA_n \otimes_k B\}.$$

This is a closed subgroup of the automorphism group scheme $Aut_k(A_n)$ defined by a nilpotent ideal. The groups G_n form a filtering projective system $\cdots \to G_2 \to G_1 \to G_0$; we call such a system an algebraic pro-group. Note that $G_\bullet$ induces a set-valued functor $G(B) = \lim_n G_n(B)$ on the category of k-algebras, and $G(k) = \mathrm{Aut}_k(A)$.

REMARK 3.6.2. Gabber also considered more complicated filtering families, but we stick to the simplest case we need.

3.6.3. *Stabilization.* We say that an algebraic pro-group $G_\bullet$ is **stable** if the homomorphisms $\pi_n : G_{n+1} \to G_n$ are surjective for large enough n. Any algebraic group can be stabilized as follows. For each G_n and $i \geq 0$ let $G_{n,i}$ denote the scheme-theoretic image of G_{n+i} in G_n. Then $G_{n,0} \supset G_{n,1} \supset \ldots$ is a decreasing sequence of algebraic subgroups of G, hence it stabilizes on a subgroup $G_n^{\mathrm{st}} \subset G_n$. The family $G_\bullet^{\mathrm{st}}$ with obvious transition morphisms is an algebraic pro-subgroup of $G_\bullet$, and it is clear from the definition that $G_\bullet^{\mathrm{st}}$ is stable. Note also that $G_\bullet^{\mathrm{st}}$ is isomorphic to $G_\bullet$ at least in the sense that $G(B) \xrightarrow{\sim} \lim_n G_n^{\mathrm{st}}(B)$ for any k-algebra B.

3.6.4. *Maximal pro-tori.* By a **pro-torus** $T_\bullet$ in $G_\bullet$ we mean a *compatible* family of tori $T_n \hookrightarrow G_n$ for all $n \geq 0$, in the sense that $\pi_n(T_{n+1}) = T_n$. It is called a **torus** if the π_n's are isomorphisms for $n \gg 0$. A pro-torus is **componentwise maximal** if for $n \gg 0$ all tori T_n are maximal. A pro-torus $T_\bullet$ is **maximal** if for any inclusion of pro-tori $T_\bullet \subset T'_\bullet$ we have that $T_n = T'_n$ for $n \gg 0$. In particular, any componentwise maximal pro-torus is maximal. If k is algebraically closed and $G_{n+1} \to G_n$ is surjective then any torus $S_n \subset G_n$ is the image of a torus $S_{n+1} \subset G_{n+1}$, and hence componentwise maximal pro-tori exist whenever $G_\bullet$ is stable. For shortness, given an element $c \in G_n(k)$ we will denote the corresponding conjugation by $c\colon G_n \to G_n$. Pro-tori $T_\bullet$ and $T'_\bullet$ are **conjugate** if there exists a *compatible* family of elements $c_n \in G_n(k)$ such that for $n \gg 0$ the conjugation $c_n\colon G_n \to G_n$ takes T_n to T'_n.

Proposition 3.6.5. *Assume that k is an algebraically closed field, $G_\bullet$ is a stable pro-algebraic k-group, and $T_\bullet, T'_\bullet \hookrightarrow G_\bullet$ are pro-tori. If $T_\bullet$ is componentwise maximal then $T'_\bullet$ is conjugate to a sub-pro-torus of $T_\bullet$. In particular, a pro-torus is maximal if and only if it is componentwise maximal, and any two maximal pro-tori are conjugate.*

Proof. It is a classical result that maximal tori in algebraic k-groups are conjugate. In particular, for each n we can move T'_n into T_n by a conjugation, and the only issue is compatibility of the conjugations. Naturally, to achieve compatibility we should lift conjugations inductively from G_n to G_{n+1}. It suffices to prove that if π_n is surjective and $c_n\colon G_n \to G_n$ conjugates T'_n into T_n then it lifts to $c_{n+1}\colon G_{n+1} \to G_{n+1}$ that conjugates T'_{n+1} into T_{n+1}. By the stability assumption, we can lift c_n to a conjugation c' of G_{n+1}. It takes T'_{n+1} to the subgroup $H = KT_{n+1}$, where K is the kernel of π_n. Since maximal tori in H are conjugate and conjugation by elements of T_{n+1} preserves T_{n+1}, we can find a conjugation c'' by an element of K that takes $c'(T'_{n+1})$ to T_{n+1}. Then $c_{n+1} = c''c'$ is a lifting of c_n as required. □

Corollary 3.6.6. *Assume that k is an algebraically closed field, $G_\bullet$ is a pro-algebraic k-group, and $T_\bullet, T'_\bullet \hookrightarrow G_\bullet$ are pro-tori. If $T_\bullet$ is maximal then $T'_\bullet$ is conjugate to a sub-pro-torus of $T_\bullet$.*

Proof. Obviously, $T_n, T'_n \subset G_n^{\mathrm{st}}$ for all $n \geq 0$. So, $T'_\bullet$ is conjugate to a subtorus of $T_\bullet$ already inside of G^{st} by Proposition 3.6.5. □

3.6.7. *Certain tori in $Aut_{k-\mathrm{aug}}(A)$.* Any k-isomorphism $C[[P]] \xrightarrow{\sim} A$, where C is a complete local k-algebra and P is a sharp fine monoid, induces a pro-algebraic action of the split torus $T_P = \mathrm{Spec}(k[P^{\mathrm{gp}}])$ on A: a character $\chi\colon P \to k^*$ acts on C trivially and acts on $p \in P$ by $p \mapsto \chi(p)p$ (and the action of B-points $\chi\colon P \to B^*$ is analogous). Thus we obtain homomorphisms $\psi\colon T_P \to G_n$ which are monomorphisms for $n > 0$. In particular, the image is a split torus of G. Furthermore, we claim that if $C = k$ then the torus is maximal (as a pro-torus). Indeed, if $\phi \in \mathrm{Aut}_k(A)$ commutes with T_P then its action on $k[[P]]$ preserves the T_P-eigenspaces pk for $p \in P$ and on each pk it acts by multiplication by a number $\lambda(p)$. Clearly, $\lambda\colon P \to k^*$ is a homomorphism and we obtain that ϕ belongs to $\psi(T_P(k))$ and corresponds to $\lambda \in T_P(k)$.

Theorem 3.6.8. *Assume that k is an algebraically closed field, P is a sharp fine monoid and $A = k[[P]]$. If C is a complete local k-algebra, Q is a sharp fine monoid and $C[[Q]] \to A$ is a k-isomorphism then $C \simeq k[[R]]$ and $P \simeq Q \times R$ for a sharp fine monoid R. In particular, P is uniquely determined by A.*

Proof. Consider $G = Aut_{k-\mathrm{aug}}(A)$ with split tori $T_P, T_Q \hookrightarrow G$ corresponding to these isomorphisms. By maximality of T_P and Corollary 3.6.6 there exists a conjugation of G that maps T_Q into T_P. This produces a new isomorphism $C[[Q]] \xrightarrow{\sim} A = k[[P]]$ that respects the grading, i.e. each pk lies in some qC, and we obtain a surjective map $f\colon P \to Q$, which is clearly a homomorphism. If $C = k$ then f is an isomorphism and we obtain that P is determined by A.

Set $R_q = \prod_{p \in f^{-1}(q)} pk$. We have natural embeddings $\prod_{p \in f^{-1}(q)} pk \hookrightarrow qC$ which are all isomorphisms because $A = \prod_{q \in Q} R_q$ is isomorphic to $C[[Q]] = \prod_{q \in Q} qC$. In particular, for $R = f^{-1}(1)$ we have that $C = \prod_{q \in R} qk = k[[R]]$. Therefore, $A \simeq k[[R]][[Q]] \simeq k[[R \times Q]]$, and since the monoid is determined by A we obtain that $P \simeq Q \times R$. □

Corollary 3.6.9. *Assume that P and Q are sharp fine monoids and k is a field such that $k[[P]]$ is k-isomorphic to $k[[Q]]$. Then P is isomorphic to Q.*

Proof. Observe that $\overline{k}[[P]]$ is isomorphic to $\overline{k}[[Q]]$ and use the above theorem. □

4. Proof of Theorem 1.1 — preliminary steps

The goal of §4 is to reduce the proof of Theorem 1.1 to the case when the following conditions are satisfied: (1) X is regular, (2) the log structure is given by an snc divisor Z which is G**-strict** in the sense that for any $g \in G$ and a component Z_i either $gZ_i = Z_i$ or $gZ_i \cap Z_i = \emptyset$, (3) G acts freely on $X \setminus Z$ and for any geometric point $\overline{x} \to X$ the inertia group $G_{\overline{x}}$ is abelian.

4.1. Plan. A general method for constructing a G-equivariant morphism f as in Theorem 1.1 is to construct a tower of G-equivariant morphisms of log schemes $X' = X_n \dashrightarrow X_0 = X$, where the underlying morphisms of schemes are normalized blow ups along G-stable centers sitting over $Z \cup T$, such that various properties of the log scheme X_i with the action of G gradually improve to match all assertions of the Theorem. To simplify notation, we will, as a rule, replace X with the new log scheme after each step. The three conditions above will be achieved in three steps as follows.

4.1.1. *Step 1. Making X regular.* This is achieved by the saturated log blow up tower $\widetilde{\mathscr{F}}(X,Z)\colon X' \dashrightarrow X$ from Theorem 3.4.9. In particular, the morphism $X' \to X$ is even log smooth. In the sequel, we assume that X is regular, in particular, Z is a normal crossings divisor by **VI**-1.7. We will call Z the **boundary** of X. In the sequel, these conditions will be preserved, so let us describe an appropriate restriction on further modifications.

4.1.2. *Permissible blow ups.* After Step 1 any modification in the remaining tower will be of the form $f\colon (X',Z') \to (X,Z)$ where $X' = \mathrm{Bl}_V(X)$, $Z' = f^{-1}(Z \cup V)$ and V has **normal crossings** with Z, i.e. étale locally on X there exist regular parameters $t_1,\dots,t_d$ such that $Z = V(\prod_{i=1}^{l} t_i)$ and $V = V(t_{i_1},\dots,t_{i_m})$. We call such modification of log schemes **permissible** and use the convention that the center V is part of the data. A blow up of schemes $f\colon X' \to X$ is called **permissible** (with respect to the boundary Z) if it underlies a permissible modification. Since there is an obvious bijective correspondence between permissible modifications and blow ups we will freely pass from one to another. Note that $Z' = f^{\mathrm{st}}(Z) \cup E'$, where $E' = f^{-1}(V)$ is the exceptional divisor.

4.1.3. *Permissible towers.* A **permissible modification tower** $(X_d,Z_d) \dashrightarrow (X_0,Z_0) = (X,Z)$ is defined in the obvious way and we say that a blow up tower $X_d \dashrightarrow X$ is permissible if it underlies such a modification tower. Again, we will freely pass between permissible towers of these types. Note that $Z_i = Z_i^{\mathrm{st}} \cup E_i$, where Z_i^{st} is the strict transform of Z under $h_i\colon X_i \dashrightarrow X$ and E_i is the exceptional divisor of h_i (i.e. the union of the preimages of the centers of h_i).

Remark 4.1.4. (i) Consider a permissible tower as above. It is well known that for any i one has that X_i is regular, Z_i is normal crossings and E_i is even snc. For completeness, let us outline the proof. Both claims follow from the following: if Z is snc then Z_i is snc. Indeed, the claim about E_i follows by taking $Z = \emptyset$ and the claim about Z_i can be checked étale locally, so we can assume that Z is snc. Finally, if Z is snc then Z_i is snc by Lemma 4.2.9 below.

(ii) Permissible towers are very common in embedded desingularization because they do not destroy regularity of the ambient scheme and the normal crossings (or snc) property of the boundary (or accumulated exceptional divisor). Even when one starts with an empty boundary, a non-trivial boundary appears after the first step, and this restricts the choice of further centers. Actually, any known self-contained proof of embedded desingularization constructs a permissible resolution tower.

4.1.5. *G-permissible towers.* In addition, we will only blow up G-equivariant centers V. So, $f\colon X' = \mathrm{Bl}_V(X) \to X$ is G-equivariant and the exceptional divisor $E = f^{-1}(V)$ is regular and G-equivariant and hence G-strict. Such a blow up (or their tower) will be called ***G*-permissible**. It follows by induction that the exceptional divisor of such a tower is G-strict.

4.1.6. *Step 2. Making Z snc and G-strict.* Consider the stratification of Z by multiplicity: a point $z \in Z$ is in Z^n if it has exactly n preimages in the normalization of Z. Note that $\{Z^n\}$ is precisely the log stratification as defined in 3.3.1. By **depth** of the stratification we mean the maximal d such that $Z^d \neq \emptyset$. In particular, Z^d is the only closed stratum. Step 2 proceeds as follows: $X_{i+1} \to X_i$ is the blow up along the closed stratum of Z_i^{st}.

Remark 4.1.7. What we use above is the standard algorithm that achieves the following two things: Z' is snc and $Z^{\mathrm{st}} = \emptyset$ (see, for example, [**de Jong, 1996**, 7.2]). Even when Z is snc, the second property is often used in the embedded desingularization algorithms to get rid of the old components of the boundary.

4.1.8. *Justification of Step 2.* Since the construction is well known, we just sketch the argument. First, observe that Z^d has normal crossings with Z, that is, $X' = \mathrm{Bl}_{Z^d}(X) \to X$ is permissible. Thus, Z' is normal crossings and hence Z^{st} is also normal crossings. A simple computation with blow up charts shows that the depth of Z^{st} is $d-1$ (for example, one can work étale locally, and then this follows from Lemma 4.2.8 below). It follows by induction that the tower produced by Step 2 is permissible, of length d and with $Z^{\mathrm{st}}_d = \emptyset$. So, $Z_d = E_d$ is snc by Remark 4.1.4 and G-strict by 4.1.5.

4.1.9. *Step 3. Making the inertia groups abelian and the action of G on $X \setminus Z$ free.* Recall (**VI**-4.1) that the inertia strata are of the form $X_H = X^H \setminus \bigcup_{H \subsetneq H'} X^{H'}$. Step 3 runs analogously to Step 2, but this time we will work with the inertia stratification of X instead of the log stratification, and will have to apply the same operation a few times. Let us first describe the blow up algorithm used in this step; its justification will be given in §4.2.

Let $f_{\{X^H\}} : X' \dashrightarrow X$ denote the following blow up tower. First we blow up the union of all closed strata X_H. In other words, V_0 is the union of all non-empty minimal X^H, i.e. non-empty X^H that do not contain X^K with $\emptyset \subsetneq X^K \subsetneq X^H$. Next, we consider the family of all strict transforms of X^H and blow up the union of the non-empty minimal ones, etc. Obviously, the construction is G-equivariant. We will prove in Proposition 4.2.11 that $f_{\{X^H\}}$ is permissible of length bounded by the length of the maximal chains of subgroups. Also, we will show in 4.2.13 that $f_{\{X^H\}}$ decreases all non-abelian inertia groups, so the algorithm of Step 3 goes as follows: until all inertia groups become abelian, apply $f_{\{X^H\}} : X' \dashrightarrow X$ (i.e. replace (X, Z) with (X', Z')).

4.2. Justification of Step 3.

4.2.1. *Weakly snc families.* Assume that X is regular, $Z \hookrightarrow X$ is an snc divisor, and $\{X_i\}_{i \in I}$ is a finite collection of closed subschemes of X. For any $J \subset I$ we denote by X_J the scheme-theoretic intersection $\bigcap_{j \in J} X_j$. The family $\{X_i\}$ is called **weakly snc** if each X_i is nowhere dense and X_J is regular. The family $\{X_i\}$ is called **weakly Z-snc** if it is weakly snc and each X_J has normal crossings with Z. In particular, $\{X_i\}_{i \in I}$ is weakly snc (resp. weakly Z-snc) if and only if the family $\{X_J\}_{\emptyset \neq J \subset I}$ is weakly snc (resp. weakly Z-snc).

Remark 4.2.2. (i) Here is a standard criterion of being an snc divisor, which is often taken as a definition. Let $D \hookrightarrow X$ be a divisor with irreducible components $\{D_i\}_{i \in I}$. Then D is snc if and only if the set of its irreducible components is weakly snc and each irreducible component of D_J is of codimension $|J|$ in X.

(ii) The condition on the codimension is essential. For example, $xy(x + y) = 0$ defines a weakly snc but not snc family of irreducible divisors in $\mathbf{A}^2_k = \mathrm{Spec}(k[x, y])$.

(iii) The criterion from (i) implies that if X is qe then the snc locus of D is open—it is the complement of the union of singular loci of D_J's and the loci where D_J is of codimension smaller than $|J|$. (Note that this makes sense for all points of X because D is snc at a point $x \in X \setminus D$ if and only if $X = D_\emptyset$ is regular at x.)

Lemma 4.2.3. *Let (X, Z) and G be as achieved in Step 2. Then the family $\{X^H\}_{1 \neq H \subset G}$ is weakly Z-snc.*

Proof. Recall that for any subgroup $H \subset G$ the fixed point subscheme X^H is regular by Proposition **VI**-4.2, and X^H is nowhere dense for $H \neq 1$ by generic freeness of the action of G. Since for any pair of subgroups $K, H \subset G$ we have that $X^H \times_X X^K = X^{KH}$, the family is weakly snc. It remains to show that $Y = X^H$ has normal crossings with Z. Note that it is enough to consider the case when X is local with closed point x and $G = H$. The cotangent spaces at x will be denoted $T^*X = m_{X,x}/m^2_{X,x}$, T^*Y, etc. Their dual spaces will be called the tangent spaces, and denoted TX, TY, etc. Let $\phi^* : T^*X \twoheadrightarrow T^*Y$ denote the natural map and let $\phi : TY \hookrightarrow TX$ denote its dual. We will systematically use without mention that $|G|$ is coprime to char $k(x)$, in particular, the action of G on T^*X is semi-simple.

The proof of **VI**-4.2 also shows that for any point $x \in X^H$, the tangent space $T_x(X^H)$ is isomorphic to $(T_xX)^H$. In particular, $TY \xrightarrow{\sim} (TX)^G$ and hence ϕ^* maps $(T^*X)^G \subset T^*X$ isomorphically onto T^*Y. Therefore, $U = \mathrm{Ker}(\phi^*)$ is the G-orthogonal complement to $(T^*X)^G$, i.e. the only G-invariant subspace such that $(T^*X)^G \oplus U \xrightarrow{\sim} T^*X$. Let $Z_i = V(t_i)$, $1 \leq i \leq n$ be the components of Z and let $dt_i \in T^*X$ denote the image of t_i. By Z-strictness of G, each line $L_i = \mathrm{Span}(dt_i)$ is G-invariant, so G acts on dt_i by a character χ_i. Without restriction of generality, $\chi_1, \dots, \chi_l$ for some $0 \le l \le n$ are the only trivial characters. In particular, $L = \mathrm{Span}(dt_1, \dots, dt_n)$ is the direct sum of $L^G = \mathrm{Span}(dt_1, \dots, dt_l)$ and its G-orthogonal complement $L \cap U$, which (by uniqueness of the complement) coincides with $\mathrm{Span}(dt_{l+1}, \dots, dt_n)$. Complete the basis of L to a basis $\{dt_1, \dots, dt_n, e_1, \dots, e_m\}$ of T^*X such that $\{dt_{l+1}, \dots, dt_n, e_1, \dots, e_r\}$ for some $r \le m$ is a basis of U and choose functions $s_1, \dots, s_m$ on X so that $ds_j = e_j$ and $s_1, \dots, s_r$ vanish on Y. Clearly, $t_1, \dots, t_n, s_1, \dots, s_m$ is a regular family of parameters of $\mathcal{O}_{X,x}$, so the lemma would follow if we prove that $Y = V(t_{l+1}, \dots, t_n, s_1, \dots, s_r)$.

Since Y is regular and $\mathrm{Ker}(\phi^*)$ is spanned by the images of $t_{l+1}, \dots, t_n, s_1, \dots, s_r$, we should only check that these functions vanish on Y. The s_j's vanish on Y by the construction, so we should check that t_i vanishes on Y whenever $l < i \le n$. Using the functorial definition from **VI**-4.1 of the subscheme of fixed points, we obtain that $Z_i^G = Z_i \times_X X^G$, hence $Z_i \times_X Y$ is regular by **VI**-4.2. However, TY is contained in TZ_i, which is the vanishing space of dt_i, hence we necessarily have that $Y \hookrightarrow Z_i$. □

4.2.4. *Snc families.* A family of nowhere dense closed subschemes $\{X_i\}_{i \in I}$ is called **snc** (resp. Z-**snc**) at a point x if X is regular at x and there exists a regular family of parameters $t_j \in \mathcal{O}_{X,x}$ such that in a neighborhood of x each X_i (resp. and each irreducible component Z_k of Z) passing through x is given by the vanishing of a subfamily $t_{j_1}, \dots, t_{j_l}$. Note that the family $\{X_i\}_{i \in I}$ is Z-snc if and only if the union $\{X_i\} \cup \{Z_k\}$ is snc. A family is **snc** if it is so at any point of X (in particular, X is regular).

Remark 4.2.5. (i) It is easy to see that the family $\{X^H\}_{H \subset G}$ is snc whenever G is abelian. Indeed, it suffices to show that for any point x there exists a basis of T_xX such that each $T_x(X^H)$ is given by vanishing of some of the coordinates. But this is so because the action of G on T_xX is (geometrically) diagonalizable and $T_x(X^H) = (T_xX)^H$. In general, the family $\{X^H\}_{H \subset G}$ does not have to be snc, as the example of a dihedral group D_n with $n \ge 3$ acting on the plane shows.

(ii) If $Z \hookrightarrow X$ is an snc divisor with components Z_i and $V \hookrightarrow X$ is a closed subscheme then the family $\{Z_i, V\}$ is snc if and only if V has normal crossings with Z.

The transversal case of the following lemma can be deduced from [**ÉGA** IV$_4$ §19.1], but we could not find the general case in the literature (although it seems very probable that it should have appeared somewhere).

Lemma 4.2.6. *Any weakly snc family with $|I| = 2$ is snc.*

Proof. We should prove that if $X = \mathrm{Spec}(A)$ is a regular local scheme and Y, Z are regular closed subschemes such that $T = Y \times_X Z$ is regular then there exists a regular family of parameters $t_1, \dots, t_n \in A$ such that Y and Z are given by vanishing of some set of these parameters. Let m be the maximal ideal of A, and let I, J and $K = I + J$ be the ideals defining Y, Z and T, respectively. By $T^*X = m/m^2$, T^*Y, etc., we denote the cotangent spaces at the closed point of X. Note that $I/mI \xrightarrow{\sim} \mathrm{Ker}(T^*X \to T^*Y)$, and similar formulas hold for J/mJ and K/mK. Indeed, we can choose the parameters so that $Y = V(t_1, \dots, t_l)$ and then the images of $t_1, \dots, t_l$ form a basis both of I/mI and $\mathrm{Ker}(T^*X \to T^*Y)$.

Now, let us prove the lemma. Assume first that Y and Z are transversal, i.e. $T^*X \hookrightarrow T^*Y \oplus T^*Z$. Choose elements $t_1, \dots, t_{l+k}$ such that $Y = V(t_1, \dots, t_l)$, $Z = V(t_{l+1}, \dots, t_{l+k})$, $l = \mathrm{codim}(Y)$ and $k = \mathrm{codim}(Z)$. Then the images $dt_i \in T^*X$ of t_i are linearly independent because $dt_1, \dots, dt_l$ span $\mathrm{Ker}(T^*X \to T^*Y)$ and $dt_{l+1}, \dots, dt_{l+k}$ span $\mathrm{Ker}(T^*X \to T^*Z)$. Hence we can complete t_i's to a regular family of parameters by choosing $t_{l+k+1}, \dots, t_n$ such that $dt_1, \dots, dt_n$ is a basis of T^*X. This proves the transversal case, and to establish the general case it now suffices to show that if Y and Z are not transversal then there exists an element $t_1 \in m \setminus m^2$ which vanishes both on Y and Z. (The we can replace X with $X_1 = V(t_1)$ and repeat this process until Y and Z are transversal in $X_a = V(t_1, \dots, t_a)$.) Tensoring the exact sequence $0 \to I \cap J \to I \oplus J \to K \to 0$ with A/m we obtain an exact sequence

$$(I \cap J)/m(I \cap J) \to I/mI \oplus J/mJ \xrightarrow{\phi} K/mK \to 0$$

The failure of transversality is equivalent to non-injectivity of ϕ, hence there exists an element $f \in I \cap J$ with a non-zero image in $I/mI \oplus J/mJ$. Thus, $f \in m \setminus m^2$ and we are done. □

4.2.7. *Blowing up the minimal strata of a weakly snc family.* Given a weakly snc family $\{X_i\}_{i\in I}$, we say that a scheme X_J with $J \subset I$ is **minimal** if it is non-empty and any $X_{J'} \subsetneq X_J$ is empty. Also, we will need the following notation: if $Z \hookrightarrow X$ is a closed subscheme and $D \hookrightarrow X$ is a Cartier divisor with the corresponding ideals $\mathscr{I}_Z, \mathscr{I}_D \subset \mathscr{O}_X$, then $Z + D$ is the closed subscheme defined by the ideal $\mathscr{I}_Z\mathscr{I}_D$.

Proposition 4.2.8. *Assume that X is regular, $Z \hookrightarrow X$ is an snc divisor with irreducible components $Z_1, \dots, Z_l$, and $\{X_i\}_{i\in I}$ is a Z-snc (resp. weakly Z-snc) family of subschemes. Let V be the union of all non-empty minimal subschemes X_J, $f : X' = \mathrm{Bl}_V(X) \to X$, $X'_i = f^{\mathrm{st}}(X_i)$ and $Z' = f^{-1}(Z \cup V)$. Then*

(i) X' is regular and Z' is snc.

(ii) The family $\{X'_i\}_{i\in I}$ is Z'-snc (resp. weakly Z'-snc).

(iii) For any $J \subset I$, the scheme-theoretical intersection $X'_J = \bigcap_{j\in J} X'_j$ coincides with $f^{\mathrm{st}}(X_J)$.

(iv) For any $J \subset I$ the total transform $X_J \times_X X'$ is of the form $X'_J + D'_J$ where D'_J is the divisor consisting of all connected components of $E' = f^{-1}(V)$ contained in $f^{-1}(X_J)$.

Proof. We start with the following lemma.

Lemma 4.2.9. *Assume that X is regular, Z is an snc divisor and $V \hookrightarrow Y$ are closed subschemes having normal crossings with Z. Let $f : X' \to X$ be the blow up along V, $Y' = f^{\mathrm{st}}(Y)$, $Z' = f^{-1}(Z \cup V)$, and $E' = f^{-1}(V)$. Then Z' is snc, Y' has normal crossings with Z' and $Y \times_X X' = Y' + E'$.*

Proof. The proof is a usual local computation with charts. Take any point $u \in V$ and choose a regular family of parameters $t_1, \dots, t_n$ at u such that Y (resp. V, resp. Z) are given by the vanishing of $t_1, \dots, t_m$ (resp. $t_1, \dots, t_l$, resp. $\prod_{i\in I} t_i$), where $0 \le m \le l \le n$ and $I \subset \{1, \dots, n\}$. Locally over u the blow up is covered by l charts, and the local coordinates on the i-th chart are t'_j such that $t'_j = t_j$ for $j > l$ or $j = i$ and $t'_j = \frac{t_j}{t_i}$ otherwise. On this chart, $Y \times_X X'$ (resp. Y', resp. E', resp. Z') is given by the vanishing of $t_1, \dots, t_m$ (resp. $t'_1, \dots, t'_m$, resp. t'_i, resp. $\prod_{j\in I\cup\{i\}} t'_j$), hence the lemma follows. ☐

The lemma implies (i). In addition, it follows from the lemma that $f^{\mathrm{st}}(X_J)$ has normal crossings with Z' and $X_J \times_X X' = f^{\mathrm{st}}(X_J) + D'_J$. Thus, (iii) implies (iv), and (iii) implies (ii) in the case when the family $\{X_i\}_{i\in I}$ is weakly Z-snc. Note also that if this family is Z-snc then locally at any point $x \in X$ there exists a family of regular parameters $\underline{t} = \{t_1, \dots, t_n\}$ such that each X_J and each component of Z is given by the vanishing of a subfamily of $\underline{t}$ locally in a neighborhood of x. Then the same local computation as was used in the proof of Lemma 4.2.9 proves also claims (ii) and (iii) of the proposition. So, it remains to prove (iii) when the family is weakly snc. It suffices to prove that if (iii) holds for X_J and X_K then it holds for $X_{J\cup K}$. Moreover, (iii) does not involve the boundary so we can assume that $Z = \varnothing$. It remains to note that $\{X_J, X_S\}$ is an snc family by Lemma 4.2.6, hence our claim follows from the snc case. ☐

4.2.10. *Blow up tower of a weakly Z-snc family.* Let X be a regular scheme, Z be an snc divisor and $\{X_i\}_{i\in I}$ be a weakly Z-snc family. By the blow up tower $f_{\{X_i\}}$ of $\{X_i\}$ we mean the following tower: the first blow up $h_1 : X_1 \to X_0 = X$ is along the union of all non-empty minimal schemes of the form X_J for $\varnothing \ne J \subset I$, the second blow up is along the union of all non-empty minimal schemes of the form $h_1^{\mathrm{st}}(X_J)$ for $\varnothing \ne J \subset I$, etc.

Proposition 4.2.11. *Keep the above notation. Then the tower $f_{\{X_i\}}$ is permissible with respect to Z and its length equals to the maximal length of chains $\varnothing \ne X_{J_1} \subsetneq \dots \subsetneq X_{J_d}$ with $\varnothing \ne J_d \subsetneq \dots \subsetneq J_1 \subset I$. Furthermore, the strict transform of any scheme X_J is empty and the total transform of X_J is a Cartier divisor.*

Proof. The claim about the length is obvious. Let $f_{\{X_i\}} : X_d \dashrightarrow X_0 = X$ and let $h_n : X_n \dashrightarrow X_0 = X$ be its n-th truncation. For each $i \in I$ set $X_{n,i} = h_n^{\mathrm{st}}(X_i)$ and for each $J \subset I$ set $X_{n,J} = \bigcap_{i\in J} X_{n,i}$. Using Proposition 4.2.8 and straightforward induction on length, we obtain that the family $\{X_{n,i}\}_{i\in I}$ is weakly Z_n-snc, $X_{n,J} = h_n^{\mathrm{st}}(X_J)$, the blow up $X_{n+1} \to X_n$ is along the union of non-empty minimal $X_{n,J}$'s, and $X_J \times_X X_n = X_{n,J} + D_{J,n}$, where $D_{J,n}$ is a divisor. So, the tower is permissible, and since $X_{d,J} = \varnothing$ we also have that $X_J \times_X X_d$ is a divisor. ☐

Remark 4.2.12. We will not need this, but it is easy to deduce from the proposition that on the level of morphisms the modification $X_d \to X$ is isomorphic to the blow up along $\prod_{\varnothing\ne J\subset I} \mathscr{I}_{X_J}$.

4.2.13. *Justification of Step 3.* The blow up tower $f_{\{X^H\}}: X' \dashrightarrow X$ from Step 3 is G-equivariant in an obvious way, and it is permissible by Proposition 4.2.11. In addition, $Z' = f^{-1}(Z) \cup f^{-1}(\bigcup_{1\neq H\subset G} X^H)$ and, since G acts freely on $X' \setminus f^{-1}(\bigcup_{1\neq H\subset G} X^H) \xrightarrow{\sim} X \setminus \bigcup_{1\neq H\subset G} X^H$, it also acts freely on $X' \setminus Z'$. It remains to show that applying $f_{\{X^H\}}$ we decrease all non-abelian inertia groups. Namely, for any $x' \in X'$ mapped to $x \in X$ we want to show that either $G_{\overline{x}}$ is abelian or the inclusion $G_{\overline{x}'} \subsetneq G_{\overline{x}}$ is strict.

Let $H \subset G$ be any non-abelian subgroup with commutator $K = [H,H]$. Since $X^K \times_X X'$ is a divisor by Proposition 4.2.11, the universal property of blow ups implies that $X' \to X$ factors through $Y = \mathrm{Bl}_{X^K}(X)$. On the other hand, it is proved in **VI**-4.8 that $Y \to X$ is an H-equivariant blow up, and if a geometric point $\overline{x} \to X$ with $G_{\overline{x}} = H$ lifts to a geometric point $\overline{y} \to Y$ then $G_{\overline{y}} \subsetneq H$. Therefore, the same is true for the G-equivariant modification $X' \to X$, and we are done.

5. Proof of Theorem 1.1 — abelian inertia

5.1. Conventions. Throughout §5 we assume that (X,Z) and G satisfy all conditions achieved at Steps 1, 2, 3, and our aim is to construct a modification $f_{(G,X,Z)}: X' \to X$ as in Theorem 1.1. Unless specially mentioned, we do not assume that X is qe. This is done in order to isolate the only place where this assumption is needed (existence of rigidifications).

5.2. Outline of our method and other approaches.

5.2.1. *Combinatorial nature of the problem.* On the intuitive level it is natural to expect that "everything relevant to our problem" should be determined by the following "combinatorial" data: the log structure of X, the inertia stratification of X by $X_H := X^H \setminus \bigcup_{H'\subsetneq H} X^{H'}$ and the representations of the inertia groups on the tangent spaces (which are essentially constant along X_H). This combinatorial nature is manifested in both approaches to the problem that we describe below.

5.2.2. *Combinatorial algorithm.* A natural approach is to seek for a "combinatorial algorithm" that iteratively blows up disjoint unions of a few closures of connected components of **log-inertia strata** (i.e. intersections of a log stratum with an inertia stratum). For example, our (very simple) algorithms in §4 were of this type. The choice of the centers should be governed by the following combinatorial data: the number of components of Z through a point x and the history of their appearance (similarly to the desingularization algorithms), and the representation of $G_{\overline{x}}$ on T_x plus the history of representations (i.e. the list of representations for all predecessors of x in the blow up tower).

It is natural to expect that building such an algorithm would lead to a relatively simple proof of Theorem 1.1. In particular, it would be non-sensitive to quasi-excellence issues. Unfortunately, despite partial positive results, we could not construct such an algorithm. Thus, the question whether it exists remains open.[(ii)]

5.2.3. *Our method.* A general plan of our method is as follows. In §5.3 we will show that such a modification f exists étale locally on the base if X is qe. A priori, our construction will be canonical up to an auxiliary choice, but then we will prove in §5.5 that actually it is independent of the choice and hence descends to a modification f as required. To prove independence we will show in §5.4 that the construction is functorial with respect to strict **inert** morphisms (i.e. morphisms that preserve both the log and the inertia structures, see 5.3.6). The latter is a manifestation of the "combinatorial nature" of our algorithm.

5.3. Local construction.

5.3.1. *Very tame action and Zariski topology.* Note that the log structure on (X,Z) is Zariski since Z is snc. Thus, it will be convenient to describe very tame action in terms of Zariski topology. We say that the action of G is **very tame at** a point $x \in X$ if for any geometric point $\overline{x}$ over x the action is very tame at $\overline{x}$. Clearly, the $G_{\overline{x}}$-equivariant log scheme $\mathrm{Spec}(\mathscr{O}_{X,\overline{x}})$ is independent of the choice of $\overline{x}$ up to an isomorphism. In particular, the action is very tame at x if and only if it is very tame at a single geometric point $\overline{x}$ above x.

(ii) F. Pop told the second author that he has a plan for constructing such a combinatorial algorithm.

LEMMA 5.3.2. *Assume that (G, X, Z) is as in §5.1, $x \in X$ is a point, and T is the set of points of X at which the action is very tame. Then*
(i) T is open,
(ii) $x \in T$ if and only if the local log stratification of X (see Remark 3.3.2) is finer than the inertia stratification in a neighborhood of x.

Proof. The first assertion follows from **VI**-3.9. To prove the second claim we note that the conditions (i) and (ii) from **VI**-3.1 are automatically satisfied at $\overline{x}$. Indeed, (i) is satisfied because the action is tame by assumption of 1.1 and (ii) is satisfied because Z is snc and G-strict. Since log and inertia stratifications are compatible with the strict henselization morphism, condition (iii) from **VI**-3.1 is satisfied at $\overline{x}$ if and only if the local log stratification is finer than the inertia stratification at x. This proves (ii). □

5.3.3. *Admissibility.* In the sequel, by saying that X is **admissible** we mean that the action of G on X is admissible in the sense of [**SGA 1** v 1.7] (e.g. X is affine). This is needed to ensure that X/G exists as a scheme. An alternative would be to allow X/G to be an algebraic space (and $(X/G, Z/G)$ to be a log algebraic space).

5.3.4. *Rigidification.* By a **rigidification** of X we mean a G-equivariant normal crossings divisor $\overline{Z}$ that contains Z and such that the action of G on the log regular log scheme $\overline{X} = (X, \overline{Z})$ is very tame. If $\overline{Z}$ is snc then we say that the rigidification is **strict**. Sometimes, by a rigidification of $X = (X, Z)$ we will mean the log scheme $\overline{X}$ itself. Our construction of a modification f uses a rigidification, so let us first establish local results on existence of the latter.

In the sequel, we say that μ_N is **split** over a scheme X if it is isomorphic to the discrete group $(\mathbf{Z}/N\mathbf{Z})^*$ over X. This happens if and only if X admits a morphism to $\mathrm{Spec}(\mathbf{Z}[\frac{1}{N}, \mu_N])$, where we use the notation $\mathbf{Z}[\frac{1}{N}, \mu_N] = \mathbf{Z}[\frac{1}{N}, x]/(x^n - 1)$.

LEMMA 5.3.5. *Let X, Z, G be as in §5.1. Assume that $X = \mathrm{Spec}(A)$ is a local scheme with closed point x, and μ_N is split over X, where N is the order of $G = G_{\overline{x}}$. Then X possesses a strict rigidification.*

Proof. Choose $t_1, \dots, t_n \in A$ such that $Z_i = (t_i)$ are the components of Z. By G-strictness of Z, for any $g \in G$ we have that $Z_i = (gt_i)$ locally at x, in particular, the tangent space to each Z_i at x is G-invariant. Now, we can use averaging by the G-action to make the parameters G-equivariant. Namely, G acts on dt_i by a character χ_i and replacing t_i with $\frac{1}{|G|}\sum_{g\in G}\frac{gt_i}{\chi_i(g)}$ we do not change dt_i and achieve that $gt_i = \chi_i(g)t_i$.

The action of G on the cotangent space at x is diagonalizable because G is abelian and μ_N is split over $k(x)$. In particular, we can complete the family $dt_1, \dots, dt_n$ to a basis $dt_1, \dots, dt_l$ such that $t_i \in \mathcal{O}_{X,x}$ and G acts on each dt_i by a character χ_i. Then $t_1, \dots, t_l$ is a regular family of parameters of $\mathcal{O}_{X,x}$ and using the same averaging procedure as above we can make them G-equivariant. Take now $\overline{Z}$ to be the union of all divisors (t_i) with $1 \le i \le l$. The action of G on $(X, \overline{Z})$ is very tame at x because it is the only point of its log stratum and so Lemma 5.3.2 applies. □

5.3.6. *Inert morphisms.* Let (X, Z) and G be as in §5.1. Our next aim is to find an étale cover $f : (Y, T) \to (X, Z)$ which "preserves" the log-inertia structure of (X, Z) and such that (Y, T) admits a rigidification. The condition on the log structure is obvious: we want f to be strict, i.e. $f^{-1}(Z) = T$. Let us introduce a restriction related to the inertia groups.

Assume that (Y, T) with an action of H is another such triple, and let $\lambda : H \to G$ be a homomorphism. A λ-equivariant morphism $f : (Y, T) \to (X, Z)$ will be called **inert** if for any point $y \in Y$ with $x \in X$ the induced homomorphism of inertia groups $H_{\overline{y}} \to G_{\overline{x}}$ is an isomorphism. In particular, the inertia stratification of Y is the preimage of the inertia stratification of X.

REMARK 5.3.7. When λ is an identity, inert morphisms are usually called "fixed point reflecting". We prefer to change the terminology since "inert" is brief and adequate.

Lemma 5.3.8. *Let X, Z and G be as above, and assume that X is qe. Then there exists a G-equivariant surjective étale inert strict morphism $h\colon (Y,T)\to(X,Z)$ such that Y is affine and (Y,T) possesses a strict rigidification.*

Proof. First, we note that the problem is local on X. Namely, it suffices for any point $x\in X$ to find a $G_{\overline{x}}$-equivariant étale inert strict morphism $h\colon (Y,T)\to(X,Z)$ such that Y is affine, $x\in h(Y)$, any point $x'\in h(Y)$ satisfies $G_{\overline{x}'}\subset G_{\overline{x}}$, and (Y,T) admits a strict rigidification. Indeed, h can be extended to a G-equivariant morphism $Y\times_X(X\times G/G_{\overline{x}})=\coprod_{g\in G/G_{\overline{x}}}Y_g\to X$, where each Y_g is isomorphic to Y and the morphism $Y_g\to X$ is obtained by composing $Y\to X$ with $g\colon X\to X$. Clearly, the latter morphism is étale, inert, and strict, and by quasi-compactness of X we can combine finitely many such morphism to obtain a required cover of (X,Z).

Now, fix $x\in X$ and consider the $G_{\overline{x}}$-invariant neighborhood $X'=X\setminus\bigcup_{H\not\subseteq G_{\overline{x}}}X^H$ of x. We will work over X', so the condition $G_{\overline{x}'}\subset G_{\overline{x}}$ will be automatic. Let N be the order of $G_{\overline{x}}$, and consider the $G_{\overline{x}}$-equivariant morphism $f\colon Y=X'\times\mathrm{Spec}(\mathbf{Z}[\frac{1}{N},\mu_N])\to X$ (with $G_{\overline{x}}$ acting trivially on the second factor). Let $T=f^{-1}(Z)$ and let y be any lift of x. It suffices to show that (Y,T) admits a rigidification in an affine $G_{\overline{x}}$-invariant neighborhood of y (such neighborhoods form a fundamental family of neighborhoods of y). By Lemma 5.3.5, the localization $Y_y=\mathrm{Spec}(\mathcal{O}_{Y,y})$ with the restriction T_y of T possesses a strict rigidification $\overline{T}_y$. Clearly, $\overline{T}_y$ extends to a divisor $\overline{T}$ with $T\hookrightarrow\overline{T}\hookrightarrow Y$ and we claim that it is a rigidification in a neighborhood of y. Indeed, $\overline{T}$ is snc at y, hence it is snc in a neighborhood of y by Remark 4.2.2, and it remains to use Lemma 5.3.2. □

5.3.9. *Main construction.* Assume, now, that $X=(X,Z)$ is admissible and admits a rigidification $\overline{Z}$. We are going to construct a G-equivariant modification

$$f_{(G,X,Z,\overline{Z})}\colon (X',Z')\to(X,Z)$$

such that G acts very tamely on the target and $f_{(G,X,Z,\overline{Z})}$ is independent of the rigidification. The latter is a subtle property (missing in the obvious modification $(X,\overline{Z})\to(X,Z)$), and it will take us a couple of pages to establish it.

The quotient log scheme $\overline{Y}=(Y,\overline{T})=(X/G,\overline{Z}/G)$ is log regular by Theorem **VI**-3.2, hence by Theorem 3.3.16 there exists a functorial saturated log blow up tower $\overline{h}=\widetilde{\mathcal{F}}^{\log}(\overline{Y})\colon \overline{Y}'=(Y',\overline{T}')\to\overline{Y}$ with a regular and log regular source. Let $\overline{f}\colon\overline{X}'=(X',\overline{Z}')\to\overline{X}$ be the pullback of $\overline{h}$ (as a saturated log blow up tower, see 3.3.11), then $\alpha'\colon\overline{X}'\to\overline{Y}'$ is a Kummer étale G-cover because $\alpha\colon\overline{X}\to\overline{Y}$ is so by **VI**-3.2 as the square

$$\begin{tikzcd}
(X',\overline{Z}') \arrow[r, "\overline{f}"] \arrow[d, "\alpha'"] & (X,\overline{Z}) \arrow[d, "\alpha"] \\
(Y',\overline{T}') \arrow[r, "\overline{h}"] & (Y,\overline{T})
\end{tikzcd}$$

is cartesian in the category of fs log schemes.

Since G acts freely on $U=X-Z$, $V=U/G$ is regular and $\overline{T}|_V$ is snc. In particular, $\overline{h}$ is an isomorphism over V and hence $\overline{f}$ is an isomorphism over U. We claim that the Weil divisor $T=Z/G$ of Y is $\mathbf{Q}$-Cartier (it does not have to be Cartier, as the orbifold case with $X=\mathbf{A}^2$, $Z=\mathbf{A}^1$ and $G=\{\pm1\}$ shows). Indeed, it suffices to check this étale locally at a point $y\in Y$. In particular, we can assume that μ_N is split over $\mathcal{O}_{Y,y}$, where $N=|G|$. Then, as we showed in the proof of Lemma 5.3.5, Z can be locally defined by equivariant parameters, in particular, $Z=V(f)$ where G acts on f by characters. Therefore, f^N is G-fixed, and we obtain that T is the reduction of the Cartier divisor C of Y given by $f^N=0$. (The same argument applied to $T\times_Y X$ shows that NT is Cartier, so T is $\mathbf{Q}$-Cartier.) So, $C'=C\times_Y Y'$ is a Cartier divisor whose reduction is $T'=\overline{h}^{-1}(T)$. Since Y' is regular and T' lies in the snc divisor $\overline{T}'$, we obtain that T' is itself an snc divisor.

Let Z' denote the divisor $\alpha'^{-1}(T') = \overline{f}^{-1}(Z)$. Since $X' \to Y'$ is étale over $V = Y' - T'$, the morphism of log schemes $(X', Z') \to (Y', T')$ is a Kummer étale G-cover. This follows from a variant of the classical Abhyankar's lemma (**IX**-2.1), which is independent of the results of the present exposé.

In particular, $X' = (X', Z')$ is log regular and it follows that the action of G on X' is very tame. We define $f_{(G,X,Z,\overline{Z})}$ to be the modification $X' \to X$.

Remark 5.3.10. (i) Note that $X' \to X$ satisfies all conditions of Theorem 1.1 because the action is very tame and G acts freely on $X' \setminus f^{-1}(Z)$. So, we completed the proof in the case when (X, Z) admits a rigidification $\overline{Z}$. Our last task will be to get rid of the rigidification.

(ii) The only dependence of our construction on the rigidification is when we construct the resolution of $(Y, \overline{T})$. Conjecturally, it depends only on the scheme Y, and then (Y', T'), and hence also (X', Z'), would depend only on (X, Z). Recall that we established in Theorem 3.4.15 the particular case of this conjecture when all maximal points of the log strata of $(Y, \overline{T})$ are of characteristic zero. Hence independence of the rigidification is unconditional in this case, and, fortunately, this will suffice.

5.3.11. *Finer structure of $f_{(G,X,Z,\overline{Z})}$.* Obviously, the saturated log blow up tower $\overline{f}\colon (X', \overline{Z}') \to (X, \overline{Z})$ depends on the rigidification, and this is the reason why we prefer to consider the modification $f_{(G,X,Z,\overline{Z})}\colon (X', Z') \to (X, Z)$ instead. However, there is an additional structure on $f_{(G,X,Z,\overline{Z})}$ that has a chance to be independent of $\overline{Z}$, and which should be taken into account. By §3.4.8, the modification of schemes $f\colon X' \to X$ has a natural structure of a normalized blow up tower $X_\bullet$ with $X = X_0$ and $X' = X_n$. Note also that the tower contains no empty blow ups because this is true for $\widetilde{\mathscr{F}}^{\log}(X/G, \overline{Z}/G)$ and $f_{(G,X,Z,\overline{Z})}$ is its strict transform with respect to the surjective morphism $X \to X/G$.

Note also that the log structure on (X', Z') is reconstructed uniquely from f because $Z' = f^{-1}(Z)$ and (X', Z') is saturated and log regular. Therefore, it is safe from now on to view $f_{(G,X,Z,\overline{Z})}$ as a normalized blow up tower of X, but the modification of log schemes $(X', Z') \to (X, Z)$ will also be denoted as $f_{(G,X,Z,\overline{Z})}$.

Remark 5.3.12. Although we do not assume that X is qe, all normalizations in the tower $f_{(G,X,Z,\overline{Z})}$ are finite. This happens because they underly saturations of fine log schemes, which are always finite morphisms.

5.4. Functoriality. Clearly, the construction of f depends canonically on $(G, X, Z, \overline{Z})$, i.e. is compatible with any automorphism of such quadruple. Our next aim is to establish functoriality with respect to strict inert λ-equivariant morphisms $\phi\colon (H, Y, T, \overline{T}) \to (G, X, Z, \overline{Z})$ (i.e. morphisms that "preserve the combinatorial structure"). For this one has first to study the quotient morphism of log schemes $\widetilde{\phi}\colon (Y/H, \overline{T}/H) \to (X/G, \overline{Z}/G)$.

5.4.1. *Log structure of the quotients.* Recall the following facts from Proposition **VI**-3.4(b) and its proof. Assume that $X = (X, M_X)$ is an fs log scheme provided with a very tame action of a group G. After replacing X with its strict localization at a geometric point $\overline{x}$, it admits an equivariant chart $X \to \operatorname{Spec}(\Lambda[Q])$, where $\Lambda = \mathbf{Z}[1/N, \mu_N]$ for the order N of $G_{\overline{x}}$, Q is an fs monoid and the action of $\overline{G}_x$ is via a pairing $\chi\colon G_{\overline{x}} \otimes Q \to \mu_N$. Moreover, if $P \subset Q$ is the maximal submonoid with $\chi(G_{\overline{x}} \otimes P) = 1$ then $\operatorname{Spec}(\Lambda[Q]) \to \operatorname{Spec}(\Lambda[P])$ is a chart of $X \to X/G_{\overline{x}}$. Now, let us apply this description to the study of $\widetilde{\phi}$.

Proposition 5.4.2. *Assume that fs log schemes X, Y are provided with admissible very tame actions of groups G and H, respectively, $\lambda\colon H \to G$ is a homomorphism, and $\phi\colon Y \to X$ is a strict inert λ-equivariant morphism. Then the quotient morphism $\widetilde{\phi}\colon Y/H \to X/G$ is strict.*

Proof. Fix a geometric point $\overline{y}$ of Y and let $\overline{x}$ be its image in X. It suffices to show that $\widetilde{\phi}$ is strict at the image of $\overline{y}$ in Y/H. The morphism $Y/H_{\overline{y}} \to Y/H$ is strict (and étale) over the image of $\overline{y}$, and the same is true for X. Therefore we can replace H and G with $H_{\overline{y}} \xrightarrow{\sim} G_{\overline{x}}$, and then we can also replace X and Y with their strict localizations at $\overline{x}$ and $\overline{y}$. Now, the morphism $X \to \widetilde{X} = X/G_{\overline{x}}$ admits an equivariant chart $h\colon \operatorname{Spec}(\Lambda[Q]) \to \operatorname{Spec}(\Lambda[P])$ as explained before the proposition. Since ϕ is strict, the induced morphism $Y \to \mathrm{Spec}(\Lambda[Q])$ is also a chart and hence h is also a chart of $Y \to Y/G_{\overline{y}}$. Thus, $\widetilde{\phi}$ is strict. $\square$

5.4.3. *An application to functoriality of $f_{\bullet}$.* Assume that $(G, X, Z, \overline{Z})$ is as earlier, and let $(H, Y, T, \overline{T})$ be another such quadruple (i.e. (Y, T) with the action of H satisfies conditions of Steps 1, 2, 3 and $(Y, \overline{T})$ is its rigidification).

Corollary 5.4.4. *Assume that $\lambda\colon H \to G$ is a homomorphism and $\phi\colon Y \to X$ is a λ-equivariant inert morphism underlying strict morphisms of log schemes $\psi\colon (Y, T) \to (X, Z)$ and $\overline{\psi}\colon (Y, \overline{T}) \to (X, \overline{Z})$. Then $f_{\bullet}$ is compatible with ϕ in the sense that $f_{(H,Y,T,\overline{T})}$ is the contraction of $\phi^{\mathrm{st}}(f_{(G,X,Z,\overline{Z})})$. In addition, $\phi^{\mathrm{st}}(f_{(G,X,Z,\overline{Z})}) = f_{(G,X,Z,\overline{Z})} \times_X Y$.*

Proof. Since $\overline{\psi}$ is strict its quotient is strict by Proposition 5.4.2, and by functoriality of saturated monoidal desingularization we obtain that $\mathscr{F}^{\log}(Y/H, \overline{T}/H)$ is the contracted pullback of $\mathscr{F}^{\log}(X/G, \overline{Z}/G)$. So, both $f_{(H,Y,T,\overline{T})}$ and $f_{(G,X,Z,\overline{Z})}$ are obtained as the contraction of the strict transform of $\mathscr{F}^{\log}(X/G, \overline{Z}/G)$. The first claim of the Corollary follows.

Furthermore, $f_{(G,X,Z,\overline{Z})}$ underlies a log blow up tower of $(X, \overline{Z})$ which is the strict transform of $\mathscr{F}^{\log}(X/G, \overline{Z}/G)$, and the same is true for $f_{(H,Y,T,\overline{T})}$. Since $\overline{\psi}$ is strict it follows from Lemma 3.4.6(ii) that the strict transform is a pullback, i.e. $\phi^{\mathrm{st}}(f_{(G,X,Z,\overline{Z})}) = f_{(G,X,Z,\overline{Z})} \times_X Y$. $\square$

5.4.5. *Localizations and completions.* In particular, it follows that the construction of $f_{\bullet}$ is compatible with localizations and completions. Namely, if $x \in X$ is a point, $X_x = \mathrm{Spec}(\mathscr{O}_{X,x})$, $Z_x = Z \times_X X_x$ and $\overline{Z}_z = \overline{Z} \times_X X_x$, then $f_{(G_{\overline{x}}, X_x, Z_z, \overline{Z}_x)}$ is the contraction of $f_{(G,X,Z,\overline{Z})} \times_X X_x$. Similarly, if $\widehat{X}_x = \mathrm{Spec}(\widehat{\mathscr{O}}_{X,x})$, $\widehat{Z}_x = Z \times_X \widehat{X}_x$ and $\widehat{\overline{Z}}_x = \overline{Z} \times_X \widehat{X}_x$, then $f_{(G_{\overline{x}}, \widehat{X}_x, \widehat{Z}_x, \widehat{\overline{Z}}_x)}$ is the contraction of $f_{(G,X,Z,\overline{Z})} \times_X \widehat{X}_x$.

5.5. Globalization. To complete the proof of Theorem 1.1 it suffices to show that $f_{(G,X,Z,\overline{Z})}$ is independent of $\overline{Z}$, and hence the local constructions glue to a global normalized blow up tower. The main idea is to simultaneously lift two rigidifications to characteristic zero and apply Theorem 3.4.15.

5.5.1. *Independence of rigidification.* We start with the case of complete local rings. Then the problem is solved by lifting to characteristic zero and referencing to 3.4.15. The general case will follow rather easily.

Lemma 5.5.2. *Keep assumptions on (X, Z) and G as in §5.1 and assume, in addition, that $X = \coprod_{i=1}^m \mathrm{Spec}(A_i)$ where each A_i is a complete noetherian regular local ring with a separably closed residue field. Then for any pair of rigidifications $\overline{Z}$ and $\overline{Z}'$ the equality $f_{(G,X,Z,\overline{Z})} = f_{(G,X,Z,\overline{Z}')}$ holds.*

Proof. Almost the whole argument runs independently on each irreducible component, so assume first that $X = \mathrm{Spec}(A)$ is irreducible. Set $k = A/m_A$. By Remark 5.3.10(ii), it suffices to consider the case when $\mathrm{char}(k) = p > 0$, so let $C(k)$ be a Cohen ring of k. Note that we can work with $H = G_{\overline{x}}$ instead of G because $f_{(H,X,Z,\overline{Z})} = f_{(G,X,Z,\overline{Z})}$ by Corollary 5.4.4. Since H acts trivially on k, for any element $t \in A$ its H-averaging is an element of A^H with the same image in the residue field. Hence k is the residue field of A^H and the usual theory of Cohen rings provides a homomorphism $C(k) \to A^H$ that lifts $C(k) \to A^H/m_{A^H}$. Note that $\overline{Z}$ and $\overline{Z}'$ are snc because each A_i is strictly henselian. Using averaging on the action of H again, we can find regular families of H-equivariant parameters $\underline{z} = (z_1, \dots, z_d)$ and $\underline{z}' = (z'_1, \dots, z'_d)$ such that $Z = V(\prod_{i=1}^l z_i)$, $z'_i = z_i$ for $1 \le i \le l$, $\overline{Z} = V(\prod_{i=1}^n z_i)$ and $\overline{Z}' = V(\prod_{i=1}^{n'} z'_i)$. Explicitly, the action on z_i (resp. z'_i) is by a character $\chi_i\colon H \to \mu_N$ (resp. $\chi'_i\colon H \to \mu_N$).

Since the image of $\underline{z}$ is a basis of the cotangent space at x, we obtain a surjective homomorphism $f\colon B = C(k)[[t_1, \dots, t_d]] \to A$ taking t_i to z_i. Provide B with the action of H which is trivial on $C(k)$ and acts on t_i via χ_i, in particular, f is H-equivariant. Let us also lift each z'_i to an H-equivariant parameter $t'_i \in B$. For $i \le l$ we take $t'_i = t_i$, and for $i > l$ we first choose any lift and then replace it with its

χ_i-weighted H-averaging. Consider the regular scheme $Y = \mathrm{Spec}(B)$ with H-equivariant snc divisors $T = V(\prod_{i=1}^{l} t_i)$, $\overline{T} = V(\prod_{i=1}^{n} t_i)$ and $\overline{T}' = V(\prod_{i=1}^{n'} t_i')$.

Since H acts vary tamely on $(X, \overline{Z})$, it acts trivially on $V(z_1, \dots, z_n) = \mathrm{Spec}(k[[z_{n+1}, \dots, z_d]])$ and we obtain that $\chi_i = 1$ for $i > n$. Therefore, H also acts trivially on $\mathrm{Spec}(B/(t_1, \dots, t_n)) = \mathrm{Spec}(C(k)[[t_{n+1}, \dots, t_d]])$ and we obtain that the action on $(Y, \overline{T})$ is very tame. Since the closed immersion $j : X \to Y$ is H-equivariant and strict, and $\overline{Z} = \overline{T} \times_Y X$, Corollary 5.4.4 implies that $f_\bullet$ is compatible with j, i.e., $f_{(H,X,Z,\overline{Z})}$ is the contracted strict transform of $f_{(H,Y,T,\overline{T})}$. The same argument applies to the rigidifications $\overline{Z}'$ and $\overline{T}'$, so it now suffices to show that $f_{(H,Y,T,\overline{T})} = f_{(H,Y,T,\overline{T}')}$. For this we observe that maximal points of log strata of the log schemes $(Y/H, \overline{T}/H)$ and $(Y/H, \overline{T}'/H)$ are of characteristic zero, hence the latter equality holds by Theorem 3.4.15.

Finally, let us explain how one deals with the case of $m > 1$. First one finds an H-equivariant strict closed immersion $i : X \to Y$ such that $\overline{Z}$ and $\overline{Z}'$ extend to rigidifications $\overline{T}$ and $\overline{T}'$ of (Y, T), and the maximal points of the log strata of $(Y/H, \overline{T}/H)$ and $(Y/H, \overline{T}'/H)$ are of characteristic zero. For this we apply independently the above construction to the connected components of X. Once i is constructed, the same reference to 3.4.15 shows that $f_{(H,Y,T,\overline{T})} = f_{(H,Y,T,\overline{T}')}$ and hence $f_{(H,X,Z,\overline{Z})} = f_{(H,X,Z,\overline{Z}')}$ □

Corollary 5.5.3. *Let (X, Z) and G be as in §5.1 and assume that the action is admissible. Then for any choice of rigidifications $\overline{Z}$ and $\overline{Z}'$ we have that $f_{(G,X,Z,\overline{Z})} = f_{(G,X,Z,\overline{Z}')}$.*

Proof. For a point $x \in X$ let $\widehat{\mathcal{O}^{\mathrm{sh}}_{X,x}}$ denote the completion of the strict henselization of $\mathcal{O}_{X,x}$. It suffices to check that for any point x the normalized blow up towers $f_{(G,X,Z,\overline{Z})}$ and $f_{(G,X,Z,\overline{Z}')}$ pull back to the same normalized blow up towers of $\widehat{X}^{\mathrm{sh}}_x = \mathrm{Spec}(\widehat{\mathcal{O}^{\mathrm{sh}}_{X,x}})$ (with respect to the morphism $\widehat{X}^{\mathrm{sh}}_x \to X$). Indeed, any normalized blow up tower $\mathscr{X} = (X_\bullet, V_\bullet)$ is uniquely determined by its centers V_i. For each i the morphism $Y_i = \coprod_{x \in X} X_i \times_X \widehat{X}^{\mathrm{sh}}_x \to X_i$ is faithfully flat, hence V_i is uniquely determined by $V_i \times_{X_i} Y_i$, which is the center of $\mathscr{X} \times_X \coprod_{x \in X} \widehat{X}^{\mathrm{sh}}_x$.

By 5.4.5, $f_{(G_{\overline{x}}, \widehat{X}^{\mathrm{sh}}_x, Z \times_X \widehat{X}^{\mathrm{sh}}_x, \overline{Z} \times_X \widehat{X}^{\mathrm{sh}}_x)}$ is the contracted pullback of $f_{(G,X,Z,\overline{Z})}$, and an analogous result is true for $f_{(G,X,Z,\overline{Z}')}$. Thus the contracted pullbacks are equal by Lemma 5.5.2. We have, however, to worry also for the synchronization, i.e. to establish equality of non-contracted pullbacks. For this we will use the following trick. Consider the finite set $S = \mathrm{Ass}(f_{(G,X,Z,\overline{Z})}) \cup \mathrm{Ass}(f_{(G,X,Z,\overline{Z}')})$ (see 2.2.11). Set $\widehat{X}^{\mathrm{sh}}_S = \coprod_{s \in S} \widehat{X}^{\mathrm{sh}}_s$, then the pullbacks of $f_{(G,X,Z,\overline{Z})}$ and $f_{(G,X,Z,\overline{Z}')}$ to $\widehat{X}^{\mathrm{sh}}_S$ are already contracted. Now, in order to compare the pullbacks to $\widehat{X}^{\mathrm{sh}}_x$, consider the pullbacks to $\widehat{X}^{\mathrm{sh}}_x \coprod \widehat{X}^{\mathrm{sh}}_S$. They are contracted, so Lemma 5.5.2 (which covers disjoint unions) implies that these pullbacks are equal. Restricting them onto $\widehat{X}^{\mathrm{sh}}_x$ we obtain equality of non-contracted pullbacks to $\widehat{X}^{\mathrm{sh}}_x$. □

Remark 5.5.4. (i) The above corollary implies that the modification $f_{(G,X,Z,\overline{Z})}$ depends only on (G, X, Z), so it will be denoted $f_{(G,X,Z)}$ in the sequel. At this stage, $f_{(G,X,Z)}$ is defined only when X is admissible and (X, Z) admits a rigidification.

(ii) Corollaries 5.4.4 and 5.5.3 imply that $f_{(G,X,Z)}$ is functorial with respect to equivariant strict inert morphisms.

5.5.5. *Theorem 1.1 — end of proof.* Let $X = (X, Z)$ be as assumed in §5.1, and suppose that X is qe. By Lemma 5.3.8 there exists a surjective étale inert strict morphism $h : X_0 \to X$ such that X_0 is affine and possesses a rigidification. Then $X_1 = X_0 \times_X X_0$ is affine and also admits a rigidification (e.g. the preimage of that of X_0 by one of the canonical projections). By Remark 5.5.4(i), X_0 and X_1 possess normalized blow up towers $f_{(G,X_0,Z_0)}$ and $f_{(G,X_1,Z_1)}$, which are compatible with both projections $X_1 \to X_0$ by Remark 5.5.4(ii). It follows that $f_{(G,X_0,Z_0)}$ is induced from a unique normalized blow up tower of X that we denote as $f_{(G,X,Z)}$. This modification satisfies all assertions of Theorem 1.1 because $f_{(G,X_0,Z_0)}$ does so by Remark 5.3.10(i).

5.6. Additional properties of $f_{(G,X,Z)}$. Finally, let us formulate an addendum to Theorem 1.1 where we summarize additional properties of the constructed modification of (X,Z). At this stage we drop any assumptions on (X,Z) beyond the assumptions of 1.1. By $f_{(G,X,Z)}$ we denote below the entire modification from Theorem 1.1 that also involves the modifications of Steps 1, 2, 3.

THEOREM 5.6.1. *Keep assumptions of Theorem 1.1. In addition to assertions of the theorem, the modifications $f_{(G,X,Z)}$ can be constructed uniformly for all triples (G,X,Z) such that the following properties are satisfied:*

(i) Each $f_{(G,X,Z)}$ is provided with a structure of a normalized blow up tower and its centers are contained in the preimages of $Z \cup T$.

(ii) For any homomorphism $\lambda : H \to G$ the construction is functorial with respect to λ-equivariant inert strict regular morphisms $(Y,T) \to (X,Z)$.

Proof. The total modification $f_{(G,X,Z)}$ is obtained by composing four modifications f_1, f_2, f_3 and f_4: the modifications from Steps 1, 2, 3 and the modification we have constructed in §5. Recall that f_1 and f_4 are constructed as normalized blow up towers. Modifications f_2 and f_3 are permissible blow up towers, hence they are also normalized blow up towers with the same centers. This establishes the first part of (i).

Concerning claim (ii), recall that normalized blow ups are compatible with regular morphisms by Lemma 2.2.9, hence we should check that the centers of $f_{(H,Y,T)}$ are the pullbacks of the centers of $f_{(G,X,Z)}$. For f_1, f_2 and f_3 this is clear, so it remains to prove that if (G,X,Z) is as in §5.1 and $(Y,T) \to (X,Z)$ is λ-equivariant, inert, strict and regular then $f_{(H,Y,T)}$ is the pullback of $f_{(G,X,Z)}$. We can work étale locally on X (replacing Y with the base change). Thus, using Lemma 5.3.8 we can assume that X possesses a rigidification $\overline{Z}$. Since $Y \to X$ is regular and inert, the preimage $\overline{T}$ of $\overline{Z}$ is a rigidification of Y. Thus, $f_{(H,Y,T)} = f_{(H,Y,T,\overline{T})}$ and $f_{(G,X,Z)} = f_{(G,X,Z,\overline{Z})}$ and it remains to use that $f_{(H,Y,T,\overline{T})}$ is the pullback of $f_{(G,X,Z,\overline{Z})}$ by Lemma 5.4.4.

To prove the second part of (i) we use (ii) to restrict $f_{(G,X,Z)}$ onto $U = X \setminus Z \cup T$. Then U is a regular scheme with a trivial log structure which is acted freely by G. It follows from the definitions of f_1, f_2, f_3 and f_4 that they are trivial for such U. So, $f_{(G,U,\varnothing)}$ is the trivial tower, and hence all centers of $f_{(G,X,Z)}$ are disjoint from the preimage of U. □

REMARK 5.6.2. One may wonder if the functoriality in Theorem 5.6.1(ii) holds for λ-equivariant strict inert morphisms which are not necessarily regular. In this case, the normalized blow up tower of X does not have to pullback to Y, but one can hope that $f_{(H,Y,T)}$ is the strict transform of $f_{(G,X,Z)}$. We indicate without proofs what can be done in this direction.

A careful examination of the argument shows that f_1 and f_2 are functorial with respect to all strict morphisms, while f_3 is functorial with respect to all schematically inert morphisms as defined below. A λ-equivariant morphism of separated schemes $Y_1 \to X_1$ is called **schematically inert** if for any subgroup $G' \subset G$ the pullback $X_1^{G'} \times_{X_1} Y_1$ of $X_1^{G'}$ is a disjoint union of $Y_1^{H'_i}$ with $\lambda(H'_i) = G'$. For example, if μ_2 acts on the coordinate x of $X_1 = \operatorname{Spec} \mathbf{Q}[x]$ by the non-trivial character then the μ_2-equivariant morphism $X_1 \to X_1$ sending x to x^3 is inert but not schematically inert.

It remains to examine the construction of f_4. Note that we only used the regularity of $h\colon Y \to X$ to construct compatible rigidifications of X and Y, since Lemma 5.4.4 holds for arbitrary strict inert morphisms. In fact, one can formulate a necessary and sufficient criterion guaranteeing that étale locally on X there exist compatible rigidifications $\overline{Z}$ and $\overline{T}$. Note that if $\overline{Z}$ and $\overline{T}$ exist then the inertia strata of X and Y are (set-theoretic) disjoint unions of connected components of log-strata of $\overline{Z}$ and $\overline{T}$. Since $\overline{T}^i = \overline{Z}^i \times_X Y$, this implies that h is schematically inert.

In addition, $\overline{Z}^i$ and $\overline{T}^i$ are of the same codimension hence the following condition holds: $(*)$ for any connected component V of $X^{G'}$ with non-empty preimage in Y, the codimension of V in X equals to the codimension of $V \times_X Y$ in Y. This motivates a further strengthening of the notion of inertness: a schematically inert morphism $h_1 : Y_1 \to X_1$ is called **derived inert** if the morphisms h_1 and $X_1^{G'} \to X_1$ are Tor-independent for any subgroup $G' \subset G$. One can show that in the case of regular X_1 and Y_1 this happens if and only if $(*)$ is satisfied. For example, if μ_n acts on the coordinates of $X = \operatorname{Spec} \mathbf{Q}[x, y]$ by characters then the μ_n-equivariant closed immersion $\operatorname{Spec} \mathbf{Q}[x] \hookrightarrow X$ is always schematically inert, and it is derived inert if and only if the action on y is through the trivial character.

As we saw, existence of compatible rigidifications guarantees that $h\colon Y \to X$ is derived inert, and strengthening Lemma 5.3.8 one can show that, conversely, if h is derived inert then compatible rigidifications exist étale locally. In particular, functoriality holds for arbitrary morphisms which are equivariant, strict and derived inert.

TALK IX

Local Uniformization Prime to ℓ

Luc Illusie

1. Statement recall and first reductions

Let us recall the statement of the local uniformization theorem prime to ℓ (**II**-4.3.1, **III**-6.1) :

Théorème 1.1. *Let X be a noetherian quasi-excellent scheme, Z a rare closed subset of X and ℓ a prime number invertible on X. There exists a finite family of morphisms $(p_i : X_i \to X)_{i \in I}$, covering for the topology of ℓ'-alterations (**II**-2.3) and such that, for every $i \in I$:*
(i) X_i is regular and integral,
(ii) $p_i^{-1}(Z)$ is the support of a strict normal crossings divisor.

The first essential ingredient of the proof of 1.1 is the following result, a weak form of a result by de Jong [**de Jong, 1997**, 2.4] :

Théorème 1.2. *Let $f : X \to Y$ be a proper morphism of noetherian excellent integral schemes, and Z a rare closed subscheme of X. Let η be the generic point of Y. We assume that X_η is smooth, geometrically irreducible and of dimension 1, and that Z_η is étale. There exists then a finite group G, a commutative diagram of G-schemes*

$$\begin{tikzcd} X' \arrow[r, "a"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "b"] & Y \end{tikzcd},$$

an effective G-equivariant divisor D in X', and a rare G-equivariant closed subset T' of Y' having the following properties :
(i) f' is projective ;
(ii) G acts trivially on X and Y, freely on $Y' - T'$;
(iii) a and b are projective alterations generically étale, and $Y'/G \to Y$ (resp. $X'/G \to X$) induces an isomorphism at η (resp. at the generic point of X_η) ;
(iv) f' is a nodal curve, smooth outside T' ;
(v) D is étale over Y', and contained in the smooth locus of f' ;
(vi) $Z' := a^{-1}(Z)$ is contained in $D \cup f'^{-1}(T')$.

Recall that saying that f' is a nodal curve means that f' is flat, with connected geometric fibers of dimension 1, having as only singularities ordinary quadratic points.

It suffices indeed to apply *loc. cit.* to the pair (f, Z), with the group G of *loc. cit.* equal to $\{1\}$. The hypotheses made on X_η and Z_η ensure that (f, Z) satisfies conditions (2.1.1) and (2.1.5) of [**de Jong, 1997**], and thus that the pair (a, b) satisfies (2.2.1) and (2.2.5) of *loc. cit.*, which implies (iii). The closed subset T' is given by the closed subset denoted D in (*loc. cit.*, 2.5), possibly enlarged so that G acts freely on $Y' - T'$. Note that, if Y is separated, so is Y', and (ii) implies that G acts faithfully on X' and Y'.

1.3. The first reductions of the proof of 1.1 are analogous to those of the proof of the weak local uniformization theorem. It suffices to prove 1.1 for X of finite dimension. We reason by induction on the dimension of X. The theorem is known in dimension ≤ 1 (normalization). Let d be an integer ≥ 2. Suppose the theorem established in dimension $< d$. According to (**III**-6.2), we can assume X is local noetherian complete, and even normal. According to **V**-3.1.3, up to making a finite extension of X of generic degree prime to ℓ, we can assume that there exists a diagram

$$\begin{tikzcd} X \arrow[r, "g"] & X' \arrow[d, "f"] \\ & Y \end{tikzcd},$$

with Y local noetherian regular complete of dimension $d-1$ and f of finite type, with fibers of dimension 1, and a closed point x' of X' and a rare closed subset Z' of X' such that $f(x')$ is the closed point of Y, g induces an isomorphism from X onto the completion of X' at x', and finally $Z = g^{-1}(Z')$. Since X' is excellent, g is regular. Base change by g preserves regularity, normal crossings divisors, and covering families for the topology of ℓ'-alterations (**II**-2.3). We can thus replace X by X', so, up to changing notation, assume X integral of dimension d, equipped with a finite type morphism $f : X \to Y$, with fibers of dimension 1. The problem being local for the topology of ℓ'-alterations, hence *a fortiori* for the Zariski topology, we can assume X is affine. Compactifying f, we reduce to assuming f proper. Up to blowing up in X a closed subscheme having Z as underlying space, we can assume that Z is a divisor in X. The morphism f is no longer necessarily a relative curve, but its generic fiber remains of dimension 1. Let η be the generic point of Y. According to ([**ÉGA** IV 4.6.6]) there exists a finite radical extension η' of η such that $(X_{\eta'})_{\mathrm{red}}$ is geometrically reduced, and $(Z_{\eta'})_{\mathrm{red}}$ is étale over η'. Up to replacing Y by its normalization in η', X by its normalization in the fraction field of $(X_{\eta'})_{\mathrm{red}}$, and Z by its reduced inverse image, we can thus assume that X_η is smooth and that Z_η is étale. The scheme Y is no longer necessarily regular, but remains affine, normal, integral and excellent. Consider the Stein factorization $X \xrightarrow{f_1} Y_1 \xrightarrow{q} Y$ of f : q is finite surjective, generically étale, f_1 is proper and surjective, and its geometric fibers are connected. Since $f_{1\star}\mathscr{O}_X = \mathscr{O}_{Y_1}$ and X is integral, Y_1 is integral as well. Up to replacing Y by Y_1 (and f by f_1), we can thus assume that the generic fiber of f is smooth, geometrically connected, and that Z_η is étale. We are then in the situation of 1.2, with Z a divisor, and Y affine.

1.4. Let us apply 1.2 to the situation we have just obtained. According to (iii), the morphism $X'/G \to X$ is $\mathrm{alt}_{\ell'}$ covering. Replacing X by X'/G, Z by its inverse image, and Y by Y'/G, and changing notation, we can thus assume that $X = X'/G$, $Y = Y'/G$. Note that, since Y is affine and b, f' are projective, the actions of G on X' and Y' are admissible, and X'/G and Y'/G are still integral and excellent. Let H be an ℓ-Sylow subgroup of G. Consider the factorization

$$\begin{tikzcd} X' \arrow[r, "a_1"] \arrow[d, "f'"] & X'/H \arrow[r, "a_2"] \arrow[d] & X \arrow[d, "f"] \\ Y' \arrow[r, "b_1"] & Y'/H \arrow[r, "b_2"] & Y \end{tikzcd}.$$

Since a_2 is $\mathrm{alt}_{\ell'}$-covering, we can (using the admissibility of the action of H) replace X by X'/H, Z by its inverse image in X'/H, Y by Y'/H, and finally G by H, so that we can assume that G is an ℓ-group.

Let us apply the induction hypothesis to the pair $(Y = Y'/G, T := T'/G)$. There exists a finite $\mathrm{alt}_{\ell'}$-covering family $(Y_i \to Y)_{i\in I}$, with Y_i regular connected, and (the space underlying) $T_i = Y_i \times_Y T$ is the support of a strict normal crossings divisor. For each $i \in I$, let Y'_i be the normalization of an irreducible component of $Y' \times_Y Y_i$ and $G_i \subset G$ be the decomposition group of this component. Replacing Y by Y_i, Y' by Y'_i, G by G_i, and the other data by their inverse images under $Y_i \to Y$, $Y'_i \to Y'$, and working separately on each Y_i, we reduce to assuming that, in the diagram of 1.2, we have the following additional properties :

$(*)$ $Y = Y'/G$ is affine, regular, connected, $T = T'/G$ is a strict normal crossings divisor in Y, $Y' - T' = Y' \times_Y (Y - T)$ is a Galois étale cover of $Y - T$ with group G, Y' is the normalization of Y in $Y' - T'$, $X = X'/G$.

2. Log regularity, end of the proof

We will need the following result, a special case of a theorem by Fujiwara-Kato [**Fujiwara & Kato, 1995**, 3.1] :

Proposition 2.1. *Let Y be a noetherian regular scheme, $T \subset Y$ a strict normal crossings divisor, $V = Y - T$. Equip Y with the log structure associated to the pair (Y, T) (**VI**-1.4). Then :*

(i) The restriction functor from the category of Kummer étale covers of Y to that of étale covers of V tamely ramified along T is an equivalence of categories.

(ii) If Y' is a Kummer étale cover of Y, Y' is the normalization of Y in $Y' \times_Y V$.

(iii) If V' is an étale cover of V, tamely ramified along T, there exists a unique fs log structure on the normalization Y' of Y in V' making Y' a Kummer étale cover of Y.

It suffices to prove (i) and (ii). The question is local for the étale topology on Y. We can thus assume Y is strictly local, $Y = \operatorname{Spec} A$, with closed point $y = \operatorname{Spec} k$, and $T = \sum_{1 \leq i \leq r} \operatorname{div}(t_i)$, where the t_i are part of a regular system of parameters of A. The log scheme Y admits the chart $(\overline{M}_y = \mathbf{N}^r \to A, e_i \mapsto t_i)$. According to the local structure theorem for Kummer étale covers (**VI**-2.2), every Kummer étale cover Y' of Y is a sum of standard Kummer étale covers, of the form $Z = Y \times_{\operatorname{Spec} \mathbf{Z}[\mathbf{N}^r]} \operatorname{Spec} \mathbf{Z}[Q]$, where $\mathbf{N}^r \to Q$ is a Kummer morphism such that $nQ \subset \mathbf{N}^r$ for an integer n prime to the characteristic of k. We thus have $Q = \mathbf{N}^r \cap L$, for a subgroup L of $\mathbf{Z}^r$ such that $n[\mathbf{Z}^r : L] = 0$, and $Z = (\operatorname{Spec} A[x_1, \cdots, x_r]/(x_1^n - t_1, \cdots, x_r^n - t_r))^G$, where $G = \operatorname{Hom}(\mathbf{Z}^r/L, \mathbf{Q}/\mathbf{Z})$ is a subgroup of μ_n^r acting on $\operatorname{Spec} A[x_1, \cdots, x_r]/(x_1^n - t_1, \cdots, x_r^n - t_r)$ in the natural way. It follows that Z is the normalization of Y in the étale cover $Z \times_Y V$ of V. We deduce (ii), and full faithfulness in (i), from the consideration of graphs of morphisms between Kummer étale covers of Y. Essential surjectivity follows from Abhyankar's lemma ([**SGA 1** XIII 5.3]) giving the structure of the tame fundamental group of V, which implies that the normalization of Y in a connected tame étale cover of V is of the form Z described previously.

2.2. Starting from the situation obtained at the end of 1.4, according to 2.1, in view of $(*)$, there exists on Y' a unique log structure making Y' a Kummer étale cover of Y (equipped with the log structure defined by T), Galois with group G. In particular, the pair (Y', T') is log regular. According to (**VI**-1.9), the pair $(X', f'^{-1}(T') \cup D)$ is log regular, and for the corresponding log structure on X', f' is log smooth. Moreover, the inverse image Z' of Z in X' is a divisor contained in $D' = f'^{-1}(T') \cup D$.

The action of G on X' is tame (G is an ℓ-group), but not necessarily very tame (**VI**-3.1). If it were, the pair $(X = X'/G, D'/G)$ would then be log regular (**VI**-3.2), and one could finish the proof of 1.1 as in (**VII**-4), using resolution of singularities of log regular pairs. We reduce to this case thanks to the modification theorem (**VIII**-1.1), whose statement we recall :

Théorème 2.3. *Let (X, Z) be a log regular pair (**VI**-1.4), equipped with an action of a finite group G. We assume that X is noetherian, separated, and that the action of G on X is tame and generically free. Let T be the complement of the largest G-stable open subset of X where G acts freely. There exists then a projective G-equivariant modification $f : X' \to X$ such that, if $Z' = f^{-1}(Z \cup T)$, the pair (X', Z') is log regular, and the action of G on X' is very tame.*

2.4. The pair (X', D') of 2.2 satisfies the hypotheses on (X, Z) of 2.3 : (X', D') is log regular, X' is separated (being projective over Y), the action of G on X' is tame, and free on $X' - f'^{-1}(T')$, in particular generically free. Moreover, the action of G on X' is admissible. There exists thus a commutative G-equivariant diagram

$$\begin{tikzcd}
(X'', D'') \arrow[r] \arrow[d, "p"] & (X''/G, D''/G) \arrow[d, "q"] \\
(X', D') \arrow[r] & (X, D) = (X'/G, D'/G)
\end{tikzcd} ,$$

where the horizontal arrows are the canonical projections, p is a projective modification (G-equivariant), (X'', D'') is log regular, with $X'' - D'' \subset p^{-1}(X' - D')$, and the action of G on (X'', D'') is very tame. According to **VI**-3.2, the pair $(X''/G, D''/G)$ is thus log regular. Let us apply to this pair the desingularization theorem of Kato-Nizioł ([**Kato, 1994**, 10.3, 10.4], [**Nizioł, 2006**, 5.7], [**Gabber & Ramero, 2013**, 9.6.32]) : there exists a log étale morphism $e : \tilde{X} \to X''/G$, projective, surjective and birational on the underlying schemes, with $\tilde{X}$ regular, and a strict normal crossings divisor $\tilde{D}$ such that $\tilde{X} - \tilde{D} = e^{-1}(X''/G - D''/G)$. Then $\tilde{Z} := (qe)^{-1}(Z)$ has as support a divisor contained in $\tilde{D}$, hence with strict normal crossings. Since q is a modification, qe is one as well, and 1.1 is proven.

EXPOSÉ X

Gabber's modification theorem (log smooth case)

Luc Illusie and Michael Temkin [(i)]

In this exposé we state and prove a variant of the main theorem of VIII (see **VIII**-1.1) for schemes X which are log smooth over a base S with trivial G-action. See 1.1 for a precise statement. The proof is given in §1 and in the remaining part of the exposé we deduce refinements of classical theorems of de Jong, for schemes of finite type over a field or a trait, where the degree of the alteration is made prime to a prime ℓ invertible on the base. Sections 2 and 3 are independent and contain two different proofs of such a refinement, so let us outline the methods briefly.

For concreteness, assume that k is a field, $S = \mathrm{Spec}(k)$, and X is a separated S-scheme of finite type. Two methods to construct regular ℓ'-alterations of X are: (1) use a pluri-nodal fibration to construct a regular G-alteration $X' \to X$ and then factor X' by an ℓ-Sylow subgroup of G, and (2) construct a regular ℓ'-alteration by induction on $\dim(S)$ so that one factors by an ℓ-Sylow subgroup at each step of the induction. The first approach is presented in §2. It is close in spirit to the approach of [**de Jong, 1997**] and its strengthening by Gabber-Vidal, see [**Vidal, 2004**, §4]. The weak point of this method is that one uses inseparable Galois alterations. In particular, even when k is perfect, one cannot obtain a separable alteration of X.

The second approach is realized in §3, using [**Temkin, 2010**]; it outperforms the method of §2 when k is perfect. Moreover, developing this method the second author discovered Theorem 3.5 that generalizes Gabber's theorems 2.1 and 2.4 to the case of a general base S satisfying a certain resolvability assumption (see §3.3). In addition, if S is of characteristic zero then the same method allows to use modifications instead of ℓ'-alterations, see Theorem 3.9. As an application, in Theorem 3.10 we generalize Abramovich-Karu's weak semistable reduction theorem. Finally, we minimize separatedness assumptions in §3, and for this we show in §3.1 how to weaken the separatedness assumptions in Theorems 1.1 and **VIII**-1.1.

1. The main theorem

Theorem 1.1. *Let $f : X \to S$ be an equivariant log smooth map between fs log schemes endowed with an action of a finite group G. Assume that:*

(i) G acts trivially on S;

*(ii) X and S are noetherian, qe, separated, log regular, and f defines a map of log regular pairs $(X, Z) \to (S, W)$ (see **VI**-1.4: (X, Z) and (S, W) are log regular pairs and $f(X - Z) \subset S - W$));*

(iii) G acts tamely and generically freely on X.

*Let T be the complement of the largest open subset of X over which G acts freely. Then there exists an equivariant projective modification $h : X' \to X$ such that, if $Z' = h^{-1}(Z \cup T)$, the pair (X', Z') is log regular, the action of G on X' is very tame, and (X', Z') is log smooth over (S, W) as well as the quotient $(X'/G, Z'/G)$ when G acts admissibly on X ([**SGA 1** v 1.7]).*

Remark 1.1.1. (a) In the absence of the hypothesis (i) it may not be possible to find a modification h satisfying the properties of 1.1, as the example at the end of **VIII**-1.2 shows.

(b) By [**Kato, 1994**, 8.2] the log smoothness of f and the log regularity of S imply the log regularity of X. Conversely, according to Gabber (private communication), if X is log regular and f is log smooth and surjective, then S is log regular.

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(c) We will deduce Theorem 1.1 from Theorem **VIII**-1.1. Recall that in the latter theorem we assumed that X is qe, though Gabber has a subtler argument that works for a general X. This forces us to assume that S (and hence X) is qe in Theorem 1.1. However, our argument also shows that once one removes the quasi-excellence assumption from **VIII**-1.1, one also obtains the analogous strengthening of Theorem 1.1.

For the proof of 1.1 we will use the following result on the local structure of equivariant log smooth maps.

Proposition 1.2 (Gabber's preparation lemma). *Let $f : X \to Y$ be an equivariant log smooth map between fine log schemes endowed with an action of a finite group G. Let x be a geometric point of X, with image y in Y. Assume that G is the inertia group at x and is of order invertible on Y. Assume furthermore that G acts trivially on $\overline{M}_x$ and $\overline{M}_y$*[ii] *and we are given an equivariant chart $a : Y \to \operatorname{Spec} \Lambda[Q]$ at y, modeled on some pairing $\chi : G^{\mathrm{ab}} \otimes Q^{\mathrm{gp}} \to \mu = \mu_N(\mathbf{C})$ (in the sense of (**VI**-3.3)), where Q is fine, $\Lambda = \mathbf{Z}[1/N, \mu]$, with N the exponent of G. Then, up to replacing X by an inert equivariant étale neighborhood of x, there is an equivariant chart $b : X \to \operatorname{Spec} \Lambda[P]$ extending a, such that $Q^{\mathrm{gp}} \to P^{\mathrm{gp}}$ is injective, the torsion of its cokernel is annihilated by an integer invertible on X, and the resulting map $b' : X \to X' = Y \times_{\operatorname{Spec} \Lambda[Q]} \operatorname{Spec} \Lambda[P]$ is smooth. Moreover, up to further shrinking X around x, b' lifts to an inert equivariant étale map $c : X \to X' \times_{\operatorname{Spec} \Lambda} \operatorname{Spec} \operatorname{Sym}_\Lambda(V)$, where V is a finitely generated projective Λ-module equipped with a G-action. If X, Y, and Q are fs, with Q sharp, then P can be chosen to be fs with its subgroup of units P^* torsionfree.*

Proof of 1.2. This is an adaptation of the proof of [**Kato, 1988**, 3.5] to the equivariant case. Consider the canonical homomorphism of *loc. cit.*

$$k(x) \otimes_{\mathscr{O}_{X,x}} \Omega^1_{X/Y,x} \to k(x) \otimes_{\mathbf{Z}} \overline{M}^{\mathrm{gp}}_{X/Y,x} \tag{1.2.1}$$

sending $1 \otimes d\log t$ to the class of $1 \otimes t$, where

$$\overline{M}^{\mathrm{gp}}_{X/Y,x} = M^{\mathrm{gp}}_{X,x} / (\mathscr{O}^*_{X,x} + \operatorname{Im} f^{-1}(M^{\mathrm{gp}}_{Y,y})).$$

It is surjective, and as G fixes x, it is G-equivariant. As G is of order invertible in $k(x)$ and acts trivially on the right hand side, (1.2.1) admits a G-equivariant decomposition

$$k(x) \otimes_{\mathscr{O}_{X,x}} \Omega^1_{X/Y,x} = V_0 \oplus (k(x) \otimes_{\mathbf{Z}} \overline{M}^{\mathrm{gp}}_{X/Y,x}), \tag{1.2.2}$$

where V_0 is a finite dimensional $k(x)$-vector space, endowed with an action of G. Let $(t_i)_{1 \le i \le r}$ be elements of M^{gp}_x such that the classes of $1 \otimes t_i$ form a basis of $k(x) \otimes_{\mathbf{Z}} \overline{M}^{\mathrm{gp}}_{X/Y,x}$. By the method of (**VI**-3.5) we can modify the t_i's to make them eigenfunctions of G. More precisely, for $g \in G$, we have

$$g t_i = z_i(g) t_i,$$

with $z_i(g) \in \mathscr{O}^*_x$, and $g \mapsto z_i(g)$ is a 1-cocycle of G with values in $\mathscr{O}^*_x$. By reduction mod $\mathfrak{m}_x$, it gives a 1-cocycle $\psi_i \in Z^1(G, \mu) = \operatorname{Hom}(G, \mu)$, as μ is naturally embedded in $k(x)^*$ since X is over Λ. Lifting μ in $\mathscr{O}^*_x$, $g \mapsto z_i(g)/\psi_i(g)$ is a 1-cocycle of G with values in $1 + \mathfrak{m}_x$, hence a coboundary $\delta_i \in B^1(G, 1 + \mathfrak{m}_x)$, $g \mapsto \delta_i(g) = g u_i / u_i$, for $u_i \in 1 + \mathfrak{m}_x$. Replacing t_i by $t_i u_i^{-1}$, we may assume that $z_i = \psi_i$, *i.e.*

$$g t_i = \psi_i(g) t_i,$$

for characters

$$\psi_i : G \to \mu.$$

Let Z be the free abelian group with basis $(e_i)_{1 \le i \le r}$, and $h : Z \to M^{\mathrm{gp}}_x$ the homomorphism sending e_i to t_i. As in the proof of [**Kato, 1988**, 3.5], consider the homomorphism

$$u : Z \oplus Q^{\mathrm{gp}} \to M^{\mathrm{gp}}_x$$

defined by h on Z and the composition $Q^{\mathrm{gp}} \to M^{\mathrm{gp}}_y \to M^{\mathrm{gp}}_x$ on the second factor. We have

$$g u(a) = \psi(g \otimes a) u(a)$$

[ii] If M is the sheaf of monoids of a log scheme, $\overline{M}$ denotes, as usual, the quotient $M/\mathscr{O}^*$.

for some homomorphism

$$\psi : G^{\mathrm{ab}} \otimes (Z \oplus Q^{\mathrm{gp}}) \to \mu$$

extending χ and such that $\psi(g \otimes e_i)u(e_i) = \psi_i(g)h(e_i)$. As in *loc. cit.*, if $\overline{u}$ denotes the composition

$$\overline{u} : Z \oplus Q^{\mathrm{gp}} \to M_x^{\mathrm{gp}} \to \overline{M}_x^{\mathrm{gp}} (= M_x^{\mathrm{gp}}/\mathscr{O}_x^*)$$

we see that $k(x) \otimes \overline{u}$ is surjective, hence the cokernel C of $\overline{u}$ is killed by an integer m invertible in $k(x)$. Using that $\mathscr{O}_{X,x}^*$ is m-divisible, one can choose elements $a_i \in M_x^{\mathrm{gp}}$ and $b_i \in Z \oplus Q^{\mathrm{gp}}$ $(1 \leq i \leq n)$ such that the images of the a_i's generate $\overline{M}_x^{\mathrm{gp}}$ and $a_i^m = u(b_i)$. Let E be the free abelian group with basis e_i $(1 \leq i \leq n)$, and let F be the abelian group defined by the push-out diagram

(1.2.3)
$$\begin{tikzcd} E \arrow[r, "m"] \arrow[d] & E \arrow[d] \\ Z \oplus Q^{\mathrm{gp}} \arrow[r, "w"] & F \end{tikzcd},$$

where the left vertical arrow sends e_i to b_i. The lower horizontal map is injective and its cokernel is isomorphic to E/mE, in particular, killed by m. The relation $a_i^m = u(b_i)$ implies that u extends to a homorphism

$$v : F \to M_x^{\mathrm{gp}}$$

whose composition $\overline{v} : F \to M_x^{\mathrm{gp}} \to \overline{M}_x^{\mathrm{gp}}$ is surjective. Associated with v is a morphism

$$\varphi : G^{\mathrm{ab}} \otimes F \to \mu$$

extending ψ, such that $gv(a) = \varphi(g \otimes a)v(a)$ for $a \in F$. Let $P := v^{-1}(M_x) \subset F$. Then P is a fine monoid containing Q, $P^{\mathrm{gp}} = F$, and v sends P to M_x. As in **VI**-3.5, **VI**-3.10 we get a G-equivariant chart of $X_{(x)}$ associated with φ, which, up to replacing X by an inert equivariant étale neighborhood at x, extends to an equivariant chart

$$b : X \to \operatorname{Spec} \Lambda[P]$$

extending the chart $a : Y \to \operatorname{Spec} \Lambda[Q]$. The homomorphism $Q^{\mathrm{gp}} \to P^{\mathrm{gp}}$ is injective, and the torsion part of its cokernel injects into the cokernel of $w : Z \oplus Q^{\mathrm{gp}} \to F$ in (1.2.3), which is killed by m. Consider the resulting map

$$b' : X \to X' = Y \times_{\operatorname{Spec} \Lambda[Q]} \operatorname{Spec} \Lambda[P].$$

This map is strict. Showing that the underlying schematic map is smooth at x is equivalent to showing that b' is log smooth at x. To do this, as X and X' are log smooth over Y, by the jacobian criterion [**Kato, 1988**, 3.12] it suffices to show that the map

$$k(x) \otimes \Omega^1_{X'/Y} \to k(x) \otimes \Omega^1_{X/Y}$$

induced by b' is injective. We have

$$k(x) \otimes \Omega^1_{X'/Y} = k(x) \otimes P^{\mathrm{gp}}/Q^{\mathrm{gp}} = k(x) \otimes Z$$

(the last equality by the fact that $F/(Z \oplus Q^{\mathrm{gp}})$ is killed by m), and by construction (cf. (1.2.2)), we have

$$k(x) \otimes Z = k(x) \otimes \overline{M}^{\mathrm{gp}}_{X/Y,x},$$

which by the map induced by b' injects into $k(x) \otimes \Omega^1_{X/Y}$.

Let us now prove the second assertion. For this, as b' is strict, we may forget the log structures of X and X', and by changing notations, we may assume that $X' = Y$ and the log structures of X and Y are trivial. In particular, we have

$$k(x) \otimes \Omega^1_{X/Y} = V_0,$$

with the notation of (1.2.2). As the question is étale local on X, and closed points are very dense in the fiber X_y, in particular, any point has a specialization at a closed point of X_y, we may assume that

x sits over a closed point of X_y, and even, up to base changing Y by a finite radicial extension, that x is a rational point of X_y. We then have

$$k(x) \otimes \Omega^1_{X/Y} = \mathfrak{m}_x/(\mathfrak{m}_x^2 + \mathfrak{m}_y \mathscr{O}_x),$$

where $\mathfrak{m}$ denotes a maximal ideal. By a classical result in representation theory (see 1.3 below) there is a finitely generated projective $\Lambda[G]$-module V such that $V_0 = k(x) \otimes V$. The homomorphism $V \to \mathfrak{m}_x/(\mathfrak{m}_x^2 + \mathfrak{m}_y \mathscr{O}_x)$ therefore lifts to a homomorphism of $\Lambda[G]$-modules

$$V \to \mathfrak{m}_x,$$

inducing an isomorphism $k(x) \otimes V \to k(x) \otimes \Omega^1_{X/Y}$. By the jacobian criterion, it follows that the (G-equivariant) map

$$X \to Y \times_{\operatorname{Spec} \Lambda} \operatorname{Spec} \operatorname{Sym}_\Lambda(V)$$

is étale at x, and as in **VI**-3.10 can be made inert by shrinking X.

Let us prove the last assertion. Now X and Y are fs, and Q is fs and sharp. First of all, as M_x is saturated, $P = \nu^{-1}(M_x)$ is fs. Then (cf. [**Gabber & Ramero, 2013**, 3.2.10]) we have a split exact sequence

$$0 \to H \to P \to P_0 \to 0$$

with P_0^* torsionfree and H a finite group. As Q is fs and sharp, Q^{gp} is torsionfree, so the composition $Q^{\mathrm{gp}} \to P^{\mathrm{gp}} \to P_0^{\mathrm{gp}}$ is still injective, as well as the composition $H \to P^{\mathrm{gp}} \to P^{\mathrm{gp}}/Q^{\mathrm{gp}}$, hence H is contained in the torsion part of $P^{\mathrm{gp}}/Q^{\mathrm{gp}}$, and we have an exact sequence

$$0 \to H \to (P^{\mathrm{gp}}/Q^{\mathrm{gp}})_{\mathrm{tors}} \to (P_0^{\mathrm{gp}}/Q^{\mathrm{gp}})_{\mathrm{tors}} \to 0,$$

where the subscript *tors* denotes the torsion part. Thus $(P_0^{\mathrm{gp}}/Q^{\mathrm{gp}})_{\mathrm{tors}}$ is killed by an integer invertible on X. As $\overline{M}_x$ is torsionfree, the composition $P \to M_x \to \overline{M}_x$ factors through P_0, into a map $v_0 : P_0 \to \overline{M}_x$. Consider the diagram

$$\begin{tikzcd} & M_x \arrow[r] \arrow[d] & M_x^{\mathrm{gp}} \arrow[d] \\ P_0 \arrow[r, "v_0"] & \overline{M}_x \arrow[r] & \overline{M}_x^{\mathrm{gp}} \end{tikzcd},$$

where the square is cartesian. As P_0^{gp} is torsionfree, the map $P_0^{\mathrm{gp}} \to \overline{M}_x^{\mathrm{gp}}$ defined by the lower row admits a lifting $s : P_0^{\mathrm{gp}} \to M_x^{\mathrm{gp}}$, sending P_0 to M_x. One can adjust s to make it compatible with the morphism $\tilde{a} : Q^{\mathrm{gp}} \to M_y^{\mathrm{gp}} \to M_x^{\mathrm{gp}}$ given by the chart $a : Q \to M_Y$. Indeed, if $j : Q^{\mathrm{gp}} \hookrightarrow P_0^{\mathrm{gp}}$ is the inclusion, the homomorphisms $sj/\tilde{a} : Q^{\mathrm{gp}} \to \mathscr{O}^*_{X,x}$ can be extended to P_0^{gp} as the torsion part of $P_0^{\mathrm{gp}}/Q^{\mathrm{gp}}$ is killed by an integer invertible on X. Assume that this adjustment is done. As v is a chart, $P/v^{-1}(\mathscr{O}_x^*) \to \overline{M}_x$ is an isomorphism, and since H is contained in $v^{-1}(\mathscr{O}_x^*)$, $P_0/s^{-1}(\mathscr{O}_x^*) \to \overline{M}_x$ is an isomorphism as well, hence $s : P_0 \to M_x$ is a chart at x compatible with a. A second adjustment is needed to make it G-equivariant. To do so, one can proceed as above, by considering the 1-cocycle z of G with values in $\operatorname{Hom}(P_0^{\mathrm{gp}}, \mathscr{O}_x^*)$ given by

$$gs(p) = z(g,p)s(p).$$

The image of z in $Z^1(G, \operatorname{Hom}(P_0^{\mathrm{gp}}, k(x)^*))$ is a homomorphism

$$\varphi_0 : G^{\mathrm{ab}} \otimes P_0^{\mathrm{gp}} \to \mu.$$

The quotient $g \mapsto (p \mapsto z(g,p)/\varphi_0(g,p))$ belongs to $B^1(G, \operatorname{Hom}(P_0^{\mathrm{gp}}/Q^{\mathrm{gp}}, 1 + \mathfrak{m}_x))$, hence can be written $g \mapsto (p \mapsto g\rho(p)/\rho(p))$ for $\rho : P_0^{\mathrm{gp}}/Q^{\mathrm{gp}} \to 1 + \mathfrak{m}_x$. So, replacing z by $g \mapsto z(g,p)\rho(p)^{-1}$, we may assume that $z = \varphi_0$, in other words, the map

$$b_0 : X \to \operatorname{Spec} \Lambda[P_0]$$

defined by the pair (s, φ_0) is an equivariant chart of X at x (extending a).

One can give an alternate, shorter proof of the last assertion which does not use the above decomposition of P into $H \oplus P_0$. Consider again the cokernel C of the map $\overline{u}$ introduced a few lines above diagram (1.2.3). Write C as a direct sum of cyclic groups of orders $m_i | m$. Choose $a_i \in M_x^{\mathrm{gp}}$ and $b_i \in Z \oplus Q^{\mathrm{gp}}$ $(1 \leq i \leq n)$ such that $a_i^{m_i} = u(b_i)$, and the a_i's induce an isomorphism

$$\bigoplus_i \mathbf{Z}/m_i\mathbf{Z} \xrightarrow{\sim} C.$$

Replace diagram (1.2.3) by the following push-out diagram

(1.2.4)
$$\begin{tikzcd}
\bigoplus_i \mathbf{Z}e_i \arrow[r] \arrow[d] & \bigoplus_i \mathbf{Z}(\frac{1}{m_i}e_i) \arrow[d] \\
Z \oplus Q^{\mathrm{gp}} \arrow[r, "w"] & F
\end{tikzcd},$$

where the upper horizontal map is the natural inclusion and the left vertical one sends e_i to b_i. In this way, we have $F = P^{\mathrm{gp}} \supset Z \oplus Q^{\mathrm{gp}}$ and $P^{\mathrm{gp}}/(Z \oplus Q^{\mathrm{gp}}) \xrightarrow{\sim} C$. As X is fs, $\overline{M}_x^{\mathrm{gp}}$ is torsionfree, so the map $\overline{\nu} : F \to \overline{M}_x^{\mathrm{gp}}$, defined similarly as above (using (1.2.4) instead of (1.2.3)), sends $(P^{\mathrm{gp}})_{\mathrm{tors}}$ to 0, hence

$$(P^{\mathrm{gp}})_{\mathrm{tors}} = (Z \oplus Q^{\mathrm{gp}})_{\mathrm{tors}} = (Q^{\mathrm{gp}})_{\mathrm{tors}},$$

which finishes the proof. □

Lemma 1.3. *Let G be a finite group of exponent n, let $\Lambda = \mathbf{Z}[\mu_n][1/n]$, let k be a field over Λ, and let L be a finitely generated $k[G]$-module. There exists a finitely generated projective $\Lambda[G]$-module V such that $L = k \otimes_\Lambda V$.*

Proof. First, observe that since n is invertible in Λ, any $\Lambda[G]$-module which is finitely generated and projective over Λ is projective over $\Lambda[G]$ [**Serre, 1978**, §14.4, Lemme 20].

Suppose first that $\mathrm{char}(k) = 0$, and let $\overline{k}$ be an algebraic closure of k. Then, L descends to a $\mathbf{Q}[\mu_n][G]$-module W, as $\overline{k} \otimes_k L$ descends [**Serre, 1978**, §12.3] and the homomorphism $R_k(G) \to R_{\overline{k}}(G)$ given by extension of scalars is injective [**Serre, 1978**, §14.6]. One can then take for V a G-stable Λ-lattice in W (projective over Λ), which is necessarily projective over $\Lambda[G]$ by the above remark.

Suppose now that $\mathrm{char}(k) = p > 0$. Let $I \twoheadrightarrow k$ be a Cohen ring for k. As Λ is étale over $\mathbf{Z}$, $\Lambda \to k$ lifts (uniquely) to $\Lambda \to I$. On the other hand, as L is projective of finite type over $k[G]$, by [**Serre, 1978**, §14.4, Prop. 42, Cor. 3] L lifts to a finitely generated projective $I[G]$-module E, free over I. Let K be the fraction field of I. Then $E \otimes K$ descends to a $\mathbf{Q}[\mu_n][G]$-module E'. Choose a G-stable Λ-lattice V in E' (projective over Λ, hence, projective of finite type over $\Lambda[G]$). By [**Serre, 1978**, §15.2, Th. 32], $k \otimes_\Lambda V$ has the same class in $R_k(G)$ as L. But, as $k[G]$ is semisimple by Maschke's theorem, L and $k \otimes_\Lambda V$ are isomorphic as $k[G]$-modules. □

Proof of 1.1 (beginning).

The strategy is to check that, at each step of the proof of the absolute modification theorem (**VIII**-1.1), the log smoothness of X/S is preserved, and, at the end, that of the quotient $(X/G)/S$ as well. For some of them, this is trivial, as the modifications performed are log blow ups. Others require a closer inspection.

1.4. *Preliminary reductions.* We may assume that conditions (1) and (2) at the beginning of (**VIII**-4) are satisfied, namely:

(1) *X is regular,*

(2) *Z is a G-strict snc divisor in X.*

Indeed, these conditions are achieved by G-equivariant saturated log blow up towers (**VIII**-4.1.1, **VIII**-4.1.6).

We will now exploit Gabber's preparation lemma 1.2 to give a local picture of f displaying both the log stratification and the inertia stratification of X. We work étale locally at a geometric point x in X with image s in S. Up to replacing X by the G_x-invariant neighborhood X' constructed at the beginning of the proof of **VIII**-5.3.8, and G by G_x, where G_x is the inertia group at x, we may assume that $G = G_x$. Indeed, the morphism $(X', G_x) \to (X, G)$ is strict and inert, and by **VIII**-5.4.4 the tower $f_{(G,X,Z)}$ is functorial with respect to such morphisms.

We now apply 1.2. Let N be the exponent of G. Assume S strictly local at s. We may replace $\Lambda = \mathbf{Z}[1/N, \mu]$ by its localization at the (Zariski) image of s, so that Λ is either the cyclotomic field $\mathbf{Q}(\mu)$ or its localization at a finite place of its ring of integers, of residue characteristic $p = \mathrm{char}(k(s))$ not dividing n. Choose a chart

$$a : S \to \mathrm{Spec}\ \Lambda[Q]$$

with Q fs and the inverse image of $\mathscr{O}^*_{S,s}$ in Q equal to $\{1\}$, so that Q is sharp and $Q \xrightarrow{\sim} \overline{M}_s$. Let C denote $k(s)$ if $\mathscr{O}_{S,s}$ contains a field, and a Cohen ring of $k(s)$ otherwise. Let $(y_i)_{1 \le i \le m}$ be a family of elements of $\mathfrak{m}_s$ such that the images of the y_i's in $\mathscr{O}_{S,s}/I_s$ form a regular system of parameters, where $I_s = I(s, M_s)$ is the ideal generated by the image of $M_s - \mathscr{O}^*_{S,s}$ by the canonical map $\alpha : M_s \to \mathscr{O}_{S,s}$. By [**Kato, 1994**, 3.2], the chart a extends to an isomorphism

$$C[[y_1, \cdots, y_m]][[Q]]/(g) \xrightarrow{\sim} \widehat{\mathscr{O}}_{S,s}, \tag{1.4.1}$$

where $g \in C[[y_1, \cdots, y_m]][[Q]]$ is 0 if $C = k(s)$, and congruent to $p = \mathrm{char}(k(s)) > 0$ modulo the ideal generated by $Q - \{1\}$ and $(y_1, \cdots, y_m)$ otherwise. By 1.2, up to shrinking X around x, we can find a G-equivariant commutative diagram (with trivial action of G on the bottom row)

$$\begin{tikzcd} X \arrow[r, "c"] \arrow[dr] & X' \arrow[r, "b"] \arrow[d] & \mathrm{Spec}\,(\Lambda[P] \otimes_\Lambda \mathrm{Sym}_\Lambda(V)) \arrow[d] \\ & S \arrow[r, "a"] & \mathrm{Spec}\ \Lambda[Q] \end{tikzcd} \tag{1.4.2}$$

where:

(i) the square is cartesian;

(ii) a, b, and c are strict, where the log structure on Spec $\Lambda[Q]$ (resp. Spec $(\Lambda[P] \otimes_\Lambda \mathrm{Sym}_\Lambda(V))$) is the canonical one, given by Q (resp. P); P is an fs monoid, with P^* torsionfree; G acts on $\Lambda[P]$ by $g(\lambda p) = \lambda\chi(g, p)p$, for some homomorphism

$$\chi : G^{\mathrm{ab}} \otimes P^{\mathrm{gp}} \to \mu$$

(iii) V is a free, finitely generated Λ-module, equipped with a G-action;

(iv) the right vertical arrow is the composition of the projection onto the factor Spec $\Lambda[P]$ and Spec $\Lambda[h]$, for a homomorphisme $h : Q \to P$ such that h^{gp} is injective and the torsion part of $\mathrm{Coker} h^{\mathrm{gp}}$ is annihilated by an integer invertible on X;

(v) c is étale and inert.

(vi) Consider the map

$$v : P \to M_x$$

defined by the chart $X \to$ Spec $\Lambda[P]$ induced by bc. Up to localizing on X' around x, we may assume that v factors through the localization $P_{(\mathfrak{p})}$ of P at the prime ideal $\mathfrak{p}$ complementary of the face $v^{-1}(\mathscr{O}^*_{X,x})$. Replacing P by $P_{(\mathfrak{p})}$, P decomposes into

$$P = P^* \oplus P_1, \tag{1.4.3}$$

with $P^* = v^{-1}(\mathscr{O}^*_{X,x})$ free finitely generated over $\mathbf{Z}$, and P_1 sharp, and the image of x by bc in the factor Spec $\Lambda[P_1]$ is the rational point at the origin. Then v induces an isomorphism $P_1 \xrightarrow{\sim} \overline{M}_x$. By the assumptions (1), (2), we have $\overline{M}_x \xrightarrow{\sim} \mathbf{N}^r$. One can therefore choose $(e_i \in P_1)$ $(1 \le i \le r)$ forming a basis of P_1. Then $v(e_i) = t_i \in M_x \subset \mathscr{O}_{X,x}$ is a local equation for a branch Z_i of Z at x, $(Z_i)_{1 \le i \le r}$ is the set of branches of Z at x, and G acts on t_i through the character $\chi_i = \chi(-, e_i) : G \to \mu$.

Furthermore:

(vii) The square in (1.4.2) is tor-independent.

Indeed, by the log regularity of S and the choice of the chart a, we have, by [**Kato, 1994**, 6.1], $\mathrm{Tor}_i^{\mathbf{Z}[\mathbf{Q}]}(\mathscr{O}_{S,s}, \mathbf{Z}[P]) = 0$ for $i > 0$.

Though this will not be needed, one can describe the local structure of (1.4.2) more precisely as follows. Let

(1.4.4) $$Y := \mathrm{Spec}\,(\Lambda[P] \otimes_\Lambda \mathrm{Sym}_\Lambda(V)) = \mathrm{Spec}\,(\Lambda[P^*] \otimes_\Lambda \Lambda[P_1] \otimes_\Lambda \mathrm{Sym}_\Lambda(V))$$

and let $Y' := \mathrm{Spec}\ C[[y_1, \cdots, y_m]][[Q]] \times_{\mathrm{Spec}\,\Lambda} Y$, with the notation of 1.4.1. We may assume that $X = X'$. Then the completion of X at x is either isomorphic to the completion of Y' at x, or a regular divisor in it, defined by the equation $g' = 0$, where g' is the image of g in $\widehat{\mathscr{O}}_{Y',x}$, with the notation of 1.4.1.

1.5. *Step 3 and log smoothness* (*beginning*). We will now analyze the modifications performed in the proof of Step 3 in **VIII**-4.1.9, **VIII**-4.2.13. The permissible towers used in *loc. cit.* are iterations of operations of the form: for a subgroup H of G, blow up the fixed point (regular) subscheme X^H, and replace Z by the union of its strict transform Z^{st} and the exceptional divisor E. Though such a blow up is not a log blow up in general, we will see that it still preserves the log smoothness of X over S.

We work étale locally around x, so we can assume $X = X'$ in 1.4.2. We then have a cartesian square

(1.5.1) $$\begin{tikzcd} X^H \arrow[r, "b^H"] \arrow[d, "f"'] & Y^H \arrow[d] \\ X \arrow[r, "b"] & Y \end{tikzcd},$$

with Y as in (1.4.4). We also have cartesian squares

(1.5.2) $$\begin{tikzcd} Z \arrow[r, "b"] \arrow[d, "f"'] & T \arrow[d] \\ X \arrow[r, "b"] & Y \end{tikzcd},$$

where $T \subset Y$ is the snc divisor $\sum T_i$, T_i defined by the equation $e_i \in P_1$ (1.4.3), and

(1.5.3) $$\begin{tikzcd} Z \times_X X^H \arrow[r] \arrow[d] & T \times_Y Y^H \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}.$$

LEMMA 1.6. *The squares* (*1.5.1*), (*1.5.2*), *and* (*1.5.3*) *are tor-independent.*

Proof. For (1.5.2), this is because Z (resp. T) is a divisor in X (resp. Y) (cf. [**SGA 6** VII 1.2]). For (1.5.1), as the square (1.4.2) is tor-independent (by 1.4 (vii)), it is enough to show that the composite (cartesian) square

(1.6.1) $$\begin{tikzcd} X^H \arrow[r] \arrow[d] & Y^H \arrow[d] \\ S \arrow[r] & \mathrm{Spec}\ \Lambda[Q] \end{tikzcd}$$

is tor-independent. We have a decomposition

(1.6.2) $$Y^H = (\mathrm{Spec}\ \Lambda[P^*])^H \times (\mathrm{Spec}\ \Lambda[P_1])^H \times (\mathrm{Spec}\ \mathrm{Sym}_\Lambda(V))^H,$$

(products taken over Spec Λ), and the map to Spec $\Lambda[Q]$ is the composition of the projection onto $(\mathrm{Spec}\ \Lambda[P^*])^H \times (\mathrm{Spec}\ \Lambda[P_1])^H$ and the canonical map induced by $\mathrm{Spec}\ \Lambda[Q] \to \mathrm{Spec}\ \Lambda[P]$, which factors through the fixed points of H, G acting trivially on the base. Let us examine the three factors.

(a) We have

$$(\mathrm{Spec}\ \mathrm{Sym}_\Lambda(V))^H = \mathrm{Spec}\ \mathrm{Sym}_\Lambda(V_H),$$

where V_H is the module of coinvariants, a free module of finite type over Λ, as H is of order invertible in Λ. Therefore $\mathrm{Spec}\ \Lambda[Q] \times_{\mathrm{Spec}\,\Lambda} (\mathrm{Spec}\ \mathrm{Sym}_\Lambda(V))^H$ is flat over Spec $\Lambda[Q]$, and its enough to check that $(\mathrm{Spec}\ \Lambda[P^*])^H \times (\mathrm{Spec}\ \Lambda[P_1])^H$ is tor-independent of S over Spec $\Lambda[Q]$.

(b) The restriction to $P^* = v^{-1}(\mathscr{O}^*_{X,x})$ of the 1-cocycle $z(v) \in Z^1(H, Hom(P, k(x)^*))$ associated with $v : P \to M_x$ ($hv(a) = z(v)(h,a)v(a)$ for $h \in H$, $a \in P$, see the proof of 1.2 and (**VI**-3.5), is a 1-coboundary, hence trivial, as $B^1(H, Hom(P, k(x)^*)) = 0$. Therefore

$$(\operatorname{Spec} \Lambda[P^*])^H = \operatorname{Spec} \Lambda[P^*].$$

(c) Recall that

$$P_1 = \bigoplus_{1\le i\le r} \mathbf{N}e_i,$$

with e_i sent by v to a local equation of the branch Z_i of Z, and that G acts on $\Lambda[\mathbf{N}e_i]$ through the character $\chi_i : G \to \mu$. Let $A \subset \{1, \cdots, r\}$ be the set of indices i such that $\chi_i|H$ is trivial. Then

$$(\operatorname{Spec} \Lambda[P_1])^H = \operatorname{Spec} \Lambda[\bigoplus_{i\in A} \mathbf{N}e_i].$$

Let I be the ideal of P generated by $\{e_i\}_{i\notin A}$. It follows from (b) and (c) that

$$(\operatorname{Spec} \Lambda[P])^H = \operatorname{Spec} \Lambda[P]/(I),$$

where (I) is the ideal of $\Lambda[P]$ generated by I. By [**Kato, 1994**, 6.1], $\operatorname{Tor}_i^{\Lambda[Q]}(\mathscr{O}_S, \Lambda[P]/(I)) = 0$ for $i > 0$, and therefore (1.6.1), hence (1.5.1) is tor-independent. It remains to show the tor-independence of (1.5.3). For this, again it is enough to show the tor-independence of

(1.6.3)
$$\begin{tikzcd} Z \times_X X^H \arrow[r] \arrow[d] & T \times_Y Y^H \arrow[d] \\ S \arrow[r] & \operatorname{Spec} \Lambda[Q] \end{tikzcd} .$$

By (a), (b), (c), we have

$$T \times_Y Y^H = \sum_{i\in A} \operatorname{Spec} \Lambda[P]/(J_i) \times \operatorname{Spec} \operatorname{Sym}_\Lambda(V_H),$$

where $J_i \subset P$ is the ideal generated by $e_i \in P_1$, and (J_i) the ideal generated by J_i in $\Lambda[P]$. The desired tor-independence follows from the vanishing of $\operatorname{Tor}_i^{\Lambda[Q]}(\mathscr{O}_S, \Lambda[P]/(J_B))$, where for a subset B of A, J_B denotes the ideal generated by the e_i's for $i \in B$. □

Lemma 1.7. *Consider a cartesian square*

(1.7.1)
$$\begin{tikzcd} V' \arrow[r] \arrow[d] & V \arrow[d] \\ X' \arrow[r, "g"] & X \end{tikzcd} ,$$

where the right vertical arrow is a regular immersion. If (1.7.1) is tor-independent, then the left vertical arrow is a regular immersion, and

$$\mathrm{Bl}_{V'}(X') = X' \times_X \mathrm{Bl}_V(X).$$

Let $W \to X$ be a second regular immersion, such that $V \times_X W \to W$ is a regular immersion, and let $W' = X' \times_X W$. If moreover the squares

(1.7.2)
$$\begin{tikzcd} V' \arrow[r] \arrow[d] & V \arrow[d] \\ X' \arrow[r, "g"] & X \end{tikzcd} ,$$

and

(1.7.3)
$$\begin{tikzcd} V' \times_{X'} W' \arrow[r] \arrow[d] & V \times_X W \arrow[d] \\ X' \arrow[r, "g"] & X \end{tikzcd} ,$$

are tor-independent, then the left vertical arrows are regular immersions, and

$$W'^{\mathrm{st}} = X' \times_X W^{\mathrm{st}},$$

where W^{st} *(resp.* W'^{st}*) is the strict transform of* W *(resp.* W'*) in* $\mathrm{Bl}_V(X)$ *(resp.* $\mathrm{Bl}_{V'}(X')$*).*

Proof. Let I (resp. I') be the ideal of V (resp. V') in X (resp. X'). By the tor-independence of (1.7.1), if $u : E \to I$ is a local surjective regular homomorphism [**SGA 6** VII 1.4], the Koszul complex $g^\star K(u)$ is a resolution of $\mathscr{O}_{V'}$, hence $V' \to X'$ is a regular immersion. Moreover, by [**SGA 6** VII 1.2], for any $n \geq 0$, the natural map $g^\star I^n \to I'^n$ is an isomorphism, and therefore $\mathrm{Bl}_{V'}(X') = X' \times_X \mathrm{Bl}_V(X)$. The tor-independence of (1.7.2) and (1.7.3) imply that of

$$\begin{tikzcd} V' \times_{X'} W' \arrow[r] \arrow[d] & V \times_X W . \arrow[d] \\ W' \arrow[r] & W \end{tikzcd}$$

The second assertion then follows from the first one and the formulas (**VIII**-2.1.3 (ii))

$$W^{\mathrm{st}} = \mathrm{Bl}_{V \times_X W} W,$$

$$W'^{\mathrm{st}} = \mathrm{Bl}_{V' \times_{X'} W'} W'.$$

□

1.8. *Step 3 and log smoothness (end).* As recalled at the beginning of 1.5, we have to show that, if H is a subgroup of G, then the log regular pair (X_1, Z_1) is log smooth over S, where $X_1 := \mathrm{Bl}_{X^H}(X)$ and Z_1 is the snc divisor $Z^{\mathrm{st}} \cup E$, Z^{st} (resp. E) denoting the strict transform of Z (resp. the exceptional divisor) in the blow-up $h : X_1 \to X$.

The question is again étale local above X around x, so we may assume that $X = X'$ and we look at the cartesian square (1.4.2)

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ S \arrow[r] & \operatorname{Spec} \Lambda[Q] \end{tikzcd} ,$$

with Y as in (1.4.4), and the associated cartesian squares (1.5.1), (1.5.2), and (1.5.3).

Claim. We have

$$\mathrm{Bl}_{X^H}(X) = X \times_Y \mathrm{Bl}_{Y^H}(Y), \tag{1.8.1}$$

$$Z^{\mathrm{st}} = X \times_Y T^{\mathrm{st}}. \tag{1.8.2}$$

Proof. In view of 1.6 and 1.7, (1.8.1) follows from the fact that the immersion $Y^H \to Y$ is regular. For (1.8.2) recall that

$$T = T_0 \times_{\operatorname{Spec} \Lambda} \operatorname{Spec} \mathrm{Sym}_\Lambda(V),$$

where $T_0 \subset \operatorname{Spec} \Lambda[P]$ is the snc divisor

$$T_0 = \sum_{1 \leq i \leq r} \mathrm{div}(z_i)$$

with $z_i \in \Lambda[P]$ the image of $e_i \in P_1$ as in 1.4.3. Hence

$$T = \sum_{1 \leq i \leq r} T_i, \tag{1.8.3}$$

where $T_i = \mathrm{div}(z_i) \times_{\operatorname{Spec} \Lambda} \mathrm{Sym}_\Lambda(V)$, and $T^{\mathrm{st}} = \sum_{1 \leq i \leq r} T_i^{\mathrm{st}}$. We have (1.6.2)

$$Y^H = (\operatorname{Spec} \Lambda[P])^H \times \operatorname{Spec} \mathrm{Sym}_\Lambda(V_H),$$

with $(\operatorname{Spec} \Lambda[P])^H$ defined by the equations $(z_i = 0)_{i \notin A}$, with the notations of 1.6 (c). In particular, the immersion $Y^H \times_Y Z_i \to Z_i$ is regular, hence, by 1.7, we have $Z_i^{\mathrm{st}} = X \times_Y T_i^{\mathrm{st}}$, hence (1.8.2), which finishes the proof of the claim. □

Since the map $S \to \operatorname{Spec} \Lambda[Q]$ is strict, in order to prove the desired log smoothness, we may, by this claim, replace the triple (X, X^H, Z) over S by (Y, Y^H, T) over $\operatorname{Spec} \Lambda[Q]$. We choose coordinates on P^*, $P_1 = \mathbf{N}^r$, V:

$$P^* = \bigoplus_{1 \le i \le t} \mathbf{Z} f_i, \ P_1 = \bigoplus_{1 \le i \le r} \mathbf{N} e_i, \ V = \bigoplus_{1 \le i \le s} \Lambda y_i$$

$$\Lambda[P] = \Lambda[u_1^{\pm 1}, \cdots, u_t^{\pm 1}, z_1, \cdots, z_r], \ \operatorname{Sym}_\Lambda(V) = \Lambda[y_1, \cdots, y_s],$$

with u_i (resp. z_i) the image of f_i (resp. e_i) in $\Lambda[P]$, in such a way that

$$\Lambda[P]^H = \Lambda[u_1^{\pm 1}, \cdots, u_t^{\pm 1}, z_{m+1}, \cdots, z_r],$$

i.e. is defined in $\Lambda[P]$ by the equations $(z_1 = \cdots = z_m = 0)$, for some m, $1 \le m \le r$, and

$$\Lambda[V_H] = \Lambda[y_{n+1}, \cdots, y_s],$$

i.e. is defined in $\Lambda[V]$ by the equations $y_1 = \cdots = y_n = 0$ for some n, $1 \le n \le s$. Then

$$Y^H \subset Y = \operatorname{Spec} \Lambda[u_1^{\pm 1}, \cdots, u_t^{\pm 1}, z_1, \cdots, z_r, y_1, \cdots, y_s]$$

is defined by the equations

$$z_1 = \cdots = z_m = y_1 = \cdots = y_n = 0.$$

Then

$$Y' := \mathrm{Bl}_{Y^H}(Y)$$

is covered by affine open pieces:

$$U_i = \operatorname{Spec} \Lambda[(u_j^{\pm 1})_{1 \le j \le t}, z'_1, \cdots, z'_{i-1}, z_i, z'_{i+1}, \cdots, z'_m, z_{m+1}, \cdots, z_r, y'_1, \cdots, y'_n, y_{n+1}, \cdots, y_s]$$

$(1 \le i \le m)$, with $U_i \to Y$ given by $z_j \to z_i z'_j$ for $1 \le j \le m$, $j \ne i$, $y_j \to z_i y'_j$, $1 \le j \le n$, and the other coordinates unchanged, and

$$V_i = \operatorname{Spec} \Lambda[(u_j^{\pm 1})_{1 \le j \le t}, z'_1, \cdots, z'_m, z_{m+1}, \cdots, z_r, y'_1, \cdots, y'_{i-1}, y_i, y'_{i+1}, \cdots, y'_n, y_{n+1}, \cdots, y_s]$$

$(1 \le i \le n)$, with $V_i \to Y$ given by $z_j \mapsto y_i z'_j$ for $1 \le j \le m$, $y_j \mapsto y_i y'_j$, $1 \le j \le n$, $j \ne i$, and the other coordinates unchanged. Recall that Y has the log structure defined by the log regular pair (Y, T), where T is the snc divisor

$$T = (z_1 \cdots z_r = 0),$$

and Y' is given the log structure defined by the log regular pair (Y', T'), where T' is the snc divisor

$$T' = F \cup T^{\mathrm{st}},$$

where F is the exceptional divisor of the blow up of Y^H and T^{st} the strict transform of T. Consider the canonical morphisms

$$Y' \xrightarrow{b} Y \xrightarrow{g} \Sigma := \operatorname{Spec} \Lambda[Q] \ .$$

They are both morphisms of log schemes. The morphism g is given by the homomorphism of monoids $\gamma : Q \to P$, i.e.

$$q \in Q \mapsto (\gamma_1(q), \cdots, \gamma_t(q), \gamma_{t+1}(q), \cdots, \gamma_{t+r}(q), 0, \cdots, 0) \in \Lambda[u_1^{\pm 1}, \cdots, u_t^{\pm 1}, z_1, \cdots, z_r, y_1, \cdots, y_s].$$

The blow up b has been described above in the various charts. Note that b is not log étale, or even log smooth, in general. However, the composition $gb : Y' \to \Sigma$ is log smooth. We will check this on the charts (U_i), (V_i).

(a) *Chart of type U_i*. We have $F = (z_i = 0)$, $T^{\mathrm{st}} = (\prod_{1 \le j \le r, j \ne i} z'_i = 0)$. Hence the log strucure of U_i is given by the canonical log structure of $\Lambda[\mathbf{N}^r]$ in the decomposition

$$U_i = \operatorname{Spec} \Lambda[\mathbf{Z}^t] \times \operatorname{Spec} \Lambda[\mathbf{N}^r] \times \operatorname{Spec} \Lambda[y'_1, \cdots, y'_n, y_{n+1}, \cdots, y_s]$$

with the basis element e_k of $\mathbf{N}^r$ sent to the k-th place in $(z'_1, \cdots, z'_{i-1}, z_i, z'_{i+1}, \cdots, z'_m, z_{m+1}, \cdots, z_r)$ (and the basis element f_k of $\mathbf{Z}^t$ sent to u_k), the third factor having the trivial log structure. Checking the log smoothness of $gb : U_i \to \Sigma$ amounts to checking the log smoothness of its factor Spec $\Lambda[P] \to \Sigma =$ Spec $\Lambda(Q)$, which is defined by the composition of homomorphisms of monoids

$$Q \xrightarrow{\gamma} \mathbf{Z}^t \oplus \mathbf{N}^r \xrightarrow{Id \oplus \beta} \mathbf{Z}^t \oplus \mathbf{N}^r ,$$

where β is the homomorphism $\mathbf{N}^r \to \mathbf{N}^r$ sending e_j to $e_j + e_i$ for $1 \leq j \leq m$, $j \neq i$, e_i to e_i, and e_j to e_j for $m+1 \leq j \leq r$. Recall ((1.4.2), (iv)) that γ^{gp} is injective and the torsion part of its cokernel is invertible in Λ. As β^{gp} is an isomorphism, the same holds for the composition $(Id \oplus \beta)\gamma$, hence $gb : U_i \to \Sigma$ is log smooth.

(b) *Chart of type V_i.* We have $F = (y_i = 0)$, $T^{\mathrm{st}} = \prod_{1 \leq j \leq m} z'_j \prod_{j \geq m+1} z_i$. Hence the log structure of V_i is given by the canonical log structure of $\Lambda[\mathbf{N}^{r+1}]$ in the decomposition

$$V_i = \operatorname{Spec} \Lambda[\mathbf{Z}^t] \times \operatorname{Spec} \Lambda[\mathbf{N}^{r+1}] \times \operatorname{Spec} \Lambda[(y'_j)_{1 \leq j \leq n, j \neq i}, y_{n+1}, \cdots, y_s]$$

with the basis element e_k of $\mathbf{N}^{r+1}$ sent to the k-th place in $(z'_1, \cdots, z'_m, z_{m+1}, \cdots, z_r)$ if $k \leq r$, and e_{r+1} sent to y_i (and the basis element f_k of $\mathbf{Z}^t$ sent to u_i), the third factor having the trivial log structure. Again, checking the log smoothness of $gb : V_i \to \Sigma$ amounts to checking the log smoothness of its factor Spec $\Lambda[\mathbf{Z}^t] \times$ Spec $\Lambda[\mathbf{N}^{r+1}] \to$ Spec $\Lambda(Q)$. This factor is defined by the composition of homomorphisms of monoids

$$Q \xrightarrow{\gamma} \mathbf{Z}^t \oplus \mathbf{N}^r \xrightarrow{Id \oplus \beta} \mathbf{Z}^t \oplus \mathbf{N}^{r+1}$$

where $\beta : \mathbf{N}^r \to \mathbf{N}^{r+1}$ sends e_j to $e_j + e_{r+1}$ for $1 \leq j \leq m$, and to e_j for $m+1 \leq j \leq r$. Then β^{gp} is injective, and its cokernel is isomorphic to $\mathbf{Z}$, hence $(\beta\gamma)^{\mathrm{gp}}$ is injective, and we have an exact sequence

$$0 \to \operatorname{Coker} \gamma^{\mathrm{gp}} \to \operatorname{Coker} (\beta\gamma)^{\mathrm{gp}} \to \mathbf{Z} \to 0.$$

In particular, the torsion part of $\operatorname{Coker}(\beta\gamma)^{\mathrm{gp}}$ is isomorphic to that of $\operatorname{Coker} \gamma^{\mathrm{gp}}$, hence of order invertible in Λ, which implies that $gb : V_i \to \Sigma$ is log smooth.

This finishes the proof that Step 3 preserves log smoothness.

1.9. *End of proof of 1.1.* We may now assume that in addition to conditions (1) and (2) of 1.4, condition (3) is satisfied as well, namely

(3) *G acts freely on $X - Z$ (i.e. $Z = Z \cup T$ in the notation of 1.1 or (**VIII**-1.1)), and, for any geometric point $x \to X$, the inertia group G_x is abelian.*

We have to check:

*Claim. If $f_{(G,X,Z)} : (X', Z') \to (X, Z)$ is the modification of (**VIII**-5.4.4), then (X', Z') and $(X'/G, Z'/G)$ are log smooth over S.*

Working étale locally around a geometric point x of X, we will first choose a strict rigidification $(X, \overline{Z})$ of (X, Z) such that $(X, \overline{Z})$ is log smooth over S. We will define $(X, \overline{Z})$ as the pull-back by $S \to \Sigma =$ Spec $\Lambda[Q]$ of a rigidification $(Y, \overline{T})$ of (Y, T) which is log smooth over Σ, with the notation of (1.4.4). Using that G $(= G_x)$ is abelian, one can decompose V into a sum of G-stable lines, according to the characters of G:

$$V = \bigoplus_{1 \leq i \leq s} \Lambda y_i$$

with G acting on Λy_i through a character $\chi_i : G \to \mu_N$, i.e. $g y_i = \chi_i(g) y_i$. We define $\overline{T}$ to be the divisor $z_1 \cdots z_r y_1 \cdots y_s = 0$ in $Y = \operatorname{Spec} \Lambda[u_1^{\pm 1}, \cdots, u_t^{\pm 1}, z_1, \cdots, z_r, y_1, \cdots, y_s]$. The action of G on $(Y, \overline{T})$ is very tame at x because the log stratum at x is Spec $\Lambda[u_1^{\pm 1}, \cdots, u_t^{\pm 1}]$, hence very tame in a neighborhood of x by (**VIII**-5.3.2) (actually on the whole of Y, cf. (**VIII**-4.6, **VIII**-4.7(a)). On the other hand, (Spec $\Lambda[y_1, \cdots, y_s], y_1 \cdots y_s = 0$) is log smooth over Spec Λ, and as Spec $\Lambda[P]$ is log smooth over Σ, $(Y, \overline{T})$ is log smooth over Σ. Since $f_{(G,X,Z)}$ is compatible with base change by strict inert morphisms, it is enough to check that if $f_{(G,Y,T)} = f_{(G,Y,T,\overline{T})} : (Y',T') \to (Y,T)$ is the modification of (**VIII**-5.4.4) then (Y',T') is log smooth over Σ. Recall (**VIII**-5.3.9) that we have a cartesian G-equivariant diagram

(1.9.1)
$$\begin{tikzcd}
(Y',\overline{T'}) \arrow[r, "\overline{h'}"] \arrow[d, "\alpha'"] & (Y,\overline{T}) \arrow[d, "\alpha"] \\
(Y'_1,\overline{T'_1}) \arrow[r, "\overline{h_1}"] & (Y_1,\overline{T_1})
\end{tikzcd} ,$$

where the horizontal maps are the compositions of saturated log blow up towers, and the vertical ones Kummer étale G-covers. From (1.9.1) is extracted the relevant diagram involving $h := f_{(G,Y,T,\overline{T})}$,

$$\begin{tikzcd}
(Y',T') \arrow[r, "h"] \arrow[d, "\beta"] & (Y,T) \\
(Y'_1,T'_1) &
\end{tikzcd} ,$$

where $T'_1 = \overline{h_1}^{-1}(T_1)$, with $T_1 = T/G$, $T' = \alpha'^{-1}(T'_1)$, and h (resp. β) is the restriction of $\overline{h'}$ (resp. α') over (Y,T) (resp. (Y'_1,T'_1)). In particular, β is a Kummer étale G-cover (as Kummer étale G-covers are stable under any fs base change). As G acts trivially on S, this diagram can be uniquely completed into a commutative diagram

(1.9.2)
$$\begin{tikzcd}
(Y',T') \arrow[r, "h"] \arrow[d, "\beta"] & (Y,T) \arrow[d, "f"] \\
(Y'_1,T'_1) \arrow[r, "g"] & \Sigma
\end{tikzcd} .$$

Here f is log smooth and β is a Kummer étale G-cover. Though $\overline{h'}$ and $\overline{h_1}$ are log smooth, h and h_1 are not, in general. However, it turns out that:

$(*)$ $g : (Y'_1,T'_1) \to \Sigma$, *hence* $g\beta = fh : (Y',T') \to \Sigma$, *are log smooth*,

which will finish the proof of the claim, hence of 1.1. We first prove

$(**)$ *With the notation of* (1.9.2), $(Y_1,\overline{T_1})$ *is log smooth over* Σ.

Let us write $Y = \operatorname{Spec}\Lambda[\overline{P}]$, with

(1.9.3)
$$\overline{P} = P \times \mathbf{N}^s = \mathbf{Z}^t \times \mathbf{N}^r \times \mathbf{N}^s.$$

As G acts very tamely on $(Y,\overline{T})$, the quotient pair $(Y_1 = Y/G, \overline{T_1} = \overline{T}/G)$ is log regular. More precisely, by the calculation in (**VI**-3.4(b)), this pair consists of the log scheme $Y_1 = \operatorname{Spec}\Lambda[\overline{R}]$ with its canonical log structure, where

$$\overline{R} = \operatorname{Ker}(\overline{P}^{\mathrm{gp}} \to \operatorname{Hom}(G,\mu_N)) \cap \overline{P},$$

$\overline{P}^{\mathrm{gp}} \to \operatorname{Hom}(G,\mu_N)$ being the homomorphism defined by the pairing $\chi : G \otimes \overline{P}^{\mathrm{gp}} \to \mu_N$. The inclusion $\overline{R} \subset \overline{P}$ is a Kummer morphism, and $\overline{P}^{\mathrm{gp}}/\overline{R}^{\mathrm{gp}}$ is annihilated by an integer invertible in Λ. As $Q^{\mathrm{gp}} \to P^{\mathrm{gp}}$ is injective, with the torsion part of its cokernel annihilated by an integer invertible in Λ, the same is true for $Q^{\mathrm{gp}} \to \overline{P}^{\mathrm{gp}}$, hence also for $Q^{\mathrm{gp}} \to \overline{R}^{\mathrm{gp}}$. Thus $(Y_1,\overline{T_1}) = \operatorname{Spec}\Lambda[\overline{R}]$ is log smooth over Σ.

Finally, let us prove $(*)$. It is enough to work locally on Y'_1 so we can replace the log blow up sequence $(Y'_1,\overline{T}'_1) \to (Y_1,\overline{T}_1)$ with an affine chart (i.e. we replace the first log blow up with a chart, then do the same for the second one, etc.). Then $Y'_1 = \operatorname{Spec}\Lambda[\overline{R}']$, and $\overline{R}^{\mathrm{gp}} \xrightarrow{\sim} \overline{R}'^{\mathrm{gp}}$ by **VIII**-3.1.19. Note that $\overline{R}' \xrightarrow{\sim} \mathbf{Z}^a \times \mathbf{N}^b$ where $D_1,\dots,D_b$ are the components of $\overline{T}'_1$. We can assume that $D_1,\dots,D_c \subset T'_1$ and $D_{c+1},\dots,D_b$ are not contained in T'_1. Let $R' \xrightarrow{\sim} \mathbf{Z}^a \times \mathbf{N}^c$ denote the submonoid $\overline{R}'$ that defines the log structure of (Y'_1,T'_1). Note that R' consists of all elements $g' \in \overline{R}'$ such that $(g' = 0) \subset T'_1$ (as a set). Also, by $\nu : \overline{R} \to \overline{R}'$ we will denote the homomorphism that defines $(Y'_1,\overline{T}'_1) \to (Y_1,\overline{T}_1)$.

We showed in **VIII**-5.3.9 that $T_1 = T/G$ is a **Q**-Cartier divisor in Y_1 and observed that therefore T_1' is a Cartier divisor in Y_1'. Note that the inclusion $R \subset \overline{R}$, where

$$R = \mathrm{Ker}(P^{\mathrm{gp}} \to \mathrm{Hom}(G, \mu_N)) \cap \overline{P}$$

defines a log structure on Y_1. Denote the corresponding log scheme (Y_1, T_1). We obtain the following diagram of log schemes (on the left). The corresponding diagram of groups is placed on the right; we will use it to establish log smoothness of g. Existence of dashed arrows requires an argument; we will construct them later.

(1.9.4)
$$\begin{tikzcd} (Y_1', \overline{T}_1') \arrow[r, "\overline{h}_1"] \arrow[d] & (Y_1, \overline{T}_1) \arrow[d] \arrow[dr] & \\ (Y_1', T_1') \arrow[r, dashed] & (Y_1, T_1) \arrow[r] & \Sigma \end{tikzcd} \qquad \begin{tikzcd} \overline{R}'^{\mathrm{gp}} & \overline{R}^{\mathrm{gp}} \arrow[l, "\sim"', "\nu^{\mathrm{gp}}"] & \\ R'^{\mathrm{gp}} \arrow[u, hook] & R^{\mathrm{gp}} \arrow[l, dashed] \arrow[u, hook] & Q^{\mathrm{gp}} \arrow[l] \arrow[ul] \end{tikzcd}$$

Part (ii) of the following remark clarifies the notation (Y_1, T_1). It will not be used so we only sketch the argument.

Remark 1.10. (i) Note that (Y_1, T_1) may be not log smooth over Σ. For example, even when Σ is log regular, e.g. $\mathrm{Spec}\, k$ with trivial log structure, (Y_1, T_1) does not have to be log regular, as T_1 may even be non-Cartier. Nevertheless, as $\overline{h_1}$ is log smooth (even log étale), $(Y_1', \overline{T_1'})$ is log smooth over Σ. Moreover, Y_1' is regular, and $\overline{T_1'}$ an snc divisor in it.

(ii) Although T_1 may be bad, one does have that $R\mathscr{O}_{Y_1}^* = \mathscr{O}_{Y_1} \cap i_\star \mathscr{O}_{Y_1 \setminus T_1}^*$ for the embedding $i : Y_1 \setminus T_1 \hookrightarrow Y_1$. This can be deduced from the formulas for R and $\overline{R}$ and the fact that $\overline{R}\mathscr{O}_{Y_1}^* = \mathscr{O}_{Y_1} \cap j_\star \mathscr{O}_{Y_1 \setminus \overline{T}_1}^*$ by log regularity of $(Y_1, \overline{T}_1)$.

Note that $Q \to \overline{P}$ factors through P, hence $Q \to \overline{R}$ factors through $R = \overline{R} \cap P$. It follows from (1.9.3) that P consists of all elements $f \in \overline{P}$ whose divisor $(f = 0)$ is contained in T (as a set). Therefore $g \in \overline{R}$ lies in R if and only if $(g = 0) \subset T_1$ (as a set). This fact and the analogous description of R' observed earlier imply that $\nu : \overline{R} \to \overline{R}'$ takes R to R'. Thus, we have established the dashed arrows in (1.9.4).

Let $\varphi : Q \to \overline{R}'$ be the homomorphism defining the composition $(Y_1', \overline{T_1'}) \to (Y_1, \overline{T_1}) \to \Sigma$. Since the latter is log smooth, φ is injective, and the torsion part of $\mathrm{Coker}(\varphi^{\mathrm{gp}})$ is annihilated by an integer m invertible in Λ. Note that $R^{\mathrm{gp}} \hookrightarrow R'^{\mathrm{gp}} \hookrightarrow \overline{R}'^{\mathrm{gp}}$, and therefore we also have that $Q^{\mathrm{gp}} \hookrightarrow R'^{\mathrm{gp}}$ and the torsion of its cokernel is annihilated by m. Therefore, (Y_1', T_1') is log smooth over Σ, which finishes the proof of $(*)$, hence of 1.1.

Remark 1.11. In the proof of $(*)$ above, we first proved that g is log smooth, and deduced that $g\beta$ is, too. In fact, as β is a Kummer étale G-cover, the log smoothness of $g\beta$ implies that of g. More generally, we have the following descent result, due to Kato-Nakayama ([**Nakayama, 2009**, 3.4]):

Theorem 1.12. *Let* $X' \xrightarrow{g} X \xrightarrow{f} Y$ *be morphisms of fs log schemes. If g is surjective, log étale and exact, and fg is log smooth, then f is log smooth.*

The assumption on g is equivalent to saying that g is a Kummer étale cover (cf. [**Illusie, 2002**, 1.6]).

2. Prime to ℓ variants of de Jong's alteration theorems

Let X be a noetherian scheme, and ℓ be a prime number. Recall that a morphism $h : X' \to X$ is called an **ℓ'-alteration** if h is proper, surjective, generically finite, maximally dominating (i.e., (**II**-1.1.2) sends each maximal point to a maximal point) and the degrees of the residual extensions $k(x')/k(x)$ over each maximal point x of X are prime to ℓ. The next theorem was stated in **Intro.**-3 (1):

THEOREM 2.1. *Let k be a field, ℓ a prime number different from the characteristic of k, X a separated and finite type k-scheme, $Z \subset X$ a nowhere dense closed subset. Then there exists a finite extension k' of k, of degree prime to ℓ, and a projective ℓ'-alteration $h : \tilde{X} \to X$ above* $\mathrm{Spec}\ k' \to \mathrm{Spec}\ k$, *with $\tilde{X}$ smooth and quasi-projective over k', and $h^{-1}(Z)$ is the support of a relative strict normal crossings divisor.*

Recall that a relative strict normal crossings divisor in a smooth scheme T/S is a divisor $D = \sum_{i \in I} D_i$, where I is finite, $D_i \subset T$ is an S-smooth closed subscheme of codimension 1, and for every subset J of I the scheme-theoretic intersection $\bigcap_{i \in J} D_i$ is smooth over S of codimension $|J|$ in T.

We will need the following variant, due to Gabber-Vidal (proof of [**Vidal, 2004**, 4.4.1]), of de Jong's alteration theorems [**de Jong, 1997**, 5.7, 5.9, 5.11], cf. [**Zheng, 2009**, 3.8]:

LEMMA 2.2. *Let X be a proper scheme over $S =$ Spec k, normal and geometrically reduced and irreducible, $Z \subset X$ a nowhere dense closed subset. We assume that a finite group H acts on $X \to S$, faithfully on X, and that Z is H-stable. Then there exists a finite extension k_1 of k, a finite group H_1, a surjective homomorphism $H_1 \to H$, and an H_1-equivariant diagram with a cartesian square* (*where $S =$* Spec k, *$S_1 =$* Spec k_1)

(2.2.1)
$$\begin{tikzcd} X \arrow[d] & X_1 \arrow[l, "b"'] \arrow[d] & X_2 \arrow[l, "a"'] \arrow[dl] \\ S & S_1 \arrow[l] & \end{tikzcd}$$

satisfying the following properties:

(*i*) *$S_1/\mathrm{Ker}(H_1 \to H) \to S$ is a radicial extension;*

(*ii*) *X_2 is projective and smooth over S_1;*

(*iii*) *$a : X_2 \to X_1$ is projective and surjective, maximally dominating and generically finite and flat, and there exists an H_1-admissible dense open subset $W \subset X_2$ over a dense open subset U of X, such that if $U_1 = S_1 \times_S U$ and $K = \mathrm{Ker}(H_1 \to \mathrm{Aut}(U_1))$, $W \to W/K$ is a Galois étale cover of group K and the morphism $W/K \to U_1$ induced by a is a universal homeomorphism;*

(*iv*) *$(ba)^{-1}(Z)$ is the support of a strict normal crossings divisor in X_2.*

Proof. We may assume X of dimension $d \geq 1$. We apply [**Vidal, 2004**, 4.4.3] to X/S, Z, and $G = H$. We get the data of *loc. cit.*, namely an equivariant finite extension of fields $(S_1, H_1) \to (S, H)$ such that $S_1/\mathrm{Ker}(H_1 \to H) \to S$ is radicial, an H_1-equivariant pluri-nodal fibration $(Y_d \to \cdots \to Y_1 \to S_1, \{\sigma_{ij}\}, Z_0 = \varnothing)$, and an H_1-equivariant alteration $a_1 : Y_d \to X$ over S, satisfying the conditions (i), (ii), (iii) of *loc. cit.* (in particular $a_1^{-1}(Z) \subset Z_d$). Then, as in the proof of [**Vidal, 2004**, 4.4.1], successively applying [**Vidal, 2004**, 4.4.4] to each nodal curve $f_i : Y_i \to Y_{i-1}$, one can replace Y_i by an H_1-equivariant projective modification Y'_i of it such that Y'_i is regular, and the inverse image Z'_i of $Z_i := \bigcup_j \sigma_{ij}(Y_{i-1}) \cup f_i^{-1}(Z_{i-1})$ in Y'_i is an H_1-equivariant strict snc divisor. Then, $X_2 := Y'_d$ is smooth over S_1 and Z'_d is a relative snc divisor over S_1. This follows from the analog of the remark following [**Vidal, 2004**, 4.4.4] with "semistable pair over a trait" replaced by "pair consisting of a smooth scheme and a relative snc divisor over a field". In particular, $(ba)^{-1}(Z)_{\mathrm{red}}$ is a subdivisor of Z'_d, hence an snc divisor. After replacing H_1 by $H_1/\mathrm{Ker}(H_1 \to \mathrm{Aut}(X_2))$ the open subsets U and V of (iii) are obtained as at the end of the proof of [**Vidal, 2004**, 4.4.1]. □

2.3. *Proof of 2.1.* There are three steps.

Step 1. Preliminary reductions. By Nagata's compactification theorem [**Conrad, 2007**], there exists a dense open immersion $X \subset \overline{X}$ with $\overline{X}$ proper over S. Up to replacing X by $\overline{X}$ and Z by its closure $\overline{Z}$, we may assume X proper over S. By replacing X by the disjoint sum of its irreducible components, we may further assume X irreducible, and geometrically reduced (up to base changing by a finite radicial extension of k). Up to blowing up Z in X me may further assume that Z is a (Cartier) divisor in X. Finally, replacing X by its normalization X', which is finite over X, and Z by its inverse image in X', we may assume X normal.

Step 2. Use of 2.2. Choose a finite Galois extension k_0 of k such that the irreducible components of $X_0 = X \times_S S_0$ ($S_0 =$ Spec k_0) are geometrically irreducible. Let $G = \mathrm{Gal}(k_0/k)$ and $H \subset G$ the decomposition subgroup of a component Y_0 of X_0. We apply 2.2 to $(Y_0/S_0, Z_0 \cap Y_0)$, where $Z_0 = S_0 \times_S Z$. We find a surjection $H_1 \to H$ and an H_1-equivariant diagram of type 2.2.1:

(2.3.1)
$$\begin{tikzcd} Y_0 \arrow[d] & Y_1 \arrow[l, "b'"'] \arrow[d] & Y_2 \arrow[l, "a'"'] \arrow[dl] \\ S_0 & S_1 \arrow[l] & \end{tikzcd} ,$$

satisfying conditions (i), (ii), (iii), (iv) with S replaced by S_0, and $X_2 \to X_1 \to X$ by $Y_2 \to Y_1 \to Y_0$. As G transitively permutes the components of X_0, X_0 is, as a G-scheme over S_0, the contracted product

$$X_0 = Y_0 \times^H G,$$

i.e. the quotient of $Y_0 \times G$ by H acting on Y_0 on the right and on G on the left (cf. proof of **VIII**-5.3.8), and similarly $Z = Z_0 \times^H G$. Choose an extension of the diagram $H_1 \xrightarrow{u} H \xrightarrow{i} G$ into a commutative diagram of finite groups

$$\begin{tikzcd} H_1 \arrow[r, "i_1"] \arrow[d, "u"] & G_1 \arrow[d, "v"] \\ H \arrow[r, "i"] & G \end{tikzcd}$$

with i_1 injective and v surjective (for example, take i_1 to be the graph of iu and v the projection). Define

$$X_1 := Y_1 \times^{H_1} G_1, \;\; X_2 := Y_2 \times^{H_1} G_1.$$

Then (2.3.1) extends to a G_1-equivariant diagram of type 2.2.1

(2.3.2)
$$\begin{tikzcd} X_0 \arrow[d] & X_1 \arrow[l, "b"'] \arrow[d] & X_2 \arrow[l, "a"'] \arrow[dl] \\ S_0 & S_1 \arrow[l] & \end{tikzcd} ,$$

satisfying again (i), (ii), (iii), (iv). In particular, the composition $h : X_2 \to X_1 \to X_0 \to X$ is an alteration above the composition $S_1 \to S_0 \to S$, X_2 is projective and smooth over S_1, and $h^{-1}(Z)$ is the support of an snc divisor. However, as regard to 2.1, the diagram

$$\begin{tikzcd} X \arrow[d] & X_2 \arrow[l, "h"'] \arrow[d] \\ S & S_1 \arrow[l] \end{tikzcd}$$

deduced from (2.3.2) has two defects:

(a) the extension $S_1 \to S$ is not necessarily of degree prime to ℓ,

(b) the alteration h is not necessarily an ℓ'-alteration.

We will first repair (a) and (b) at the cost of temporarily losing the smoothness of X_2/S_1 and the snc property of $h^{-1}(Z)$. By (i), $S_1/\mathrm{Ker}(G_1 \to G) \to S_0$ is a radicial extension, hence $S_1/G_1 \to S = S_0/G$ is a radicial extension, too. Similarly, by (iii), $X_2/G_1 \to X$ is an alteration over $S_1/G_1 \to S$, which is a universal homeomorphism over a dense open subset. Now let L be an ℓ-Sylow subgroup of G_1. Then $S_1/L \to S_1/G_1$ is of degree prime to ℓ, and $X_2/L \to X_2/G_1$ is a finite surjective morphism of generic degree prime to ℓ. Let $S' := \mathrm{Spec}\, k' = S_1/L$, $X' = S' \times_S X$. We get a commutative diagram with cartesian square

(2.3.3)
$$\begin{tikzcd} X \arrow[d] & X' \arrow[l] \arrow[d] & X_2/L \arrow[l] \arrow[dl] & X_2 \arrow[l] \arrow[dl] \\ S & S' \arrow[l] & S_1 \arrow[l] & \end{tikzcd} ,$$

where $S' \to S$ is an extension of degree prime to ℓ, $S_1 \to S'$ a Galois extension of group L,

$$h_2 : X_2/L \to X$$

an ℓ'-alteration, X_2/S_1 is projective and smooth, and if now h denotes the composition $X_2 \to X$, $Z_1 := h^{-1}(Z)$ is an snc divisor in X_2. If X_2/L was smooth over S' and Z_1/L an snc divisor in X_2/L, we would be finished. However, this is not the case in general. We will use Gabber's theorem 1.1 to fix this.

Step 3. Use of 1.1. Let Y be a connected component of X_2, $(Z_1)_Y = h^{-1}(Z) \cap Y$, D the stabilizer of Y in L, $I \subset D$ the inertia group at the generic point of Y. Then D acts on Y through $K := D/I$, and this action is generically free. As Y is smooth over S_1 and $(Z_1)_Y$ is an snc divisor in Y, $(Y, (Z_1)_Y)$ makes a log regular pair, log smooth over S_1, hence over $S' = S_1/L$ (equipped with the trivial log structure). We have a K-equivariant commutative diagram

$$\begin{tikzcd} (Y/K, (Z_1)_Y/K) \arrow[d] & (Y, (Z_1)_Y) \arrow[l] \arrow[dl, "f"] \\ S' & \end{tikzcd} \tag{2.3.4}$$

where K acts trivially on S', and f is projective, smooth, and log smooth (S' having the trivial log structure). We now apply 1.1 to $(f : (Y, (Z_1)_Y) \to S' = (S', \emptyset), K)$, which satisfies conditions (i) - (iii) of *loc. cit.* We get a D-equivariant projective modification $g : Y_1 \to Y$ (D acting through K) and a D-strict snc divisor E_1 on Y_1, containing $Z_1 := g^{-1}((Z_1)_Y)$ as a subdivisor, such that the action of D on (Y_1, E_1) is very tame, and (Y_1, E_1) and $(Y_1/D, E_1/D)$ are log smooth over S'. Pulling back g to the orbit $Y \times^D L$ of Y under L, i.e. replacing g by $g \times^D L$, and working separately over each orbit, we eventually get an L-equivariant commutative square

$$\begin{tikzcd} (Y_2/L, E_2/L) \arrow[d, "v"'] & (Y_2, E_2) \arrow[l] \arrow[d, "u"] \\ (X_2/L, Z_1/L) & (X_2, Z_1) \arrow[l] \end{tikzcd} \tag{2.3.5}$$

where u, v are projective modifications (and $Z_1 = h^{-1}(Z)$, $Z_1/L = h_2^{-1}(Z)$ as above), with the property that the pair $(Y_2/L, E_2/L)$ is an fs log scheme log smooth over S' $(= S_1/L)$, and $v^{-1}(h_2^{-1}(Z)) \subset E_2/L)$. Let $w : (\tilde{X}, \tilde{E}) \to (Y_2/L, E_2/L)$ be a projective, log étale modification such that $\tilde{X}$ is regular, and $\tilde{E} = w^{-1}(E_2/L)$ is an snc divisor in $\tilde{X}$. For example, one can take for w the saturated monoidal desingularization $\mathscr{F}^{\log}$ of (**VIII**-3.4.9). We then apply 1.2 to the log smooth morphism $\tilde{X} \to S'$. By a special case of the (1.4.2), with $Q = \{1\}$, $G = \{1\}$, as P^* is torsion free, $\tilde{X}$ is not only regular, but smooth over S', and $\tilde{E}$ a relative snc on $\tilde{X}$. Let

$$\tilde{h} : \tilde{X} \to X$$

be the composition

$$\tilde{X} \xrightarrow{w} Y_2/L \xrightarrow{v} X_2/L \xrightarrow{h_2} X .$$

This is a projective ℓ'-alteration, and it fits in the commutative diagram

$$\begin{tikzcd} X \arrow[d] & \tilde{X} \arrow[l, "\tilde{h}"'] \arrow[d] \\ S & S' \arrow[l] \end{tikzcd} ,$$

where S' is an extension of S of degree prime to ℓ, $\tilde{X}$ is projective and smooth over S', and $\tilde{h}^{-1}(Z)_{\mathrm{red}}$ is a sub-divisor of the S'-relative snc divisor $\tilde{E}$, hence a relative snc divisor as well. This finishes the proof of 2.1.

Recall now the theorem stated in **Intro.**-3 (2):

THEOREM 2.4. *Let S be a separated, integral, noetherian, excellent, regular scheme of dimension* 1, *with generic point η, X a scheme separated, flat and of finite type over S, ℓ a prime number invertible on S, $Z \subset X$ a nowhere dense closed subset. Then there exists a finite extension η' of η of degree prime to ℓ and a projective ℓ'-alteration $h : \tilde{X} \to X$ above $S' \to S$, where S' is the normalization of S in η', with $\tilde{X}$ regular and quasi-projective over S', a strict normal crossings divisor $\tilde{T}$ on $\tilde{X}$, and a finite closed subset Σ of S' such that:*

(i) *outside Σ, $\tilde{X} \to S'$ is smooth and $\tilde{T} \to S'$ is a relative divisor with normal crossings;*

(ii) *étale locally around each geometric point x of $\tilde{X}_{s'}$, where $s' = \operatorname{Spec} k'$ belongs to Σ, the pair $(\tilde{X}, \tilde{T})$ is isomorphic to the pair consisting of*

$$X' = S'[u_1^{\pm 1}, \cdots, u_s^{\pm 1}, t_1, \cdots, t_n]/(u_1^{b_1} \cdots u_s^{b_s} t_1^{a_1} \cdots t_r^{a_r} - \pi),$$

$$T' = V(t_{r+1} \cdots t_m) \subset X'$$

around the point $(u_i = 1), (t_j = 0)$, with $1 \leq r \leq m \leq n$, for positive integers $a_1, \cdots, a_r, b_1, \cdots, b_s$ satisfying $\gcd(p, a_1, \cdots, a_r, b_1, \cdots, b_s) = 1$, *$p$ the characteristic exponent of k', π a local uniformizing parameter at s';*

(iii) *$\tilde{h}^{-1}(Z)_{\mathrm{red}}$ is a sub-divisor of $\bigcup_{s' \in \Sigma} (\tilde{X}_{s'})_{\mathrm{red}} \cup \tilde{T}$.*

The proof follows the same lines as that of 2.1. We need again a Gabber-Vidal variant of de Jong's alteration theorems (cf. [**Zheng, 2009**, 3.8]). This is essentially [**Vidal, 2004**, 4.4.1]), except for the additional data of $Z \subset X$ and the removal of the hypothesis that S is a strictly local trait:

LEMMA 2.5. *Let X be a normal, proper scheme over S, whose generic fiber is geometrically reduced and irreducible, $Z \subset X$ a nowhere dense closed subset. We assume that a finite group H acts on $X \to S$, faithfully on X, and that Z is H-stable. Then there exists a finite group H_1, a surjective homomorphism $H_1 \to H$, and an H_1-equivariant diagram with a cartesian square*

(2.5.1)
$$\begin{tikzcd} X \arrow[d] & X_1 \arrow[l, "b"'] \arrow[d] & X_2 \arrow[l, "a"'] \arrow[dl] \\ S & S_1 \arrow[l] & \end{tikzcd} ,$$

satisfying the following properties:

(i) *$S_1 \to S$ is the normalization of S in a finite extension η_1 of η such that $\eta_1/\mathrm{Ker}(H_1 \to H) \to \eta$ is a radicial extension* (*where η is the generic point of S*);

(ii) *X_2 is projective and strictly semistable over S_1* (*i.e. is strictly semistable over the localizations of S_1 at closed points* [**de Jong, 1996**, 2.16]);

(iii) *$a : X_2 \to X_1$ is projective and surjective, maximally dominating and generically finite and flat, and there exists an H_1-admissible dense open subset $W \subset X_{2\eta_1}$ over a dense open subset U of X_η, such that if $U_1 = \eta_1 \times_\eta U$ and $K = \mathrm{Ker}(H_1 \to \mathrm{Aut}(U_1))$, $W \to W/K$ is a Galois étale cover of group K and the morphism $W/K \to U_1$ induced by a is a universal homeomorphism;*

(iv) *$(ba)^{-1}(Z)$ is the support of a strict normal crossings divisor in X_2, and $(X_2, (ba)^{-1}(Z))$ is a strict semistable pair over S_1* (*i.e. over the localizations of S_1 at closed points* [**de Jong, 1996**, 6.3]).

Note that (ii) and (iv) imply that there exists a finite closed subset Σ of S_1 such that, outside Σ, the pair $(X_2, (ba)^{-1}(Z))$ consists of a smooth morphism and a relative strict normal crossings divisor.

Proof. Up to minute modifications the proof is the same as that of 2.2. We may assume the generic fiber X_η is of dimension $d \geq 1$. We apply [**Vidal, 2004**, 4.4.3] to X/S, Z, and $G = H$. We get the data of *loc. cit.*, namely an equivariant finite surjective morphism $(S_1, H_1) \to (S, H)$, with S_1 regular (hence equal to the normalization of S in the extension η_1 of the generic point η of S) such that $\eta_1/\mathrm{Ker}(H_1 \to H) \to \eta$ is radicial, an H_1-equivariant pluri-nodal fibration $(Y_d \to \cdots \to Y_1 \to S_1, \{\sigma_{ij}\}, Z_0)$, and an H_1-equivariant alteration $a_1 : Y_d \to X$ over S, satisfying the conditions (i), (ii), (iii) of *loc. cit.* (in particular $a_1^{-1}(Z) \subset Z_d$). Then, as in the proof of [**Vidal, 2004**, 4.4.1], successively applying [**Vidal, 2004**, 4.4.4] to each nodal curve $f_i : Y_i \to Y_{i-1}$, one can replace Y_i by an H_1-equivariant projective modification Y_i' of it such that Y_i' is regular, and the inverse image of $Z_i := \bigcup_j \sigma_{ij}(Y_{i-1}) \cup f_i^{-1}(Z_{i-1})$ in Y_i' is an H_1-equivariant strict snc divisor. Then, by the remark following [**Vidal, 2004**, 4.4.4] $X_2 := Y_d'$ is strict semistable over S_1 and (X_2, Z_d) is a strict semistable pair over S_1. In particular, $(ba)^{-1}(Z)_{\mathrm{red}}$ is a subdivisor of Z_d, hence an snc divisor, and $(X_2, (ba)^{-1}(Z))_{\mathrm{red}}$ is a strict semistable pair over S_1. The open subsets U and W as in (iii) are constructed as at the end of the proof of [**Vidal, 2004**, 4.4.1]. □

2.6. *Proof of 2.4.* It is similar to that of 2.1. There are again three steps. We will indicate which modifications should be made.

Step 1. Preliminary reductions. Up to replacing X by the disjoint union of the schematic closures of the reduced components of its generic fiber, and working separately with each of them, we may assume X integral (and $X_\eta \neq \varnothing$). Applying Nagata's compactification theorem, we may further assume X proper and integral. Base changing by the normalization of S in a finite radicial extension of η and taking the reduced scheme, we reduce to the case where, in addition, X_η is irreducible and geometrically reduced. Then we blow up Z in X and normalize as in the previous step 1. Here we used the excellency of S to guarantee the finiteness of the normalizations.

Step 2. Use of 2.5. Let S_0 be the normalization of S in a finite Galois extension η_0 of η such that the irreducible components of the generic fiber of $X_0 = X \times_S S_0$ ($S_0 = \operatorname{Spec} k_0$) are geometrically irreducible. Let $G = \mathrm{Gal}(\eta_0/\eta)$ and $H \subset G$ the decomposition subgroup of a component Y_0 of X_0. We apply 2.5 to $(Y_0/S_0,\ Z_0 \cap Y_0)$, where $Z_0 = S_0 \times_S Z$. We find a surjection $H_1 \to H$ and an H_1-equivariant diagram of type (2.5.1) satisfying conditions (i), (ii), (iii), (iv) with S replaced by S_0, and $X_2 \to X_1 \to X$ by $Y_2 \to Y_1 \to Y_0$. We then, as above, extend $H_1 \to H$ to a surjection $G_1 \to G$ and obtain a G_1-equivariant diagram of type (2.5.1)

(2.6.1)
$$\begin{tikzcd} X_0 \arrow[d] & X_1 \arrow[l, "b"'] \arrow[d] & X_2 \arrow[l, "a"'] \arrow[dl] \\ S_0 & S_1 \arrow[l] & \end{tikzcd},$$

satisfying again (i), (ii), (iii), (iv). In particular, the composition $h : X_2 \to X_1 \to X_0 \to X$ is an alteration above the composition $S_1 \to S_0 \to S$, X_2 is projective and strictly semistable over S_1, and $h^{-1}(Z)$ is the support of an snc divisor forming a strict semistable pair with X_2/S_1. It follows from (i) that $S_1/G_1 \to S = S_0/G$ is generically radicial, and by (iii) that $X_2/G_1 \to X$ is an alteration over $S_1/G_1 \to S$, which is a universal homeomorphism over a dense open subset. As above, choose an ℓ-Sylow subroup L of G_1. Then S_1/L is regular, $S_1/L \to S_1/G_1$ is finite surjective of generic degree prime to ℓ, and $X_2/L \to X_2/G_1$ is a finite surjective morphism of generic degree prime to ℓ. Putting $S' = S_1/L$, $X' = S' \times_S X$, we get a commutative diagram with cartesian square

(2.6.2)
$$\begin{tikzcd} X \arrow[d] & X' \arrow[l] \arrow[d] & X_2/L \arrow[l] \arrow[dl] & X_2 \arrow[l] \arrow[dl] \\ S & S' \arrow[l] & S_1, \arrow[l] & \end{tikzcd}$$

where S' is regular, $S' \to S$ is finite surjective of generic degree prime to ℓ, $S_1 \to S'$ generically étale of degree the order of L,

$$h_2 : X_2/L \to X$$

an ℓ'-alteration, X_2/S_1 is projective and strictly semistable, and if h denotes the composition $X_2 \to X$, $Z_1 := h^{-1}(Z)_{\mathrm{red}}$ is an snc divisor in X_2, forming a strictly semistable pair with X_2/S_1.

Step 3. Use of 1.1. Defining Y, $(Z_1)_Y$, $I \subset D$, $K = D/I$ as in the former step 3, K acts generically freely on Y. As the pair $(Y, (Z_1)_Y)$ is strictly semistable over S_1, there exists a finite closed subset Σ_1 of S_1 such that $(Y, Y_{\Sigma_1} \cup (Z_1)_Y)$ forms a log regular pair, log smooth over S_1 equipped with the log structure defined by Σ_1. As $S_1 \to S' = S_1/L$ is Kummer étale, $(Y, (Y_{\Sigma'})_{\mathrm{red}} \cup (Z_1)_Y)$ (where Σ' is the image of Σ_1) is also log smooth over S' (equipped with the log structure given by Σ'), and we get a K-equivariant commutative diagram (2.3.4), with trivial action of K on S' and f projective and log smooth over S'. We then apply 1.1 to $f : (Y, (Y_{\Sigma'})_{\mathrm{red}} \cup (Z_1)_Y) \to S'$, and the proof runs as above. We get a D-equivariant projective modification $g : Y_1 \to Y$ (D acting through K) and a D-strict snc divisor

E_1 on Y_1, containing $(g^{-1}((Z_1)_Y) \cup (Y_1)_{\Sigma'})_{\mathrm{red}}$ as a subdivisor, such that the action of D on (Y_1, E_1) is very tame, and (Y_1, E_1) and $(Y_1/D, E_1/D)$ are log smooth over S'. After extending from D to L we get an L-equivariant commutative square of type (2.3.5), with $(Y_2/L, E_2/L)$ log smooth over S' $(= S_1/L)$, and $(v^{-1}(h_2^{-1}(Z)) \cup (Y_2/L)_{\Sigma'})_{\mathrm{red}} \subset E_2/L$. As above, we take a projective, log étale modification such that $\tilde{X}$ is regular, and $\tilde{E} = w^{-1}(E_2/L)$ is an snc divisor in $\tilde{X}$.

We now apply 1.2 to the log smooth morphism $(\tilde{X}, \tilde{E}) \to S'$. It's enough to work étale locally on $\tilde{X}$ around some geometric point x of $\tilde{X}_{s'}$, with $s' \in \Sigma'$. We replace S' by its strict localization at the image of x, and consider (1.4.2), with $Q = \mathbf{N}$, $G = \{1\}$, $\Lambda = \mathbf{Z}_{(p)}$ if $p > 1$ and $\mathbf{Q}$ otherwise, and the chart $a : \mathbf{N} \to M_{S'}, 1 \mapsto \pi$, π a uniformizing parameter of S'. In (1.4.3) we have $P^* = \mathbf{Z}^{\nu}$, $P_1 = \mathbf{N}^{\mu}$, for nonnegative integers μ, ν, hence

$$P = \mathbf{Z}^{\nu} \oplus \mathbf{N}^{\mu}.$$

Let $((b_i), (a_i))$ be the image of $1 \in \mathbf{N}$ in P in the above decomposition, and let $(a_1, \cdots, a_r)$, $(b_1, \cdots, b_s)$ be the sets of those a_i's and b_i's which are $\neq 0$. We may assume $b_i > 0$ if $b_i \neq 0$. As the torsion part of $\mathrm{Coker}(\mathbf{Z} \to P^{\mathrm{gp}})$ is annihilated by an integer invertible on $\tilde{X}$, we have $\gcd(p, a_1, \cdots, a_r, b_1, \cdots, b_s) = 1$, where p is the characteristic exponent of k. We have $P = \mathbf{Z}^s \oplus \mathbf{N}^r \oplus \mathbf{Z}^{\nu - s} \oplus \mathbf{N}^{\mu - r}$. Choosing a basis $t_{\mu+1}, \ldots, t_n$ of V, we get that étale locally around x, $\tilde{X}$ is given by a cartesian square

$$\begin{tikzcd}
\tilde{X} \arrow[r] \arrow[d] & \mathrm{Spec}\ \Lambda[u_1^{\pm 1}, \cdots, u_s^{\pm 1}, t_1, \cdots, t_n] \arrow[d] \\
S' \arrow[r] & \mathrm{Spec}\ \Lambda[z]
\end{tikzcd}$$

with x going to the point $(u_i = 1), (t_j = 0)$, and $z \mapsto \pi$, $z \mapsto u_1^{b_1} \cdots u_s^{b_s} t_1^{a_1} \cdots t_r^{a_r}$, in other words,

$$\tilde{X} = S'[u_1^{\pm 1}, \cdots, u_s^{\pm 1}, t_1, \cdots, t_n]/(u_1^{b_1} \cdots u_s^{b_s} t_1^{a_1} \cdots t_r^{a_r} - \pi),$$

Finally, $\tilde{E}$ is the union of the special fiber $\tilde{X}_{s'}$ and a horizontal divisor $\tilde{T}$, étale locally given by the equation $t_{r+1} \cdots t_m = 0$, where $m = \mu$ and $1 \leq r \leq m \leq n$. As $\tilde{h}^{-1}(Z)_{\mathrm{red}}$ is a sub-divisor of $(\tilde{X}_{s'})_{\mathrm{red}} \cup \tilde{T}$, this finishes the proof of 2.4.

3. Resolvability, log smoothness, and weak semistable reduction

3.1. Elimination of separatedness assumptions. The main aim of §3.1 is to weaken the separatedness assumptions in Theorems 1.1 and **VIII**-1.1.

3.1.1. Recall, see **VI**-4.1, that if a finite group G acts on a scheme X then the fixed point subscheme X^G is the intersection of graphs of the translations $g : X \to X$. In particular, X^G is closed whenever X is separated. The definition obviously makes sense for non-separated schemes, and the only novelty is that X^G is a subscheme that does not have to be closed.

3.1.2. *Inertia specializing actions.* An action of a finite group G on a scheme X is **inertia specializing** if for any point $x \in X$ with a specialization $y \in X$ one has that $G_x \subset G_y$.

Lemma 3.1.3. *An action of G on X is inertia specializing if and only if for each subgroup $H \subset G$ the subscheme X^H is closed.*

Proof. Note that a subscheme is closed if and only if it is closed under specializations. If X^H is not closed then there exists a point $x \in X^H$ with a specialization $y \notin X^H$. Thus, $H \subset G_x$ and $H \nsubseteq G_y$, and the action is not inertia specializing. The opposite direction is proved similarly. □

Remark 3.1.4. (i) A large class of examples of inertia specializing actions can be described as follows. The following conditions are equivalent and imply that the action is inertia specializing: (a) any G orbit is contained in an open separated subscheme of X, (b) X admits a covering by G-equivariant separated open subschemes X_i. In particular, any admissible action is inertia specializing.

(ii) If (G, X, Z) is as in Theorem **VIII**-1.1, but instead of separatedness of X one only assumes that it possesses a covering by G-equivariant separated open subschemes X_i, then the assertion of the theorem still holds true. Indeed, the theorem applies to the G-equivariant log schemes $(X_i, Z_i = Z|_{X_i})$, and by Theorem **VIII**-5.6.1 the modifications $f_{(G,X_i,Z_i)}$ agree on the intersections and hence glue to a required modification $f_{(G,X,Z)}$ of X.

A quick analysis of the proof of **VIII**-1.1 is required to obtain the following stronger result.

THEOREM 3.1.5. (*i*) *Theorem* **VIII**-*1.1 and its complement* **VIII**-*5.6.1 hold true if the assumption that X is separated is weakened to the assumption that the action of G on X is inertia specializing.*

(*ii*) *Theorem 1.1 holds true if the assumptions that X and S are separated are replaced with the single assumption that the action of G on X is inertia specializing.*

Proof. The construction of modification $f_{(G,X,Z)}$ in the proof of **VIII**-1.1 runs in four steps. The first two steps are determined by X and Z, see **VIII**-4. These steps do not use any separatedness assumption. In Step 3, one blows up the inertia strata, see **VIII**-4.1.9. Here one only needs to know that the inertia strata are closed, and by Lemma 3.1.3 this happens if and only if the action is inertia specializing. Finally, let us discuss the main part of the construction, see **VIII**-5 and **VIII**-5.5.5. Using Lemma **VIII**-5.3.8, one finds an appropriate equivariant covering $(X', Z', G) \to (X, Z, G)$ with an affine X' and reduces the problem to studying the source. Thus, the separatedness assumption is only used in Lemma **VIII**-5.3.8. In fact, the only property of the G-action used in the proof of the latter is that for any $x \in X$ the set $X \setminus \bigcup_{H \not\subseteq G_{\overline{x}}} X^H$ is open. Thus, in this case too, one only uses that the action is inertia specializing.

The proof of Theorem 1.1 runs as follows. One considers the modification $f_{(G,X,Z)}$ from **VIII**-1.1 and checks that it satisfies the additional properties asserted by 1.1. This check is local on X and hence applies to non-separated schemes as well. Since by part (i) of 3.1.5, $f_{(G,X,Z)}$ is well defined whenever G acts inertia specializing on X, we obtain (ii). □

3.1.6. *Pseudo-projective morphisms and non-separated Chow's lemma.* We conclude §3.1 with recalling a non-separated version of Chow's lemma due to Artin-Raynaud-Gruson, see [**Raynaud & Gruson, 1971**, I 5.7.13]. It will be needed to avoid unnecessary separatedness assumptions in the future. We prefer to use the following non-standard terminology: a finite type morphism $f : X \to S$ is **pseudo-projective** if it can be factored into a composition of a **local isomorphism** $X \to \overline{X}$ (i.e. X admits an open covering $X = \bigcup_i X_i$ such that the morphisms $X_i \to \overline{X}$ are open immersions) and a projective morphism $\overline{X} \to S$.

REMARK 3.1.7. (i) We introduce pseudo-projective morphisms mainly for terminological convenience. Although pseudo-projectivity is preserved by base changes, it can be lost under compositions. Moreover, even if X is pseudo-projective over a field k, its blow up X' does not have to be pseudo-projective over k (thus giving an example of a projective morphism $f : X' \to X$ and a pseudo-projective one $X \to \mathrm{Spec}(k)$ so that the composition is not pseudo-projective). Indeed, let X be an affine plane with a doubled origin $\{o_1, o_2\}$, and let X' be obtained by blowing up o_1. By η we denote the generic point of $C_1 = f^{-1}(o_1)$. The ring $\mathrm{Spec}(\mathcal{O}_{X',\eta})$ is a DVR and its spectrum has two different k-morphisms to X': one takes the closed point to η and another one takes it to o_2. It then follows from the valuative criterion of separatedness that any k-morphism $g : X' \to Y$ with a separated target takes o_2 and η to the same point of Y. In particular, such g cannot be a local isomorphism, and hence X' is not pseudo-projective over k.

(ii) Note that a morphism f is separated (resp. proper) and pseudo-projective if and only if it is quasi-projective (resp. projective). So, the following result extends the classical Chow's lemma to non-separated morphisms.

PROPOSITION 3.1.8. *Let $f : X \to S$ be a finite type morphism of quasi-compact and quasi-separated schemes, and assume that X has finitely many maximal points. Then there exists a projective modification $g : X' \to X$ (even a blow up along a finitely generated ideal with a nowhere dense support) such that the morphism $X' \to S$ is pseudo-projective.*

Proof. As a simple corollary of the flattening theorem, it is proved in [**Raynaud & Gruson, 1971**, I 5.7.13] that there exists a modification $X' \to X$ such that $X' \to S$ factors as a composition of an étale morphism $X' \to \overline{X}$ that induces an isomorphism of dense open subschemes and a projective morphism $\overline{X} \to S$. (In loc.cit. one works with algebraic spaces and assumes that f is locally separated, but the latter is automatic for any morphism of schemes.) Our claim now follows from the following lemma (which fails for locally separated morphisms between algebraic spaces). □

Lemma 3.1.9. *Assume that $\phi: Y \to Z$ is an étale morphism of schemes that restricts to an open embedding on a dense open subscheme $Y_0 \hookrightarrow Y$. Then ϕ is a local isomorphism.*

Proof. Let us prove that if, in addition, ϕ is separated then ϕ is an open immersion. Since Y possesses an open covering by separated subschemes, this will imply the lemma. The diagonal $\Delta_\phi: Y \to Y \times_Z Y$ is an open immersion, and by the separatedness of ϕ, it is also a closed immersion. Thus, Y is open and closed in $Y \times_Z Y$, and since both Y and $Y \times_Z Y$ have dense open subschemes that map isomorphically onto Y_0, Δ_ϕ is an isomorphism. This implies that ϕ is a monomorphism, but any étale monomorphism is an open immersion by [**ÉGA** IV$_4$ 17.9.1]. □

3.2. Semistable curves and log smoothness.

3.2.1. *Log structure associated to a closed subset.* Let S be a reduced scheme. Any closed nowhere dense subset $W \subset S$ induces a log structure $j_\star \mathcal{O}_U^* \cap \mathcal{O}_S \hookrightarrow \mathcal{O}_S$, where $j: U \hookrightarrow S$ is the embedding of the complement of W. The associated log scheme will be denoted (S, W). By **VI**-1.4, any log regular log scheme is of the form (S, W), where W is the non-triviality locus of the log structure.

3.2.2. *Semistable relative curves.* Following the terminology of [**Temkin, 2010**], by a **semistable multipointed relative curve** over a scheme S we mean a pair (C, D), where C is a flat finitely presented S-scheme of pure relative dimension one and with geometric fibers having only ordinary nodes as singularities, and $D \hookrightarrow C$ is a closed subscheme which is étale over S and disjoint from the singular locus of $C \to S$. We do not assume C to be neither proper nor even separated over S.

Proposition 3.2.3. *Assume that (S, W) is a log regular log scheme and (C, D) is a semistable multipointed relative S-curve such that the morphism $f: C \to S$ is smooth over $S \setminus W$. Then the morphism of log schemes $(C, D \cup f^{-1}(W)) \to (S, W)$ is log smooth.*

Proof. See **VI**-1.9. □

3.3. ℓ'-resolvability.

3.3.1. *Alterations.* Assume that S' and S are reduced schemes with finitely many maximal points and let $\eta' \subset S'$ and $\eta \subset S$ denote the subschemes of maximal points. Let $f: S' \to S$ be an **alteration**, i.e. a proper, surjective, generically finite, and maximally dominating morphism. Recall that f is an **ℓ'-alteration** if one has that $([k(x) : k(f(x))], l) = 1$ for any $x \in \eta'$. We say that f is **separable** if $k(\eta')$ is a separable $k(\eta)$-algebra (i.e. $\eta' \to \eta$ has geometrically reduced fibers). If, in addition, S' and S are provided with an action of a finite group G such that f is G-equivariant, the action on S is trivial, the action on η' is free, and $\eta'/G \xrightarrow{\sim} \eta$, then we say that f is a **separable Galois alteration** of group G, or just **separable G-alteration**.

Remark 3.3.2. We add the word "separable" to distinguish our definition from Galois alterations in the sense of de Jong (see [**de Jong, 1997**]) or Gabber-Vidal (see [**Vidal, 2004**, p. 370]). In the latter cases, one allows alterations that factor as $S' \to S'' \to S$, where $S' \to S''$ is a separable Galois alteration and $S'' \to S$ is purely inseparable.

3.3.3. *Universal ℓ'-resolvability.* Let X be a locally noetherian scheme and let ℓ be a prime invertible on X. Assume that for any alteration $Y \to X_{\mathrm{red}}$ and nowhere dense closed subset $Z \subset Y$ there exists a surjective projective morphism $f: Y' \to Y$ such that Y' is regular and $Z' = f^{-1}(Z)$ is an snc divisor. (By a slight abuse of language, by saying that a closed subset is an snc divisor we mean that it is the support of an snc divisor.) If, furthermore, one can always choose such f to be a modification, separable ℓ'-alteration, ℓ'-alteration, or alteration, then we say that X is **universally resolvable**, **universally separably ℓ'-resolvable**, **universally ℓ'-resolvable**, or **universally Q-resolvable**, respectively.

Remark 3.3.4. (i) Due to resolution of singularities in characteristic zero, any qe scheme over Spec(**Q**) is universally resolvable. This is essentially due to Hironaka, [**Hironaka, 1964**], though an additional work was required to treat qe schemes that are not algebraic in Hironaka's sense, see [**Temkin, 2008**] for the noetherian case and [**Temkin, 2012**] for the general case.

(ii) It is hoped that all qe schemes admit resolution of singularities (in particular, are universally resolvable). However, we are, probably, very far from proving this. Currently, it is known that any qe scheme of dimension at most two admits resolution of singularities (see [**Cossart et al., 2009**] for a modern treatment). In particular, any qe scheme of dimension at most two is universally resolvable.

(iii) One can show that any universally $\mathbf{Q}$-resolvable scheme is qe, but we prefer not to include this proof here, and will simply add quasi-excellence assumption to the theorems below.

(iv) On the negative side, we note that there exist regular (hence resolvable) but not universally $\mathbf{Q}$-resolvable schemes X. They can be constructed analogously to examples from **I**-11.5. For instance, there exists a discretely valued field K whose completion $\widehat{K}$ contains a non-trivial finite purely inseparable extension K'/K (e.g. take an element $y \in k((x))$ which is transcendental over $k(x)$ and set $K = k(x, y^p) \subset K' = k(x, y) \subset k((x))$ with the induced valuation). The valued extension K'/K has a defect in the sense that $e_{K'/K} = f_{K'/K} = 1$. In other words, the DVR's A' and A of K' and K, have the same residue field and satisfy $m_{A'} = m_A A'$. Since A' is A-flat, it cannot be A-finite. On the other hand, A' is the integral closure of A in K', and we obtain that A is not qe. In addition, although $X = \operatorname{Spec}(A)$ is regular, any X-finite integral scheme X' with $K' \subset k(X')$ possesses a non-finite normalization and hence does not admit a desingularization. Thus, X is not universally $\mathbf{Q}$-resolvable.

Our main goal will be to show that universal ℓ'-resolvability of a qe base scheme S is inherited by finite type S-schemes whose structure morphism $X \to S$ is maximally dominating (see Theorem 3.5 below, where a more precise result is formulated). The proof will be by induction on the relative dimension, and the main work is done when dealing with the case of generically smooth relative curves.

Theorem 3.4. *Let S be an integral, noetherian, qe scheme with generic point $\eta = \operatorname{Spec}(K)$, let $f : X \to S$ be a maximally dominating* (**II**-1.1.2) *morphism of finite type, and let $Z \subset X$ be a nowhere dense closed subset. Assume that S is universally ℓ'-resolvable* (*resp. universally separably ℓ'-resolvable*), *$X_\eta = X \times_S \eta$ is a smooth curve over K, and $Z_\eta = Z \times_S \eta$ is étale over K. Then there exist a projective ℓ'-alteration* (*resp. a separable projective ℓ'-alteration*) *$a : S' \to S$, a projective modification $b : X' \to (X \times_S S')^{\mathrm{pr}}$, where $(X \times_S S')^{\mathrm{pr}}$ is the proper transform of X, i.e. the schematic closure of $X_\eta \times_S S'$ in $X \times_S S'$,*

$$\begin{tikzcd} X' \arrow[r, "b"] \arrow[drr, "f'"'] & (X \times_S S')^{\mathrm{pr}} \arrow[r] & X \times_S S' \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ & & S' \arrow[r, "a"] & S \end{tikzcd}$$

and divisors $W' \subset S'$ and $Z' \subset X'$ such that S' and X' are regular, W' and Z' are snc, the morphism $f' : X' \to S'$ is pseudo-projective (§3.1.6), *$(X', Z') \to (S', W')$ is log smooth, and $Z' = c^{-1}(Z) \cup f'^{-1}(W')$, where c denotes the alteration $X' \to X$.*

We also note if f is separated (resp. proper) then f' is even quasi-projective (resp. projective) by Remark 3.1.7(ii).

Proof. It will be convenient to represent Z as $Z_h \cup Z_v$, where the horizontal component Z_h is the closure of Z_η and the vertical component Z_v is the closure of $Z \setminus Z_h$. The following observation will be used freely: if $a_1 : S_1 \to S$ is a (resp. separable) projective ℓ'-alteration with an integral S_1 and $b_1 : X_1 \to (X \times_S S_1)^{\mathrm{pr}}$ is a projective modification, then it suffices to prove the theorem for $f_1 : X_1 \to S_1$ and the preimage $Z_1 \subset X_1$ of Z (note that the generic fiber of f_1 is smooth because it is a base change of that of f). So, in such a situation we can freely replace f by f_1, and Z will be updated automatically without mentioning, as a rule. We will change S and X a few times during the proof. We start with some preliminary steps.

Step 1. *We can assume that f is quasi-projective.* By Proposition 3.1.8, replacing X with its projective modification we can achieve that f factors through a local isomorphism $X \to \overline{X}$, where $\overline{X}$ is S-projective. Let $X_1 \subset \overline{X}$ be the image of X. Then the induced morphism $X_1 \to S$ is quasi-projective and with smooth generic fiber. If the theorem holds for f_1 and the image $Z_1 \subset X_1$ of Z, i.e., there exist $a : S' \to S$ and $b_1 : X_1' \to (X_1 \times_S S')^{\mathrm{pr}}$ that satisfy all assertions of the theorem, then the theorem also holds for f and Z because we can keep the same a and take $b = b_1 \times_{X_1} X$. This completes the step, and in the sequel we assume that f is quasi-projective. As we will only use projective modifications $b_1 : X_1 \to (X \times_S S_1)^{\mathrm{pr}}$, the quasi-projectivity of f will be preserved automatically.

Step 2. *We can assume that f and $Z_h \to S$ are flat.* Indeed, due to the flattening theorem of Raynaud-Gruson, see [**Raynaud & Gruson, 1971**, I 5.2.2], this can be achieved by replacing S with an appropriate projective modification S', replacing X with the proper transform, and replacing Z with its preimage. From now on, the proper transforms of X will coincide with the base changes.

Step 3. *Use of the stable modification theorem.* By the stable modification theorem[**Temkin, 2010**, 1.5 and 1.1] there exist a separable alteration $\overline{a} : \overline{S} \to S$ with an integral $\overline{S}$ and a projective modification $\overline{X} \to X \times_S \overline{S}$ such that $(\overline{X}, \overline{Z}_h)$ is a *semistable multipointed* $\overline{S}$-curve (see §3.2.2), where $\overline{Z}_h \subset \overline{X}$ is the horizontal part of the preimage $\overline{Z}$ of Z. Enlarging $\overline{S}$ we can assume that it is integral and normal.

$$\begin{tikzcd}
\overline{X} \arrow[r] \arrow[dr, "\overline{f}"'] & X \times_S \overline{S} \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
& \overline{S} \arrow[r, "\overline{a}"] & S
\end{tikzcd}$$

Step 4. *We can assume that $\overline{a}$ is a separable projective G-alteration, where G is an ℓ-group.* Since semistable multipointed relative curves are preserved by base changes, we can just enlarge $\overline{S}$ by replacing it with any separable projective Galois alteration that factors through $\overline{S}$. Once $\overline{S} \to S$ is Galois, let $\overline{G}$ denote its Galois group and let $G \subset \overline{G}$ be any Sylow ℓ-subgroup. Since $\overline{S} \to \overline{S}/G$ is a separable G-alteration and $\overline{S}/G \to S$ is a separable projective ℓ'-alteration, we can replace S with $\overline{S}/G$, replace X with $X \times_S (\overline{S}/G)$, and update Z accordingly, accomplishing the step.

Step 5. *The action of G on $X \times_S \overline{S}$ via $\overline{S}$ lifts equivariantly to $\overline{X}$. In particular, $\overline{f}$ becomes G-equivariant and $\overline{X} \to X$ becomes a separable projective G-alteration.* This follows from [**Temkin, 2010**, 1.6].

Step 6. *The action of G on $\overline{X}$ is inertia specializing.* Indeed, any covering of X by separated open subschemes induces a covering of $\overline{X}$ by G-equivariant separated open subschemes. So, it remains to use Remark 3.1.4(i).

Step 7. *We can assume that $\overline{S} \to S$ is finite.* By Raynaud-Gruson there exists a projective modification $S' \to S$ such that the proper transform $\overline{S}'$ of $\overline{S}$ is flat over S'. Let η denote the generic point of S and S' and let η' denote the generic point of $\overline{S}$ and $\overline{S}'$. Since the morphisms $\overline{S} \times_S S' \to S'$ and $\eta' \to \eta$ are G-equivariant and $\overline{S}'$ is the schematic closure of η' in $\overline{S} \times_S S'$, we obtain that the morphism $\overline{S}' \to S'$ is a separable projective G-alteration. Replacing $\overline{S} \to S$ with $\overline{S}' \to S'$, and updating X and $\overline{X}$ accordingly, we achieve that $\overline{S} \to S$ becomes flat, and hence finite. All conditions of steps 1–6 are preserved with the only exception that $\overline{S}$ may be non-normal. So, we replace $\overline{S}$ with its normalization and update $\overline{X}$. This operation preserves the finiteness of $\overline{S} \to S$, so we complete the step.

Step 8. *Choice of W.* Fix a closed subset $W \subsetneq S$ such that $\overline{S} \to S$ is étale over $S \setminus W$, $\overline{f}(\overline{Z}_v) \subset \overline{W}$, where $\overline{Z}_v$ is the vertical part of $\overline{Z}$ and $\overline{W} = \overline{a}^{-1}(W)$, and $\overline{f}$ is smooth over $\overline{S} \setminus \overline{W}$.

Step 9. *We can assume that S is regular and W is snc.* Indeed, by our assumptions on S there exists a projective ℓ'-alteration (resp. a separable projective ℓ'-alteration) $a : S' \to S$ such that S' is regular and $a^{-1}(W)$ is snc. Choose any preimage of η in $S' \times_S \overline{S}$ and let $\overline{S}'$ be the normalization of its closure. Then $\overline{S}' \to S'$ is a separable projective Galois alteration with Galois group $G' \subset G$, so we can replace $S, \overline{S}, G$ and X with $S', \overline{S}', G'$ and $X \times_S S'$, respectively, and update W, $\overline{W}$ and Z accordingly (i.e. replace them with their preimages). Note that step 9 is the only step where a non-separable alteration of S may occur.

Step 10. *The morphism $(\overline{S}, \overline{W}) \to (S, W)$ is Kummer étale.* Indeed, $\overline{S} \to S$ is an étale G-covering outside of W, and $\overline{S}$ is the normalization of S in this covering, so the assertion follows from **IX**-2.1.

Consider the G-equivariant subscheme $\overline{T} = \overline{Z} \cup \overline{f}^{-1}(\overline{W})$ of $\overline{X}$. The morphism $(\overline{X}, \overline{T}) \to (\overline{S}, \overline{W})$ is log smooth by Proposition 3.2.3, hence so is the composition $(\overline{X}, \overline{T}) \to (S, W)$ and we obtain that $(\overline{X}, \overline{T})$ is log regular. The group G acts freely on $\overline{S} \setminus \overline{W}$ and hence also on $\overline{X} \setminus \overline{T}$. Also, its action on $\overline{X}$ is tame and inertia specializing (step 6), hence we can apply Theorem 1.1 to $(\overline{X}, \overline{T}) \to (S, W)$. As a result, we obtain a projective G-equivariant modification $(\overline{X}', \overline{T}') \to (\overline{X}, \overline{T})$ such that $\overline{T}'$ is the preimage of $\overline{T}$, G acts very tamely on $(\overline{X}', \overline{T}')$, and $(X', Z') = (\overline{X}'/G, \overline{T}'/G)$ is log smooth over (S, W) (the quotient exists as a scheme as f is quasi-projective by Step 1 and the morphisms $\overline{S} \to S$ and $\overline{X}' \to \overline{X} \to X \times_S \overline{S}$ are projective). Clearly, X' is a projective modification of X and Z' is the union of the preimages of W and Z, hence it only remains to achieve that X' is regular and Z' is snc. For this it suffices to replace (X', Z') with its projective modification $\mathscr{F}^{\log}(X', Z')$ introduced in **VIII**-3.4.9. $\square$

Remark 3.4.1. It is natural to compare Theorem 3.4 and the classical de Jong's result recalled in **IX**-1.2. The main differences are as follows.

(i) One considers non-proper relative curves in 3.4, and this is the only point that requires to use the stable modification theorem instead of de Jong's result. The reason is that although the problem easily reduces to the case of a quasi-projective f (see Step 2), one cannot make f projective, as the compactified generic fiber $\overline{X}_\eta$ does not have to be smooth (i.e. *geometrically* regular) at the added points.

(ii) One uses ℓ'-alterations in 3.4. This is more restrictive than in **IX**-1.2, but the price one has to pay is that the obtained log smooth morphism $(X', Z') \to (S', W')$ does not have to be a nodal curve (e.g. $X' \to S'$ may have non-reduced fibers). The construction of such $b : X' \to X$ involves a quotient by a Sylow subgroup, and is based on Theorem 1.1. (Note also that it seems probable that instead of 1.1 one could use the torification argument of Abramovich-de Jong, see [**Abramovich & de Jong, 1997**, §1.4.2].)

Now, we are going to use Theorem 3.4 to prove the main result of §3.

Theorem 3.5. *Let $f : X \to S$ be a maximally dominating (**II**-1.1.2) morphism of finite type between noetherian qe schemes, let $Z \subset X$ be a nowhere dense closed subset, and assume that S is universally ℓ'-resolvable, then:*

(i) X is universally ℓ'-resolvable.

(ii) There exist projective ℓ'-alterations $a : S' \to S$ and $b : X' \to X$ with regular sources, a pseudo-projective (§3.1.6) morphism $f' : X' \to S'$ compatible with f

$$\begin{tikzcd} X' \arrow[r, "b"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ S' \arrow[r, "a"] & S \end{tikzcd}$$

and snc divisors $W' \subset S'$ and $Z' \subset X'$ such that $Z' = b^{-1}(Z) \cup f'^{-1}(W')$ and the morphism $(X', Z') \to (S', W')$ is log smooth.

(iii) If $S = \operatorname{Spec}(k)$, where k is a perfect field, then one can achieve in addition to (ii) that a is an isomorphism and the alteration b is separable. In particular, X is universally separably ℓ'-resolvable in this case.

Proof. Note that (i) follows from (ii) because any alteration X_1 of X is also of finite type over S, so we can apply (ii) to X_1 as well. Thus, our aim is to prove (ii) and its complement (iii). We will view Z both as a closed subset and a reduced closed subscheme. We start with a few preliminary steps, that reduce the theorem to a special case. We will tacitly use that if $S_1 \to S$ and $X_1 \to X$ are projective ℓ'-alterations, separable in case (iii), and $f_1 : X_1 \to S_1$ is compatible with f, then it suffices to prove the theorem for f_1 and the preimage $Z_1 \subset X_1$ of Z. So, in such situation we can freely replace f with f_1, and Z will be updated automatically without mentioning, as a rule.

Step 1. *We can assume that X and S are integral and normal.* For a noetherian scheme Y, let Y^{nor} denote the normalization of its reduction. Since f is maximally dominating, it induces a morphism

$f^{\mathrm{nor}}: X^{\mathrm{nor}} \to S^{\mathrm{nor}}$, and replacing f with f^{nor} we can assume that S and X are normal. Since we can work separately with the connected components, we can now assume that S and X are integral.

Step 2. *We can assume that f is projective.* By Proposition 3.1.8 there exists a projective modification $X_1 \to X$ such that the morphism $X_1 \to S$ factors into a composition of a local isomorphism $X_1 \to \overline{X}$ and a projective morphism $\overline{f}: \overline{X} \to S$. Replacing X with X_1 we can assume that X itself admits a local isomorphism $g: X \to \overline{X}$ with an S-projective target. Let $\overline{Z}$ be the closure of $g(Z)$. Then it suffices to solve our problem for $\overline{f}$ and $\overline{Z}$, as the corresponding alteration of $\overline{X}$ will induce an alteration of X as required. Thus, replacing X and Z with $\overline{X}$ and $\overline{Z}$, we can assume that f is projective.

Step 3. *It suffices to find f' which satisfies all assertions of the theorem except the formula for Z', while the latter is weakened as $b^{-1}(Z) \cup f'^{-1}(W') \subset Z'$.* Given such an f' note that $Z'' = b^{-1}(Z) \cup f'^{-1}(W')$ is a subdivisor of Z', hence it is an snc divisor too. We claim that X', Z'', S', W' satisfy all assertions of the theorem, and the only thing one has to check is that the morphism $(X', Z'') \to (S', W')$ is log smooth. The latter follows from Lemma 3.5.3 whose proof will be given below.

Step 4. *In the situation of (iii) we can assume that the field k is infinite.* Assume that $S = \mathrm{Spec}(k)$ where k is a finite field and fix an infinite algebraic ℓ'-prime extension $\overline{k}/k$ (i.e. it does not contain the extension of k of degree ℓ). We claim that it suffices to prove (ii) and (iii) for $\overline{S} = \mathrm{Spec}(\overline{k})$ and the base changes $\overline{X} = X \times_S \overline{S}$ and $\overline{Z} = Z \times_S \overline{S}$. Indeed, assume that $a: \overline{X}' \to \overline{X}$ is a separable ℓ'-alteration with a regular source and such that $\overline{Z}' = a^{-1}(\overline{Z})$ is an snc divisor (obviously, we can take $\overline{S}' = \overline{S}$ and $\overline{W}' = \emptyset$). Since $\overline{S} = \lim_i S_i$ where $S_i = \mathrm{Spec}(k_i)$ and k_i/k run over finite subextensions of $\overline{k}/k$, [**ÉGA** IV$_3$ 8.8.2(ii)] implies that there exists i and a finite type morphism $X'_i \to X_i = X \times_S S_i$ such that $\overline{X}' \xrightarrow{\sim} X'_i \times_{S_i} \overline{S}$. For any finite subextension $k_i \subset k_j \subset \overline{k}$ set $X'_j = X'_i \times_{S_i} S_j$ and $X_j = X_i \times_{S_i} S_j$, and let $Z'_j \subset X'_j$ be the preimage of Z. Then it follows easily from [**ÉGA** IV$_3$ 8.10.5] and [**ÉGA** IV$_4$ 17.7.8] that $X'_j \to X_j$ is an ℓ'-alteration and $X'_j \to S_j$ is smooth for large enough k_j. In the same manner one achieves that Z'_j is an snc divisor. Now, it is obvious that $(X'_j, Z'_j) \to (S, \emptyset)$ is log smooth and $X'_j \to X_j \to X$ is an ℓ'-alteration.

Now we are in a position to prove the theorem. We will use induction on $d = \mathrm{tr.deg.}(k(X)/k(S))$, with the case of $d = 0$ being obvious. Assume that $d \geq 1$ and the theorem holds for smaller values of d.

Step 5. *Factorizing f through a relative curve.* After replacing X by a projective modification, we can factor f as $h \circ g$, where $h: Y \to S$ is projective, Y is integral, $g: X \to Y$ is maximally dominating and $\mathrm{tr.deg.}(k(X)/k(Y)) = 1$. Indeed, one can obviously construct such a rational map $g': X \dashrightarrow Y$ even without modifying X (i.e. g' is well defined on a non-empty open subscheme $U \subset X$). Let X' be the schematic image of the morphism $U \hookrightarrow X \times Y$. Then $X' \to X$ is a projective modification (an isomorphism over U), and the morphism $X' \to S$ factors through $g: X \to Y$.

Let $\eta = \mathrm{Spec}(k(Y))$ denote the generic point of Y, $X_\eta = X \times_Y \eta$ and $Z_\eta = Z \times_Y \eta$. We claim that in addition to factoring f through Y one can achieve that the following condition is satisfied:

$(*)$ X_η and Z_η are smooth over η.

In general, this is achieved by replacing X and Y by inseparable alterations. Pick up any finite purely inseparable extension $K/k(Y)$ such that $Z_K = (Z \times_\eta \mathrm{Spec}(K))^{\mathrm{nor}}$ (i.e. just the reduction) and $X_K = (X \times_\eta \mathrm{Spec}(K))^{\mathrm{nor}}$ are smooth, extend $K/k(Y)$ to a projective alteration $Y' \to Y$, and replace Y and X with Y' and the schematic closure of X_K in $X \times_Y Y'$, respectively. Clearly, $(*)$ holds after this replacement.

It remains to deal with the case (iii). This time we should avoid inseparable alterations, so g and Y should be chosen more carefully. If $k = \overline{k}$ is algebraically closed and $S = \mathrm{Spec}(k)$ then such g and Y exist by [**de Jong, 1996**, 4.11], and the general assertion of (iii) will be proven similarly. Let us recall the main line of the proof of [**de Jong, 1996**, 4.11]. Fix a closed immersion $X \hookrightarrow \mathbf{P}^N_k$ and for each linear subspace L of dimension $N - d$ consider the classical projection $\mathrm{Bl}_L(\mathbf{P}^N_k) \to Y = \mathbf{P}^{d-1}_k$. If L is general then it does not contain X and hence the strict transform $X'_L \hookrightarrow \mathrm{Bl}_L(\mathbf{P}^N_k)$ is a modification of X. de Jong shows that if $k = \overline{k}$ then for a general choice of L the projection $X'_L \to Y$ satisfies $(*)$.

In the general case, the schemes $\overline{X} = X \otimes_k \overline{k}$ and $\overline{Z} = Z \otimes_k \overline{k}$ are reduced since k is perfect. Hence a general $\overline{L} \hookrightarrow \mathbf{P}^N_{\overline{k}}$ induces a modification $\overline{X}'_{\overline{L}} \to \overline{X}$ and a curve fibration $g_{\overline{L}} : \overline{X}'_{\overline{L}} \to \mathbf{P}^{d-1}_{\overline{k}}$ that satisfies $(*)$. Since k is infinite we can choose $\overline{L}$ to be defined over k, i.e. $\overline{L} = L \otimes_k \overline{k}$ for $L \hookrightarrow \mathbf{P}^N_k$. We obtain thereby a modification $X'_L \to X$ and a curve fibration $g_L : X'_L \to \mathbf{P}^{d-1}_k$. Since $g_{\overline{L}}$ is the flat base change of g_L, the latter satisfies $(*)$ by descent.

Step 6. *Use of Theorem 3.4.* So far, we have constructed the right column of the following diagram

$$\begin{tikzcd}
(X',Z') \arrow[r, "\mathscr{F}^{\log}(L,M_L)"] \arrow[ddr, "f'"'] & (L,M_L) \arrow[r] \arrow[d, "g'"] \arrow[dr, phantom, "\square"] & (X'',Z'') \arrow[r, "b''"] \arrow[d, "g''"] & X \arrow[d, "g"] \arrow[dd, bend left, "f"] \\
& (Y',V') \arrow[r, "c'"] \arrow[d, "h'"] & (Y'',V'') \arrow[r, "c''"] \arrow[d, "h''"] & Y \arrow[d, "h"] \\
& (S',W') \arrow[r, "a"] & S \arrow[r] & S
\end{tikzcd}$$

By Theorem 3.4 there exists a projective ℓ'-alteration $c'' : Y'' \to Y$ with regular source, a projective modification $X'' \to (X \times_Y Y'')^{\mathrm{pr}}$ with regular source, a projective morphism $g'' : X'' \to Y''$ compatible with g, and snc divisors $V'' \subset Y''$ and $Z'' \subset X''$ such that $(X'', Z'') \to (Y'', V'')$ is log smooth and $b''^{-1}(Z) \subset Z''$. In case (iii), Y is universally separably ℓ'-resolvable by the induction assumption, hence we can take c'' to be separable, and then $b'' : X'' \to X$ is also separable. In addition, by the induction assumption applied to $h'' : Y'' \to S$ and $V'' \subset Y''$ there exist projective ℓ'-alterations $a : S' \to S$ and $c' : Y' \to Y''$ with regular sources and snc divisors $W' \subset S'$ and $V' \subset Y'$ and a projective morphism $h' : Y' \to S'$ compatible with h'' such that $(Y', V') \to (S', W')$ is log smooth, $c'^{-1}(V'') \subset V'$, and c' is separable if the assumption of (iii) is satisfied.

Set $(L, M_L) = (X'', Z'') \times^{\mathrm{fs}}_{(Y'',V'')} (Y', V')$, where the product is taking place in the category of fs log schemes. To simplify notation we will write P^{sat} instead of $(P^{\mathrm{int}})^{\mathrm{sat}}$ for monoids and log schemes. Recall that $(L, M_L) = (F, M_F)^{\mathrm{sat}}$, where (F, M_F) is the usual log fibered product, and $F = X'' \times_{Y''} Y'$ by [**Kato, 1988**, 1.6]. Furthermore, we have local Zariski charts for c' and g'' modeled, say, on $P_i \to P'_i$ and $P_i \to Q_i$. Hence (F, M_F) is a Zariski log scheme with charts modeled on $R_i = P'_i \oplus_{P_i} Q_i$, and (L, M_L) is a Zariski log scheme with charts modeled on R_i^{sat}. Furthermore, the saturation morphism $L \to F$ is finite hence the composition $L \to F \to X''$ is projective. The morphism $g' : (L, M_L) \to (Y', V')$ is a saturated base change of the log smooth morphism $g'' : (X'', Z'') \to (Y'', V'')$, hence it is log smooth. As (Y', V') is log regular, (L, M_L) is also log regular. Applying to (L, M_L) the saturated monoidal desingularization functor $\mathscr{F}^{\log}$ from **VIII**-3.4.9 we obtain a log regular Zariski log scheme (X', Z') with a regular X'. Then Z' is a normal crossings divisor, which is even an snc divisor since the log structure is Zariski.

We claim that (X', Z') and (S', W') are as asserted by the theorem except of the weakening dealt with in Step 3. Indeed, the morphism $(X', Z') \to (S', W')$ is log smooth because it is the composition $(X', Z') \to (L, M_L) \to (Y', V') \to (S', W')$ of log smooth morphisms. The preimage of Z in X'' is contained in Z'', which is the non-triviality locus of the log structure of (X'', Z''), hence its preimage in X' is also contained in the non-triviality locus of the log structure of (X', Z'), which is Z'. Clearly, Z' also contains the preimage of W'. By the construction, $S' \to S$ is a projective ℓ'-alteration, and it remains to check that $X' \to X$ is also a projective ℓ'-alteration. Since $\mathscr{F}^{\log}(L, M_L)$ is a saturated log blow up tower and (L, M_L) is log regular, the underlying morphism of schemes $X' \to L$ is a projective modification by **VIII**-3.4.6(i). The projective morphism $L \to X''$ is an ℓ'-alteration because generically (where the log structures are trivial) it is a base change of the projective ℓ'-alteration $Y' \to Y''$. And $X'' \to X$ is a projective ℓ'-alteration by the construction. It remains to recall that in the situation of (iii) the alterations $c' : Y' \to Y''$ and $b'' : X'' \to X$ are separable, hence so are $(L, M_L) \to X''$ and the total composition $X' \to X$. $\square$

Remark 3.5.1. The only place where inseparable alterations are used is the argument at step 5, where we had to choose an inseparable extension $K/k(Y)$ when X_η or Z_η is not geometrically regular.

REMARK 3.5.2. Analogs of Theorems 3.4 and 3.5 hold also for the class of universally **Q**-resolvable schemes. In a sense, this is the "$\ell = 1$" case of these theorems. One can prove this by the same argument but with ℓ replaced by 1. In fact, few arguments become vacuous (though formally true); for example, in steps 4-6 in the proof of Theorem 3.4, an ℓ-group G should be replaced by the trivial group, so the steps 5 and 6 collapse.

LEMMA 3.5.3. *Assume that S and X are regular schemes, $W \subset S$ and $Z \subset X$ are snc divisors, and $f : X \to S$ is a morphism such that $f^{-1}(W) \subset Z$ and the induced morphism of log schemes $h\colon (X,Z) \to (S,W)$ is log smooth. Then for any intermediate divisor $f^{-1}(W) \subset Z' \subset Z$ the morphism $h' : (X,Z') \to (S,W)$ is log smooth.*

Proof. We can work locally at a geometric point $\overline{x} \to X$. Let $x \in X$ and $s \in S$ be the images of $\overline{x}$, and let $q_1, \dots, q_r \in \mathcal{O}_{S,s}$ define the irreducible components of W through s. Set $Q = \bigoplus_{i=1}^r q_i^{\mathbf{N}}$. Shrinking S we obtain a chart $c\colon S \to \operatorname{Spec}(\mathbf{Z}[Q])$ of (S,W). By Proposition 1.2 applied to c, h, and $G = 1$, after localizing X along $\overline{x}$ one can find an fs chart of h consisting of c, $X \to \operatorname{Spec}(\mathbf{Z}[P])$, and $\phi\colon Q \to P$ such that the morphism $X \to S \times_{\operatorname{Spec}(\mathbf{Z}[Q])} \operatorname{Spec}(\mathbf{Z}[P])$ is smooth, P^* is torsion free, ϕ is injective, and the torsion of $\operatorname{Coker}(\phi^{\mathrm{gp}})$ is annihilated by an integer n invertible on S.

Let $p_1, \dots, p_t \in \mathcal{O}_{X,x}$ define the irreducible components of Z through x. Our next aim is to adjust the chart similarly to 1.4.2(vi) to achieve that $\overline{P} \xrightarrow{\sim} N = \bigoplus_{j=1}^t p_j^{\mathbf{N}}$. Note that $\overline{M}_{X,x} \xrightarrow{\sim} N$, where $M_X \hookrightarrow \mathcal{O}_X$ is the log structure of (X,Z). The homomorphism $\psi\colon P \to M_{X,x}$ factors through the fs monoid $R = P[T^{-1}]$ where $T = \psi^{-1}(M^*_{X,x})$. Clearly, R^* is torsion free, $\overline{R} \xrightarrow{\sim} N$, and shrinking X around x we obtain a chart $X \to \operatorname{Spec}(\mathbf{Z}[R])$. Since $\operatorname{Spec}(\mathbf{Z}[R])$ is open in $\operatorname{Spec}(\mathbf{Z}[P])$ the morphism $X \to S \times_{\operatorname{Spec}(\mathbf{Z}[Q])} \operatorname{Spec}(\mathbf{Z}[R])$ is smooth. So, we can replace P with R achieving that $\overline{P} \xrightarrow{\sim} N$, and hence $P \xrightarrow{\sim} \mathbf{N}^t \oplus \mathbf{Z}^u$.

Without restriction of generality Z' is defined by the vanishing of $\prod_{j=1}^{t'} p_j$ for $t' \le t$. Since $f^{-1}(W) \subset Z'$, the image of Q in $\overline{P}$ is contained in $\overline{P}' = \bigoplus_{j=1}^{t'} p_j^{\mathbf{N}}$. Hence ϕ factors through a homomorphism $\phi' : Q \to P' = \overline{P}' \oplus P^*$, and we obtain a chart of h' consisting of c, $X \to \operatorname{Spec}(\mathbf{Z}[P'])$, and ϕ'. By [**Kato, 1994**, 3.5, 3.6], to prove that h' is log smooth it remains to observe that ϕ' is injective, the torsion of $\operatorname{Coker}(\phi'^{\mathrm{gp}})$ is annihilated by n, and the morphism

$$X \to S \times_{\operatorname{Spec}(\mathbf{Z}[Q])} \operatorname{Spec}(\mathbf{Z}[P]) \to S \times_{\operatorname{Spec}(\mathbf{Z}[Q])} \operatorname{Spec}(\mathbf{Z}[P'])$$

is smooth because $\operatorname{Spec}(\mathbf{Z}[P]) \to \operatorname{Spec}(\mathbf{Z}[P'])$ is so. ☐

3.5.4. *Comparison with Theorems 2.1 and 2.4.* Theorems 2.1 and 2.4 follow by applying Theorem 3.5(ii) to $X \to S$ and Z (where one takes $S = \operatorname{Spec}(k)$ in 2.1). Indeed, the main part of the proofs of 2.1 and 2.4 was to construct ℓ'-alterations $X' \to X$ and $S' \to S$ with regular sources, snc divisors $Z' \subset X'$ and $W' \subset S'$, and a log smooth morphism $f' : (X',Z') \to (S',W')$ compatible with f. Then, in the last paragraphs of both proofs, Proposition 1.2 was used to obtain a more detailed description of X', Z', and f'. In particular, for a zero-dimensional base this amounted to saying that X' is S-smooth and Z' is relatively snc over S, and for a one-dimensional base this amounted to the conditions (i) and (ii) of 2.4.

Conversely, Theorem 2.1 (resp. 2.4) implies assertion (ii) of Theorem 3.5 under the assumptions of 2.1 (resp. 2.4) on X and S. Moreover, the non-separated Chow's lemma could be used in their proofs as well, so the separatedness assumption there could be easily removed. In such case, Theorems 2.1 and 2.4 would simply become the low dimensional (with respect to S) cases of Theorem 3.5(ii) plus an explicit local description of the log smooth morphism f'. The strengthening 3.5(iii), however, was not achieved in 2.1, and required a different proof of the whole theorem.

3.6. Saturation. In Theorems 3.4 and 3.5 we resolve certain morphisms $f : X \to S$ with divisors $Z \subset X$ by log smooth morphisms $f' : (X',Z') \to (S',W')$. However, as we insisted to use only ℓ'-alterations and to obtain regular X' and snc Z', we had to compromise a little on the "quality" of f'. For example, our f' may have non-reduced fibers. Due to de Jong's theorem, if the relative dimension is one, then one can make f' a nodal curve. We will see that a similar improvement of f' is possible in general if one uses arbitrary alterations and allows non-regular X'. The procedure reduces to saturating f' and is essentially due to Tsuji and Illusie-Kato-Nakayama ([**Illusie et al., 2005**, A.4.4 and A.4.3]).

3.6.1. *Saturated morphisms.* Recall that a homomorphism $P \to Q$ of fs (resp. fine) monoids is **saturated** (resp. **integral**) if for any homomorphism $P \to P'$ with fs (resp. fine) target the pushout $Q \oplus_P P'$ is fs (resp. fine). A morphism of fs (resp. fine) log schemes $f: (Y, M_Y) \to (X, M_X)$ is **saturated** (resp. **integral**) if so are the homomorphisms $\overline{M}_{X,f(y)} \to \overline{M}_{Y,y}$.

Remark 3.6.2. (i) Integral morphism were introduced already by Kato in [**Kato, 1988**, §4]. Kato also introduced the notion of saturated morphisms, which was first seriously studied by Tsuji in [**Tsuji, 1997**]. Actually, one can define saturated morphisms for arbitrary fine log schemes, but the definition is more involved than we use. For fs log schemes our definition coincides with the usual one due to [**Tsuji, 1997**, II 2.13(2)].

(ii) The following two basic properties of saturated morphisms follow from the definition: (a) a composition of saturated morphisms between fs log schemes is saturated, (b) if $f: Y \to X$ is a saturated morphism between fs log schemes, then, for any morphism of fs log schemes $X' \to X$, the base change $f': Y' \to X'$ of f in the category of log schemes is a saturated morphism of fs log schemes. (Also, it is proved in [**Tsuji, 1997**, II 2.11] that analogous properties hold for saturated morphisms between arbitrary integral log schemes.)

(iii) Let $f: Y \to X$ be a morphism of fs log schemes. It is shown in [**Tsuji, 1997**, II 3.5] that if f can be modeled on charts corresponding to saturated homomorphisms of fs monoids $P_i \to Q_i$ then f is saturated. Let us remark that the converse is also true: if f is saturated then it can be modeled on charts corresponding to $P_i \to Q_i$ as above.

3.6.3. *Integrality and saturatedness for log smooth morphisms.* We recall the following result that relates the notions of integral and saturated morphisms to certain properties of the underlying morphisms of schemes.

Proposition 3.6.4. *Let $f: (Y, M_Y) \to (X, M_X)$ be a log smooth morphism between fs log schemes and assume that f is integral. Then,*

(i) the morphism $Y \to X$ is flat,

(ii) f saturated if and only if $Y \to X$ has reduced fibers.

Proof. The first claim is proved in [**Kato, 1988**, 4.5] and the second one is proved in [**Tsuji, 1997**, II 4.2]. □

One can also go in the opposite direction: from flatness to integrality.

Proposition 3.6.5. *Let $f: (Y, M_Y) \to (X, M_X)$ be a log smooth morphism between fs log schemes and assume that the morphism $Y \to X$ is flat and (X, M_X) is log smooth over a field k with the trivial log structure. Then f is integral.*

Proof. It suffices to show that if $\overline{y} \to Y$ is a geometric point and $\overline{x} = f(\overline{y})$ then the homomorphism $\overline{\phi}: \overline{M}_{X,\overline{x}} \to \overline{M}_{Y,\overline{y}}$ is integral. By Proposition 1.2 and the argument in 1.4(vi), localizing X and Y along these points we can assume that X possesses a chart $a: X \to X_0 = \mathrm{Spec}(k[Q])$ with smooth a and f is modeled on a chart $Y_0 = \mathrm{Spec}(k[P]) \to X_0$ corresponding to a homomorphism $\phi: Q \to P$ so that the morphism $g: Y \to Z = X \times_{X_0} Y_0$ is smooth (in particular, flat), and ϕ has the following properties: $Q = \overline{M}_{X,\overline{x}}$, P is fs, P^* is torsion free, the composition $Q \to P \to \overline{P} = P/P^*$ coincides with $\overline{\phi}$, the kernel of ϕ^{gp} is finite, killed by an integer invertible at x, as well as the torsion part of its cokernel (but we will not need these last two properties). Since Q is sharp and saturated, Q^{gp} is torsion free, so ϕ is injective. We claim that $\overline{\phi}$ is integral if and only if ϕ is integral. To see this note that if $Q \to R$ is a homomorphism of monoids, then $R \oplus_Q \overline{P}$ is isomorphic to the quotient of $R \oplus_Q P$ by the image of P^*, and hence either both pushouts are integral or neither of them is integral. Thus, we only need to prove that ϕ is integral.

Note that the morphism $h: Z \to X$ is flat at the (Zariski) image $z \in Z$ of $\overline{y}$ because f and g are flat. Note that a takes $x = h(z)$ to the origin of X_0 and $Y_0 \to X_0$ is flat at the image $y_0 \in Y_0$ of z by flat descent with respect to a. In other words, if $I \subset k[P]$ is the ideal corresponding to y_0 then the homomorphism $k[Q] \to k[P]_I$ is flat. The preimage of m_y under $k[P] \to \mathcal{O}_{Y,y}$ contains the set $m_P = P \setminus P^*$, hence $m_P \subset I$ and we obtain that I contains $J = k[m_P]$. Note that the ideal J is prime as $k[P]/J \xrightarrow{\sim} k[P^*]$ is a domain due to P^* being torsion free. Thus, the localization $k[P]_J$ makes sense, and we obtain a flat homomorphism $\psi\colon k[Q] \to k[P]_J$.

It is proved in [**Kato, 1988**, 4.1], that if the homomorphisms $K[\phi]\colon K[Q] \to K[P]$ are flat for any field K then ϕ is integral. The proof consists of two parts. First one checks that ϕ is injective, which is automatic in our case. This is the only argument in loc.cit. where a play with different fields is needed. We claim that the second part of the proof of the implication (iii) $\Longrightarrow$ (v) in [**Kato, 1988**, 4.1] works fine with a single field k, and, moreover, it suffices to only use that $k[Q] \to k[P]_J$ is flat. Let us indicate how the argument in loc.cit. should be adjusted.

Assume that, as in the proof of [**Kato, 1988**, 4.1], we are given $a_1, a_2 \in Q$ and $b_1, b_2 \in P$ such that $\phi(a_1)b_1 = \phi(a_2)b_2$. Let S be the kernel of the homomorphism of $k[Q]$-modules $k[Q] \oplus k[Q] \to k[Q]$ given by $(x, y) \mapsto a_1x - a_2y$. By the flatness, the kernel of $k[P]_J \oplus k[P]_J \to k[P]_J$, $(x, y) \mapsto \phi(a_1)x - \phi(a_2)y$ is generated by the image of S. Hence there exist representations $b_1 = \sum_{i=1}^r \phi(c_i)\frac{f_i}{s}$ and $b_2 = \sum_{i=1}^r \phi(d_i)\frac{f_i}{s}$ with $c_i, d_i \in k[Q]$, $f_i \in k[P]$, $s \in k[P] \setminus J$, and $a_1c_i = a_2d_i$. Moreover, multiplying s and f_i's by an appropriate unit $u \in P^*$ we can assume that $s = 1 + s'$ for $s' \in \mathrm{Span}_k(P \setminus \{1\})$. Then $b_1 + \sum_{1\le\alpha\le m} \lambda_\alpha t_\alpha = \sum_{1\le i\le r} \phi(c_i)f_i$, with $\lambda_\alpha \in k^*$, and the $t_\alpha \in P$ pairwise distinct and distinct from b_1, so we see that there exist $a_3 \in Q$, $b \in P$, and $1 \le i \le r$, such that a_3 appears in c_i, b appears in f_i, and $b_1 = \phi(a_3)b$. The remaining argument copies that of the loc.cit. verbatim, and one obtains in the end that ϕ satisfies the condition (v) from [**Kato, 1988**, 4.1]. Thus, ϕ is integral and we are done. $\square$

Before going further, let us discuss an incarnation of saturated morphisms in (more classical) toroidal geometry.

Remark 3.6.6. In toroidal geometry an analog of saturated morphisms was introduced by Abramovich and Karu in [**Abramovich & Karu, 2000**]. In the language of log schemes toroidal morphisms can be interpreted as log smooth morphisms $f\colon (X, Z) \to (S, W)$ between log regular schemes (with the toroidal structure given by the triviality loci of the log structures). If f is a toroidal morphism as above then Abramovich-Karu called it weakly semistable when the following conditions hold: S is regular, f is locally equidimensional, and the fibers of f are reduced. Furthermore, they remarked that the equidimensionality condition is equivalent to flatness of f whenever S is regular, see [**Abramovich & Karu, 2000**, 4.6]. Thus, the weak semistability condition is nothing else but saturatedness of f and regularity of the target. In particular, saturated log smooth morphisms between log regular log schemes may be viewed as the generalization of weakly semistable morphisms to the case of an arbitrary log regular (or toroidal) base.

Now, we are going to prove our main result about saturation.

Theorem 3.7. *Assume that $f\colon (X, Z) \to (S, W)$ is a log smooth morphism such that (S, W) is log regular and S is universally* **Q***-resolvable (§3.3.3). Then there exists an alteration $h\colon S' \to S$ such that S' is regular, $W' = g^{-1}(W)$ is an snc divisor, and the fs base change $f'\colon (X', Z') \to (S', W')$ is a saturated morphism.*

Recall that $(X', Z') = (X, Z) \times^{\mathrm{fs}}_{(S,W)} (S', W')$ and f' is log smooth because the saturation morphism is log smooth.

Proof. By **VIII**-3.4.9, applying to (S, W) an appropriate saturated log blow up tower and replacing (X, Z) with the fs base change we can achieve that S is regular and W is normal crossings. By an additional sequence of log blow ups we can also make W snc (see **VIII**-4.1.6), so (S, W) becomes a Zariski log scheme. Now, we can étale-locally cover f by charts $f_i\colon (X_i, Z_i) \to (S_i, W_i)$ modeled on $P_i \to Q_i$ such that S_i are open subschemes in S. By [**Illusie et al., 2005**, A.4.4, A.4.3], for each i there exists a morphism $h_i\colon (S'_i, W'_i) \to (S_i, W_i)$ such that h_i is a composition of a Kummer morphism and a log blow up, and the fs base change of f_i is saturated. (Although the proof in loc.cit. is written in the context of log analytic spaces, it translates to our situation almost verbatim. The only changes are that we have to distinguish étale and Zariski topology on the base (in order to construct log blow ups), and h_i does not have to be log étale as there might be inseparable Kummer morphisms.)

Note that $W_i' = h_i^{-1}(W_i)$. In addition, $S_i' \to S_i$ is a projective alteration by **VIII**-3.4.6. Extend each h_i to a projective alteration $g_i : T_i \to S$, and let $h : S' \to S$ be a projective alteration that factors through each T_i. By the universal **Q**-resolvability assumption we can enlarge S' so that it becomes regular and $Z' = h^{-1}(Z)$ becomes snc. We claim that h is as claimed. It suffices to check that the fs base change of each morphism $f_i' : (X_i, Z_i) \to (S, W)$ is saturated. However, already the fs base change of f_i' to $(T_i, g_i^{-1}(W))$ is saturated by the construction, hence so is its further base change to (S', W'). □

REMARK 3.7.1. Our proof is an easy consequence of [**Illusie et al., 2005**, A.4.4 and A.4.3]. The first cited result shows that (locally) any log smooth morphism can be made exact by an appropriate log blow up of the base. This result is somewhat analogous to the flattening theorem of Raynaud-Gruson. The second cited result shows that by a Kummer extension of the base one can (locally) saturate an exact log smooth morphism. It is somewhat analogous to the reduced fiber theorem of Bosch-Lütkebohmert-Raynaud ([**Bosch et al., 1995**]) which implies that if $f : Y \to X$ is a finite type morphism between reduced noetherian schemes then there exists an alteration $X' \to X$ such that the normalized base change $f' : (Y \times_X X')^{\mathrm{nor}} \to X'$ has reduced fibers. Although the proof of the latter is far more difficult.

3.8. Characteristic zero case. Theorem 3.5 can be substantially strengthened when S is of characteristic zero, i.e., the morphism $S \to \mathrm{Spec}(\mathbf{Z})$ factors through $\mathrm{Spec}(\mathbf{Q})$.

THEOREM 3.9. *Assume that S is a reduced, noetherian, qe scheme of characteristic zero, $f : X \to S$ is a maximally dominating morphism of finite type with reduced source, and $Z \subset X$ is a nowhere dense closed subset. Then there exist projective modifications $a : S' \to S$ and $b : X' \to X$ with regular sources, a pseudo-projective morphism $f' : X' \to S'$ compatible with f, and snc divisors $W' \subset S'$ and $Z' \subset X'$ such that $Z' = b^{-1}(Z) \cup f'^{-1}(W')$ and the morphism $(X', Z') \to (S', W')$ is log smooth.*

Proof. The proof is very close to the proof of Theorem 3.5, so we will just say which changes in that proof should be made. First, we note that any S-scheme Y of finite type is noetherian and qe. Thus, if Y is reduced and $T \subset Y$ is a nowhere dense closed subset then the pair (Y, T) can be desingularized by [**Temkin, 2008**] in the following sense: there exists a projective modification $h : Y' \to Y$ with regular source and such that $h^{-1}(T)$ is an snc divisor. This result replaces the ℓ'-resolvability assumption in Theorem 3.5, and it allows to apply the proof of that theorem to our situation with the only changes that one always uses projective modifications instead of projective ℓ'-alterations, and Theorem 3.4 is replaced with Lemma 3.9.1 below. (Note that Lemma 3.9.1 is weaker than Theorem 3.9, while Theorem 3.4 does not follow from Theorem 3.5.) □

LEMMA 3.9.1. *Let S be an integral, noetherian, qe scheme with generic point $\eta = \mathrm{Spec}(K)$, let $f : X \to S$ be a maximally dominating morphism of finite type, and let $Z \subset X$ be a nowhere dense closed subset. Assume that $X_\eta = X \times_S \eta$ is a smooth curve over K, and $Z_\eta = Z \times_S \eta$ is étale over K. Then there exist projective modifications $a : S' \to S$ and $b : X' \to X$ with regular sources, a pseudo-projective morphism $f' : X' \to S'$ compatible with f and snc divisors $W' \subset S'$ and $Z' \subset X'$ such that $Z' = b^{-1}(Z) \cup f'^{-1}(W')$ and the morphism $(X', Z') \to (S', W')$ is log smooth.*

Proof. The proof copies the proof of Theorem 3.4 with the only difference that instead of an ℓ-Sylow subgroup $G \subset \overline{G}$ one simply takes $G = \overline{G}$. The latter is possible because the schemes are of characteristic zero and hence any action of $\overline{G}$ is tame. □

Combining Theorem 3.9 and 3.7 we obtain the following weak semistable reduction theorem.

THEOREM 3.10. *Assume that S is a reduced, noetherian, qe scheme of characteristic zero, $f : X \to S$ is a maximally dominating morphism of finite type with reduced source, and $Z \subset X$ is a nowhere dense closed subset. Then there exists an alteration $S' \to S$, a modification $X' \to (X \times_S S')^{\mathrm{pr}}$ of the proper transform of X, a pseudo-projective morphism $f' : X' \to S'$ compatible with f, and divisors $W' \subset S'$ and $Z' \subset X'$ such that S' is regular, W' is snc, $Z' = b^{-1}(Z) \cup f'^{-1}(W')$, and the morphism $(X', Z') \to (S', W')$ is log smooth and saturated (i.e. $X' \to S'$ is weakly semistable).*

Remark 3.10.1. (i) In the case when X and S are integral proper varieties over an algebraically closed field k of characteristic zero, this theorem becomes the weak semistable reduction theorem of Abramovich-Karu. Our proof has many common lines with their arguments. In particular, the first step of their proof was to make f toroidal, and it was based on de Jong's theorem. (Note also that in a recent work [**Abramovich et al., 2013**] of Abramovich-Denef-Karu, the toroidalization theorem was extended to separated schemes of finite type over an arbitrary ground field of characteristic zero.) Our Theorem 3.9 can be viewed as a generalization of the toroidalization theorem of Abramovich-Karu.

(ii) The second stage in the proof of the weak semistable reduction theorem of Abramovich-Karu (the combinatorial stage) is analogous to Theorem 3.7. It obtains as an input a toroidal morphism $f: (X, Z) \to (S, W)$ between proper varieties of characteristic zero and outputs an alteration $h: S' \to S$ such that S' is regular, $W' = h^{-1}(W)$ is snc, and the saturated base change of f is weakly semistable. The proof is similar to the arguments used in the proofs of [**Illusie et al., 2005**, A.4.4 and A.4.3]. First, one constructs a toroidal blow up of the base that makes the fibers equidimensional (i.e. makes the log morphism integral), and then an appropriate normalized finite base change is used to make the fibers reduced.

EXPOSITION XI

Oriented Products

Luc Illusie

We fix a universe $\mathscr{U}$. Unless otherwise specified, the sites (resp. topoi) considered will be $\mathscr{U}$-sites (resp. topoi), and "small" will mean $\mathscr{U}$-small.

1. Construction of Oriented Products

The following construction is due to Deligne [**Laumon, 1983**]:

1.1. Let $f : X \to S$ and $g : Y \to S$ be morphisms of topoi. We assume that X, Y, S have defining sites C_1, C_2, D, admitting finite projective limits, and that $f^\star, g^\star$ extend continuous functors between sites, and commute with finite projective limits. Let C be the following site:

(i) C is the category of pairs of morphisms $U \to V \leftarrow W$ over $X \to S \leftarrow Y$, where $U \to V$ (resp. $V \leftarrow W$) denotes a morphism $U \to f^\star V$ (resp. $g^\star V \leftarrow W$) in C_1 (resp. C_2) and V is an object of D.

(ii) C is equipped with the topology generated (cf. [**SGA 4** II 1.1.6]) by covering families $(U_i \to V_i \leftarrow W_i) \to (U \to V \leftarrow W)$ $(i \in I)$ of the following type:

(a) $V_i = V$, $W_i = W$ for all i, and the family $(U_i \to U)$ is covering;

(b) $U_i = U$, $V_i = V$ for all i, and the family $(W_i \to W)$ is covering;

(c) $(U' \to V' \leftarrow W') \to (U \to V \leftarrow W)$, where $U' = U$ and $W' \to W$ is obtained by base change from a morphism $V' \to V$ in D.

Note that finite projective limits are representable in C.

We denote $\widetilde{C}$ as the topos of sheaves on C.

Lemme 1.2. *Let F be a presheaf on C. For F to be a sheaf, it is necessary and sufficient that the following two conditions are satisfied:*

(i) for any covering family $(Z_i \to Z)$ in C of type (a) or (b), the sequence $F(Z) \to \prod_{i\in I} F(Z_i) \rightrightarrows \prod_{(i,j)\in I\times I} F(Z_i \times_Z Z_j)$ is exact;

(ii) for any covering family $(U' \to V' \leftarrow W') \to (U \to V \leftarrow W)$ of type (c), the map

$$F(U \to V \leftarrow W) \to F(U' \to V' \leftarrow W')$$

is bijective.

In particular, if we denote by $(-)^a$ the associated sheaf functor, for any covering family $(Z' \to Z)$ of type (c), the morphism of associated sheaves $Z'^a \to Z^a$ is an isomorphism.

Necessity is trivial for (i), and for (ii), it suffices to observe that the diagonal morphism

$$(U \to V' \leftarrow W') \to (U \to V' \times_V V' \leftarrow W' \times_W W')$$

is a covering morphism (of type (c)), which equalizes the double arrow

$$(U \to V' \times_V V' \leftarrow W' \times_W W') \rightrightarrows (U \to V' \leftarrow W').$$

For sufficiency, we note that covering families $(Z_i \to Z)$ of type (a), (b), or (c) are stable under base change $Z' \to Z$, and we apply [**SGA 4** II 2.3].

1.3. Let e_X (resp. e_Y, e_S) denote the final object of C_1 (resp. C_2, D). We have natural projections

$$p_1 : \widetilde{C} \to X, p_2 : \widetilde{C} \to Y$$

given by

$$p_1^\star(U) = (U \to e_S \leftarrow e_Y), p_2^\star(W) = (e_X \to e_S \leftarrow W).$$

Furthermore, we have a canonical morphism

$$\tau : gp_2 \to fp_1$$

given by the morphism of functors $\tau : (gp_2)_\star \to (fp_1)_\star$ defined as follows: for a sheaf F on C, and an object V of S,

$$\tau : ((gp_2)_\star F)(V) \to ((fp_1)_\star F)(V)$$

is the composite

$$F(e_X \to e_S \leftarrow g^\star V) \to F(f^\star V \to V \leftarrow g^\star V) \to F(f^\star V \to e_S \leftarrow e_Y),$$

where the first arrow is induced by the localization $(f^\star V \to V \leftarrow g^\star V) \to (e_X \to e_S \leftarrow g^\star V)$, and the second is the inverse of the isomorphism given by 1.2, relative to the type (c) morphism $(f^\star V \to V \leftarrow g^\star V \to (f^\star V \to e_S \leftarrow e_Y)$.

Théorème 1.4. *Let T be a topos equipped with morphisms $a : T \to X$, $b : T \to Y$ and a morphism $t : gb \to fa$. Then there exists a triplet $(h : T \to \widetilde{C}, \alpha : p_1 h \xrightarrow{\sim} a, \beta : p_2 h \xrightarrow{\sim} b)$, unique up to unique isomorphism, such that the composite*

$$gb \xrightarrow{\beta^{-1}} gp_2h \xrightarrow{\tau} fp_1h \xrightarrow{\alpha} fa$$

is equal to t.

For the proof, we will need the following lemma:

Lemme 1.5. *Let $Z = (U \to V \leftarrow W)$ be an object of C. With the notation of 1.2, the following square is Cartesian:*

(1.5.1)
$$\begin{tikzcd} Z^a \arrow[r] \arrow[d] & (p_2^\star W)^a \arrow[d, "v"] \\ (p_1^\star U)^a \arrow[r, "u"] & ((gp_2^\star)V)^a \end{tikzcd} .$$

In this square, v and the arrows originating from Z^a are the obvious arrows, and u is the composite arrow

$$(p_1^\star U)^a \to ((fp_1)^\star V)^a \xrightarrow{\tau} ((gp_2)^\star V)^a ,$$

where τ is the composite

$$(f^\star V \to e_S \leftarrow e_Y)^a \xrightarrow{r^{-1}} (f^\star V \to V \leftarrow g^\star V)^a \to (e_X \to e_S \leftarrow g^\star V)^a ,$$

r denoting the isomorphism $(f^\star V \to V \leftarrow g^\star V)^a \xrightarrow{\sim} (f^\star V \to e_S \leftarrow e_Y)^a$ from 1.2.

Let $z : Z \to Z' = (f^\star V \to V \leftarrow g^\star V)$ be the canonical projection. The composite $Z \to p_1^\star U \to (fp_1)^\star V$ factors through z. Consequently, and by definition of τ, the diagram

$$\begin{tikzcd} Z^a \arrow[r, "z"] \arrow[d] & Z'^a \arrow[d, "r"] \arrow[dr] & \\ (p_1^\star U)^a \arrow[r] & ((fp_1^\star)V)^a \arrow[r, "\tau"] & ((gp_2)^\star V)^a \end{tikzcd}$$

is commutative. Since the composite $Z \to p_2^\star W \to (gp_2)^\star V$ also factors through z, the square 1.5.1 is therefore commutative. This is the outer boundary of the following diagram, where the arrows other than τ are the obvious ones:

$$\begin{tikzcd}
(U \to V \leftarrow W)^a \arrow[r] \arrow[d] & (f^\star V \to V \leftarrow W)^a \arrow[r] \arrow[d] & (e_X \to e_S \leftarrow W)^a \arrow[d] \\
(U \to V \leftarrow g^\star V)^a \arrow[r] \arrow[d] & (f^\star V \to V \leftarrow g^\star V)^a \arrow[r] \arrow[d, "r"] & (e_X \to e_S \leftarrow g^\star V)^a \arrow[d, "\mathrm{Id}"] \\
(U \to e_S \leftarrow e_Y)^a \arrow[r] & (f^\star V \to e_S \leftarrow e_Y)^a \arrow[r, "\tau"] & (e_X \to e_S \leftarrow g^\star V)^a
\end{tikzcd}$$

Each of the squares composing it is Cartesian. The same is therefore true for 1.5.1.

1.6. Let us prove 1.4. We can assume that a and b are given by morphisms of sites ([**SGA 4** IV 4.9.4]). By 1.5, uniqueness is clear: for $Z = (U \to V \leftarrow W)$ in C, we must have

(1.6.1) $$h^\star Z = a^\star U \times_{(gb)^\star V} b^\star W,$$

where $a^\star U \to (gb)^\star V$ is the composite $a^\star U \to (fa)^\star V \xrightarrow{t} (gb)^\star V$. The isomorphisms α and β are then tautological; we will neglect them in the rest of the proof. Let us verify that the functor $h^\star$ given by 1.6.1 defines a morphism of topoi h satisfying the property stated in 1.4. Since $h^\star$ commutes with finite projective limits, to verify that $h^\star$ induces a morphism of topoi, it suffices to verify that $h^\star$ is continuous ([**SGA 4** IV 4.9.1, 4.9.2]). It is trivial that $h^\star$ transforms covering families of type (a) or (b) into covering families. Furthermore, if $(U' \to V' \leftarrow W') \to (U \to V \leftarrow W)$ is a covering family of type (c), the square

$$\begin{tikzcd}
b^\star W' \arrow[r] \arrow[d] & b^\star W \arrow[d] \\
(gb)^\star V' \arrow[r] & (gb)^\star V
\end{tikzcd}$$

is Cartesian, and consequently

$$a^\star U \times_{(gb)^\star V'} b^\star W' \to a^\star U \times_{(gb)^\star V} b^\star W$$

is an isomorphism. It remains to verify that τ induces t. But by definition, the morphism of sheaves defined by $h^\star(\tau)$ applied to V is the composite

$$((fa)^\star V)^a \xrightarrow{r^{-1}} ((fa)^\star V \times_{(gb)^\star V} (gb)^\star V)^a \to ((gb)^\star V)^a ,$$

and is therefore equal to the one defined by t applied to $(fa)^\star V$, which completes the proof.

Definition 1.7. The topos $\widetilde{C}$ constructed in 1.1 is called the **left oriented product of** X **and** Y **over** S, and is denoted $X \overleftarrow{\times}_S Y$. The morphisms in the diagram

$$\begin{tikzcd}
& X \overleftarrow{\times}_S Y \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\
X \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\
& S &
\end{tikzcd}$$

are related by the 2-arrow $\tau : gp_2 \to fp_1$. It follows from 1.4 that the quadruplet $(X \overleftarrow{\times}_S Y, p_1, p_2, \tau)$ is independent (up to unique isomorphism) of the choice of defining sites C_1, C_2, D.

For an object $Z = (U \to V \leftarrow W)$ in C, we will sometimes denote $U\overleftarrow{\times}_V W$ as the object Z^a in $X\overleftarrow{\times}_S Y$.

We similarly define the **right oriented product** $X \overrightarrow{\times}_S Y$, with its canonical projections $p_1 : X \overrightarrow{\times}_S Y \to X$, $p_2 : X \overrightarrow{\times}_S Y \to Y$ and the 2-arrow $\tau' : fp_1 \to gp_2$, which possesses the universal property of 1.4, with X and Y exchanged.

1.8. Let pt denote a punctual topos (category of sheaves of sets on a space reduced to a single point). Let $x : \mathrm{pt} \to X$, $y : \mathrm{pt} \to Y$ be points of X and Y respectively, and $u : gy \to fx$ be a 2-arrow. By 1.4, the triplet (x, y, u) defines a point $z : \mathrm{pt} \to X\overleftarrow{\times}_S Y$ such that $p_1 z \simeq x$, $p_2 z \simeq y$. This point will be denoted (x, y, u) (or sometimes (x, y) if no confusion is to be feared). Every point of $X\overleftarrow{\times}_S Y$ is of this form.

1.9. Consider a diagram of 1-morphisms of topoi

(1.9.1)
$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "u"] & S' \arrow[d, "h"] & Y' \arrow[l, "g'"'] \arrow[d, "v"] \\ X \arrow[r, "f"] & S & Y \arrow[l, "g"'] \end{tikzcd}$$

and 2-arrows $a : hf' \to fu$, $b : gv \to hg'$. Let $T = X\overleftarrow{\times}_S Y$, $T' = X'\overleftarrow{\times}_{S'} Y'$, $p_1 : T \to X$, $p_2 : T \to Y$, $p_1' : T' \to X'$, $p_2' : T' \to Y'$ be the canonical projections, $\tau : gp_2 \to fp_1$, $\tau' : g'p_2' \to f'p_1'$ be the canonical 2-arrows. Consider the composite 2-arrow

$$c :\ gvp_2' \xrightarrow{b} hg'p_2' \xrightarrow{\tau'} hf'p_1' \xrightarrow{a} fup_1' .$$

According to 1.4, c defines a diagram of 1-morphisms

(1.9.2)
$$\begin{tikzcd} X' \arrow[d, "u"] & T' \arrow[l, "p_1'"'] \arrow[r, "p_2'"] \arrow[d, "t"] & Y' \arrow[d, "v"] \\ X & T \arrow[l, "p_1"'] \arrow[r, "p_2"] & Y \end{tikzcd}$$

and 2-isomorphisms $\alpha : p_1 t \xrightarrow{\sim} up_1'$, $\beta : p_2 t \xrightarrow{\sim} vp_2'$ making the square commutative

(1.9.3)
$$\begin{tikzcd} gp_2t \arrow[r, "\tau"] \arrow[d, "\beta"] & fp_1t \arrow[d, "\alpha"] \\ gvp_2' \arrow[r, "a\tau'b"] & fup_1' \end{tikzcd} .$$

The triplet (t, α, β) (or simply $t : T' \to T$) from 1.9.2 is said to be derived from 1.9.1 by *functoriality*. We will denote

$$t = u\overleftarrow{\times}_h v.$$

There is an obvious compatibility for a composite of two data 1.9.1.

1.10. Here are some examples.

(a) In the situation of 1.4, the triplet (a, b, t) defines a diagram of type 1.9.1

$$\begin{tikzcd} T \arrow[r, "\mathrm{Id}"] \arrow[d, "a"] & T \arrow[d, "fa"] & T \arrow[l, "\mathrm{Id}"'] \arrow[d, "b"] \\ X \arrow[r, "f"] & S & Y \arrow[l, "g"'] \end{tikzcd} ,$$

with $t : gb \to fa$, yielding a morphism

$$a\overleftarrow{\times}_{fa} b : T\overleftarrow{\times}_T T \to X\overleftarrow{\times}_S T.$$

Furthermore, according to 1.4, the identity arrows of T define a canonical morphism, called **diagonal**

$$\Delta : T \to T\overleftarrow{\times}_T T.$$

The 1-arrow h of 1.4 is the composite

$$h = (a\overleftarrow{\times}_{fa} b)\Delta : T \to X\overleftarrow{\times}_S Y.$$

In particular, taking T to be a punctual topos, so that Δ is an isomorphism, we have, with the notations of 1.8

$$(x, y, u) = x\overleftarrow{\times}_{fx} y : \mathrm{pt} \to X\overleftarrow{\times}_S Y.$$

(b) In the situation of 1.7, let X', S', Y' be objects of X, S, Y respectively, and $f' : X' \to S'$, (resp. $g' : Y' \to S'$) an arrow over f (resp. g), *i.e.* an arrow $f' : X' \to f^\star(S')$, (resp. $g' : Y' \to g^\star(S')$). We denote $X'\overleftarrow{\times}_{S'} Y'$ as the object $(X' \to S' \leftarrow Y')^a = p_1^\star(X') \times_{(gp_2)^\star(S')} p_2^\star Y'$ of $X\overleftarrow{\times}_S Y$, cf. 1.5. From this, we deduce a 2-commutative diagram of natural 1-arrows

(1.10.1)
$$\begin{tikzcd}
& (X\overleftarrow{\times}_S Y)_{/(X'\overleftarrow{\times}_{S'} Y')} \arrow[dl, "p'_1"'] \arrow[dr, "p'_2"] & & \\
X_{/X'} \arrow[r, "f_{/S'}"] \arrow[d] & S_{/S'} \arrow[d] & Y_{/Y'} \arrow[l, "g_{/S'}"'] \arrow[d] \\
X \arrow[r] & S & Y \arrow[l]
\end{tikzcd},$$

where the notation $(-)_{/-}$ denotes a localized topos, and the 2-arrow $\tau' : g_{/S'} p'_2 \to f_{/S'} p'_1$ is deduced by localization from the 2-arrow $\tau : gp_2 \to fp_1$. According to 1.4, τ' therefore defines a 1-morphism

(1.10.2)
$$m : (X\overleftarrow{\times}_S Y)_{/(X'\overleftarrow{\times}_{S'} Y')} \to X_{/X'}\overleftarrow{\times}_{S_{/S'}} Y_{/Y'}.$$

It is shown (cf. ([**Abbes & Gros, 2011b**, 3.15]) that m is an equivalence, by which, in what follows, we will identify the two sides. On the other hand, the 2-commutative squares of 1.10.1 define, according to 1.9, a functoriality arrow

$$X_{/X'}\overleftarrow{\times}_{S_{/S'}} Y_{/Y'} \to X\overleftarrow{\times}_S Y,$$

This, or its composite with m,

(1.10.3)
$$(X\overleftarrow{\times}_S Y)_{/(X'\overleftarrow{\times}_{S'} Y')} \to X\overleftarrow{\times}_S Y$$

is called the **localization arrow**.

Proposition 1.11. *Suppose that X' is the final object of X and that g' is Cartesian over g,* i.e. $g' : Y' \xrightarrow{\sim} g^\star(S')$. *Then the arrow 1.10.3 is an equivalence.*

Indeed, with the notations of 1.10.1, it follows from 1.2 that the arrow $e_X\overleftarrow{\times}_{S'} Y' \to e_X\overleftarrow{\times}_{e_S} e_Y$ in $X\overleftarrow{\times}_S Y$ is an isomorphism.

1.12. Consider in particular the case where $S = Y$ is a scheme equipped with the étale topology, $g = \mathrm{Id}$ and $f : X \to S = Y$ is the inclusion of a closed subscheme of Y. The topos $T = X\overleftarrow{\times}_Y Y$ plays the role of an *étale tubular neighborhood* of X in Y. The points of T are the triplets (x, y, t), where x (resp. y) is a geometric point of X (resp. Y) and $t : y \to x$ is a specialization arrow (cf. [**SGA 4** VIII 7.9]). In other words, (x, y, t) is the data of a geometric point x of X, a generization y_0 of the closed point (still abusively denoted x) of the strict localization $X_{(x)}$ of X at x and a geometric point of $Y_{(x)}$ localized at

y_0, or else a separably closed extension $y \to y_0$ of the generic point of $\overline{\{y_0\}}$. Furthermore, if $v : Y' \to Y$ is an étale neighborhood of X in Y, *i.e.* a commutative diagram

$$\begin{tikzcd} & Y' \arrow[d, "v"] \\ X \arrow[ur] \arrow[r] & Y \end{tikzcd},$$

where v is étale, then, according to 1.11, the canonical morphism

$$X\overleftarrow{\times}_{Y'}Y' \to X\overleftarrow{\times}_{Y}Y$$

is an equivalence. Thus, T depends only on the Henselization of Y along X (when it is defined, in particular, for affine Y, cf. [**Raynaud, 1970**]). We will see in the next section and in exposé $\mathrm{XII_A}$ other properties of T clarifying this analogy with a tubular neighborhood.

1.13. Here is a last example of functoriality of oriented products, which will only be used in 2.7. Let I be a small cofiltered category and let

$$(X_i \xrightarrow{f_i} S_i \xleftarrow{g_i} Y_i)_{i\in I}$$

be morphisms of topoi fibered over I. From this, we deduce a diagram of projective limit topoi ([**SGA 4** VI 8.1.3])

$$\begin{tikzcd} & \operatorname{Limproj}(X_i\overleftarrow{\times}_{S_i}Y_i) \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ \operatorname{Limproj} X_i \arrow[dr, "f"'] & & \operatorname{Limproj} Y_i \arrow[dl, "g"] \\ & \operatorname{Limproj} S_i & \end{tikzcd},$$

where $f = \operatorname{Limproj} f_i$, $g = \operatorname{Limproj} g_i$, with a 2-arrow $\tau : gp_2 \to fp_1$ deduced from the arrows $\tau_i : g_ip_1 \to f_ip_2$, and consequently, a morphism

$$\operatorname{Limproj}(X_i\overleftarrow{\times}_{S_i}Y_i) \to \operatorname{Limproj} X_i\overleftarrow{\times}_{\operatorname{Limproj} S_i}\operatorname{Limproj} Y_i. \tag{1.13.1}$$

It follows from the definitions that 1.13.1 is an equivalence.

2. Tubes and Base Change

2.1. Let (S, s) be a pointed topos, *i.e.* a pair consisting of a topos S and a point $s : \mathrm{pt} \to S$ of S. If (S, s) and (T, t) are pointed topoi, a (pointed) morphism from (S, s) to (T, t) is a pair (f, a) of a morphism $f : S \to T$ and a 2-arrow $a : fs \to t$. A 2-arrow $c : (f, a) \to (g, b)$ is a 2-arrow $c : f \to g$ such that $b(cs) = a$. If (S, s) is a pointed topos, we denote $F \mapsto F_s = s^\star F$ as the fiber functor at s.

Recall that a pointed topos (S, s) is said to be **local with center** s ([**SGA 4** VI 8.4.6]) if, for every object F of S, the natural arrow $\Gamma(S, F) \to F_s$ is bijective. A morphism $(f, a) : (S, s) \to (T, t)$ of pointed topoi is said to be **local** if the 2-arrow $a : fs \to t$ is an isomorphism.

2.2. The following construction is due to Gabber. Let (S, s) be a local topos with center s. Let $\varepsilon : S \to \mathrm{pt}$ be the projection. By definition, the canonical arrow $\varepsilon_\star \to s^\star$ is an isomorphism. From this, we deduce an isomorphism

$$\varepsilon^\star\varepsilon_\star \xrightarrow{\sim} (s\varepsilon)^\star. \tag{2.2.1}$$

The adjunction arrow $\varepsilon^\star\varepsilon_\star \to \mathrm{Id}$ therefore identifies, via 2.2.1, with a morphism $(s\varepsilon)^\star \to \mathrm{Id}$, *i.e.* with a 2-arrow

$$c_s : \mathrm{Id} \to s\varepsilon \tag{2.2.2}$$

between the 1-morphisms $\mathrm{Id} : S \to S$ and $s\varepsilon : S \to S$. If F is an object of S, $(s\varepsilon)^\star F$ is the constant sheaf on S with value $F_s = \varepsilon_\star F = \Gamma(S, F)$. If U is a connected object of S, the morphism $(s\varepsilon)^\star F \to F$ induces on $\Gamma(U, -)$ the restriction morphism $\Gamma(S, F) \to \Gamma(U, F)$. The composite $c_s s : s \to s$ is the identity: $(s\varepsilon)^\star F \to F$ induces the identity on the fibers at s.

Let $f : X \to S$, $g : Y \to S$ be morphisms of topoi, $x : \mathrm{pt} \to X$ a point of X, $s = fx : \mathrm{pt} \to S$ its image in S. The diagram

(2.2.3)
$$\begin{tikzcd} \mathrm{pt} \arrow[d, "x"] & \mathrm{pt} \arrow[l, "\mathrm{Id}"'] \arrow[d, "s"] & Y \arrow[l, "\varepsilon g"'] \arrow[d, "\mathrm{Id}"] \\ X \arrow[r, "f"] & S & Y \arrow[l, "g"'] \end{tikzcd},$$

(where the left square is 2-commutative) and the 2-arrow

(2.2.4)
$$c_s g : g \to s\varepsilon g$$

are data of type 1.9.1. For an object F of S, $(s\varepsilon g)^\star F$ is the constant sheaf on Y with value $F_s = \Gamma(S, F)$, and the arrow $(s\varepsilon g)^\star F \to g^\star F$ is the composite $\Gamma(S, F)_Y \to \Gamma(Y, g^\star F)_Y \to g^\star F$. Note that, by 1.4, the oriented product $\mathrm{pt}\overleftarrow{\times}_{\mathrm{pt}} Y$ canonically identifies with Y, with $p_1 = \mathrm{Id} : Y \to Y$. From 2.2.3 and 2.2.4 we therefore deduce a diagram of type 1.9.2:

(2.2.5)
$$\begin{tikzcd} \mathrm{pt} \arrow[d, "x"] & Y \arrow[l, "\varepsilon g"'] \arrow[r, "\mathrm{Id}"] \arrow[d, "\sigma"] & Y \arrow[d, "\mathrm{Id}"] \\ X & X\overleftarrow{\times}_S Y \arrow[l, "p_1"'] \arrow[r, "p_2"] & Y \end{tikzcd},$$

in other words, a section $\sigma : Y \to X\overleftarrow{\times}_S Y$ of p_2 such that $p_1\sigma = x\varepsilon g$. This section can be seen as being defined, via 1.4, by the pair of morphisms $x\varepsilon g : Y \to X$, $\mathrm{Id} : Y \to Y$ and the 2-arrow $c_s g : g \to fx\varepsilon g = s\varepsilon g$. We say that σ is the **canonical section** defined by the point x. By composition with $p_{2\star}$, the adjunction arrow $\mathrm{Id} \to \sigma_\star\sigma^\star$ gives a canonical arrow

(2.2.6)
$$\gamma : p_{2\star} \to \sigma^\star.$$

The following result is due to Gabber:

Proposition 2.3. *Let $f : (X, x) \to (S, s)$ be a local morphism of local topoi ($fx = s$), and $g : Y \to S$ be a morphism of topoi. Let $y : \mathrm{pt} \to Y$ be a point of Y. For every object F of $X\overleftarrow{\times}_S Y$, γ (2.2.6) induces an isomorphism*

$$\gamma_y : (p_{2\star}F)_y \xrightarrow{\sim} (\sigma^\star F)_y.$$

Let $t = \sigma y : \mathrm{pt} \to T$ be the point of $T = X\overleftarrow{\times}_S Y$ image of y by σ. This point is defined (cf. 1.4) by the triplet (x, y, u), where $u : gy \to fx = fx\varepsilon gy = s\varepsilon gy = s$ is deduced from 2.2.4. We have $(\sigma^\star F)_y = F_t$. Let us choose defining sites C_1, C_2, D as in 1.1. By definition,

$$F_t = \mathrm{colim}_z\, F(U \to V \leftarrow W),$$

where $z : \mathrm{pt} \to (U \to V \leftarrow W)$ ranges over the neighborhoods of t in T, with $(U \to V \leftarrow W)$ in C. Since X and S are local, the neighborhoods of t of the form $\sigma w : \mathrm{pt} \to (e_X \to e_S \leftarrow W)$, where $w : \mathrm{pt} \to W$ is a neighborhood of y in Y, form a cofinal system. Thus

$$F_t = \mathrm{colim}_w\, F(e_X \to e_S \leftarrow W),$$

where $z = \sigma w : \mathrm{pt} \to (e_X \to e_S \leftarrow W)$ ranges over the preceding neighborhoods, with $U = e_X$, $V = e_S$. Furthermore,

$$(p_{2\star}F)_y = \mathrm{colim}_w\, F(e_X \to e_S \leftarrow W),$$

where $w : \mathrm{pt} \to W$ ranges over the neighborhoods of y in Y. The arrow γ_y is the natural restriction. It is therefore an isomorphism.

Corollaire 2.3.1. *Under the hypotheses of 2.3, if Y has enough points, in particular, if Y is locally coherent* ([**SGA 4** vi 9.0]), *γ (2.2.6) is an isomorphism.*

It is plausible that the hypothesis of having enough points is superfluous. However, it will be satisfied in the applications we have in mind.

Corollaire 2.3.2. *Under the hypotheses of 2.3, suppose Y is local with center y. Then $X\overleftarrow{\times}_S Y$ is local with center $\sigma(y)$.*

Indeed, we then have $(p_{2\star}F)_y = \Gamma(Y, p_{2\star}F) = \Gamma(X\overleftarrow{\times}_S Y, F)$, and γ_y identifies with the restriction $\Gamma(X\overleftarrow{\times}_S Y, F) \to F_{\sigma(y)}$.

Note that, if $f : X \to S$ is a local morphism of strictly local schemes, and $g : Y \to S$ is a morphism of strictly local schemes, the schematic fibered product $X \times_S Y$ is not generally strictly local, nor even local, even if g is local.

The following result is also due to Gabber:

Théorème 2.4. *Let $f : X \to S$, $g : Y \to S$ be morphisms of topoi, $T = X\overleftarrow{\times}_S Y$, $p_1 : T \to X$, $p_2 : T \to Y$ the canonical projections, $\tau : gp_2 \to fp_1$ the canonical 2-arrow. We assume X, Y, S are coherent and f, g are coherent* ([**SGA 4** vi 2.3, 2.4.5, 3.1]). *Let Λ be a ring. Then, for any $F \in D^+(Y, \Lambda)$, the base change arrow, deduced from τ,*

$$f^\star \mathrm{R}g_\star F \to \mathrm{R}p_{1\star}p_2^\star F \tag{2.4.1}$$

is an isomorphism (in $D^+(X, \Lambda)$).

We will need the following lemma, which generalizes [**Orgogozo, 2006**, 9.1]:

Lemme 2.5. *Under the hypotheses of 2.4, T is coherent, and the projections p_1, p_2 are coherent morphisms.*

The topos X (resp. Y, resp. S) admits a (small) generating family C_1 (resp. C_2, resp. D) consisting of coherent objects, stable under finite projective limits ([**SGA 4** vi 2.4.5]). Since f and g are coherent, $f^\star V$ (resp. $g^\star V$) is coherent if V is in D ([**SGA 4** vi 3.2]). We can therefore assume that $f^\star D \subset C_1$, $g^\star D \subset C_2$. The corresponding full subcategory of X (resp. Y, resp. S), equipped with the induced topology, is a defining site for X (resp. Y, resp. S). Let C be the category defined as in 1.1, equipped with the topology defined by the pretopology $\mathscr{P}$ generated by *finite* families of type (a) and (b) and families of type (c). It is stable under finite projective limits, and is a defining site for T, since any covering family in C of type (a) or (b) is refined by a finite covering family. It is therefore sufficient to show that every object of C is quasi-compact ([**SGA 4** vi 2.4.5]). Let us describe $\mathscr{P}$. For each object $Z = (U \to V \leftarrow W)$ of C, let $\mathrm{Cov}(Z)$ denote the set of families $(Z_i \to Z)_{i\in I}$ obtained by composition of a finite number of families of type (c) and (finite) families of type (a) and (b). In particular, the set I is finite. By definition, the data of $\mathrm{Cov}(Z)$ satisfies axioms PT0, PT2 and PT3 of [**SGA 4** ii 1.3]. Axiom PT1 (stability under base change) is also satisfied, since families of type (c), as well as finite families of type (a) (resp. (b)), are stable under base change, and base change commutes with the composition of families. The data of $\mathrm{Cov}(Z)$ is therefore a pretopology, and by definition, it is the pretopology $\mathscr{P}$. Since the families belonging to $\mathrm{Cov}(Z)$ are finite, every object of C is automatically quasi-compact, as announced. The coherence of the projections p_1 and p_2 follows.

Remarques 2.6. (a) Gabber knows how to show that the conclusion of 2.5 holds under the sole hypotheses that X, Y, S and g are coherent. We will not need this generalization. (b) The reader will find in $\mathbf{XII_A}$-2.3.3 another application of 2.3 to base change theorems, in the context of schemes and the étale topology.

2.7. Let us prove 2.4. Since X is coherent, and thus has enough points, it suffices to verify that, for every point $x : \mathrm{pt} \to X$ of X, the fiber at x of 2.4.1

$$(f^\star \mathrm{R}g_\star F)_x \to (\mathrm{R}p_{1\star}p_2^\star F)_x \tag{2.7.1}$$

is an isomorphism. Let $s : \mathrm{pt} \to S$ be the image of x by f. Let $X_{(x)}$ (resp. $S_{(s)}$) be the localization of X (resp. S) at x (resp. s). Recall ([**SGA 4** VI 8.4.2]) that $X_{(x)}$ (resp. $Y_{(y)}$) (denoted $\mathrm{Loc}_x(X)$ (resp. $\mathrm{Loc}_s(S)$) in *loc. cit.*) is the projective limit

$$X_{(x)} = \mathrm{Limtop}_{U \in \mathrm{Vois}(x)} X_{/U}$$

(resp.

$$S_{(s)} = \mathrm{Limtop}_{V \in \mathrm{Vois}(s)} S_{/V}),$$

where U (resp. V) ranges over the cofiltered category $\mathrm{Vois}(x)$ (resp. $\mathrm{Vois}(s)$) of neighborhoods of x (resp. s) in X (resp. S). Since X (resp. S) is coherent, we can moreover restrict to U (resp. V) that are coherent, the transition morphisms then being automatically coherent. This is what we will do in the following, still denoting $\mathrm{Vois}(x)$ (resp. $\mathrm{Vois}(s)$) the full subcategory consisting of coherent U (resp. V). The topos $X_{(x)}$ (resp. $S_{(s)}$) is a local topos, over X (resp. S), whose center's image is x (resp. s) ([**SGA 4** VI 8.4.6]). The morphism f induces a local morphism $f_{(x)} : X_{(x)} \to S_{(s)}$. Similarly define

$$T_{(x)} = \mathrm{Limtop}_{U \in \mathrm{Vois}(x)} T_{/p_1^\star U},$$

$$Y_{(s)} = \mathrm{Limtop}_{V \in \mathrm{Vois}(s)} Y_{/g^\star V},$$

such that we obtain a square

(2.7.2)
$$\begin{tikzcd} T_{(x)} \arrow[r, "p_2"] \arrow[d, "p_1"'] & Y_{(s)} \arrow[d, "g"] \\ X_{(x)} \arrow[r, "f"'] & S_{(s)} \end{tikzcd},$$

with a 2-arrow $\tau : gp_2 \to fp_1$. By the compatibility of the formation of oriented products with localization (1.10.2) and projective limits (1.13.1), the arrow

(2.7.3)
$$T_{(x)} \to X_{(x)} \overleftarrow{\times}_{S_{(s)}} Y_{(s)},$$

deduced from this square by 1.4 is an equivalence. In fact, as the reviewer observes, for any $V \in \mathrm{Vois}(s)$, the arrow $X_{(x)} \overleftarrow{\times}_{S_{/V}} Y_{/g^\star V} \to X_{(x)} \overleftarrow{\times}_S Y$ is an equivalence, so that localization on S is superfluous. According to 2.5, the topoi $X_{/U}$, $S_{/V}$, $T_{/p_1^\star U}$, $Y_{/g^\star V}$ are coherent, the transition arrows of the projective systems $X_{/U}$, $S_{/V}$ are coherent, and the morphisms $g : Y_{/g^\star V} \to S_{/V}$, $p_1 : T_{/p_1^\star U} \to X_{/U}$ are coherent. We are therefore in the conditions for applying [**SGA 4** VI 8.7.3], which, considering that $S_{(s)}$ and $X_{(x)}$ are local, implies that the canonical arrows

(2.7.4)
$$(\mathrm{R}g_\star F)_s \to \mathrm{R}\Gamma(S_{(s)}, \mathrm{R}g_\star F) \to \mathrm{R}\Gamma(Y_{(s)}, F),$$

(2.7.5)
$$(\mathrm{R}p_{1\star} p_2^\star F)_x \to \mathrm{R}\Gamma(X_{(x)}, \mathrm{R}p_{1\star} p_2^\star F) \to \mathrm{R}\Gamma(T_{(x)}, p_2^\star F)$$

are isomorphisms. With the identifications 2.7.3, 2.7.4 and 2.7.5, the arrow 2.7.1 identifies with the fiber at x of the base change arrow (deduced from τ) of the square 2.7.2. This arrow is written as

(2.7.6)
$$\mathrm{R}\Gamma(Y_{(s)}, F) \to \mathrm{R}\Gamma(T_{(x)}, p_2^\star F).$$

We have:

$(*)$: *The arrow 2.7.6 canonically identifies with the functoriality arrow defined by p_2.*

To verify this, we can assume S and X are local, with respective centers s and x, and f is local. By definition, 2.7.6 is the composite arrow

$$\begin{tikzcd} \mathrm{R}\Gamma(S, \mathrm{R}g_\star F) \arrow[r, "2.7.6"] \arrow[d, "\alpha"'] & \mathrm{R}\Gamma(T, p_2^\star F) \\ \mathrm{R}\Gamma(S, \mathrm{R}g_\star \mathrm{R}p_{2\star} p_2^\star F) \arrow[r, "\mathrm{R}\Gamma(S,\tau)"'] & \mathrm{R}\Gamma(S, \mathrm{R}f_\star \mathrm{R}p_{1\star} p_2^\star F) \arrow[u, "\beta"'] \end{tikzcd},$$

where the arrow α is defined by the adjunction arrow $adj : F \to \mathrm{R}p_{2\star}p_2^\star F$ (and is thus the functoriality arrow $\mathrm{R}\Gamma(Y,F) \to \mathrm{R}\Gamma(T, p_2^\star F)$ defined by p_2), and β is the canonical transitivity isomorphism relative to $fp_1 : T \to S$. Now, by definition of τ (1.3), for any sheaf G on T, the arrow

$$\Gamma(S,\tau) : \Gamma(S,(gp_2)_\star G) \to \Gamma(S,(fp_1)_\star G)$$

is the identity. The same is therefore true for the lower horizontal arrow of the diagram above, which proves $(*)$. It remains to prove that 2.7.6 is an isomorphism. In fact, the arrow

$$adj : F \to \mathrm{R}p_{2\star}p_2^\star F \tag{2.7.7}$$

is an isomorphism. To see this, it suffices to observe that, given the description of γ in 2.3.2, the composite

$$F \xrightarrow{adj} \mathrm{R}p_{2\star}p_2^\star F \xrightarrow{\gamma} \sigma^\star p_2^\star F = F\ ,$$

where σ is the section of p_2 defined in 2.2.5 and γ is the isomorphism from 2.3.1, is the identity. This completes the proof of 2.4.

Remarques 2.8. (1) Suppose that the data in 2.4 come from morphisms of schemes, equipped with the étale topology, with X, S, Y coherent and f and g coherent. The points x, s are geometric points, and the localizations $X_{(x)}$, $S_{(s)}$ are strict localizations. If f is a closed immersion, the adjunction arrow 2.7.7 is an isomorphism (one can indeed assume S is strictly local, and then X is likewise). Suppose further that $Y = S$, $g = \mathrm{Id}_S$ as in 1.12. We saw in *loc. cit.* that $T = X\overleftarrow{\times}_S S$ plays the role of a tubular neighborhood of X in S. Let $j : S^\star = S - X \to S$ be the open complement of X, and $T^\star = X\overleftarrow{\times}_S S^\star = T_{/(e_X \to e_S \leftarrow S^\star)}$ be the induced topos. Then $T^\star$ plays the role of a *punctured tubular neighborhood* of X in S: for $F \in D^+(S^\star, \Lambda)$, we have, by 2.4,

$$f^\star \mathrm{R}j_\star F \xrightarrow{\sim} \mathrm{R}p_{1\star}p_2^\star F.$$

(2) Without the coherence hypothesis on g, the conclusion of 2.4 may fail, as shown by the following example, due to Gabber. Let X be a connected, non-empty topological space, $i : Y \to X$ the inclusion of a non-empty closed subset distinct from X, $j : U = X - Y \to X$ the inclusion of the open complement. Then $i^\star j_\star \mathbf{Z}$ is non-zero. But, if every point of U has a neighborhood whose closure in X does not meet Y, then the oriented product $Y\overleftarrow{\times}_X U$ is empty. This is the case, for example, if X is the segment $[0,1]$ and Y is the point $\{0\}$.

(3) Under the hypotheses of 2.4, it is shown in an analogous manner that:

(a) For any sheaf of sets F on Y, the base change arrow

$$f^\star g_\star F \to p_{1\star}p_2^\star F$$

is an isomorphism.

Higher non-abelian variants can be expected:

(b) For any sheaf of groups F on Y, the base change arrow

$$f^\star \mathrm{R}^1 g_\star F \to \mathrm{R}^1 p_{1\star}p_2^\star F,$$

is an isomorphism (of pointed sheaves of sets).

(c) More generally, for any stack F on Y, the base change arrow

$$f^\star g_\star F \to p_{1\star}p_2^\star F$$

is an equivalence.

The verification of (b) and (c) seems to require, in addition to techniques for reducing stacks to gerbes from [**Giraud, 1971**, III 2.1.5], results on passing to the limit for non-abelian cohomology analogous to those in [**SGA 4** vi 8.7], for which we know no reference.

(4) Gabber knows how to demonstrate the following generalization of 2.4. Let $f : X \to S$, $g : Y \to S$ be morphisms of topoi, $T = X\overleftarrow{\times}_S Y$. We assume that Y and S are locally coherent, and that, for every coherent algebraic object V of S, $g^\star V$ is coherent algebraic ([**SGA 4** VI 2.1, 2.3]). Then, for any $F \in D^+(Y, \Lambda)$, the base change arrow 2.4.1 is an isomorphism, and analogous results should hold in the non-abelian case, as in (3) (a), (b), (c) above. Gabber deduces these results from a general base change theorem for certain fibered topoi.

3. Fibered Products

The complements given in this and the following section will not be used in the rest of the volume.

3.1. Fibered products of topoi were constructed by Giraud [**Giraud, 1972**, 3.4]. The following construction is due to Gabber. Let $f : X \to S$, $g : Y \to S$ be morphisms of topoi as in 1.1. Let D be the following site:

(i) The category underlying D is the category C considered in 1.1 (i).

(ii) D is equipped with the topology defined by the pretopology generated by covering families $(U_i \to V_i \leftarrow W_i) \to (U \to V \leftarrow W)$ $(i \in I)$ of type (a), (b), (c) of 1.1 (ii) and of the form
(d) $(U' \to V' \leftarrow W') \to (U \to V \leftarrow W)$, where $W' = W$ and $U' \to U$ is deduced by base change from a morphism $V' \to V$ in the defining site of S.

In other words, the topology on D is the supremum of the topologies on C defining the oriented products $X\overleftarrow{\times}_S Y$ and $X \overrightarrow{\times}_S Y$.

According to 1.2, for a presheaf F on D to be a sheaf, it is necessary and sufficient that F satisfies the usual exactness conditions relative to covering families of type (a) and (b), and that, for any covering family $Z' \to Z$ of type (c) or (d), $F(Z) \to F(Z')$ is an isomorphism.

Let $\widetilde{D}$ be the topos of sheaves on D. We have natural projections

$$p_1 : \widetilde{D} \to X, p_2 : \widetilde{D} \to Y$$

given by the same formulas as in 1.3, and the construction of τ in *loc. cit.* gives an *isomorphism*

$$\varepsilon : gp_2 \xrightarrow{\sim} fp_1. \tag{3.1.1}$$

Théorème 3.2. *Let T be a topos equipped with morphisms $a : T \to X$, $b : T \to Y$ and an isomorphism $t : gb \xrightarrow{\sim} fa$. Then there exists a triplet $(h : T \to \widetilde{D}, \alpha : p_1h \xrightarrow{\sim} a, \beta : p_2h \xrightarrow{\sim} b)$, unique up to unique isomorphism, such that the composite*

$$gb \xrightarrow{\beta^{-1}} gp_2h \xrightarrow{\varepsilon} fp_1h \xrightarrow{\alpha} fa$$

is equal to t.

The proof is analogous to that of 1.4. The functor $h^\star$ is still given by formula 1.6.1. Since t is an isomorphism, we have

$$a^\star U \times_{(gb)^\star V} b^\star W = a^\star U \times_{(fa)^\star V} b^\star W,$$

where $b^\star V \to (fa)^\star V$ is the composite $b^\star V \longrightarrow (gb)^\star V \xrightarrow{t^{-1}} (fa)^\star V$. It follows that $h^\star$ transforms a covering family of type (d) into a covering family, and we conclude as in 1.6.

Definition 3.3. The topos $\widetilde{D}$ is called the **fibered product of** X **and** Y **over** S, and is denoted $X \times_S Y$. The morphisms in the diagram

$$\begin{tikzcd} & X \times_S Y \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & S & \end{tikzcd}$$

are related by the 2-isomorphism $\varepsilon : gp_2 \to fp_1$.

EXEMPLES 3.4. (1) *Topological Spaces*. Let $f : X \to S$, $g : Y \to S$ be continuous maps between topological spaces. We have a canonical morphism

$$\widetilde{X \times_S Y} \to \widetilde{X} \times_{\widetilde{S}} \widetilde{Y}, \tag{3.4.1}$$

where $\widetilde{Z}$ denotes the topos of sheaves on a topological space Z. If X, Y, S are sober, (3.4.1) induces a bijection on isomorphism classes of points ([**SGA 4** IV 4.2.3]). If we further assume that $\widetilde{X} \times_{\widetilde{S}} \widetilde{Y}$ has enough points (a condition satisfied for example if X, Y, S, f, g are coherent, cf. 2.5), then, since the family of sub-objects of the final object of $\widetilde{X} \times_{\widetilde{S}} \widetilde{Y}$ is generating, it follows from [**SGA 4** IV 7.1.9] that (3.4.1) is an equivalence of topoi. It is unknown if this second hypothesis is necessary.

(2) *Schemes*. Let $f : X \to S$, $g : Y \to S$ be morphisms of schemes. Let the index zar (resp. ét) denote the associated Zariski (resp. étale) topos. Due to (1), the natural morphism $(X \times_S Y)_{\mathrm{zar}} \to X_{\mathrm{zar}} \times_{S_{\mathrm{zar}}} Y_{\mathrm{zar}}$ is not an equivalence in general. Similarly, the natural morphism $(X \times_S Y)_{\text{ét}} \to X_{\text{ét}} \times_{S_{\text{ét}}} Y_{\text{ét}}$ is not an equivalence in general, even if X, Y, S are spectra of fields: if $S = \operatorname{Spec} k$, $S_{\text{ét}}$ is equivalent to the classifying topos BG of the profinite group $G = \mathrm{Gal}(\overline{k}/k)$, where $\overline{k}$ is a separable closure of k, and the formation of BG commutes with fibered products.

4. Evanescent and Co-evanescent Topoi

4.1. Let $f : X \to S$, $g : Y \to S$ be morphisms of topoi as in 1.1. The oriented product

$$X \overleftarrow{\times}_S S, \tag{4.1.1}$$

where $S \to S$ is the identity morphism, is called the **evanescent topos** of f. It is studied in [**Laumon, 1983**] and [**Orgogozo, 2006**]. The oriented product

$$S \overleftarrow{\times}_S Y, \tag{4.1.2}$$

where $S \to S$ is the identity morphism, plays an important role in Faltings' works on p-adic comparison theorems and the p-adic Simpson correspondence (**Faltings topos**) (cf. [**Faltings, 2002**], [**Faltings, 2005**], [**Abbes & Gros, 2011a**], [**Deligne, 1995**], [**Abbes & Gros, 2011b**]). We propose to call it here the **co-evanescent topos** of g. From the evanescent topos of Id_S,

$$\overleftarrow{S} = S \overleftarrow{\times}_S S, \tag{4.1.3}$$

which is also the co-evanescent topos of Id_S, the oriented products $X \overleftarrow{\times}_S Y$ are deduced by base change. Consider indeed the iterated fibered product

$$Z = X \times_S \overleftarrow{S} \times_S Y, \tag{4.1.4}$$

where the left (resp. right) arrow from $\overleftarrow{S}$ to S is p_1 (resp. p_2). We have natural projections $q_1 : Z \to X$, $q_2 : Z \to Y$ and $m : Z \to \overleftarrow{S}$, with isomorphisms $p_1 m \xrightarrow{\sim} f q_1$, $g q_2 \xrightarrow{\sim} p_2 m$. By composition with the structural arrow $\tau : p_2 \to p_1$ of $\overleftarrow{S}$, we deduce a 2-arrow $z : g q_2 \to f p_1$. According to 1.4, the triplet (q_1, q_2, z) therefore defines an arrow

$$h : Z \to X \overleftarrow{\times}_S Y \tag{4.1.5}$$

and isomorphisms $p_1 h \xrightarrow{\sim} q_1$, $p_2 h \xrightarrow{\sim} q_2$, by which z identifies with τh, where p_1, p_2 denote the canonical projections of $X \overleftarrow{\times}_S Y$ onto X and Y.

PROPOSITION 4.2. *The morphism h (4.1.5) is an isomorphism. In particular, it defines canonical isomorphisms*

$$X \times_S \overleftarrow{S} \xrightarrow{\sim} X \overleftarrow{\times}_S S, \tag{4.2.1}$$

$$\overleftarrow{S} \times_S Y \xrightarrow{\sim} S \overleftarrow{\times}_S Y. \tag{4.2.2}$$

It suffices to show that Z, equipped with (q_1, q_2, z), satisfies the universal property of the oriented product. Let T be a topos equipped with morphisms $a : T \to X$, $b : T \to Y$, and a 2-arrow $t : gb \to fa$. By the universal property of $\overleftarrow{S}$, we first deduce a unique triplet, consisting of a morphism $k : T \to \overleftarrow{S}$ and isomorphisms $p_1 k \to fa$, $p_2 k \to gb$ such that $t = \tau k$ modulo these identifications. Then, by the universal property of fibered products, we deduce a unique quadruplet consisting of a morphism $s : T \to Z$ and isomorphisms $ms \xrightarrow{\sim} k$, $q_1 s \xrightarrow{\sim} a$, $q_2 s \xrightarrow{\sim} b$ such that $zs = t$ modulo these identifications.

4.3. Let $f : X \to S$ be a morphism of topoi. According to 1.4, the morphisms Id_X and f define a morphism

(4.3.1) $$\Psi : X \to X \overleftarrow{\times}_S S$$

such that

(4.3.2) $$p_1 \Psi = \mathrm{Id}_X, p_2 \Psi = f, \tau\Psi = \mathrm{Id}_f,$$

where $\tau : p_2 \to f p_1$ is the structural 2-arrow of $X \overleftarrow{\times}_S S$:

(4.3.3)
$$\begin{tikzcd}
 & X \arrow[dl, "\mathrm{Id}_X"'] \arrow[d, "\Psi"] \arrow[dr, "f"] & \\
X \arrow[dr, "f"'] & X \overleftarrow{\times}_S S \arrow[l, "p_1"'] \arrow[r, "p_2"] & S \arrow[dl, "\mathrm{Id}_S"] \\
 & S &
\end{tikzcd}$$

The functor $\Psi_\star$ is called the **nearby cycles functor**. For an object $(U \to V \leftarrow W)$ of the site C defining $X \overleftarrow{\times}_S S$, we have $\Psi^\star(U \to V \leftarrow W) = U \times_V W$, where $U \times_V W := U \times_{f^\star(V)} f^\star W$. If Λ is a ring and $F \in D^+(X, \Lambda)$, the complex $R\Psi_\star F \in D^+(X \overleftarrow{\times}_S S, \Lambda)$ (also denoted $R\Psi F$) is called the **nearby cycles complex** (of f relative to F).

The identity $p_{1\star}\Psi_\star = \mathrm{Id}$ defines, by adjunction, a canonical morphism

(4.3.4) $$p_1^\star \to \Psi_\star.$$

For $F \in D^+(X, \Lambda)$, the cone of the morphism $p_1^\star F \to R\Psi F$ that is deduced from it is called the **evanescent cycles complex** (of f relative to F) and is denoted $R\Phi F$. In the case of schemes (equipped with the étale topology), these functors, which generalize Grothendieck's $R\Psi$ and $R\Phi$ functors ([**SGA 7** XIII]), are studied in [**Laumon, 1983**] and [**Orgogozo, 2006**].

Consider the base change morphism

(4.3.5) $$p_{1\star} \to \Psi^\star$$

deduced from the identity $p_1\Psi = \mathrm{Id}_X$, in other words, the morphism deduced, by application of $p_{1\star}$, from the adjunction arrow $\mathrm{Id} \to \Psi_\star\Psi^\star$, taking into account that $p_{1\star}\Psi_\star = \mathrm{Id}$. The following result is given without proof in [**Laumon, 1983**]:

PROPOSITION 4.4. *The morphism 4.3.5 is an isomorphism.*

We will define a morphism

(4.4.1) $$\Psi^\star \to p_{1\star},$$

which we will show is the inverse of 4.3.5. To do this, we define a morphism

(4.4.2) $$\mathrm{Id} \to \Psi_\star p_{1\star}$$

as follows. For a sheaf F on $X \overleftarrow{\times}_S S$ and an object $Z = (U \to V \leftarrow W)$ of the site C in 1.1, the arrow $F(Z) \to (\Psi_\star p_{1\star} F)(Z)$ is the composite

(4.4.3) $$F(Z) \to F(U \times_{f^\star V} f^\star W \to W \leftarrow W) \to F(U \times_{f^\star V} f^\star W \to e_S \leftarrow e_S),$$

where $W \to W$ is the identity, the first arrow is the restriction and the second, the inverse of the isomorphism relative to the type (c) covering

$$(U \times_{f^\star V} f^\star W \to W \leftarrow W) \to (U \times_{f^\star V} f^\star W \to e_S \leftarrow e_S).$$

The morphism 4.4.1 is adjoint to 4.4.2. Let u (resp. v) denote the morphism 4.3.5 (resp. 4.4.1). We will show that u and v are inverses of each other. The following argument is due to Orgogozo. It is about showing that, for any sheaf F on $X\overleftarrow{\times}_S S$ and any sheaf G on X, the maps $\alpha(F,G) = \mathrm{Hom}(u(F),G) : \mathrm{Hom}(\Psi^\star F, G) \to \mathrm{Hom}(p_{1\star}F, G)$ and $\beta(F,G) = \mathrm{Hom}(v(F),G) : \mathrm{Hom}(p_{1\star}F,G) \to \mathrm{Hom}(\Psi^\star F, G)$ are inverses of each other.

The map

$$\alpha(F,G) : \mathrm{Hom}(\Psi^\star F, G) = \mathrm{Hom}(F, \Psi_\star G) \to \mathrm{Hom}(p_{1\star}F, G)$$

sends $a : F \to \Psi_\star G$ to $p_{1\star}a : p_{1\star}F \to p_{1\star}\Psi_\star G = G$. The map a is the data of a compatible family of maps $a_{(U\to V\leftarrow W)} : F(U \to V \leftarrow W) \to G(U \times_V W)$, for $(U \to V \leftarrow W)$ ranging over the objects of C, "compatible" meaning compatible with restriction arrows. The map $\alpha(a)$ is the family $a_{(U\to e_S \leftarrow e_S)} : F(U \to e_S \leftarrow e_S) \to G(U)$, for U ranging over the objects of X.

The map

$$\beta(F,G) : \mathrm{Hom}(p_{1\star}F, G) \to \mathrm{Hom}(\Psi_\star F, G) = \mathrm{Hom}(F, \Psi_\star G)$$

sends $b : p_{1\star}F \to G$ to the composite $F \to \Psi_\star p_{1\star}F \to \Psi_\star G$, where the first arrow is 4.4.2 and the second is $\Psi_\star b$. The map $\beta(b)$ is the family of $\beta(b)_{(U\to V\leftarrow W)} : F(U \to V \leftarrow W) \to G(U \times_V W)$, where $\beta(b)_{(U\to V\leftarrow W)}$ is the composite of 4.4.3 and of

$$(\Psi_\star b)(U \times_V W \to e_S \leftarrow e_S) : F(U \times_V W \to e_S \leftarrow e_S) = (p_{1\star}F)(U \times_V W) \to G(U \times_V W).$$

For each $a : F \to \Psi_\star G$, we have a commutative diagram

$$\begin{tikzcd}
F(U \to V \leftarrow W) \arrow[r, "a_{(U\to V\leftarrow W)}"] \arrow[d] & G(U \times_V W) \\
F(U \times_V W \to W \leftarrow W) \arrow[r] & F(U \times_V W \to e_S \leftarrow e_S) \arrow[u, "a_{(U\times_V W \to e_S \leftarrow e_S)}"']
\end{tikzcd},$$

where the lower horizontal arrow is the isomorphism appearing in 4.4.3. In this diagram, the composite of the arrows other than the upper horizontal arrow is $\beta\alpha(a)_{(U\to V\leftarrow W)}$, so $\beta\alpha = \mathrm{Id}$. The verification of $\alpha\beta = \mathrm{Id}$ is also trivial.

Remarque 4.5. When the topoi X and S are locally coherent and $f : X \to S$ is locally coherent, hence in particular in the case of schemes, one can prove 4.4 more simply by reducing to the local case. Since X has enough points, it suffices to show that, for every point $x : \mathrm{pt} \to X$, and every sheaf F on $X\overleftarrow{\times}_S S$, the fiber at x of 4.3.5, is an isomorphism. By replacing X with its localization at x (cf. 1.10.2 and 1.13.1), we can assume X is local with center x. Let $s : \mathrm{pt} \to S$ be the image of x by f. Let F be a sheaf on $T = X\overleftarrow{\times}_S S$. We must show that

$$(p_{1\star}F)_x = \Gamma(T,F) \to (\Psi_\star F)_x = F_{(x,s)}$$

is an isomorphism. Let $S_{(s)}$ be the localization of S at s and $T_{(s)} = X\overleftarrow{\times}_{S_{(s)}} S_{(s)}$. According to the limit passage results invoked in 2.7, we have

$$\Gamma(T_{(s)}, F) = \operatorname{colim} \Gamma(T_U, F),$$

where U ranges over the coherent neighborhoods of s and $T_U = X\overleftarrow{\times}_{S_{/U}} S_{/U}$. According to 1.11, the restriction maps $\Gamma(T,F) \to \Gamma(T_U, F)$ are isomorphisms (in fact, $T_U \to T$ is an equivalence). We can therefore assume S is local with center s. According to 2.3.2, T is then local, with center (x,s), and $\Gamma(T,F) = F_{(x,s)} = (\Psi^\star F)_x$.

4.6. Let $g : Y \to S$ be a morphism of topoi. The sheaves on the co-evanescent topos $T = S\overleftarrow{\times}_S Y$ (4.1.2) have a simple description, due to Deligne [**Deligne, 1995**]. For a sheaf F on T, the restriction arrow $F(U \to V \leftarrow W) \to F(U \to U \to U \times_V W)$ is an isomorphism. This suggests considering the site C_0 following. The category C_0 is that of arrows $(V \leftarrow W)$ over g, i.e. arrows $W \to g^\star V$ in Y. We equip C_0 with the topology defined by the pretopology generated by covering families of types (a) and (b) below:

(a) $(V \leftarrow W_i)_{i\in I} \to (V \leftarrow W)$, where the family $(W_i \to W)_{i\in I}$ is covering,

(b) $(V_i \leftarrow W_i)_{i\in I} \to (V \leftarrow W)$, where the family $(V_i \to V)_{i\in I}$ is covering, and $W_i = V_i \times_V W$.

It is shown ([**Abbes & Gros, 2011b**, 4.10]) that C_0 is a defining site for T. An object F of $S\overleftarrow{\times}_S Y$, described as a sheaf on C_0, is interpreted as the data of a family of sheaves $F_V : W \mapsto F(V \leftarrow W)$ on $g^\star V$ and restriction arrows $F_V \to j_{V'V\star} F_{V'}$ for $V' \to V$, defining $j_{V'V} : g^\star V' \to g^\star V$, satisfying the descent condition that, for a covering family $(V_i \to V)_{i\in I}$, the sequence

$$F_V \to \prod_i j_{V_i V\star} F_{V_i} \rightrightarrows \prod_{ii'} j_{V_{ii'} V\star} F_{V_{ii'}}$$

is exact, where $V_{ii'} = V_i \times_V V_{i'}$. According to 1.4, the morphisms $g : Y \to S$ and Id_Y define a morphism

$$\Psi : Y \to S\overleftarrow{\times}_S Y \tag{4.6.1}$$

such that

$$p_1\Psi = g,\ p_2\Psi = \mathrm{Id}_Y,\ \tau\Psi = \mathrm{Id}_g, \tag{4.6.2}$$

where $\tau : gp_2 \to p_1$ is the structural 1-arrow of $S\overleftarrow{\times}_S Y$:

$$\begin{tikzcd}
 & Y \arrow[dl, "g"'] \arrow[d, "\Psi"] \arrow[dr, "\mathrm{Id}_Y"] & \\
S \arrow[dr, "\mathrm{Id}_S"'] & S\overleftarrow{\times}_S Y \arrow[l, "p_1"'] \arrow[r, "p_2"] & Y \arrow[dl, "g"] \\
 & S &
\end{tikzcd} \quad . \tag{4.6.3}$$

The functor $\Psi_\star$, which could be called the **co-nearby cycles functor**, behaves very differently from the nearby cycles functor of 4.3.1. Indeed, from the identity $p_2\Psi = \mathrm{Id}_Y$ we deduce, by adjunction, a canonical morphism

$$p_2^\star \to \Psi_\star, \tag{4.6.4}$$

analogous to 4.3.4, and we have:

Proposition 4.7. *The morphism 4.6.4 is an isomorphism.*

In particular, *the functor $\Psi_\star$ is exact.* Here, it is the base change arrow, deduced from the identity $p_2\Psi = \mathrm{Id}_Y$,

$$p_{2\star} \to \Psi^\star, \tag{4.7.1}$$

analogous to 4.3.5, which is not, in general, an isomorphism. One can give a proof of 4.7 analogous to that of 4.4. It is simpler to deduce this result from the following explicit descriptions of the functors $p_1^\star$, $p_2^\star$, Ψ and the morphism τ. These descriptions are due to Deligne [**Deligne, 1995**] (see [**Abbes & Gros, 2011b**, 4.12, 4.14] for details).

4.8. (a) *Description of* $p_1^\star$. We have $p_1^\star V = (V \leftarrow g^\star V)$ (the object $(V \to e_S \leftarrow e_Y)$ in C corresponding to the object $(V \leftarrow g^\star V)$ in C_0). If F is a sheaf on S, $p_1^\star F$ is the sheaf associated with the presheaf whose value at $(V \leftarrow W)$ is the inductive limit of $F(V')$ following the category of arrows $(V \leftarrow W) \to p_1^\star V'$. This category having $(V \leftarrow W) \to (V \leftarrow g^\star V)$ as initial object, this limit is equal to $F(V)$. In other words, $p_1^\star F$ is the sheaf associated with the presheaf whose value at $(V \leftarrow W)$ is $F(V)$. In the description given above of a sheaf on $S\overleftarrow{\times}_S Y$ in terms of a family of sheaves on $g^\star V$, $p_1^\star F$ is the sheaf associated with the family of constant sheaves G_V on $g^\star V$ with value $F(V)$.

(b) *Description of* $p_2^\star$. We have $p_2^\star W = (e_S \leftarrow W)$. If F is a sheaf on Y, $p_2^\star F$ is the sheaf associated with the presheaf whose value at $(V \leftarrow W)$ is the inductive limit of $F(W')$ following the category of arrows $(V \leftarrow W) \to p_2^\star W'$. This category having $(V \leftarrow W) \to (e_S \leftarrow W)$ as initial object, this limit is equal to $F(W)$. In other words, $p_2^\star F$ is the sheaf associated with the family of sheaves H_V on $g^\star V$, where H_V is the sheaf $W \mapsto F(W)$ (in fact, this family is already a sheaf, the descent condition being automatically satisfied).

(c) *Description of* τ. If F is a sheaf on S, the morphism $\tau : p_1^\star F \to (gp_2)^\star F$ is deduced from the presheaf morphism which, for $(V \leftarrow W)$ in C_0, sends $F(V)$ to $(gp_2)^\star F(V \leftarrow W) = (g^\star F)(W)$ by the composite morphism $F(V) \to (g^\star F)(g^\star V) \to (g^\star F)(W)$. This can be seen using (a) and (b), by explicitly describing the morphism $(f p_1)^\star \to (gp_2)^\star$ adjoint to the morphism τ described in 1.3.

(d) *Description of* $\Psi_\star$. If F is a sheaf on Y, and $(U \to V \leftarrow W)$ an object of C, we have $(\Psi_\star F)(U \to V \leftarrow W) = F(U \times_V W$. In the description of $S\overleftarrow{\times}_S Y$ using site C_0, we therefore have

$$(\Psi_\star F)(V \leftarrow W) = F(W).$$

Taking into account (b), we thus have

$$\Psi_\star F = p_2^\star F$$

This identification is the one given by 4.6.4, which proves 4.7.

Note further that the points of $T = S\overleftarrow{\times}_S Y$ are the arrows $s \leftarrow y$, where s (resp. y) is a point of S (resp. Y), and that, for a sheaf F on Y, if $(s \leftarrow y)$ is a point of T, we have

$$(p_2^\star F)_{(s\leftarrow y)} = F_y = (\Psi_\star F)_{(s\leftarrow y)}.$$

Remarque 4.9. As Gabber observes, the isomorphism 4.6.4 implies the base change theorem 2.4 for $X = S$, $f = \mathrm{Id}_S$ without any coherence hypothesis on S, Y, and g (see [**Abbes & Gros, 2011b**, 4.15]).

EXPOSÉ $\mathrm{XII}_{\mathrm{A}}$

Oriented Cohomological Descent

Fabrice Orgogozo

1. Oriented Acyclicity of Proper Morphisms

The purpose of this section is to prove Theorem 1.1.2 below, which generalizes the cohomological invariance of the tubular neighborhood, defined using the oriented fibered product, by admissible blow-up (cf. **XI**-1.7 and **XI**-2.8).

1.1. Notations and Statement.

1.1.1. Consider a commutative diagram

$$
\begin{tikzcd}
X \arrow[d, "f"'] \arrow[dr, phantom, "\square"] & X' \arrow[l, "r'"'] \arrow[d, "f'"] & \\
S & S' \arrow[l, "r"] & Y \arrow[l, "g'"'] \arrow[ll, bend left, "g"]
\end{tikzcd}
\tag{1.1.1.1}
$$

of coherent schemes — with necessarily coherent morphisms ([**ÉGA** ɪ′ 6.1.10 (i),(iii)]) —, to which we associate a topos morphism

$$
\begin{tikzcd}
T' = X' \overleftarrow{\times}_{S'} Y \arrow[d, "\overleftarrow{r}"'] \\
T = X \overleftarrow{\times}_{S} Y
\end{tikzcd}
$$

by functoriality of the oriented product (**XI**-1.9). (Abuse of notation : X for $X_{\text{ét}}$, etc.)

Théorème 1.1.2. *If the morphism r is* proper, *the morphism $\overleftarrow{r}$ is acyclic for torsion sheaves : for every integer $n \geq 1$, the adjunction unit $\eta_{\overleftarrow{r}} : \mathrm{Id} \to \mathrm{R}\overleftarrow{r}_\star \overleftarrow{r}^\star$ is an isomorphism between endofunctors of $\mathrm{D}^+(T, \mathbf{Z}/n\mathbf{Z})$.*

1.2. Recollections on Base Change Maps.

We advise the reader to only read this paragraph if necessary. Recall that if

$$
\begin{tikzcd}
B \arrow[d, "a"'] & B' \arrow[l, "b'"'] \arrow[d, "a'"] \arrow[dl, bend right] \arrow[dl, bend left] \arrow[dl, phantom, "\Searrow \tau"] \\
A & A' \arrow[l, "b"]
\end{tikzcd}
$$

is a diagram of topoi equipped with a 2-arrow $\tau : ab' \Rightarrow ba'$, corresponding to transitivity arrows $\tau_\star : a_\star b'_\star \Rightarrow b_\star a'_\star$ and $\tau^\star : a'^\star b^\star \Rightarrow b'^\star a^\star$, the base change map $b^\star a_\star \to a'_\star b'^\star$ is the composite $\mathrm{cb}^\tau_{BAA'B'} = (a'_\star b'^\star \star \varepsilon_a) \odot (a'_\star \star \tau^\star \star a_\star) \odot (\eta_{a'} \star b^\star a_\star)$, where $\eta_?$ (resp. $\varepsilon_?$) denotes the unit (resp. the counit) of the adjunction $?^\star \dashv ?_\star$. The derived variant is defined similarly. Note that in [**SGA 4** xɪɪ § 4] — where the diagram considered is essentially commutative —, the base change map is defined as the composite $(\varepsilon_b \star a'_\star b'^\star)\odot(b^\star \star \tau_\star \star b'^\star)\odot(b^\star a_\star \star \eta_{b'})$; according to P. Deligne, "Artin's perplexity" does not need to exist : these two definitions coincide ([**SGA 4** XVII § 2.1]).

In Joseph Ayoub's language ([**Ayoub, 2007**, 1.1.10]), this 2-base change map is *obtained from* $\tau^\star$ *by adjunction following* $(a'^\star, a'_\star)$ *and* $(a^\star, a_\star)$. He shows (*op. cit.*, 1.1.9) that, given a diagram

$$\begin{tikzcd} A \arrow[r, "b^\star"] \arrow[d, "a^\star"'] \arrow[dr, phantom, "\Swarrow"] \arrow[dr, bend left] \arrow[dr, bend right] & A' \arrow[d, "a'^\star"] \\ B \arrow[r, "b'^\star"'] & A' \end{tikzcd}$$

in any 2-category, the base change map defined by the above formula is the unique 2-arrow of the diagram obtained by reversing the vertical arrows

$$\begin{tikzcd} A \arrow[r, "b^\star"] \arrow[dr, phantom, "\Searrow"] & A' \\ B \arrow[u, "a_\star"] \arrow[r, "b'^\star"'] \arrow[ur, bend left] \arrow[ur, bend right] & A' \arrow[u, "a'_\star"'] \end{tikzcd}$$

satisfying one of the two equalities (written in simplified form) $\tau^\star \odot \eta_a = \mathrm{cb} \odot \eta_{a'}$ and $\varepsilon_{a'} \odot \tau^\star = \varepsilon_a \odot \mathrm{cb}$. He also shows (*op. cit.*, 1.1.11 and 1.1.12), that the expected compatibility with horizontal and vertical compositions holds. (A particular case is stated without proof in [**SGA 4** XII 4.4] ; see also [**Lipman & Hashimoto, 2009**, 3.7.2]).

1.3. The remainder of this paragraph is devoted to the proof of Theorem 1.1.2.

1.3.1. *Reduction to the local case.* Let $\mathcal{K} \in \mathrm{Ob}\, \mathrm{D}^+(T, \mathbf{Z}/n\mathbf{Z})$ whose pullback by $\overleftarrow{r}$ on T' is denoted $\mathcal{K}'$. Recall ([**SGA 4** VI 2.9.2, 3.10]) that the étale topos of any scheme (resp. coherent) is locally coherent (resp. coherent) and that a coherent scheme morphism induces a coherent morphism between the étale topoi. It follows (**XI**-2.5) that the topos T is coherent and consequently it suffices to verify that the fiber at each point $t \to T$ of the unit $\mathcal{K} \to \mathrm{R}\overleftarrow{r}_\star \mathcal{K}'$ is an isomorphism. (*Locally* coherent would suffice, cf. [**SGA 4** VI 9.0]). According to **XI**-1.8, any such point t comes from a pair of (geometric) points x, y of X and Y, equipped with a specialization morphism $g(y) \rightsquigarrow f(x) = s$. On the other hand, it is verified as in **XI**-2.7 that if $a : (X_2 \to S_2 \leftarrow Y_2) \to (X_1 \to S_1 \leftarrow Y_1)$ is a morphism of diagrams of coherent schemes, the induced morphism $\overleftarrow{a} : X_2 \overleftarrow{\times}_{S_2} Y_2 \to X_1 \overleftarrow{\times}_{S_1} Y_1$ is a coherent morphism of coherent topoi and that the fiber at a point t_1 of a derived direct image by $\overleftarrow{a}$ is naturally isomorphic to the complex of derived global sections of the topos $X_{2(x_1)} \overleftarrow{\times}_{S_{2(s_1)}} Y_{2(y_1)}$, where x_1 and y_1 (resp. s_1) are the geometric points of X_1 and Y_1 (resp. S_1) deduced from t_1 (resp. is the image of x_1), and where we set $X_{2(x_1)} = X_2 \times_{X_1} X_{1(x_1)}$, etc.

This leads us to consider the strict localizations $X_{(x)}$, $S_{(s)}$ and $Y_{(y)}$ as well as the diagram

$$\begin{tikzcd} X_{(x)} \arrow[d, "f_{(x)}"'] \arrow[dr, phantom, "\square"] & X'_{(x)} \arrow[l, "r'_{(x)}"'] \arrow[d, "f'_{(x)}"] & \\ S_{(s)} \arrow[d] \arrow[dr, phantom, "\square"] & S'_{(s)} \arrow[l, "r_{(s)}"'] \arrow[d] & Y_{(y)} \arrow[l, "g'_{(y)}"'] \arrow[d] \arrow[ll, bend left] \\ S & S' \arrow[l] & Y \arrow[l] \end{tikzcd}$$

above 1.1.1.1. The morphism $f_{(x)}$ is *local* while $r_{(s)}$ is *proper*. As stated above, the fiber at t of the unit $\eta_{\overleftarrow{r}}$ considered identifies with the functoriality morphism of cohomology :

$$\mathcal{K}_t = \mathrm{R}\Gamma(T_{(t)}, \mathcal{K}) \to \mathrm{R}\Gamma(T'_{(t)}, \mathcal{K}')$$

where $T_{(t)}$ is the topos $X_{(x)}\overleftarrow{\times}_{S_{(s)}}Y_{(y)}$ — local (in the sense of **XI**-2.1) by *loc. cit.*, 2.3.2 — and $T'_{(t)}$ is the oriented product $X'_{(x)}\overleftarrow{\times}_{S'_{(s)}}Y'_{(y)}$. By replacing X (resp. S, Y) with $X_{(x)}$ (resp. $S_{(s)}$, $Y_{(y)}$), we can therefore assume the schemes X, Y, S to be strictly local, as well as the morphism f and the topos T to be local. We do this to simplify the notation in the proof below. We have also seen that it is sufficient, under these additional hypotheses, to verify that the complex morphism

$$\eta : \mathrm{R}\Gamma(T, \mathcal{K}) \to \mathrm{R}\Gamma(T', \mathcal{K}')$$

is an isomorphism.

1.3.2. *Case of a complex coming from the second factor.* In this paragraph, we show that the map $\eta : \mathrm{R}\Gamma(T, \mathcal{K}) \to \mathrm{R}\Gamma(T', \mathcal{K}')$ is an isomorphism when there exists a complex $\mathcal{K}_Y \in \mathrm{D}^+(Y, \mathbf{Z}/n\mathbf{Z})$ such that $\mathcal{K} = p_2^\star \mathcal{K}_Y$. This amounts to showing that the functoriality map $\mathrm{R}\Gamma(T, p_2^\star -) = \mathrm{R}\Gamma(X, \mathrm{R}p_{1\star}p_2{}^\star -) \to \mathrm{R}\Gamma(T', p_2'{}^\star -) = \mathrm{R}\Gamma(X', \mathrm{R}p'_{1\star}p_2'{}^\star -)$ is an isomorphism (for any object $\mathcal{K}_Y \in \mathrm{Ob}\,\mathrm{D}^+(Y, \mathbf{Z}/n\mathbf{Z})$). The different morphisms are represented in the following cubic diagram :

(1.3.2.1)
$$\begin{tikzcd}
 & Y \arrow[dl, equal] \arrow[dd, dotted, "g'"] & & T' \arrow[ll, "p_2'"'] \arrow[dl, "\overleftarrow{r}"'] \arrow[dd, "p_1'"] \\
Y \arrow[dd, "g"'] & & T \arrow[ll, "p_2"'] \arrow[dd, "p_1"] & \\
 & S' \arrow[dl, dotted, "r"'] \arrow[dr, phantom, "\square"] & & X' \arrow[ll, dotted, "f'"'] \arrow[dl, "r'"] \\
S & & X \arrow[ll, "f"] &
\end{tikzcd}$$

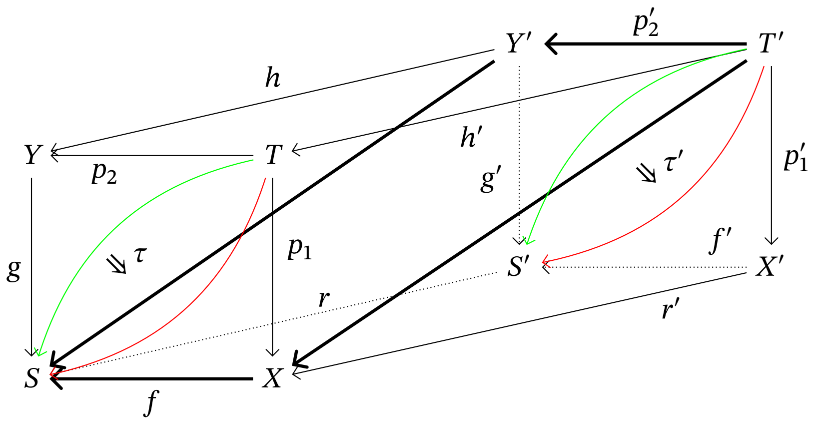
To show that $\mathrm{R}\Gamma(X, \mathrm{R}p_{1\star}p_2{}^\star -) \to \mathrm{R}\Gamma(X', \mathrm{R}p'_{1\star}p_2'{}^\star -)$ is an isomorphism, it suffices to establish the *a priori* stronger result following : the diagram

(1.3.2.2)
$$\begin{tikzcd}
p_{1\star}p_2{}^\star \arrow[r, "p_{1\star}\star\eta_{\overleftarrow{r}}\star p_2{}^\star"] & r'_\star p'_{1\star}p_2'{}^\star \\
f^\star g_\star \arrow[u, "\mathrm{cb}^{\tau}_{YSXT}"] \arrow[r, "\mathrm{cb}_{S'SXX'}\star g'_\star"] & r'_\star f'^\star g'_\star \arrow[u, "r'_\star\star\mathrm{cb}^{\tau'}_{YS'X'T'}"']
\end{tikzcd}$$

is a *commutative* diagram of *isomorphisms*, where $\tau : gp_2 \Rightarrow fp_1$ and $\tau' : g'p_2' \Rightarrow f'p_1'$ are as in **XI**-1.3. We already know that the vertical arrows are isomorphisms (**XI**-2.4) ; by properness of r, the same is true for the lower horizontal arrow.

1.3.3. *Commutativity.* For lack of a metatheorem (to be stated) guaranteeing the commutativity of diagram 1.3.2.2, we will proceed *ad hoc* to prove a slightly more general result. To lighten the notation, we will henceforth omit compositions with 1-arrows. Let

(1.3.3.1)



a cubic diagram of topoi, four faces of which are (essentially) commutative and two (those in the plane of the text) are equipped with 2-arrows τ and τ'. We assume that these arrows are compatible in the following sense : the two 2-arrows induced by τ and τ' in the diagonal square with solid lines $Y'SXT'$ coincide. This means that we have the equality $r \star \tau' = \tau \star h'$; under the hypotheses of the previous paragraph (see diagram 1.3.2.1), this is verified directly on the defining sites of the topoi.

By compatibility with vertical compositions of base change maps, recalled in 1.2, the two triangles below are commutative, where the horizontal arrow is the base change map induced by the diagonal square :

(1.3.3.2)
$$\begin{tikzcd} & p_{1\star}p_2{}^{\star}h_{\star} \arrow[dr, dashed, "\mathrm{cb}_{Y'YTT'}"] & \\ f^{\star}r_{\star}g'_{\star} = f^{\star}g_{\star}h_{\star} \arrow[ur, "\mathrm{cb}_{YSXT}"] \arrow[rr, "\mathrm{cb}_{Y'SXT'}"] \arrow[dr, "\mathrm{cb}_{X'S'SX}"'] & & r'_{\star}p'_{1\star}p'^{\star}_2 = p_{1\star}h'_{\star}p'^{\star}_2 \\ & r'_{\star}f'^{\star}g'_{\star} \arrow[ur, "\mathrm{cb}_{Y'S'X'T'}"'] & \end{tikzcd}$$

is *commutative*[(i)]. Note that when h is the identity, the dotted 2-arrow is none other than (the 2-arrow induced by) the counit $\eta_{h'}$. The commutativity of 1.3.2.2 follows.

1.3.4. In this paragraph, we show that the map $\eta : \mathrm{R}\Gamma(T, \mathcal{K}) \to \mathrm{R}\Gamma(T', \mathcal{K}')$ considered at the end of paragraph 1.3.1 — whose hypotheses we retain — is an isomorphism. We proceed by reduction to the previous case (1.3.2). The morphism p_2 has a section, which we will denote σ, defined in **XI**-2.2. Let $\mathcal{K}_Y = \sigma^{\star}\mathcal{K}$ — this notation is compatible with that of the previous paragraph —, and $\widetilde{\mathcal{K}} = p_2^{\star}\mathcal{K}_Y$. According to **XI**-2.3 and **XI**-2.3.2, the base change morphism $\mathrm{cb}_{TYYY} : \mathrm{R}p_{2\star} \to \sigma^{\star}$ deduced from the 2-commutative diagram

$$\begin{tikzcd} T \arrow[d, "p_2"'] & Y \arrow[l, "\sigma"'] \arrow[d, equal] \\ Y \arrow[r, equal] & Y \end{tikzcd}$$

is an isomorphism and the counit $\widetilde{\mathcal{K}} \to \mathcal{K}$ induces an isomorphism $\mathrm{R}\Gamma(T, \widetilde{\mathcal{K}}) \xrightarrow{\sim} \mathrm{R}\Gamma(T, \mathcal{K})$. (We recall that T is local.) As we saw in the previous paragraph, the functoriality morphism $\mathrm{R}\Gamma(T, \widetilde{\mathcal{K}}) \to \mathrm{R}\Gamma(T', \widetilde{\mathcal{K}}')$ is an isomorphism, where $\widetilde{\mathcal{K}}'$ denotes the pullback by $\overleftarrow{r}$ of $\widetilde{\mathcal{K}}$ (or equivalently, by p'_2 of $\mathcal{K}_Y$) on the topos T'. To conclude, we must show that the map $\mathrm{R}\Gamma(T', \widetilde{\mathcal{K}}') \to \mathrm{R}\Gamma(T', \mathcal{K}')$ deduced from $\widetilde{\mathcal{K}} \to \mathcal{K}$ is an isomorphism. Calculating these complexes by projection onto the first factor X'

[(i)] The author thanks Joseph Ayoub for his help on this matter.

and using the proper base change theorem for the morphism $X' \to X$, it suffices to show that the morphism $\mathrm{R}p_1{}'_{\star}\widetilde{\mathcal{K}}' \to \mathrm{R}p_1{}'_{\star}\mathcal{K}'$ of complexes on X' is an isomorphism above the closed point x of X. Evaluation at a geometric point $x' \to X'_x$ is

$$\mathrm{R}\Gamma(T'_{(x')}, \widetilde{\mathcal{K}}') \to \mathrm{R}\Gamma(T'_{(x')}, \mathcal{K}'),$$

where $T'_{(x')} = X'_{(x')} \overleftarrow{\times}_{S'_{(s')}} Y_{(s')}$, with $s' \to S'$ the image of $x' \to X'$ and $Y_{(s')} := Y \times_{S'} S'_{(s')}$. As recalled above, this map identifies by projection onto the second factor with the map

$$\mathrm{R}\Gamma(Y_{(s')}, \sigma'_{(x')}{}^{\star}\widetilde{\mathcal{K}}') \to \mathrm{R}\Gamma(Y_{(s')}, \sigma'_{(x')}{}^{\star}\mathcal{K}'),$$

where $\sigma'_{(x')} : Y_{(s')} \to X'_{(x')} \overleftarrow{\times}_{S'_{(s')}} Y_{(s')}$ is the section defined in **XI**-2.2. We will show more precisely that the counit $\widetilde{\mathcal{K}} \to \mathcal{K}$ induces an isomorphism $\sigma'_{(x')}{}^{\star}\widetilde{\mathcal{K}}' \to \sigma'_{(x')}{}^{\star}\mathcal{K}'$. By functoriality of inverse images, it suffices to show that we have the equalities

$$\sigma \circ p_2 \circ \overleftarrow{r}_{(x')} \circ \sigma'_{(x')} = \overleftarrow{r}_{(x')} \circ \sigma'_{(x')} = \sigma \circ \mathrm{Id}_{(s')},$$

— the second implying the first —, where the morphisms are as in the diagram below :

$$\begin{tikzcd}
(x', y) \arrow[ddd, bend right=40, mapsto] & y \arrow[l, mapsto] \arrow[ddd, bend left=40, mapsto] \\
T'_{(x')} \arrow[r, "p'_{2(x')}"'] \arrow[d, "\overleftarrow{r}_{(x')}"'] & Y_{(s')} \arrow[l, bend right, "\sigma'_{(x')}"'] \arrow[d, "Id_{(s')}"] \\
T \arrow[r, "p_2"] & Y \arrow[l, bend left, "\sigma"] \\
(x, y) & y \arrow[l, mapsto]
\end{tikzcd}$$

The identity $\overleftarrow{r}_{(x')} \circ \sigma'_{(x')} = \sigma \circ \mathrm{Id}_{(s')}$ is a consequence of the fact that the morphism $X'_{(x')} \to X$ is *local*.

1.4. Remarks on the non-abelian case. By using the non-abelian proper base change theorem (**XX**-7.1), the same proof should allow us to show that, under the hypotheses of 1.1.2, the morphism $\overleftarrow{r}$ is **1-aspherical** : for any ind-finite stack $\mathcal{C}$ on T, the adjunction unit $\mathcal{C} \to \overleftarrow{r}_{\star}\overleftarrow{r}^{\star}\mathcal{C}$ is an equivalence of categories. Let us note, however, that, as pointed out in **XI**-2.8 (3), we are not aware of any published reference allowing to justify the passage to the limit necessary for the calculation of fibers.

2. Oriented Cohomological Descent

2.1. Oriented h-topology.

2.1.1. Let $\mathcal{B}$ denote the category whose objects are diagrams of schemes $X \xrightarrow{f} S \xleftarrow{g} Y$ and whose morphisms are triples of morphisms $(X' \to X, S' \to S, Y' \to Y)$ making the two associated squares commutative. Finite (projective) limits exist in $\mathcal{B}$ and are calculated "term by term". For example, the fibered product of $X_1 \to S_1 \leftarrow Y_1$ and $X_2 \to S_2 \leftarrow Y_2$ above $X \to S \leftarrow Y$ is $(X_1 \times_X X_2) \to (S_1 \times_S S_2) \leftarrow (Y_1 \times_Y Y_2)$.

2.1.2. Let $\overleftarrow{\mathcal{B}} \to \mathcal{B}$ denote the category bifibered in duals of topoi induced by the pseudo-functor "oriented product of étale topoi" from $\mathcal{B}$ to the 2-category of topoi, $(X \to S \leftarrow Y) \rightsquigarrow X \overleftarrow{\times}_S Y$. Condition [**SGA 4** $\mathrm{v}^{\mathrm{bis}}$ 3.0.0] is satisfied. Finally, note that if $\overleftarrow{\mathcal{B}}$ is annular by constant coefficients, all arrows in B are *flat* in the sense of *loc. cit.*, § 1.3.

2.1.3. We equip the category of schemes with the **h-topology**, generated by proper and surjective morphisms as well as Zariski coverings. In the preceding definition, we can restrict to proper and surjective morphisms between *coherent* schemes, as follows from the following facts : every scheme is Zariski-locally coherent; every coherent scheme over a coherent scheme is coherent (see [**ÉGA** I′ 6.1.10 (ii)] for quasi-separation, the analogue for quasi-compactness being tautological); a universally closed morphism (resp. proper) is quasi-compact (resp. coherent). Recall that the h-topology is finer than the étale topology (cf. for example **II**-3.1.1).

2.1.4. We consider the **oriented h-topology** on $\mathcal{B}$ generated by the families below :

(**i**). $\big((X_\alpha \to S \leftarrow Y) \to (X \to S \leftarrow Y)\big)_{\alpha\in A}$, where $(X_\alpha \to X)_{\alpha\in A}$ is a covering family for the h-topology;

(**ii**). $\big((X \to S \leftarrow Y_\alpha) \to (X \to S \leftarrow Y)\big)_{\alpha\in A}$, where $(Y_\alpha \to Y)_{\alpha\in A}$ is a covering family for the h-topology;

(**iii**). $\big((X \times_S S_\alpha \to S_\alpha \leftarrow Y \times_S S_\alpha) \to (X \to S \leftarrow Y)\big)_{\alpha\in A}$, where $(S_\alpha \to S)_{\alpha\in A}$ is a covering family for the h-topology;

(**iv**). $\big((X \times_S S' \to S' \leftarrow Y) \to (X \to S \leftarrow Y)\big)$, where $S' \to S$ is a *proper* morphism;

(**v**). $\big((X \to S' \leftarrow Y \times_S S') \to (X \to S \leftarrow Y)\big)$, where $S' \to S$ is an *étale* morphism.

Note that the properties of the families of morphisms (**i**)–(**v**) are stable under base change in the category $\mathcal{B}$.

2.1.5. It is formally verifiable that the oriented h-topology is generated by :

— families of type (**i**)–(**iii**), where the h-covering families are Zariski coverings;

— families of type (**i**)–(**iii**), where the h-covering families are given by a single proper and surjective morphism;

— families of type (**iv**) and (**v**).

One can also observe that families of type (**iii**) with $(S_\alpha \to S)_{\alpha\in A}$ a Zariski covering (resp. a proper and surjective morphism) are obtained by composition from types (**i**) and (**v**) (resp. from types (**iv**) and (**ii**)).

2.2. Statements.

2.2.1. Fix an object $B = (X \to S \leftarrow Y)$ of $\mathcal{B}$. In accordance with general definitions, a simplicial object augmented $(X_\bullet \to S_\bullet \leftarrow Y_\bullet) \to B$ in $\mathcal{B}$ is called an **hypercovering for the oriented h-topology** if for every integer $i \geq -1$, the canonical morphism $(X_{i+1} \to S_{i+1} \leftarrow Y_{i+1}) \to \big(\mathrm{cosq}_i(X_\bullet \to S_\bullet \leftarrow Y_\bullet)\big)_{i+1}$ of the category $\mathcal{B}_{/B}$ is a covering for the oriented h-topology defined in 2.1.4 above. The terms of the coskeleton, which are finite limits, exist and are calculated in $\mathcal{B}_{/B}$.

2.2.2. To the preceding data is associated a simplicial topos — that is, fibered over Δ, cf. [**SGA 4** v$^{\mathrm{bis}}$ § 1.2] —, which we denote $X_\bullet \overleftarrow{\times}_{S_\bullet} Y_\bullet$, and its total topos $\mathrm{Tot}(X_\bullet \overleftarrow{\times}_{S_\bullet} Y_\bullet)$. We refer the reader to [**Illusie, 1972**, VI, § 5.1] and [**SGA 4** vi § 7] for general definitions on the total topos, as well as [**Deligne, 1974**, § 5.1] for a summary in the particular case of simplicial topological spaces. We have the following result on cohomological descent.

Théorème 2.2.3. *Let $(X_\bullet \to S_\bullet \leftarrow Y_\bullet) \to (X \to S \leftarrow Y)$ be an hypercovering for the oriented h-topology and let $\overleftarrow{\varepsilon} : \mathrm{Tot}(X_\bullet \overleftarrow{\times}_{S_\bullet} Y_\bullet) \to X \overleftarrow{\times}_S Y$ be the associated augmentation morphism. For every integer $n \geq 1$ and every complex $\mathcal{K} \in \mathrm{Ob}\,\mathrm{D}^+(X \overleftarrow{\times}_S Y, \mathbf{Z}/n\mathbf{Z})$, the adjunction unit $\mathcal{K} \to \mathrm{R}\overleftarrow{\varepsilon}_\star \overleftarrow{\varepsilon}^\star \mathcal{K}$ is an isomorphism.*

Remarque 2.2.4. Be careful that, contrary to the usual (non-oriented) case, a Cartesian object of $\mathrm{Tot}(X_\bullet \overleftarrow{\times}_{S_\bullet} Y_\bullet)$ does not necessarily come from an object of $X \overleftarrow{\times}_S Y$: the morphism $\overleftarrow{\varepsilon}$ is generally not a 2-cohomological descent in the sense of [**SGA 4** v$^{\mathrm{bis}}$ 2.2.6].

The main application we will make of the previous theorem is the following base change formula, another proof of which can be found in **XII$_{\mathrm{B}}$-??**.

Théorème 2.2.5 ("Hyper-base change"). *Let $X \xrightarrow{f} S \xleftarrow{g} Y$ be an object of $\mathcal{B}$ with g coherent, and $S_\bullet \to S$ an hypercovering for the h-topology. Let $f_\bullet : X_\bullet \to S_\bullet$ and $g_\bullet : Y_\bullet \to S_\bullet$ respectively denote the morphisms* *deduced from f and g by the base change $S_\bullet \to S$, and $X_\bullet \to X$ the hypercovering of X for the h-topology deduced from $S_\bullet \to S$. Finally, let ε_X denote the topos morphism* $\mathrm{Tot}(X_\bullet) \to X$. *For every integer $n \geq 1$ and every complex* $\mathcal{K} \in \mathrm{Ob}\,\mathrm{D}^+(Y, \mathbf{Z}/n\mathbf{Z})$, *the morphism*

$$f^\star \mathrm{R}g_\star \mathcal{K} \to \mathrm{R}\varepsilon_{X_\star}(f_\bullet^{\ \star} \mathrm{R}g_{\bullet\star} \mathcal{K}_{|Y_\bullet})$$

is an isomorphism.

We write $f_\bullet$, etc., for what we could have written $\mathrm{Tot}(f_\bullet)$, etc. This should not lead to confusion.

The coherence hypothesis on g is made to be able to apply Theorem **XI**-2.4, which can fail without this hypothesis(ii). (For example, if S is a Dedekind scheme having infinitely many closed points $\xi_0, \xi_1, \ldots$, if we set $X = \xi_0$ and $Y = \coprod_{i>0} \xi_i$, the topos $X \overleftarrow{\times}_S Y$ — where f and g are the obvious morphisms — is the empty topos while the sheaf $f^\star g_\star \mathbf{Z}/n\mathbf{Z}$ is non-zero for any $n > 1$.)

Démonstration. Admitting the preceding descent theorem (2.2.3), it is not difficult to prove the base change formula above. Let $\mathcal{K}$ be a complex on Y as in the statement, whose pullback by the second projection $p_2 : X \overleftarrow{\times}_S Y \to Y$ is denoted $\overleftarrow{\mathcal{K}}$. It follows from Theorem 2.2.3 (i) that the unit $\mathcal{K} \to \mathrm{R}\overleftarrow{\varepsilon}_\star \overleftarrow{\varepsilon}^\star \mathcal{K}$ is an isomorphism, where $\overleftarrow{\varepsilon}$ denotes the augmentation morphism $\mathrm{Tot}(X_\bullet \overleftarrow{\times}_{S_\bullet} Y_\bullet) \to X \overleftarrow{\times}_S Y$. The conclusion results from the chain of isomorphisms below, the first and last resulting from **XI**-2.4, the third from the functoriality of direct images, and the second — as we have seen — from the oriented h-cohomological descent.

$$f^\star \mathrm{R}g_\star \mathcal{K} \xrightarrow{\sim} \mathrm{R}p_{1\star} \overleftarrow{\mathcal{K}}$$
$$\mathrm{R}p_{1\star} \overleftarrow{\mathcal{K}} \xrightarrow{\sim} \mathrm{R}p_{1\star} \mathrm{R}\overleftarrow{\varepsilon}_\star \overleftarrow{\varepsilon}^\star \overleftarrow{\mathcal{K}}$$
$$\mathrm{R}p_{1\star} \mathrm{R}\overleftarrow{\varepsilon}_\star \overleftarrow{\varepsilon}^\star \overleftarrow{\mathcal{K}} = \mathrm{R}\varepsilon_{X_\star} \mathrm{R}p_{1\bullet\star} \overleftarrow{\mathcal{K}}_\bullet, \text{ where } \overleftarrow{\mathcal{K}}_\bullet = \overleftarrow{\varepsilon}^\star \overleftarrow{\mathcal{K}}$$
$$\mathrm{R}\varepsilon_{X_\star} \mathrm{R}p_{1\bullet\star} \overleftarrow{\mathcal{K}}_\bullet \xleftarrow{\sim} \mathrm{R}\varepsilon_{X_\star} f_\bullet^\star \mathrm{R}g_{\bullet\star} \overleftarrow{\mathcal{K}}_\bullet$$

□

2.2.6. *Remarks.*

2.2.6.1. In the two preceding theorems, one could replace the ring $\mathbf{Z}/n\mathbf{Z}$ with any torsion ring Λ; the proof given below applies without change. More generally, one could consider complexes from $\mathrm{D}^+(X \overleftarrow{\times}_S Y, \Lambda)$ or $\mathrm{D}^+(Y, \Lambda)$ with torsion cohomology (iii) when Λ is any ring. In this context, the hypothesis [**SGA 4** v$^{\mathrm{bis}}$ 2.4.1.1] of stability under direct image is not always satisfied; this explains why it sometimes seems useful to restrict to the category of schemes with coherent morphisms (cf. $\mathbf{XII_B}$-**??** and [**SGA 4** v$^{\mathrm{bis}}$ 4.3.1]). It is nevertheless plausible that a precise analysis of the proofs in *op. cit.* would allow one to dispense with this hypothesis and consequently prove the preceding theorems with weaker torsion hypotheses.

2.2.6.2. Using 1.4 and [**Orgogozo, 2003**, 2.5, 2.8] it is plausible that one could adapt the proofs given below to obtain the following non-abelian cohomology statements :

- (under the hypotheses of 2.2.3) the adjunction unit $\mathcal{C} \to \overleftarrow{\varepsilon}_\star \overleftarrow{\varepsilon}^\star \mathcal{C}$ is an equivalence for any ind-finite stack $\mathcal{C}$ on $X \overleftarrow{\times}_S Y$;
- (under the hypotheses of 2.2.5) $f^\star g_\star \mathcal{C} \to \varepsilon_{X_\star}(f_\bullet^{\ \star} g_{\bullet\star} \mathcal{C}_{|Y_\bullet})$ is an equivalence for any ind-finite stack $\mathcal{C}$ on Y.

For the meaning to be given to these statements see [**Giraud, 1971**] (especially Chap. VII, § 2.2) and [**Orgogozo, 2003**, § 2]. See also $\mathbf{XII_B}$-**??** for another proof sketch of the second statement.

2.3. The remainder of this exposé is devoted to the proof of Theorem 2.2.3.

(ii) It is probable that the techniques of Exposé $\mathrm{XII_B}$ allow to prove the preceding theorem without assuming g coherent.

(iii) An abelian sheaf $\mathcal{F}$ is *of torsion* if the map $\mathrm{colim}_n \mathrm{Ker}([n] : \mathcal{F} \to \mathcal{F}) \to \mathcal{F}$ is an isomorphism, where n ranges over the non-zero integers ordered by divisibility; see ([**SGA 4** IX § 1]).

2.3.1. *Reductions.*

2.3.1.1. According to the general theory of descent — cf. [**SGA 4** v$^{\mathrm{bis}}$ § 3.3] or [**Deligne, 1974**, 5.3.5] for a summary —, it suffices to prove that a morphism $B' = (\coprod X_\alpha \to \coprod S_\alpha \leftarrow \coprod Y_\alpha) \to B$ deduced from a family of type (**i**) to (**v**) is a cohomological descent. (By stability under base change, we will automatically have *universal* cohomological descent.) Furthermore, for the same reason, we can content ourselves with the particular case where the h-covering families of X, Y and S considered in (**i**)–(**iii**) are either a Zariski covering or a proper and surjective morphism.

2.3.1.2. Let $\mathcal{S}$ be a site, with associated topos $\mathcal{T}$ and the pseudo-functor $V \mapsto \mathcal{T}_{/V}$. For any covering $V \to U$, we have cohomological descent for $\mathrm{Tot}\,\mathcal{T}_{/V_\bullet} \to \mathcal{T}_{/U}$, where $\mathcal{T}_{/V_\bullet}$ is the simplicial topos deduced from the coskeleton of V over U. This results for example from the local existence of a section to the morphism $V \to U$. Applying this observation to our category $\mathcal{B}$, we observe that we have (universal) cohomological descent for families of type (**i**), (**ii**), (**iii**) and (**v**) when we assume the morphisms $X_\alpha \to X$, $Y_\alpha \to Y$ and $S_\alpha \to S$ to be *étale*. According to observation 2.1.5, this allows us to reduce to the case where the h-covering families of 2.1.4 are given by a single proper and surjective morphism. This also allows us to assume the schemes are coherent and, consequently, to use Theorem 1.1.2 whose given proof uses finiteness hypotheses (allowing the calculation of fibers of direct images).

2.3.1.3. In the following paragraphs, we fix an object $B = (X \to S \leftarrow Y)$ of $\mathcal{B}$ with coherent objects (and morphisms), and we denote T the topos $X \overleftarrow{\times}_S Y$. Given a simplicial object $B_\bullet = (X_\bullet \to S_\bullet \leftarrow Y_\bullet)$ augmented to B, we denote $T_\bullet$ the total topos of the simplicial topos $X_\bullet \overleftarrow{\times}_{S_\bullet} Y_\bullet$: to lighten the notation, we will henceforth omit the "Tot". We also fix an integer $n \geq 1$ and a complex $\mathcal{K} \in \mathrm{Ob}\,\mathrm{D}^+(T, \mathbf{Z}/n\mathbf{Z})$, whose pullback by the topos morphism $\overleftarrow{\varepsilon} : T_\bullet \to T$ we denote $\mathcal{K}_\bullet$. We wish to show that if $B_\bullet$ is the 0-coskeleton of a map $B' \to B$ as in 2.3.1.1 above and with coherent objects, the adjunction unit

$$\mathcal{K} \to \mathrm{R}\overleftarrow{\varepsilon}_\star \mathcal{K}_\bullet$$

is an isomorphism.

2.3.2. *Family of type* (**i**) : $(X' \to S \leftarrow Y) \to (X \to S \leftarrow Y)$. Let $r' : X' \to X$ be a coherent morphism covering for the h-topology, and $X_\bullet = \mathrm{cosq}_0^X(X')$ its coskeleton. (As explained in 2.3.1, we could assume r' proper and surjective ; this is not necessary.) By coherence and passing to fibers at the points of the topos T, it suffices to show that the functoriality morphism

$$\mathrm{R}\Gamma(T, \mathcal{K}) \to \mathrm{R}\Gamma(T_\bullet, \mathcal{K}_\bullet)$$

is an isomorphism, when the schemes Y, X, S, and the morphism $f : X \to S$, are local (for the étale topology). As in § 1, let's calculate the adjunction unit by projection on the first factor, i.e. on the lower horizontal line of the 2-commutative diagram

$$\begin{tikzcd} T \arrow[d, "p_1"'] & T_\bullet \arrow[l, "\overleftarrow{\varepsilon}"'] \arrow[d, "p_{1\bullet}"] \\ X & X_\bullet \arrow[l, "\varepsilon_X"'] \end{tikzcd}$$

We want to show that the map

$$\mathrm{R}\Gamma(X, \mathrm{R}p_{1_\star}\mathcal{K}) \to \mathrm{R}\Gamma(X_\bullet, \mathrm{R}p_{1\bullet_\star}\mathcal{K}_\bullet)$$

is an isomorphism. Since, by usual cohomological descent, the morphism $\mathrm{R}\Gamma(X, \mathrm{R}p_{1_\star}\mathcal{K}) \to \mathrm{R}\Gamma(X_\bullet, (\mathrm{R}p_{1_\star}\mathcal{K})_\bullet)$ is an isomorphism, it suffices to show that the base change map $\varepsilon_X^\star \mathrm{R}p_{1_\star}\mathcal{K} \to \mathrm{R}p_{1\bullet_\star}\overleftarrow{\varepsilon}^\star\mathcal{K}$ induces an isomorphism on global sections (on $X_\bullet$). The simplicial direct and inverse images are calculated stage by stage, so it suffices to show that, for each $i \geq 0$, the base change morphisms $\varepsilon_i^\star \mathrm{R}p_{1_\star}\mathcal{K} \to \mathrm{R}p_{1i_\star}\overleftarrow{\varepsilon}_i^\star\mathcal{K}$ associated with the 2-commutative diagrams

$$\begin{tikzcd} T & T_i = X_i \overleftarrow{\times}_S Y \arrow[l, "\overleftarrow{\varepsilon}_i"'] \arrow[d, "p_{1i}"] \\ X \arrow[from=1-1, "p_1"'] & X_i \arrow[l, "\varepsilon_i"'] \end{tikzcd}$$

are isomorphisms. This follows from Lemma 2.3.3 (i), applied to $\varepsilon_i : X_i \to X$.

LEMME 2.3.3. *Consider an object $X \to S \leftarrow Y$ of $\mathcal{B}$ and a complex $\mathcal{K}$ as in 2.3.1.3.*

(i) *Let $\gamma : Z \to X$ be a coherent morphism, making Z an S-scheme. Let $\overleftarrow{\gamma} : Z \overleftarrow{\times}_S Y \to X \overleftarrow{\times}_S Y$ denote the induced morphism between oriented topoi. The base change morphism*

$$\gamma^\star(\mathrm{R}p^X_{1\,\star}\mathcal{K}) \to \mathrm{R}p^Z_{1\,\star}\overleftarrow{\gamma}^\star\mathcal{K}$$

associated with the diagram

$$\begin{tikzcd} X \overleftarrow{\times}_S Y \arrow[d, "p_1^X"'] & Z \overleftarrow{\times}_S Y \arrow[l, "\overleftarrow{\gamma}"'] \arrow[d, "p_1^Z"] \\ X & Z \arrow[l, "\gamma"'] \end{tikzcd}$$

is an isomorphism.

(ii) *Let $\delta : Z \to Y$ be a coherent morphism, making Z an S-scheme. Let $\overleftarrow{\delta} : X \overleftarrow{\times}_S Z \to X \overleftarrow{\times}_S Y$ denote the induced morphism between oriented topoi. Under the additional hypothesis that $X \to S$ is a* local *morphism between* strictly local *schemes, the base change morphism*

$$\delta^\star(\mathrm{R}p^Y_{2\,\star}\mathcal{K}) \to \mathrm{R}p^Z_{2\,\star}\overleftarrow{\delta}^\star\mathcal{K}$$

associated with the diagram

$$\begin{tikzcd} X \overleftarrow{\times}_S Y \arrow[d, "p_2^Y"'] & X \overleftarrow{\times}_S Z \arrow[l, "\overleftarrow{\delta}"'] \arrow[d, "p_2^Z"] \\ Y & Z \arrow[l, "\delta"'] \end{tikzcd}$$

is an isomorphism.

Démonstration. (i) By passing to fibers at a geometric point of Z and its image by γ, it suffices to show that the morphism $\overleftarrow{\gamma}$ induces an isomorphism on *global sections* when $Z \to X$ is a local morphism between strictly local schemes. Let s be the geometric point of S image of the central points of Z and X; the topos morphisms $Z \overleftarrow{\times}_{S_{(s)}} Y_{(s)} \to Z \overleftarrow{\times}_S Y$ and $X \overleftarrow{\times}_{S_{(s)}} Y_{(s)} \to X \overleftarrow{\times}_S Y$ being equivalences (**XI**-1.11), we can further assume that the morphisms $Z \to S$ and $X \to S$ are local (between strictly local schemes). Let σ_X and σ_Z denote the canonical sections of the second projections $Z \overleftarrow{\times}_S Y \to Y$ and $Z \overleftarrow{\times}_S Y \to Y$. To conclude, we observe that the vertical arrows of restriction in the diagram below are isomorphisms by the equalities $p_{2\star} = \sigma^\star$ already used.

$$\begin{tikzcd} \mathrm{R}\Gamma(X \overleftarrow{\times}_S Y, \mathcal{K}) \arrow[r, "\mathrm{R}\Gamma(\overleftarrow{\gamma}, \mathcal{K})"] \arrow[d, "\mathrm{R}\Gamma(\sigma_X, \mathcal{K})"'] & \mathrm{R}\Gamma(Z \overleftarrow{\times}_S Y, \mathcal{K}) \arrow[d, "\mathrm{R}\Gamma(\sigma_Y, \mathcal{K})"] \\ \mathrm{R}\Gamma(Y, \mathcal{K}) \arrow[r, dash, "="] & \mathrm{R}\Gamma(Y, \mathcal{K}) \end{tikzcd}$$

(ii) By passing to fibers at a geometric point of Z and its image by δ, it suffices to show that the morphism $\overleftarrow{\delta}$ induces an isomorphism on *global sections* when $Z \to Y$ is a local morphism between strictly local schemes. As we assume, furthermore, that the morphism $X \to S$ is a local morphism between strictly local schemes, the topoi $X\overleftarrow{\times}_S Y$ and $X\overleftarrow{\times}_S Z$ are local and the morphism $X\overleftarrow{\times}_S Z \to X\overleftarrow{\times}_S Y$ is local. The conclusion is then immediate. □

2.3.4. *Family of type* (***ii***) *:* $(X \to S \leftarrow Y') \to (X \to S \leftarrow Y)$. Same argument, modulo exchanging p_1 and p_2. More precisely : we can assume T is local, and we project onto Y and $Y_\bullet$ to reduce to showing, by usual cohomological descent for $Y_\bullet = \mathrm{cosq}_0^Y(Y') \to Y$, that the morphism $[(\mathrm{R}p_{2\star}\mathcal{K})_\bullet \to \mathrm{R}p_{2\bullet\star}\mathcal{K}_\bullet]$ induces an isomorphism by application of the functor $\mathrm{R}\Gamma(Y_\bullet, -)$. We then use (ii) of the lemma above from which it follows that the base change morphism $(\mathrm{R}p_{2\star}\mathcal{K})_\bullet \to \mathrm{R}p_{2\bullet\star}\mathcal{K}_\bullet$ is an isomorphism.

2.3.5. *Family of type* (***iii***) *:* $(X' = X \times_S S' \to S' \leftarrow Y' = Y \times_S S') \to (X \to S \leftarrow Y)$. Let $S_\bullet = \mathrm{cosq}_0^S(S')$ denote the coskeleton of the proper and surjective morphism $S' \to S$ and $Y_\bullet \to Y, X_\bullet \to X$ the augmented simplicial schemes derived from it. (The reduction to the case of a proper and surjective morphism $S' \to S$ results from 2.1.5.) We factor $\overleftarrow{\varepsilon}$ as

$$\begin{tikzcd} X_\bullet\overleftarrow{\times}_{S_\bullet} Y_\bullet \arrow[r, "\overleftarrow{r}_\bullet"] \arrow[rr, bend right, "\overleftarrow{\varepsilon}"'] & X\overleftarrow{\times}_S Y_\bullet \arrow[r, "\overleftarrow{\varepsilon_Y}"] & X\overleftarrow{\times}_S Y \end{tikzcd}$$

We saw in the previous paragraph (2.3.4) that the morphism $\mathcal{K} \to \mathrm{R}\overleftarrow{\varepsilon_Y}_\star\overleftarrow{\varepsilon_Y}^\star\mathcal{K}$ is an isomorphism. To conclude, it suffices to show that the unit $\mathrm{Id} \to \mathrm{R}\overleftarrow{r}_{\bullet\star}\overleftarrow{r}_\bullet^\star$ is an isomorphism, evaluated at a complex in $\mathrm{D}^+(X\overleftarrow{\times}_S Y_\bullet, \mathbf{Z}/n\mathbf{Z})$. Since simplicial direct and inverse images are calculated stage by stage, it suffices to show that, for each $i \geq 0$, the morphism

$$\mathrm{Id} \to \mathrm{R}\overleftarrow{r}_{i\star}\overleftarrow{r}_i^\star$$

associated with the morphisms

$$\begin{tikzcd} X \arrow[d] \arrow[dr, phantom, "\square"] & X_i \arrow[l, "r'_i"'] \arrow[d] & \\ S & S_i \arrow[l, "r_i"'] & Y_i \arrow[l] \end{tikzcd} \qquad \begin{tikzcd} X_i\overleftarrow{\times}_{S_i} Y_i \arrow[d, "\overleftarrow{r}_i"] \\ X\times_S Y_i \end{tikzcd}$$

is an isomorphism. This is asserted by Theorem 1.1.2.

2.3.6. *Family of type* (***iv***) *:* $(X' = X \times_S S' \to S' \leftarrow Y) \to (X \to S \leftarrow Y)$. Here we assume that the morphism $Y \to S$ factors through a *proper* morphism $S' \to S$, and we define $S_\bullet$ and $X_\bullet$ as above. Note that the coskeleton over $B = (X \to S \leftarrow Y)$ of $(X' \to S' \leftarrow Y)$ is, at stage i, the object $X_i \to S_i \leftarrow Y$, where $Y \to S_i$ is the obvious morphism factoring through the diagonal. Let's decompose the morphism $\overleftarrow{\varepsilon}$ as

$$\begin{tikzcd} X_\bullet\overleftarrow{\times}_{S_\bullet} Y \arrow[r, "\overleftarrow{r}_\bullet"] \arrow[rr, bend right, "\overleftarrow{\varepsilon}"'] & (X\overleftarrow{\times}_S Y)_\bullet \arrow[r] & X\overleftarrow{\times}_S Y \end{tikzcd}$$

where $(X\overleftarrow{\times}_S Y)_\bullet$ is the total topos associated with the constant simplicial object of value $(X \to S \leftarrow Y)$ in $\mathcal{B}$. (This is the topos of cosimplicial sheaves on $X\overleftarrow{\times}_S Y$.) Cohomological descent is trivial for $(X\overleftarrow{\times}_S Y)_\bullet \to X\overleftarrow{\times}_S Y$: the identity has a section. (Alternatively : the direct image functor is derived to the associated simple complex functor, $\mathrm{Tot} : \mathrm{D}^+((X\overleftarrow{\times}_S Y)_\bullet, \mathbf{Z}/n\mathbf{Z}) \to \mathrm{D}^+(X\overleftarrow{\times}_S Y, \mathbf{Z}/n\mathbf{Z})$.) The conclusion results from the fact that the unit $\mathrm{Id} \to \mathrm{R}\overleftarrow{r}_{\bullet\star}\overleftarrow{r}_\bullet^\star$ is an isomorphism, by a new application of Theorem 1.1.2.

2.3.7. *Family of type* (v) *:* $(X \to S' \leftarrow Y' = Y \times_S S') \to (X \to S \leftarrow Y)$. Here we assume that the morphism $X \to S$ factors through an *étale* morphism $S' \to S$. We have already given in 2.3.1 a proof of the desired result. We can also proceed as before ; here is a brief argument. Let $S_\bullet$ again denote the 0-coskeleton of the morphism $S' \to S$, and $Y_\bullet = Y \times_S S_\bullet$ the augmented simplicial scheme to Y derived from it. The coskeleton of $(X \to S' \leftarrow Y')$ over B is, at stage i, the object $X \to S_i \leftarrow Y_i$, where $X \to S_i$ is the obvious morphism factoring through the diagonal. Since the morphism $X \overleftarrow{\times}_{S_i} Y_i \to X \overleftarrow{\times}_S Y$ is an equivalence, the simplicial topos $X \overleftarrow{\times}_{S_\bullet} Y_\bullet$ is constant : its total topos is equivalent to the (total) topos of $(X \overleftarrow{\times}_S Y)_\bullet$. As in the previous paragraph, we are reduced to the trivial case where $S' = S$.

EXPOSÉ $\mathrm{XII_B}$

Dimension functions

Vincent Pilloni and Benoît Stroh

We define the notion of a dimension function on a scheme X and we show the existence of such functions locally for the étale topology if X is quasi-excellent.

1. Universal catenarity of Henselian schemes

In this part, we recall the notions of *catenarity* and *universal catenarity*. The reader may consult exposé I for more details.

1.1. Universally catenary schemes. Let S be a topological space and $X \subset Y$ be irreducible closed subsets of S. We denote $\mathrm{codim}(X, Y)$ the supremum of the set of lengths of strictly increasing chains of irreducible closed subsets $X \subset Z \subset Y$ (cf. [**ÉGA** 0_{IV} 14.2.1 & 14.1.1]). If S is a scheme, X and Y are integral closed subschemes and x is the generic point of X, we have

$$\mathrm{codim}(X, Y) = \dim(\mathscr{O}_{Y,x}).$$

Définition 1.1.1 ([**ÉGA** 0_{IV} 14.3.2]). A scheme S is **catenary** if it is locally Noetherian and if for every chain $X \subset Y \subset Z$ of irreducible closed subsets of S, we have

$$\mathrm{codim}(X, Z) = \mathrm{codim}(Y, Z) + \mathrm{codim}(X, Y).$$

A scheme S is **universally catenary** if every scheme of finite type over S is catenary.

The notion of catenarity is stable under localization and restriction to closed subschemes. Thus, S is universally catenary if and only if for every integer $n \geq 0$, the scheme $\mathbf{A}^n_S$ is catenary.

Lemme 1.1.2. *A Cohen-Macaulay scheme is universally catenary.*

Démonstration. If S is Cohen-Macaulay, it is catenary according to [**Matsumura, 1980a**, 16.B]. Since for all $n \geq 0$, the scheme $\mathbf{A}^n_S$ remains Cohen-Macaulay, the scheme S is indeed universally catenary. □

Exemple 1.1.3. Every regular scheme is universally catenary because it is Cohen-Macaulay. In particular, the spectrum of a field, a trait, and the spectrum of an algebra of formal series over a field or over a discrete valuation ring are universally catenary. Every scheme of finite type over a regular scheme is universally catenary.

Proposition 1.1.4 ([**Matsumura, 1980a**], 28.P). *A complete Noetherian local scheme is universally catenary.*

Démonstration. Cohen's structure theorem [**ÉGA** 0_{IV} 19.8.8] allows any complete Noetherian local scheme to be written as a closed subscheme in the spectrum of a formal power series algebra over a Cohen ring. Universal catenarity results from the previous example and the stability of this notion by passing to a closed subscheme. □

1.2. A theorem of Ratliff. A Noetherian scheme is said to be **equidimensional** if all its irreducible components have the same (finite) dimension. Let S be a Noetherian local scheme. We denote $\hat{S}$ the spectrum of the completion of the ring of S at its maximal ideal.

Définition 1.2.1. The local scheme S is **formally equidimensional** if $\hat{S}$ is equidimensional. It is **formally catenary** if for every $s \in S$, the closure $\overline{\{s\}}$ is formally equidimensional.

Proposition 1.2.2. *Let S be a Noetherian local scheme. The scheme S, its completion $\hat{S}$, its Henselization S^{h} and its strict Henselization all have the same dimension.*

Démonstration. This results from the following general statement : if $A \to A'$ is a local and flat morphism between Noetherian local rings with respective maximal ideals $\mathfrak{m}$ and $\mathfrak{m}'$ and if $\mathfrak{m}' = \mathfrak{m}A'$, then for any natural integer n, the lengths $\lg_A(A/\mathfrak{m}^n)$ and $\lg_{A'}(A'/\mathfrak{m}'^n)$ are equal. The equality of these Hilbert-Samuel functions implies the equality $\dim A = \dim A'$ (cf. [**Zariski & Samuel, 1975**, chap. VIII, §9]). □

According to this proposition, if S is an integral Noetherian local scheme, the irreducible components of $\hat{S}$ have dimension $\leq \dim(S)$ and one of them has dimension $\dim(S)$. The scheme S is therefore formally equidimensional if and only if all irreducible components of $\hat{S}$ have dimension $\dim(S)$.

Let S be a Noetherian local scheme. Ratliff proved the following fundamental theorem, which has already been mentioned in proposition **I**-7.1.1.

Théorème 1.2.3 ([**Matsumura, 1989**], 31.7). *For a Noetherian local scheme S, the following conditions are equivalent :*

- *S is formally catenary,*
- *S is universally catenary,*
- *$\mathbf{A}^1_S$ is catenary,*
- *S is catenary and for all $s \in S$, for every integral scheme S' equipped with a finite and dominant map $S' \to \overline{\{s\}}$ and every closed point s' of S', we have $\dim(\mathscr{O}_{S',s'}) = \dim \overline{\{s\}}$.*

We have added a fourth condition equivalent to statement [**Matsumura, 1989**, Theorem 31.7]. It results from [**ÉGA** IV$_2$ 5.6.10] that the first three equivalent conditions imply the fourth. The converse is proved during the proof of [**Matsumura, 1989**, Theorem 31.7] (in the second paragraph of page 255).

Corollaire 1.2.4 ([**Matsumura, 1989**], 31.2). *Every Noetherian scheme of dimension ≤ 2 is catenary. Every Noetherian scheme of dimension ≤ 1 is universally catenary.*

1.3. Henselian schemes and catenarity. We have seen that every complete Noetherian local scheme is universally catenary in proposition 1.1.4. Henselian local schemes also enjoy good catenarity properties :

Proposition 1.3.1. *Every catenary Henselian local scheme is universally catenary.*

Démonstration. Let $S = \mathrm{Spec}(A)$ be a catenary Henselian local scheme, let P be a prime ideal of A, let L be a finite extension of $\mathrm{Frac}(A/P)$ and let B be a finite extension of A/P contained in L. According to theorem 1.2.3, it suffices to prove that the dimension of the localization of B at each of its maximal ideals is equal to the dimension of A/P. Every finite algebra over a Henselian ring is semi-local according to [**ÉGA** IV$_4$ 18.5] and [**ÉGA** IV$_4$ 18.6]. Since the scheme B is integral, it is local. The "going-up" theorem ([**Matsumura, 1989**, 9.3 and 9.4]) shows that we indeed have $\dim(B) = \dim(A/P)$. □

Let us also recall the following result, a consequence of corollaire **I**-6.3 (ii).

Proposition 1.3.2. *Every quasi-excellent Henselian local scheme is universally catenary.*

Thus, every quasi-excellent Henselian local scheme is excellent.

2. Immediate specializations and dimension functions

2.1. Definitions. Let X be a scheme. For every point x of X and every geometric point $\bar{x}$ above x, we denote $X_{(x)}, X^{\mathrm{h}}_{(x)}$ and $\widehat{X}_{(x)}$ the localization, the Henselization and the completion of X at x. Similarly, we denote $X_{(\bar{x})}$ the strict Henselization of X at $\bar{x}$.

Let x and y be two points of X, and $\bar{x}$ and $\bar{y}$ be two geometric points above x and y.

Définition 2.1.1 ([**SGA 4** vii 7.2]). A **specialization morphism** $\bar{x} \rightsquigarrow \bar{y}$ is the data of an X-morphism $X_{(\bar{x})} \to X_{(\bar{y})}$ between strict Henselizations.

According to [**SGA 4** vii 7.4], the data of a specialization $\bar{x} \rightsquigarrow \bar{y}$ is equivalent to the data of an X-morphism $\bar{x} \to X_{(\bar{y})}$.

Définition 2.1.2. Let $r \in \mathbf{N}$. A specialization $\bar{x} \rightsquigarrow \bar{y}$ is called a **specialization of codimension** r if the closure of the image of $\bar{x}$ in $X_{(\bar{y})}$ is a scheme of dimension r.
We say that y is an **immediate étale specialization** of x if there exists a specialization $\bar{x} \rightsquigarrow \bar{y}$ which is of codimension 1.
We say that y is an **immediate Zariski specialization** of x if $y \in \overline{\{x\}}$ and if the localization at y of the closure of x is of dimension 1.

2.1.3. If y is an immediate étale specialization of x, we also say that x is an **immediate étale generization** of y. Let $f : X_{(\bar{y})} \to X_{(y)}$ denote the strict Henselization morphism. The immediate étale generizations of y are then the images under f of the points $x' \in X_{(\bar{y})}$ such that $\dim \overline{\{x'\}} = 1$.

Before examining these notions in more detail, we recall the following simple fact (**II**-1.1.3) which we will use implicitly below : if $f : X \to S$ is a flat morphism, f sends maximal points of X to maximal points of S, in other words every irreducible component of X dominates an irreducible component of S.

2.1.4. If x and y are two points of a Noetherian scheme X such that $y \in \overline{\{x\}}$ (in the usual sense, i.e., y is a specialization of x or x is a generization of y), then y is an immediate Zariski (resp. étale) specialization of x if and only if this is the case in $\overline{\{x\}}_{(y)}$. For certain considerations, this allows us to assume that X is integral local with generic point x and closed point y. In this case, y is an immediate Zariski specialization of x if and only if $\dim(X) = 1$. In the étale case, this can be read on the strict Henselization :

Proposition 2.1.5. *If x and y are two points of a Noetherian scheme X, the point y is an immediate étale specialization of x if and only if $y \in \overline{\{x\}}$ and the strict Henselization at a geometric point above y of the closure of x has an irreducible component of dimension* 1.

Démonstration. We reduce to the particular case $X = \overline{\{x\}}_{(y)}$ considered above. The point y is an immediate étale specialization if and only if there exists a point $\tilde{x}$ of $X_{(\bar{y})}$ above x such that the closure of $\tilde{x}$ in $X_{(\bar{y})}$ is of dimension 1. As stated here, this is equivalent to the fact that $X_{(\bar{y})}$ has an irreducible component C of dimension 1. Indeed, if we denote $\tilde{x}$ the generic point of C, by the flatness argument stated above, C dominates X, meaning that $\tilde{x}$ is above x. Conversely, if $\tilde{x}$ is a point above x whose closure in $X_{(\bar{y})}$ is of dimension 1, we can denote C an irreducible component of $X_{(\bar{y})}$ containing $\tilde{x}$. The generic point of C and $\tilde{x}$ being both above x, they are equal since one is a generization of the other and the fibers of $X_{(\bar{y})} \to X$ are discrete. □

Proposition 2.1.6. *Let X be a Noetherian scheme. An immediate Zariski specialization between points of X is an immediate étale specialization, and the converse is true if X is universally catenary.*

We can assume that $X = \overline{\{x\}}_{(y)}$ as before. For the implication, we assume that $\dim(X) = 1$ and we want to show that $X_{(\bar{y})}$ has an irreducible component of dimension 1. According to proposition 1.2.2, $X_{(\bar{y})}$ is of dimension 1 and it is obvious that the irreducible components of a local scheme of dimension 1 are all of dimension 1.

For the converse, we will use two lemmas :

Lemme 2.1.7. *Let X be a Henselian Noetherian local scheme with closed point y. Let $\bar{y}$ be a geometric point above y. Then, X has an irreducible component of dimension* 1 *if and only if its strict Henselization $X_{(\bar{y})}$ has one.*

Démonstration. If C is an irreducible component of dimension 1 of $X_{(\bar{y})}$, its image in X is closed because $p\colon X_{(\bar{y})} \to X$ is integral. Since p is flat, $p(C)$ is an irreducible component of X containing exactly two points, so $\dim(p(C)) = 1$. Conversely, the surjectivity and flatness of p imply that if $D \subset X$ is an irreducible component of dimension 1, there exists an irreducible component C of $X_{(\bar{y})}$ such that $p(C) = D$. We of course have $\dim(C) \geq 1$. Let $z \in C$ be a point which is not the generic point of C. The point $p(z)$ cannot be the generic point of D because otherwise the generic fiber of p would not be discrete. Thus $p(z)$ is the closed point of C. The fact that $p^{-1}(y)$ is discrete then implies that z can only be the closed point of C. The integral local scheme C therefore has exactly two points : $\dim(C) = 1$. □

Lemme 2.1.8. *Let X be a Noetherian local scheme. If X has an irreducible component of dimension* 1*, then its completion $\widehat{X}$ also does, and the converse is true if X is universally catenary.*

Démonstration. Let's start with the case where X is integral. If $\dim(X) = 1$, as at the beginning of the proof of proposition 2.1.6, $\dim(\widehat{X}) = \dim(X) = 1$ and all irreducible components of $\widehat{X}$ are of dimension 1. Conversely, if X is universally catenary, according to theorem 1.2.3, the irreducible components of $\widehat{X}$ all have the same dimension. If one of them is of dimension 1, the scheme $\widehat{X}$ is also of dimension 1, and then $\dim(X) = \dim(\widehat{X}) = 1$.

In the general case, let X_i be the irreducible components of X. For all i, the fiber product $X_i \times_X \widehat{X}$ can be identified with $\widehat{X}_i$ (see [**SGA 1** IV 3]). It is a fact that the irreducible components of the different $\widehat{X}_i$ are exactly the irreducible components of $\widehat{X}$: they are irreducible closed parts covering $\widehat{X}$ and none of them is contained in another (this follows from the fact that each irreducible component of $\widehat{X}_i$ dominates X_i). It is then obvious that the statement for X results from the statement for the integral local schemes X_i. □

Let us show the converse stated in proposition 2.1.6. As observed above (2.1.4), it suffices to show that if X is an integral Noetherian local scheme universally catenary with closed point y and generic point x (i.e., $X = \overline{\{x\}}$), and if x is an immediate étale generization of y, then $\dim(X) = 1$. According to proposition 2.1.5, the strict Henselization of X at a geometric point above y has an irreducible component of dimension 1, which, according to lemma 2.1.7, is equivalent to saying that the Henselization X^{h} of X has an irreducible component of dimension 1. The completion $\widehat{X}$ of X also being that of X^{h}, the easy direction of lemma 2.1.8 applied to X^{h} shows that $\widehat{X}$ has an irreducible component of dimension 1. The converse of this lemma applied to the universally catenary scheme X shows that X has an irreducible component of dimension 1 ; so we have $\dim(X) = 1$ and y is an immediate Zariski specialization of x.

One can read the étale specializations of a point x of X in the completion of X at x :

Proposition 2.1.9. *Let X be a Noetherian scheme. Let x and y be two points of X. We assume that $y \in \overline{\{x\}}$. Let $c\colon \widehat{X}_{(y)} \to X_{(y)}$ denote the completion morphism. The point y is an immediate étale specialization of x if and only if $c^{-1}(\overline{\{x\}})$ has an irreducible component of dimension* 1.

Démonstration. We can assume that $X = \overline{\{x\}}_{(y)}$. The point y is an immediate étale specialization of X if and only if $X_{(\bar{y})}$ has an irreducible component of dimension 1, i.e., according to lemma 2.1.7 that $X^{\mathrm{h}}_{(y)}$ has one. We want to show that this is equivalent to the completion $\widehat{X}$ having one.

If we make the additional hypothesis that X is quasi-excellent (hence universally catenary according to proposition 1.3.2), the desired equivalence results from lemma 2.1.8. Let's show this equivalence without the quasi-excellence hypothesis. If $\dim(X) = 0$, $X_{(\bar{y})}$ and $\widehat{X}$ are also of dimension 0, so neither of these schemes has an irreducible component of dimension 1. If $\dim(X) = 1$, all irreducible components of $X_{(\bar{y})}$ and $\widehat{X}$ are of dimension 1. We can therefore assume $\dim(X) = \dim(X_{(\bar{y})}) = \dim(\widehat{X}) \geq 2$. The non-existence of an irreducible component of dimension 1 of $X_{(\bar{y})}$ (resp. of $\widehat{X}$) is equivalent to saying that all irreducible components of $X_{(\bar{y})}$ (resp. of $\widehat{X}$) are of dimension ≥ 2. The desired equivalence then results from **XX**-3.3 (ii) $\Leftrightarrow$ (iii) applied to the inclusion of the closed point of X. $\square$

Définition 2.1.10. We call a **dimension function** on X any function $\delta : X \to \mathbf{Z}$ such that for any immediate étale specialization $x \rightsquigarrow y$ between points of X, we have

$$\delta(y) = \delta(x) - 1.$$

The notion of a dimension function does not see nilpotent elements : δ is a dimension function on X if and only if it induces a dimension function on the reduced subscheme $X^{\text{réd}}$. Furthermore, if $U \overset{i}{\hookrightarrow} X$ is an étale morphism and δ is a dimension function on X, $\delta \circ i$ defines a dimension function on U. More precisely, the set of dimension functions on étale X-schemes defines an étale sheaf on X. The difference between two dimension functions on X is a function invariant under any specializations, hence a locally constant function. We will show later that if X is quasi-excellent, dimension functions exist locally for the étale topology on X so that dimension functions form a $\mathbf{Z}$-torsor over the étale site.

Proposition 2.1.11. *Let $f : X \to Y$ be a morphism between Noetherian schemes. Suppose given $\bar{x}$ and $\bar{x}'$ two geometric points of X. Let $\bar{y}$ (resp. $\bar{y}'$) be the geometric point of Y above which $\bar{x}$ (resp. $\bar{x}'$) is located. To any étale specialization $\bar{x} \rightsquigarrow \bar{x}'$ is canonically associated an étale specialization $\bar{y} \rightsquigarrow \bar{y}'$. If f is quasi-finite, the specializations $\bar{x} \rightsquigarrow \bar{x}'$ and $\bar{y} \rightsquigarrow \bar{y}'$ have the same codimension.*

Démonstration. The first part of the statement is trivial (see also **XVII**-2.3). We only need to show the equality of codimensions in the case where f is quasi-finite. For that, according to Zariski's Main Theorem, we can assume that f is finite and that X and Y are strictly Henselian local with respective closed points $\bar{x}'$ and $\bar{y}'$. We can further assume that X and Y are integral with respective generic points x and y where x and y are the respective images of the geometric points $\bar{x}$ and $\bar{y}$. Stating the equality of codimensions of $\bar{x} \rightsquigarrow \bar{x}'$ and $\bar{y} \rightsquigarrow \bar{y}'$ then amounts to saying that X and Y have the same dimension, which results from the "going-up" theorem (cf. [**Matsumura, 1980a**, 13.C]). $\square$

Corollaire 2.1.12. *Let $f : X \to Y$ be a quasi-finite morphism between Noetherian schemes. If $\delta : Y \to \mathbf{Z}$ is a dimension function on Y, then $f^{\star}\delta := \delta \circ f : X \to \mathbf{Z}$ is a dimension function on X.*

2.2. Dimension functions and universal catenarity. The purpose of this paragraph is to prove the following result.

Théorème 2.2.1. *A Noetherian scheme is universally catenary if and only if it has a dimension function locally for the Zariski topology.*

The theorem results from the conjunction of corollaire 2.2.4 and proposition 2.2.6 below.

Proposition 2.2.2. *Let X be an integral universally catenary scheme. The function $\delta : X \to \mathbf{Z}$ defined by $\delta(x) = -\dim(\mathscr{O}_{X,x})$ is a dimension function on X.*

Démonstration. By virtue of proposition 2.1.6, since X is universally catenary, it suffices to show that $\delta(y) = \delta(x) - 1$ for any immediate Zariski specialization $x \rightsquigarrow y$. Since X is integral catenary, we have

$$\dim(\mathscr{O}_{X,y}) = \dim(\mathscr{O}_{X,x}) + \dim(\mathscr{O}_{\overline{\{y\}},x}) = \dim(\mathscr{O}_{X,x}) + 1.$$

$\square$

Remarque 2.2.3. If X is not assumed integral, the function $\delta(x) = -\dim(\mathscr{O}_{X,x})$ is not necessarily a dimension function, as shown by the example where X is obtained by gluing at a point a line and a plane.

Corollaire 2.2.4. *Every universally catenary scheme admits dimension functions locally for the Zariski topology.*

Démonstration. Let X be a universally catenary scheme. Let $x \in X$. It is a matter of showing that there exists an open neighborhood of x that can be endowed with a dimension function. The topological space X is a union of its irreducible components $X_1, ..., X_n$. By replacing X with the open complement of the components X_i not containing x, we can assume that x belongs to all components X_i. For all $1 \leq i \leq n$, let $\mathscr{F}_i$ be the set of dimension functions on X_i. According to proposition 2.2.2, this set is non-empty and is a torsor under $\mathbf{Z}$. We choose an element $\delta_i \in \mathscr{F}_i$ which is 0 at the point x. For all $1 \leq i,\ j \leq n$, the function $\delta_i - \delta_j$ is locally constant on $X_i \cap X_j$ and is 0 at the point x. Let $F_{i,j}$ be the closed subset of X, the union of the connected components of $X_i \cap X_j$ not containing x. Let U be the complement in X of the union of the $F_{i,j}$. The functions δ_i glue together to form a dimension function on U. □

Let us prove a partial converse of corollaire 2.2.4.

Lemme 2.2.5. *A Noetherian scheme that has a dimension function locally for the Zariski topology is catenary.*

Démonstration. To show catenarity, we can assume that the scheme S has a dimension function δ. Suppose that $X \subset Y$ are irreducible closed subsets with generic points respectively x and y. We choose a chain of immediate Zariski specializations $y = x_0 \rightsquigarrow x_1 \rightsquigarrow \cdots \rightsquigarrow x_d = x$ of maximal length. By definition of codimension, we have $\operatorname{codim}(X, Y) = d$ and by definition of dimension functions, taking into account the easy direction of proposition 2.1.6, we obtain $\delta(x) = \delta(y) - d$, hence $\operatorname{codim}(X, Y) = \delta(y) - \delta(x)$.

Now, if $X \subset Y \subset Z$ are irreducible closed subsets, we have :

$$\begin{aligned} \delta(y) - \delta(x) &= \operatorname{codim}(X, Y), \\ \delta(z) - \delta(y) &= \operatorname{codim}(Y, Z), \\ \delta(z) - \delta(x) &= \operatorname{codim}(X, Z). \end{aligned}$$

From this we deduce $\operatorname{codim}(X, Z) = \operatorname{codim}(X, Y) + \operatorname{codim}(Y, Z)$, which proves catenarity. □

Thanks to theorem 1.2.3, we can replace « catenary » with « universally catenary » in lemma 2.2.5 :

Proposition 2.2.6. *A Noetherian scheme that has a dimension function locally for the Zariski topology is universally catenary.*

Démonstration. We can assume that S is local and endowed with a dimension function δ, which induces a dimension function δ^{h} on the Henselization S^{h}, which is therefore catenary according to lemma 2.2.5, then universally catenary thanks to proposition 1.3.1 and finally formally catenary according to theorem 1.2.3.

Let's show that S is formally catenary. Let Z be an integral closed subscheme of S. It is a matter of showing that Z is formally equidimensional. The local schemes Z and Z^{h} having the same completion, it suffices to show that the irreducible components C of Z^{h} are formally equidimensional and all of the same dimension. Since Z^{h} is a closed subscheme of S^{h} which is formally catenary, every irreducible component C of Z^{h} is indeed formally equidimensional. Let us now express the dimension of C using dimension functions. Let s be the closed point of S, η_C (resp. η_Z) the generic point of C (resp. Z). We have $\dim(C) = \delta^{\mathrm{h}}(\eta_C) - \delta^{\mathrm{h}}(s) = \delta(\eta_Z) - \delta(s) = \dim(Z)$. The dimension $\dim(C)$ is therefore independent of C. We have thus shown that Z is formally equidimensional. Finally, S is formally catenary and theorem 1.2.3 shows that S is universally catenary. □

2.3. Local existence for the étale topology. In this paragraph we will prove the following theorem.

Théorème 2.3.1. *Every quasi-excellent scheme possesses dimension functions locally for the étale topology.*

A repeated application of the following lemma (a variant of the argument in corollaire 2.2.4) shows that if the statement of the theorem is true for the irreducible components of a Noetherian scheme X, then the theorem also holds for X. Later, we can thus assume that X is integral.

LEMME 2.3.2. *Let X be a Noetherian scheme whose underlying topological space is a union of two closed subschemes X_1 and X_2. Let $\overline{x}$ be a geometric point of $X_1 \cap X_2$. We assume that for all $i \in \{1,2\}$, there exists an étale neighborhood U_i of $\overline{x}$ in X_i such that U_i admits a dimension function. Then, there exists an étale neighborhood U of $\overline{x}$ in X such that U admits a dimension function.*

Démonstration. For each $i \in \{1,2\}$, we choose an étale neighborhood U_i of $\overline{x}$ in X_i such that U_i admits a dimension function δ_i. We are given a distinguished geometric point $\overline{u}_i$ above $\overline{x}$ and we can assume that $\delta_i(u_i) = 0$. According to [**ÉGA** IV$_4$ 18.1.1], by replacing U_i with an open neighborhood of u_i, we can assume that there exists an étale morphism $\widetilde{U_i} \to X$ and an isomorphism $\widetilde{U_i} \times_X X_i \simeq U_i$. We can form the fiber product $V = \widetilde{U_1} \times_X \widetilde{U_2}$. Let $\pi : V \to X$ be the projection and $\overline{v}$ a geometric point of V above $\overline{u}_1$ and $\overline{u}_2$. For each $i \in \{1,2\}$, the projection from V to the factor $\widetilde{U_i}$ induces an étale morphism $\pi^{-1}(X_i) \to U_i$. By composition with this étale morphism, the dimension function δ_i on U_i induces a dimension function $\tilde{\delta}_i$ on the closed subscheme $\pi^{-1}(X_i)$ of V and it satisfies $\tilde{\delta}_i(v) = 0$. These dimension functions $\tilde{\delta}_i$ for $i \in \{1,2\}$ glue on the open set U complementary in V to the union of the connected components of $\pi^{-1}(X_1 \cap X_2)$ not containing v. □

Before treating the case of integral schemes, let's start with that of normal schemes :

PROPOSITION 2.3.3. *Let X be a normal quasi-excellent scheme. The function $\delta : X \to \mathbf{Z}$ defined by $\delta(x) = -\dim(\mathscr{O}_{X,x})$ is a dimension function.*

Démonstration. We can further assume that X is local. Let Y be its Henselization and $h : Y \to X$ be the Henselization morphism. According to theorem **I**-8.1 and the subsequent comments, Y is also quasi-excellent. According to proposition 1.3.2, Y is universally catenary. Moreover, since the morphism $Y \to X$ is regular, the normality of X implies that of Y (cf. [**ÉGA** IV$_2$ 6.5.4]). The local scheme Y is therefore integral and universally catenary, and the opposite of the codimension defines a dimension function $\delta' : Y \to \mathbf{Z}$. Since an immediate étale specialization between points of X is, by definition, lifted to an immediate étale specialization of points of Y, to show that δ is a dimension function on X, it suffices to show that for every $y \in Y$, if we denote $x = h(y)$, we have $\delta(x) = \delta'(y)$, which results from proposition 1.2.2. □

Let us return to the case of theorem 2.3.1 where X is assumed integral and quasi-excellent, and let Y be its normalization. The morphism $p : Y \to X$ is finite and surjective, and therefore of universal cohomological descent. Let δ be a dimension function on Y; its existence is guaranteed by proposition 2.3.3. Let p_1 and p_2 be the two projections $Y \times_X Y \to Y$. According to corollaire 2.1.12 applied to the morphisms p_1 and p_2, it follows that $p_i^\star \delta := \delta \circ p_i$ for $i \in \{1,2\}$ are two dimension functions on $Y \times_X Y$. The difference $p_1^\star\delta - p_2^\star\delta : Y \times_X Y \to \mathbf{Z}$ is a locally constant function which defines a 1-Čech cocycle, hence a class $[p_1^\star\delta - p_2^\star\delta]$ in $\mathrm{H}^1_{\text{Čech}}(Y \to X, \mathbf{Z})$. According to the theory of cohomological descent, there exists a natural injection

$$\mathrm{H}^1_{\text{Čech}}(Y \to X, \mathbf{Z}) \hookrightarrow \mathrm{H}^1(X, \mathbf{Z}).$$

The class $[p_1^\star\delta - p_2^\star\delta]$ therefore defines an isomorphism class of étale $\mathbf{Z}$-torsors on X. It then immediately follows from the following proposition that X admits a dimension function locally for the étale topology :

PROPOSITION 2.3.4. *Let U be an étale scheme over X (quasi-excellent). The vanishing of the class $[p_1^\star\delta - p_2^\star\delta]|_U$ in $\mathrm{H}^1(U, \mathbf{Z})$ implies the existence of a dimension function on U.*

Démonstration. Using the compatibility of constructions with étale base change $U \to X$, we can assume that $U = X$. The vanishing of $[p_1^\star\delta - p_2^\star\delta]$ in $\mathrm{H}^1_{\text{Čech}}(Y \to X, \mathbf{Z}) \hookrightarrow \mathrm{H}^1(X, \mathbf{Z})$ means that there exists a *locally constant* function $\gamma : Y \to \mathbf{Z}$ such that $p_1^\star\delta - p_2^\star\delta = p_1^\star\gamma - p_2^\star\gamma$. In other words, by replacing δ with $\delta - \gamma$, we can assume that $p_1^\star\delta = p_2^\star\delta$. Thus, $\delta : Y \to \mathbf{Z}$ descends to a function $\delta' : X \to \mathbf{Z}$.

To conclude, it is a matter of showing that if $p : Y \to X$ is a finite surjective morphism between quasi-excellent schemes, that $\delta' : X \to \mathbf{Z}$ is a function and $\delta = \delta' \circ p$, then δ' is a dimension function on X if δ is a dimension function on Y. For this statement, we can assume that X is Henselian quasi-excellent local, thus universally catenary (cf. proposition 1.3.2). According to corollaire 2.2.4, there exists a dimension function δ'' on X. We have two dimension functions δ and $p^\star\delta'' := \delta'' \circ p$ on Y. The difference $\delta - p^\star\delta'' : Y \to \mathbf{Z}$ is therefore a locally constant function. As $\delta - p^\star\delta'' = (\delta' - \delta'') \circ p$ and p is finite surjective, we easily deduce that the function $\delta' - \delta'' : X \to \mathbf{Z}$ is locally constant. As δ'' is a dimension function, we deduce that δ' is a dimension function.

□

2.4. Global existence of dimension functions. Following [**ÉGA** 0_{IV} 14.2.1], a Noetherian scheme X is said to be **equicodimensional** if its closed points all have the same codimension (which is then equal to $\dim(X)$).

Exemple 2.4.1. Equidimensional schemes of finite type over a field k or over $\mathbf{Z}$ are equicodimensional : it is classical that in this situation, we have $\dim(X) = \dim(\mathscr{O}_{X,x})$ for every closed point x. Local schemes are equicodimensional because they have a unique closed point. If $S = \mathrm{Spec}(R)$ is a trait with uniformizer π, the scheme $\mathbf{A}^1_S$ is not equicodimensional. Indeed, there exists a closed point of $\mathbf{A}^1_S$ above the generic point of S : it suffices to write $\mathbf{A}^1_S = \mathrm{Spec}(R[t])$ and consider $\mathfrak{m} = (\pi t - 1)$, which is a maximal ideal of residual field $\mathrm{Frac}(R)$.

The following lemma is inspired by [**ÉGA** 0_{IV} 14.3.3] (i).

Lemme 2.4.2. *Let X be a Noetherian equidimensional catenary scheme whose irreducible components are equicodimensional. For every $x \in X$, we have*

$$\dim(X) = \dim \overline{\{x\}} + \dim(\mathscr{O}_{X,x}).$$

Remarque 2.4.3. In particular, this equality is satisfied for every integral local catenary scheme. According to [**Matsumura, 1989**, th. 31.4], if X is integral Noetherian local and if for every $x \in X$, we have $\dim(X) = \dim \overline{\{x\}} + \dim(\mathscr{O}_{X,x})$, then X is catenary.

Lemma 2.4.2 and proposition 2.2.2 imply the following result.

Corollaire 2.4.4. *Let X be an integral Noetherian scheme, equicodimensional and universally catenary. The function $\delta : X \to \mathbf{Z}$ defined by $\delta(x) = \dim \overline{\{x\}}$ is a dimension function on X.*

The conclusions of the corollary are false if X is not equicodimensional. For example, let $S = \mathrm{Spec}(R)$ be a trait with uniformizer π and $X = \mathbf{A}^1_S = \mathrm{Spec}(R[t])$. If we denote x the closed point of X corresponding to the maximal ideal $(\pi t - 1)$ and η the generic point of $\mathbf{A}^1_S$, then the specialization $\eta \rightsquigarrow x$ is immediate and yet $\dim \overline{\{x\}} = 0$ and $\dim \overline{\{\eta\}} = 2$.

Corollaire 2.4.5. *Let X be a scheme which is either of finite type over a field, or of finite type over $\mathbf{Z}$, or a universally catenary local scheme. The function defined by $\delta(x) = \dim \overline{\{x\}}$ is a dimension function on X.*

Démonstration. The scheme X is universally catenary. According to corollaire 2.4.4, the function δ is a dimension function on each irreducible component of X. This function is globally defined and is therefore a dimension function on X. □

2.5. Induced dimension function. Let $Y \to X$ be a morphism of schemes and δ_X a dimension function on X. In certain cases we can construct an induced dimension function δ_Y on Y. We admit the following proposition.

(i) Gabber notes that proposition [**ÉGA** 0_{IV} 14.3.3] is false. Assertions a, c and d of *loc. cit.* are equivalent to each other and imply b but are not equivalent to it. One must replace b by the condition " X is catenary equidimensional and its irreducible components are equicodimensional ". Gabber gives as a counter-example the spectrum of the localization of $k[x, y, z, w]/(xz, xw)$ at the complement of the union of the prime ideals $(x-1, y)$ and (x, z, w) with k a field. The same error was noted, independently, by Huayi Chen (letter to Luc Illusie dated 2005-9-26).

Proposition 2.5.1 ([**Matsumura, 1980a**], 14.C). *Let X be an integral universally catenary Noetherian scheme, Y an integral scheme and $Y \to X$ a dominant morphism of finite type. Let $k(X)$ and $k(Y)$ be the respective fraction fields of X and Y, let y be a point of Y and x its image in X, and let $k(y)$ and $k(x)$ be their residual fields. We have*

$$\dim(\mathscr{O}_{Y,y}) - \deg.\mathrm{tr}.(k(Y)/k(X)) = \dim(\mathscr{O}_{X,x}) - \deg.\mathrm{tr}.(k(y)/k(x)).$$

Corollaire 2.5.2. *Let X be a Noetherian scheme which possesses a dimension function δ_X and $f : Y \to X$ a morphism of finite type. The function $\delta_Y : Y \to \mathbf{Z}$ defined by*

$$\delta_Y(y) = \delta_X(f(y)) + \deg.\mathrm{tr}.(k(f(y))/k(y))$$

is a dimension function on Y.

Démonstration. We can assume that X and Y are integral and that f is dominant. According to proposition 2.2.6, X is universally catenary and according to proposition 2.2.2, $x \mapsto -\dim(\mathscr{O}_{X,x})$ is a dimension function on X. Since dimension functions form a $\mathbf{Z}$-torsor, we can assume that $\delta_X(x) = -\dim(\mathscr{O}_{X,x})$ for every $x \in X$.

Corollaire 2.5.1 shows that $\delta_Y(y) = -\dim(\mathscr{O}_{Y,y}) + \deg.\mathrm{tr}.(k(Y)/k(X))$ and proposition 2.2.2 shows that $y \mapsto -\dim(\mathscr{O}_{Y,y})$ is a dimension function on Y. Thus, δ_Y is a dimension function on Y. □

Before establishing the functoriality of dimension functions with respect to regular morphisms between excellent schemes, let's prove a base change statement for regular morphisms in étale cohomology. This lemma is a simple consequence of Popescu's theorem **I**-10.3 and the smooth base change theorem [**SGA 4** xvi 1.2].

Lemme 2.5.3. *Let*

$$\begin{tikzcd} T' \arrow[r, "g'"] \arrow[d, "f'"'] & T \arrow[d, "f"] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$

be a Cartesian diagram of schemes, n an integer invertible on S and $\mathscr{F}$ an étale sheaf of $\mathbf{Z}/n\mathbf{Z}$-modules on T. Suppose that f is coherent and that g is a regular morphism between Noetherian schemes. The natural base change map

$$g^\star \mathrm{R} f_\star(\mathscr{F}) \xrightarrow{\sim} \mathrm{R} f'_\star g'^\star(\mathscr{F})$$

is an isomorphism.

Démonstration. The question being local on S and S', we can assume that S and S' are affine. According to Popescu's theorem, there exists a filtered ordered set I (non-empty) and a family of affine schemes S_i indexed by I, such that S_i is smooth over S for all $i \in I$ and that $S' = \lim_i S_i$. Therefore, for all $i \in I$, there exists a commutative diagram with Cartesian squares

$$\begin{tikzcd} T' \arrow[r, "h'_i"] \arrow[d, "f'"] & T_i \arrow[r, "g'_i"] \arrow[d, "f_i"] & T \arrow[d, "f"] \\ S' \arrow[r, "h_i"] & S_i \arrow[r, "g_i"] & S \end{tikzcd}$$

We conclude thanks to the following sequence of isomorphisms for all $q \geq 0$:

$$\begin{aligned} \mathrm{R}^q f'_\star g'^\star(\mathscr{F}) &\xleftarrow{\sim} \mathrm{colim}_i h_i^\star \mathrm{R}^q f_{i\star} g_i'^\star(\mathscr{F}) \\ &\xleftarrow{\sim} \mathrm{colim}_i g^\star \mathrm{R}^q f_\star(\mathscr{F}) \\ &\xleftarrow{\sim} g^\star \mathrm{R}^q f_\star(\mathscr{F}) \end{aligned}$$

The first of these isomorphisms results from the limit theorem [**SGA 4** vii 5.11], and the second from the base change theorem for the smooth morphism g_i [**SGA 4** xvi 1.2]. □

We now prove that a regular morphism between excellent schemes allows to induce dimension functions.

Proposition 2.5.4. *Let $f : Y \to X$ be a regular morphism between quasi-excellent schemes and δ_X a dimension function on X. The function $\delta_Y : Y \to \mathbf{Z}$ defined by*

$$\delta_Y(y) = \delta_X(f(y)) - \dim(\mathscr{O}_{Y_{f(y)},y})$$

is a dimension function on Y.

Démonstration. As the verification is local, there is no harm in assuming X and Y strictly local and f local. The quasi-excellent schemes X and Y are then excellent (cf. proposition 1.3.2). Let δ be a dimension function on Y; its existence is guaranteed by theorem 2.2.1. It suffices to show that $\delta_Y - \delta$ is a constant function on Y. The fibers of f are regular, hence universally catenary according to 1.1.3. Proposition 2.2.2 shows that the function which assigns to $y \in Y - \dim(\mathscr{O}_{Y_{f(y)},y})$ induces a dimension function on each of the fibers of f. The function $\delta_Y - \delta$ is therefore locally constant on each fiber of f. It results from lemma 2.5.3 that these fibers are connected : indeed, we have $\mathrm{H}^0(f^{-1}(x), \mathbf{Z}/n\mathbf{Z}) = \mathrm{H}^0(x, \mathbf{Z}/n\mathbf{Z}) = \mathbf{Z}/n\mathbf{Z}$ for all $x \in X$ and all integer n invertible on X. The function $\delta_Y - \delta$ is therefore constant on the fibers of f and descends to X. It suffices to show that $\gamma = \delta_Y - \delta$ is locally constant on X. One way to calculate the value of γ at a point s of X is to consider the generic point η_s of the regular connected scheme $f^{-1}(s)$, such that $\gamma(s) = \delta_X(s) - \delta(\eta_s)$. Let $s' \rightsquigarrow s$ be an immediate Zariski specialization between two points of X. We need to show that $\gamma(s) = \gamma(s')$. Given that δ_X and δ are dimension functions on X and Y respectively, to show this, it suffices to know that $\eta_{s'}$ is an immediate specialization of η_s. To show this, by replacing X with the localization at s of the closure of s', we can assume that X is local integral of dimension 1, with generic point s' and closed point s. It is then a matter of showing that the fiber $f^{-1}(s)$ is of codimension 1 in Y, which easily results from the *Hauptidealsatz*. □

2.6. Counterexample.

2.6.1. Recall the example from [**ÉGA** IV$_2$ 5.6.11] of a catenary scheme which is not universally catenary. Let k_0 be a field and k a purely transcendental extension of k_0 of infinite transcendence degree. Let $S = k[X]_{(X)}$ be the localization of the ring of the affine line over k at the origin and $V = S[T]$. The maximal ideals $\mathfrak{m} = (X, T)$ and $\mathfrak{m}' = (XT - 1)$ of V are respectively of height 2 and 1, and there exists an isomorphism $\phi : V/\mathfrak{m} \xrightarrow{\sim} V/\mathfrak{m}'$. Let v and v' be the closed points of $\mathrm{Spec}(V)$ corresponding to the maximal ideals $\mathfrak{m}$ and $\mathfrak{m}'$. Let $C = \{f \in V | \phi(f \mod \mathfrak{m}) = f \mod \mathfrak{m}'\}$. This is a subring of V which is not of finite type over k. The morphism $\mathrm{Spec}(V) \to \mathrm{Spec}(C)$ is finite and induces an isomorphism over the dense open set $\mathrm{Spec}(C) - \{c\}$ where c is the closed point of C corresponding to the maximal ideal $\mathfrak{n} = \mathfrak{m} \cap \mathfrak{m}' \subset C$. The topological space $\mathrm{Spec}(C)$ can be identified with the quotient of $\mathrm{Spec}(V)$ by the equivalence relation that identifies v and v'.

Proposition 2.6.2. *The scheme* $\mathrm{Spec}(C)$ *is Noetherian, quasi-excellent, catenary but not universally catenary. The closed point corresponding to the maximal ideal $\mathfrak{n}$ of C is an immediate étale specialization but not an immediate Zariski specialization of the generic point of* $\mathrm{Spec}(C)$.

Démonstration. The Noetherian character is shown in [**ÉGA** IV$_2$ 5.6.11] and the quasi-excellent character in [**ÉGA** IV$_2$ 7.8.4 (ii)]. The scheme $\mathrm{Spec}(C)$ is catenary according to corollaire 1.2.4 because its dimension is 2. The points v and v' are identified with the two closed points of $\mathrm{Spec}(V \otimes_C C_{\mathfrak{n}})$ and the corresponding localizations $\mathrm{Spec}(V_{\mathfrak{m}})$ and $\mathrm{Spec}(V_{\mathfrak{m}'})$ are of dimensions 2 and 1 respectively, which violates the last condition of theorem 1.2.3 : the local ring $C_{\mathfrak{n}}$ is not universally catenary.

The local ring $C_{\mathfrak{n}}$ being of dimension 2, the point $c \in \mathrm{Spec}(C)$ is not an immediate Zariski specialization of the generic point. However, it is an immediate étale specialization thanks to proposition 2.1.11 applied to the finite morphism $\mathrm{Spec}(V) \to \mathrm{Spec}(C)$ and to the obvious immediate étale specialization $\bar{\eta}_{\mathrm{Spec}(V)} \rightsquigarrow \bar{v}'$ of geometric points of $\mathrm{Spec}(V)$. □

PRESENTATION XIII

The Finitude Theorem

Fabrice Orgogozo

1. Introduction

1.1. The purpose of this presentation is to demonstrate the following theorem (**Intro.**-1).

THÉORÈME 1.1.1. *Let X be a Noetherian* quasi-excellent *scheme* (***I***-*2.10*), $f : Y \to X$ *a finite type morphism,* $n \geq 1$ *an integer invertible on* X *and* $\mathcal{F}$ *a constructible sheaf of* $\mathbf{Z}/n\mathbf{Z}$*-modules on* Y*. Then :*

(i) *For every integer* $q \geq 0$*, the sheaf* $\mathrm{R}^q f_\star \mathcal{F}$ *is constructible.*

(ii) *There exists an integer* N *such that* $\mathrm{R}^q f_\star \mathcal{F} = 0$ *for* $q \geq N$*.*

1.1.2. Equivalently, the morphism $\mathrm{R}f_\star : \mathsf{D}^+(Y_{\text{ét}}, \mathbf{Z}/n\mathbf{Z}) \to \mathsf{D}^+(X_{\text{ét}}, \mathbf{Z}/n\mathbf{Z})$ induces a morphism $\mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(Y_{\text{ét}}, \mathbf{Z}/n\mathbf{Z}) \to \mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X_{\text{ét}}, \mathbf{Z}/n\mathbf{Z})$ between the subcategories of complexes with bounded and constructible cohomology.

1.2. Remarks.

1.2.1. *Organization of the Presentation.* The above statement is the conjunction of a *constructibility* result (i) and an *annulation* (vanishing) result (ii). In § 2, we present a proof of constructibility that does not require the strong form of the uniformization theorem but only the weak form (**VII**-1.1). The key additional ingredients are the absolute purity theorem, Deligne's generic constructibility theorem, and the hyper-base change formula. In paragraph 2.3, we provide a proof of an annulation result for *finite-dimensional* schemes, which completes the proof of Theorem 1.1.1 for these schemes. The general case is treated in § 3, relying on the uniformization theorem prime to ℓ (**IX**-1.1), where ℓ is a prime number dividing n. Finally, we sketch extensions of this result, first to the case of ℓ-adic coefficients, where ℓ is a prime number invertible on the schemes considered, then to the case of stacks (as coefficients).

1.2.2. *Terminology and Notation.* We will say that a complex $\mathcal{K} \in \mathsf{Ob}\, \mathsf{D}^+(Y_{\text{ét}}, \Lambda)$, where Λ is a finite ring, is **constructible** if it belongs to $\mathsf{Ob}\, \mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(Y_{\text{ét}}, \Lambda)$, meaning that its cohomology sheaves are constructible and vanish in high degrees. When $n \geq 1$ is fixed and this does not seem to create confusion, we denote Λ the ring $\mathbf{Z}/n\mathbf{Z}$. Similarly, given a complex $\mathcal{K}$ on a scheme X, we will often still denote $\mathcal{K}$ its inverse images on different X-schemes.

2. Constructibility via Weak Local Uniformization

In this section, we prove 1.1.1 (i), adopting its notation.

2.1. Reductions. The following reductions are standard : cf., for example, [**SGA 4** XVI 4.5].

2.1.1. *Reduction to the case where the sheaf* $\mathcal{F}$ *is constant.* According to [**SGA 4** IX 2.14 (ii)], the sheaf $\mathcal{F}$ injects into a finite sum $\mathcal{G} = \bigoplus_{i \in I} g_{i\star} C_i$ of direct images by finite morphisms g_i of sheaves of constant constructible $\mathbf{Z}/n\mathbf{Z}$-modules C_i. We can assume $\mathcal{F} = \mathcal{G}$. This follows partly from the fact that a subquotient of a constructible sheaf is constructible[(i)] and partly from the long exact cohomology sequence associated with the triangle

$$\mathrm{R}f_\star \mathcal{F} \to \mathrm{R}f_\star \mathcal{G} \to \mathrm{R}f_\star(\mathcal{G}/\mathcal{F}) \xrightarrow{+1} .$$

Finally, we can assume $\mathcal{F}$ is constant constructible because we can assume the index set I to be a singleton and the equality $\mathrm{R}f_\star(g_\star C) = \mathrm{R}(f \circ g)_\star C$, where g is a finite morphism, allows us to assume

[(i)] To see this, one can use the fact that a sheaf is constructible if and only if it is Noetherian, cf. [**SGA 4** IX 2.9 (i)].

$g = \mathrm{Id}$. Decomposing n into a product, we reduce to the case where $\mathcal{F}$ is a constant sheaf $\mathbf{F}_\ell$, with the prime number ℓ being invertible on X (cf., for example, [**SGA 4½** [Th. finitude] 2.2 b)]).

2.1.2. *Reduction to the case where the morphism f is an open immersion.* A sheaf on scheme X being constructible if and only if it is locally constructible for the Zariski topology ([**SGA 4** IX 2.4 (ii)]), we can assume X is affine. Here we use the trivial fact that the formation of direct images commutes with base change by a Zariski open set. We can also assume Y is affine ; this results, for example, from the sheaf-theoretic analogue

$$E_1^{p,q} = \mathrm{R}^q f_{p\star}(\mathcal{F}_{|Y_p}) \Rightarrow \mathrm{R}^{p+q} f_\star \mathcal{F}$$

of the Leray spectral sequence ([**Deligne, 1974**, 5.2.7.1]), where $f_p : Y_p \to X$ are derived from f and a Zariski hypercover $Y_\bullet \to Y$. The morphism $f : Y \to X$ is then affine, hence quasi-projective, and since the constructibility theorem is known for proper morphisms ([**SGA 4** XIV 1.1]), we can assume that f is a dominant *open immersion*. (One could also use Nagata's compactification theorem when f is separated.) As is customary, we will henceforth denote the morphism f as $j : U \to X$.

2.2. End of the Proof of Theorem 1.1.1 (i).

2.2.1. Let $q \geq 0$ be an index for which we want to show that the sheaf $\mathrm{R}^q j_\star \mathbf{F}_\ell$ is constructible. We recall that j is an open immersion $U \hookrightarrow X$ and ℓ is a prime number invertible on X. It follows from the constructibility criterion [**SGA 4** IX 2.4(v)] that it suffices to show that for any closed immersion $g : Z \hookrightarrow X$, the sheaf $g^\star \mathrm{R}^q j_\star \mathbf{F}_\ell$ is constructible *on a dense open subset* of Z. The local uniformization theorem (**VII**-1.1), combined with the classical method for constructing hypercovers ([**Deligne, 1974**, § 6.2]), has the following immediate corollary.

Théorème 2.2.2. *There exists a hypercover for the h-topology on finite type X-schemes $\varepsilon_\bullet : X_\bullet \to X$ satisfying the following conditions :*

(i) *for each $i \geq 0$, the scheme X_i is regular ;*
(ii) *for each $i \geq 0$, the inverse image of U in each connected component of X_i is either the complement of the support of a strict normal crossing divisor, or empty.*

The h-topology on the category of finite type X-schemes is defined similarly to $\mathbf{XII_A}$-2.1.3. (See also **XV**-2.2.1 and [**Goodwillie & Lichtenbaum, 2001**].) We use the fact that, by **II**-3.2.1, a morphism in alt/X which is a covering for the alteration topology is an h-cover.

2.2.3. Let $Z_\bullet$, $g_\bullet$ and $j_\bullet$ denote the simplicial scheme and morphisms derived from Z, g and j respectively by the base change $X_\bullet \to X$. It follows from the absolute purity theorem (**XVI**-3.1.1) that the complex $g_\bullet^\star \mathrm{R} j_{\bullet\star} \Lambda$ on $Z_\bullet$ has constructible cohomology. Furthermore, it follows from the generic constructibility theorem [**SGA 4½** [Th. finitude] 1.9 (i)] — applied to the morphisms $\varepsilon_p : Z_p \to Z$ and to the complexes $g_p^\star \mathrm{R} j_{p_\star} \mathbf{F}_\ell$ — and from the spectral sequence recalled above that there exists a dense open subset of Z above which the truncation in degrees less than or equal to q of the direct image $\mathrm{R}\varepsilon_{Z_\bullet\star}(g_\bullet^\star \mathrm{R} j_{\bullet\star} \mathbf{F}_\ell)$ is *constructible*. According to the hyper-base change theorem ($\mathbf{XII_A}$-2.2.5 or $\mathbf{XII_B}$-**??**), this direct image is isomorphic to $g^\star \mathrm{R} j_\star \mathbf{F}_\ell$. The sheaf $g^\star \mathrm{R}^q j_\star \mathbf{F}_\ell$ is therefore constructible on a dense open subset of Z. Q.E.D.

2.3. Complements.

Théorème 2.3.1. *Let S be a Noetherian scheme, $f : X \to Y$ a finite type morphism between finite type S-schemes, and n an integer invertible on S. Assume one of the following two conditions is satisfied :*

(i) *the scheme S is of dimension* 1 ;
(ii) *the scheme S is* local *of dimension* 2.

Then, for any constructible sheaf $\mathcal{F}$ of $\mathbf{Z}/n\mathbf{Z}$-modules on X, the direct image sheaves $\mathrm{R}^q f_\star \mathcal{F}$ are constructible and vanish for $q \gg 0$.

Remarque 2.3.2. We will see in **XIX**-2.5 that there exists a counterexample to constructibility when S is Noetherian of dimension 2 (non-local). This results from the existence of a regular surface and a divisor possessing an infinite number of double points. It would be interesting to construct a counterexample to the preceding constructibility statement when S is *local* (Noetherian) of dimension 3, or to show that none exists.

Proof Sketch. (i) According to the generic constructibility theorem ([**SGA 4½** [Th. finitude] 1.9 (i)]), there exists a dense open subset of S above which the result is established. We can therefore assume the scheme S is *local*. It is also permissible to assume it is strictly Henselian. By restricting to its irreducible components, we can finally assume S is local *integral* (of dimension 1). Let $S' \to S$ be the normalization morphism. This is a universal homeomorphism, so we can replace S with S'. Now, this latter scheme is Noetherian regular, of dimension 1. The conclusion then follows from the constructibility theorem [**SGA 4½** [Th. finitude] 1.1] and the finiteness theorem for cohomological dimension [**SGA 4** x 3.2, 4.4]. (See also [**Illusie, 2003**, 2.4] and **XVIII$_{\mathbf{A}}$**-1.1.)

(ii) Let s be the closed point of S. Since the scheme $S-\{s\}$ is of dimension 1, the result is established over this open set. Let $\mathcal{F}$ be a constructible $\mathbf{Z}/n\mathbf{Z}$-module on X and consider a distinguished triangle

$$\mathcal{K} \to \mathcal{F} \to \mathrm{R}j_\star j^\star \mathcal{F},$$

where j is the open immersion $X - X_s \hookrightarrow X$. Note that $\mathcal{K}$ is supported in X_s and constructible if $\mathrm{R}j_\star j^\star\mathcal{F}$ is. Applying the functor $\mathrm{R}f_\star$ to the preceding triangle and using finiteness on $X - X_s$ (resp. X_s), we are reduced to showing the constructibility of direct images by the open immersions $X - X_s \hookrightarrow X$ and $Y - Y_s \hookrightarrow Y$. We then use the completion morphism $\hat{S} \to S$ and the Gabber-Fujiwara comparison theorem ([**Fujiwara, 1995**, 6.6.4]) to reduce to the case where the local scheme S is complete, and thus excellent. □

2.3.3. Let's resume the notation of Theorem 1.1.1 and assume the scheme X is finite-dimensional. Part (i) of *loc. cit.* combined with the affine Lefschetz theorem **XV**-1.1.2 [(ii)] leads to the following complement, which will be improved in the next section.

PROPOSITION 2.3.4. *Let X be a Noetherian quasi-excellent scheme of* finite dimension *and $f : Y \to X$ a finite type morphism. For any integer $n \geq 1$ invertible on X, the functor $\mathrm{R}f_\star : \mathrm{D}^+(Y_{\text{ét}}, \mathbf{Z}/n\mathbf{Z}) \to \mathrm{D}^+(X_{\text{ét}}, \mathbf{Z}/n\mathbf{Z})$ has finite cohomological dimension. In particular, it induces a functor from $\mathrm{D}^{\mathrm{b}}(Y_{\text{ét}}, \mathbf{Z}/n\mathbf{Z})$ to $\mathrm{D}^{\mathrm{b}}(X_{\text{ét}}, \mathbf{Z}/n\mathbf{Z})$.*

Démonstration. We can assume X is affine. It follows from the alternating Čech spectral sequence ([**Gabber & Ramero, 2013**, 7.2.20]) associated with a finite cover by affine open sets of Y that we can assume f is separated and then — by a new application of this spectral sequence — affine (see also **XVIII$_{\mathbf{A}}$**-1.4). Now let N be an upper bound for the dimension of the fibers of f. The cohomological dimension of the direct image functor by f is at most $d + N$, where $d = \dim(X)$. Indeed, if $\overline{x}$ is a geometric point of X, and $\mathcal{F}$ is a constructible $\mathbf{Z}/n\mathbf{Z}$-sheaf on Y, we have $(\mathrm{R}f_\star\mathcal{F})_{\overline{x}} = \mathrm{R}\Gamma(Y \times_X X_{(\overline{x})}, \mathcal{F})$. The schemes $X' = X_{(\overline{x})}$ and $Y' = Y \times_X X_{(\overline{x})}$ admit respectively the dimension functions $\delta_{X'} : x' \mapsto \dim(\overline{\{x'\}})$ and the induced function $\delta_{Y'}$ defined in **XIV**-2.5.2. Note that $\delta_{X'}$ is bounded by d and $\delta_{Y'}$ by $d + N$. It therefore follows from the affine Lefschetz theorem (in the form **XV**-1.2.4) that $\mathrm{H}^q(Y', \mathcal{F}) = 0$ for $q > d + N$. □

REMARQUE 2.3.5. We will see in **XVIII$_{\mathbf{A}}$**-1.1 that an annulation result holds under the sole assumption that X is Noetherian of finite dimension : if X is a strictly local Henselian Noetherian scheme of dimension $d > 0$ and n is invertible on X, then any open subset of X has n-cohomological dimension at most $2d - 1$.

2.3.6. *Constructibility of Direct Images in the Non-Abelian Case.* By replacing the reduction 2.1.1 with [**SGA 1** xiii § 3, 4)], the use of [**SGA 4** xiv 1.1] in 2.1.2 with [**SGA 1** xiii 6.2], the absolute purity theorem with [**SGA 1** xiii 2.4], the finiteness theorem [**SGA 4½** [Th. finitude] 1.9 (i)] with [**Orgogozo, 2003**, 2.2] and finally **XII$_{\mathbf{A}}$**-2.2.5 with the results stated in **XII$_{\mathbf{A}}$**-2.2.6.2 or **XII$_{\mathbf{B}}$-??**, we essentially obtain a proof of the following theorem by the same method.

THÉORÈME 2.4 (**XXI**-1.2). *Let X be a Noetherian* quasi-excellent *scheme, $f : Y \to X$ a finite type morphism, and L the set of prime numbers invertible on X. For every constructible ind-L-finite groupoid stack on $Y_{\text{ét}}$, the stack $f_\star\mathcal{C}$ is constructible.*

(ii) The reader will note that this reference to a later exposé does not generate a vicious circle.

3. Constructibility and Annulation via Local Uniformization Prime to ℓ

In this section, we prove Theorem 1.1.1. Constructibility (i) and annulation (ii) are established simultaneously.

3.1. Reduction to the case of an open immersion and finiteness outside a locus of given codimension.

3.1.1. As in 1.2.2, let $\Lambda = \mathbf{Z}/n\mathbf{Z}$, where n is the integer invertible on X in the statement. For each integer $c \geq 0$, consider the following property (P_c) :

> *For any quasi-excellent Noetherian scheme X, any dominant open immersion $j : U \hookrightarrow X$ and any complex $\mathcal{K} \in \operatorname{Ob} \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(U_{\text{ét}}, \Lambda)$, there exists a closed subset $T \hookrightarrow X$ of codimension strictly greater than c such that $(\mathrm{R}j_\star\mathcal{K})_{|X-T}$ belongs to $\operatorname{Ob} \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}((X-T)_{\text{ét}}, \Lambda)$.*

3.1.2. Let's verify that the conjunction of statements (P_c) for each $c \geq 0$, implies the theorem. We can assume the target scheme, say Y, is *affine*. Then, proceeding as in the proof of 2.3.4, we reduce to the case where the considered morphism $f : X \to Y$ is *separated* of finite type. According to Nagata's compactification theorem, there exists an open immersion $j : X \hookrightarrow \overline{X}$ and a proper morphism $\overline{f} : \overline{X} \to Y$ such that $f = \overline{f} j$. The composition formula $\mathrm{R}f_\star = \mathrm{R}\overline{f}_\star \mathrm{R}j_\star$ and the finiteness theorem for proper morphisms reduce us to proving the constructibility of the complex $\mathcal{K} = \mathrm{R}j_\star\mathcal{F}$. The conclusion then follows from the following lemma.

Lemme 3.1.3. *Let X be a Noetherian scheme and $\mathcal{K} \in \operatorname{Ob} \mathrm{D}^+(X_{\text{ét}}, \Lambda)$. Suppose that for every integer $c \geq 0$, there exists a closed subset T_c of codimension strictly greater than c such that $\mathscr{K}_{|X-T_c} \in \operatorname{Ob} \mathrm{D}^+((X-T_c)_{\text{ét}}, \Lambda)$. Then, $\mathscr{K} \in \operatorname{Ob} \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X_{\text{ét}}, \Lambda)$.*

Démonstration. Since the scheme X is Noetherian, its localizations are finite-dimensional, and for any sequence of closed subsets $(T_c)_{c\in\mathbf{N}}$ as in the statement, we have $X = \bigcup_c (X - T_c)$. Furthermore, since the scheme X is quasi-compact, it is covered by a *finite* number of the open sets $X - T_c$. The conclusion then follows from the fact that if U, U' are two open sets of X such that $\mathscr{K}_{|U} \in \operatorname{Ob} \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(U_{\text{ét}}, \Lambda)$, $\mathscr{K}_{|U'} \in \operatorname{Ob} \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(U'_{\text{ét}}, \Lambda)$, then we also have $\mathscr{K}_{|U\cup U'} \in \operatorname{Ob} \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}((U\cup U')_{\text{ét}}, \Lambda)$. □

3.1.4. We will prove property (P_c) above by induction on c. We emphasize that *the scheme X and the complex $\mathcal{K}$ are variable*. For $c = 0$, this property is trivial : take $T = X - U$. Let $c \geq 1$ and assume the property is established for $c - 1$. We wish to prove it for c.

3.2. Recursion : the Key Ingredient and a First Reduction.

3.2.1. Suppose, as we can, that the scheme X is *reduced*. According to the uniformization theorem prime to ℓ (**IX**-1.1) and the standard form theorem (**II**-3.2.3) there exists a finite family, indexed by a set I of elements i, of commutative diagrams

$$\begin{tikzcd}
X_i''' \arrow[rr] \arrow[d, "\text{finite, flat, surjective}"', "\text{degree prime to } \ell"] & & Y = \coprod_{j\in J} Y_j \arrow[d] & \\
X_i'' \arrow[r, "\text{étale}"'] & X' \arrow[r, "p", "\text{proper, birational}"'] & X & U \arrow[l, hook', "j"']
\end{tikzcd}$$

where, in addition to the properties indicated above,

— the family $(X_i'' \to X')$ is a covering for the completely decomposed étale topology ;

— the schemes Y_j, $j \in J$, are *regular* ;

— the inverse image of U in Y_j is the complement of a strict normal crossing divisor.

3.2.2. Let $(j, \mathcal{K})$ be a pair as in the statement of property (P_c) (3.1). We will only see later that we can assume $\mathcal{K} = \Lambda$. According to the induction hypothesis applied to pairs $(j, \mathcal{K})$ and $(j', \mathcal{K})$, where j' is the open immersion of $U' = U \times_X X'$ into X', there exist two closed subsets $T \hookrightarrow X$ and $T' \hookrightarrow X'$ of codimension $\geq c$ such that the complexes $\mathrm{R}j_\star\mathcal{K}$ and $\mathrm{R}j'_\star\mathcal{K}$ are constructible on the corresponding complementary open sets. Since the closed set T has only a finite number of maximal points and the statement to be proven — constructibility outside a closed set of codimension $> c$ — is a local problem in the neighborhood of these points, we can assume T is irreducible, of codimension c, with generic point denoted η_T. Let η' be a maximal point of T'. If the image of η' by p is not equal to η_T, the corresponding irreducible component of T' disappears after localization (Zariski) in the neighborhood of η_T. (Indeed, it follows from the dimension formula [**ÉGA** IV_1 5.5.8.1] that $p(\eta')$ cannot be a strict generalization of η_T.) Given that T is of codimension c and T' is of at least equal codimension, any irreducible component T'_α of T' dominating T is necessarily of codimension equal to that of T — by virtue of the dimension formula (*loc. cit.*) —, and the induced morphism $T'_\alpha \to T$ is generically finite. By restricting to a suitable open neighborhood of η_T in X, we can finally assume that T' is a sum $\coprod_\alpha T'_\alpha$, where the T'_α are irreducible and the morphisms $T'_\alpha \to T$ are *finite*, surjective.

3.3. Notation : the complex $\psi_f(g, \mathcal{K})$.

3.3.1. For any X-scheme $f : X_1 \to X$ and any X_1-scheme $g : X_2 \to X_1$, let h denote the composite morphism $X_2 \to X_1 \to X$ and j_1 the open immersion $U_1 = X_1 \times_X U \hookrightarrow X_1$ derived from j by base change. For any complex $\mathcal{K} \in \operatorname{Ob} \mathsf{D}^+(U_{\text{ét}}, \Lambda)$, consider the complex of sheaves on X,

$$\psi_f(g, \mathcal{K}) := \mathrm{R}h_\star\big(g^\star(\mathrm{R}j_{1\star}\mathcal{K})\big)$$

$$\begin{tikzcd} X_2 \arrow[d, "g"'] \arrow[dd, bend right=60, "h"'] & \\ X_1 \arrow[d, "f"'] \arrow[dr, phantom, "\square"] & U_1 \arrow[l, hook', "j_1"'] \arrow[d] \\ X & U \arrow[l, hook', "j"] \end{tikzcd}$$

where $\mathcal{K}$ is abusively denoted for its inverse image on U_1. Below, the morphism g will most often be a closed immersion, which will sometimes be omitted from the notation, as will f, if this does not seem to cause confusion. For example, $\psi(X, \mathcal{K}) = \mathrm{R}j_\star\mathcal{K}$.

3.3.2. The formation of the complex ψ is functorial in the following sense : for any commutative diagram

$$\begin{tikzcd} & X_2 \arrow[d, "g"'] & & X'_2 \arrow[ll, "m"'] \arrow[d, "g'"] & \\ U_1 \arrow[r, hook, "j_1"] & X_1 \arrow[dr, "f"'] & & X'_1 \arrow[ll, "n"'] \arrow[dl, "f'"] & U'_1 = n^{-1}(U_1) \arrow[l, hook', "j'_1"'] \\ & & X & & \end{tikzcd}$$

the base change morphism (adjunction) $n^\star \mathrm{R}j_{1\star}\mathcal{K} \to \mathrm{R}j_{1'\star}\mathcal{K}$ induces a morphism

$$\psi_f(g, \mathcal{K}) \to \psi_{f'}(g', \mathcal{K}).$$

3.4. Second Localization.

3.4.1. We will say that a morphism in a derived category $\mathsf{D}^+(\mathcal{T}, \Lambda)$, where $\mathcal{T}$ is the étale topos of a scheme, is a $\mathsf{D}^{\mathsf{b}}_{\mathsf{c}}$**-isomorphism** or **isomorphism modulo** $\mathsf{D}^{\mathsf{b}}_{\mathsf{c}}$, if it has a cone in $\mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(\mathcal{T}, \Lambda)$. This amounts, according to [**Neeman, 2001**, 2.1.35], to assuming that the induced arrow in the quotient triangulated category $\mathsf{D}^+(\mathcal{T}, \Lambda) / \mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(\mathcal{T}, \Lambda)$ is an *isomorphism*. Note that in the terminology of *op. cit.*, the subcategory $\mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(\mathcal{T}, \Lambda)$ is *thick*. The localization considered here (due to J.-L. Verdier) is the triangulated analogue of that considered by J.-P. Serre in the case of abelian categories.

PROPOSITION 3.4.2. *By restricting to a neighborhood of η_T, we can assume that the adjunction morphism*

$$\psi_{\mathrm{Id}}(T \hookrightarrow X, \mathcal{K}) \to \psi_p(T' \hookrightarrow X', \mathcal{K})$$

is a $\mathsf{D}^{\mathsf{b}}_{\mathsf{c}}$-isomorphism.

Note that the right-hand term, $\psi_p(T' \hookrightarrow X', \mathcal{K})$, is isomorphic to the direct sum $\bigoplus_\alpha \psi_p(T'_\alpha \hookrightarrow X', \mathcal{K})$.

Démonstration. Let p_U be the morphism induced by p above the open set U of X; it is an isomorphism above a dense open set W of U. Let i be the closed immersion of the complement $Z = U - W$ into U. On U, we have a distinguished triangle

$$\mathcal{K} \to \mathrm{R}p_{U\star}p_U{}^\star\mathcal{K} \to i_\star\mathcal{H} \xrightarrow{+1},$$

where $\mathcal{H}$ is constructible on Z, according to the finiteness theorem for the proper morphism p_U. It follows from the proper base change theorem for p that the preceding distinguished triangle becomes, after application of the functor $\psi_{\mathrm{Id}}(T \hookrightarrow X, -)$, the following distinguished triangle of complexes supported on T :

$$\psi_{\mathrm{Id}}(T \hookrightarrow X, \mathcal{K}) \to \psi_p(p^{-1}(T) \hookrightarrow X', \mathcal{K}) \to \psi_{\mathrm{Id}}(T \hookrightarrow X, i_\star\mathcal{H}) \xrightarrow{+1} .$$

First step. We will begin by showing that the first arrow is generically on T a $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}$-isomorphism. ("Generically on T" : by restricting to a suitable Zariski neighborhood of η_T.) Indeed, let $\overline{Z}$ be the closure of Z in X, $\bar{j} : Z \hookrightarrow \overline{Z}$ the open immersion and $\bar{i} : \overline{Z} \hookrightarrow X$ the closed immersion, represented in the diagram below.

$$\begin{tikzcd} U \arrow[d, hook, "j"'] & Z = U - W \arrow[l, hook', "i"'] \arrow[d, hook, "\bar{j}"] \\ X & \overline{Z} \arrow[l, hook', "\bar{i}"'] \end{tikzcd}$$

The restriction to T of the complex $\psi(T, i_\star\mathcal{H})$ — which we want to show is generically $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}$-null — is isomorphic to the restriction of the complex $\bar{i}_\star\mathrm{R}\bar{j}_\star\mathcal{H}$. Since the closed set $\overline{Z}$ is of codimension ≥ 1 in X, because W is everywhere dense in X, the induction hypothesis for the pair $(\bar{j}, \mathcal{H})$ immediately yields the result.

Second step. To conclude, we must now show that the adjunction morphism $\psi_p(p^{-1}(T), \mathcal{K}) \to \psi_p(T', \mathcal{K})$, through which the morphism $\psi_{\mathrm{Id}}(T, \mathcal{K}) \to \psi_p(T', \mathcal{K})$ from the statement factors, is, generically on T, a $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}$-isomorphism. On the closed subset $T'_p = p^{-1}(T)$ of X', consider the restriction $\mathcal{L} = (\mathrm{R}j'_\star\mathcal{K})_{|p^{-1}(T)}$ of the direct image by j' of $\mathcal{K}$, and the distinguished triangle

$$(T'_p - T' \hookrightarrow T'_p)_!\mathcal{L}_{|T'_p - T'} \to \mathcal{L} \to (T' \hookrightarrow T'_p)_\star\mathcal{L}_{|T'} \xrightarrow{+1}$$

composed of its zero extensions. Recall that j' denotes the open immersion of U' into X'. By definition of T', the first complex is constructible; therefore, its direct image (derived) by the *proper* morphism p_T is also constructible. Now, the direct image of the second arrow by p_T is precisely the adjunction morphism $\psi_p(T'_p, \mathcal{K}) \to \psi_p(T', \mathcal{K})$. Q.E.D. □

3.5. Construction of a Retraction.

3.5.1. By further shrinking X, we can assume that for every α — we recall that $T' = \coprod_\alpha T'_\alpha$ —, there exists an index i_α such that the étale morphism $X''_{i_\alpha} \to X'$ has a section σ_α above T'_α. This follows from the fact that the family $(X''_i \to X')_i$ is completely decomposed, so a section exists in the neighborhood of the generic point of T'_α (**II**-2.2.3). The properness of the dominant morphism $X' \to X$ allows us to deduce the existence of a suitable open subset of X from that of an open subset of X'.

3.5.2. To simplify notation, for each index α, we set $X''_\alpha = X''_{i_\alpha}$, $X'''_\alpha = X'''_{i_\alpha}$ and denote $T''_\alpha \subset X''_\alpha$ the image of T'_α by a section σ_α as above, and finally $T'''_\alpha \subset X'''_\alpha$ the inverse image of T''_α by the finite morphism $X'''_\alpha \to X''_\alpha$.

Proposition 3.5.3. *The adjunction morphism* $\psi(T'_\alpha \hookrightarrow X'_\alpha, \mathcal{K}) \to \psi(T''_\alpha \hookrightarrow X''_\alpha, \mathcal{K})$ *is an* isomorphism.

Of course, the complexes above are calculated by equipping the schemes X'_α and X''_α with the obvious X-scheme structure. We will henceforth allow this abuse of notation.

Démonstration. Results from the fact that the morphism $X''_\alpha \to X'$ is étale. □

PROPOSITION 3.5.4. *The adjunction morphism $\psi(T''_\alpha \hookrightarrow X'_\alpha, \mathcal{K}) \to \psi(T'''_\alpha \hookrightarrow X''_\alpha, \mathcal{K})$ has a left inverse.*

Démonstration. Consider the following diagram with Cartesian squares :

$$\begin{tikzcd} T'''_\alpha \arrow[r] \arrow[d, "\pi_T"'] & X'''_\alpha \arrow[d, "\pi"] & U'''_\alpha \arrow[l, "j'''_\alpha"'] \arrow[d, "\pi_U"] \\ T''_\alpha \arrow[r] & X''_\alpha & U''_\alpha \arrow[l, "j''_\alpha"'] \end{tikzcd}$$

where $U''_\alpha = U \times_X X''_\alpha$, similarly for U'''_α, and $\pi : X'''_\alpha \to X''_\alpha$ is as in 3.2.1. In particular, the morphism π_U is finite, flat, and of generic degree prime to ℓ, so that the composite morphism

$$\mathcal{K} \to \pi_{U\star}\pi_U{}^\star\mathcal{K} \xrightarrow{\mathrm{Tr}} \mathcal{K}$$

is multiplication by the degree, and therefore invertible. Let's apply the functor $\mathrm{R}j''_{\alpha\star}$. By composition of direct images, the middle term is $\pi_\star \mathrm{R}j'''_{\alpha\star}\mathcal{K}$, where we omit the inverse image functor from the notation (1.2.2). According to the base change theorem for finite morphisms, its restriction to the closed subset T''_α is isomorphic to $\pi_{T\star}\big((\mathrm{R}j'''_{\alpha\star}\mathcal{K})_{|T'''_\alpha}\big)$. By pushing the sheaves onto X via the morphism $T''_\alpha \to X$, the preceding sequence thus becomes

$$\psi(T''_\alpha \hookrightarrow X''_\alpha, \mathcal{K}) \to \psi(T'''_\alpha \hookrightarrow X'''_\alpha, \mathcal{K}) \to \psi(T''_\alpha \hookrightarrow X''_\alpha, \mathcal{K})$$

and the composition of these arrows is an isomorphism. □

3.6. Case of Constant Coefficients : Use of the Purity Theorem.

3.6.1. Let $T''' = \coprod T'''_\alpha$, $X''' = \coprod X'''_\alpha$ and consider the commutative diagram of adjunction morphisms, completed with the trace morphism :

$$\begin{tikzcd} \psi(T \hookrightarrow X, \mathcal{K}) \arrow[r, dashed] \arrow[drrr, bend right] & \psi(T' \hookrightarrow X', \mathcal{K}) \arrow[r, dashed] & \psi(T'' \hookrightarrow X'', \mathcal{K}) \arrow[r] \arrow[rr, dashed, bend left] & \psi(T''' \hookrightarrow X''', \mathcal{K}) \arrow[r, "\mathrm{Tr}"'] & \psi_p(T'' \hookrightarrow X'', \mathcal{K}). \\ & & & \psi(T''' \to Y, \mathcal{K}) \arrow[u] & \end{tikzcd}$$

According to the three preceding propositions, the dashed arrows become isomorphisms modulo $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}$. If the complex $\psi(T''' \to Y, \mathcal{K})$ is *constructible*, that is, null modulo $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}$, it follows that $\psi(T \hookrightarrow X, \mathcal{K})$ — or, equivalently, $(\mathrm{R}j_\star\mathcal{K})_{|T}$ — is also constructible.

PROPOSITION 3.6.2. *The complex $\psi(T''' \to Y, \Lambda)$ is constructible.*

Démonstration. Since the composite morphism $T''' \to X$ is finite, it suffices to show that the complex $\mathrm{R}j_{Y\star}\Lambda$ is constructible. This follows from the assumptions made in 3.2.1 and the purity theorem **XVI**-3.1.1. □

3.7. Reduction to the Case of Constant Coefficients.

3.7.1. To complete the proof of Theorem 1.1.1, we must now show that property (P_c) from § 3.1, where c is fixed, follows from the particular case where $\mathcal{K} = \Lambda$ and from (P_{c-1}).

3.7.2. Let's begin by observing that we can assume $\mathcal{K}$ is concentrated in degree 0, i.e., it is a constructible *sheaf*, which we will now denote $\mathcal{F}$. The set of constructible sheaves satisfying the property to be established at rank c is, for fixed X, stable under extension and direct factor. According to [**SGA 5** I 3.1.2], we can assume $\mathcal{F} = \pi_\star k_! \Lambda$ where $\pi : U' \to U$ is a *finite* morphism and $k : W \hookrightarrow U'$ an open immersion, with U' *integral*. According to Zariski's main theorem ([**ÉGA** IV$_3$ 8.12.6]), the composite morphism $U' \to X$, which is *quasi-finite*, factors into an open immersion $j' : U' \hookrightarrow X'$ followed by a finite morphism $\overline{\pi} : X' \to X$.

$$\begin{tikzcd} W \arrow[r, hook, "k"] & U' \arrow[r, hook, "j'"] \arrow[d, "\pi"'] & X' \arrow[d, "\overline{\pi}"] \\ & U \arrow[r, hook, "j"] & X \end{tikzcd}$$

The complex $\mathrm{R}j_\star\pi_\star k_!\Lambda$, whose constructibility is in question, is isomorphic to the complex $\overline{\pi}_\star\mathrm{R}j'_\star k_!\Lambda$. By virtue of the following lemma, we can assume $X' = X$.

LEMME 3.7.3. *Let $f : Y \to X$ be a finite morphism of schemes, T_Y a closed subset of Y and $T_X = f(T_Y)$ its image.*

(i) *We have the inequality :* $\mathrm{codim}(T_X, X) \geq \mathrm{codim}(T_Y, Y)$.
(ii) *Let* $K \in \mathrm{Ob}\,\mathrm{D}^+(Y_{\text{ét}}, \Lambda)$ *such that* $K_{|Y-T_Y} \in \mathrm{Ob}\,\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}((Y - T_Y)_{\text{ét}}, \Lambda)$. *Then,* $(\mathrm{R}f_\star K)_{|X-T_X} \in \mathrm{Ob}\,\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}((X - T_X)_{\text{ét}}, \Lambda)$.

Démonstration. The first statement is well-known. The second is an immediate corollary of the preservation of constructibility by the composite, finite morphism, $Y - f^{-1}(T_X) \hookrightarrow Y - T_Y \to X - T_X$. □

3.7.4. Let $j : U \to X$ and $k : W \to U$ be two open immersions, with U integral. We now wish to deduce the constructibility of the complex $\mathrm{R}j_\star k_!\Lambda$ outside a closed subset of codimension at least c from the analogous property for the complexes $\mathrm{R}j_\star\Lambda$. Admitting this result for the latter, it follows from the distinguished triangle $k_!\Lambda \to \Lambda \to i_\star\Lambda \xrightarrow{+1}$, where i is the closed immersion of the complement F of W in U, that it suffices to demonstrate the constructibility of $\mathrm{R}j_\star i_\star\Lambda$ outside a closed subset of codimension at least c. Since scheme U is integral, the closure $\overline{F}$ of F in X has strictly positive codimension in X. Let $m : F \hookrightarrow \overline{F}$ be the corresponding open immersion and $n : \overline{F} \hookrightarrow X$ the closed immersion. We tautologically have :

$$\mathrm{R}j_\star i_\star\Lambda = n_\star\mathrm{R}m_\star\Lambda,$$

by commutativity of the diagram

$$\begin{tikzcd} U \arrow[r, hook, "j"] & X \\ F \arrow[u, "i"] \arrow[r, hook, "m"] & \overline{F}. \arrow[u, "n"'] \end{tikzcd}$$

By the induction hypothesis (P_{c-1}), there exists a closed subset $T_{\overline{F}}$ of $\overline{F}$, of codimension at least c in $\overline{F}$, such that the restriction of $\mathrm{R}m_\star\Lambda$ to the open set $\overline{F} - T_{\overline{F}}$ of $\overline{F}$ is constructible. The direct image by the closed immersion n of the complex $\mathrm{R}m_\star\Lambda$ is therefore constructible on the open set $X - T_{\overline{F}}$ of X. The conclusion now follows from the fact that the codimension of $T_{\overline{F}}$ in X is *strictly greater* than c.

REMARQUE 3.7.5. O. Gabber also knows how to demonstrate a result of *uniform constructibility*, in the spirit of those in [**Katz & Laumon, 1985**, § 3] but without hypothesis on the characteristic. Cf. email to Luc Illusie, April 3, 2007; see also [**Orgogozo, 2011**].

4. ℓ-adic Coefficients

4.1. Definitions. Here we recall the construction, due to Torsten Ekedahl ([**Ekedahl,**]), of the triangulated category of bounded constructible ℓ-adic complexes. See also [**Fargues, 2009**], § 5 for a summary and some improvements. For other approaches, see [**Bhatt & Scholze, 2013**] or [**Liu & Zheng, 2014**] (for stacks).

Here we fix a Noetherian scheme X, on which a prime number ℓ is invertible.

4.1.1. *Projective Systems.* Let $X^{\mathbf{N}}$ denote the topos of projective systems indexed by $\mathbf{N}$ of étale sheaves on X; we make it a ringed topos via $\mathbf{Z}/\ell^{\mathbf{N}}\mathbf{Z} := (\mathbf{Z}/\ell^n\mathbf{Z})_n$. An **$\ell$-adic projective system** of sheaves on X is a $\mathbf{Z}/\ell^{\mathbf{N}}\mathbf{Z}$-module on $X^{\mathbf{N}}$, i.e., a projective system of abelian sheaves $\mathcal{F} = (\ldots \to \mathcal{F}_{n+1} \to \mathcal{F}_n \to \ldots)$ on X, where $\mathcal{F}_n$ is a $\mathbf{Z}/\ell^n\mathbf{Z}$-module on X. These form an abelian category, whose derived category is denoted $\mathsf{D}(X^{\mathbf{N}}, \mathbf{Z}/\ell^{\mathbf{N}}\mathbf{Z})$ and whose full subcategory of *uniformly* bounded systems of complexes is denoted $\mathsf{D}^{\mathrm{b}}(X^{\mathbf{N}}, \mathbf{Z}/\ell^{\mathbf{N}}\mathbf{Z})$. An ℓ-adic projective system is said to be **essentially null** if, for every n, there exists an integer $m \geq n$ such that the corresponding transition morphism $\mathcal{F}_m \to \mathcal{F}_n$ is null. Similarly, a complex $\mathcal{K} \in \operatorname{Ob} \mathsf{D}(X^{\mathbf{N}}, \mathbf{Z}/\ell^{\mathbf{N}}\mathbf{Z})$ is essentially null if each projective system $\mathrm{H}^i(\mathcal{K})$ of sheaves is.

4.1.2. *$\mathbf{Z}_\ell$-complexes.* Let $\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(X^{\mathbf{N}}, \mathbf{Z}/\ell^{\mathbf{N}}\mathbf{Z})$ be the full subcategory of $\mathsf{D}^{\mathrm{b}}(X^{\mathbf{N}}, \mathbf{Z}/\ell^{\mathbf{N}}\mathbf{Z})$ consisting of complexes

$$\mathcal{K} = \big(\mathcal{K}_n \in \operatorname{Ob} \mathsf{D}^{\mathrm{b}}(X, \mathbf{Z}/\ell^n\mathbf{Z})\big)_n$$

whose "reduction modulo ℓ"

$$\mathbf{F}_\ell \overset{\mathbb{L}}{\otimes}_{\mathbf{Z}/\ell^{\mathbf{N}}\mathbf{Z}} \mathcal{K} = (\mathbf{F}_\ell \overset{\mathbb{L}}{\otimes}_{\mathbf{Z}/\ell^n\mathbf{Z}} \mathcal{K}_n)_n \in \operatorname{Ob} \mathsf{D}^{-}(X^{\mathbf{N}}, \mathbf{Z}/\ell^{\mathbf{N}}\mathbf{Z})$$

is *essentially constant constructible*, meaning isomorphic, modulo essentially null complexes, to a projective system originating from $\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(X, \mathbf{Z}/\ell\mathbf{Z})$. Such an object is called a **bounded constructible $\mathbf{Z}_\ell$-complex**; they form a triangulated category.

Note that the complexes $\mathbf{F}_\ell \overset{\mathbb{L}}{\otimes}_{\mathbf{Z}/\ell^{\mathbf{N}}\mathbf{Z}} \mathcal{K}$ are not necessarily bounded below even if $\mathcal{K}$ is bounded. In [**Fargues, 2009**, 5.7] a subcategory denoted $\mathsf{D}^{e+}(X^{\mathbf{N}}, \mathbf{Z}/\ell^{\mathbf{N}}\mathbf{Z})$ is defined, preserved by the above tensor product; this relies on the fact that the constant sheaf $\mathbf{Z}/\ell\mathbf{Z} \in \operatorname{Ob} \mathsf{D}(X^{\mathbf{N}}, \mathbf{Z}/\ell^{\mathbf{N}}\mathbf{Z})$ has finite tor-dimension *modulo essentially null objects*.

4.1.3. *Triangulated Category of $\mathbf{Z}_\ell$-sheaves.* We denote $\mathcal{D}^{\mathrm{b}}_{\mathrm{c}}(X, \mathbf{Z}_\ell)$ the triangulated category obtained from the category $\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(X^{\mathbf{N}}, \mathbf{Z}/\ell^{\mathbf{N}}\mathbf{Z})$ by inverting **essential isomorphisms**, i.e., arrows with an essentially null cone. (According to *op. cit.* 5.18, it amounts to the same to invert arrows u such that $\mathbf{F}_\ell \overset{\mathbb{L}}{\otimes}_{\mathbf{Z}/\ell^{\mathbf{N}}\mathbf{Z}} u$ has an essentially null cone.) Similarly, one can define unbounded and non-constructible variants.

4.1.4. As explained in [**Fargues, 2009**, § 5.9], when X is of finite type over a separably closed or finite field, the category obtained is equivalent to the category $2-\lim_n \mathsf{D}^{\mathrm{b}}_{\mathrm{ctf}}(X, \mathbf{Z}/\ell^n\mathbf{Z})$ considered by P. Deligne in Weil II. Note that the constituents of a bounded constructible $\mathbf{Z}_\ell$-complex $(\mathcal{K}_n) \in \operatorname{Ob} \mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(X^{\mathbf{N}}, \mathbf{Z}/\ell^{\mathbf{N}}\mathbf{Z})$ are not necessarily of finite tor-dimension but there exists a representative (by a "normalized complex", [**Fargues, 2009**, 5.14]) for which this is true. We recall ([**SGA 4** XVII 4.1.9]) that a complex $\mathcal{K} \in \operatorname{Ob} \mathsf{D}^{\mathrm{b}}(T, A)$, where T is a topos and A a commutative ring, is said to have **tor-dimension** less than or equal to n if for every complex $\mathcal{L}$ of A-modules concentrated in non-negative degrees, $\mathrm{H}^i(\mathcal{K} \overset{\mathbb{L}}{\otimes} \mathcal{L}) = 0$ for all $i < -n$.

4.1.5. One of the key points of the theory is the *Nakayama-Ekedahl lemma* according to which the triangulated functor denoted $\mathbf{F}_\ell \overset{\mathbb{L}}{\otimes}_{\mathbf{Z}_\ell} -$, derived from the above modulo ℓ reduction functor, is *conservative* : $\mathcal{K} \in \operatorname{Ob} \mathcal{D}^{\mathrm{b}}_{\mathrm{c}}(X, \mathbf{Z}_\ell)$ is null if and only if the complex $\mathbf{F}_\ell \overset{\mathbb{L}}{\otimes}_{\mathbf{Z}_\ell} \mathcal{K} \in \operatorname{Ob} \mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(X, \mathbf{Z}/\ell\mathbf{Z})$ is. We refer the reader to [**Ekedahl, 1984**, prop. 1.1, p. 214] or [**Illusie, 1983**, 2.3.7, 2.4.5] for a first appearance of this lemma, and [**Ekedahl,** , 3.6 (ii)] for the preceding result.

4.1.6. According to this lemma, and its corollaries (*op. cit.*, th. 5.1 (ii) and th. 6.3), the ℓ-adic finiteness theorem below results from a finiteness theorem for finite coefficients.

4.2. Theorems : Statements.

THÉORÈME 4.2.1. *Let X be a Noetherian quasi-excellent scheme of finite dimension, ℓ a prime number invertible on X and $f : Y \to X$ a finite type morphism. For every integer $n \geq 1$, the functor $\mathrm{R}f_\star : \mathsf{D}^+(Y_{\text{ét}}, \mathbf{Z}/\ell^n\mathbf{Z}) \to \mathsf{D}^+(X_{\text{ét}}, \mathbf{Z}/\ell^n\mathbf{Z})$ maps $\mathsf{D}^{\mathrm{b}}_{\mathrm{ctf}}(Y_{\text{ét}}, \mathbf{Z}/\ell^n\mathbf{Z})$ to $\mathsf{D}^{\mathrm{b}}_{\mathrm{ctf}}(X_{\text{ét}}, \mathbf{Z}/\ell^n\mathbf{Z})$.*

Démonstration. According to the result in paragraph § 2.3, the functor $\mathrm{R}f_\star$ has finite cohomological dimension. The conclusion follows from [**SGA 4** XVII 5.2.11] (finite tor-dimension) and Theorem 1.1.1 (constructibility). ☐

Remarque 4.2.2. It should be possible to show, by reduction to the case of schemes of finite type over $\mathbf{Z}$, that for any coherent morphism f, the functor $\mathrm{R}f_\star$ maps a flat constructible sheaf to a complex of tor-dimension at most zero.

According to the results sketched in [**Ekedahl,** , § 5-6], the following ℓ-adic theorem can be deduced from the preceding theorem.

Théorème 4.2.3. *Let X be a Noetherian quasi-excellent scheme of finite dimension, ℓ a prime number invertible on X and $f : Y \to X$ a finite type morphism. The functor* $\mathrm{R}f_\star : \mathrm{D}^+(Y^{\mathbf{N}}, \mathbf{Z}/\ell^{\mathbf{N}}\mathbf{Z}) \to \mathrm{D}^+(X^{\mathbf{N}}, \mathbf{Z}/\ell^{\mathbf{N}}\mathbf{Z})$, $\mathcal{K} = (\mathcal{K}_n)_n \mapsto \mathrm{R}f_\star\mathcal{K} = (\mathrm{R}f_\star\mathcal{K}_n)_n$ *induces a functor* $\mathrm{R}f_\star : \mathcal{D}^b_c(Y, \mathbf{Z}_\ell) \to \mathcal{D}^b_c(X, \mathbf{Z}_\ell)$.

EXPOSÉ XIV

Dimension Functions

Vincent Pilloni and Benoît Stroh

We define the notion of a dimension function on a scheme X and we show the existence of such functions locally for the étale topology if X is quasi-excellent.

1. Universal catenarity of Henselian schemes

In this part, we recall the notions of *catenarity* and *universal catenarity*. The reader may consult exposé I for more details.

1.1. Universally catenary schemes. Let S be a topological space and $X \subset Y$ irreducible closed subsets of S. Let $\operatorname{codim}(X, Y)$ denote the supremum of the lengths of strictly increasing chains of irreducible closed subsets $X \subset Z \subset Y$ (cf. [**ÉGA** 0_{IV} 14.2.1 & 14.1.1]). If S is a scheme, X and Y integral closed subschemes and x the generic point of X, we have

$$\operatorname{codim}(X, Y) = \dim(\mathscr{O}_{Y,x}).$$

Definition 1.1.1 ([**ÉGA** 0_{IV} 14.3.2]). A scheme S is **catenary** if it is locally Noetherian and if for any chain $X \subset Y \subset Z$ of irreducible closed subsets of S, we have

$$\operatorname{codim}(X, Z) = \operatorname{codim}(Y, Z) + \operatorname{codim}(X, Y).$$

A scheme S is **universally catenary** if every scheme of finite type over S is catenary.

The notion of catenarity is stable under localization and under restriction to closed subschemes. Thus, S is universally catenary if and only if for every integer $n \geq 0$, the scheme $\mathbf{A}^n_S$ is catenary.

Lemme 1.1.2. *A Cohen-Macaulay scheme is universally catenary.*

Proof. If S is Cohen-Macaulay, it is catenary by [**Matsumura, 1980a**, 16.B]. As for any $n \geq 0$, the scheme $\mathbf{A}^n_S$ remains Cohen-Macaulay, the scheme S is indeed universally catenary. □

Exemple 1.1.3. Every regular scheme is universally catenary because it is Cohen-Macaulay. In particular, the spectrum of a field, a curve and the spectrum of a power series algebra over a field or over a discrete valuation ring are universally catenary. Every scheme of finite type over a regular scheme is universally catenary.

Proposition 1.1.4 ([**Matsumura, 1980a**], 28.P). *A complete Noetherian local scheme is universally catenary.*

Proof. Cohen's structure theorem [**ÉGA** 0_{IV} 19.8.8] allows us to write any complete Noetherian local scheme as a closed subscheme in the spectrum of a power series algebra over a Cohen ring. Universal catenarity follows from the preceding example and the stability of this notion under passing to a closed subscheme. □

1.2. A theorem by Ratliff. A Noetherian scheme is said to be **equidimensional** if all its irreducible components have the same (finite) dimension. Let S be a Noetherian local scheme. We denote by $\hat{S}$ the spectrum of the completion of the ring of S at its maximal ideal.

Definition 1.2.1. The local scheme S is **formally equidimensional** if $\hat{S}$ is equidimensional. It is **formally catenary** if for every $s \in S$, the closure $\overline{\{s\}}$ is formally equidimensional.

PROPOSITION 1.2.2. *Let S be a Noetherian local scheme. The scheme S, its completion $\hat{S}$, its Henselization S^{h} and its strict Henselization all have the same dimension.*

Proof. This results from the following general statement: if $A \to A'$ is a local and flat morphism between Noetherian local rings with respective maximal ideals $\mathfrak{m}$ and $\mathfrak{m}'$ and if $\mathfrak{m}' = \mathfrak{m}A'$, then for every natural integer n, the lengths $\lg_A(A/\mathfrak{m}^n)$ and $\lg_{A'}(A'/\mathfrak{m}'^n)$ are equal. The equality of these Hilbert-Samuel functions implies the equality $\dim A = \dim A'$ (cf. [**Zariski & Samuel, 1975**, chap. VIII, §9]). □

According to this proposition, if S is an integral Noetherian local scheme, the irreducible components of $\hat{S}$ have dimension $\leq \dim(S)$ and one of them has dimension $\dim(S)$. The scheme S is therefore formally equidimensional if and only if all irreducible components of $\hat{S}$ have dimension $\dim(S)$.

Let S be a Noetherian local scheme. Ratliff proved the following fundamental theorem, which has already been mentioned in Proposition **I**-7.1.1.

THÉORÈME 1.2.3 ([**Matsumura, 1989**], 31.7). *For a Noetherian local scheme S, the following conditions are equivalent:*

- *S is formally catenary,*
- *S is universally catenary,*
- *$\mathbf{A}^1_S$ is catenary,*
- *S is catenary and for every $s \in S$, every integral scheme S' equipped with a finite and dominant map $S' \to \overline{\{s\}}$ and every closed point s' of S', we have $\dim(\mathcal{O}_{S',s'}) = \dim \overline{\{s\}}$.*

We have added a fourth equivalent condition to statement [**Matsumura, 1989**, Theorem 31.7]. It follows from [**ÉGA** IV$_2$ 5.6.10] that the first three equivalent conditions imply the fourth. The converse is proven during the demonstration of [**Matsumura, 1989**, Theorem 31.7] (in the second paragraph of page 255).

COROLLAIRE 1.2.4 ([**Matsumura, 1989**], 31.2). *Every Noetherian scheme of dimension ≤ 2 is catenary. Every Noetherian scheme of dimension ≤ 1 is universally catenary.*

1.3. Henselian schemes and catenarity. We have seen that every complete Noetherian local scheme is universally catenary in Proposition 1.1.4. Henselian local schemes also enjoy good catenarity properties:

PROPOSITION 1.3.1. *Every catenary Henselian local scheme is universally catenary.*

Proof. Let $S = \mathrm{Spec}(A)$ be a catenary Henselian local scheme, let P be a prime ideal of A, let L be a finite extension of $\mathrm{Frac}(A/P)$ and let B be a finite extension of A/P contained in L. According to Theorem 1.2.3, it suffices to prove that the dimension of the localization of B at each of its maximal ideals is equal to the dimension of A/P. Every finite algebra over a Henselian ring is semi-local by [**ÉGA** IV$_4$ 18.5] and [**ÉGA** IV$_4$ 18.6]. As the scheme B is integral, it is local. The "going-up" theorem ([**Matsumura, 1989**, 9.3 and 9.4]) shows that we indeed have $\dim(B) = \dim(A/P)$. □

Let us also recall the following result, a consequence of Corollary **I**-6.3 (ii).

PROPOSITION 1.3.2. *Every quasi-excellent Henselian local scheme is universally catenary.*

Thus, every quasi-excellent Henselian local scheme is excellent.

2. Immediate specializations and dimension functions

2.1. Definitions. Let X be a scheme. For any point x of X and any geometric point $\bar{x}$ above x, we denote by $X_{(x)}$, $X^{\mathrm{h}}_{(x)}$ and $\widehat{X}_{(x)}$ the localization, Henselization and completion of X at x. Similarly, we denote by $X_{(\bar{x})}$ the strict Henselization of X at $\bar{x}$.

Let x and y be two points of X, and $\bar{x}$ and $\bar{y}$ two geometric points above x and y.

DEFINITION 2.1.1 ([**SGA 4** VII 7.2]). A **specialization morphism** $\bar{x} \rightsquigarrow \bar{y}$ is the data of an X-morphism $X_{(\bar{x})} \to X_{(\bar{y})}$ between strict Henselizations.

According to [**SGA 4** VII 7.4], the data of a specialization $\bar{x} \rightsquigarrow \bar{y}$ is equivalent to the data of an X-morphism $\bar{x} \to X_{(\bar{y})}$.

Definition 2.1.2. Let $r \in \mathbf{N}$. We say that a specialization $\bar{x} \rightsquigarrow \bar{y}$ is a **specialization of codimension** r if the closure of the image of $\bar{x}$ in $X_{(\bar{y})}$ is a scheme of dimension r.
We say that y is an **immediate étale specialization** of x if there exists a specialization $\bar{x} \rightsquigarrow \bar{y}$ which is of codimension 1.
We say that y is an **immediate Zariski specialization** of x if $y \in \overline{\{x\}}$ and if the localization at y of the closure of x is of dimension 1.

2.1.3. If y is an immediate étale specialization of x, we also say that x is an **immediate étale generization** of y. Let $f : X_{(\bar{y})} \to X_{(y)}$ denote the strict Henselization morphism. The immediate étale generizations of y are then the images by f of points $x' \in X_{(\bar{y})}$ such that $\dim \overline{\{x'\}} = 1$.

Before examining these notions in more detail, we recall the following easy fact (**II**-1.1.3) which we will use implicitly below: if $f : X \to S$ is a flat morphism, f sends maximal points of X to maximal points of S, in other words every irreducible component of X dominates an irreducible component of S.

2.1.4. If x and y are two points of a Noetherian scheme X such that $y \in \overline{\{x\}}$ (in the usual sense, i.e. y is a specialization of x or equivalently x is a generization of y), then y is an immediate Zariski (resp. étale) specialization of x if and only if this is the case in $\overline{\{x\}}_{(y)}$. For certain considerations, this allows us to assume that X is integral local with generic point x and closed point y. In this case, y is an immediate Zariski specialization of x if and only if $\dim(X) = 1$. In the étale case, this can be read on the strict Henselization:

Proposition 2.1.5. *If x and y are two points of a Noetherian scheme X, the point y is an immediate étale specialization of x if and only if $y \in \overline{\{x\}}$ and the strict Henselization at a geometric point above y of the closure of x has an irreducible component of dimension* 1.

Proof. We reduce to the particular case $X = \overline{\{x\}}_{(y)}$ considered above. The point y is an immediate étale specialization if and only if there exists a point $\tilde{x}$ of $X_{(\bar{y})}$ above x such that the closure of $\tilde{x}$ in $X_{(\bar{y})}$ is of dimension 1. As stated here, this is equivalent to $X_{(\bar{y})}$ having an irreducible component C of dimension 1. Indeed, if we denote by $\tilde{x}$ the generic point of C, by the flatness argument stated above, C dominates X, i.e. $\tilde{x}$ is above x. Conversely, if $\tilde{x}$ is a point above x whose closure in $X_{(\bar{y})}$ is of dimension 1, we can denote by C an irreducible component of $X_{(\bar{y})}$ containing $\tilde{x}$. The generic point of C and $\tilde{x}$ being both above x, they are equal since one is a generization of the other and the fibers of $X_{(\bar{y})} \to X$ are discrete. □

Proposition 2.1.6. *Let X be a Noetherian scheme. An immediate Zariski specialization between points of X is an immediate étale specialization, and the converse is true if X is universally catenary.*

We can assume that $X = \overline{\{x\}}_{(y)}$ as before. For the implication, we assume that $\dim(X) = 1$ and we want to show that $X_{(\bar{y})}$ has an irreducible component of dimension 1. By Proposition 1.2.2, $X_{(\bar{y})}$ is of dimension 1 and it is clear that the irreducible components of a local scheme of dimension 1 are all of dimension 1.

For the converse, we will use two lemmas:

Lemme 2.1.7. *Let X be a Henselian Noetherian local scheme with closed point y. Let $\bar{y}$ be a geometric point above y. Then, X has an irreducible component of dimension* 1 *if and only if the strict Henselization $X_{(\bar{y})}$ has one.*

Proof. If C is an irreducible component of dimension 1 of $X_{(\bar{y})}$, its image in X is closed because $p : X_{(\bar{y})} \to X$ is integral. As p is flat, $p(C)$ is an irreducible component of X containing exactly two points, so $\dim(p(C)) = 1$. Conversely, the surjectivity and flatness of p imply that if $D \subset X$ is an irreducible component of dimension 1, there exists an irreducible component C of $X_{(\bar{y})}$ such that $p(C) = D$. We of course have $\dim(C) \geq 1$. Let $z \in C$ be a point which is not the generic point of C. The point $p(z)$ cannot be the generic point of D because otherwise the generic fiber of p would not be discrete.

Therefore, $p(z)$ is the closed point of C. The fact that $p^{-1}(y)$ is discrete then implies that z can only be the closed point of C. The integral local scheme C therefore has exactly two points: $\dim(C) = 1$. □

Lemme 2.1.8. *Let X be a Noetherian local scheme. If X has an irreducible component of dimension 1, then its completion $\widehat{X}$ also has one, and the converse is true if X is universally catenary.*

Proof. Let's start with the case where X is integral. If $\dim(X) = 1$, similarly to the beginning of the proof of Proposition 2.1.6, $\dim(\widehat{X}) = \dim(X) = 1$ and all irreducible components of $\widehat{X}$ are of dimension 1. Conversely, if X is universally catenary, according to Theorem 1.2.3, the irreducible components of $\widehat{X}$ all have the same dimension. If one of them is of dimension 1, the scheme $\widehat{X}$ is itself of dimension 1, and then $\dim(X) = \dim(\widehat{X}) = 1$.

In the general case, let X_i denote the irreducible components of X. For every i, the fiber product $X_i \times_X \widehat{X}$ is identified with $\widehat{X_i}$ (see [**SGA 1** IV 3]). It is a fact that the irreducible components of the various $\widehat{X_i}$ are exactly the irreducible components of $\widehat{X}$: these are irreducible closed subsets covering $\widehat{X}$ and none of them is contained in another (this follows from the fact that each irreducible component of $\widehat{X_i}$ dominates X_i). It is therefore clear that the statement for X results from the statement for the integral local schemes X_i. □

Let us show the converse stated in Proposition 2.1.6. As observed above (2.1.4), it suffices to show that if X is an integral Noetherian local scheme universally catenary with closed point y and generic point x (i.e. $X = \overline{\{x\}}$), and if x is an immediate étale generization of y, then $\dim(X) = 1$. By Proposition 2.1.5, the strict Henselization of X at a geometric point above y has an irreducible component of dimension 1, which by Lemma 2.1.7 is equivalent to saying that the Henselization X^{h} of X has an irreducible component of dimension 1. The completion $\widehat{X}$ of X being also that of X^{h}, the easy direction of Lemma 2.1.8 applied to X^{h} shows that $\widehat{X}$ has an irreducible component of dimension 1. The converse of this lemma applied to the universally catenary scheme X shows that X has an irreducible component of dimension 1; we therefore have $\dim(X) = 1$ and y is an immediate Zariski specialization of x.

We can read the étale specializations of a point x of X in the completion of X at x:

Proposition 2.1.9. *Let X be a Noetherian scheme. Let x and y be two points of X. We assume that $y \in \overline{\{x\}}$. Let $c : \widehat{X}_{(y)} \to X_{(y)}$ denote the completion morphism. The point y is an immediate étale specialization of x if and only if $c^{-1}(\overline{\{x\}})$ has an irreducible component of dimension 1.*

Proof. We can assume that $X = \overline{\{x\}}_{(y)}$. The point y is an immediate étale specialization of X if and only if $X_{(\bar{y})}$ has an irreducible component of dimension 1, i.e., by Lemma 2.1.7, that $X^{\mathrm{h}}_{(y)}$ has one. We want to show that this is equivalent to the completion $\widehat{X}$ having one.

If we make the additional assumption that X is quasi-excellent (hence universally catenary by Proposition 1.3.2), the desired equivalence results from Lemma 2.1.8. Let us show this equivalence without the quasi-excellence assumption. If $\dim(X) = 0$, $X_{(\bar{y})}$ and $\widehat{X}$ are also of dimension 0, so none of these schemes has an irreducible component of dimension 1. If $\dim(X) = 1$, all irreducible components of $X_{(\bar{y})}$ and $\widehat{X}$ are of dimension 1. We can therefore assume $\dim(X) = \dim(X_{(\bar{y})}) = \dim(\widehat{X}) \geq 2$. The non-existence of an irreducible component of dimension 1 of $X_{(\bar{y})}$ (resp. of $\widehat{X}$) is equivalent to saying that all irreducible components of $X_{(\bar{y})}$ (resp. of $\widehat{X}$) are of dimension ≥ 2. The desired equivalence then results from **XX**-3.3 (ii) ⇔ (iii) applied to the inclusion of the closed point of X. □

Definition 2.1.10. A **dimension function** on X is any function $\delta : X \to \mathbf{Z}$ such that for any immediate étale specialization $x \rightsquigarrow y$ between points of X, we have

$$\delta(y) = \delta(x) - 1.$$

The notion of a dimension function does not see nilpotent elements: δ is a dimension function on X if and only if it induces a dimension function on the reduced subscheme $X^{\text{réd}}$. Moreover, if $U \overset{i}{\hookrightarrow} X$ is an étale morphism and δ is a dimension function on X, $\delta \circ i$ defines a dimension function on U. More precisely, the set of dimension functions on étale X-schemes defines an étale sheaf on X. The difference between two dimension functions on X is a function invariant under any specializations, thus a locally constant function. We will show later that if X is quasi-excellent, dimension functions exist locally for the étale topology on X such that dimension functions form an étale $\mathbf{Z}$-torsor.

Proposition 2.1.11. *Let $f : X \to Y$ be a morphism between Noetherian schemes. Suppose given $\bar{x}$ and $\bar{x}'$ two geometric points of X. Let $\bar{y}$ (resp. $\bar{y}'$) be the geometric point of Y above which $\bar{x}$ (resp. $\bar{x}'$) lies. To any étale specialization $\bar{x} \rightsquigarrow \bar{x}'$ is canonically associated an étale specialization $\bar{y} \rightsquigarrow \bar{y}'$. If f is quasi-finite, the specializations $\bar{x} \rightsquigarrow \bar{x}'$ and $\bar{y} \rightsquigarrow \bar{y}'$ have the same codimension.*

Proof. The first part of the statement is trivial (see also **XVII-??**). We only need to show the equality of codimensions in the case where f is quasi-finite. For this, by Zariski's Main Theorem, we can assume that f is finite and that X and Y are strictly Henselian local schemes with closed points $\bar{x}'$ and $\bar{y}'$ respectively. We can further assume that X and Y are integral with generic points x and y respectively where x and y are the respective images of the geometric points $\bar{x}$ and $\bar{y}$. Stating the equality of codimensions of $\bar{x} \rightsquigarrow \bar{x}'$ and $\bar{y} \rightsquigarrow \bar{y}'$ then amounts to saying that X and Y have the same dimension, which results from the "going-up" theorem (cf. [**Matsumura, 1980a**, 13.C]). □

Corollaire 2.1.12. *Let $f : X \to Y$ be a quasi-finite morphism between Noetherian schemes. If $\delta : Y \to \mathbf{Z}$ is a dimension function on Y, then $f^\star\delta := \delta \circ f : X \to \mathbf{Z}$ is a dimension function on X.*

2.2. Dimension functions and universal catenarity. The aim of this paragraph is to prove the following result.

Théorème 2.2.1. *A Noetherian scheme is universally catenary if and only if it possesses a dimension function locally for the Zariski topology.*

The theorem results from the conjunction of Corollary 2.2.4 and Proposition 2.2.6 below.

Proposition 2.2.2. *Let X be an integral universally catenary scheme. The function $\delta : X \to \mathbf{Z}$ defined by $\delta(x) = -\dim(\mathscr{O}_{X,x})$ is a dimension function on X.*

Proof. By virtue of Proposition 2.1.6, since X is universally catenary, it suffices to show that $\delta(y) = \delta(x) - 1$ for any immediate Zariski specialization $x \rightsquigarrow y$. As X is an integral catenary scheme, we have

$$\dim(\mathscr{O}_{X,y}) = \dim(\mathscr{O}_{X,x}) + \dim(\mathscr{O}_{\overline{\{y\}},x}) = \dim(\mathscr{O}_{X,x}) + 1.$$

□

Remarque 2.2.3. If X is not assumed integral, the function $\delta(x) = -\dim(\mathscr{O}_{X,x})$ is not necessarily a dimension function, as shown by the example where X is obtained by gluing a line and a plane at a point.

Corollaire 2.2.4. *Every universally catenary scheme admits dimension functions locally for the Zariski topology.*

Proof. Let X be a universally catenary scheme. Let $x \in X$. We need to show that there exists an open neighborhood of x which can be endowed with a dimension function. The topological space X is a union of its irreducible components $X_1, ..., X_n$. By replacing X with the open complement of the components X_i not containing x, we can assume that x belongs to all components X_i. For every $1 \le i \le n$, let $\mathscr{F}_i$ be the set of dimension functions on X_i. By Proposition 2.2.2, this set is non-empty and is a torsor under $\mathbf{Z}$. We choose an element $\delta_i \in \mathscr{F}_i$ which is 0 at point x. For all $1 \le i,\ j \le n$, the function $\delta_i - \delta_j$ is locally constant on $X_i \cap X_j$ and is 0 at point x. Let $F_{i,j}$ be the closed subset of X, the union of the connected components of $X_i \cap X_j$ not containing x. Let U be the complement in X of the union of the $F_{i,j}$. The functions δ_i glue together to a dimension function on U. □

Let us prove a partial converse of Corollary 2.2.4.

LEMME 2.2.5. *A Noetherian scheme that possesses a dimension function locally for the Zariski topology is catenary.*

Proof. To prove catenarity, we can assume that the scheme S possesses a dimension function δ. Suppose that $X \subset Y$ are irreducible closed subsets with generic points x and y respectively. Choose a chain of immediate Zariski specializations $y = x_0 \rightsquigarrow x_1 \rightsquigarrow \cdots \rightsquigarrow x_d = x$ of maximal length. By definition of codimension, we have $\mathrm{codim}(X, Y) = d$ and by definition of dimension functions, taking into account the easy direction of Proposition 2.1.6, we obtain $\delta(x) = \delta(y) - d$, hence $\mathrm{codim}(X, Y) = \delta(y) - \delta(x)$.

Now, if $X \subset Y \subset Z$ are irreducible closed subsets, we have:

$$\begin{aligned} \delta(y) - \delta(x) &= \mathrm{codim}(X, Y), \\ \delta(z) - \delta(y) &= \mathrm{codim}(Y, Z), \\ \delta(z) - \delta(x) &= \mathrm{codim}(X, Z). \end{aligned}$$

From this we deduce $\mathrm{codim}(X, Z) = \mathrm{codim}(X, Y) + \mathrm{codim}(Y, Z)$, which proves catenarity. □

Thanks to Theorem 1.2.3, we can replace "catenary" by "universally catenary" in Lemma 2.2.5:

PROPOSITION 2.2.6. *A Noetherian scheme that possesses a dimension function locally for the Zariski topology is universally catenary.*

Proof. We can assume that S is local and endowed with a dimension function δ, which induces a dimension function δ^{h} on the Henselization S^{h}, which is therefore catenary by Lemma 2.2.5, then universally catenary thanks to Proposition 1.3.1 and finally formally catenary by Theorem 1.2.3.

Let us show that S is formally catenary. Let Z be an integral closed subscheme of S. We need to show that Z is formally equidimensional. Since the local schemes Z and Z^{h} have the same completion, it suffices to show that the irreducible components C of Z^{h} are formally equidimensional and all have the same dimension. As Z^{h} is a closed subscheme of S^{h} which is formally catenary, every irreducible component C of Z^{h} is indeed formally equidimensional. Now let's express the dimension of C using dimension functions. Let s be the closed point of S, η_C (resp. η_Z) the generic point of C (resp. Z). We have $\dim(C) = \delta^{\mathrm{h}}(\eta_C) - \delta^{\mathrm{h}}(s) = \delta(\eta_Z) - \delta(s) = \dim(Z)$. The dimension $\dim(C)$ is therefore independent of C. We have thus shown that Z is formally equidimensional. Finally, S is formally catenary and Theorem 1.2.3 shows that S is universally catenary. □

2.3. Local existence for the étale topology. In this paragraph we will prove the following theorem.

THÉORÈME 2.3.1. *Every quasi-excellent scheme possesses dimension functions locally for the étale topology.*

A repeated application of the following lemma (a variant of the argument of Corollary 2.2.4) shows that if the statement of the theorem is true for the irreducible components of a Noetherian scheme X, then the theorem also holds for X. Further on, we can thus assume that X is integral.

LEMME 2.3.2. *Let X be a Noetherian scheme whose underlying topological space is a union of two closed subschemes X_1 and X_2. Let $\overline{x}$ be a geometric point of $X_1 \cap X_2$. We assume that for every $i \in \{1, 2\}$, there exists an étale neighborhood U_i of $\overline{x}$ in X_i such that U_i admits a dimension function. Then, there exists an étale neighborhood U of $\overline{x}$ in X such that U admits a dimension function.*

Proof. For every $i \in \{1, 2\}$, we choose an étale neighborhood U_i of $\overline{x}$ in X_i such that U_i admits a dimension function δ_i. We are given a distinguished geometric point $\overline{u}_i$ above $\overline{x}$ and we can assume that $\delta_i(u_i) = 0$. By [**ÉGA** IV$_4$ 18.1.1], by replacing U_i with an open neighborhood of u_i, we can assume that there exists an étale morphism $\widetilde{U_i} \to X$ and an isomorphism $\widetilde{U_i} \times_X X_i \simeq U_i$. We can form the fiber product $V = \widetilde{U_1} \times_X \widetilde{U_2}$. Let $\pi : V \to X$ denote the projection and $\overline{v}$ a geometric point of V above $\overline{u}_1$ and $\overline{u}_2$. For every $i \in \{1, 2\}$, the projection of V onto the factor $\widetilde{U_i}$ induces an étale morphism $\pi^{-1}(X_i) \to U_i$. By composition with this étale morphism, the dimension function δ_i on U_i induces a dimension function $\tilde{\delta}_i$ on the closed subscheme $\pi^{-1}(X_i)$ of V and it satisfies $\tilde{\delta}_i(v) = 0$. These dimension functions $\tilde{\delta}_i$ for $i \in \{1, 2\}$ glue on the open subset U complement in V of the union of the connected components of $\pi^{-1}(X_1 \cap X_2)$ not containing v. □

Before treating the case of integral schemes, let us start with that of normal schemes:

Proposition 2.3.3. *Let X be a normal quasi-excellent scheme. The function $\delta : X \to \mathbf{Z}$ defined by $\delta(x) = -\dim(\mathscr{O}_{X,x})$ is a dimension function.*

Proof. We can furthermore assume that X is local. Let Y denote its Henselization and $h : Y \to X$ the Henselization morphism. By Theorem **I**-8.1 and subsequent comments, Y is also quasi-excellent. By Proposition 1.3.2, Y is universally catenary. Furthermore, since the morphism $Y \to X$ is regular, the normality of X implies that of Y (cf. [**ÉGA** IV$_2$ 6.5.4]). The local scheme Y is therefore integral and universally catenary, and the opposite of the codimension defines a dimension function $\delta' : Y \to \mathbf{Z}$. As an immediate étale specialization between points of X is lifted, so to speak by definition, to an immediate étale specialization of points of Y, to show that δ is a dimension function on X, it suffices to show that for every $y \in Y$, if we denote $x = h(y)$, we have $\delta(x) = \delta'(y)$, which results from Proposition 1.2.2. □

Let us return to the case of Theorem 2.3.1 where X is assumed integral and quasi-excellent, and let Y denote its normalization. The morphism $p : Y \to X$ is finite and surjective, hence of universal cohomological descent. Let δ denote a dimension function on Y; its existence is ensured by Proposition 2.3.3. Let p_1 and p_2 be the two projections $Y \times_X Y \to Y$. By Corollary 2.1.12 applied to the morphisms p_1 and p_2, it follows that $p_i^\star\delta := \delta \circ p_i$ for $i \in \{1, 2\}$ are two dimension functions on $Y \times_X Y$. The difference $p_1^\star\delta - p_2^\star\delta : Y \times_X Y \to \mathbf{Z}$ is a locally constant function which defines a 1-Čech cocycle, hence a class $[p_1^\star\delta - p_2^\star\delta]$ in $\mathrm{H}^1_{\text{Čech}}(Y \to X, \mathbf{Z})$. By cohomological descent theory, there exists a natural injection

$$\mathrm{H}^1_{\text{Čech}}(Y \to X, \mathbf{Z}) \hookrightarrow \mathrm{H}^1(X, \mathbf{Z}).$$

The class $[p_1^\star\delta - p_2^\star\delta]$ therefore defines an isomorphism class of étale $\mathbf{Z}$-torsors on X. It then immediately follows from the following proposition that X admits a dimension function locally for the étale topology:

Proposition 2.3.4. *Let U be an étale scheme over X (quasi-excellent). The vanishing of the class $[p_1^\star\delta - p_2^\star\delta]|_U$ in $\mathrm{H}^1(U, \mathbf{Z})$ implies the existence of a dimension function on U.*

Proof. By using the compatibility of constructions with étale base change $U \to X$, we can assume that $U = X$. The vanishing of $[p_1^\star\delta - p_2^\star\delta]$ in $\mathrm{H}^1_{\text{Čech}}(Y \to X, \mathbf{Z}) \hookrightarrow \mathrm{H}^1(X, \mathbf{Z})$ means that there exists a *locally constant* function $\gamma : Y \to \mathbf{Z}$ such that $p_1^\star\delta - p_2^\star\delta = p_1^\star\gamma - p_2^\star\gamma$. In other words, by replacing δ with $\delta - \gamma$, we can assume that $p_1^\star\delta = p_2^\star\delta$. Thus, $\delta : Y \to \mathbf{Z}$ descends to a function $\delta' : X \to \mathbf{Z}$.

To conclude, we need to show that if $p : Y \to X$ is a finite surjective morphism between quasi-excellent schemes, that $\delta' : X \to \mathbf{Z}$ is a function and $\delta = \delta' \circ p$, then δ' is a dimension function on X if δ is a dimension function on Y. For this statement, we can assume that X is a Henselian quasi-excellent local scheme, hence universally catenary (cf. Proposition 1.3.2). By Corollary 2.2.4, there exists a dimension function δ'' on X. We have two dimension functions δ and $p^\star\delta'' := \delta'' \circ p$ on Y. The difference $\delta - p^\star\delta'' : Y \to \mathbf{Z}$ is therefore a locally constant function. As $\delta - p^\star\delta'' = (\delta' - \delta'') \circ p$ and p is finite surjective, it easily follows that the function $\delta' - \delta'' : X \to \mathbf{Z}$ is locally constant. As δ'' is a dimension function, it follows that δ' is a dimension function.

□

2.4. Global existence of dimension functions. Following [**ÉGA** 0_{IV} 14.2.1], a Noetherian scheme X is said to be **equicodimensional** if its closed points all have the same codimension (which is then equal to $\dim(X)$).

Exemple 2.4.1. Equidimensional schemes of finite type over a field k or over $\mathbf{Z}$ are equicodimensional: it is classical that in this situation, we have $\dim(X) = \dim(\mathscr{O}_{X,x})$ for every closed point x. Local schemes are equicodimensional because they possess a unique closed point. If $S = \mathrm{Spec}(R)$ is a curve with uniformizer π, the scheme $\mathbf{A}^1_S$ is not equicodimensional. Indeed, there exists a closed point of

$\mathbf{A}^1_S$ above the generic point of S: it suffices to write $\mathbf{A}^1_S = \mathrm{Spec}(R[t])$ and consider $\mathfrak{m} = (\pi t - 1)$, which is a maximal ideal with residue field $\mathrm{Frac}(R)$.

The following lemma is inspired by [**ÉGA** 0_{IV} 14.3.3] [(i)].

Lemme 2.4.2. *Let X be an equidimensional catenary Noetherian scheme whose irreducible components are equicodimensional. For every $x \in X$, we have*

$$\dim(X) = \dim \overline{\{x\}} + \dim(\mathscr{O}_{X,x}).$$

Remarque 2.4.3. In particular, this equality is verified for every integral local catenary scheme. By [**Matsumura, 1989**, Th. 31.4], if X is an integral Noetherian local scheme and if for every $x \in X$, we have $\dim(X) = \dim \overline{\{x\}} + \dim(\mathscr{O}_{X,x})$, then X is catenary.

Lemma 2.4.2 and Proposition 2.2.2 imply the following result.

Corollaire 2.4.4. *Let X be an integral equicodimensional and universally catenary Noetherian scheme. The function $\delta : X \to \mathbf{Z}$ defined by $\delta(x) = \dim \overline{\{x\}}$ is a dimension function on X.*

The conclusions of the corollary are false if X is not equicodimensional. For example, let $S = \mathrm{Spec}(R)$ be a curve with uniformizer π and $X = \mathbf{A}^1_S = \mathrm{Spec}(R[t])$. If we denote by x the closed point of X corresponding to the maximal ideal $(\pi t - 1)$ and η the generic point of $\mathbf{A}^1_S$, then the specialization $\eta \rightsquigarrow x$ is immediate, yet $\dim \overline{\{x\}} = 0$ and $\dim \overline{\{\eta\}} = 2$.

Corollaire 2.4.5. *Let X be a scheme which is either of finite type over a field, or of finite type over $\mathbf{Z}$, or a universally catenary local scheme. The function defined by $\delta(x) = \dim \overline{\{x\}}$ is a dimension function on X.*

Proof. The scheme X is universally catenary. By Corollary 2.4.4, the function δ is a dimension function on each irreducible component of X. This function is defined globally, so it is a dimension function on X. □

2.5. Induced dimension function. Let $Y \to X$ be a morphism of schemes and δ_X a dimension function on X. In certain cases we can construct an induced dimension function δ_Y on Y. We admit the following proposition.

Proposition 2.5.1 ([**Matsumura, 1980a**], 14.C). *Let X be an integral universally catenary Noetherian scheme, Y an integral scheme and $Y \to X$ a dominant morphism of finite type. Let $k(X)$ and $k(Y)$ be the respective fraction fields of X and Y, let y be a point of Y and x its image in X, and let $k(y)$ and $k(x)$ be their residue fields. We have*

$$\dim(\mathscr{O}_{Y,y}) - \mathrm{deg.\,tr.}(k(Y)/k(X)) = \dim(\mathscr{O}_{X,x}) - \mathrm{deg.\,tr.}(k(y)/k(x)).$$

Corollaire 2.5.2. *Let X be a Noetherian scheme which possesses a dimension function δ_X and $f : Y \to X$ a morphism of finite type. The function $\delta_Y : Y \to \mathbf{Z}$ defined by*

$$\delta_Y(y) = \delta_X(f(y)) + \mathrm{deg.\,tr.}(k(f(y))/k(y))$$

is a dimension function on Y.

Proof. We can assume that X and Y are integral and that f is dominant. By Proposition 2.2.6, X is universally catenary and by Proposition 2.2.2, $x \mapsto -\dim(\mathscr{O}_{X,x})$ is a dimension function on X. Since dimension functions form a $\mathbf{Z}$-torsor, we can assume that $\delta_X(x) = -\dim(\mathscr{O}_{X,x})$ for every $x \in X$.

Corollary 2.5.1 shows that $\delta_Y(y) = -\dim(\mathscr{O}_{Y,y}) + \mathrm{deg.\,tr.}(k(Y)/k(X))$ and Proposition 2.2.2 shows that $y \mapsto -\dim(\mathscr{O}_{Y,y})$ is a dimension function on Y. Thus, δ_Y is a dimension function on Y. □

(i) Gabber remarks that Proposition [**ÉGA** 0_{IV} 14.3.3] is false. Assertions a, c and d of *loc. cit.* are equivalent to each other and imply b but are not equivalent to it. Condition b should be replaced by " X is catenary equidimensional and its irreducible components are equicodimensional ". Gabber gives as a counterexample the spectrum of the localization of $k[x, y, z, w]/(xz, xw)$ at the complement of the union of the prime ideals $(x-1, y)$ and (x, z, w) with k a field. The same error was noted, independently, by Huayi Chen (letter to Luc Illusie dated 2005-9-26).

Before establishing the functoriality of dimension functions with respect to regular morphisms between excellent schemes, let us prove a base change statement for regular morphisms in étale cohomology. This lemma is a simple consequence of Popescu's theorem **I**-10.3 and the smooth base change theorem [**SGA 4** xvi 1.2].

Lemme 2.5.3. *Let*

$$\begin{tikzcd} T' \arrow[r, "g'"] \arrow[d, "f'"'] & T \arrow[d, "f"] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$

be a Cartesian diagram of schemes, n an invertible integer over S and $\mathscr{F}$ an étale sheaf of $\mathbf{Z}/n\mathbf{Z}$-modules on T. Suppose that f is coherent and that g is a regular morphism between Noetherian schemes. The natural base change map

$$g^\star \mathrm{R} f_\star(\mathscr{F}) \xrightarrow{\sim} \mathrm{R} f'_\star g'^\star(\mathscr{F})$$

is an isomorphism.

Proof. The question being local on S and S', we can assume that S and S' are affine. By Popescu's theorem, there exists a filtered ordered set I (non-empty) and a family of affine schemes S_i indexed by I, such that S_i is smooth over S for every $i \in I$ and that $S' = \lim_i S_i$. Therefore, for every $i \in I$ there exists a commutative diagram with Cartesian squares

$$\begin{tikzcd} T' \arrow[r, "h'_i"] \arrow[d, "f'"'] & T_i \arrow[r, "g'_i"] \arrow[d, "f_i"] & T \arrow[d, "f"] \\ S' \arrow[r, "h_i"] & S_i \arrow[r, "g_i"] & S \end{tikzcd}$$

We conclude thanks to the following sequence of isomorphisms for every $q \geq 0$:

$$\begin{aligned} \mathrm{R}^q f'_\star g'^\star(\mathscr{F}) &\xleftarrow{\sim} \mathrm{colim}_i h_i^\star \mathrm{R}^q f_{i\star} g_i'^\star(\mathscr{F}) \\ &\xleftarrow{\sim} \mathrm{colim}_i g^\star \mathrm{R}^q f_\star(\mathscr{F}) \\ &\xleftarrow{\sim} g^\star \mathrm{R}^q f_\star(\mathscr{F}) \end{aligned}$$

The first of these isomorphisms results from the limit theorem [**SGA 4** vii 5.11], and the second from the smooth base change theorem for the smooth morphism g_i [**SGA 4** xvi 1.2]. □

We now prove that a regular morphism between excellent schemes allows us to induce dimension functions.

Proposition 2.5.4. *Let $f : Y \to X$ be a regular morphism between quasi-excellent schemes and δ_X a dimension function on X. The function $\delta_Y : Y \to \mathbf{Z}$ defined by*

$$\delta_Y(y) = \delta_X(f(y)) - \dim(\mathscr{O}_{Y_{f(y)},y})$$

is a dimension function on Y.

Proof. As the verification is local, there is no harm in assuming X and Y strictly local and f local. The quasi-excellent schemes X and Y are then excellent (cf. Proposition 1.3.2). Let δ be a dimension function on Y; its existence is ensured by Theorem 2.2.1. It suffices to show that $\delta_Y - \delta$ is a constant function on Y. The fibers of f are regular hence universally catenary by 1.1.3. Proposition 2.2.2 shows that the function that assigns to $y \in Y - \dim(\mathscr{O}_{Y_{f(y)},y})$ induces a dimension function on each of the fibers of f. The function $\delta_Y - \delta$ is therefore locally constant on each fiber of f. It results from Lemma 2.5.3 that these fibers are connected: indeed, we have $\mathrm{H}^0(f^{-1}(x), \mathbf{Z}/n\mathbf{Z}) = \mathrm{H}^0(x, \mathbf{Z}/n\mathbf{Z}) = \mathbf{Z}/n\mathbf{Z}$ for every $x \in X$ and every integer n invertible on X. The function $\delta_Y - \delta$ is therefore constant on the fibers of f and descends to X. It suffices to show that $\gamma = \delta_Y - \delta$ is locally constant on X. One way to calculate the value of γ at a point s of X is to consider the generic point η_s of the connected regular scheme $f^{-1}(s)$, so that $\gamma(s) = \delta_X(s) - \delta(\eta_s)$. Let $s' \rightsquigarrow s$ be an immediate Zariski specialization between two points of X. We need to show that $\gamma(s) = \gamma(s')$. Given that δ_X and δ are dimension functions on X and Y respectively, to show this, it suffices to know that $\eta_{s'}$ is an immediate specialization of η_s. To show this, by replacing X by the localization at s of the closure of s', we can assume that X is local integral of dimension 1, with generic point s' and closed point s. It is then a matter of showing that the fiber $f^{-1}(s)$ has codimension 1 in Y, which easily follows from the *Hauptidealsatz*. $\square$

2.6. Counterexample.

2.6.1. Let us recall the example from [**ÉGA** IV$_2$ 5.6.11] of a catenary scheme that is not universally catenary. Let k_0 be a field and k a purely transcendental extension of k_0 of infinite transcendence degree. Let $S = k[X]_{(X)}$ be the localization of the affine line ring over k at the origin and $V = S[T]$. The maximal ideals $\mathfrak{m} = (X, T)$ and $\mathfrak{m}' = (XT - 1)$ of V are respectively of height 2 and 1, and there exists an isomorphism $\phi : V/\mathfrak{m} \xrightarrow{\sim} V/\mathfrak{m}'$. Let v and v' be the closed points of $\mathrm{Spec}(V)$ corresponding to the maximal ideals $\mathfrak{m}$ and $\mathfrak{m}'$. Let $C = \{f \in V \mid \phi(f \bmod \mathfrak{m}) = f \bmod \mathfrak{m}'\}$. This is a subring of V which is not of finite type over k. The morphism $\mathrm{Spec}(V) \to \mathrm{Spec}(C)$ is finite and induces an isomorphism over the dense open subset $\mathrm{Spec}(C) - \{c\}$ where c is the closed point of C corresponding to the maximal ideal $\mathfrak{n} = \mathfrak{m} \cap \mathfrak{m}' \subset C$. The topological space $\mathrm{Spec}(C)$ is identified with the quotient of $\mathrm{Spec}(V)$ by the equivalence relation that identifies v and v'.

Proposition 2.6.2. *The scheme* $\mathrm{Spec}(C)$ *is Noetherian, quasi-excellent, catenary but not universally catenary. The closed point corresponding to the maximal ideal* $\mathfrak{n}$ *of* C *is an immediate étale specialization but not an immediate Zariski specialization of the generic point of* $\mathrm{Spec}(C)$.

Proof. The Noetherian character is shown in [**ÉGA** IV$_2$ 5.6.11] and the quasi-excellent character in [**ÉGA** IV$_2$ 7.8.4 (ii)]. The scheme $\mathrm{Spec}(C)$ is catenary by Corollary 1.2.4 because its dimension is 2. The points v and v' are identified with the two closed points of $\mathrm{Spec}(V \otimes_C C_\mathfrak{n})$ and the corresponding localizations $\mathrm{Spec}(V_\mathfrak{m})$ and $\mathrm{Spec}(V_{\mathfrak{m}'})$ are of dimensions 2 and 1 respectively, which contradicts the last condition of Theorem 1.2.3: the local ring $C_\mathfrak{n}$ is not universally catenary.

Since the local ring $C_\mathfrak{n}$ is of dimension 2, the point $c \in \mathrm{Spec}(C)$ is not an immediate Zariski specialization of the generic point. However, it is an immediate étale specialization thanks to Proposition 2.1.11 applied to the finite morphism $\mathrm{Spec}(V) \to \mathrm{Spec}(C)$ and to the obvious immediate étale specialization $\bar{\eta}_{\mathrm{Spec}(V)} \rightsquigarrow \bar{v}'$ of geometric points of $\mathrm{Spec}(V)$. $\square$

EXPOSÉ XV

Affine Lefschetz Theorem

Vincent Pilloni and Benoît Stroh

1. Statement of the theorem and first reductions

1.1. Statement.

1.1.1. Let X be a scheme endowed with a dimension function δ_X (**XIV**-2.1.10) and n an integer invertible on X. For any étale sheaf $\mathcal{F}$ of $\mathbf{Z}/n\mathbf{Z}$-modules on X,

$$\delta_X(\mathscr{F}) = \sup\left\{\delta_X(x)\ ,\ x \in X \mid \mathscr{F}_{\bar{x}} \neq 0\right\}.$$

Recall (**XIV**-2.5.2) that a finite type morphism $f : Y \to X$ induces a dimension function on Y; we will denote it here $f^\star\delta_X$. The main theorem of this exposé is the following (see also **Intro.**-7).

Théorème 1.1.2. *Assume the scheme X is quasi-excellent and the morphism $f : Y \to X$ is* affine *of finite type. Then, for any constructible sheaf $\mathcal{F}$ of $\mathbf{Z}/n\mathbf{Z}$-modules on Y, we have:*

$$\delta_X(\mathrm{R}^q f_\star(\mathscr{F})) \leq f^\star\delta_X(\mathscr{F}) - q.$$

Remarque 1.1.3. This theorem was already demonstrated in 1994 by O. Gabber when X is of finite type over a trait, cf. [**Illusie, 2003**]. The proof of the preceding theorem proceeds notably by reduction to this case.

1.2. Reformulation and reductions.

1.2.1. Let f and $\mathcal{F}$ be as above. The conclusion of the theorem means that for any geometric point $\bar{x}$ of X localized at a point x and any integer $q > f^\star\delta_X(\mathcal{F}) - \delta_X(x)$, we have

$$(\mathrm{R}f_\star\mathcal{F})_{\bar{x}} = H^q(Y_{(\bar{x})}, \mathcal{F}) = 0,$$

where $Y_{(\bar{x})}$ denotes the fiber product $Y \times_X X_{(\bar{x})}$. Recall (**XIV**-2.4.5) that the strictly local scheme $X_{(\bar{x})}$ can be endowed with the dimension function $\delta_{X_{(\bar{x})}} : t \mapsto \dim\overline{\{t\}}$ (**XIV**-2.4.5); this is the unique dimension function that is zero at x. Note the inequality $f^\star\delta_X(\mathcal{F}) - \delta_X(x) \geq f^\star_{(\bar{x})}\delta_{X_{(\bar{x})}}(\mathcal{F})$, which is trivial in the case where $\delta_X(x) = 0$, to which we can reduce. Thus, Theorem 1.1.2 is equivalent to the following statement.

Corollaire 1.2.2. *Let X be a quasi-excellent,* strictly local *scheme, endowed with the dimension function $\delta_X : t \mapsto \dim\overline{\{t\}}$. Then, for any constructible sheaf $\mathcal{F}$ of $\mathbf{Z}/n\mathbf{Z}$-modules on Y, we have:*

$$\mathrm{H}^q(Y, \mathscr{F}) = 0 \text{ if } q > f^\star\delta_X(\mathscr{F}).$$

1.2.3. Proceeding as in [**SGA 4** xiv 4.4] to reduce to the case of an affine open immersion and then using the trace method ([**SGA 4** ix 5.5] or [**SGA 5** i 3.1.2]) to reduce to the case of constant coefficients (see also **XIII**-3.7), we show that the theorem is also equivalent to the following corollary. (One could moreover assume the integer n is prime.)

Corollaire 1.2.4. *Let X be a strictly local quasi-excellent scheme of dimension d, an affine open U of X, and an integer n invertible on X. Then,*

$$\mathrm{H}^q(U, \mathbf{Z}/n\mathbf{Z}) = 0 \text{ if } q > d.$$

1.2.5. *Reduction to the complete case.* Let us now assume X is strictly local quasi-excellent, with completion $\widehat{X}$ at the closed point, and fix an affine open U of X, whose inverse image on $\widehat{X}$ we denote by $\widehat{U}$. The natural morphism from $\widehat{X}$ to X is regular because X is quasi-excellent. By applying the base change lemma for a regular morphism (**XIV**-2.5.3) to the Cartesian diagram

$$\begin{tikzcd} \widehat{U} \arrow[r] \arrow[d] & U \arrow[d] \\ \widehat{X} \arrow[r] & X \end{tikzcd}$$

we obtain $\mathrm{H}^q(\widehat{U}, \mathbf{Z}/n\mathbf{Z}) = \mathrm{H}^q(U, \mathbf{Z}/n\mathbf{Z})$ for all $q \geq 0$. See also [**Fujiwara, 1995**, 7.1.1] for another approach, as well as **XX**-4.4.

1.2.6. In the two following sections, we will demonstrate statement 1.2.4 in the particular case where X is a complete Noetherian local ring with separably closed residue field.

2. Purity, combinatorics of branches and descent

2.1. Purity: recall and an application.

2.1.1. We recall the absolute purity theorem demonstrated by O. Gabber ([**Fujiwara, 2002**]). By convention, the empty scheme is considered as a strict normal crossings divisor whose set of branches is indexed by the empty set.

Théorème 2.1.2 (**XVI**-3.1.1). *Let X be a regular scheme, Z a strict normal crossings divisor with complement $j : U = X - Z \hookrightarrow X$ and branches $\{Z_i\}_{i \in I}$, and n an integer invertible on X. There exist canonical isomorphisms*

$$\mathrm{R}^1 j_\star (\mathbf{Z}/n\mathbf{Z}) \xrightarrow{\sim} \bigoplus_{i \in I} (\mathbf{Z}/n\mathbf{Z})_{Z_i}(-1)$$
$$\mathrm{R}^q j_\star (\mathbf{Z}/n\mathbf{Z}) \xrightarrow{\sim} \bigwedge^q \mathrm{R}^1 j_\star (\mathbf{Z}/n\mathbf{Z})$$

2.1.3. *Combinatorics of branches: definitions.* Let $g : X' \to X$ be a morphism between schemes, and U a retrocompact open subset of X. Let $j : U \hookrightarrow X$ be the open immersion, $j' : U' \hookrightarrow X'$ the open immersion derived from it by base change, and Z and Z' the respective complementary closed subsets. Finally, we are given a closed subset $F \subset Z$ whose inverse image we denote by F'. (The retrocompactness hypothesis for U — i.e., quasi-compactness of j — is automatically satisfied if X is locally Noetherian; it allows for the computation of fibers of direct images $\mathrm{R}^p j_\star$ below.)

Définition 2.1.4. We say that $(Z \hookrightarrow X)$ and $(Z' \hookrightarrow X')$ have the **same combinatorics along** F if for every geometric point $\overline{z}'$ of F' whose image is the geometric point $\overline{z}$ of F, the following properties are satisfied:

(i) the schemes $X_{(\overline{z})}$ and $X'_{(\overline{z}')}$ are regular;

(ii) (a) either $X_{(\overline{z})} = Z_{(\overline{z})}$,

(b) or the closed set $Z_{(\overline{z})}$ is a strict normal crossings divisor, whose components are defined by equations $f_1, \ldots, f_r$, and the functions $g^\star f_1, \ldots, g^\star f_r$ form a free family in $\mathfrak{m}/\mathfrak{m}^2$, where $\mathfrak{m}$ is the maximal ideal of $X'_{(\overline{z}')}$.

2.1.5. Note that, in the second case ($Z_{(\overline{z})}$ divisor), the closed set $Z'_{(\overline{z}')}$ is a normal crossings divisor in $X'_{(\overline{z}')}$, having the same number of branches.

2.1.6. When X is a scheme over a base S, and F is a closed subset of the latter, we allow ourselves to say "... along F" for "... along the inverse image $F \times_S X$".

Proposition 2.1.7. *Assume that $(Z \hookrightarrow X)$ and $(Z' \hookrightarrow X')$ have the same combinatorics along a closed set F of X. Then, for any integer n invertible on X, the adjunction morphism*

$$(\mathrm{R}j_\star \mathbf{Z}/n\mathbf{Z})_{|F'} \to (\mathrm{R}j'_\star \mathbf{Z}/n\mathbf{Z})_{|F'}$$

is an isomorphism.

Proof. By localizing at geometric points $\overline{z}'$ and $\overline{z}$, we can assume the schemes are strictly local and the morphism $X' \to X$ is local. We then need to show that $\mathrm{R}\Gamma(U, \mathbf{Z}/n\mathbf{Z}) \to \mathrm{R}\Gamma(U', \mathbf{Z}/n\mathbf{Z})$ is an isomorphism. According to purity theorem 2.1.2, it suffices to show that the induced morphism on H^1 is an isomorphism, which immediately follows from the structure of these groups and from the fact that the class associated with a branch $Z_i = V(f_i)$ of Z is sent by restriction to the class of the branch $g^{-1}(Z_i) = V(g^\star f_i)$ (cf. **XVI**-2). □

2.2. Application of the hyper-base change theorem.

2.2.1. Let X be a Noetherian scheme. In this exposé, we use a variant of the h-topology on X defined in **XII$_\mathbf{A}$**-2.1.3: we only want to consider here X-schemes of finite type (in order to be able to apply the results of III for example) while in XII$_\mathrm{A}$ it was necessary to allow infinite coproducts (in order to be able to apply the formalism of cohomological descent; see [**SGA 4** v$^{\mathrm{bis}}$ 3.0.0]).

We will thus say that a morphism $Y' \to Y$ in the category Sch.tf/X of finite type X-schemes is ***h*-covering** if it is dominated by a composition (finite, in an arbitrary order) of covering families (called "elementary") of one of the two following types (in Sch.tf/X):

— a proper and surjective morphism $Z' \to Z$;

— a Zariski covering $(Z_i \to Z)_{i \in I}$, where I is a *finite* set.

Observe that an h-hypercovering $X_\bullet \to X$ (in the sense of the definition above) is also a hypercovering for the h-topology (on Sch/X) of XII$_\mathrm{A}$.

2.2.2. Now assume X is strictly local (Noetherian) with closed point x, which we denote by $i_x : x \hookrightarrow X$ for the closed immersion. Consider an open immersion $j : U \hookrightarrow X$ and $\varepsilon : X_\bullet \to X$ an h-hypercovering. The following proposition — where the morphisms obtained by base change are denoted in an obvious way — is an immediate corollary of the hyper-base change theorem (**XII$_\mathbf{A}$**-2.2.5, or **XII$_\mathbf{B}$-??**) and the fact that the cohomology of U is the fiber at x of the direct image by j.

Proposition 2.2.3. *Under the preceding hypotheses, the adjunction morphism*

$$\mathrm{R}\Gamma(U, \mathbf{Z}/n\mathbf{Z}) \to \mathrm{R}\varepsilon_{x_\star}(i^\star_{x\bullet} \mathrm{R}j_{\bullet_\star} \mathbf{Z}/n\mathbf{Z})$$

is an isomorphism.

2.2.4. Now assume a *local* morphism $X' \to X$ of strictly local Noetherian schemes is given. As before, we denote U an open subset of X, Z its complement and x the closed point of X. To every h-hypercovering $X_\bullet \to X$ of X is associated by base change a hypercovering of X'. (We use here the stability under base change of the elementary covering families of 2.2.1 above.)

Proposition 2.2.5. *Assume that for each integer $q \geq 0$ the closed sets $(Z_q \hookrightarrow X_q)$ and $(Z'_q \hookrightarrow X'_q)$ have the same combinatorics along the special fiber $(X_q)_x$. Then the adjunction morphism*

$$\mathrm{R}\Gamma(U, \mathbf{Z}/n\mathbf{Z}) \to \mathrm{R}\Gamma(U', \mathbf{Z}/n\mathbf{Z})$$

is an isomorphism. Moreover, if we make the preceding hypothesis for only integers $q \leq N+1$, where N is any integer, the morphisms $\mathrm{H}^q(U, \mathbf{Z}/n\mathbf{Z}) \to \mathrm{H}^q(U', \mathbf{Z}/n\mathbf{Z})$ are isomorphisms for each $q \leq N$.

Proof. The first point is an immediate consequence of the two preceding propositions. Indeed, according to 2.2.3 and the invariance of cohomology under base change of a separably closed field, i.e.,

$$\left(\mathrm{R}\varepsilon_{x_\star}(i^\star_{x\bullet} \mathrm{R}j_{\bullet_\star} \mathbf{Z}/n\mathbf{Z})\right)_{|x'} = \mathrm{R}\varepsilon'_{x'_\star}\left(\left(i_{x\bullet}{}^\star \mathrm{R}j_{\bullet_\star} \mathbf{Z}/n\mathbf{Z}\right)_{|x'_\bullet}\right),$$

it suffices to show that for each q, the adjunction $(\mathrm{R}j_{q_\star} \mathbf{Z}/n\mathbf{Z})_{|X'_{q_x}} \to (\mathrm{R}j'_{q_\star} \mathbf{Z}/n\mathbf{Z})_{|X'_{q_x}}$ is an isomorphism. This follows from the combinatorial hypothesis and 2.1.7. The truncated variant is a corollary of the preceding proof and the fact that cohomology in degree q only depends on levels $\leq q+1$. □

Remarque 2.2.6. In this criterion, hypotheses are only made at points of the special fibers of hypercoverings; this is what makes it so powerful.

3. Uniformization and approximation of data

3.1. Notations.

3.1.1. Let X, U, Z and n be as at the end of paragraph 1.2: the scheme X is strictly local complete, U is a strict *affine* open subset (otherwise there is nothing to prove), $Z = X - U$ (endowed with the reduced structure), and n is an integer invertible on X. We want to demonstrate 1.2.4 in this case. Fix an integer N.

3.1.2. It follows from the uniformization theorem (**VII**-1.1), completed by **XIII**-2.2.2, that there exists an h-hypercovering $\varepsilon : X_\bullet \to X$ such that each X_q is regular and, in each connected component of X_q, the closed subscheme $Z_q = Z \times_X X_q$ is either the whole scheme, or a strict normal crossings divisor.

3.1.3. Let k be the residue field of X, a Cohen ring C with residue field k (**IV**-4.1.7) and $S = \mathrm{Spec}(C)$. It follows from the structure theorem of Noetherian local rings ([**ÉGA** 0_{IV} 19.8.8]) that there exists an integer m and a closed immersion of X into the completion at the origin (of the special fiber over S) of the affine space $\mathbf{A}^m_S$. Let $\widehat{E}$ denote this completion, E its Henselian analogue (Henselization of the affine space at the origin) and finally e the closed point of the latter.

3.1.4. Since the scheme E is quasi-excellent, the completion morphism $\widehat{E} \to E$ is (local) regular so that, according to Popescu's theorem (**I**-10.3), we can write:

$$\widehat{E} = \lim_\alpha E_\alpha,$$

where $E_\alpha \to E$ are essentially smooth morphisms between strictly local schemes, the limit being filtered. Note that the schemes E_α are essentially of finite type over S.

3.2. Passage to the limit.

3.2.1. By restricting the index set, i.e., by assuming $\alpha \geq \alpha_0$ for a suitable α_0, the general principles of [**ÉGA** IV□ § 8], combined with the fact that the schemes X, Z, U and the X_q for $q \leq N$ are of finite presentation over $\widehat{E}$, imply the existence of diagrams with Cartesian squares of finite type E_α-schemes

$$\begin{tikzcd}
U_{\leq N,\alpha} \arrow[r] \arrow[d] & X_{\leq N,\alpha} \arrow[d] & Z_{\leq N,\alpha} \arrow[l] \arrow[d] \\
U_\alpha \arrow[r] \arrow[dr] & X_\alpha \arrow[d] & Z_\alpha \arrow[l] \arrow[dl] \\
 & E_\alpha &
\end{tikzcd}$$

derived by base change $E_\alpha \to E_{\alpha_0}$ from the diagram for α_0, and whose analogue over $\widehat{E}$ is derived by base change $\widehat{E} \to E_\alpha$. Moreover, we can assume that for each $\alpha \geq \alpha_0$, $X_\alpha \to E_\alpha$ is a closed immersion — so that X_α is strictly local —, and $U_\alpha \to X_\alpha$ an affine open immersion with complement Z_α.

Remarque 3.2.2. The schemes X_q and $X_{q\alpha}$ have the same special fiber for all $q \leq N$.

3.2.3. It follows from [**ÉGA** IV□ 8.10.5] and the description (2.2.1) of h-covering morphisms that we can assume that $X_{\leq N,\alpha} \to X_\alpha$ are (truncated) h-hypercoverings for $\alpha \geq \alpha_0$.

3.2.4. Let's verify that we can assume that for each α and each $q \leq N$, the "model" $X_{q\alpha}$ of X_q is regular along its special fiber. Fix q then disregard it. The scheme X is now regular, of finite type over $\widehat{E}$. The problem being local for the Zariski topology, we can assume, by quasi-compactness, that X is a subscheme of $Y = \mathbf{A}^m_{\widehat{E}}$ of the form $V(f_1, \ldots, f_c) \cap D(g)$, where $f_1, \ldots, f_c, g \in \Gamma(Y, \mathcal{O}_Y)$, purely of codimension c in $D(g)$[(i)].

For α sufficiently large, the functions $f_1, \ldots, f_c, g$ are descended to functions $f_{i\alpha}, g_\alpha$ on $Y_\alpha = \mathbf{A}^m_{E_\alpha}$. Let x be a point of Y belonging to the special fiber of X and let x_α be its image by $Y \to Y_\alpha$. Let $\mathfrak{m}$ (resp. $\mathfrak{m}_\alpha$) denote the maximal ideal of $\mathcal{O}_{Y,x}$ (resp. $\mathcal{O}_{Y_\alpha,x_\alpha}$). By regularity of X at x, the images $\overline{f_1}, \ldots, \overline{f_c}$ of the f_i in $\mathfrak{m}/\mathfrak{m}^2$ are linearly independent over $\kappa(x) = \mathcal{O}_{Y,x}/\mathfrak{m}$. The diagram

(i) Although not useful here, note that such an X has a regular immersion of codimension $c+1$ in $\mathbf{A}^{m+1}_{\widehat{E}}$, where it is defined by the equations $f_1, \ldots, f_c, 1 - gT_{m+1}$.

$$\begin{tikzcd} f_i & \mathfrak{m} \arrow[r, two heads] & \mathfrak{m}/\mathfrak{m}^2 \\ f_{i\alpha} \arrow[u, mapsto] & \mathfrak{m}_\alpha \arrow[u] \arrow[r, two heads] & \mathfrak{m}_\alpha/\mathfrak{m}_\alpha^2 \arrow[u] \end{tikzcd}$$

being commutative, it follows from the equality $\kappa(x) = \kappa(x_\alpha)$ that the images $\overline{f_{i\alpha}}$ of the $f_{i\alpha}$ in $\mathfrak{m}_\alpha/\mathfrak{m}_\alpha^2$ are linearly independent over $\kappa(x_\alpha)$. The locally closed subscheme $X_\alpha = V(f_{1\alpha}, \dots, f_{c\alpha}) \cap D(g_\alpha)$ of Y_α is therefore regular at x_α.

3.2.5. We similarly show that we can assume that for each α and each $q \leq N$, the immersions $(Z_q \hookrightarrow X_q)$ and $(Z_{q\alpha} \hookrightarrow X_{q\alpha})$ have the *same combinatorics* along the closed point $e \in E$, i.e., along the special fibers.

3.2.6. It follows from Proposition 2.2.5 that the adjunction morphisms

$$\mathrm{H}^q(U_\alpha, \mathbf{Z}/n\mathbf{Z}) \to \mathrm{H}^q(U, \mathbf{Z}/n\mathbf{Z})$$

are isomorphisms for $q < N$. We will show that if $q > d = \dim(X)$ and α is sufficiently large, we have $\mathrm{H}^q(U_\alpha, \mathbf{Z}/n\mathbf{Z}) = 0$. This will complete the proof of the affine Lefschetz theorem. Note that in general the X_α are of much higher dimension than $d = \dim(X)$.

3.3. Use of a section.

3.3.1. Let $\sigma : E \to E_\alpha$ be a section of $E_\alpha \to E$ and X_α^σ (resp. $U_\alpha^\sigma, Z_\alpha^\sigma$) the E-scheme derived from X_α (resp. U_α, Z_α) by base change. Let $X_{\leq N,\alpha}^\sigma$ also denote the h-hypercovering of X_α^σ obtained from $X_{\leq N,\alpha} \to X_\alpha$ by base change. Finally $U_{\leq N,\alpha}^\sigma$ (resp. $Z_{\leq N,\alpha}^\sigma$) is the obvious simplicial open (resp. closed) subset.

3.3.2. It follows from **III**-5.1 and **III**-5.4 (see also **III**-6.2, proof) that if α is sufficiently large and σ is sufficiently close to the identity, then the closed immersions $(Z_{q\alpha}^\sigma \hookrightarrow X_{q\alpha}^\sigma)$ and $(Z_{q\alpha} \hookrightarrow X_{q\alpha})$ have the *same combinatorics* along the special fiber above E for each $q \leq N$, and $\dim(X_\alpha^\sigma) = d$. It follows as above that the morphism

$$\mathrm{H}^q(U_\alpha, \mathbf{Z}/n\mathbf{Z}) \to \mathrm{H}^q(U_\alpha^\sigma, \mathbf{Z}/n\mathbf{Z})$$

is an isomorphism for $q < N$. Since the open subset U_α^σ is affine in X_α^σ of dimension d and essentially of finite type over S, it follows by passing to the limit from the Lefschetz theorem over S that

$$\mathrm{H}^q(U_\alpha^\sigma, \mathbf{Z}/n\mathbf{Z}) = 0 \text{ if } d < q < N.$$

If S is a *trait*, the Lefschetz theorem used is due to O. Gabber; see [**Illusie, 2003**, 2.4]. If S is the spectrum of a field, see [**SGA 4** XIV]. Finally, $\mathrm{H}^q(U, \mathbf{Z}/n\mathbf{Z}) = 0$ if $q > d = \dim(X)$. QED.

EXPOSITION XVI

Chern classes, Gysin morphisms, absolute purity

Joël Riou

In these notes, we present Ofer Gabber's new proof of the absolute cohomological purity theorem, announced in [**Gabber, 2005b**]. Section 1 recalls the construction of Chern classes in étale cohomology. These are used in Section 2, which consists of the construction and study of the properties of Gysin morphisms associated with smoothable complete intersection morphisms. In Section 3, these Gysin morphisms are used to give a precise formulation of the absolute purity theorem (Theorem 3.1.1). The proof of the purity theorem (different from the one written in [**Fujiwara, 2002**]) relies in particular on the results of logarithmic geometry established in Expositions VI, VIII, and X. We have endeavored to be careful with signs in the cohomological calculations : the conventions used and some remarks concerning them are detailed in Section 4.

Throughout this exposition, we fix a natural number $n \geq 1$. All schemes will be assumed to be schemes over $\operatorname{Spec}\left(\mathbf{Z}\left[\frac{1}{n}\right]\right)$. We denote by Λ the constant sheaf of rings with value $\mathbf{Z}/n\mathbf{Z}$, by $\Lambda(1)$ the sheaf of n-th roots of unity (for the étale topology) and by $\Lambda(r)$ its tensor powers, which can be given a meaning for any $r \in \mathbf{Z}$.

1. Chern classes

In this section, we recall the construction of Chern classes of vector bundles on general schemes with values in étale cohomology. We rely on the computation of the étale cohomology of projective bundles from [**SGA 5** VII 2] and on the method of [**Grothendieck, 1958a**]. The proofs are sometimes different from those of [**SGA 5** VII 3] : we have tried to provide a presentation that is as « economical » as possible.

Unlike the oral presentation, which used geometric language, in these notes, a vector bundle is a locally free Module $\mathscr{E}$ of finite rank, and the projective bundle of $\mathscr{E}$ is the bundle of hyperplanes defined in [**ÉGA** II 4.1.1] : $\mathbf{P}(\mathscr{E}) = \operatorname{Proj}(\mathbf{S}^\star\mathscr{E})$ where $\mathbf{S}^\star\mathscr{E}$ is the symmetric Algebra of $\mathscr{E}$.

Définition 1.1. Let X be a $\mathbf{Z}\left[\frac{1}{n}\right]$-scheme. Let $\mathscr{L}$ be a line bundle on X. By letting the invertible functions act by multiplication on the invertible sections of $\mathscr{L}$, we give the sheaf of invertible sections of $\mathscr{L}$ the structure of a torsor under the group scheme $\mathbf{G}_{\mathrm{m}}$. The isomorphism class of $\mathscr{L}$ thus defines an element in $\mathrm{H}^1_{\text{ét}}(X, \mathbf{G}_{\mathrm{m}})$. We denote by $c_1(\mathscr{L}) \in \mathrm{H}^2_{\text{ét}}(X, \Lambda(1))$ the image of this element under the boundary map $\delta \colon \mathrm{H}^1_{\text{ét}}(X, \mathbf{G}_{\mathrm{m}}) \to \mathrm{H}^2_{\text{ét}}(X, \Lambda(1))$ derived from the short exact Kummer sequence :

$$0 \to \Lambda(1) \to \mathbf{G}_{\mathrm{m}} \xrightarrow{[n]} \mathbf{G}_{\mathrm{m}} \to 0 \text{ (i)}.$$

If $\mathscr{L}$ and $\mathscr{L}'$ are two line bundles on X, we have the additivity relation :

$$c_1(\mathscr{L} \otimes \mathscr{L}') = c_1(\mathscr{L}) + c_1(\mathscr{L}') \in \mathrm{H}^2(X, \Lambda(1)) \text{ (ii)}.$$

Note that the Chern classes of line bundles lie in the even degrees of étale cohomology, so they commute with all cohomology classes. Note also that if $f \colon Y \to X$ is a morphism and $\mathscr{L}$ is a line bundle on X, then $f^\star(c_1(\mathscr{L})) = c_1(f^\star\mathscr{L})$.

(i) The sign conventions used in this exposition are specified in Section 4 (see in particular 4.3 for the cohomology class associated with a torsor and 4.2 for the morphism δ).

(ii) There exist « oriented » cohomology theories for which this property of the first Chern class is not satisfied, cf. [**Morel & Levine, 2001**].

Théorème 1.2 (Projective bundle formula). *Let X be a $\mathbf{Z}\left[\frac{1}{n}\right]$-scheme. Let $\mathscr{E}$ be a vector bundle of constant rank r on X. We denote by $\pi : \mathbf{P}(\mathscr{E}) \to X$ the projective bundle of $\mathscr{E}$. Let $\xi = c_1(\mathscr{O}(1)) \in \mathrm{H}^2(\mathbf{P}(\mathscr{E}), \Lambda(1))$*[(iii)]. *Then, the powers $\xi^i \in \mathrm{H}^{2i}(\mathbf{P}(\mathscr{E}), \Lambda(i))$ of ξ define an isomorphism in $\mathrm{D}^+(X_{\text{ét}}, \Lambda)$:*

$$(1, \xi, \dots, \xi^{r-1}) : \bigoplus_{i=0}^{r-1} \Lambda(-i)[-2i] \xrightarrow{\sim} \mathrm{R}\pi_\star\Lambda$$

By the base change theorem for a proper morphism, we can assume that X is the spectrum of an algebraically closed field k. We are thus reduced to computing the étale cohomology algebra of projective spaces over k, cf. [**SGA 5** vii 2].

Théorème 1.3. *There is a unique way to define, for any $\mathbf{Z}\left[\frac{1}{n}\right]$-scheme X and any vector bundle $\mathscr{E}$, elements $c_i(\mathscr{E}) \in \mathrm{H}^{2i}_{\text{ét}}(X, \Lambda(i))$ for all $i \in \mathbf{N}$, called Chern classes, such that if we define the formal series $c_t(\mathscr{E}) = \sum_{i \geq 0} c_i(\mathscr{E})t^i$, the following properties hold :*

- *the formal series $c_t(\mathscr{E})$ depends only on the isomorphism class of the vector bundle $\mathscr{E}$ on the $\mathbf{Z}\left[\frac{1}{n}\right]$-scheme X;*
- *if $f : Y \to X$ is a morphism of $\mathbf{Z}\left[\frac{1}{n}\right]$-schemes and $\mathscr{E}$ is a vector bundle on X, then $f^\star(c_t(\mathscr{E})) = c_t(f^\star\mathscr{E})$;*
- *if $0 \to \mathscr{E}' \to \mathscr{E} \to \mathscr{E}'' \to 0$ is a short exact sequence of vector bundles on a $\mathbf{Z}\left[\frac{1}{n}\right]$-scheme X, we have the Cartan-Whitney relation :*
$$c_t(\mathscr{E}) = c_t(\mathscr{E}')c_t(\mathscr{E}'') ;$$
- *if $\mathscr{L}$ is a line bundle on a $\mathbf{Z}\left[\frac{1}{n}\right]$-scheme X, the class $c_1(\mathscr{L})$ is that of Definition 1.1 and*
$$c_t(\mathscr{L}) = 1 + c_1(\mathscr{L})t .$$

We then have the relations $c_0(\mathscr{E}) = 1$ and $c_i(\mathscr{E}) = 0$ for $i > \operatorname{rang} \mathscr{E}$ for any vector bundle $\mathscr{E}$ on a $\mathbf{Z}\left[\frac{1}{n}\right]$-scheme X.

The proof uses several geometric constructions :

Proposition 1.4 (Splitting principle I). *Let X be a $\mathbf{Z}\left[\frac{1}{n}\right]$-scheme. Let $\mathscr{E}$ be a vector bundle of rank $\mathbf{r}$. We denote by $\pi : \mathbf{Drap}(\mathscr{E}) \to X$ the bundle of complete flags of $\mathscr{E}$. The following properties are satisfied :*

- *the vector bundle $\pi^\star\mathscr{E}$ admits a (canonical) filtration $0 = \mathscr{M}_0 \subset \mathscr{M}_1 \subset \cdots \subset \mathscr{M}_r = \pi^\star\mathscr{E}$ by vector bundles such that for any integer $1 \leq i \leq r$, the quotient $\mathscr{L}_i = \mathscr{M}_i/\mathscr{M}_{i-1}$ is a line bundle;*
- *the canonical morphism $\Lambda \to \mathrm{R}\pi_\star\Lambda$ is a split monomorphism in $\mathrm{D}^+(X_{\text{ét}}, \Lambda)$.*

The only non-trivial property lies in the fact that $\Lambda \to \mathrm{R}\pi_\star\Lambda$ is a split monomorphism. By noting that the projection $\mathbf{Drap}(\mathscr{E}) \to X$ can be written as a composition of r projective bundle projections, this follows from the projective bundle formula (Theorem 1.2)[(iv)].

Proposition 1.5 (Splitting principle II). *Let X be a $\mathbf{Z}\left[\frac{1}{n}\right]$-scheme. Let $(E) : 0 \to \mathscr{E}' \to \mathscr{E} \xrightarrow{p} \mathscr{E}'' \to 0$ be a short exact sequence of vector bundles on X. We denote by $\mathbf{Sect}(E)$ the X-scheme defined by the fact that for any X-scheme $f : Y \to X$, the set of X-morphisms $Y \to \mathbf{Sect}(E)$ identifies naturally with the set of sections of the vector bundle surjection $f^\star(p) : f^\star\mathscr{E} \to f^\star\mathscr{E}''$ on Y*[(v)]. *The Y-scheme $\mathbf{Sect}(E)$ is naturally equipped with the structure of a torsor under the Y-scheme of vector groups of homomorphisms $\mathbf{Hom}(\mathscr{E}'', \mathscr{E}')$. Let $\pi : \mathbf{Sect}(E) \to X$ be the projection. The following properties are satisfied :*

(iii) The sheaf $\mathscr{O}(1)$ is the fundamental sheaf on $\mathbf{P}(\mathscr{E})$: it is the invertible quotient of $\pi^\star\mathscr{E}$ by the universal hyperplane.

(iv) More precisely, Grothendieck showed (cf. [**Grothendieck, 1958b**], or [**SGA 6** vi 4.6] for the same argument in the case of algebraic K-theory) that the theory of Chern classes allows for the computation of the cohomology algebra of flag bundles, even incomplete ones.

(v) I thank Dennis Eriksson for pointing out this construction to me.

- *the inverse image under π : $\mathbf{Sect}(E) \to X$ of the exact sequence of vector bundles (E) is (canonically) split;*
- *the canonical morphism $\Lambda \to \mathrm{R}\pi_\star\Lambda$ is an isomorphism in $\mathsf{D}^+(X_{\text{ét}}, \Lambda)$.*

The existence of $\mathbf{Sect}(E)$ is clear, as the question is of a local nature on X. Locally for the Zariski topology on X, the projection π is the projection from an affine space, so the isomorphism $\Lambda \xrightarrow{\sim} \mathrm{R}\pi_\star\Lambda$ results from the homotopy invariance of étale cohomology for $\mathbf{Z}\left[\frac{1}{n}\right]$-schemes [**SGA 4** xv 2.2].

Let us prove Theorem 1.3. Thanks to Propositions 1.4 and 1.5, uniqueness is clear. It is therefore a matter of constructing a theory of Chern classes satisfying the required properties. Let $\mathscr{E}$ be a vector bundle (which we can assume to be of constant rank r) on a $\mathbf{Z}\left[\frac{1}{n}\right]$-scheme X. We consider the projective bundle $\mathbf{P}(\mathscr{E})$ over X. We denote $\xi = c_1(\mathscr{O}(1))$. By the projective bundle formula (Theorem 1.2), there exist unique elements, denoted $c_i(\mathscr{E}) \in \mathrm{H}^{2i}(X, \Lambda(i))$ for $1 \leq i \leq r$, such that we have the relation

$$\xi^r - c_1(\mathscr{E})\xi^{r-1} + c_2(\mathscr{E})\xi^{r-2} + \cdots + (-1)^r c_r(\mathscr{E}) = 0 \in \mathrm{H}^{2r}(\mathbf{P}(\mathscr{E}), \Lambda(r)) .$$

We set $c_0(\mathscr{E}) = 1$ and $c_i(\mathscr{E}) = 0$ for $i > r$. In the case where $\mathscr{E}$ is a line bundle, $\mathbf{P}(\mathscr{E}) \simeq X$ and $\mathscr{O}(1) \simeq \mathscr{E}$, which shows that this definition extends the previous one for line bundles. The only non-obvious property is the Cartan-Whitney formula. By the splitting principle (Propositions 1.4 and 1.5), it suffices to establish the following formula :

Lemme 1.6. *Let X be a $\mathbf{Z}\left[\frac{1}{n}\right]$-scheme. Let $(\mathscr{L}_i)_{1\leq i\leq r}$ be a finite family of line bundles on X, and let $\mathscr{E} = \bigoplus_{1\leq i\leq r} \mathscr{L}_i$ be their direct sum. In $\mathrm{H}^{2r}(\mathbf{P}(\mathscr{E}), \Lambda(r))$, we have the relation :*

$$\prod_{i=1}^{r}(\xi - c_1(\mathscr{L}_i)) = 0$$

where $\xi = c_1(\mathscr{O}(1))$. In other words,

$$c_t(\mathscr{E}) = \prod_{i=1}^{r} c_t(\mathscr{L}_i) .$$

The following argument is inspired by [**Panin & Smyrnov, 2003**]. For $1 \leq i \leq r$, we denote by $H_i \simeq \mathbf{P}(\mathscr{E}/\mathscr{L}_i)$ the projective hyperplane of $\mathbf{P}(\mathscr{E})$ defined by the inclusion $\mathscr{L}_i \to \mathscr{E}$. Let π : $\mathbf{P}(\mathscr{E}) \to X$ be the projection. The canonical morphism $\pi^\star\mathscr{L}_i \to \mathscr{O}(1)$ induces an isomorphism on the open complement of H_i in $\mathbf{P}(\mathscr{E})$. From this, we deduce that the element $\xi - c_1(\mathscr{L}_i)$ of $\mathrm{H}^2(X, \Lambda(1))$ can be lifted to an element x_i of the cohomology group with supports $\mathrm{H}^2_{H_i}(X, \Lambda(1))$[(vi)]. The product of the elements x_i lives naturally in the cohomology group with support $\mathrm{H}^{2r}_{\cap_{1\leq i\leq r} H_i}(\mathbf{P}(E), \Lambda(i))$, which is zero since the intersection of these r hyperplanes is empty; we then deduce the desired formula by forgetting the support.

Proposition 1.7. *Let $\mathscr{E}$ be a vector bundle on a $\mathbf{Z}\left[\frac{1}{n}\right]$-scheme X. For any natural number i, we have the equality :*

$$c_i(\mathscr{E}^\vee) = (-1)^i c_i(\mathscr{E}) ;$$

in other words, we have a change of variables formula :

$$c_t(\mathscr{E}^\vee) = c_{-t}(\mathscr{E}) .$$

Thanks to the Cartan-Whitney relation and the splitting principle, we can reduce to the case where $\mathscr{E}$ is a line bundle. This then results from the fact that c_1 : $\mathrm{Pic}(X) \to \mathrm{H}^2(X, \Lambda(1))$ is a group homomorphism.

(vi) For the moment, it is not important to fix a canonical lift.

2. Gysin morphisms

Given a complete intersection morphism $X \xrightarrow{f} S$ between $\mathbf{Z}\left[\frac{1}{n}\right]$-schemes satisfying certain technical hypotheses, we will construct a Gysin morphism $\mathrm{Cl}_f : \Lambda \to f^? \Lambda$ where $f^? = f^!(-d)[-2d]$ (d is the virtual relative dimension of f). These Gysin morphisms will be compatible with the composition of complete intersection morphisms.

The main part of this section is devoted to the construction of these Gysin morphisms in the case of regular immersions. The trace map will allow for the construction in the case of smooth morphisms. These two definitions will be glued together to give Definition 2.5.11 in the general case and Theorem 2.5.12 will establish the compatibility of these Gysin morphisms with composition.

2.1. First Chern class of a pseudo-divisor. Let $\mathscr{L}$ be a line bundle on a $\mathbf{Z}\left[\frac{1}{n}\right]$-scheme X, Z a closed subscheme of X, and U the open complement. We assume we are given an invertible section $s : \mathscr{O}_U \xrightarrow{\sim} \mathscr{L}_{|U}$. To the pair $(\mathscr{L}, s)$ is canonically associated a class $c_1(\mathscr{L}, s) \in \mathrm{H}^2_Z(X, \Lambda(1))$ inducing $c_1(\mathscr{L}) \in \mathrm{H}^2(X, \Lambda(1))$ by forgetting the support (construct an element of $\mathrm{H}^1_Z(X, \mathbf{G}_{\mathrm{m}})$ and use the Kummer exact sequence).

The class $c_1(\mathscr{L}, s)$ corresponds to a morphism $\Lambda_Z = \Lambda_X/\Lambda_U \to \Lambda_X(1)\,[2]$ in $\mathrm{D}^+(X_{\text{ét}}, \Lambda)$. By « composing » such a morphism with a cohomology class of Z represented by a morphism $\Lambda_Z \to \Lambda_Z(q)\,[p]$ (cf. 4.5.3), it follows that $c_1(\mathscr{L}, s)$ induces Gysin morphisms

$$\mathrm{Gys}_{(\mathscr{L},s)} : \mathrm{H}^p(Z, \Lambda(q)) \to \mathrm{H}^{p+2}_Z(X, \Lambda(q+1))\,.$$

Définition 2.1.1. If $Z \to X$ is a regular immersion of codimension 1 defined by an (invertible) Ideal $\mathscr{I}$, we set $\mathrm{Gys}_{Z \subset X} = -\,\mathrm{Gys}_{(\mathscr{I}, 1_{X-Z})} = \mathrm{Gys}_{-(\mathscr{I}, 1_{X-Z})}$.

Here we have denoted by $-(\mathscr{I}, 1_{X-Z})$ the opposite of the pseudo-divisor $(\mathscr{I}, 1_{X-Z})$, cf. [**Fulton, 1998**, §2.2]. *Via* the usual identifications, $-(\mathscr{I}, 1_{X-Z})$ corresponds to the effective divisor Z.

2.2. Generalized fundamental classes. To study the compatibility with composition of the fundamental classes defined in [**Fujiwara, 2002**, § 1] in the case of regular immersions (cf. [**SGA 6** vii 1.4]), Ofer Gabber defines a generalized fundamental class for a closed immersion $Y \to X$ defined by a finitely generated Ideal $\mathscr{I}$. This construction is no longer limited to regular immersions and is compatible with arbitrary base changes, but it depends on an additional piece of data, namely that of a vector bundle on Y surjecting onto the conormal sheaf $\mathscr{N}_{X/Y} = \mathscr{I}/\mathscr{I}^2$.

2.2.1. *Modified blow-up.* Let $Y \to X$ be a closed immersion between $\mathbf{Z}\left[\frac{1}{n}\right]$-schemes defined by a finitely generated Ideal $\mathscr{I}$. We denote by U the open complement. Let $\mathscr{E} \to \mathscr{N}_{X/Y}$ be an epimorphism of Modules over Y where $\mathscr{E}$ is a locally free Module of finite rank. We define a quasi-coherent graded $\mathscr{O}_X$-Algebra $\mathscr{A}_\star$ by fiber product so as to have a Cartesian square of $\mathscr{O}_X$-Modules, for any natural number n :

$$\begin{tikzcd} \mathscr{A}_n \arrow[r] \arrow[d] & \mathscr{I}^n \arrow[d] \\ \mathbf{S}^n\mathscr{E} \arrow[r] & \mathscr{I}^n/\mathscr{I}^{n+1} \end{tikzcd}$$

where the symmetric algebra $\mathbf{S}^\star\mathscr{E}$ is taken over the sheaf of rings $\mathscr{O}_Y = \mathscr{O}_X/\mathscr{I}$.

Définition 2.2.1.1. We set $\mathrm{Écl}_{Y,\mathscr{E}}(X) = \mathrm{Proj}(\mathscr{A}_\star)$ and denote by $\pi : \mathrm{Écl}_{Y,\mathscr{E}}(X) \to X$ the projection.

Remarque 2.2.1.2. If $Y \to X$ is a regular closed immersion and $\mathscr{E} \to \mathscr{N}_{X/Y}$ is an isomorphism, then $\mathrm{Écl}_{Y,\mathscr{E}}(X)$ identifies with the blow-up of Y in X. It is this particular case that we generalize here with the aim of obtaining a construction compatible with base changes.

Proposition 2.2.1.3. *The Algebra $\mathscr{A}_0$ is isomorphic to $\mathscr{O}_X$, the Modules $\mathscr{A}_n$ are finitely generated, the graded Algebra $\mathscr{A}_\star$ is generated by $\mathscr{A}_1$, and we have a canonical isomorphism of graded $\mathscr{O}_Y$-Algebras $\mathscr{A}_\star \otimes_{\mathscr{O}_X} (\mathscr{O}_X/\mathscr{I}) \xrightarrow{\sim} \mathbf{S}^\star\mathscr{E}$.*

The assertion concerning $\mathscr{A}_0$ is tautological. Let n be a natural number. Since $\mathscr{I}^n \to \mathscr{I}^n/\mathscr{I}^{n+1}$ is an epimorphism, the projection $\mathscr{A}_n \to \mathbf{S}^n\mathscr{E}$ is also an epimorphism, and if we denote by $\mathscr{K}_n$ its kernel, we have an isomorphism $\mathscr{K}_n \xrightarrow{\sim} \mathscr{I}^{n+1}$. By dévissage, it follows that $\mathscr{A}_n$ is a finitely generated Module.

Since $\mathbf{S}^n\mathscr{E}$ is an $\mathscr{O}_X/\mathscr{I}$-Module, $\mathscr{K}_n$ contains $\mathscr{I} \cdot \mathscr{A}_n$. Since $\mathscr{E} \to \mathscr{I}/\mathscr{I}^2$ is an epimorphism, the morphism $\mathbf{S}^n\mathscr{E} \to \mathscr{I}^n/\mathscr{I}^{n+1}$ is also an epimorphism, which implies that the projection $\mathscr{A}_n \to \mathscr{I}^n$ is an epimorphism. The induced morphism $\mathscr{I} \cdot \mathscr{A}_n \to \mathscr{I} \cdot \mathscr{I}^n = \mathscr{I}^{n+1} \simeq \mathscr{K}_n$ is therefore both a monomorphism and an epimorphism : $\mathscr{K}_n = \mathscr{I} \cdot \mathscr{A}_n$. This allows us to obtain the isomorphism $\mathscr{A}_\star \otimes_{\mathscr{O}_X} (\mathscr{O}_X/\mathscr{I}) \xrightarrow{\sim} \mathbf{S}^\star\mathscr{E}$.

To show that the obvious morphism $\mathscr{A}_1^{\otimes n} \to \mathscr{A}_n$ of Modules is an epimorphism, it suffices, by Nakayama's lemma, to test it after passing to the residue fields of X. Above the open set U, this is clear; above Y, it results from the isomorphism $\mathscr{A}_\star \otimes_{\mathscr{O}_X} (\mathscr{O}_X/\mathscr{I}) \xrightarrow{\sim} \mathbf{S}^\star\mathscr{E}$.

Corollaire 2.2.1.4. *The morphism $\pi\colon \text{Écl}_{Y,\mathscr{E}}(X) \to X$ is projective, and we have canonical isomorphisms $\pi^{-1}(U) \xrightarrow{\sim} U$ and $\pi^{-1}(Y) \simeq \mathbf{P}(\mathscr{E})$.*

The isomorphism above U is clear. Taking into account [**ÉGA** II 3.5.3], the one describing $\pi^{-1}(Y)$ is deduced from the isomorphism of graded $\mathscr{O}_Y$-Algebras $\mathscr{A}_\star \otimes_{\mathscr{O}_X} \mathscr{O}_Y \xrightarrow{\sim} \mathbf{S}^\star\mathscr{E}$.

Proposition 2.2.1.5. *Let $p\colon X' \to X$ be a morphism. We set $Y' = Y \times_X X'$ and $\mathscr{E}' = p^\star\mathscr{E}$. We have an obvious epimorphism $\mathscr{E}' \to \mathscr{N}_{X'/Y'}$. The canonical morphism*

$$\text{Écl}_{Y',\mathscr{E}'}(X') \to \text{Écl}_{Y,\mathscr{E}}(X) \times_X X'$$

is a nil-immersion.

Let $\mathscr{A}'_\star$ be the quasi-coherent graded $\mathscr{O}_{X'}$-Algebra giving rise to $\text{Écl}_{Y',\mathscr{E}'}(X')$. We have an obvious morphism $p^\star\mathscr{A}_\star \to \mathscr{A}'_\star$ of quasi-coherent graded $\mathscr{O}_{X'}$-Algebras. For any integer, the morphism $p^\star\mathscr{A}_n \to \mathscr{A}'_n$ is a morphism between finitely generated $\mathscr{O}_{X'}$-Modules; to show that it is an epimorphism, by Nakayama's lemma, it suffices to check that this morphism induces an isomorphism on one hand above $U' = X' - Y'$ (this is clear) and on the other hand modulo the ideal $\mathscr{I}'$ defining Y' in X' (this results from the description given in Proposition 2.2.1.3). The morphism

$$\text{Écl}_{Y',\mathscr{E}'}(X') \to \text{Écl}_{Y,\mathscr{E}}(X) \times_X X'$$

identifies with the obvious X'-morphism $\text{Proj}(\mathscr{A}'_\star) \to \text{Proj}(p^\star\mathscr{A}_\star)$ ([**ÉGA** II 3.5.3]); by what precedes, this is a closed immersion. The fact that this morphism induces an isomorphism above $p^{-1}(U)$ and $p^{-1}(Y)$ allows us to deduce immediately that the induced morphism at the level of the associated reduced schemes

$$\text{Écl}_{Y',\mathscr{E}'}(X')_{\text{réd}} \to (\text{Écl}_{Y,\mathscr{E}}(X) \times_X X')_{\text{réd}}$$

is an isomorphism.

2.2.2. *Definition of the classes.* We are still given a closed immersion $i\colon Y \to X$ defined by a finitely generated Ideal $\mathscr{I}$. We denote by $j\colon U \to X$ the inclusion of the open complement ($\mathscr{I}$ being finitely generated, j is a morphism of finite type). We assume we are given an epimorphism of $\mathscr{O}_Y$-Modules $\mathscr{E} \to \mathscr{N}_{X/Y}$ where $\mathscr{E}$ is locally free of finite rank. We denote by $\pi\colon \text{Écl}_{Y,\mathscr{E}}(X) \to X$ the projection of the modified blow-up, $j'\colon U \to \text{Écl}_{Y,\mathscr{E}}(X)$ the obvious open immersion, and $i'\colon \mathbf{P}(\mathscr{E}) \to \text{Écl}_{Y,\mathscr{E}}(X)$ the closed immersion given by Corollary 2.2.1.4. We denote by r the rank of the vector bundle $\mathscr{E}$, which we assume to be of constant rank for simplicity, and we assume $r > 0$.

We thus have the following diagram of schemes, where the squares are Cartesian :

$$\begin{tikzcd}
\mathbf{P}(\mathscr{E}) \arrow[r, "i'"] \arrow[d, "\pi'"] & \text{Écl}_{Y,\mathscr{E}}(X) \arrow[d, "\pi"] & U \arrow[l, "j'"'] \arrow[d, equal] \\
Y \arrow[r, "i"] & X & U \arrow[l, "j"']
\end{tikzcd}$$

Proposition 2.2.2.1. *The obvious morphism $\Lambda \to \mathrm{R}\pi'_\star\Lambda$ in $\mathsf{D}^+(Y_{\text{ét}}, \Lambda)$ is a split monomorphism : the projective bundle formula identifies its cokernel with*

$$\bigoplus_{k=1}^{r-1} \Lambda(-k)\,[-2k]\ .$$

The obvious morphisms define a distinguished triangle :

$$\Lambda \to \mathrm{R}\pi_\star\Lambda \to i_\star\mathrm{Coker}(\Lambda \to \mathrm{R}\pi'_\star\Lambda) \xrightarrow{0} \Lambda\,[1]$$

in $\mathsf{D}^+(X_{\text{ét}}, \Lambda)$. We can rewrite it in the form

$$\Lambda \to \mathrm{R}\pi_\star\Lambda \xrightarrow{\rho} \bigoplus_{k=1}^{r-1} i_\star\Lambda(-k)\,[-2k] \xrightarrow{0} \Lambda\,[1]\ ,$$

where the morphism ρ admits a canonical section given by the elements $\mathrm{c}_1(\mathscr{O}(1), 1_U)^k$ of $\mathrm{H}^{2k}_{\mathbf{P}(\mathscr{E})}(\mathrm{Écl}_{Y,\mathscr{E}}(X), \Lambda(k))$, identified with morphisms $i_\star\Lambda(-k)\,[-2k] \to \mathrm{R}\pi_\star\Lambda$ in $\mathsf{D}^+(X_{\text{ét}}, \Lambda)$.

We denote by L an injective resolution of the constant sheaf Λ viewed as a sheaf of Λ-modules on the big étale site of schemes of finite type over X. For any morphism of finite type $W \xrightarrow{p} X$, we denote by $L_{|W}$ the complex of sheaves of Λ-modules on $W_{\text{ét}}$ induced by L; it can be seen as an object of $\mathsf{D}^+(W_{\text{ét}}, \Lambda)$ isomorphic to Λ.

Lemme 2.2.2.2. *The obvious commutative square of complexes of sheaves on $X_{\text{ét}}$ is homotopy bicartesian :*

$$\begin{tikzcd} L_{|X} \arrow[r] \arrow[d] & i_\star L_{|Y} \arrow[d] \\ \pi_\star L_{|\mathrm{Écl}_{Y,\mathscr{E}}(X)} \arrow[r] & \pi_\star i'_\star L_{|\mathbf{P}(\mathscr{E})} \end{tikzcd}$$

(*this means, for example, that the total complex associated with this diagram, identified with a 3-tuple complex, is acyclic*).

The total complexes associated with the double complexes

$$j_! L_{|U} \to L_{|X} \to i_\star L_{|Y}$$

and

$$j'_! L_{|U} \to L_{|\mathrm{Écl}_{Y,\mathscr{E}}(X)} \to i'_\star L_{|\mathbf{P}(\mathscr{E})}$$

of sheaves on X and $\mathrm{Écl}_{Y,\mathscr{E}}(X)$ respectively are acyclic. Let us choose an additive "flasque" resolution functor r on the category of sheaves of Λ-modules on $\mathrm{Écl}_{Y,\mathscr{E}}(X)_{\text{ét}}$ and denote, by abuse of notation, $\mathrm{R}\pi_\star$ the (additive) functor from the category of (bounded below) complexes of sheaves of Λ-modules on $\mathrm{Écl}_{Y,\mathscr{E}}(X)$ to the category of complexes of sheaves of Λ-modules on X defined by the formula $\mathrm{R}\pi_\star K = \mathrm{Tot}(\pi_\star r K)$. This functor preserves quasi-isomorphisms and induces the usual functor $\mathrm{R}\pi_\star :\ \mathsf{D}^+(\mathrm{Écl}_{Y,\mathscr{E}}(X)_{\text{ét}}, \Lambda) \to \mathsf{D}^+(X_{\text{ét}}, \Lambda)$.

We thus obtain a commutative diagram of complexes of sheaves of Λ-modules on X :

$$\begin{tikzcd} j_! L_{|U} \arrow[r] \arrow[d] & L_{|X} \arrow[r] \arrow[d] & i_\star L_{|Y} \arrow[d] \\ \mathrm{R}\pi_\star j'_! L_{|U} \arrow[r] & \mathrm{R}\pi_\star L_{|\mathrm{Écl}_{Y,\mathscr{E}}(X)} \arrow[r] & \mathrm{R}\pi_\star i'_\star L_{|\mathbf{P}(\mathscr{E})} \end{tikzcd}$$

The rows of this diagram constitute double complexes whose associated total complexes are acyclic. By the base change theorem for a proper morphism, the morphism $j_! L_{|U} \to \mathrm{R}\pi_\star j'_! L_{|U}$ is a quasi-isomorphism. We deduce that the right square is homotopy bicartesian, which allows us to conclude.

Let us return to the proof of Proposition 2.2.2.1. The projective bundle formula for $\mathbf{P}(\mathscr{E})$ implies that we have a distinguished triangle in $\mathrm{D}^+(X_{\text{ét}}, \Lambda)$:

$$i_\star\Lambda \to \mathrm{R}\pi_\star i'_\star\Lambda \to \bigoplus_{i=1}^{r-1} i_\star\Lambda(-i)\,[-2i] \xrightarrow{0} i_\star\Lambda\,[1] \ .$$

By considering the columns of the homotopy bicartesian square given by the lemma, we can conclude that there exists a distinguished triangle

$$\Lambda \to \mathrm{R}\pi_\star\Lambda \to \bigoplus_{i=1}^{r-1} i_\star\Lambda(-i)\,[-2i] \to \Lambda\,[1] \ .$$

This triangle is split by the powers of the element $c_1(\mathscr{O}(1), 1_U)$; the morphism on the right is therefore zero, which completes the proof of the proposition.

Corollaire 2.2.2.3. *The following sequence, whose morphisms are obvious, is exact :*

$$0 \to \mathrm{H}^{2r}_Y(X, \Lambda(r)) \to \mathrm{H}^{2r}_{\mathbf{P}(\mathscr{E})}(\mathrm{\acute{E}cl}_{Y,\mathscr{E}}(X), \Lambda(r)) \to \mathrm{Coker}(\mathrm{H}^{2r}(Y, \Lambda(r)) \to \mathrm{H}^{2r}(\mathbf{P}(\mathscr{E}), \Lambda(r))) \to 0$$

The statement of this corollary holds, of course, in any bidegree (p, q) and not just in bidegree $(2r, r)$, but we will essentially only use this particular case.

We denote by $\mathrm{Gys} : \mathrm{H}^p(\mathbf{P}(\mathscr{E}), \Lambda(q)) \to \mathrm{H}^{p+2}_{\mathbf{P}(\mathscr{E})}(\mathrm{\acute{E}cl}_{Y,\mathscr{E}}(X), \Lambda(q+1))$ the Gysin morphism associated with the pseudo-divisor $-(\mathscr{O}(1), 1_U)$ on $\mathrm{\acute{E}cl}_{Y,\mathscr{E}}(X)$ and $\xi = c_1(\mathscr{O}(1)) \in \mathrm{H}^2(\mathbf{P}(\mathscr{E}), \Lambda(1))$. The following lemma is clear :

Lemme 2.2.2.4. *The composite morphism*

$$\mathrm{H}^p(\mathbf{P}(\mathscr{E}), \Lambda(q)) \xrightarrow{\mathrm{Gys}} \mathrm{H}^{p+2}_{\mathbf{P}(\mathscr{E})}(\mathrm{\acute{E}cl}_{Y,\mathscr{E}}(X), \Lambda(q+1)) \to \mathrm{H}^{p+2}(\mathbf{P}(\mathscr{E}), \Lambda(q+1)) \ ,$$

where the right arrow is the restriction morphism, is multiplication by $-\xi$.

Définition 2.2.2.5. We define an element $\mathrm{Clf}_{i,\mathscr{E}}$ of $\mathrm{H}^{2r-2}(\mathbf{P}(\mathscr{E}), \Lambda(r-1))$ by the formula :

$$\mathrm{Clf}_{i,\mathscr{E}} = \xi^{r-1} - c_1(\mathscr{E})\xi^{r-2} + \cdots + (-1)^{r-1}c_{r-1}(\mathscr{E}) \ .$$

Lemme 2.2.2.6. *In* $\mathrm{H}^{2r}(\mathbf{P}(\mathscr{E}), \Lambda(r))$, *we have the equality*

$$-\xi\mathrm{Clf}_{i,\mathscr{E}} = (-1)^r c_r(\mathscr{E}) \ .$$

If $\mathscr{E}$ is a vector bundle of rank r on X, we can introduce the polynomial $P_t(\mathscr{E}) = \sum_{i=0}^r (-1)^i c_i(\mathscr{E}) t^{r-i}$ in an indeterminate t with coefficients in the commutative ring $\oplus_n H^{2n}(X, \Lambda(n))$. We can write :

$$P_t(\xi) = tG_t(\mathscr{E}) + (-1)^r c_r(\mathscr{E}) \text{ where } G_t(\mathscr{E}) = \sum_{i=0}^{r-1} (-1)^i c_i(\mathscr{E}) t^{r-1-i} \ .$$

When we perform the substitution $t := \xi \in H^2(\mathbf{P}(\mathscr{E}), \Lambda(1))$, by definition of $\mathrm{Clf}_{i,\mathscr{E}}$ we have $\mathrm{Clf}_{i,\mathscr{E}} = G_\xi(\mathscr{E})$, and the definition of the Chern classes gives the relation $0 = P_\xi(\mathscr{E}) = \xi\mathrm{Clf}_{i,\mathscr{E}} + (-1)^r c_r(\mathscr{E})$, so that $-\xi\mathrm{Clf}_{i,\mathscr{E}} = (-1)^r c_r(\mathscr{E})$.

Définition 2.2.2.7. Taking into account Corollary 2.2.2.3, Lemmas 2.2.2.4 and 2.2.2.6 show that the element $\mathrm{Gys}(\mathrm{Clf}_{i,\mathscr{E}}) \in \mathrm{H}^{2r}_{\mathbf{P}(\mathscr{E})}(\mathrm{\acute{E}cl}_{Y,\mathscr{E}}(X), \Lambda(r))$ comes by restriction from a unique element of $\mathrm{H}^{2r}_Y(X, \Lambda(r))$, denoted by $\mathrm{Cl}_{i,\mathscr{E}}$.

2.2.3. *Properties of generalized classes.*

Proposition 2.2.3.1. *The formation of the generalized classes* $\mathrm{Cl}_{i,\mathscr{E}}$ *and* $\mathrm{Clf}_{i,\mathscr{E}}$ *is compatible with any base change* $X' \to X$.

Taking into account Proposition 2.2.1.5, this results immediately from the definitions.

Proposition 2.2.3.2. *Let* $\mathscr{E}' \to \mathscr{E}$ *be an epimorphism of locally free Modules on* Y. *Let* $\mathscr{K}$ *be the kernel of this epimorphism. Assume that* $\mathscr{E}'$ *has constant rank* $\mathbf{r}'$. *We then have the relation*

$$\mathrm{Cl}_{i,\mathscr{E}'} = (-1)^{r'-r} c_{r'-r}(\mathscr{K}) \cdot \mathrm{Cl}_{i,\mathscr{E}}$$

in $\mathrm{H}^{2r'}_Y(X, \Lambda(r'))$, *where we have used the canonical pairings*

$$\mathrm{H}^a(Y, \Lambda(b)) \otimes \mathrm{H}^{a'}_Y(X, \Lambda(b')) \to \mathrm{H}^{a+a'}_Y(X, \Lambda(b+b'))\,.$$

We have a closed immersion of $\mathrm{\acute{E}cl}_{Y,\mathscr{E}}(X)$ into $\mathrm{\acute{E}cl}_{Y,\mathscr{E}'}(X)$, which allows us to consider the following composition of restriction arrows :

$$\mathrm{H}^{2r'}_Y(X, \Lambda(r')) \to \mathrm{H}^{2r'}_{\mathbf{P}(\mathscr{E}')}(\mathrm{\acute{E}cl}_{Y,\mathscr{E}'}(X), \Lambda(r')) \to \mathrm{H}^{2r'}_{\mathbf{P}(\mathscr{E})}(\mathrm{\acute{E}cl}_{Y,\mathscr{E}}(X), \Lambda(r'))\,.$$

Since this composition is injective, we need to show that the images of the two elements considered in $\mathrm{H}^{2r'}_{\mathbf{P}(\mathscr{E})}(\mathrm{\acute{E}cl}_{Y,\mathscr{E}}(X), \Lambda(r'))$ are equal. But since these two elements are naturally defined as being the images of elements of $\mathrm{H}^{2r'-2}(\mathbf{P}(\mathscr{E}), \Lambda(r'-1))$ by the Gysin morphism Gys associated with the line bundle $\mathscr{O}(-1)$ on $\mathrm{\acute{E}cl}_{Y,\mathscr{E}}(X)$ canonically trivialized on $X - Y$, we are reduced to showing the equality

$$\mathrm{Clf}_{i,\mathscr{E}'_{|\mathbf{P}(\mathscr{E})}} = (-1)^{r'-r} c_{r'-r}(\mathscr{K}) \cdot \mathrm{Clf}_{i,\mathscr{E}}$$

in $\mathrm{H}^{2r'}(\mathbf{P}(\mathscr{E}), \Lambda(r'))$.

We reuse the notations of Lemma 2.2.2.6. The Cartan-Whitney formula applied to the short exact sequence $0 \to \mathscr{K} \to \mathscr{E}' \to \mathscr{E} \to 0$ gives the following relation : $P_t(\mathscr{E}') = P_t(\mathscr{K}) P_t(\mathscr{E})$, or

$$tG_t(\mathscr{E}') + (-1)^{r'} c_{r'}(\mathscr{E}') = tG_t(\mathscr{K}) P_t(\mathscr{E}) + (-1)^{r'-r} c_{r'-r}(K)(tG_t(\mathscr{E}) + (-1)^r c_r(\mathscr{E}))\,.$$

This implies the identity $G_t(\mathscr{E}') = G_t(\mathscr{K}) P_t(\mathscr{E}) + (-1)^{r'-r} c_{r'-r}(\mathscr{K}) G_t(\mathscr{E})$. By making the substitution $t := \xi \in H^2(\mathbf{P}(\mathscr{E}), \Lambda(1))$, we obtain the equality $G_\xi(\mathscr{E}') = (-1)^{r'-r} c_{r'-r}(\mathscr{K}) G_\xi(\mathscr{E})$, which is none other than the desired relation.

2.3. Regular immersions. We recall that the notion of a regular immersion is defined in [**SGA 6** vii 1.4].

Définition 2.3.1. Let $i\colon Y \to X$ be a regular immersion between $\mathbf{Z}\left[\frac{1}{n}\right]$-schemes. We set $i^? = i^\star(c)\,[2c] : \mathrm{D}^+(X_{\mathrm{\acute{e}t}}, \Lambda) \to \mathrm{D}^+(Y_{\mathrm{\acute{e}t}}, \Lambda)$ where c is the codimension of i. We define a morphism $\mathrm{Cl}_i\colon \Lambda \to i^?\Lambda$ in $\mathrm{D}^+(Y_{\mathrm{\acute{e}t}}, \Lambda)$ in the following way. By decomposing Y into a disjoint union of open-closed subschemes, we can assume that the codimension c of i is constant. If $c = 0$, i is the inclusion of an open subscheme, and Cl_i is the obvious isomorphism. In the case where $c > 0$, let us choose an open set U of X in which Y is a closed subscheme, and let $i'\colon Y \to U$ be this closed immersion. The conormal sheaf $\mathscr{N}_{X/Y}$ of Y in X is a vector bundle of rank c on Y equipped with the tautological epimorphism $\mathscr{N}_{X/Y} \to \mathscr{N}_{X/Y}$; we can therefore consider the class $\mathrm{Cl}_{i'} = \mathrm{Cl}_{i',\mathscr{N}_{X/Y}} \in \mathrm{H}^{2c}_Y(U, \Lambda(c))$, which we identify with a morphism $\mathrm{Cl}_i\colon \Lambda \to i'^?\Lambda \simeq i^?\Lambda$ in $\mathrm{D}^+(Y_{\mathrm{\acute{e}t}}, \Lambda)$; it is clear that the construction does not depend on the intermediate open set U. If the immersion i is closed, we denote by Gys_i or $\mathrm{Gys}_{Y\subset X}$ the morphisms $\mathrm{H}^p(Y, \Lambda(q)) \to \mathrm{H}^{p+2c}_Y(X, \Lambda(q+c))$ induced by multiplication by $\mathrm{Cl}_i \in \mathrm{H}^{2c}_Y(X, \Lambda(c))$[(vii)]. We similarly denote the versions with supports $\mathrm{H}^p_Z(Y, \Lambda(q)) \to \mathrm{H}^{p+2c}_Z(X, \Lambda(q+c))$ defined in the same way for any closed subscheme Z of Y.

Propositions 2.2.3.1 and 2.2.3.2 immediately imply the "excess intersection formula" (analogous to [**Fulton, 1998**, theorem 6.3] where it is stated in Chow theory) :

(vii) This definition is of course compatible with the one already given in codimension 1 in Definition 2.1.1.

Proposition 2.3.2 (Excess intersection formula). *Suppose we have a Cartesian square in the category of* $\mathbf{Z}\left[\frac{1}{n}\right]$*-schemes :*

$$\begin{tikzcd} Y' \arrow[r, "i'"] \arrow[d, "q"] & X' \arrow[d, "f"] \\ Y \arrow[r, "i"] & X \end{tikzcd}$$

where $i\colon Y \to X$ *is a regular closed immersion of codimension* c*. Suppose that* $i'\colon Y' \to X'$ *is a regular immersion of codimension* c'*, and let* $\mathscr{K} := \mathrm{Ker}(q^\star \mathscr{N}_{X/Y} \to \mathscr{N}_{X'/Y'})$ *be the excess conormal sheaf. It is a vector bundle of rank* $e := c - c'$ *on* Y'*. We then have the equality* $f^\star \mathrm{Cl}_i = (-1)^e c_e(\mathscr{K}) \cdot \mathrm{Cl}_{i'} \in H^{2c}_{Y',\text{ét}}(X', \Lambda(c))$*.*

In particular, if $i'\colon Y' \to X'$ *is a regular closed immersion of the same codimension as* $i\colon Y \to X$*, then* $f^\star \mathrm{Cl}_i = \mathrm{Cl}_{i'} \in H^{2c}_{Y',\text{ét}}(X', \Lambda(c))$*.*

The following theorem generalizes the statement established in [**Fujiwara, 2002**, proposition 1.2.1] :

Théorème 2.3.3. *If* $Z \xrightarrow{i} Y$ *and* $Y \xrightarrow{j} X$ *are two composable regular immersions, the following diagram is commutative in* $\mathrm{D}^+(Z_{\text{ét}}, \Lambda)$ *:*

$$\begin{tikzcd} \Lambda \arrow[r, "\mathrm{Cl}_i"] \arrow[dr, "\mathrm{Cl}_{j\circ i}"'] & i^?\Lambda \arrow[d, "i^?(\mathrm{Cl}_j)"] \\ & i^?j^?\Lambda \end{tikzcd}$$

We can obviously assume that the immersions i and j are closed immersions and that the codimensions of i and j are constant, with respective values m and n. If $m = 0$ or $n = 0$, it is trivial; we therefore assume that $m > 0$ and $n > 0$.

Lemme 2.3.4. *We can assume that* $n = 1$ *(i.e.,* j *is of codimension 1).*

We blow up Y in X to obtain the following diagram where the squares are Cartesian :

$$\begin{tikzcd} P' \arrow[r, "i'"] \arrow[d, "p'"] & P \arrow[r, "j'"] \arrow[d, "p"] & \mathrm{\acute{E}cl}_Y(X) \arrow[d, "\pi"] \\ Z \arrow[r, "i"] & Y \arrow[r, "j"] & X \end{tikzcd}$$

The idea of the proof will be to use the excess intersection formula 2.3.2 for the immersions $j \circ i\colon Z \to X$ and $j\colon Y \to X$ relative to the base change $\pi\colon \mathrm{\acute{E}cl}_Y(X) \to X$, which will reduce the codimension of these regular closed immersions.

We have canonical isomorphisms $P = \mathbf{P}(\mathscr{N}_{X/Y})$ and $P' = \mathbf{P}(\mathscr{N}_{X/Y|Z})$. We easily check that $P \to \mathrm{\acute{E}cl}_Y(X)$ is a regular closed immersion of codimension 1. By smooth base change, $P' \to P$ is a regular closed immersion of codimension m. We assume that $i'^?(\mathrm{Cl}_{j'}) \circ \mathrm{Cl}_{i'} = \mathrm{Cl}_{j'\circ i'}$ and we want to show that $i^?(\mathrm{Cl}_j) \circ \mathrm{Cl}_i = \mathrm{Cl}_{j\circ i}$. The morphisms to be compared can be identified with elements of $\mathrm{H}^{2(m+n)}_Z(X, \Lambda(m+n))$ (we will make this type of identification until the end of the proof). Proposition 2.2.2.1 implies that the map

$$\pi^\star\colon \mathrm{H}^{2(m+n)}_Z(X, \Lambda(m+n)) \to \mathrm{H}^{2(m+n)}_{P'}(\mathrm{\acute{E}cl}_Y(X), \Lambda(m+n))$$

is injective, so it suffices to compare the classes after applying $\pi^\star$.

According to the excess intersection formula 2.3.2, we have the equality $\pi^\star(\mathrm{Cl}_{j\circ i}) = c_{n-1}(\mathscr{E}'^\vee) \cdot \mathrm{Cl}_{j'\circ i'} \in \mathrm{H}^{2(m+n)}_{P'}(\mathrm{\acute{E}cl}_Y(X), \Lambda(m+n))$ where $\mathscr{E}'$ is the vector bundle of rank $n-1$ that is the kernel of the epimorphism $p'^\star \mathscr{N}_{X/Z} \to \mathscr{N}_{\mathrm{\acute{E}cl}_Y(X)/P'}$.

The composition of classes temporarily assumed for the immersions j' and i' gives the equality

$$\mathrm{Cl}_{j'\circ i'} = \mathrm{Cl}_{i'} \cdot \mathrm{Cl}_{j'}$$

via the pairing

$$\mathrm{H}^{2m}_{P'}(P,\Lambda(m))\times\mathrm{H}^{2}_{P}(\mathrm{Écl}_Y(X),\Lambda(1))\to\mathrm{H}^{2(m+1)}_{P'}(\mathrm{Écl}_Y(X),\Lambda(n+1))\ .$$

We have thus obtained :

$$\pi^\star\mathrm{Cl}_{j\circ i}=c_{n-1}(\mathscr{E}'^\vee)\cdot\mathrm{Cl}_{i'}\cdot\mathrm{Cl}_{j'}\ .$$

Let $\mathscr{E}$ be the kernel of the epimorphism $p^\star\mathscr{N}_{X/Y}\to\mathscr{N}_{\mathrm{Écl}_Y(X)/P}$. It follows immediately that in the obvious diagram of Modules on P' that follows, the rows and columns are exact :

$$\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & i'^\star\mathscr{E} \arrow[r] \arrow[d] & \mathscr{E}' \arrow[r] \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & i'^\star p^\star\mathscr{N}_{X/Y} \arrow[r] \arrow[d] & p'^\star\mathscr{N}_{X/Z} \arrow[r] \arrow[d] & p'^\star\mathscr{N}_{Y/Z} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & i'^\star\mathscr{N}_{\mathrm{Écl}_Y(X)/P} \arrow[r] \arrow[d] & \mathscr{N}_{\mathrm{Écl}_Y(X)/P'} \arrow[r] \arrow[d] & \mathscr{N}_{P/P'} \arrow[r] \arrow[d] & 0 \\
& 0 & 0 & 0 &
\end{tikzcd}$$

In particular, we obtain a canonical isomorphism $i'^\star\mathscr{E}\xrightarrow{\sim}\mathscr{E}'$, hence $i'^\star c_{n-1}(\mathscr{E}^\vee)=c_{n-1}(\mathscr{E}'^\vee)\in\mathrm{H}^{2(n-1)}(P',\Lambda(n-1))$. We deduce :

$$\pi^\star\mathrm{Cl}_{j\circ i}=c_{n-1}(\mathscr{E}'^\vee)\cdot\mathrm{Cl}_{i'}\cdot\mathrm{Cl}_{j'}=\mathrm{Cl}_{i'}\cdot c_{n-1}(\mathscr{E}^\vee)\cdot\mathrm{Cl}_{j'}\ .$$

We implicitly use in these notations the associativity of the multiplicative structures, allowing, for example, to define a map

$$\mathrm{H}^{2m}_{P'}(P,\Lambda(m))\times\mathrm{H}^{2(n-1)}(P,\Lambda(n-1))\times\mathrm{H}^{2}_{P}(\mathrm{Écl}_Y(X),\Lambda(1))\to\mathrm{H}^{2(m+n)}_{P'}(\mathrm{Écl}_Y(X),\Lambda(m+n))$$

without having to worry about the order in which the multiplications are performed. The excess intersection formula 2.3.2 implies the following equality :

$$\pi^\star\mathrm{Cl}_j=c_{n-1}(\mathscr{E}^\vee)\cdot\mathrm{Cl}_{j'}\in\mathrm{H}^{2n}_{P}(\mathrm{Écl}_Y(X),\Lambda(n))\ .$$

The morphism p being smooth, we immediately have $\mathrm{Cl}_{i'}=\pi^\star\mathrm{Cl}_i$. We have thus obtained the desired equality :

$$\pi^\star\mathrm{Cl}_{j\circ i}=\pi^\star\mathrm{Cl}_i\cdot\pi^\star\mathrm{Cl}_j\ ,$$

which completes the proof of the lemma.

We are reduced to establishing Theorem 2.3.3 in the case where j has codimension 1. We now set $P=\mathbf{P}(\mathscr{N}_{X/Z})$ and $P'=\mathbf{P}(\mathscr{N}_{Y/Z})$. The following diagram summarizes the situation :

$$\begin{tikzcd}
P \arrow[r] \arrow[d] & \pi^{-1}(Y) \arrow[r] \arrow[d] & \mathrm{Écl}_Z(X) \arrow[dd, "\pi"] \\
P' \arrow[r] \arrow[d] & \mathrm{Écl}_Z(Y) \arrow[d] & \\
Z \arrow[r, "i"] & Y \arrow[r, "j"] & X
\end{tikzcd}$$

We want to establish the following equality in $\mathrm{H}^{2m+2}_{Z}(X,\Lambda(m+1))$:

$$\mathrm{Cl}_{j\circ i}=\mathrm{Cl}_i\cdot\mathrm{Cl}_j\ .$$

By Proposition 2.2.2.1, it suffices to check this equality in $\mathrm{H}^{2m+2}_{P}(\mathrm{Écl}_Z(X),\Lambda(m+1))$ after applying $\pi^\star$.

By definition, the class $\mathrm{Cl}_{j\circ i} \in \mathrm{H}^{2m+2}_Z(X, \Lambda(m+1))$ « restricts » to an element

$$\gamma := \pi^\star \mathrm{Cl}_{j\circ i} = \mathrm{Gys}_{P\subset \mathrm{\acute{E}cl}_Z(X)}(\mathrm{Clf}_{j\circ i})$$

in $\mathrm{H}^{2m+2}_P(\mathrm{\acute{E}cl}_Z(X), \Lambda(m+1))$ where $\mathrm{Clf}_{j\circ i} \in \mathrm{H}^{2m}(P, \Lambda(m))$.

Let $\mathscr{I}$ be the Ideal of Y in X, $\mathscr{I}_P$ that of P in $\mathrm{\acute{E}cl}_Z(X)$, and $\tilde{\mathscr{I}}$ that of $\mathrm{\acute{E}cl}_Z(Y)$ in $\mathrm{\acute{E}cl}_Z(X)$. We have a canonical isomorphism of invertible sheaves on $\mathrm{\acute{E}cl}_Z(X)$:

$$\pi^\star \mathscr{I} \simeq \mathscr{I}_P \otimes \tilde{\mathscr{I}} .$$

This isomorphism is compatible with the given trivializations on $\pi^{-1}(V)$ where $V = X - Y$. We thus obtain an equality in the group of equivalence classes of such pseudo-divisors, which allows us to decompose $\pi^\star \mathrm{Cl}_j = -\pi^\star(c_1(\mathscr{I}, 1_{X-Y})) \in \mathrm{H}^2_{\pi^{-1}(V)}(\mathrm{\acute{E}cl}_Z(X), \Lambda(1))$ into a sum of two components :

$$\begin{aligned} \pi^\star \mathrm{Cl}_j &= -\pi^\star(c_1(\mathscr{I}, 1_{X-Y})) = -c_1(\pi^\star \mathscr{I}, 1_{\pi^{-1}(X-Y)}) \\ &= -c_1(\mathscr{I}_P, 1_{\mathrm{\acute{E}cl}_Z(X)-P}) - c_1(\tilde{\mathscr{I}}, 1_{\mathrm{\acute{E}cl}_Z(X)-\mathrm{\acute{E}cl}_Z(Y)}) . \end{aligned}$$

We deduce a decomposition

$$\pi^\star \mathrm{Cl}_i \cdot \pi^\star \mathrm{Cl}_j = \alpha + \beta$$

in $\mathrm{H}^{2m+2}_P(\mathrm{\acute{E}cl}_Z(X), \Lambda(m+1))$ where

$$\begin{aligned} \alpha &= \mathrm{Gys}_{P\subset \mathrm{\acute{E}cl}_Z(X)}(\mathrm{Cl}_{i|P}) , \\ \beta &= \mathrm{Gys}_{\mathrm{\acute{E}cl}_Z(Y)\subset \mathrm{\acute{E}cl}_Z(X)}(\mathrm{Gys}_{P'\subset \mathrm{\acute{E}cl}_Z(Y)}(\mathrm{Clf}_i)) . \end{aligned}$$

The computation of Cl_k where k is the inclusion of the intersection of Cartier divisors intersecting transversely in the ambient scheme, carried out in [**Fujiwara, 2002**, proposition 1.1.4], allows us to obtain the following equality of operators :

$$\mathrm{Gys}_{\mathrm{\acute{E}cl}_Z(Y)\subset \mathrm{\acute{E}cl}_Z(X)} \circ \mathrm{Gys}_{P'\subset \mathrm{\acute{E}cl}_Z(Y)} = \mathrm{Gys}_{P\subset \mathrm{\acute{E}cl}_Z(X)} \circ \mathrm{Gys}_{P'\subset P} .$$

Our goal is to establish the equality $\gamma = \alpha + \beta$. The preceding calculations allow us to write each of the elements α, β, and γ as the images under the morphism $\mathrm{Gys}_{P\subset \mathrm{\acute{E}cl}_Z(X)}$ of classes $\tilde{\alpha}$, $\tilde{\beta}$, and $\tilde{\gamma}$ in $\mathrm{H}^{2m}(P, \Lambda(m))$:

$$\begin{aligned} \tilde{\alpha} &= \mathrm{Cl}_{i|P} , \\ \tilde{\beta} &= \mathrm{Gys}_{P'\subset P}(\mathrm{Clf}_i) , \\ \tilde{\gamma} &= \mathrm{Clf}_{j\circ i} . \end{aligned}$$

We are thus reduced to establishing the equality $\tilde{\gamma} = \tilde{\alpha} + \tilde{\beta}$ in $\mathrm{H}^{2m}(P, \Lambda(m))$.

By Lemmas 2.2.2.4 and 2.2.2.6, we have $\mathrm{Cl}_{i|Z} = (-1)^m c_m(\mathscr{N}_{Y/Z})$. We deduce the equality

$$\tilde{\alpha} = (-1)^m c_m(\mathscr{N}_{Y/Z}) .$$

To compute $\tilde{\beta}$, we observe that the Ideal of P' in P identifies with the invertible sheaf $\mathscr{K} \otimes_{\mathscr{O}_Z} \mathscr{O}(-1)$ where $\mathscr{K} = \mathscr{N}_{X/Y|Z}$ is the kernel of the epimorphism $\mathscr{N}_{X/Z} \to \mathscr{N}_{Y/Z}$. We deduce

$$\tilde{\beta} = (\xi - c_1(\mathscr{K})) \cdot \left[\xi^{m-1} - c_1(\mathscr{N}_{Y/Z})\xi^{m-2} + \cdots + (-1)^{m-1} c_{m-1}(\mathscr{N}_{Y/Z})\right] .$$

Furthermore, the definition of $\tilde{\gamma}$ gives the equality :

$$\tilde{\gamma} = \xi^m - c_1(\mathscr{N}_{X/Z})\xi^{m-1} + \cdots + (-1)^m c_m(\mathscr{N}_{X/Z}) .$$

The Cartan-Whitney formula applied to the short exact sequence

$$0 \to \mathscr{K} \to \mathscr{N}_{X/Z} \to \mathscr{N}_{Y/Z} \to 0$$

of vector bundles on Z immediately yields the desired relation $\tilde{\gamma} = \tilde{\alpha} + \tilde{\beta}$, which completes the proof of the theorem.

Lemme 2.3.5. *Suppose we have a Cartesian square in the category of* $\mathbf{Z}\left[\frac{1}{n}\right]$*-schemes :*

$$\begin{tikzcd} Y_{12} \arrow[r, "i_1'"] \arrow[d, "i_2'"'] & Y_2 \arrow[d, "i_2"] \\ Y_1 \arrow[r, "i_1"] & X \end{tikzcd}$$

Suppose further that i_1 *and* i_1' *(resp.* i_2 *and* i_2'*) are two regular closed immersions* of the same codimension c_1 *(resp.* c_2*). Then, we have the following equality between operators* $H^p(Y_{12}, \Lambda(q)) \to H^{p+2c_1+2c_2}_{Y_{12}}(X, \Lambda(q + c_1 + c_2))$ *:*

$$\mathrm{Gys}_{Y_1 \subset X} \circ \mathrm{Gys}_{Y_{12} \subset Y_1} = \mathrm{Gys}_{Y_2 \subset X} \circ \mathrm{Gys}_{Y_{12} \subset Y_2} \ .$$

By Proposition 2.3.2, the image of Cl_{i_1} under $i_2^\star : H^{2c_1}_{Y_1}(X, \Lambda(c_1)) \to H^{2c_1}_{Y_{12}}(Y_2, \Lambda(c_1))$ is $\mathrm{Cl}_{i_1'}$. This allows us to show that the operator $\mathrm{Gys}_{Y_2 \subset X} \circ \mathrm{Gys}_{Y_{12} \subset Y_2}$ is induced by the product $\mathrm{Cl}_{i_1} \cdot \mathrm{Cl}_{i_2} \in H^{2c_1+2c_2}_{Y_{12}}(X, \Lambda(c_1 + c_2))$. By symmetry of the roles of Y_1 and Y_2, we obtain the identity stated in the lemma.

Remarque 2.3.6. Once Theorem 2.3.3 is known, we can observe that the two operators appearing in the preceding lemma are equal to $\mathrm{Gys}_{Y_{12} \subset X}$, which also means that the product $\mathrm{Cl}_{i_1} \cdot \mathrm{Cl}_{i_2} \in H^{2c_1+2c_2}_{Y_{12}}(X, \Lambda(c_1 + c_2))$ is equal to Cl_k where $k : Y_{12} \to X$ is the inclusion of the intersection of Y_1 and Y_2 in X.

The class we have defined is compatible with the one defined locally in [**SGA 4½** [Cycle] 2.2] :

Proposition 2.3.7. *Let* $i : Y \to X$ *be a regular immersion of codimension* c *between* $\mathbf{Z}\left[\frac{1}{n}\right]$*-schemes. The morphism of sheaves* $\Lambda \to \mathscr{H}^{2c}(i^!\Lambda(c))$ *induced by the morphism* $\mathrm{Cl}_i : \Lambda \to i^?\Lambda$ *is given by the class* $\mathrm{cl}\, Y$ *of* [**SGA 4½** [Cycle] 2.2].

The reader interested in sign issues may consult 4.8...

2.4. Smooth morphisms. Let $p : X \to S$ be a compactifiable smooth morphism of $\mathbf{Z}\left[\frac{1}{n}\right]$-schemes of relative dimension d. By [**SGA 4** xviii 2.9], we have a trace morphism

$$\mathrm{Tr}_p : \mathrm{R}^{2d} p_! \Lambda(d) \to \Lambda \ ,$$

which we can reinterpret as a morphism

$$\mathrm{R}p_! \Lambda(d)\,[2d] \to \Lambda$$

in $\mathrm{D}^+(S_{\text{ét}}, \Lambda)$ (indeed, by the base change theorem for a proper morphism and [**SGA 4** x 4.3], the sheaves $\mathrm{R}^i p_! \Lambda$ are zero for $i > 2d$).

Définition 2.4.1. Let $p : X \to S$ be a compactifiable smooth morphism of $\mathbf{Z}\left[\frac{1}{n}\right]$-schemes. The morphism $\mathrm{Cl}_p : \Lambda \to p^?\Lambda$ [(viii)] in $\mathrm{D}^+(X_{\text{ét}}, \Lambda)$ is the morphism deduced by adjunction from the morphism $\mathrm{R}p_!\Lambda(d)\,[2d] \to \Lambda$ defined above.

According to [**SGA 4** xviii 3.2.4], this morphism Cl_p is an isomorphism : this is Poincaré duality.

Proposition 2.4.2. *If* $f : Z \to Y$ *and* $g : Y \to X$ *are composable compactifiable smooth morphisms, the following diagram is commutative in* $\mathrm{D}^+(Z_{\text{ét}}, \Lambda)$ *:*

$$\begin{tikzcd} \Lambda \arrow[r, "\mathrm{Cl}_f"] \arrow[dr, "\mathrm{Cl}_{g \circ f}"'] & f^?\Lambda \arrow[d, "f^?(\mathrm{Cl}_g)"] \\ & f^? g^? \Lambda \end{tikzcd}$$

(viii) We recall that we set $p^? = p^!(-d)\,[-2d]$ where d is the relative dimension of p.

This is stated in [**SGA 4** XVIII 3.2.4] and results from the compatibility of trace morphisms with composition, cf. property (Var 3) in [**SGA 4** XVIII 2.9].

REMARQUE 2.4.3. If this theory had been available to us, it might have been more convenient to use here the construction of the functors $f^!$ for smoothable f mentioned in the introduction of [**SGA 4** XVIII 0.4]. In the axiomatic framework of "stable homotopy functors", this is realized in [**Ayoub, 2007**].

2.5. Smoothable complete intersection morphisms.

DÉFINITION 2.5.1. A **complete intersection** morphism is a morphism $X \xrightarrow{f} S$ that locally admits a factorization of the form $X \xrightarrow{i} T \xrightarrow{p} S$ where p is smooth and i is a regular immersion (cf. [**SGA 6** VII 1.4]). We set $\dim.\mathrm{rel.virt.}\, f = \dim p - \operatorname{codim} i$: this is the virtual relative dimension of f (cf. [**SGA 6** VIII 1.9]). It is a locally constant function $X \to \mathbf{Z}$.

DÉFINITION 2.5.2. We denote by $\mathscr{S}$ the category whose objects are quasi-compact $\mathbf{Z}\left[\frac{1}{n}\right]$-schemes admitting an ample invertible sheaf and whose morphisms are morphisms of finite type between such schemes. We denote by $\mathscr{S}^{\mathrm{ic}}$ the subcategory of $\mathscr{S}$ having the same objects but whose morphisms are complete intersection morphisms.

In $\mathscr{S}$, any morphism $X \to Y$ can be factored as $X \xrightarrow{i} \mathbf{P}^n_Y \xrightarrow{\pi} Y$ where i is an immersion and π is the canonical projection. All morphisms in $\mathscr{S}$ are therefore compactifiable, and we can apply to them the formalism of the functors $\mathrm{R}f_!$ and $f^!$.

The morphisms of $\mathscr{S}^{\mathrm{ic}}$ admit global factorizations in $\mathscr{S}^{\mathrm{ic}}$ as a regular closed immersion followed by a smooth morphism.

DÉFINITION 2.5.3. For any morphism $f\colon X \to Y$ in $\mathscr{S}^{\mathrm{ic}}$, we can define a functor

$$f^?\colon \mathrm{D}^+(Y_{\text{ét}}, \Lambda) \to \mathrm{D}^+(X_{\text{ét}}, \Lambda)$$

by the formula $f^? = f^!(-d)[-2d]$ where $d = \dim.\mathrm{rel.virt.}\, f$ [(ix)].

The functors $f^?$ are the inverse image functors for a suitable fibered category structure over the category $\mathscr{S}^{\mathrm{ic}}$: we will implicitly use the transitivity isomorphisms $f^?g^? \simeq (gf)^?$ associated with the composition of two composable morphisms in $\mathscr{S}^{\mathrm{ic}}$.

DÉFINITION 2.5.4. Let $f\colon X \to S$ be a morphism in $\mathscr{S}^{\mathrm{ic}}$. Suppose we are given a factorization of f in $\mathscr{S}^{\mathrm{ic}}$ of the form $X \xrightarrow{i} Y \xrightarrow{p} S$ where i is a regular immersion and p is a smooth morphism. We define a morphism

$$\mathrm{Cl}_{p,i}\colon \Lambda \to f^?\Lambda$$

in $\mathrm{D}^+(X_{\text{ét}}, \Lambda)$ as the composite morphism

$$\begin{tikzcd} \Lambda \arrow[r, "\mathrm{Cl}_i"] \arrow[dr, "\mathrm{Cl}_{p,i}"'] & i^?\Lambda \arrow[d, "i^?(\mathrm{Cl}_p)"] \\ & f^?\Lambda \end{tikzcd}$$

where Cl_i is the morphism from Definition 2.3.1 and Cl_p is that from Definition 2.4.1.

THÉORÈME 2.5.5. *Let $f\colon X \to S$ be a morphism in $\mathscr{S}^{\mathrm{ic}}$. If $X \xrightarrow{i} Y \xrightarrow{p} S$ and $X \xrightarrow{i'} Y' \xrightarrow{p'} S$ are two factorizations of the type considered in Definition 2.5.4, then the following two morphisms in the category $\mathrm{D}^+(X_{\text{ét}}, \Lambda)$ are equal :*

$$\mathrm{Cl}_{p,i} = \mathrm{Cl}_{p',i'}\colon \Lambda \to f^?\Lambda\,.$$

(ix) We can give meaning to this definition even if the virtual relative dimension is not constant. We then define $f^?K$ for any $K \in \mathrm{D}^+(X_{\text{ét}}, \Lambda)$ by gluing the objects $(f^!K)_{|U_i}(-i)[-2i]$ on the disjoint open-closed subschemes $U_i := \{x \in X, d(x) = i\}$ where f has virtual relative dimension i, for all $i \in \mathbf{Z}$.

The following notation proves to be quite convenient for this proof :

Définition 2.5.6. If $f : Z \to Y$ and $g : Y \to X$ are composable morphisms in $\mathscr{S}^{\mathrm{ic}}$, and $a : \Lambda \to g^{?}\Lambda$ and $b : \Lambda \to f^{?}\Lambda$ are morphisms in $\mathrm{D}^+(Y_{\text{ét}}, \Lambda)$ and $\mathrm{D}^+(Z_{\text{ét}}, \Lambda)$ respectively, we set $a \bigstar b = f^{?}(a) \circ b : \Lambda \to (g \circ f)^{?}\Lambda$.

This law $\bigstar$ satisfies an obvious associativity property, so we will omit parentheses.
By definition, we thus have : $\mathrm{Cl}_{p,i} = \mathrm{Cl}_p \bigstar \mathrm{Cl}_i$. We want to check the equality $\mathrm{Cl}_p \bigstar \mathrm{Cl}_i = \mathrm{Cl}_{p'} \bigstar \mathrm{Cl}_{i'}$. By introducing the fiber product of Y and Y' over S, we can assume that "Y' covers Y", namely that there exists a smooth morphism $q : Y' \to Y$ such that $i = q \circ i'$ and $p' = p \circ q$:

$$\begin{tikzcd} & Y' \arrow[d, "q"] \arrow[dr, "p'"] & \\ X \arrow[ur, "i'"] \arrow[r, "i"'] & Y \arrow[r, "p"'] & S \end{tikzcd}$$

We thus have

$$\mathrm{Cl}_{p',i'} = \mathrm{Cl}_{p'} \bigstar \mathrm{Cl}_{i'} = \mathrm{Cl}_p \bigstar \mathrm{Cl}_q \bigstar \mathrm{Cl}_{i'} \ ,$$

the last equality resulting from Proposition 2.4.2. We are reduced to showing the equality $\mathrm{Cl}_i = \mathrm{Cl}_q \bigstar \mathrm{Cl}_{i'}$. For this, we introduce the fiber product X' of X and Y' over Y :

$$\begin{tikzcd} X' \arrow[r, "j"] \arrow[d, "q'"] & Y' \arrow[d, "q"] \\ X \arrow[u, bend left, "s"] \arrow[ur, "i'"] \arrow[r, "i"'] & Y \end{tikzcd}$$

The morphism i' gives rise to the section s of the projection $q' : X' \to X$. The morphism q being smooth, the immersion $j : X' \to Y'$ is regular. Let us temporarily assume the following equalities :

$$\mathrm{Cl}_{q'} \bigstar \mathrm{Cl}_s = \mathrm{Id}_\Lambda \ , \qquad \mathrm{Cl}_q \bigstar \mathrm{Cl}_j = \mathrm{Cl}_i \bigstar \mathrm{Cl}_{q'} \ .$$

It follows :

$$\begin{aligned} \mathrm{Cl}_i &= \mathrm{Cl}_i \bigstar \mathrm{Cl}_{q'} \bigstar \mathrm{Cl}_s \\ &= \mathrm{Cl}_q \bigstar \mathrm{Cl}_j \bigstar \mathrm{Cl}_s \ . \end{aligned}$$

We then use the composition of Gysin morphisms associated with regular immersions (cf. Theorem 2.3.3). This gives the equality $\mathrm{Cl}_j \bigstar \mathrm{Cl}_s = \mathrm{Cl}_{i'}$, which allows us to conclude that $\mathrm{Cl}_i = \mathrm{Cl}_q \bigstar \mathrm{Cl}_{i'}$.
The two lemmas that follow allow us to obtain the two equalities admitted above :

Lemme 2.5.7. *Let there be a Cartesian diagram in $\mathscr{S}$:*

$$\begin{tikzcd} X' \arrow[r, "j"] \arrow[d, "q'"] & Y' \arrow[d, "q"] \\ X \arrow[r, "i"] & Y \end{tikzcd}$$

Assume that q is smooth and that i is a regular immersion (so j is as well). Then we have the equality

$$\mathrm{Cl}_q \bigstar \mathrm{Cl}_j = \mathrm{Cl}_i \bigstar \mathrm{Cl}_{q'} \ .$$

We can assume that i is a closed immersion. We identify Cl_i (resp. Cl_j) with a class in $\mathrm{H}^{2d}_X(Y, \Lambda(d))$ (resp. $\mathrm{H}^{2c}_{X'}(Y', \Lambda(c))$) where c is the codimension of the regular immersion i. By Proposition 2.3.2, we have $q^\star(\mathrm{Cl}_i) = \mathrm{Cl}_j$.

We can identify $\mathrm{D}^+(X'_{\text{ét}}, \Lambda)$ with the full subcategory of $\mathrm{D}^+(Y'_{\text{ét}}, \Lambda)$ formed by complexes K such that $K \xrightarrow{\sim} j_\star j^\star K$: the functor $j_\star$ is then interpreted as an inclusion functor. We similarly identify $\mathrm{D}^+(X_{\text{ét}}, \Lambda)$ with a full subcategory of $\mathrm{D}^+(Y_{\text{ét}}, \Lambda)$. The functor $q'_! : \mathrm{D}^+(X'_{\text{ét}}, \Lambda) \to \mathrm{D}^+(X_{\text{ét}}, \Lambda)$ is then induced by $q_! : \mathrm{D}^+(Y'_{\text{ét}}, \Lambda) \to \mathrm{D}^+(Y_{\text{ét}}, \Lambda)$, and $q'^!$ is also induced by $q^!$. Let d be the relative dimension of q. The morphism $\mathrm{Tr}_q\colon\ q_!\Lambda_Y(d)[2d] \to \Lambda_Y$ extends *via* the projection formula [**SGA 4** XVII 5.2.9] to a functorial morphism in $K \in \mathrm{D^b}(Y_{\text{ét}}, \Lambda)$:

$$\mathrm{Tr}_q\colon\ q_! q^\star K(d)[2d] \simeq q_!\Lambda_{Y'}(d)[2d] \overset{\mathrm{L}}{\otimes} K \overset{\mathrm{Tr}_q \overset{\mathrm{L}}{\otimes} K}{\to} \Lambda_Y \overset{\mathrm{L}}{\otimes} K \simeq K\,.$$

The compatibility of the trace morphism with base change by i (property (Var 2) of [**SGA 4** XVIII 2.9]) amounts to saying that for $K = \Lambda_X = i_\star\Lambda_X \in \mathrm{D^b}(Y_{\text{ét}}, \Lambda)$, the morphism $\mathrm{Tr}_q\colon\ q_! q^\star\Lambda_X(d)[2d] \to \Lambda_X$ is $\mathrm{Tr}_{q'}$. The functoriality of the natural transformation Tr_q above applied to the morphism $\mathrm{Cl}_i\colon\ \Lambda_X \to \Lambda_Y(c)[2d]$ then provides the following commutative diagram :

$$\begin{tikzcd}
q_! q^\star\Lambda_X(d)[2d] \arrow[r, "\mathrm{Tr}_{q'}"] \arrow[d, "q_! q^\star \mathrm{Cl}_i"] & \Lambda_X \arrow[d, "\mathrm{Cl}_i"] \\
q_! q^\star\Lambda_Y(d+c)[2d+2c] \arrow[r, "\mathrm{Tr}_q(c)[2c]"] & \Lambda_Y(c)[2c]
\end{tikzcd}$$

Taking into account the previously obtained identity $q^\star\mathrm{Cl}_i = \mathrm{Cl}_j$, the commutativity of the above diagram means precisely that $\mathrm{Cl}_q \bigstar \mathrm{Cl}_j = \mathrm{Cl}_i \bigstar \mathrm{Cl}_{q'}$.

LEMME 2.5.8. *Let $p\colon\ X \to S$ be a smooth morphism in $\mathscr{S}$ admitting a section $s\colon\ S \to X$ (which is a regular immersion). Then, $\mathrm{Cl}_p \bigstar \mathrm{Cl}_s = \mathrm{Id}_\Lambda$ in $\mathrm{D}^+(S_{\text{ét}}, \Lambda)$.*

The endomorphisms of Λ in $\mathrm{D}^+(S_{\text{ét}}, \Lambda)$ are given by sections of the sheaf Λ in S, so it suffices to check that the numbers obtained by passing to the generic points of S are equal to 1. Since we can assume that S is reduced and that the construction is compatible with passing to generic points, we can assume that S is the spectrum of a field k. Let x be the image of $\mathrm{Spec}(k)$ in X. By replacing X with an open neighborhood, we can assume that there exists an étale morphism $\pi\colon\ X \to \mathbf{A}^d_k$ identifying x with the inverse image of the origin in $\mathbf{A}^d_k$. Using the obvious isomorphism $\mathrm{H}^{2d}_{(0,\dots,0)}(\mathbf{A}^d_k, \Lambda(d)) \overset{\sim}{\to} \mathrm{H}^{2d}_x(X, \Lambda(d))$, we are reduced to the following lemma :

LEMME 2.5.9. *For any natural number d and any scheme $S \in \mathscr{S}$, if we denote by $p\colon\ \mathbf{A}^d_S \to S$ the projection and by $s\colon\ S \to \mathbf{A}^d_S$ the inclusion of the origin, we have the equality*

$$\mathrm{Cl}_p \bigstar \mathrm{Cl}_s = \mathrm{Id}_\Lambda$$

in $\mathrm{D}^+(S_{\text{ét}}, \Lambda)$.

The statement is obvious for $d = 0$. An obvious induction based on Theorem 2.3.3 and Proposition 2.4.2 allows us to reduce to the case where $d = 1$, and as before, we can assume that $S = \mathrm{Spec}(k)$ where k is a field that we can assume to be separably closed. We are finally reduced to the following lemma :

LEMME 2.5.10. *For any separably closed field k, if we denote by $p\colon\ \mathbf{P}^1_k \to \mathrm{Spec}(k)$ the projection and by $s\colon\ \mathrm{Spec}(k) \to \mathbf{P}^1_k$ the inclusion of 0, we have the equality*

$$\mathrm{Cl}_p \bigstar \mathrm{Cl}_s = \mathrm{Id}_\Lambda$$

in $\mathrm{D}^+(\mathrm{Spec}(k)_{\text{ét}}, \Lambda)$.

The ideal of the closed immersion s identifies with the invertible sheaf $\mathscr{O}(-1)$ on the projective line. The image $\mathrm{Cl}_{s|\mathbf{P}^1_k}$ of Cl_s in $\mathrm{H}^2(\mathbf{P}^1_k, \Lambda(1))$ is $-c_1(\mathscr{O}(-1)) = c_1(\mathscr{O}(1))$ (cf. Definitions 2.1.1 and 2.3.1). The degree of the line bundle $\mathscr{O}(1)$ being 1, we can conclude by using the commutativity of the following diagram (cf. [**SGA 4** XVIII 1.1.6]) :

$$\begin{tikzcd}
\mathrm{Pic}(\mathbf{P}^1_k) \arrow[r, "c_1"] \arrow[dr, "\deg"'] & \mathrm{H}^2(\mathbf{P}^1_k, \Lambda(1)) \arrow[d, "\mathrm{Tr}_p", "\sim"'] \\
 & \Lambda
\end{tikzcd}$$

Définition 2.5.11. Let $f : X \to S$ be a morphism in $\mathscr{S}^{\mathrm{ic}}$. We denote by $\mathrm{Cl}_f : \Lambda \to f^?\Lambda$ the morphism $\mathrm{Cl}_{p,i}$ in $\mathrm{D}^+(X_{\text{ét}}, \Lambda)$ defined from a factorization of f in $\mathscr{S}^{\mathrm{ic}}$ of the form $f = p \circ i$ with i a regular immersion and p a smooth morphism. By Theorem 2.5.5, this definition is independent of the factorization.

Théorème 2.5.12. *If $X \xrightarrow{f} Y$ and $Y \xrightarrow{g} Z$ are composable morphisms in $\mathscr{S}^{\mathrm{ic}}$, the following diagram is commutative in $\mathrm{D}^+(X_{\text{ét}}, \Lambda)$.*

$$\begin{tikzcd}
\Lambda \arrow[r, "\mathrm{Cl}_f"] \arrow[dr, "\mathrm{Cl}_{g\circ f}"'] & f^?\Lambda \arrow[d, "f^?(\mathrm{Cl}_g)"] \\
& (g\circ f)^?\Lambda
\end{tikzcd}$$

Paraphrasing [**SGA 6** viii 2.6], we choose a factorization $Y \xrightarrow{j} V' \xrightarrow{p'} Z$ in $\mathscr{S}^{\mathrm{ic}}$ with j a regular immersion and p' smooth, and a regular immersion $X \xrightarrow{i} \mathbf{P}^n_Y$, so as to obtain the following diagram :

$$\begin{tikzcd}
X \arrow[r, "i"] \arrow[dr, "f"'] & \mathbf{P}^n_Y \arrow[r, "j'"] \arrow[d, "p"] & \mathbf{P}^n_{V'} \arrow[d, "p''"] \\
& Y \arrow[r, "j"] \arrow[dr, "g"'] & V' \arrow[d, "p'"] \\
& & Z
\end{tikzcd}$$

Using Theorem 2.3.3 and Proposition 2.4.2, we obtain

$$\mathrm{Cl}_{g\circ f} = (\mathrm{Cl}_{p'} \bigstar \mathrm{Cl}_{p''}) \bigstar (\mathrm{Cl}_{j'} \bigstar \mathrm{Cl}_i) .$$

Lemma 2.5.7 gives the equality :

$$\mathrm{Cl}_{p''} \bigstar \mathrm{Cl}_{j'} = \mathrm{Cl}_j \bigstar \mathrm{Cl}_p ,$$

which allows us to obtain :

$$\mathrm{Cl}_{g\circ f} = (\mathrm{Cl}_{p'} \bigstar \mathrm{Cl}_j) \bigstar (\mathrm{Cl}_p \bigstar \mathrm{Cl}_i) ,$$

in which we recognize the equality $\mathrm{Cl}_{g\circ f} = \mathrm{Cl}_g \bigstar \mathrm{Cl}_f$.

Proposition 2.5.13. *Let $f : X \to S$ be a morphism in $\mathscr{S}^{\mathrm{ic}}$. Assume that f is flat of relative dimension d. Then the morphism $\mathrm{Cl}_f : \Lambda \to f^?\Lambda$ corresponds by adjunction to the morphism $\mathrm{R}f_!\Lambda(d)[2d] \to \Lambda$ given by the trace morphism $\mathrm{Tr}_f : \mathrm{R}^{2d} f_!\Lambda(d) \to \Lambda$.*

Taking into account Proposition 2.3.7, this results from [**SGA 4½** [Cycle] 2.3.8 (i)].

Remarque 2.5.14. If $f : X \to Y$ is a proper morphism in $\mathscr{S}^{\mathrm{ic}}$ of (constant) virtual relative dimension d, the morphism Cl_f allows to define, for any $K \in \mathrm{D}^+(Y_{\text{ét}}, \Lambda)$, a morphism $f_\star : \mathrm{H}^p(X, f^\star K) \to \mathrm{H}^{p-2d}(Y, K(-d))$, compatible with composition. One can also define a version with supports $f_\star : \mathrm{H}^p_Z(X, f^\star K) \to \mathrm{H}^{p-2d}_{Z'}(Y, K(-d))$ if $f : X \to Y$ is a morphism in $\mathscr{S}^{\mathrm{ic}}$, Z and Z' are closed subschemes of X and Y respectively, $f(Z) \subset Z'$, and the induced morphism $f_{|Z} : Z \to Z'$ is proper.

3. Purity theorem

3.1. Statements. The objective of this section is to give a proof of the following theorem :

Théorème 3.1.1. *Let X be a regular $\mathbf{Z}\left[\frac{1}{n}\right]$-scheme. Let Y be a (closed) subscheme of X which is also regular. We denote by $i : Y \to X$ the immersion, and by c its codimension. Then, the Gysin morphism $\mathrm{Cl}_i : \Lambda \to i^?\Lambda = i^!\Lambda(c)[2c]$ is an isomorphism in $\mathrm{D}^+(Y_{\text{ét}}, \Lambda)$.*

Corollaire 3.1.2. *Let $f: X \to S$ be a morphism of finite type between regular $\mathbf{Z}\left[\frac{1}{n}\right]$-schemes. Assume that X and S admit an ample sheaf. Then, the Gysin morphism $\mathrm{Cl}_f(d)[2d]: \Lambda(d)[2d] \to f^!\Lambda$ is an isomorphism in $\mathsf{D}^+(X_{\text{ét}}, \Lambda)$, where d denotes the virtual relative dimension of f.*

Corollaire 3.1.3. *Let X be a regular $\mathbf{Z}\left[\frac{1}{n}\right]$-scheme. Let D be a regular divisor in X. We denote by $j: X - D \to X$ the inclusion of its complement. Then, we have canonical isomorphisms $j_\star\Lambda = \Lambda$, $\mathrm{R}^1 j_\star\Lambda = \Lambda_Z(-1)$ and $\mathrm{R}^q j_\star\Lambda = 0$ for $q \geq 2$.*

This corollary results from Theorem 3.1.1 applied to the closed immersion $i: D \to X$ and from the long exact cohomology sequence applied to the following canonical distinguished triangle in $\mathsf{D}^+(X_{\text{ét}}, \Lambda)$:

$$i_\star i^\star\Lambda \to \Lambda \to \mathrm{R}j_\star\Lambda \xrightarrow{\delta} i_\star i^!\Lambda[1]$$

The isomorphism $\Lambda_Z(-1) \simeq \mathrm{R}^1 j_\star\Lambda_Z(-1)$ is normalized so that the composition $\Lambda_Z(-1) \simeq \mathrm{R}^1 j_\star\Lambda_Z(-1) \underset{\sim}{\xrightarrow{\delta}} i_\star\mathrm{R}^2 i^!\Lambda$ is given by the class Cl_i. Alternatively, this identification is induced by a global section of the sheaf $\mathrm{R}^1 j_\star\Lambda(1)$ which is given locally by the opposite of the class of the μ_n-torsor of the n-th roots of f where f is a local equation of D in X. (See the proof of Lemma 3.4.8 for more details on this compatibility.)

Corollaire 3.1.4. *Let X be a regular $\mathbf{Z}\left[\frac{1}{n}\right]$-scheme. Let D be a normal crossings divisor in X. We denote by $j: X - D \to X$ the inclusion of its complement. Then, $\mathrm{R}j_\star\Lambda$ belongs to $\mathsf{D}^{\mathrm{b}}_{\mathrm{ctf}}(X_{\text{ét}}, \Lambda)$. More precisely, if $D = D_1 + \cdots + D_n$ is a strict normal crossings divisor, then $\mathrm{R}^1 j_\star\Lambda$ identifies with $\bigoplus_{1\leq i\leq n} \Lambda_{D_i}(-1)$ and $\mathrm{R}^\star j_\star\Lambda$ is the exterior algebra on $\mathrm{R}^1 j_\star\Lambda$.*

This corollary deserves a proof. For the first assertion, we can work locally for the étale topology on X; it thus suffices to establish the second assertion. We assume that $D = D_1 + \cdots + D_n$ is a strict normal crossings divisor. We denote by $j_i: X - D_i \to X$ the inclusion of the complement of D_i for all i. We will show that the Künneth morphism

$$\mathrm{R}j_{1\star}\Lambda \overset{\mathrm{L}}{\otimes} \dots \overset{\mathrm{L}}{\otimes} \mathrm{R}j_{n\star}\Lambda \to \mathrm{R}j_\star\Lambda$$

is an isomorphism in $\mathsf{D}(X_{\text{ét}}, \Lambda)$, which will imply the result since the sheaves $\mathrm{R}^q j_{i\star}\Lambda$ are known by purity (Corollary 3.1.3) and are flat.

We proceed by induction on n. The cases $n = 0$ and $n = 1$ are obvious. We assume $n \geq 2$, we set $D' = D_2 + \cdots + D_n$, and we assume that the result is known for D'. The task is therefore to show that if we denote by $j': X - D' \to X$ the inclusion of the complement of D', then the Künneth morphism

$$\mathrm{R}j_{1\star}\Lambda \overset{\mathrm{L}}{\otimes} \mathrm{R}j'_\star\Lambda \to \mathrm{R}j_\star\Lambda$$

is an isomorphism. In other words, the canonical morphism

$$\mathrm{R}\,\mathbf{Hom}(\Lambda_{X-D_1}, \Lambda) \overset{\mathrm{L}}{\otimes} \mathrm{R}\,\mathbf{Hom}(\Lambda_{X-D'}, \Lambda) \to \mathrm{R}\,\mathbf{Hom}(\Lambda_{X-D_1} \otimes \Lambda_{X-D'}, \Lambda)$$

is an isomorphism in $\mathsf{D}(X_{\text{ét}}, \Lambda)$. For a fixed K (resp. L) in $\mathsf{D}(X_{\text{ét}}, \Lambda)$, the family of L (resp. K) such that the morphism

$$\mathrm{R}\,\mathbf{Hom}(K, \Lambda) \overset{\mathrm{L}}{\otimes} \mathrm{R}\,\mathbf{Hom}(L, \Lambda) \to \mathrm{R}\,\mathbf{Hom}(K \overset{\mathrm{L}}{\otimes} L, \Lambda)$$

is an isomorphism, a property we will call (Kü), is a triangulated subcategory of $\mathsf{D}(X_{\text{ét}}, \Lambda)$.

For $K = \Lambda$ or $L = \Lambda$, the condition (Kü) is obviously satisfied. Showing it for $(\Lambda_{X-D_1}, \Lambda_{X-D'})$ thus amounts, by dévissage, to showing it for $(\Lambda_{D_1}, \Lambda_{X-D'})$ or even for $(\Lambda_{D_1}, \Lambda_{D'})$. It follows immediately from the purity theorem and the compatibilities obtained (cf. Remark 2.3.6) that if Y and Z are two regular closed subschemes of X intersecting transversely (i.e., $Y \cap Z$ is regular of codimension the sum of the codimensions of Y and Z), then (Λ_Y, Λ_Z) satisfies (Kü). In particular, $(\Lambda_{D_1}, \Lambda_{D_i})$ satisfies (Kü) for $i \geq 2$ and more generally, for any non-empty subset I of $\{2, \dots, n\}$, $(\Lambda_{D_1}, \Lambda_{D_I})$ satisfies (Kü) where D_I is the intersection of the D_i for $i \in I$. Using the standard exact sequence

$$0 \to \Lambda_{D'} \to \bigoplus_{2\leq i\leq n} \Lambda_{D_i} \to \bigoplus_{2\leq i<j\leq n} \Lambda_{D_{ij}} \to \dots ,$$

we deduce by dévissage the condition (Kü) for $(\Lambda_{D_1}, \Lambda_{D'})$, which was to be shown.

Définition 3.1.5. A **regular pair** is a pair (X, Y) where X is a regular $\mathbf{Z}\left[\frac{1}{n}\right]$-scheme and Y is a closed subscheme of X that is regular. We say that (X, Y) is **pure** if the conclusion of Theorem 3.1.1 is true for the inclusion of Y in X. If $\overline{y} \to Y$ is a geometric point of Y, we will say that (X, Y) is **pure at** $\overline{y}$ if, after passing to the germs at $\overline{y}$, the morphism $\mathrm{Cl}_i : \Lambda \to i^?\Lambda$ induces an isomorphism in the category $\mathsf{D}^+(\overline{y}_{\text{ét}}, \Lambda)$.

Theorem 3.1.1 can thus be reformulated by saying that every regular pair is pure. In Subsection 3.2, the notion of pointwise purity will be introduced, which consists in studying regular pairs of the form (X, x) where X is a regular local scheme with closed point x. To prove the purity theorem, it will suffice to know that regular couples of this form are pure. In Subsection 3.3, we will reduce to the case where the coefficient ring Λ is $\mathbf{Z}/\ell\mathbf{Z}$ with ℓ a prime number invertible on the regular schemes under consideration. In Subsection 3.4, we will establish some useful properties concerning the purity of regular pairs given by divisors. As in the proof of [**Fujiwara, 2002**], the proof of pointwise purity for arbitrary regular schemes will be reduced to that of regular schemes that are of finite type over a trait S (of unequal characteristic). In Subsection 3.5, we will obtain sufficient conditions to show that regular schemes of finite type over S are pointwise pure. Subsection 3.6 will provide the statements from logarithmic geometry that allow to establish that if (X, M) is a log-smooth log-scheme over a trait (equipped with its canonical log-structure) and if the scheme X is regular, then X is pointwise pure. The proof of Theorem 3.1.1 will be given in Subsection 3.7. It will use the results of the preceding subsections as well as three theorems of resolution of singularities which can be summarized as follows :

- use of alterations to obtain a scheme with semi-stable reduction from a (normal) scheme over S (cf. [**Vidal, 2004**, proposition 4.4.1]) ;
- modification of a tame action of a finite group on a log-regular log-scheme so as to obtain a very tame action (cf. **X**-1.1) ;
- resolution of singularities of log-regular log-schemes (Kato-Nizioł theorem, cf. [**Kato, 1994**, 10.3, 10.4] and [**Nizioł, 2006**, 5.7]).

3.2. Pointwise purity.

Définition 3.2.1. Let X be a regular local $\mathbf{Z}\left[\frac{1}{n}\right]$-scheme. We say that X is **pointwise pure at its closed point** x if the morphism $\mathrm{Cl}_i : \Lambda \to i^?\Lambda$ is an isomorphism in $\mathsf{D}^+(x_{\text{ét}}, \Lambda)$ where $i : x \to X$ is the inclusion of the closed point of X.

A regular local scheme is pointwise pure at its closed point if and only if its henselization (resp. its strict henselization) is.

Définition 3.2.2. Let X be a $\mathbf{Z}\left[\frac{1}{n}\right]$-scheme. If $x \in X$, we say that X is **pointwise pure at the point** x if the localization of X at x is pointwise pure at its closed point. We say that X is **pointwise pure** if it is so at all its points.

The following proposition is [**Fujiwara, 2002**, proposition 2.2.4]. The proof in that article seems complicated as it goes through results finer than what we need. We therefore provide a shorter proof.

Proposition 3.2.3. *Let $i : Y \to X$ be a closed immersion between regular schemes. The number of conditions satisfied among the following three cannot be two :*

(a) *The regular pair (X, Y) is pure ;*
(b) *The scheme Y is pointwise pure ;*

(c) *The scheme X is pointwise pure at the points located in the image of i.*

Let $y \in Y$, let $V(y)$ be the localization of Y at y and $V(x)$ that of the image x of y in X. We have a diagram of schemes :

$$\begin{tikzcd} y \arrow[r, "i_y"] \arrow[dr, "i_x"'] & V(y) \arrow[d, "i'"] \\ & V(x) \end{tikzcd}$$

The composition of Gysin morphisms gives the following commutative diagram in $\mathsf{D}^+(y_{\text{ét}}, \Lambda)$:

$$\begin{tikzcd} \Lambda \arrow[r, "\mathrm{Cl}_{i_y}"] \arrow[dr, "\mathrm{Cl}_{i_x}"'] & i_y^? \Lambda \arrow[d, "i_y^? \mathrm{Cl}_{i'}"] \\ & i_x^? \Lambda \end{tikzcd}$$

On this diagram, we see immediately that (a) and (b) imply (c) and that (a) and (c) imply (b). Let us show that (b) and (c) imply (a). It is a matter of showing that for any point y of Y, the morphism $i_y^\star \mathrm{Cl}_{i'}$ is an isomorphism. We can proceed by induction on the dimension of $V(y)$. We can thus assume that the support of a cone C of the morphism $\mathrm{Cl}_{i'}$ in $\mathsf{D}^+(V(y)_{\text{ét}}, \Lambda)$ is contained in $\{y\}$. But then, the canonical morphism $i_y^! C \to i_y^\star C$ is an isomorphism; the diagram above shows that $i_y^! C = 0$, which allows us to conclude that $C = 0$ and finally to obtain (a).

Let us recall some important properties concerning pointwise purity :

Proposition 3.2.4 ([**Fujiwara, 2002**, proposition 2.2.2]). *Let X be a regular strictly henselian local scheme. The completion $\hat{X}$ is pointwise pure at its closed point if and only if X is.*

Proposition 3.2.5 ([**Fujiwara, 2002**, corollary 2.2.3]). *Let k be a prime field. If X is a regular scheme that is a k-scheme, then X is pointwise pure.*

3.3. Change of coefficients.

Proposition 3.3.1. *Let n be a non-zero natural number. Let $n = \prod_{j=1}^{k} \ell_j^{\nu_j}$ be the factorization of n into a product of powers of distinct prime numbers. A regular pair (X, Y) is pure with respect to the coefficient ring $\mathbf{Z}/n\mathbf{Z}$ if and only if it is so with respect to the coefficient ring $\mathbf{Z}/\ell_j^{\nu_j}\mathbf{Z}$ for all $j \in \{1, \dots, k\}$.*

This results immediately from the Chinese Remainder Theorem and the fact that if m is a natural number dividing n, then for any regular closed immersion $i \colon Y \to X$, the obvious diagram commutes in $\mathsf{D}^+(Y_{\text{ét}}, \mathbf{Z}/n\mathbf{Z})$:

$$\begin{tikzcd} \mathbf{Z}/n\mathbf{Z} \arrow[r, "\mathrm{Cl}_i"] \arrow[d] & i^? \mathbf{Z}/n\mathbf{Z} \arrow[d] \\ \mathbf{Z}/m\mathbf{Z} \arrow[r, "\mathrm{Cl}_i"] & i^? \mathbf{Z}/m\mathbf{Z} \end{tikzcd}$$

Proposition 3.3.2. *Let ℓ be a prime number. For any integer $\nu \geq 1$, a regular pair (X, Y) is pure with respect to the coefficient ring $\mathbf{Z}/\ell\mathbf{Z}$ if and only if it is so with respect to the coefficient ring $\mathbf{Z}/\ell^\nu\mathbf{Z}$.*

Using the Godement resolution of the sheaves $\mathbf{Z}/\ell^\nu\mathbf{Z}(c)$ (where c is the codimension of the immersion $i \colon Y \to X$) for all ν, we can represent the Gysin morphisms $\mathrm{Cl}_i \colon \mathbf{Z}/\ell^\nu\mathbf{Z} \to i^? \mathbf{Z}/\ell^\nu\mathbf{Z}$ in $\mathsf{D}^+(Y_{\text{ét}}, \mathbf{Z}/\ell^\nu\mathbf{Z})$ by cocycles. Such a cocycle for a fixed ν_0 induces for any integer $\nu \leq \nu_0$ a cocycle representing the Gysin morphism with coefficients in $\mathbf{Z}/\ell^\nu\mathbf{Z}$. The elementary properties of the Godement resolution ensure that, if desired, one can in fact find a compatible family of cocycles for all $\nu \in \mathbf{N}$.

Given these observations, once these cocycles are suitably chosen, we have a preferred cone $C(\nu)$ of the morphism $\mathrm{Cl}_i : \mathbf{Z}/\ell^\nu\mathbf{Z} \to i^?\mathbf{Z}/\ell^\nu\mathbf{Z}$ in $\mathrm{D}^+(Y_{\text{ét}}, \mathbf{Z}/\ell^\nu\mathbf{Z})$ for all $\nu \in \mathbf{N}$ and triangles

$$C(\mu) \to C(\mu+\nu) \to C(\nu) \to C(\mu)[1]$$

in $\mathrm{D}^+(Y_{\text{ét}}, \mathbf{Z}/\ell^{\mu+\nu}\mathbf{Z})$ for all $(\mu,\nu) \in \mathbf{N}^2$.

Consequently, if $C(1) = 0$, it follows that for all $\nu \geq 1$, $C(\nu) = 0$. Conversely, if $C(1)$ is non-zero, its first non-zero cohomology object injects into that of $C(\nu)$ for all $\nu \geq 1$.

3.4. Regular divisors.

Définition 3.4.1. If X is a scheme and $\overline{x} \to X$ is a geometric point, we denote by $V(\overline{x})$ the strict henselization of X at $\overline{x}$ and by $i_{\overline{x}} : V(\overline{x}) \to X$ the canonical morphism.

Proposition 3.4.2. *Let X be a regular scheme. Let D be a regular divisor of X*[(x)]*. The regular pair (X, D) is pure if and only if for any geometric point $\overline{x} \to D$, we have $\mathrm{H}^q_{\text{ét}}(V(\overline{x}) - i_{\overline{x}}^{-1}(D), \Lambda) = 0$ for all $q \geq 2$.*

This results from the computation of $\mathrm{H}^q_{\text{ét}}(V(\overline{x}) - i_{\overline{x}}^{-1}(D), \Lambda)$ for $q \in \{0, 1\}$ (cf. [**SGA 4½** [Cycle] 2.1.4]).

Proposition 3.4.3. *We assume that the coefficient ring is $\mathbf{Z}/\ell\mathbf{Z}$ where ℓ is a prime number. Let $f : Y \to X$ be a finite flat morphism of constant degree prime to ℓ between regular $\mathbf{Z}\left[\frac{1}{\ell}\right]$-schemes. Let D be a regular divisor of X. We assume that $D' = f^{-1}(D)_{\text{réd}}$ is a regular divisor of Y. If the regular pair (Y, D') is pure, then so is (X, D).*

Thanks to Proposition 3.4.2, we can choose a geometric point of D and replace X by its strict henselization at this point. We therefore assume that X and D are local strictly henselian and we focus on the purity of the pair (X, D) at the closed point of D. The scheme Y is then a finite disjoint union of local strictly henselian schemes ; at least one of these has degree prime to ℓ over X. We can therefore assume that Y is also local strictly henselian. It then suffices to show that $\mathrm{H}^q(X - D, \mathbf{Z}/\ell\mathbf{Z})$ injects into $\mathrm{H}^q(Y - D', \mathbf{Z}/\ell\mathbf{Z})$, which results from the following lemma :

Lemme 3.4.4. *We assume that the coefficient ring Λ is $\mathbf{Z}/\ell\mathbf{Z}$ where ℓ is a prime number. Let $f : Y \to X$ be a finitely presented, finite, and flat morphism of rank d prime to ℓ between $\mathbf{Z}\left[\frac{1}{\ell}\right]$-schemes. Then, the canonical morphism $\Lambda \to f_\star\Lambda$ is a split monomorphism in $\mathrm{D}^+(X_{\text{ét}}, \Lambda)$.*

According to [**SGA 4** xvii 6.2.3], we have a morphism $\mathrm{Tr}_f : f_\star\Lambda \to \Lambda$ such that the composition

$$\Lambda \to f_\star\Lambda \to \Lambda$$

is multiplication by d, which gives the desired splitting since d is invertible in Λ.

Proposition 3.4.5. *We assume that the coefficient ring Λ is $\mathbf{Z}/\ell\mathbf{Z}$ where ℓ is a prime number. Let X be a regular $\mathbf{Z}\left[\frac{1}{\ell}\right]$-scheme. Let f be a function on X whose zero locus $D = V(f)$ is a regular divisor of X. We set $X' = \mathrm{Spec}\left(\mathscr{O}_X[T]/(T^\ell - f)\right)$. We denote by $\pi : X' \to X$ the projection, $D' = \pi^{-1}(D)_{\text{réd}}$ (note that $D' \to D$ is an isomorphism). Then, X' is a regular scheme, and the regular pair (X', D') is pure if and only if the regular pair (X, D) is.*

Let $\overline{x}$ be a geometric point of D (we will also identify $\overline{x}$ with a geometric point of D'). We will in fact show that (X, D) is pure at $\overline{x}$ if and only if (X', D') is. We can assume that X is the prime spectrum of a local strictly henselian ring A with maximal ideal $\mathfrak{m}$ and that $\overline{x}$ is over the closed point of X. We obviously have $f \in \mathfrak{m}$; the fact that $D = V(f)$ is regular amounts to saying that $f \notin \mathfrak{m}^2$.

Let $A' = A[T]/(T^\ell - f)$. By considering the determinant of the endomorphism of A' as an A-module given by multiplication by an element $b \in A'$, we observe that b is invertible in A' if and only if its image in the local algebra $(A/\mathfrak{m})[T]/(T^\ell)$ is invertible. It follows that A' is local with maximal

[(x)]By this we mean that D is a closed subscheme of X which is regular and purely of codimension 1. This does not exclude the case where D is empty.

ideal $\mathfrak{m}' = (T) + \mathfrak{m}A'$. Furthermore, we have an isomorphism $A/(f) \xrightarrow{\sim} A'/(T)$ (i.e., $D' \to D$ is an isomorphism). The ring $A_{(f)}[T]/(T^\ell - f)$ is a discrete valuation ring with uniformizer T; we deduce an isomorphism $A_{(f)}[T]/(T^\ell - f) \xrightarrow{\sim} A'_{(T)}$ from which it follows that the localization of A' with respect to the ideal (T) is a discrete valuation ring. The codimension of the prime ideal (T) in A' is therefore 1. We deduce that $\dim A' \geq 1 + \dim A'/(T) = 1 + \dim A/(f) = \dim A$. Since $f \in \mathfrak{m} - \mathfrak{m}^2$, we can introduce elements $(g_1, \dots, g_d)$ of $\mathfrak{m}$ such that the classes of the elements $f, g_1, \dots, g_d$ form a basis of $\mathfrak{m}/\mathfrak{m}^2$ as an $A/\mathfrak{m}$-vector space. We then have $\mathfrak{m} = (f, g_1, \dots, g_d)$ and $\dim A = d+1$ because A is regular. The maximal ideal $(T) + \mathfrak{m}A'$ of A' is generated by $(T, g_1, \dots, g_d)$, so $\dim A' \leq d+1 = \dim A$. Since we already know that $\dim A' \geq \dim A$, it follows that $\dim A' = d+1$ and that the maximal ideal of A' is generated by $\dim A'$ elements, so A' is regular.

We can consider, for any integer $\nu \geq 0$, the affine X-scheme $X^\nu = \operatorname{Spec}\big(A[T]/(T^{\ell^\nu} - f)\big)$. By raising T to the power ℓ, we obtain a tower of morphisms

$$\dots \to X^{\nu+1} \to X^\nu \to \dots \to X^1 \to X^0 ,$$

the last morphism $X^1 \to X^0$ identifying with $\pi : X' \to X$. For any integer n invertible in A, we denote $\mu_n := \mu_n(A)$. For all $\nu \geq 0$, we equip the A-algebra $A[T]/(T^{\ell^\nu} - f)$ with the left action of μ_{ℓ^ν} such that $\zeta \in \mu_{\ell^\nu}$ acts by sending T to ζT. The scheme X^ν thus inherits a right action of μ_{ℓ^ν} and $X^\nu \times_X (X-D)$ is equipped with a right μ_{ℓ^ν}-torsor structure over $(X-D)_{\text{ét}}$. According to 4.6.2, we have for all $\nu \geq 0$ a morphism of topoi $(X-D)_{\text{ét}} \to \mathbf{B}\mu_{\ell^\nu}$ such that the inverse image of the right μ_{ℓ^ν}-torsor $\mathbf{E}\mu_{\ell^\nu}$ (cf. 4.6.1) identifies with X^ν. The obvious compatibilities between the coverings constituting this tower mean that if we denote $\mathbf{Z}_\ell(1) = \lim\limits_{\nu} \mu_{\ell^\nu}$, then we in fact have a morphism of topoi $\rho_f : (X-D)_{\text{ét}} \to \mathbf{B}\mathbf{Z}_\ell(1)$ such that the functor $\rho_f^\star$ sends the projective system $\mathbf{E}\mathbf{Z}_\ell(1) := (\mathbf{E}\mu_{\ell^\nu})_\nu$ to $(X^\nu \times_X (X-D))_\nu$, and this in an equivariant way for the right actions of the groups μ_{ℓ^ν}. (If we have chosen a compatible system of geometric points $\overline{y}_\nu$ of the schemes $X^\nu \times_X (X-D)$, the construction 4.6.3 gives a compatible system of morphisms $\pi_1^{\text{ét}}(X-D, \overline{y}_0) \to \mu_{\ell^\nu}$. By passing to the projective limit, we obtain a morphism $\pi_1^{\text{ét}}(X-D, \overline{y}_0) \to \mathbf{Z}_\ell(1)$. The morphism of topoi ρ_f then identifies with the obvious composition $X_{\text{ét}} \to \mathbf{B}\pi_1^{\text{ét}}(X-D, \overline{y}_0) \to \mathbf{B}\mathbf{Z}_\ell(1)$.)

In what follows, the right étale μ_ℓ-torsor $X' - D = X^1 - D$ over $X - D$ will also be considered as a left μ_ℓ-torsor (*without taking inverses*, which is possible because μ_ℓ is commutative) : it is the μ_ℓ-torsor of the ℓ-th roots of f.

Lemme 3.4.6. *The regular pair (X, D) is pure at $\overline{x}$ if and only if the morphism*

$$\mathrm{R}(\mathbf{B}\mathbf{Z}_\ell(1), \mu_\ell) \to \mathrm{R}((X-D)_{\text{ét}}, \mu_\ell)$$

induced by the morphism of topoi ρ_f is an isomorphism in the derived category of abelian groups.

This lemma follows from the next two lemmas :

Lemme 3.4.7. *For any integer $q \geq 2$, $\mathrm{H}^q(\mathbf{B}\mathbf{Z}_\ell(1), \mu_\ell) = 0$ and we have canonical isomorphisms*

$$\mathrm{H}^0(\mathbf{B}\mathbf{Z}_\ell(1), \mu_\ell) \simeq \mu_\ell , \quad \mathrm{H}^1(\mathbf{B}\mathbf{Z}_\ell(1), \mu_\ell) \simeq \mathrm{Hom}_{cont}(\mathbf{Z}_\ell(1), \mu_\ell) \simeq \mathbf{Z}/\ell\mathbf{Z} .$$

To obtain the identification $\mathrm{H}^1(\mathbf{B}\mathbf{Z}_\ell(1), \mu_\ell) \simeq \mathrm{Hom}_{\text{cont}}(\mathbf{Z}_\ell(1), \mu_\ell)$, we use sign conventions compatible with 4.6.1. For the rest, it is a matter of showing that $\mathbf{Z}_\ell(1)$ has cohomological ℓ-dimension 1. For this, see for example [**Serre, 1994**, § 3.4, chapter I].

Lemme 3.4.8. *The composite morphism*

$$\mathbf{Z}/\ell\mathbf{Z} \simeq \mathrm{H}^1(\mathbf{B}\mathbf{Z}_\ell(1), \mu_\ell) \xrightarrow{\rho_f^\star} \mathrm{H}^1((X-D)_{\text{ét}}, \mu_\ell) \underset{\sim}{\xrightarrow{\delta}} \mathrm{H}^2_D(X, \mu_\ell)$$

sends 1 *to* $-\mathrm{Cl}_{D \subset X}$.

Thanks to 4.6.1, we obtain that the canonical generator of $\mathrm{H}^1(\mathbf{B}\mathbf{Z}_\ell(1), \mu_\ell)$ is sent by $\rho_f^\star$ to the class of the μ_ℓ-torsor $X' - D \to X - D$, which according to construction 4.3.1 is equal to $\delta_K(f)$ where $\delta_K : \mathrm{H}^i(X-D, \mathbf{G}_{\mathrm{m}}) \to \mathrm{H}^{i+1}(X-D, \mu_\ell)$ is the boundary map associated with the Kummer exact sequence. If we denote by $\delta_{X-D} : \mathrm{H}^i(X-D, \mathscr{F}) \to \mathrm{H}^{i+1}_D(X, \mathscr{F})$ the boundary maps relating cohomology and cohomology with supports (cf. 4.7.6), as in [**SGA 4½** [Cycle] 2.1.3], 4.7.5 provides the following relation in $\mathrm{H}^2_D(X,\mu_\ell)$:

$$\begin{aligned}\delta_{X-D}(\delta_K(f)) &= -\delta_K(\delta_{X-D}(f))\\ &= -c_1(\mathscr{O}_X,f) \quad \text{by 4.7.6}\\ &= c_1(\mathscr{O}_X,f^{-1})\\ &= c_1(f\mathscr{O}_X,1)\\ &= -\mathrm{Cl}_{D\subset X}\,.\end{aligned}$$

We can apply Lemma 3.4.6 to X' : it follows that the regular pair (X',D') is pure at $\overline{x}$ if and only if the morphism

$$\mathrm{R}(\mathbf{B}\mathbf{Z}_\ell(1),\mu_\ell)\to\mathrm{R}((X'-D')_{\text{ét}},\mu_\ell)$$

induced by the morphism of topoi $\rho_T\colon (X'-D')_{\text{ét}}\to\mathbf{B}\mathbf{Z}_\ell(1)$ is an isomorphism.

We have a commutative square of topoi :

$$\begin{tikzcd}(X'-D')_{\text{ét}} \arrow[r, "\rho_T"] \arrow[d, "g"] & \mathbf{B}\mathbf{Z}_\ell(1) \arrow[d, "g'"'] \\ (X-D)_{\text{ét}} \arrow[r, "\rho_f"] & \mathbf{B}\mathbf{Z}_\ell(1)\end{tikzcd}$$

where g is induced by $\pi\colon X'\to X$ and g' by multiplication by ℓ on $\mathbf{Z}_\ell(1)$. The sheaf $g_\star\mathbf{Z}/\ell\mathbf{Z}$ identifies canonically with $\rho_f^\star g'_\star(\mathbf{Z}/\ell\mathbf{Z})$. It easily follows that the regular pair (X',D') is pure at $\overline{x}$ if and only if the canonical morphism

$$\mathrm{R}(\mathbf{B}\mathbf{Z}_\ell(1),g'_\star(\mathbf{Z}/\ell\mathbf{Z}))\to\mathrm{R}((X-D)_{\text{ét}},\rho_f^\star g'_\star(\mathbf{Z}/\ell\mathbf{Z}))$$

is an isomorphism.

The relative cohomology of the morphism of topoi $\rho_f\colon (X-D)_{\text{ét}}\to\mathbf{B}\mathbf{Z}_\ell(1)$ defines a triangulated functor

$$F\colon \mathrm{D}^+(\mathbf{B}\mathbf{Z}_\ell(1),\mathbf{Z}/\ell\mathbf{Z})\to\mathrm{D}^+(\mathbf{Z}/\ell\mathbf{Z})$$

such that for any $K\in\mathrm{D}^+(\mathbf{B}\mathbf{Z}_\ell(1),\mathbf{Z}/\ell\mathbf{Z})$, $F(K)$ is isomorphic to a cone of the canonical morphism $\mathrm{R}(\mathbf{B}\mathbf{Z}_\ell(1),K)\to\mathrm{R}((X-D)_{\text{ét}},\rho_f^\star K)$.

The following lemma follows from what precedes :

Lemme 3.4.9. *The regular pair (X,D) is pure at $\overline{x}$ if and only if $F(\mathbf{Z}/\ell\mathbf{Z})=0$, while (X',D') is pure at $\overline{x}$ if and only if $F(g'_\star(\mathbf{Z}/\ell\mathbf{Z}))=0$.*

Since $\mathbf{Z}_\ell(1)$ is a pro-ℓ-group, the sheaf $g'_\star(\mathbf{Z}/\ell\mathbf{Z})$ is a successive extension of ℓ copies of $\mathbf{Z}/\ell\mathbf{Z}$. The functor F being triangulated, we immediately deduce that if $F(\mathbf{Z}/\ell\mathbf{Z})$ is zero, then so is $F(g'_\star\mathbf{Z}/\ell\mathbf{Z})$, and that if $F(\mathbf{Z}/\ell\mathbf{Z})$ is non-zero, its first non-zero cohomology object injects into that of $F(g'_\star(\mathbf{Z}/\ell\mathbf{Z}))$. This completes the proof of Proposition 3.4.5.

3.5. Schemes over a trait. Let S be a trait (in which the prime numbers dividing the cardinality of Λ are invertible). We denote by s its closed point, η its generic point, and π a uniformizer.

Proposition 3.5.1. *The trait S is pointwise pure.*

We need to show that S is pointwise pure at its closed point. We can assume that S is strictly henselian ; by Proposition 3.4.2, this then results from the fact that the fraction field of S has cohomological ℓ-dimension 1 for any prime number ℓ invertible on S (cf. [**SGA 4** x 2.2]).

Proposition 3.5.2. *For any natural number n, the affine space $\mathbf{A}^n_S$ is pointwise pure.*

By Proposition 3.2.5, the schemes $\mathbf{A}^n_s$ and $\mathbf{A}^n_\eta$ are pointwise pure. Thus, $\mathbf{A}^n_S$ is pointwise pure at the points of the generic fiber. To establish the pointwise purity of $\mathbf{A}^n_S$ at the points of the special fiber, we use Proposition 3.2.3 : it suffices to show that the regular pair $(\mathbf{A}^n_S,\mathbf{A}^n_s)$ is pure. The case $n=0$ results from Proposition 3.5.1, and the general case follows from it by virtue of the smooth base change theorem.

Corollaire 3.5.3. *A smooth S-scheme is pointwise pure.*

Définition 3.5.4. Let $p: X \to S$ be a morphism of finite type, with X regular and admitting an ample sheaf. We set $K_X = p^? \Lambda_S$ and we have a Gysin morphism $\mathrm{Cl}_{X/S} : \Lambda_X \to K_X$ in $\mathsf{D}^+(X_{\text{ét}}, \Lambda)$ (cf. Definition 2.5.11).

Proposition 3.5.5. *Let $p: X \to S$ be a morphism of finite type, with X regular and admitting an ample sheaf. The scheme X is pointwise pure if and only if the morphism $\mathrm{Cl}_{X/S} : \Lambda_X \to K_X$ is an isomorphism in $\mathsf{D}^+(X_{\text{ét}}, \Lambda)$.*

We choose a factorization $X \xrightarrow{i} Y \xrightarrow{q} S$ of p (in the category $\mathscr{S}^{\text{ic}}$) with Y smooth over S and i a (regular) closed immersion. By Theorem 2.5.12 (or rather, by definition of $\mathrm{Cl}_{X/S}$), the following diagram is commutative :

$$\begin{tikzcd}
\Lambda \arrow[r, "\mathrm{Cl}_i"] \arrow[dr, "\mathrm{Cl}_{X/S}"'] & i^?\Lambda \arrow[d, "i^?\mathrm{Cl}_q", "\sim"'] \\
& i^?q^?\Lambda
\end{tikzcd}$$

The morphism q being smooth, the Gysin morphism Cl_q is an isomorphism. Consequently, $\mathrm{Cl}_{X/S}$ is an isomorphism if and only if $\mathrm{Cl}_i : \Lambda \to i^?\Lambda$ is one. By Proposition 3.2.3, and taking into account the fact that Y is pointwise pure (cf. Corollary 3.5.3), this is still equivalent to saying that X is pointwise pure.

Corollaire 3.5.6. *Let X be an S-scheme of finite type that is regular. Let Y be a smooth X-scheme. If X is pointwise pure, then so is Y.*

Proposition 3.5.7. *Let $f: X \to Y$ be a proper and dominant morphism of S-schemes where X and Y are assumed to be of finite type over S, integral, regular, and admitting ample sheaves. We further assume that f is generically étale of degree d invertible in Λ. Then, the pointwise purity of X implies that of Y.*

The morphism f is locally a smoothable complete intersection of virtual relative dimension zero, hence $f^? = f^!$. The Gysin morphism relative to f is a morphism $\mathrm{Cl}_f : \Lambda \to f^!\Lambda$.

Lemme 3.5.8. *We can generalize the morphism $\mathrm{Cl}_f : \Lambda \to f^!\Lambda$ to morphisms $f^\star M \to f^! M$, functorially in $M \in \mathsf{D}^+(Y_{\text{ét}}, \Lambda)$.*

The morphism $\mathrm{Cl}_f : \Lambda \to f^!\Lambda$ corresponds by adjunction to a morphism $\mathrm{R}f_!\Lambda \to \Lambda$, which can be tensored with M to obtain (*via* the projection formula) a morphism $\mathrm{R}f_! f^\star M \to M$ which itself corresponds by adjunction to the morphism $f^\star M \to f^! M$ of the desired type.

By applying the functoriality of the construction in the lemma to the morphism $\mathrm{Cl}_{Y/S} : \Lambda_Y \to K_Y$, we obtain a commutative diagram in $\mathsf{D}^+(X_{\text{ét}}, \Lambda)$.

$$\begin{tikzcd}
f^\star\Lambda_Y \arrow[r] \arrow[d, "f^\star(\mathrm{Cl}_{Y/S})"'] & f^!\Lambda_Y \arrow[d, "f^!(\mathrm{Cl}_{Y/S})"] \\
f^\star K_Y \arrow[r] & f^! K_Y
\end{tikzcd}$$

Via the canonical isomorphism $f^\star\Lambda_Y \simeq \Lambda_X$, the top morphism identifies with the morphism $\mathrm{Cl}_f : \Lambda_X \to f^!\Lambda_Y$; the right one is $f^!(\mathrm{Cl}_{Y/S})$. By Theorem 2.5.12, it follows that the composite morphism $\Lambda_X \simeq f^\star\Lambda_Y \to f^! K_Y \simeq K_X$ is the Gysin morphism $\mathrm{Cl}_{X/S}$. We deduce from this a commutative diagram of the following form in $\mathsf{D}^+(X_{\text{ét}}, \Lambda)$:

$$\begin{tikzcd}
f^\star\Lambda_Y \arrow[r, "\sim"] \arrow[d, "f^\star(\mathrm{Cl}_{Y/S})"'] & \Lambda_X \arrow[r, "\mathrm{Cl}_f"] \arrow[d, "\mathrm{Cl}_{X/S}"] & f^!\Lambda_Y \arrow[d, "f^!\mathrm{Cl}_{Y/S}"] \\
f^\star K_Y \arrow[r] & K_X \arrow[r, "\sim"] & f^! K_Y
\end{tikzcd}$$

Since f is proper, we obtain by adjunction a new commutative diagram in $\mathsf{D}^{+}(Y_{\text{ét}}, \Lambda)$:

$$\begin{tikzcd}
\Lambda_Y \arrow[r] \arrow[d, "\mathrm{Cl}_{Y/S}"'] & \mathrm{R}f_\star \Lambda_X \arrow[r] \arrow[d, "\mathrm{R}f_\star(\mathrm{Cl}_{X/S})"] & \Lambda_Y \arrow[d, "\mathrm{Cl}_{Y/S}"] \\
K_Y \arrow[r] & \mathrm{R}f_\star K_X \arrow[r] & K_Y
\end{tikzcd}$$

The above diagram highlights a relation between the morphisms $\mathrm{Cl}_{Y/S}$ and $\mathrm{R}f_\star(\mathrm{Cl}_{X/S})$. As the following lemma will show, the first morphism is a direct factor of the second, which shows that the pointwise purity of X implies that of Y, completing the proof of Proposition 3.5.7.

Lemme 3.5.9. *In the preceding diagram, the composite morphisms $\Lambda_Y \to \Lambda_Y$ and $K_Y \to K_Y$ are multiplication by the degree d (in particular, they are isomorphisms).*

Since Y is connected (non-empty), we have an obvious isomorphism $\Lambda \xrightarrow{\sim} \mathsf{End}_{\mathsf{D}^{+}(Y_{\text{ét}},\Lambda)}(\Lambda_Y)$. By the local biduality theorem (cf. [**SGA 4½** [Th. finitude] 4.3]), we also have an isomorphism $\Lambda \xrightarrow{\sim} \mathsf{End}_{\mathsf{D}^{+}(Y_{\text{ét}},\Lambda)}(K_Y)$. It thus suffices to obtain the conclusion over a non-empty open set of Y. By replacing Y with a suitable non-empty open set, we can assume that f is an étale covering. We are thus reduced to the following lemma :

Lemme 3.5.10. *Let $f\colon X \to Y$ be a finite étale morphism of schemes of constant degree d. For any object $M \in \mathsf{D}^{+}(Y_{\text{ét}}, \Lambda)$, the composite morphism*

$$M \to f_\star f^\star M \to M$$

deduced from the canonical adjunctions $(f^\star, f_\star)$ and $(f_\star, f^\star)$ is multiplication by d.

Thanks to the projection formulas, we can assume that $M = \Lambda_Y$. It then suffices to establish the result after a (non-empty) étale base change trivializing the covering $X \to Y$ (for example, a Galois closure of this covering). In short, we can assume that X is a disjoint union of d copies of Y, in which case the result is trivial.

Définition 3.5.11. Let $(e_1, \dots, e_n) \in \mathbf{N}^n$. We define an S-scheme :

$$V(S, \pi, e_1, \dots, e_n) = \operatorname{Spec}\left(\mathscr{O}_S[T_1, \dots, T_n]/\left(\textstyle\prod_{i=1}^{n} T_i^{e_i} - \pi\right)\right) .$$

For each i, we denote by H_i the closed subscheme of $V(S, \pi, e_1, \dots, e_n)$ defined by the equation $T_i = 0$.

Proposition 3.5.12. *Let $(e_1, \dots, e_n)$ be an n-tuple of natural numbers not all zero. Then, the scheme $V(S, \pi, e_1, \dots, e_n)$ is regular and pointwise pure.*

We can assume that the coefficient ring is $\mathbf{Z}/\ell\mathbf{Z}$ where ℓ is a prime number invertible on S.

Lemme 3.5.13.

(i) *If $(e_1, \dots, e_n)$ is an n-tuple of non-zero integers, at least one of which is invertible in η, the S-scheme $V(S, \pi, e_1, \dots, e_n)$ is integral, regular, and has a smooth generic fiber ;*

(ii) *Let $d \geq 1$. If S' is the trait obtained by extracting a d-th root π' of the uniformizer π (cf.* [**Serre, 1968**, Proposition 17, §6, Chapter I]), *for any n-tuple $(e_1, \dots, e_n)$, we have an isomorphism of schemes*

$$V(S', \pi', e_1, \dots, e_n) = V(S, \pi, de_1, \dots, de_n) ;$$

(iii) *If $(e_1, \dots, e_n)$ is an n-tuple of non-zero integers, the scheme $V(S, \pi, e_1, \dots, e_n)$ is regular and integral ;*

(iv) *If $(e_1, \dots, e_n)$ is an n-tuple of non-zero integers, the scheme $V(S, \pi, e_1, \dots, e_n)$ is pointwise pure if and only if for each i such that $e_i > 0$, the regular pair $(V(S, \pi, e_1, \dots, e_n), H_i)$ is pure*[xi] *;*

(v) *Let $(e_1, \dots, e_n)$ be an n-tuple of non-zero integers, and let e be the l.c.m. of the e_i ; assume that ℓ does not divide e ; if $V(S, \pi, e, \dots, e)$ is pointwise pure, then so is $V(S, \pi, e_1, \dots, e_n)$;*

(vi) *If $(e_1, \dots, e_n)$ is an n-tuple of integers such that $e_1 \neq 0$, $V(S, \pi, e_1, \dots, e_n)$ is pointwise pure if and only if $V(S, \pi, \ell e_1, e_2, \dots, e_n)$ is pointwise pure.*

[xi] If $e_i = 0$, this is also true : it is a particular case of the relative purity theorem, cf. [**SGA 4** XVI 3.7].

Assertions (i) and (ii) are left as an exercise for the reader. Assertion (iii) results immediately from (i) and (ii).

To show assertion (iv), it suffices to observe that the divisors H_i for $e_i > 0$ are pointwise pure (they are affine spaces over the residue field of S) and form a covering of the special fiber of $V(S, \pi, e_1, \dots, e_n)$. The generic fiber of the scheme $V(S, \pi, e_1, \dots, e_n)$ being pointwise pure (since it is smooth over an extension of η), we can conclude by using Proposition 3.2.3.

Concerning assertion (v), raising the T_i to the power $\frac{e}{e_i}$ defines a finite flat morphism $V(S, \pi, e, \dots, e) \to V(S, \pi, e_1, \dots, e_n)$ of degree $\frac{e^n}{e_1 \dots e_n}$ (prime to ℓ); taking into account criterion (iv), Proposition 3.4.3 allows us to conclude.

To establish (vi), let us remark that raising T_1 to the power ℓ defines a finite flat morphism $V(S, \pi, \ell e_1, e_2, \dots, e_n) \to V(S, \pi, e_1, \dots, e_n)$ of degree ℓ and étale away from the zero locus of T_1. It therefore suffices to show that $(V(S, \pi, \ell e_1, e_2, \dots, e_n), H_1)$ is pure if and only if $(V(S, \pi, e_1, \dots, e_n), H_1)$ is, which results from Proposition 3.4.5.

Let us establish Proposition 3.5.12. By assertion (iii), the schemes considered are regular. To establish their pointwise purity, by Corollary 3.5.6, we can assume that none of the exponents e_i is zero. In the case where all the integers e_i are equal to 1, the result is established in [**Illusie, 2004**, theorem 1.4] (see also [**Rapoport & Zink, 1982**, Satz 2.21]). Thanks to the use of an auxiliary trait, assertion (ii) allows us to deduce that for any integer $d \geq 1$, $V(S, \pi, d, \dots, d)$ is pointwise pure. Using assertion (v), we obtain that $V(S, \pi, e_1, \dots, e_n)$ is pointwise pure if ℓ does not divide any of the integers e_i. Assertion (vi) allows us to move to the general case.

3.6. Logarithmic geometry.

Définition 3.6.1. Let S be a trait, with generic point η. The canonical log-structure on S is the direct image log-structure of the trivial log-structure on η. Any uniformizer of S defines a morphism of monoids $\mathbf{N} \to \Gamma(S, \mathscr{O}_S)$ giving rise to a chart $S \to \mathrm{Spec}(\mathbf{Z}[\mathbf{N}])$ of the log-scheme S.

The objective of this subsection is to establish the following theorem :

Théorème 3.6.2. *Let S be a trait equipped with its canonical log-structure. Let $(X, M) \to S$ be a log-smooth morphism of fs log-schemes. If the scheme X is regular, then it is pointwise pure.*

The following proposition specifies [**Kato, 1988**, theorem 3.5] in the case of fs log-schemes :

Proposition 3.6.3. *Let $(X, M) \to (Y, N)$ be a log-smooth morphism between fs log-schemes. We assume we are given a chart $Y \to \mathrm{Spec}(\mathbf{Z}[Q])$ of (Y, N) where Q is a torsion-free fs monoid*(xii)*. For any geometric point $\overline{x}$ of X, there exists an étale neighborhood U of $\overline{x}$, an injective morphism of monoids $Q \to P$ with P fs and torsion-free, and a chart $U \to \mathrm{Spec}(\mathbf{Z}[P])$ such that the torsion part of $\mathrm{Coker}(Q^{\mathrm{gp}} \to P^{\mathrm{gp}})$ has order invertible on U and such that the morphism of schemes $U \to Y \times_{\mathrm{Spec}(\mathbf{Z}[Q])} \mathrm{Spec}(\mathbf{Z}[P])$ is étale.*

In the proof of the log-smoothness criterion of [**Kato, 1988**, theorem 3.5], elements $t_1, \dots, t_r$ of $M_{\overline{x}}$ are chosen so that the family $(d \log t_1, \dots, d \log t_r)$ forms a basis of the sheaf of log-differentials $\omega^1_{X/Y, \overline{x}}$. We then consider the obvious morphism of monoids $\mathbf{N}^r \oplus Q \to M_{\overline{x}}$ given on the component $\mathbf{N}^r$ by $t_1, \dots, t_r$. It is such that the cokernel of $\mathbf{Z}^r \oplus Q^{\mathrm{gp}} \to M^{\mathrm{gp}}_{\overline{x}} / \mathscr{O}^{\times}_{X, \overline{x}}$ is finite and of exponent n invertible in the ring $\mathscr{O}_{X, \overline{x}}$ (in particular, $\mathscr{O}^{\times}_{X, \overline{x}}$ is n-divisible). There exists an injective morphism $\mathbf{Z}^r \oplus Q^{\mathrm{gp}} \to G$ with cokernel killed by a power of n and an extension $h \colon G \to M^{\mathrm{gp}}_{\overline{x}}$ of $\mathbf{Z}^r \oplus Q^{\mathrm{gp}} \to M^{\mathrm{gp}}_{\overline{x}}$ such that $G \to M^{\mathrm{gp}}_{\overline{x}} / \mathscr{O}^{\times}_{X, \overline{x}}$ is surjective. Since $M^{\mathrm{gp}}_{\overline{x}} / \mathscr{O}^{\times}_{X, \overline{x}}$ is a finitely generated abelian group and torsion-free, the following lemma shows that we can arrange for G to be a free abelian group. In the proof of [**Kato, 1988**, theorem 3.5], one then sets $P = h^{-1}(M_{\overline{x}})$ and it is shown that on an étale neighborhood U of $\overline{x}$, P generates the log-structure of (X, M) and that the morphism of schemes $U \to S \times_{\mathrm{Spec}(\mathbf{Z}[Q])} \mathrm{Spec}(\mathbf{Z}[P])$ is étale at $\overline{x}$. The monoid P thus constructed is fs and torsion-free.

(xii) If $\overline{y}$ is a geometric point of Y, there exists an étale neighborhood of $\overline{y}$ admitting such a chart with $Q = M_{\overline{y}} / \mathscr{O}^{\times}_{Y, \overline{y}}$, which is fs and sharp (cf. [**Kato, 1994**, Lemma 1.6]).

Lemme 3.6.4. *Let n be a non-zero natural number. Let A be a finitely generated free abelian group. Let $\varphi\colon A \to B$ be a morphism of abelian groups. Let $U \subset B$ be an n-divisible subgroup. Assume that B/U is torsion-free and that* $\mathrm{Coker}(A \to B/U)$ *is finite and killed by n. Then, there exists a finitely generated free abelian group A', an injective morphism of abelian groups $A \to A'$ such that A'/A is killed by a power of n, and an extension $A' \to B$ of the morphism $A \to B$ such that the composite morphism $A' \to B/U$ is surjective.*

By induction on the order of $\mathrm{Coker}(A \to B/U)$, we can assume that $\mathrm{Coker}(A \to B/U)$ is cyclic of order $d \geq 2$, generated by the class of an element $b \in B$. There thus exist $a \in A$ and $u \in U$ such that $db = \varphi(a) + u$. Since u is n-divisible, there exists $\tilde{u} \in U$ such that $u = d\tilde{u}$. By replacing b with $b - \tilde{u}$, we can assume that $u = 0$. We form the following pushout square in the category of abelian groups :

$$\begin{tikzcd} \mathbf{Z} \arrow[r, "a"] \arrow[d] & A \arrow[d] \\ \frac{1}{d}\mathbf{Z} \arrow[r] & A' \end{tikzcd}$$

Because of the relation $db = \varphi(a)$, we can define a unique morphism of abelian groups $\varphi'\colon A' \to B$ inducing $\varphi\colon A \to B$ and sending $\frac{1}{d}$ to b. We thus obtain a surjection $A' \to B/U$ inducing an isomorphism $A'/A \xrightarrow{\sim} \mathrm{Coker}(A \to B/U)$. It remains to check that A' is torsion-free. Let a' be a torsion element of A'. The image of a' in B/U *via* φ' is torsion, but since B/U is torsion-free, we have $\varphi'(a') \in U$. Since φ' induces an isomorphism $A'/A \xrightarrow{\sim} \mathrm{Coker}(A \to B/U)$, we deduce that $a' \in A$, but A is torsion-free, so $a' = 0$.

Proposition 3.6.5. *Let $(X, M) \to S$ be a log-smooth fs log-scheme over a trait S (equipped with its canonical log-structure). Assume that the scheme X is regular. Then, locally for the étale topology, X admits an étale morphism to a scheme $V(S, \pi, e_1, \dots, e_n)$ where $(e_1, \dots, e_n)$ is an n-tuple of non-zero integers (cf. Definition 3.5.11).*

Let π be a uniformizer of S; it gives rise to a chart $S \to \mathrm{Spec}(\mathbf{Z}[\mathbf{N}])$. By Proposition 3.6.3, we can assume that there exists a torsion-free fs monoid P, an injective morphism $\mathbf{N} \to P$, a chart $X \to \mathrm{Spec}(\mathbf{Z}[P])$ such that the morphism of schemes $X \to S \times_{\mathrm{Spec}(\mathbf{Z}[\mathbf{N}])} \mathrm{Spec}(\mathbf{Z}[P])$ is étale. Let $\overline{x}$ be a geometric point of X. We denote by P' the submonoid of P formed by the elements whose image in $\Gamma(X, \mathscr{O}_X)$ is invertible at the point $\overline{x}$.

We can assume that P' is a group. Indeed, if A is a finite subset of P' that generates the (finitely generated free) abelian group P'^{gp} (xiii), we can replace X by the open neighborhood of $\overline{x}$ on which the images of the elements of A (and therefore of P') are invertible in the structure sheaf and consequently, replace P by $P[-P']$, which is still fs and torsion-free.

The fact that $X \to \mathrm{Spec}(\mathbf{Z}[P])$ is a chart then implies that P' is the kernel of $P^{\mathrm{gp}} \to M^{\mathrm{gp}}_{\overline{x}}/\mathscr{O}^{\times}_{X,\overline{x}}$. In particular, we obtain an isomorphism

$$P/P' \xrightarrow{\sim} M_{\overline{x}}/\mathscr{O}^{\times}_{X,\overline{x}}\ .$$

Since X is log-regular, we recognize that X is regular from the fact that $M_{\overline{x}}/\mathscr{O}^{\times}_{X,\overline{x}}$ is a free monoid (cf. **VI**-1.7). Consequently, there exists an integer r and an isomorphism of monoids $\mathbf{N}^r \xrightarrow{\sim} P/P'$. We can lift this morphism to a morphism $\mathbf{N}^r \to P$, which allows us to construct an isomorphism $\mathbf{N}^r \oplus P' \xrightarrow{\sim} P$.

It follows that the chart morphism $X \to \mathrm{Spec}(\mathbf{Z}[P])$ has for its target a scheme isomorphic to $\mathrm{Spec}(\mathbf{Z}[\mathbf{N}^r \oplus P'])$, which is the product of an affine space and a split torus (of which P' is the character group). In the given chart of the morphism $(X, M) \to S$, the image of 1 by the morphism of monoids $\mathbf{N} \to P$ can be written as $(e_1, \dots, e_r, p')$ in $\mathbf{N}^r \oplus P'$ *via* the identifications above. We can choose a basis $a_1, \dots, a_s$ of P' as an abelian group such that $p' = \sum_{i=1}^{s} f_i a_i$ with $f_i \in \mathbf{N}$. We have thus constructed an étale morphism $X \to V(S, \pi, e_1, \dots, e_r, f_1, \dots, f_s)$ (with the $e_1, \dots, e_r, f_1, \dots, f_s$ not all zero).

(xiii) In fact, one can show that P' is a finitely generated monoid (it is a face of P).

Taking into account Proposition 3.5.12, Theorem 3.6.2 results immediately from Proposition 3.6.5.

3.7. Proof of the purity theorem. Let us prove Theorem 3.1.1. By Propositions 3.3.1 and 3.3.2, we can assume that the coefficient ring Λ is $\mathbf{Z}/\ell\mathbf{Z}$ where ℓ is a prime number. By Proposition 3.2.3, we need to show that any regular $\mathbf{Z}\left[\frac{1}{\ell}\right]$-scheme is pointwise pure. By [**Fujiwara, 2002**, corollary 6.1.5], we can assume that X is an integral regular scheme, quasi-projective, and flat over a (strictly henselian) trait S, which we can assume to be of unequal characteristic by Proposition 3.2.5. We can use the notations of Subsection 3.5. By extending the trait S if necessary, we can assume that the ring underlying S is integrally closed in the field of rational functions on X. The generic fiber X_η of X is therefore geometrically integral.

By applying [**Vidal, 2004**, proposition 4.4.1] to the normalization of the closure of X in a projective embedding, we obtain that there exists a finite group G and a G-equivariant diagram :

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & X \arrow[d] \\ S' \arrow[r] & S \end{tikzcd}$$

such that :

- G acts trivially on X and S ;
- $S' \to S$ is a finite extension of traits ;
- $X' \to X$ is projective, X' is regular, connected, and has semi-stable reduction over S' ;
- G acts faithfully on X' and $X' \to X$ is generically a Galois étale covering with group G.

We equip X' with the log-structure whose locus of triviality is the generic fiber of $X' \to S'$. Let H be an ℓ-Sylow subgroup of G. We denote $T = S'/H$. The extension of (strictly henselian) traits $S' \to T$ has order a power of ℓ, so it is tamely ramified. Consequently, for the canonical log-structures, $S' \to T$ is log-étale. Since we know that X' is log-smooth over S', it is therefore also so over T. Since H acts trivially on T and its action on X' is tame, we can apply the equivariant resolution theorem **X**-1.1 which gives a projective and birational H-equivariant morphism $X'' \to X'$ of log-schemes such that X'' is log-smooth over T and H acts very tamely on X''. The quotient log-scheme X''/H is also log-smooth over T (in particular, X''/H is log-regular). By the Kato-Nizioł resolution of singularities theorem (cf. [**Kato, 1994**, 10.3, 10.4] and [**Nizioł, 2006**, 5.7]), there exists a log-blowup (in particular, log-étale, projective, and birational) $X''' \to X''/H$ such that X''' is regular. The situation is summarized in the following diagram :

$$\begin{tikzcd} & X'' \arrow[loop above, "H"] \arrow[r, "\text{birat.}"] \arrow[d] & X' \arrow[loop above, "G"] \arrow[r] \arrow[d] & X \arrow[dd] \\ X''' \arrow[r, "\text{birat.}"] & X''/H \arrow[urr] \arrow[dr] & S' \arrow[loop left, "G"] \arrow[d] & \\ & & T \arrow[r] & S \end{tikzcd}$$

The log-scheme X''' is regular and log-smooth over T ; by Theorem 3.6.2, X''' is pointwise pure. The obvious morphism $X''' \to X$ is projective and generically an étale covering of degree prime to ℓ ; by Proposition 3.5.7, we can conclude that X is pointwise pure, which completes the proof of the purity theorem.

4. Sign conventions

Although the author of this exposition is reluctant to do so, it may be useful to specify certain sign conventions. We will rely on those of [**Conrad, 2000**, §1.3] ; they do not coincide with those of [**SGA 4** XVII 1.1]. Some additional conventions are specified below.

4.1. We recall that if $K = (\ldots \to K^n \to K^{n+1} \to \ldots)$ is a complex in an abelian category, for any $i \in \mathbf{Z}$, the complex $K[i]$ is such that $K[i]^n = K^{i+n}$ and the differentials on $K[i]$ are given by the differentials on K multiplied by $(-1)^i$.

If $f\colon K \to L$ is a morphism of complexes, $\mathrm{cone}(f)$ is the complex such that $\mathrm{cone}(f)^n := K^{n+1} \oplus L^n$ and whose differential is represented by the matrix $\begin{pmatrix} -d_K & 0 \\ f & d_L \end{pmatrix}$. The inclusions $L^n \to K^{n+1} \oplus L^n$ induce a morphism $i\colon L \to \mathrm{cone}(f)$ and the projections $K^{n+1} \oplus L^n \to K^{n+1}$ a morphism $p\colon \mathrm{cone}(f) \to K[1]$. We decree that the following triangle is distinguished[(xiv)] :

$$K \xrightarrow{f} L \xrightarrow{i} \mathrm{cone}(f) \xrightarrow{-p} K[1]\ .$$

If $0 \to M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \to 0$ is a short exact sequence in an abelian category $\mathscr{A}$, β induces a quasi-isomorphism $\mathrm{cone}(\alpha) \to M''$:

$$\begin{tikzcd}
-1 & M' \arrow[r] \arrow[d, "\alpha"] & 0 \arrow[d] \\
0 & M \arrow[r, "\beta"] & M''
\end{tikzcd}$$

$$\mathrm{cone}(\alpha) \xrightarrow{\sim} M''$$

In the derived category of $\mathscr{A}$, we thus obtain a distinguished triangle

$$M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \xrightarrow{\delta} M'[1]\ ,$$

where δ is the zigzag thus described :

$$\begin{tikzcd}
-1 & 0 \arrow[r] \arrow[d] & M' \arrow[r, "-\mathrm{Id}"] \arrow[d, "\alpha"] & M' \arrow[d] \\
0 & M'' & M \arrow[l, "\beta"] \arrow[r] & 0
\end{tikzcd}$$

$$M'' \xleftarrow{\sim} \mathrm{cone}(\alpha) \longrightarrow M'[1]$$

4.2. If we have a (covariant) cohomological functor $\mathscr{F}^0$ from one of the variants of the derived category of an abelian category $\mathscr{A}$, we can extend this functor to a sequence of functors $(\mathscr{F}^n)_{n\in\mathbf{Z}}$ by setting $\mathscr{F}^n M := \mathscr{F}^0(M[n])$. To every short exact sequence $0 \to M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \to 0$ in $\mathscr{A}$ is associated a long exact sequence :

$$\ldots \to \mathscr{F}^n M' \xrightarrow{\alpha} \mathscr{F}^n M \xrightarrow{\beta} \mathscr{F}^n M'' \xrightarrow{\delta} \mathscr{F}^{n+1} M' \to \ldots$$

where the morphisms $\delta\colon \mathscr{F}^n M'' \to \mathscr{F}^{n+1} M'$ are obtained by applying the functor $\mathscr{F}^0$ to the morphism $\delta[n]\colon M'[n] \to M''[n+1]$. This is how we equip, for example, the sequence of functors $\mathrm{H}^n(X, -)$ for a site X with a ∂-functor structure [**Grothendieck, 1957**, §2.1], which is universal, allowing us to compare cohomology classes constructed by processes involving different constructions of the universal ∂-functor (cf. §4.8).

4.3. Let $\mathscr{G}$ be a sheaf of abelian groups on a site X and $\mathscr{T}$ a (left) $\mathscr{G}$-torsor. We propose to define a class $[\mathscr{T}] \in \mathrm{H}^1(X, \mathscr{G})$.

(xiv) One should be careful that [**SGA 4** xvii 1.1] uses an opposite convention.

4.3.1. There exists a monomorphism of abelian sheaves $i\colon \mathscr{G} \to \mathscr{A}$ and a $\mathscr{G}$-equivariant morphism $\alpha\colon \mathscr{T} \to \mathscr{A}$ where we let $\mathscr{G}$ act on $\mathscr{A}$ by the formula $g.a = i(g) + a$. The image of α in the quotient $\mathscr{A}/\mathscr{G}$ identifies with an element $s \in \mathrm{H}^0(X, \mathscr{A}/\mathscr{G})$. We denote by $[\mathscr{T}] := \delta(s) \in \mathrm{H}^1(X,\mathscr{G})$ the image of s under the boundary map $\delta\colon \mathrm{H}^0(X,\mathscr{A}/\mathscr{G}) \to \mathrm{H}^1(X,\mathscr{G})$ associated with the exact sequence $0 \to \mathscr{G} \to \mathscr{A} \to \mathscr{A}/\mathscr{G} \to 0$ (cf. [**SGA 4½** [Cycle] 1.1.1]).

4.3.2. If $\mathscr{E} \to \bullet$ is an epimorphism of sheaves of sets (where $\bullet$ is the final object) and $s\colon \mathscr{E} \to \mathscr{T}$ is a morphism (which we consider as a section of $\mathscr{T}$ over $\mathscr{E}$), by considering $\mathscr{T}$ as a $\mathscr{G}$-torsor (*i.e.*, as a right torsor under $\mathscr{G}$) by letting $\mathscr{G}$ act on the right on $\mathscr{T}$ by the formula $t.g := g.t$ (which is possible since $\mathscr{G}$ is commutative), there exists a (unique) morphism $\gamma\colon \mathscr{E}\times\mathscr{E} \to \mathscr{G}$ such that $pr_2^\star(s) = pr_1^\star(s).\gamma$: this is the 1-cocycle associated with the right torsor $\mathscr{T}$ [xv]. Following the conventions of [**Conrad, 2000**, §1.3] suitably generalized to apply to sites and not just to topological spaces, the 1-cocycle γ is a Čech 1-cocycle associated with the covering $\mathscr{E} \to \bullet$; it therefore defines an element of $\mathrm{H}^1(X,\mathscr{G})$ which can be shown to coincide with the element $[\mathscr{T}]$ defined in 4.3.1.

4.3.3. It is also interesting to have a third construction of $[\mathscr{T}]$, this time identifying $\mathrm{H}^1(X,\mathscr{G})$ with the group of morphisms $\mathbf{Z} \to \mathscr{G}[1]$ in the derived category of abelian sheaves. For this, with the same notations as in 4.3.2, we introduce the simplicial sheaf of sets $\check{\mathrm{C}}(\mathscr{E})$ defined by $\check{\mathrm{C}}(E)_n := \mathbf{Hom}(\{0,\dots,n\},\mathscr{E}) \simeq \mathscr{E}^{1+n}$ for all $n \in \mathbf{N}$, the simplicial structure being obvious. By denoting by $\mathbf{Z}-$ the left adjoint functor of the forgetful functor from abelian sheaves to sheaves of sets, we obtain a simplicial abelian sheaf $\mathbf{Z}\check{\mathrm{C}}(\mathscr{E})$ which gives rise to a complex of abelian sheaves (concentrated in negative degrees), which we will still denote by abuse of notation $\mathbf{Z}\check{\mathrm{C}}(\mathscr{E})$:

$$\dots \to \mathbf{Z}(\mathscr{E}^3) \overset{d_0-d_1+d_2}{\to} \mathbf{Z}(\mathscr{E}^2) \overset{d_0-d_1}{\to} \mathbf{Z}\mathscr{E} \to 0 \to \dots$$

The projection $\mathscr{E} \to \bullet$ induces the augmentation morphism $\varepsilon\colon \mathbf{Z}\check{\mathrm{C}}(\mathscr{E}) \to \mathbf{Z}$, which is a quasi-isomorphism. We can then describe $[\mathscr{T}] \in \mathrm{H}^1(X,\mathscr{G})$ as the following zigzag :

$$\begin{tikzcd}
-2 & 0 \arrow[d] & \vdots \arrow[l] \arrow[r] \arrow[d] & 0 \arrow[d] \\
-1 & 0 \arrow[d] & \mathbf{Z}(\mathscr{E}^2) \arrow[l] \arrow[r, "-\gamma"] \arrow[d, "{(e,e') \mapsto e'-e}"] & \mathscr{G} \arrow[d] \\
0 & \mathbf{Z} & \mathbf{Z}\mathscr{E} \arrow[l] \arrow[r] & 0 \\
& \mathbf{Z} & \mathbf{Z}\check{\mathrm{C}}(\mathscr{E}) \arrow[l, "\varepsilon", "\sim"'] \arrow[r] & \mathscr{G}[1]
\end{tikzcd}$$

4.4. A bicomplex is a family $(K^{p,q})_{(p,q)\in\mathbf{Z}^2}$ of objects of an abelian category $\mathscr{A}$ equipped with horizontal differentials $d_h\colon K^{p,q} \to K^{p+1,q}$ and vertical ones $d_v\colon K^{p,q} \to K^{p,q+1}$ such that $d_h \circ d_h = 0$, $d_v \circ d_v = 0$, and $d_v \circ d_h = d_h \circ d_v$. The total complex associated with $K^{\bullet,\bullet}$ is defined by $(\mathrm{Tot}\, K^{\bullet,\bullet})^n := \bigoplus_{p+q=n} K^{p,q}$ and the differential $(\mathrm{Tot}\, K^{\bullet,\bullet})^n \to (\mathrm{Tot}\, K^{\bullet,\bullet})^{n+1}$ is defined on the term $K^{p,q}$ (for $p+q=n$) as being $d_h + (-1)^p d_v\colon K^{p,q} \to K^{p+1,q} \oplus K^{p,q+1} \subset (\mathrm{Tot}\, K^{\bullet,\bullet})^{n+1}$. (The total complex defined above is the one defined in terms of sums. There is also a version defined using products rather than sums. In both cases, one must ensure that the sums or products considered are representable in $\mathscr{A}$.)

4.5. The tensor product of complexes is defined in the usual way. If K and L are two complexes (of modules, or of sheaves of modules), $(K\otimes L)^n := \oplus_{p+q=n} K^p \otimes L^q$ and the differential is defined by the formula $d(x\otimes y) = dx \otimes y + (-1)^{|x|} x \otimes dy$ where $|x|$ is the degree of x (in other words, $K \otimes L$ is the total complex associated with the obvious bicomplex $(K^p \otimes L^q)_{(p,q)\in\mathbf{Z}^2}$, cf. 4.4).

[xv] This is the formula one would use to define the 1-cocycle associated with a right torsor under a not-necessarily-commutative sheaf of groups.

4.5.1. The symmetry automorphism $K \otimes L \simeq L \otimes K$ sends $x \otimes y$ to $(-1)^{|x| \cdot |y|} y \otimes x$. (The associativity isomorphism $(K \otimes L) \otimes M \simeq K \otimes (L \otimes M)$, on the other hand, does not involve any sign.) We can then remark that if K is a complex and $i \in \mathbf{Z}$, then $K \otimes (\Lambda[i])$ identifies canonically with K shifted by i places to the left *without changing the sign of the differentials*. On the other hand, we can observe that $\Lambda[i] \otimes K$ identifies quite canonically with $K[i]$. (Here, Λ is an arbitrary (commutative) coefficient ring.)

4.5.2. Suppose that $a : M' \otimes_\Lambda M'' \to M$ is a morphism of complexes of Λ-modules. We define for any $(i, j) \in \mathbf{Z}^2$ a morphism $a : \mathrm{H}^i(M') \otimes_\Lambda \mathrm{H}^j(M'') \to \mathrm{H}^{i+j}(M)$ in the following way. If $x \in M'^i$ and $y \in M''^j$ are cocycles, then $x \otimes y \in M'^i \otimes_\Lambda M''^j \subset (M' \otimes_\Lambda M'')^{i+j}$ is a cocycle whose image under a in M^{i+j} can be considered : we thus define the image of $[x] \otimes [y]$. With the preceding notations, if we interpret x, y, and $a(x \otimes y)$ as morphisms of complexes $x : \Lambda \to M'[i] = \Lambda[i] \otimes_\Lambda M'$, $y : \Lambda \to M''[j] = \Lambda[j] \otimes_\Lambda M''$, and $a(x \otimes y) : \Lambda \to M[i+j]$, the following diagram is commutative :

$$\begin{tikzcd}
\Lambda \arrow[r, "(-1)^{ij}", "\sim"'] \arrow[d, "a(x \otimes y)"'] & \Lambda \otimes_\Lambda \Lambda \arrow[r, "x \otimes y"] & (\Lambda[i] \otimes_\Lambda M') \otimes_\Lambda (\Lambda[j] \otimes_\Lambda M'') \arrow[d, "\sim"] \\
M[i+j] & \Lambda[i] \otimes_\Lambda \Lambda[j] \otimes_\Lambda M \arrow[l] & \Lambda[i] \otimes_\Lambda \Lambda[j] \otimes_\Lambda (M' \otimes_\Lambda M'') \arrow[l, "\mathrm{Id} \otimes a"']
\end{tikzcd}$$

(Beware the presence of multiplication by $(-1)^{ij}$ at the top left! It is related to the symmetry isomorphism appearing on the right.) This construction extends formally to the case where a is a morphism $M' \otimes^{\mathrm{L}}_\Lambda M'' \to M$ in $\mathsf{D}(\Lambda)$ (cf. **XVII**-8 for the construction of the derived tensor product on the total derived category).

4.5.3. Let X be a site equipped with the constant sheaf of rings Λ. We can apply construction 4.5.2 to the Künneth morphism $a : \mathrm{R}(X, \Lambda) \overset{\mathrm{L}}{\otimes}_\Lambda \mathrm{R}(X, \Lambda) \to \mathrm{R}(X, \Lambda)$ (cf. **XVII**-12.4.2). We thus define the product $\mathrm{H}^i(X, \Lambda) \times \mathrm{H}^j(X, \Lambda) \to \mathrm{H}^{i+j}(X, \Lambda)$. If $u \in \mathrm{H}^i(X, \Lambda)$ and $v \in \mathrm{H}^j(X, \Lambda)$, we denote by uv (or $u \cup v$) the product of the two classes. This product satisfies the relation $vu = (-1)^{ij} uv$.

It is also possible to describe this product in terms of the composition of morphisms in $\mathsf{D}(X, \Lambda)$. Let us identify $u \in \mathrm{H}^i(X, \Lambda)$ and $v \in \mathrm{H}^j(X, \Lambda)$ with morphisms $u : \Lambda \to \Lambda[i]$ and $v : \Lambda \to \Lambda[j]$ in $\mathsf{D}(X, \Lambda)$. The morphism $\Lambda \simeq \Lambda \otimes \Lambda \xrightarrow{u \overset{\mathrm{L}}{\otimes} v} \Lambda[i] \otimes \Lambda[j] \simeq \Lambda[i+j]$ corresponds to a class in $H^{i+j}(X, \Lambda)$ which is equal not to uv in general, but to $(-1)^{ij} uv$, that is, vu. Using the factorization $u \overset{\mathrm{L}}{\otimes} v = (\mathrm{Id} \overset{\mathrm{L}}{\otimes} v) \circ (u \overset{\mathrm{L}}{\otimes} \mathrm{Id})$, we show that $vu = (-1)^{ij} uv$ can also be described as the composition $\Lambda \xrightarrow{u} \Lambda[i] \xrightarrow{v[i]} \Lambda[i+j]$. A more elegant description can be obtained by identifying $u \in \mathrm{H}^i(X, \Lambda)$ with a morphism $u' : \Lambda[-i] \to \Lambda$ and $v \in \mathrm{H}^j(X, \Lambda)$ with a morphism $v' : \Lambda[-j] \to \Lambda$. The product $uv \in \mathrm{H}^{i+j}(X, \Lambda)$ then corresponds, without any parasitic sign, to the morphism $\Lambda[-(i+j)] \simeq \Lambda[-i] \otimes \Lambda[-j] \xrightarrow{u' \overset{\mathrm{L}}{\otimes} v'} \Lambda$. (Note that this description of the $\cup$-product is particularly well-suited to the definition of the variants with supports of these multiplicative structures.)

4.6. This paragraph deals with conventions regarding the cohomology of groups and the étale fundamental group.

4.6.1. Let G be a group. Let A be an abelian group (which we equip with the trivial action of G). We need to specify the identification $\mathrm{H}^1(G, A) \simeq \mathrm{Hom}(G, A)$ for the first cohomology group of the (discrete) group G. We define here the cohomology of the group G as that of the topos $\mathbf{B}G$ of (left) G-sets. Let $\varphi : G \to A$ be a homomorphism. We define a set $A^\varphi := A$ which we equip with a left action of G by the formula $g.a := \varphi(g) + a$ and a right action of A by $a.a' := a + a'$. The set A^φ equipped with the left action of G can thus be considered as a sheaf on the topos $\mathbf{B}G$. If we also take into account the right action of A, we make A^φ a right torsor under the constant sheaf A on the topos $\mathbf{B}G$. Since A is abelian, A^φ can also be seen as a left torsor under A on $\mathbf{B}G$. By 4.3, A^φ has a class $[A^\varphi] \in \mathrm{H}^1(\mathbf{B}G, A) = \mathrm{H}^1(G, A)$. The canonical isomorphism $\mathrm{Hom}(G, A) \xrightarrow{\sim} \mathrm{H}^1(G, A)$ is the one that associates $[A^\varphi]$ to φ. *Via* the comparison between cohomology and Čech cohomology (cf. 4.3.2), this definition is compatible with the identification of the cohomology of the group G calculated in terms of cochains (cf. [**Serre, 1968**, §3, Chapter VII]) and the Čech cohomology associated with the covering given by the epimorphism $\mathbf{E}G \to \mathbf{B}G$ where $\mathbf{B}G$ is here the final object of the topos $\mathbf{B}G$ and where $\mathbf{E}G$ is the sheaf on $\mathbf{B}G$ corresponding to the set G equipped with the left action of G by multiplication. We can observe that the right multiplication on G induces on $\mathbf{E}G$ a structure of a right torsor under the constant group G in the topos $\mathbf{B}G$. In the case where G is commutative, the class of this torsor $[\mathbf{E}G]$ corresponds to the identity of G *via* the identifications $\mathrm{Hom}(G,G) \simeq \mathrm{H}^1(G,G) = \mathrm{H}^1(\mathbf{B}G,G)$.

4.6.2. Let G be a finite group. Let X be a topos. Let $\mathscr{T}$ be a sheaf of sets on X equipped with a right torsor structure under the group G. For any sheaf of sets F on X, we denote by $u_\star F := \mathrm{Hom}_X(\mathscr{T},F)$ and this set inherits a (left) G-set structure from the action on $\mathscr{T}$. This functor $u_\star$ is the direct image functor for a morphism of topoi $u\colon X \to \mathbf{B}G$ and we have a canonical isomorphism of right torsors $\mathscr{T} \simeq u^\star\mathbf{E}G$. It thus follows that giving a morphism of topoi $u\colon X \to \mathbf{B}G$ is equivalent to giving a right torsor under G on X.

4.6.3. Let X be a connected noetherian scheme equipped with a geometric point $\overline{x}$. We have the étale fundamental group $\pi_1^{\text{ét}}(X,\overline{x})$, cf. [**SGA 1** v 7]. The category of discrete (left) $\pi_1^{\text{ét}}(X,\overline{x})$-sets identifies with a full subcategory of the category of sheaves of sets on $X_{\text{ét}}$ (cf. **XVII**-7.2 for more details). This inclusion functor is the inverse image functor for a canonical morphism of topoi $X_{\text{ét}} \to \mathbf{B}\pi_1^{\text{ét}}(X,\overline{x})$. Let $Y \to X$ be a Galois étale covering. Let $G := \mathrm{Aut}_{\mathscr{O}_X-\mathrm{Algebra}}(\mathscr{O}_Y)$: this is the opposite group to the group of automorphisms of the X-scheme Y. The scheme Y is thus naturally equipped with a right action of G which makes it a right torsor under G over $X_{\text{ét}}$. We thus have by 4.6.2 a morphism of topoi $X_{\text{ét}} \to \mathbf{B}G$, which is canonically isomorphic to the composite $X_{\text{ét}} \to \mathbf{B}\pi_1^{\text{ét}}(X,\overline{x}) \xrightarrow{\mathbf{B}p} \mathbf{B}G$ for a morphism $p\colon \pi_1^{\text{ét}}(X,\overline{x}) \to G$ that we will now define and which depends on the choice of a closed point $\overline{y}$ in the geometric fiber $Y_{\overline{x}}$. Let $\gamma \in \pi_1^{\text{ét}}(X,\overline{x})$. This element γ acts (on the left) on the fiber $Y_{\overline{x}}$ on which G also acts (on the right), and these two actions commute. We denote by $p(\gamma) \in G$ the unique element such that $\gamma.\overline{y} = \overline{y}.p(\gamma)$.

4.7. We define here the functor **Hom** (internal hom) on complexes (of étale sheaves of Λ-modules on a scheme X) as the bifunctor defined by an adjunction isomorphism "dear to Cartan" where Hom is the hom in the category of complexes :

$$\mathrm{Hom}(K,\mathbf{Hom}(L,M)) \simeq \mathrm{Hom}(K \otimes L, M)\ .$$

This adjunction is tautologically enriched into a "dear to Cartan" isomorphism stated in terms of the internal hom :

$$\mathbf{Hom}(K,\mathbf{Hom}(L,M)) \simeq \mathbf{Hom}(K \otimes L, M)\ .$$

4.7.1. If K, L, and M are complexes, the identity of $\mathbf{Hom}(K,L)$ induces by adjunction an "evaluation morphism" $\mathbf{Hom}(K,L) \otimes K \to L$ to which one can apply $M \otimes -$ to obtain a morphism $M \otimes \mathbf{Hom}(K,L) \otimes K \to M \otimes L$, which in turn induces by adjunction a canonical morphism $M \otimes \mathbf{Hom}(K,L) \to \mathbf{Hom}(K, M \otimes L)$. In particular, for $M = \Lambda[m]$ with $m \in \mathbf{Z}$, this morphism is a canonical isomorphism $\mathbf{Hom}(K,L)[m] \simeq \mathbf{Hom}(K,L[m])$.

4.7.2. If L and N are complexes, we can define a morphism $\gamma_L\colon \mathbf{Hom}(L,\Lambda) \otimes N \to \mathbf{Hom}(L,N)$ in the following way. The identity of $\mathbf{Hom}(L,\Lambda)$ induces by adjunction a morphism $\mathrm{ev}\colon \mathbf{Hom}(L,\Lambda) \otimes L \to \Lambda$ which allows to define a morphism :

$$\mathbf{Hom}(L,\Lambda) \otimes N \otimes L \xrightarrow{\sim} \mathbf{Hom}(L,\Lambda) \otimes L \otimes N \xrightarrow{\mathrm{ev}\otimes N} \Lambda \otimes N \xrightarrow{\sim} N\,,$$

which defines by adjunction the desired morphism $\mathbf{Hom}(L,\Lambda) \otimes N \to \mathbf{Hom}(L,N)$. Let $m \in \mathbf{Z}$. The obvious isomorphism $\mathbf{Z}[-m] \otimes \mathbf{Z}[m] \xrightarrow{\sim} \mathbf{Z}$ induces by adjunction an isomorphism $\mathbf{Z}[-m] \xrightarrow{\sim} \mathbf{Hom}(\mathbf{Z}[m],\mathbf{Z})$. The preceding construction γ_L applied to $L = \mathbf{Z}[m]$ thus provides a morphism $\gamma_m\colon N[-m] \to \mathbf{Hom}(\Lambda[m],N)$ which is an isomorphism. If $N = \mathbf{Hom}(K,L)$ with K and L two complexes, we obtain an isomorphism (still denoted by γ_m) :

$$\mathbf{Hom}(K,L)[-m] \underset{\sim}{\xrightarrow{\gamma_m}} \mathbf{Hom}(\Lambda[m],\mathbf{Hom}(K,L)) \simeq \mathbf{Hom}(\Lambda[m] \otimes K, L) \simeq \mathbf{Hom}(K[m],L)\ .$$

We define $\alpha_m\colon \mathbf{Hom}(K[m],L) \xrightarrow{\sim} \mathbf{Hom}(K,L)[-m]$ by the formula $\alpha_m := (-1)^{\frac{m(m+1)}{2}}\gamma_m^{-1}$.

4.7.3. In [**Conrad, 2000**, §1.3], Conrad explicitly defines the functor **Hom**. His construction is compatible with the one defined here by adjunction since it satisfies such an adjunction isomorphism : this isomorphism is given degree by degree by "dear to Cartan" adjunction isomorphisms at the level of the internal hom in the category of sheaves, and this *without adding signs*. With these conventions, the "commutation" isomorphisms of $\mathbf{Hom}(-,-)$ with the functors $-[m]$ in both variables (4.7.1 and 4.7.2) are the same as those in [**Conrad, 2000**, §1.3]. By deriving this functor (cf. **XVII**-8 for more details), we obtain a bifunctor $\mathrm{R}\,\mathbf{Hom}: \mathsf{D}(X_{\text{ét}},\Lambda)^{\text{opp}} \times \mathsf{D}(X_{\text{ét}},\Lambda) \to \mathsf{D}(X_{\text{ét}},\Lambda)$ which is "triangulated with respect to both variables". This means in particular that for any $K \in \mathsf{D}(X_{\text{ét}},\Lambda)$, the functor $\mathrm{R}\,\mathbf{Hom}(K,-): \mathsf{D}(X_{\text{ét}},\Lambda) \to \mathsf{D}(X_{\text{ét}},\Lambda)$ is triangulated and that for any $L \in \mathsf{D}(X_{\text{ét}},\Lambda)$, the functor $\mathrm{R}\,\mathbf{Hom}(-,L): \mathsf{D}(X_{\text{ét}},\Lambda)^{\text{opp}} \to \mathsf{D}(X_{\text{ét}},\Lambda)$ is triangulated.

4.7.4. The last assertion of 4.7.3 means that if $K' \xrightarrow{\alpha} K \xrightarrow{\beta} K'' \xrightarrow{\gamma} K'[1]$ is a distinguished triangle in $\mathsf{D}(X_{\text{ét}},\Lambda)$, the following triangle

$$\mathrm{R}\,\mathbf{Hom}(K',L)[-1] \xrightarrow{\gamma^\star} \mathrm{R}\,\mathbf{Hom}(K'',L) \xrightarrow{\beta^\star} \mathrm{R}\,\mathbf{Hom}(K,L) \xrightarrow{\alpha^\star} \mathrm{R}\,\mathbf{Hom}(K',L)$$

obtained by applying the functor $\mathrm{R}\,\mathbf{Hom}(-,L)$ and using $\alpha_1: \mathrm{R}\,\mathbf{Hom}(K'[1],L) \xrightarrow{\sim} \mathrm{R}\,\mathbf{Hom}(K',L)[-1]$, is **anti-distinguished** in $\mathsf{D}(X_{\text{ét}},\Lambda)$, which means that the following triangle is *distinguished* in $\mathsf{D}(X_{\text{ét}},\Lambda)$:

$$\mathrm{R}\,\mathbf{Hom}(K',L)[-1] \xrightarrow{-\gamma^\star} \mathrm{R}\,\mathbf{Hom}(K'',L) \xrightarrow{\beta^\star} \mathrm{R}\,\mathbf{Hom}(K,L) \xrightarrow{\alpha^\star} \mathrm{R}\,\mathbf{Hom}(K',L)$$

It is also true that the following triangle is distinguished, where we implicitly use $\alpha_{-1}: \mathrm{R}\,\mathbf{Hom}(K''[-1],L) \xrightarrow{\sim} \mathrm{R}\,\mathbf{Hom}(K'',L)[1]$:

$$\mathrm{R}\,\mathbf{Hom}(K'',L) \xrightarrow{\beta^\star} \mathrm{R}\,\mathbf{Hom}(K,L) \xrightarrow{\alpha^\star} \mathrm{R}\,\mathbf{Hom}(K',L) \xrightarrow{\gamma[-1]^\star} \mathrm{R}\,\mathbf{Hom}(K'',L)[1]$$

Let us consider this morphism $\delta := \gamma[-1]^\star: \mathrm{R}\,\mathbf{Hom}(K',L) \xrightarrow{\gamma[-1]^\star} \mathrm{R}\,\mathbf{Hom}(K'',L)[1]$. For any integer $n \in \mathbf{Z}$, the morphism δ induces after applying the functor $\mathrm{H}^n_{\text{ét}}(X,-)$ a morphism of abelian groups, where Hom is the abelian group of morphisms in the category $\mathsf{D}(X_{\text{ét}},\Lambda)$:

$$\delta^n: \mathrm{Hom}(K',L[n]) \to \mathrm{Hom}(K'',L[n+1]) \;.$$

We can then observe that if $\varphi \in \mathrm{Hom}(K',L[n])$, then $\delta^n(\varphi)$ is the composite morphism $K'' \xrightarrow{(-1)^{n+1}\gamma} K'[1] \xrightarrow{\varphi[1]} K[n+1]$.

4.7.5. If $L' \xrightarrow{a} L \xrightarrow{b} L'' \xrightarrow{c} L'[1]$ is a distinguished triangle in $\mathsf{D}(X_{\text{ét}},\Lambda)$, we also have long exact sequences for any $K \in \mathsf{D}(X_{\text{ét}},\Lambda)$:

$$\ldots \to \mathrm{Hom}(K,L'[n]) \xrightarrow{a[n]_\star} \mathrm{Hom}(K,L[n]) \xrightarrow{b[n]_\star} \mathrm{Hom}(K,L''[n]) \xrightarrow{\delta} \mathrm{Hom}(K,L'[n+1]) \to \ldots$$

where for any $\varphi \in \mathrm{Hom}(K,L''[n])$, $\delta(\varphi) = c[n] \circ \varphi$. If $K' \xrightarrow{\alpha} K \xrightarrow{\beta} K'' \xrightarrow{\gamma} K'[1]$ is a distinguished triangle in $\mathsf{D}(X_{\text{ét}},\Lambda)$, we can consider the following square :

$$\begin{tikzcd}
\mathrm{Hom}(K'',L''[n+1]) \arrow[r, "\delta"] & \mathrm{Hom}(K'',L'[n+2]) \\
\mathrm{Hom}(K',L''[n]) \arrow[u, "\delta"] \arrow[r, "\delta"] & \mathrm{Hom}(K',L'[n+1]) \arrow[u, "\delta"']
\end{tikzcd}$$

The interpretation of the morphisms δ in terms of composition in $\mathsf{D}(X_{\text{ét}},\Lambda)$ shows that this square is *anti-commutative* !

4.7.6. A particular case of 4.7.4 that will interest us is the following. If Z is a closed subscheme of a scheme X and $U = X - Z$, we have a canonical short exact sequence of étale sheaves on X :

$$0 \to \mathbf{Z}_U \to \mathbf{Z}_X \to \mathbf{Z}_Z \to 0 \;.$$

By 4.7.4, we have a long exact sequence for any $L \in \mathsf{D}(X_{\text{ét}},\Lambda)$:

$$\ldots \to \mathrm{H}^n_{Z,\text{ét}}(X,L) \to \mathrm{H}^n_{\text{ét}}(X,L) \to \mathrm{H}^n_{\text{ét}}(U,L) \xrightarrow{\delta} \mathrm{H}^{n+1}_{Z,\text{ét}}(X,L) \to \ldots$$

Suppose that L is a bounded below complex of injective sheaves. Let a class $[\gamma] \in \mathrm{H}^n_{\text{ét}}(U, L)$ be represented by a section $\gamma \in \Gamma(U, L^n)$ such that $d\gamma = 0 \in \Gamma(U, L^{n+1})$. The sheaf L^n being injective, there exists a section $\tilde{\gamma} \in \Gamma(X, L^n)$ such that $\tilde{\gamma}_{|U} = \gamma$. The section $d\tilde{\gamma} \in \Gamma(X, L^{n+1})$ vanishes on U, so it defines an element $d\tilde{\gamma} \in \Gamma_Z(X, L^{n+1})$ which is obviously an $(n+1)$-cocycle. We then have $\delta([\gamma]) = [d\tilde{\gamma}] \in \mathrm{H}^{n+1}_{Z,\text{ét}}(X, L)$.

If we now assume that $L = \mathscr{G}$ where $\mathscr{G}$ is an abelian sheaf on $X_{\text{ét}}$, it is useful to have an explicit description of the morphism $\delta\colon H^0(U, \mathscr{G}) \to H^1_Z(X, \mathscr{G})$. First, let us note that we can generalize construction 4.3.1 : if $\mathscr{T}$ is a $\mathscr{G}$-torsor on $X_{\text{ét}}$ equipped with a section $s \in \mathscr{T}(U)$, we have an element $[\mathscr{T}, s] \in \mathrm{H}^1_{Z,\text{ét}}(X, \mathscr{G})$ (inducing $[\mathscr{T}] \in \mathrm{H}^1_{\text{ét}}(X, \mathscr{G})$). It is easy to show that if $s \in \mathscr{G}(U) = H^0(U, \mathscr{G})$, then, if we denote by $(\mathscr{G}, s)$ the trivial $\mathscr{G}$-torsor ($\mathscr{G}$ acting on itself by addition) equipped with the section s, we have $[\mathscr{T}, s] = \delta(s) \in H^1_{Z,\text{ét}}(X, \mathscr{G})$.

4.7.7. If K and L are two objects of $\mathsf{D}(X_{\text{ét}}, \Lambda)$, we have already constructed (cf. 4.7.1) a morphism $\mathbf{Hom}(K, L) \otimes K \to L$. By using the symmetry isomorphism, we deduce a morphism $K \otimes \mathbf{Hom}(K, L) \to L$ which corresponds by adjunction to a morphism $K \to \mathbf{Hom}(\mathbf{Hom}(K, L), L)$ called the "biduality" morphism. It satisfies the same sign rules as those stated in [**Conrad, 2000**, §1.3]. The object of Exposition XVII will be to study a derived version of this construction...

4.8. This paragraph is a warning about the ambiguity of meaning that a statement saying two cohomological constructions using different sign conventions are equal or opposite could have. Let us consider for example the derived category $\mathsf{D}^+(X)$ of abelian sheaves on a site X. For any abelian sheaf $\mathscr{F}$, we can denote $\mathrm{H}^i(X, \mathscr{F}) := \mathrm{Hom}_{\mathsf{D}^+(X)}(\mathbf{Z}, \mathscr{F}[i])$. Using construction 4.2, we obtain a ∂-functor $(\mathrm{H}^\star(X, -), \delta)$. Let us now set $\tilde{\mathrm{H}}^i(X, \mathscr{F}) := \mathrm{H}^i(X, \mathscr{F})$ and denote $\tilde{\delta} := -\delta$. Of course, $(\tilde{\mathrm{H}}^\star(X, -), \tilde{\delta})$ is also a ∂-functor : it is the one we obtain naturally by using the conventions of [**SGA 4** XVII 1.1].

The universal character of these two ∂-functors induces a canonical isomorphism of ∂-functors $\varphi\colon (\mathrm{H}^\star(X, -), \delta) \xrightarrow{\sim} (\tilde{\mathrm{H}}^\star(X, -), \tilde{\delta})$: in degree i, it is given by multiplication by $(-1)^i$.

Let us admit that a certain cohomological construction using the first ∂-functor produces a class $x \in \mathrm{H}^i(X, \mathscr{F})$ and that another construction using the second produces a class $y \in \tilde{\mathrm{H}}^i(X, \mathscr{F})$. The statement "the cohomology classes x and y are equal" can then reasonably take two different meanings :

(a) Since set-theoretically, $\mathrm{H}^i(X, \mathscr{F})$ and $\tilde{\mathrm{H}}^i(X, \mathscr{F})$ are both equal to the set of morphisms $\mathbf{Z} \to \mathscr{F}[i]$ in $\mathsf{D}^+(X)$, we can understand that $y = x$;

(b) If we identify $\mathrm{H}^i(X, \mathscr{F})$ and $\tilde{\mathrm{H}}^i(X, \mathscr{F})$ *via* the isomorphism of ∂-functors φ, we can understand that $y = (-1)^i x$.

Note that in even degrees, the two meanings coincide. (Otherwise, the author recommends using meaning (b).)

EXPOSITION XVII

Duality

Joël Riou

This text aims to provide a write-up of the results announced by Ofer Gabber in [**Gabber, 2005b**] concerning dualizing complexes in the étale context on excellent Noetherian schemes.

We fix a natural number $n \geq 2$, and we let $\Lambda = \mathbf{Z}/n\mathbf{Z}$: this will be our ring of coefficients. If X is a Noetherian $\mathbf{Z}\left[\frac{1}{n}\right]$-scheme, we denote by $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X_{\mathrm{\acute{e}t}}, \Lambda)$ the subcategory of $\mathrm{D}^{\mathrm{b}}(X_{\mathrm{\acute{e}t}}, \Lambda)$ consisting of complexes with constructible cohomology sheaves, by $\mathrm{D}^{\mathrm{b}}_{\mathrm{tf}}(X_{\mathrm{\acute{e}t}}, \Lambda)$ the full subcategory of $\mathrm{D}^{\mathrm{b}}(X_{\mathrm{\acute{e}t}}, \Lambda)$ consisting of complexes of finite tor-dimension, and $\mathrm{D}^{\mathrm{b}}_{\mathrm{ctf}}(X_{\mathrm{\acute{e}t}}, \Lambda) := \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X_{\mathrm{\acute{e}t}}, \Lambda) \cap \mathrm{D}^{\mathrm{b}}_{\mathrm{tf}}(X_{\mathrm{\acute{e}t}}, \Lambda)$.

Définition 0.1. Let X be a Noetherian $\mathbf{Z}\left[\frac{1}{n}\right]$-scheme. A **dualizing complex** on X is an object $K \in \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X_{\mathrm{\acute{e}t}}, \Lambda)$ such that the duality functor $\mathrm{D}_K = \mathrm{R}\,\mathbf{Hom}(-, K)$ preserves $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X_{\mathrm{\acute{e}t}}, \Lambda)$ and such that for any $L \in \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X_{\mathrm{\acute{e}t}}, \Lambda)$, the biduality morphism $L \to \mathrm{D}_K\,\mathrm{D}_K\,L$ (cf. **XVI**-4.7.7 and also §12) is an isomorphism.

This definition differs from that of [**SGA 5** I 1.7] in that we do not require a dualizing complex to have finite quasi-injective dimension.

The following theorem summarizes the main results of Gabber that we will establish in these notes :

Théorème 0.2. *Let X be an excellent Noetherian $\mathbf{Z}\left[\frac{1}{n}\right]$-scheme equipped with a dimension function (cf. definition 2.1.1). Then, X admits a dualizing complex K, unique up to tensor product with an invertible object (cf. proposition 9.2). This dualizing complex K belongs to $\mathrm{D}^{\mathrm{b}}_{\mathrm{ctf}}(X_{\mathrm{\acute{e}t}}, \Lambda)$; it has finite quasi-injective dimension (in other words, it is a dualizing complex in the sense of [**SGA 5** I 1.7]) if and only if X has finite Krull dimension.*

Furthermore, we have the following results :

- *if X is regular, the constant sheaf Λ is a dualizing complex ;*
- *if $f : Y \to X$ is a flat morphism with geometrically regular fibers, with Y an excellent Noetherian scheme, then $f^!K$ is a dualizing complex ;*
- *if $f : Y \to X$ is a compactifiable morphism of finite type, then $f^!K$ is a dualizing complex.*

The proof relies on the notion of a potential dualizing complex (definition 2.1.2) on an excellent Noetherian $\mathbf{Z}\left[\frac{1}{n}\right]$-scheme X equipped with a dimension function δ : this is the data of a complex $K \in \mathrm{D}^{+}(X_{\mathrm{\acute{e}t}}, \Lambda)$ equipped with isomorphisms (called pinnings) $\mathrm{R}\Gamma_x(K) \xrightarrow{\sim} \Lambda(\delta(x))[2\delta(x)]$ for all points $x \in X$, which are compatible with the transition morphisms associated with immediate specializations of geometric points of X (cf. section 1). Gabber shows in [**Gabber, 2004**, lemma 8.1][(i)] that if a Noetherian $\mathbf{Z}\left[\frac{1}{n}\right]$-scheme admits a dualizing complex, then it admits a dimension function ; this hypothesis of theorem 0.2 is therefore necessary.

In section 2, we will see in particular that on a regular excellent Noetherian $\mathbf{Z}\left[\frac{1}{n}\right]$-scheme, the constant sheaf Λ is a potential dualizing complex for the dimension function $-\operatorname{codim}$ (cf. proposition 2.4.4.1). To do this, we will essentially use the absolute cohomological purity theorem proven by Gabber, as well as the properties of Gysin morphisms established in **XVI**-2.

[(i)] In *loc. cit.*, the hypothesis "$n > 0$" should be replaced by "$n > 1$" : when the ring of coefficients $A = \mathbf{Z}/n\mathbf{Z}$ is zero, the integers $n(x)$ are not defined.

In section 3, we will construct an isomorphism $\Lambda \xrightarrow{\sim} \mathrm{H}^{2d}_{x,\text{ét}}(X, \Lambda(d))$ where X is a strictly Henselian normal excellent local Noetherian $\mathbf{Z}\left[\frac{1}{n}\right]$-scheme of dimension d and closed point x, we will verify that this isomorphism is compatible with immediate specializations and we will use this result to construct transition morphisms $\mathrm{H}^i_{\overline{y}}(X, K) \to \mathrm{H}^{i+2c}_{\overline{x}}(X, K(c))$ for an arbitrary specialization $\overline{y} \to \overline{x}$ of codimension c between geometric points of an excellent Noetherian $\mathbf{Z}\left[\frac{1}{n}\right]$-scheme X, for any $K \in \mathrm{D}^+(X_{\text{ét}}, \Lambda)$. We will use in particular the resolution of singularities for excellent Noetherian schemes of dimension 2.

In section 5, we will show the existence and uniqueness up to unique isomorphism of a potential dualizing complex on an excellent Noetherian $\mathbf{Z}\left[\frac{1}{n}\right]$-scheme equipped with a dimension function. In the case of a normal scheme, we will rely on the results of section 3 and on Gabber's general results on the existence of t-structures defined by perversity functions. We will also show that potential dualizing complexes satisfy good stability properties with respect to flat morphisms with geometrically regular fibers and to morphisms of finite type.

In section 6, we will show that a potential dualizing complex is a dualizing complex. Once the finiteness properties are established, we will proceed by induction on the dimension, using on the one hand a generalization of an argument from [**SGA 4½** [Th. finitude] 4.3] and on the other hand the partial algebrization theorem **V**-3.1.3.

In section 7, we show that from a dualizing complex with coefficients Λ, one can construct dualizing complexes for more general coefficient rings. These results are essentially independent of the preceding sections. However, it is only by combining the results of the preceding sections on potential dualizing complexes with the uniqueness result for dualizing complexes from proposition 7.5.1.1 that one can deduce theorem 0.2. By virtue of this theorem 0.2, the constructions in this section give in particular dualizing complexes for general coefficient rings on excellent Noetherian schemes equipped with dimension functions. It seems very likely that it is possible to extend Ofer Gabber's results to duality statements with ℓ-adic coefficients. However, the author has decided not to write them up.

Finally, sections 8, 9, 10, 11 and 12 contain some results necessary for the above. In particular, they describe a construction of the derived tensor product on the entire derived category of sheaves of modules on a (commutative) ringed topos.

1. The transition morphism in codimension 1

1.1. Notations.

Définition 1.1.1. Let X be a $\mathbf{Z}\left[\frac{1}{n}\right]$-scheme, let $x \in X$, let $K \in \mathrm{D}^+(X_{\text{ét}}, \Lambda)$. We set $\mathrm{R}\Gamma_x(K) = i_x^! K_{|X_{(x)}} \in \mathrm{D}^+(x_{\text{ét}}, \Lambda)$ where i_x is the inclusion of the closed point of the local scheme $X_{(x)}$ [(ii)]. If $\overline{x}$ is a geometric point of X above x [(iii)], we denote $\mathrm{R}_{\overline{x}}(K) = \mathrm{R}\Gamma_x(K)_{\overline{x}} \in \mathrm{D}^+(\Lambda)$; this object is identified with the cohomology with supports in the closed point of the restriction of K to the strict Henselization $X_{(\overline{x})}$. The cohomology objects of $\mathrm{R}_{\overline{x}}(K)$ will be denoted $\mathrm{H}^i_{\overline{x}}(K)$ for all $i \in \mathbf{Z}$.

Définition 1.1.2 ([**SGA 4** viii 7.2], **XIV**-2.1.1, **XIV**-2.1.2). If $\overline{y}$ and $\overline{x}$ are two geometric points of a scheme X, a **specialization** $\overline{y} \to \overline{x}$ is an X-morphism $X_{(\overline{y})} \to X_{(\overline{x})}$ between the corresponding strict Henselizations, which amounts to giving an X-morphism $\overline{y} \to X_{(\overline{x})}$. We define the **codimension of a specialization** as the dimension of the closure of the point of $X_{(\overline{x})}$ below $\overline{y}$. A specialization is said to be **immediate** if it has codimension 1.

We propose here to define a transition morphism $\mathrm{sp}^X_{\overline{y}\to\overline{x}} \colon \mathrm{R}_{\overline{y}}(K) \to \mathrm{R}_{\overline{x}}(K)(1)[2]$ in $\mathrm{D}^+(\Lambda)$ for any immediate specialization $\overline{y} \to \overline{x}$ of geometric points on an excellent $\mathbf{Z}\left[\frac{1}{n}\right]$-scheme X, for any $K \in \mathrm{D}^+(X_{\text{ét}}, \Lambda)$.

(ii) An equivalent definition would be obtained by replacing the local scheme $X_{(x)}$ with its Henselization.

(iii) In what follows, so as not to unnecessarily weigh down the text, if we fix a point x of a scheme X, $\overline{x}$ will denote a geometric point above x and conversely, if we fix a geometric point $\overline{x} \to X$, we will denote by x the point of the underlying topological space of X above which $\overline{x}$ lies. In this exposition, we will always implicitly assume that the field extension $\kappa(\overline{x})/\kappa(x)$ is algebraic and separable, in other words that $\kappa(\overline{x})$ is a separable closure of $\kappa(x)$.

1.2. Case of a strictly Henselian trait. Let X be a strictly Henselian trait with generic point η and closed point s. Let $\overline{\eta}$ be a geometric point above η. We will define the transition morphism

$$\mathrm{R}_{\overline{\eta}}(K) \to \mathrm{R}_s(K)(1)[2]$$

for any $K \in \mathrm{D}^+(X_{\text{ét}}, \Lambda)$.

We denote by p the characteristic exponent of the residue field of X, which we assume to be invertible in Λ. We have a canonical exact sequence of profinite groups :

$$1 \to S \to \mathrm{Gal}(\overline{\eta}/\eta) \to G \to 1\ ,$$

where G is the tame inertia group, canonically isomorphic to $\widehat{\mathbf{Z}}'(1)$ (iv) and where S, the wild ramification group, is a pro-p-group (cf. [**Gabber & Ramero, 2003**, proposition 6.2.12]).

We recall that the order of a profinite group is a supernatural number (cf. [**Serre, 1994**, § 1.3, Chapter I]). We can then state that the order of G is a multiple of $(\#\Lambda)^\infty$: this fact will be useful to the scrupulous reader who would like to verify as an exercise the details omitted in this subsection.

1.2.1. *The group algebra of* $\widehat{\mathbf{Z}}'(1)$.

Définition 1.2.1.1. The group algebra $\Lambda[[G]]$ is the ring of endomorphisms of the forgetful functor from the category of discrete Λ-modules equipped with a continuous action of G to that of Λ-modules. This algebra is naturally topologized : it is equipped with the coarsest topology such that for any discrete Λ-module M equipped with a continuous action of G and any element $m \in M$, the map $\Lambda[[G]] \to M$ which to $a \in \Lambda[[G]]$ associates the result $a.m$ of its action on m is continuous.

We have a canonical isomorphism of topological rings

$$\Lambda[[G]] \xrightarrow{\sim} \lim \Lambda[G/H]\ ,$$

where H runs through the ordered set of open normal subgroups of G and where $\Lambda[G/H]$ is the usual (discrete) group algebra of the finite group G/H. We can identify discrete Λ-modules equipped with a continuous action of G with discrete $\Lambda[[G]]$-modules.

The natural action of $\Lambda[[G]]$ on Λ equipped with the trivial action of G defines a continuous augmentation morphism $\varepsilon\colon\ \Lambda[[G]] \to \Lambda$. We denote by I_G the kernel of ε : it is the augmentation ideal.

Proposition 1.2.1.2. *The* $\Lambda[[G]]$*-module* I_G *is free of rank* 1, *generated by* $1-\sigma$ *if* σ *is a topological generator of* G. (*More generally, if* $d \geq 1$ *is an integer prime to* p, *the kernel of the canonical ring morphism* $\Lambda[[G]] \to \Lambda[\mu_d]$ *is a free* $\Lambda[[G]]$*-module generated by* $1-\sigma^d$.)

The proof of this proposition is left as an exercise to the reader.

The following lemma will be useful to us in §1.2.2 :

Lemme 1.2.1.3. *Let M be an injective object in the category of discrete* $\Lambda[[G]]$*-modules. The evident morphism* $M \to \mathrm{Hom}_{\Lambda[[G]]}(I_G, M)$ (*which associates to x the map* $a \longmapsto ax$) *is surjective.*

Let σ be a topological generator of G. Let $\varphi\colon I_G \to M$ be a morphism of $\Lambda[[G]]$-modules. Let $m := \varphi(1-\sigma)$. Since M is discrete, there exists an integer $d \geq 1$ prime to p such that $\sigma^d(m) = m$. Let $C_d \in \mathrm{GL}_{1+d}(\Lambda)$ be the following matrix :

$$C = \begin{pmatrix} 1 & 0 & \dots & \dots & 0 \\ 1 & 0 & \dots & 0 & 1 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & 0 & 0 \\ 0 & \dots & 0 & 1 & 0 \end{pmatrix}$$

(iv) To avoid any sign ambiguity, $\mathrm{Gal}(\overline{\eta}/\eta)$ is the group $\mathrm{Aut}_{\kappa(\eta)}(\kappa(\overline{\eta}))$ and for any integer n invertible in X, the composite morphism $\mathrm{Gal}(\overline{\eta}/\eta) \to \widehat{\mathbf{Z}}'(1) \to \mu_n$ sends $\sigma \in \mathrm{Aut}_{\kappa(\eta)}(\kappa(\overline{\eta}))$ to $\frac{\sigma(\pi')}{\pi'}$ where $\pi' \in \kappa(\overline{\eta})$ is an n-th root of a uniformizer of X.

We introduce a free Λ-module V_d of rank $1+d$ with a basis denoted by $\mathscr{B} = (f, e_0, e_1, \dots, e_{d-1})$. We denote by $c \in \mathrm{Aut}_\Lambda(V_d)$ the automorphism whose matrix in the basis $\mathscr{B}$ is C, that is, $c(f) = f + e_0$, $c(e_i) = e_{i+1}$ if $0 \leq i \leq d-2$ and $c(e_{d-1}) = e_0$. From this we deduce the following identity :

$$C^d = \begin{pmatrix} 1 & 0 & \dots & \dots & 0 \\ 1 & 1 & 0 & \dots & 0 \\ 1 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & 1 & 0 \\ 1 & 0 & \dots & 0 & 1 \end{pmatrix}$$

If we set $d' := d \cdot \#\Lambda$, it then follows that $C^{d'} = \mathrm{Id}$ and that $c \in \mathrm{Aut}_\Lambda(V_d)$ has order d'. The integer d' being prime to p, we deduce a structure of a discrete $\Lambda[[G]]$-module on the Λ-module V_d such that the topological generator $\sigma \in G$ acts by $c \colon V_d \to V_d$. Let $V'_d \subset V_d$ be the sub-Λ-module generated by $(e_0, \dots, e_{d-1})$. It is also a discrete sub-$\Lambda[[G]]$-module and we can consider the Λ-linear map $\psi \colon V'_d \to M$ which to e_i associates $\sigma^i(m)$. We have thus constructed a morphism $\psi \colon V'_d \to M$ of discrete $\Lambda[[G]]$-modules. The injectivity of M means that this morphism extends to a morphism $\tilde{\psi} \colon V_d \to M$. Let $\tilde{m} := \tilde{\psi}(f)$; the $\Lambda[[G]]$-linearity of $\tilde{\psi}$ implies that $\varphi(1-\sigma) = m = \psi(e_0) = \tilde{\psi}(e_0) = \tilde{\psi}(\sigma(f)-f) = \sigma(\tilde{m})-\tilde{m}$. According to proposition 1.2.1.2), the ideal I_G is generated by $1-\sigma$; we thus have more generally $\varphi(a) = a\tilde{m}$ for all $a \in I_G$.

Remarque 1.2.1.4. Proposition 1.2.1.2 allows us to show that the $\Lambda[[G]]$-module V_d appearing in the preceding proof is isomorphic to the quotient $\Lambda[[G]]/J_d$ where J_d is the ideal of $\Lambda[[G]]$ generated by $(1-\sigma)(\sigma^d - 1)$, the isomorphism sending the basis $(f, e_0, e_1, \dots, e_{d-1})$ of V_d to $(1, \sigma-1, \sigma^2-\sigma, \dots, \sigma^d - \sigma^{d-1})$. The inclusion $V'_d \to V_d$ is then identified with the inclusion $I_G/J_d \to \Lambda[[G]]/J_d$.

1.2.2. *Description of the cohomology with supports.* It is clear that the category of sheaves (of sets) on $X_{\text{ét}}$ is naturally equivalent to the category of arrows of (discrete) $\mathrm{Gal}(\overline{\eta}/\eta)$-sets $\mathscr{M}_s \to \mathscr{M}_{\overline{\eta}}$ such that the action of $\mathrm{Gal}(\overline{\eta}/\eta)$ on $\mathscr{M}_s$ is trivial.

For any sheaf of Λ-modules $\mathscr{M}$ on $X_{\text{ét}}$, we set $F^0\mathscr{M} := \mathscr{M}_s$, $F^1\mathscr{M} := \mathscr{M}_{\overline{\eta}}^S$, $F^2\mathscr{M} := \mathrm{Hom}_{\Lambda[[G]]}(I_G, \mathscr{M}_{\overline{\eta}}^S)$ where $\mathscr{M}_{\overline{\eta}}^S$ denotes the subgroup of fixed points under S of the action of $\mathrm{Gal}(\overline{\eta}/\eta)$ on the fiber $\mathscr{M}_{\overline{\eta}}$. The canonical morphism $\overline{\eta} \to X$ induces a morphism $\alpha \colon F^0\mathscr{M} \to F^1\mathscr{M}$. We define a morphism $\beta \colon F^1\mathscr{M} \to F^2\mathscr{M}$ such that if $m \in \mathscr{M}_{\overline{\eta}}^S$ and $a \in I_G$, then $\beta(m)(a) = am$. By setting $F^q\mathscr{M} = 0$ for $q \notin \{0,1,2\}$, we thus define a complex $F\mathscr{M}$ in the category of Λ-modules :

$$\dots \to 0 \to \mathscr{M}_s \xrightarrow{\alpha} \mathscr{M}_{\overline{\eta}}^S \xrightarrow{\beta} \mathrm{Hom}_{\Lambda[[G]]}(I_G, \mathscr{M}_{\overline{\eta}}^S) \to 0 \to \dots$$

where $\mathscr{M}_s$ is placed in degree 0. We note that the functor which to $\mathscr{M}$ associates $F\mathscr{M}$ is additive.

For any complex K in the category of sheaves of Λ-modules on $X_{\text{ét}}$, we denote $FK := \mathrm{Tot}((F^qK^p)_{(p,q)\in\mathbf{Z}^2})$ where $(F^qK^p)_{(p,q)\in\mathbf{Z}^2}$ is the evident bicomplex (cf. **XVI**-4.4 for the sign conventions on the simple complex). In other words, $(FK)^n := F^0K^n \oplus F^1K^{n-1} \oplus F^2K^{n-2}$ and the differential $d_{FK} \colon (FK)^n \to (FK)^{n+1}$ is described by the following matrix

$$\begin{pmatrix} d_K & 0 & 0 \\ (-1)^n\alpha & d_K & 0 \\ 0 & (-1)^{n-1}\beta & d_K \end{pmatrix}$$

We thus define a functor F from the category of complexes of sheaves of Λ-modules on $X_{\text{ét}}$ to that of complexes of Λ-modules. The sign conventions ensure that for any K, we have a canonical isomorphism of complexes $F(K[1]) \simeq (FK)[1]$ (whose definition does not involve any sign). If $K \xrightarrow{f} L$ is a morphism of complexes, the image by F of the distinguished triangle $K \xrightarrow{f} L \to \mathrm{cône}(f) \to K[1]$ is identified (without additional signs) with $FK \xrightarrow{F(f)} FL \to \mathrm{cône}(F(f)) \to F(K)[1]$. We also observe that, since the functors F^q for $q \in \{0,1,2\}$ are exact, if K is acyclic, then so is FK. From these remarks, it follows that F preserves quasi-isomorphisms and induces a triangulated functor $F \colon \mathsf{D}^+(X_{\text{ét}}, \Lambda) \to \mathsf{D}^+(\Lambda)$.

Finally, let us observe that if $\mathscr{M}$ is a sheaf of Λ-modules on $X_{\text{ét}}$, then we have an equality $\Gamma_s(X_{\text{ét}}, \mathscr{M}) = \operatorname{Ker}(\alpha \colon F^0\mathscr{M} \to F^1\mathscr{M})$. Thus, for any complex K of sheaves of Λ-modules on $X_{\text{ét}}$, if we denote by $\Gamma_s(X, K)$ the complex obtained by applying $\Gamma_s(X_{\text{ét}}, -)$ to the sheaves constituting the complex K, we have a natural transformation $\Gamma_s(X_{\text{ét}}, K) \to FK$.

Proposition 1.2.2.1. *The natural transformation $\Gamma_s(X_{\text{ét}}, K) \to FK$ induces for any $K \in \mathsf{D}^+(X_{\text{ét}}, \Lambda)$ an isomorphism*

$$\mathrm{R}_s(X_{\text{ét}}, K) \xrightarrow{\sim} FK$$

in $\mathsf{D}^+(\Lambda)$.

We need to show that if K is a complex bounded below consisting of injective objects, then the morphism $\Gamma_s(X_{\text{ét}}, K) \to FK$ is a quasi-isomorphism. This statement is immediately reduced to the case where K consists of a single injective sheaf of Λ-modules $\mathscr{M}$ placed in degree 0. Given that $\Gamma_s(X, \mathscr{M}) = H^0(F\mathscr{M})$ for any sheaf of Λ-modules, it is sufficient to show that $H^q(F\mathscr{M}) = 0$ for $q \in \{1, 2\}$ if $\mathscr{M}$ is injective.

We note that $\operatorname{Ker}(\beta)$ is identified with $\Gamma(\eta, \mathscr{M})$ and then that we have an isomorphism $H^1(F\mathscr{M}) \simeq \operatorname{Coker}(\Gamma(X, \mathscr{M}) \to \Gamma(\eta, \mathscr{M}))$ for any sheaf of Λ-modules. An injective sheaf being flasque, we indeed obtain that $H^1(F\mathscr{M}) = 0$ if $\mathscr{M}$ is injective.

It remains to show that $H^2(F\mathscr{M}) = 0$ if $\mathscr{M}$ is injective. The restriction functor from sheaves of Λ-modules on $X_{\text{ét}}$ to $\eta_{\text{ét}}$ admits an exact left adjoint, so if $\mathscr{M}$ is injective, then so is $\mathscr{M}_{|\eta}$. Similarly, the direct image functor associated with the evident morphism of topoi $u \colon \eta_{\text{ét}} \to \mathbf{B}G$ (cf. [**SGA 4** IV 4.5.2]) preserves injectives. Since $u_\star \mathscr{M}_{|\eta} = \mathscr{M}_{\overline{\eta}}^S$, if $\mathscr{M}$ is injective, then $\mathscr{M}_{\overline{\eta}}^S$ is injective in the category of discrete $\Lambda[[G]]$-modules and we can conclude that $H^2(F\mathscr{M}) = \operatorname{Coker}(\beta)$ is zero by using lemma 1.2.1.3.

1.2.3. *Definition of the transition morphism.* Let σ be a topological generator of G. Let M be a discrete $\Lambda[[G]]$-module. We observe that we have a canonical isomorphism of abelian groups $M(-1) \simeq \operatorname{Hom}(G, M)$. We define a morphism of abelian groups $M(-1) \to \operatorname{Hom}_{\Lambda[[G]]}(I_G, M)$ *via* the following isomorphisms, cf. proposition 1.2.1.2 :

$$M(-1) \xrightarrow{\sim} \operatorname{Hom}(G, M) \xrightarrow[\sim]{\mathrm{ev}_\sigma} M \xleftarrow[\sim]{\mathrm{ev}_{1-\sigma}} \operatorname{Hom}_{\Lambda[[G]]}(I_G, M)\ .$$

If $\mathscr{M}$ is a sheaf of Λ-modules on $X_{\text{ét}}$, we can apply the above construction to the discrete $\Lambda[[G]]$-module $\mathscr{M}_{\overline{\eta}}^S$, which provides a morphism $\mathscr{M}_{\overline{\eta}}^S(-1) \to \operatorname{Hom}_{\Lambda[[G]]}(I_G, \mathscr{M}_{\overline{\eta}}^S) = F^2\mathscr{M}$. The profinite group S being a pro-p-group and p being invertible in Λ, we have a canonical projector $\mathscr{M}_{\overline{\eta}} \to \mathscr{M}_{\overline{\eta}}^S$ which after twisting provides a morphism $\mathscr{M}_{\overline{\eta}}(-1) \to \mathscr{M}_{\overline{\eta}}^S(-1)$ that we can compose with the one previously constructed to obtain a morphism $\mathscr{M}_{\overline{\eta}}(-1) \to F^2\mathscr{M}$ and thus a morphism of complexes $s_\sigma \colon \mathscr{M}_{\overline{\eta}}(-1) \to F\mathscr{M}[2]$. By tensoring this morphism with $\Lambda(1)$ (or by applying s_σ to the sheaf $\mathscr{M}(1)$, it amounts to the same thing), we obtain a morphism of complexes $s_\sigma(1) \colon \mathscr{M}_{\overline{\eta}} \to F(\mathscr{M})(1)[2]$. Without adding any additional sign, we deduce a morphism $s_\sigma(1) \colon K_{\overline{\eta}} \to F(K)(1)[2]$ for any complex of sheaves of Λ-modules K on $X_{\text{ét}}$. Taking into account proposition 1.2.2.1, we can make the following definition :

Définition 1.2.3.1. We denote by $\mathrm{sp}^X_{\overline{\eta}\to s} \colon \mathrm{R}_{\overline{\eta}}(K) \to \mathrm{R}_s(K)(1)[2]$ the morphism in $\mathsf{D}^+(\Lambda)$ defined by $s_\sigma(1)$ functorially for any object $K \in \mathsf{D}^+(X_{\text{ét}}, \Lambda)$. According to the following lemma, this morphism $\mathrm{sp}^X_{\overline{\eta}\to s}$ is independent of the generator σ of G : it is the transition morphism associated with the specialization $\overline{\eta} \to s$ of geometric points of X.

Lemme 1.2.3.2. *If σ and σ' are two topological generators of G, there exists a unique functorial homotopy in the sheaf $\mathscr{M}$ connecting the two morphisms $\mathscr{M}_{\overline{\eta}}(-1) \to F(\mathscr{M})[2]$ given by s_σ and $s_{\sigma'}$*

We immediately see that we can restrict to $\mathscr{M}$ such that $\mathscr{M}_s = 0$ and S acts trivially on $\mathscr{M}_{\overline{\eta}}$. We can identify this category of sheaves with that of discrete $\Lambda[[G]]$-modules.

Let M be a discrete $\Lambda[[G]]$-module. Let $\mathscr{M}$ be the corresponding sheaf on $X_{\text{ét}}$. We denote by $F_\sigma(M)$ the complex

$$\dots \longrightarrow 0 \longrightarrow M \xrightarrow{1-\sigma} M \longrightarrow 0 \longrightarrow \dots$$

concentrated in degrees 1 and 2. We denote by $\Psi_\sigma : F\mathscr{M} \xrightarrow{\sim} F_\sigma(M)$ the isomorphism of complexes defined in an evident way from σ :

$$\begin{tikzcd}
F\mathscr{M} \arrow[d, "\Psi_\sigma"', "\sim"] & \dots \arrow[r] & 0 \arrow[r] \arrow[d] & \mathscr{M}_{\overline{\eta}} \arrow[r] \arrow[d] & \mathrm{Hom}_{\Lambda[[G]]}(I_G, \mathscr{M}_{\overline{\eta}}) \arrow[r] \arrow[d, "\sim"', "\mathrm{ev}_{1-\sigma}"] & 0 \arrow[r] \arrow[d] & \dots \\
F_\sigma(M) & \dots \arrow[r] & 0 \arrow[r] & M \arrow[r, "1-\sigma"] & M \arrow[r] & 0 \arrow[r] & \dots
\end{tikzcd}$$

Let $\varphi_\sigma : \mathscr{M}_{\overline{\eta}}(-1) \xrightarrow{\sim} M$ be the isomorphism defined by evaluation at σ *via* the canonical isomorphism $\mathscr{M}_{\overline{\eta}}(-1) \simeq \mathrm{Hom}(G, \mathscr{M}_{\overline{\eta}})$. Let $t_\sigma : M \to F_\sigma(M)[2]$ be the morphism of complexes represented by the vertical arrows below :

$$\begin{tikzcd}
M \arrow[d, "t_\sigma"] & \dots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \dots \\
F_\sigma(M)[2] & \dots \arrow[r] & 0 \arrow[r] & M \arrow[r, "1-\sigma"] & M \arrow[r] & 0 \arrow[r] & \dots
\end{tikzcd}$$

We thus have a commutative square of complexes, functorial in M :

$$\begin{tikzcd}
\mathscr{M}_{\overline{\eta}}(-1) \arrow[r, "s_\sigma"] \arrow[d, "\sim"', "\varphi_\sigma"] & F(\mathscr{M})[2] \arrow[d, "\sim"', "\Psi_\sigma"] \\
M \arrow[r, "t_\sigma"] & F_\sigma(M)[2]
\end{tikzcd}$$

We thus have $t_\sigma = \Psi_\sigma \circ s_\sigma \circ \varphi_\sigma^{-1}$. Let $f_{\sigma,\sigma'} = \Psi_\sigma \circ s_{\sigma'} \circ \varphi_\sigma^{-1}$. The vertical arrows of the diagram above being isomorphisms of complexes, showing that the morphisms $s_\sigma, s_{\sigma'} : \mathscr{M}_{\overline{\eta}}(-1) \to F(\mathscr{M})[2]$ are (functorially) homotopic amounts to verifying that the two morphisms $t_\sigma, f_{\sigma,\sigma'} : M \to F_\sigma(M)[2]$ are.

We can thus represent the situation more concretely :

$$\begin{tikzcd}
\dots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, bend right, "\mathrm{Id}"'] \arrow[d, bend left, "g"] & 0 \arrow[r] \arrow[d] & \dots \\
\dots \arrow[r] & 0 \arrow[r] & M \arrow[r, "1-\sigma"] & M \arrow[r] & 0 \arrow[r] & \dots
\end{tikzcd}$$

where g is the natural transformation induced by $f_{\sigma,\sigma'}$.

Since $\Lambda[[G]]$ denotes precisely the ring of natural transformations $M \to M$, we can identify Id and g with elements 1 and g of $\Lambda[[G]]$ respectively. Showing the existence and uniqueness of the functorial homotopy between s_σ and $s_{\sigma'}$ is therefore reduced to showing the existence and uniqueness of $h \in \Lambda[[G]]$ such that $(1-\sigma) \cdot h = 1 - g$. According to proposition 1.2.1.2, this amounts to showing that $\varepsilon(g) = 1$. By definition of $f_{\sigma,\sigma'}$, if we denote by u the unit of $\Lambda[[G]]$ such that $(1-\sigma') = u \cdot (1-\sigma)$ and if we denote by α the element of $\widehat{\mathbf{Z}}'^{\times}$ such that $\sigma' = \sigma^\alpha$, then we have the relation $u \cdot g = \alpha$. To show $\varepsilon(g) = 1$, we are thus reduced to showing that $\varepsilon(u) = \varepsilon(\alpha)$. For this, we use the following formula :

$$\frac{1-\sigma^\beta}{1-\sigma} = \sum_{i=0}^{\beta-1} \sigma^i \; .$$

This formula is obviously correct for $\beta \in \mathbf{N}$; we can give it a meaning for any $\beta \in \widehat{\mathbf{Z}}$ by extending each of the members by continuity. By applying this formula with $\beta = \alpha$, we obtain the desired result :

$$\varepsilon(u) = \varepsilon\left(\sum_{i=0}^{\alpha-1} \sigma^i\right) = \sum_{i=0}^{\alpha-1} 1 = \varepsilon(\alpha) \; ,$$

which completes the proof of the lemma.

1.2.4. *Compatibility with the class of the closed point.*

Proposition 1.2.4.1. *In the case of the sheaf Λ, the image of $1 \in \Lambda$ by the transition morphism $\mathrm{sp}^X_{\overline{\eta}\to s}\colon \Lambda \to \mathrm{R}_s(X_{\text{ét}}, \Lambda(1))[2]$ is the class $\mathrm{Cl}_{s\to X} \in H^2_{s,\text{ét}}(X, \Lambda(1))$ from **XVI**-2.3.1.*

The only issue in the proof of this proposition is to ensure that the sign is correct. This is what justifies the precision of the sign conventions discussed in **XVI**-4, since they allow to give a precise meaning to the statement above.

Définition 1.2.4.2. For the purposes of the proof of proposition 1.2.4.1, we introduce a functor E from the category of complexes of sheaves of Λ-modules on $X_{\text{ét}}$ to that of Λ-modules in the following way. For any complex of sheaves K, we denote $E^{p,0}K := F^1K^p = (K^p_{\overline{\eta}})^S$, $E^{p,1}K := F^2K^p = \mathrm{Hom}_{\Lambda[[G]]}(I_G, (K^p_{\overline{\eta}})^S)$ and $E^{p,q}K = 0$ for $q \notin \{0,1\}$. We define a bicomplex $(E^{p,q}K)_{(p,q)\in\mathbf{Z}^2}$ by denoting $d_h\colon E^{p,q}K \to E^{p+1,q}K$ for $q \in \{0,1\}$ the morphisms induced by the differentials on K and $d_v\colon E^{p,0}K \to E^{p,1}K$ the morphism $\beta\colon F^1K^p \to F^2K^p$. We set $EK := \mathrm{Tot}((E^{p,q})_{(p,q)\in\mathbf{Z}^2})$. In other words, $(EK)^n := F^1K^n \oplus F^2K^{n-1}$ and the differential $d_{EK}\colon (EK)^n \to (EK)^{n+1}$ is described by the matrix :

$$\begin{pmatrix} d_K & 0 \\ (-1)^n\beta & d_K \end{pmatrix}$$

For any sheaf of Λ-modules $\mathscr{M}$ on $X_{\text{ét}}$ and for any $n \in \mathbf{Z}$, the kernel of $\beta\colon F^1\mathscr{M} \to F^2\mathscr{M}$ is canonically identified with $\Gamma(\eta_{\text{ét}}, \mathscr{M})$. We deduce a natural transformation $\Gamma(\eta_{\text{ét}}, K) \to EK$ for any complex K of sheaves of Λ-modules on $X_{\text{ét}}$. Just as in §1.2.2, E commutes with the functor [1] (without introducing any additional sign), preserves quasi-isomorphisms, induces a triangulated functor $E\colon \mathsf{D}^+(X_{\text{ét}}, \Lambda) \to \mathsf{D}^+(\Lambda)$ and the canonical morphism $\mathrm{R}(\eta_{\text{ét}}, K) \to EK$ is an isomorphism in $\mathsf{D}^+(\Lambda)$ for any $K \in \mathsf{D}^+(X_{\text{ét}}, \Lambda)$. (Since the construction depends only on the restriction of K to η, we can allow ourselves to use this construction for $K \in \mathsf{D}^+(\eta_{\text{ét}}, \Lambda)$.)

Définition 1.2.4.3. We define a natural transformation $\Psi\colon EK[-1] \to FK$ for any complex K of sheaves of Λ-modules on $X_{\text{ét}}$ in the following way. In degree $n \in \mathbf{Z}$, it is given by the morphism $F^1K^{n-1} \oplus F^2K^{n-2} \to F^0K^n \oplus F^1K^{n-1} \oplus F^2K^{n-2}$ described by the following matrix :

$$\begin{pmatrix} 0 & 0 \\ (-1)^n & 0 \\ 0 & (-1)^n \end{pmatrix}$$

Via the canonical isomorphisms $EK \simeq \mathrm{R}(\eta_{\text{ét}}, K)$ and $FK \simeq \mathrm{R}_s(X_{\text{ét}}, K)$ (cf. proposition 1.2.2.1), Ψ induces a morphism $\Psi\colon \mathrm{R}(\eta_{\text{ét}}, K)[-1] \to \mathrm{R}_s(X_{\text{ét}}, K)$ for any $K \in \mathsf{D}^+(X_{\text{ét}}, \Lambda)$.

Lemme 1.2.4.4. *For any $K \in \mathsf{D}^+(X_{\text{ét}}, \Lambda)$, the morphism $\Psi\colon \mathrm{R}(\eta_{\text{ét}}, K)[-1] \to \mathrm{R}_s(X_{\text{ét}}, K)$ induces for any $n \in \mathbf{Z}$ the boundary morphism $\delta\colon H^{n-1}_{\text{ét}}(\eta, K) \to H^n_{s,\text{ét}}(X, K)$ (cf. **XVI**-4.7.6).*

We can assume that K is a complex bounded below consisting of injective objects. Let $[\gamma] \in H^{n-1}_{\text{ét}}(\eta, K)$ be a cohomology class represented by an element $\gamma \in \Gamma(\eta, K^{n-1})$ such that $d\gamma = 0$. (In this proof, we will simply denote by d the differentials induced by the differentials d_K of K.) We can calculate $\delta([\gamma]) \in H^n_{s,\text{ét}}(X, K)$ by using **XVI**-4.7.6. The sheaf K^{n-1} being flasque, we can choose $\tilde{\gamma} \in \Gamma(X, K^{n-1})$ such that $\tilde{\gamma}_{|\eta} = \gamma$. We then have $d\tilde{\gamma} \in \Gamma_s(X, K^{n-1})$ and $\delta([\gamma]) = [d\tilde{\gamma}] \in H^n_{s,\text{ét}}(X, K)$. *Via* the isomorphism $\mathrm{R}_s(X_{\text{ét}}, K) \simeq FK$, the class $\delta([\gamma]) = [d\tilde{\gamma}] \in H^n_{s,\text{ét}}(X, K)$ is described by the cocycle $(d\tilde{\gamma}, 0, 0) \in F^0K^n \oplus F^1K^{n-1} \oplus F^2K^{n-2} = (FK)^n$. The class $[\gamma] \in H^{n-1}_{\text{ét}}(\eta, K)$ being represented *via* the isomorphism $EK \simeq \mathrm{R}(\eta_{\text{ét}}, K)$ by the $(n-1)$-cocycle $(\gamma_{\overline{\eta}}, 0) \in F^1K^{n-1} \oplus F^2K^{n-2} = (EK)^{n-1}$ of EK, the class $\Psi([\gamma]) \in H^n_{s,\text{ét}}(X, K)$ is described by the n-cocycle $(0, (-1)^n\gamma_{|\overline{\eta}}, 0) \in (FK)^n$. To conclude, it is sufficient to observe that the two n-cocycles $(d\tilde{\gamma}, 0, 0)$ and $(0, (-1)^n\gamma_{|\overline{\eta}}, 0)$ of the complex FK are cohomologous, which comes from the relation

$$d_{FK}(\tilde{\gamma}, 0, 0) = (d\tilde{\gamma}, (-1)^{n-1}\gamma_{|\overline{\eta}}, 0) \in F^0K^n \subset F^0K^n \oplus F^1K^{n-1} \oplus F^2K^{n-2} = (FK)^n\ .$$

Définition 1.2.4.5. We choose a uniformizer π of X and an element $\pi' \in \kappa(\overline{\eta})$ such that $\pi'^n = \pi$ (we recall that $\Lambda = \mathbf{Z}/n\mathbf{Z}$.). We denote by R the subgroup of $\kappa(\overline{\eta})^\times$ generated by μ_n and π'. We set

$\overline{R} := R \otimes \mathbf{Z}/n\mathbf{Z}$, that is, $\overline{R}$ is the quotient of R by the subgroup generated by π. (We will denote $\overline{R}$ additively.) The action of $\mathrm{Gal}(\kappa(\overline{\eta})/\kappa(\eta))$ on R factors through an action of the tame inertia group G. We deduce a structure of a discrete $\Lambda[[G]]$-module on $\overline{R}$ and we have an evident exact sequence of discrete $\Lambda[[G]]$-modules :

$$0 \to \mu_n \to \overline{R} \xrightarrow{p} \mathbf{Z}/n\mathbf{Z} \to 0 \,.$$

(The morphism $p\colon \overline{R} \to \mathbf{Z}/n\mathbf{Z}$ sends the class $\overline{\pi'}$ of π' to 1.)

Lemme 1.2.4.6. *There exists a unique morphism of $\Lambda[[G]]$-modules $Z\colon I_G \to \mu_n$ such that for any $\sigma \in G$, $Z(1-\sigma) = \frac{\pi'}{\sigma(\pi')}$. In particular, if σ is a topological generator of G and $\zeta \in \mu_n$ is the image of σ by $G \simeq \widehat{\mathbf{Z}}'(1) \to \mu_n$, then $Z(1-\sigma) = \zeta^{-1}$.*

Let $\tau \in I_G$. As an element of $\Lambda[[G]]$, τ induces an endomorphism of $\overline{R}$. That $\tau \in I_G$ means that $\tau(\overline{\pi'}) \in \mu_n \subset \overline{R}$. We can thus set $Z(\tau) := \tau(\overline{\pi'})$. We easily check that $Z\colon I_G \to \mu_n$ is a morphism of $\Lambda[[G]]$-modules and that it is the only one to satisfy the relation $Z(1-\sigma) = \frac{\pi'}{\sigma(\pi')}$ for all $\sigma \in G$ (cf. proposition 1.2.1.2).

Lemme 1.2.4.7. *Thanks to the definitions $(E\mu_n)^1 = F^2\mu_n = \mathrm{Hom}_{\Lambda[[G]]}(I_G, \mu_n)$, the element Z of lemma 1.2.4.6 can be considered as a 1-cocycle of the complex $E\mu_n$.* Via *the canonical isomorphism $\mathrm{R}(\eta_{\text{ét}}, \mu_n) \simeq E\mu_n$, the element Z corresponds to the class in $H^1_{\text{ét}}(\eta, \mu_n)$ of the torsor $\mathscr{T}$ of the n-th roots of unity of π.*

The torsor $\mathscr{T}$ of the n-th roots of π is identified with the sheaf of sets on $\eta_{\text{ét}}$ equipped with the action of μ_n corresponding to the inverse image of 1 by the morphism $p\colon \overline{R} \to \mathbf{Z}/n\mathbf{Z}$. We denote $\delta\colon H^0_{\text{ét}}(\eta, \mathbf{Z}/n\mathbf{Z}) \to H^1_{\text{ét}}(\eta, \mu_n)$ the boundary morphism associated with the exact sequence

$$0 \to \mu_n \to \overline{R} \xrightarrow{p} \mathbf{Z}/n\mathbf{Z} \to 0$$

from definition 1.2.4.5; we can indeed identify this short exact sequence of discrete $\Lambda[[G]]$-modules with a short exact sequence of sheaves of Λ-modules on $\eta_{\text{ét}}$. Thanks to the construction **XVI**-4.3.1, it follows that the class of the torsor $\mathscr{T}$ is $\delta(1) \in H^1_{\text{ét}}(\eta, \mu_n)$. To conclude, we need to describe δ in terms of the triangulated functor $E \simeq \mathrm{R}(\eta, -)$. The boundary morphism δ is indeed the one coming from the usual construction applied to the short exact sequence of complexes of Λ-modules $0 \to E\mu_n \to E\overline{R} \to E\mathbf{Z}/n\mathbf{Z} \to 0$. We start from $1 \in (E\mathbf{Z}/n\mathbf{Z})^0 = \mathbf{Z}/n\mathbf{Z}$ which we lift to $\pi' \in (E\overline{R})^0 = \overline{R}$. The differential $d\pi' := \beta(\pi') \in (E\overline{R})^1 = \mathrm{Hom}_{\Lambda[[G]]}(I_G, \overline{R})$ is by construction the image of $Z \in \mathrm{Hom}_{\Lambda[[G]]}(I_G, \mu_n) = (E\mu_n)^1$ by composition with the inclusion $\mu_n \to \overline{R}$. We deduce that $\delta(1) = [Z] \in H^1(E\mu_n)$.

We can now finally prove proposition 1.2.4.1. To determine the image of $1 \in \Gamma(\overline{\eta}_{\text{ét}}, \Lambda)$ by $\mathrm{sp}^X_{\overline{\eta}\to s}$, we need to use the definition of s_σ applied to $\Lambda(1) = \mu_n$ for σ a topological generator of G. If we follow the construction of §1.2.3, we first associate to $1 \in \Lambda \simeq \mu_n(-1)$ an element of $\mathrm{Hom}_{\Lambda[[G]]}(I_G, \mu_n)$ which is $-Z$. We then consider $-Z$ as an element of $(F\mu_n)^2 = F^2\mu_n = \mathrm{Hom}_{\Lambda[[G]]}(I_G, \mu_n)$ and it thus follows that $\mathrm{sp}^X_{\overline{\eta}\to s}(1) = [-Z] \in H^2(F\mu_n) \simeq H^2_s(X, \mu_n)$. The morphism $-Z$ can also be thought of as an element of $(E\mu_n)^1 = F^2\mu_n = \mathrm{Hom}_{\Lambda[[G]]}(I_G, \mu_n)$ and we indeed have $\Psi(-Z) = -Z$ where Ψ is the morphism $\Psi\colon E\mu_n[-1] \to F\mu_n$ from definition 1.2.4.3. By combining the statements of lemmas 1.2.4.4 and 1.2.4.7, we obtain that $\mathrm{sp}^X_{\overline{\eta}\to s}(1) = [-Z] = [\Psi(-Z)] = \delta([-Z]) = -\delta([\mathscr{T}])$ where $\delta\colon H^1_{\text{ét}}(\eta, \mu_n) \to H^2_{s,\text{ét}}(X, \mu_n)$ is the boundary morphism and $\mathscr{T}$ is the torsor of the n-th roots of π. The calculation made during the proof of **XVI**-3.4.8 allows us to conclude that $\mathrm{sp}^X_{\overline{\eta}\to s}(1) = -\delta([\mathscr{T}]) = \mathrm{Cl}_{s\to X}$.

Remarque 1.2.4.8. If $\mathscr{M}$ is a sheaf of Λ-modules on the strictly Henselian trait X, the morphism $\mathscr{H}^0(\mathrm{sp}^X_{\overline{\eta}\to s})\colon \mathscr{M}_{\overline{\eta}} \to H^2_s(X, \mathscr{M}(1))$ induces an isomorphism $(\mathscr{M}_{\overline{\eta}})_{\mathrm{Gal}(\overline{\eta}/\eta)} \xrightarrow{\sim} H^2_s(X, \mathscr{M}(1))$ where we have denoted by $-_{\mathrm{Gal}(\overline{\eta}/\eta)}$ the co-invariants under the group $\mathrm{Gal}(\overline{\eta}/\eta)$.

1.3. Case of an excellent integral strictly Henselian local scheme of dimension 1. Let X be an excellent integral strictly Henselian local $\mathbf{Z}\left[\frac{1}{n}\right]$-scheme of dimension 1. Let η be the generic point of X. Let $\overline{\eta}$ be a geometric point above η. Let s be the closed point of X. Let $\tilde{X} \xrightarrow{f} X$ be the normalization of X. The scheme $\tilde{X}$ is a strictly Henselian trait (with closed point $\tilde{s}$). The finite extension $\tilde{s}/s$ can only be purely inseparable, which allows us to remark that f is a universal homeomorphism and thus the inverse image functor $f^\star$ induces an equivalence between the categories of sheaves on $X_{\text{ét}}$ and $\tilde{X}_{\text{ét}}$ (cf. [**SGA 4** VIII 1.1]).

Définition 1.3.1. *Via* the identifications above, the transition morphism $\mathrm{sp}^X_{\overline{\eta}\to s}$ is induced by $\frac{1}{[\tilde{s}:s]}\mathrm{sp}^{\tilde{X}}_{\overline{\eta}\to\tilde{s}}$ (cf. definition 1.2.3.1).

1.4. General case.

Définition 1.4.1. Let X be an excellent $\mathbf{Z}\left[\frac{1}{n}\right]$-scheme. Let $\overline{y} \to \overline{x}$ be an immediate specialization of geometric points of X. To define the transition morphism $\mathrm{sp}^X_{\overline{y}\to\overline{x}} \colon \mathrm{R}_{\overline{y}}(K) \to \mathrm{R}_{\overline{x}}(K)(1)[2]$ for any $K \in \mathsf{D}^+(X_{\text{ét}}, \Lambda)$, up to replacing K by its inverse image *via* the canonical morphism $X_{(\overline{x})} \to X$, we can assume that X is a local strictly Henselian (excellent) scheme with closed point $\overline{x}$. We then denote by Z the closure of the point of X below $\overline{y}$ and by $i \colon Z \to X$ its immersion in X. This $\mathbf{Z}\left[\frac{1}{n}\right]$-scheme Z is an excellent integral strictly Henselian local scheme of dimension 1, the transition morphism $\mathrm{sp}^Z_{\overline{y}\to\overline{z}}$ was introduced in definition 1.3.1. For any $K \in \mathsf{D}^+(X_{\text{ét}}, \Lambda)$, we define the transition morphism $\mathrm{sp}^X_{\overline{y}\to\overline{x}}$ so as to make the following diagram commute, where the vertical arrows are the evident isomorphisms :

$$\begin{tikzcd}
\mathrm{R}_{\overline{y}}(K) \arrow[r, "\mathrm{sp}^X_{\overline{y}\to\overline{x}}"] \arrow[d, "\sim"] & \mathrm{R}_{\overline{x}}(K)(1)[2] \arrow[d, "\sim"] \\
\mathrm{R}_{\overline{y}}(i^!K) \arrow[r, "\mathrm{sp}^Z_{\overline{y}\to\overline{x}}"] & \mathrm{R}_{\overline{x}}(i^!K)(1)[2]
\end{tikzcd}$$

Remarque 1.4.2. For any specialization $\overline{x}' \to \overline{x}$ of codimension 0 between geometric points of X (essentially, an element of the absolute Galois groupoid of the residue field of one of the points of X), we have an evident isomorphism $\mathrm{R}_{\overline{x}'}(K) \xrightarrow{\sim} \mathrm{R}_{\overline{x}}(K)$, which we denote by $\mathrm{sp}^X_{\overline{x}'\to\overline{x}}$. It is evident that if $\overline{z} \to \overline{y}$ and $\overline{y} \to \overline{x}$ are composable specializations such that the codimension c of $\overline{z} \to \overline{x}$ is 0 or 1, we have an equality of morphisms

$$\mathrm{sp}^X_{\overline{z}\to\overline{x}} = \mathrm{sp}^X_{\overline{y}\to\overline{x}} \circ \mathrm{sp}^X_{\overline{z}\to\overline{y}} \colon \mathrm{R}_{\overline{z}}(K) \to \mathrm{R}_{\overline{x}}(K)(c)[2c]\ .$$

Thus, the transition morphisms associated with specializations compose well in the range where these constructions have been made so far. We will later define transition morphisms in arbitrary codimension and in a way compatible with composition, but only at the level of cohomology groups (cf. theorem 3.1.2).

2. Putative and potential dualizing complexes

2.1. Definition of putative and potential dualizing complexes.

Définition 2.1.1. A **dimension function** on a locally Noetherian scheme X is a function $\delta \colon X \to \mathbf{Z}$ such that for any immediate specialization $\overline{y} \to \overline{x}$ of geometric points of X, we have $\delta(y) = \delta(x) + 1$.

Locally for the Zariski topology, an excellent scheme admits a dimension function, cf. **XIV**-2.2.1.

Définition 2.1.2. Let X be an excellent Noetherian $\mathbf{Z}\left[\frac{1}{n}\right]$-scheme equipped with a dimension function δ. A **putative dualizing complex** consists of the data of $K \in \mathsf{D}^+(X_{\text{ét}}, \Lambda)$ and for any $x \in X$ an isomorphism (called a **pinning** at x) $\mathrm{R}\Gamma_x(K) \xrightarrow{\sim} \Lambda(\delta(x))[2\delta(x)]$ in $\mathsf{D}^+(x_{\text{ét}}, \Lambda)$. A putative dualizing complex is a potential dualizing complex if for any immediate specialization $\overline{y} \to \overline{x}$, the following diagram is commutative :

$$\begin{tikzcd} \mathrm{R}_{\overline{y}}(K) \arrow[r, "\mathrm{sp}^{X}_{\overline{y}\to\overline{x}}"] \arrow[dr, "\sim"'] & \mathrm{R}_{\overline{x}}(K)(1)[2] \arrow[d, "\sim"] \\ & \Lambda(\delta(y))[2\delta(y)] \end{tikzcd}$$

In other words, the pinnings are compatible with the transition morphisms associated with immediate specializations. (We note that since the objects appearing in the diagram above have only one non-zero cohomology object, it is sufficient to state the commutativity of the diagram after passing to the cohomology groups of degree $-2\delta(y)$.)

The notions of putative and potential dualizing complexes (with coefficients Λ) are only defined for excellent Noetherian $\mathbf{Z}\left[\frac{1}{n}\right]$-schemes equipped with a dimension function. Some of the upcoming statements will therefore implicitly contain these hypotheses on the schemes. The dimension function will also not be systematically mentioned in the statements.

Remarque 2.1.3. If X is connected and $K \in \mathrm{D}^+(X_{\text{ét}}, \Lambda)$ is equipped with two structures of a potential dualizing complex, to verify that the pinnings are the same at all points of X, it is sufficient to do so at a single point.

The objective of this section is to show that on a regular excellent scheme equipped with the dimension function $-$ codim, the constant sheaf Λ is naturally equipped with the structure of a potential dualizing complex.

2.2. Functoriality with respect to étale morphisms. Important stability properties of potential dualizing complexes with respect to certain classes of morphisms will be obtained in section 4. For the moment, let us simply mention the following compatibility for étale morphisms :

Proposition 2.2.1. *Let $f: Y \to X$ be an étale morphism between excellent Noetherian $\mathbf{Z}\left[\frac{1}{n}\right]$-schemes. Assume that X is equipped with a dimension function δ_X. We define a dimension function δ_Y on Y by setting $\delta_Y(y) = \delta_X(f(y))$ for all $y \in Y$. Let K be a putative dualizing complex on X. Then, $f^\star K$ is naturally equipped with a structure of a putative dualizing complex and it is a potential dualizing complex if K is one.*

If y is a point of Y, let $x = f(y)$ and let $g: y \to x$ be the morphism induced by f, we have a canonical isomorphism $g^\star \mathrm{R}\Gamma_x(K) \simeq \mathrm{R}\Gamma_y(f^\star K)$. This allows us to define the pinnings on $f^\star K$. We immediately check that if K is a potential dualizing complex, then so is $f^\star K$.

The construction of this proposition obviously passes to the limit : the result also holds for localizations $X_{(x)} \to X$, $X^{\mathrm{h}}_{(x)} \to X$ or $X_{(\overline{x})} \to X$. We will freely use these simple observations in what follows.

2.3. Complements on specializations.

Définition 2.3.1. Let $X' \to X$ be a morphism of schemes. Let $\overline{y'} \to \overline{x'}$ and $\overline{y} \to \overline{x}$ be specializations of geometric points of X' and X respectively. If we are given X-morphisms $\overline{y'} \to \overline{y}$ and $\overline{x'} \to \overline{x}$ such that the evident diagram below commutes, then we say that $\overline{y'} \to \overline{x'}$ is **above** $\overline{y} \to \overline{x}$.

$$\begin{tikzcd} \overline{y'} \arrow[r] \arrow[d] & X'_{(\overline{x'})} \arrow[d] \\ \overline{y} \arrow[r] & X_{(\overline{x})} \end{tikzcd}$$

Proposition 2.3.2. *Let $X' \to X$ be a morphism of schemes. Let $\overline{y'} \to \overline{x'}$ be a specialization of geometric points of X'. Then, up to unique isomorphisms, there exists a unique specialization $\overline{y} \to \overline{x}$ of geometric points of X below $\overline{y'} \to \overline{x'}$.*

This is evident.

Proposition 2.3.3. *Let $X' \to X$ be a finite morphism and $\overline{y} \to \overline{x}$ be a specialization of geometric points of X. Let $\overline{y'} \to X'$ be a geometric point of X' above $\overline{y}$ (i.e. we are given an X-morphism $\overline{y'} \to \overline{y}$). Then, up to unique isomorphisms, there exists a unique specialization of geometric points of X' above $\overline{y} \to \overline{x}$ of the form $\overline{y'} \to \overline{x'}$.*

We can assume that X is a local strictly Henselian scheme with closed point $\overline{x}$. The scheme X' being finite over X, it is a finite disjoint union of local strictly Henselian schemes. Up to replacing X' by the connected component containing $\overline{y'}$, we can assume that X' is also local strictly Henselian. There is then clearly no alternative : $\overline{x'}$ is the closed point of X'.

2.4. Construction of a potential dualizing complex in the regular case.
2.4.1. *A putative dualizing complex.*

Proposition 2.4.1.1. *Let X be an excellent regular Noetherian $\mathbf{Z}\left[\frac{1}{n}\right]$-scheme equipped with the dimension function $-\operatorname{codim}$. Then Λ is naturally equipped with a structure of a putative dualizing complex.*

Let $x \in X$. To define the pinning at x, we can assume that X is local with closed point x. We denote by $i\colon x \to X$ the inclusion of this closed point. The Gysin morphism $\mathrm{Cl}_i\colon \Lambda(\delta(x))[2\delta(x)] \to i^!\Lambda = \mathrm{R}\Gamma_x(\Lambda)$ is an isomorphism according to the absolute cohomological purity theorem (cf. **XVI**-3.1.1). The pinning at x is the inverse isomorphism.
2.4.2. *Case of a trait.*

Proposition 2.4.2.1. *Let X be an excellent trait, equipped with the dimension function $\delta = -\operatorname{codim}$. Assume that X is a $\mathbf{Z}\left[\frac{1}{n}\right]$-scheme.*

(a) *The putative dualizing complex Λ from proposition 2.4.1.1 for X is a potential dualizing complex.*
(b) *If K is a putative dualizing complex on X, there exists a unique morphism $\Lambda \to K$ compatible with the pinnings at the generic point.*
(c) *If K is a potential dualizing complex on X, the morphism $\Lambda \to K$ defined above is an isomorphism (compatible with the pinnings).*

Let $i\colon s \to X$ be the inclusion of the closed point s of X and $j\colon \eta \to X$ the inclusion of its generic point η. Proposition 1.2.4.1 states precisely compatibility (a) in the case where S is strictly Henselian; the general case follows since the cohomology class Cl_i of the closed point s in $H^2_s(X, \Lambda(1))$ induces the class of $\overline{s}$ in the strict Henselization $X_{(\overline{s})}$ (cf. **XVI**-2.3.2). Let us establish (b). Let K be a putative dualizing complex on X. We have a canonical distinguished triangle, which we can rewrite in the presence of pinnings :

$$\begin{tikzcd}
i_\star i^! K \arrow[r] \arrow[d, "\sim"] & K \arrow[r] \arrow[d, equal] & \mathrm{R}j_\star j^\star K \arrow[r] \arrow[d, "\sim"] & i_\star i^! K[1] \arrow[d, "\sim"] \\
i_\star \Lambda(-1)[-2] \arrow[r] & K \arrow[r] & \mathrm{R}j_\star \Lambda \arrow[r] & i_\star \Lambda(-1)[-1]
\end{tikzcd}$$

By applying the i-th cohomology sheaf functor $\mathscr{H}^i$, we obtain the vanishing of $\mathscr{H}^i K$ for $i < 0$ and a canonical isomorphism $\mathscr{H}^0 K \simeq \Lambda$. By virtue of the elementary properties of the canonical t-structure on $\mathrm{D}^+(X_{\text{ét}}, \Lambda)$, we obtain a unique morphism $\Lambda \simeq \mathscr{H}^0 K \to K$ compatible with the pinning at the generic point.

To obtain (c), assume that K is a potential dualizing complex. We consider the commutative diagram induced by the canonical morphism $\Lambda \to K$ from (b) and the functoriality of the specialization morphism associated with a choice of specialization $\overline{\eta} \to \overline{s}$ above the points η and s :

$$\begin{tikzcd}
\mathrm{R}_{\overline{\eta}}(\Lambda) \arrow[r] \arrow[d, "\mathrm{sp}^X_{\overline{\eta}\to\overline{s}}"] & \mathrm{R}_{\overline{\eta}}(K) \arrow[d, "\mathrm{sp}^X_{\overline{\eta}\to\overline{s}}"] \\
\mathrm{R}_{\overline{s}}(\Lambda)(1)[2] \arrow[r] & \mathrm{R}_{\overline{s}}(K)(1)[2]
\end{tikzcd}$$

Since Λ and K are potential dualizing complexes, the vertical arrows are isomorphisms. By construction, the top morphism is an isomorphism. It follows that the bottom morphism is also one. Consequently, the morphism $\Lambda \to K$ induces an isomorphism after application of $j^\star$ and of $i^!$: it is an isomorphism.

2.4.3. *Functoriality with respect to quasi-finite morphisms.*

Proposition 2.4.3.1. *Let $f : Y \to X$ be a quasi-finite morphism. Let K be a putative dualizing complex on X for some dimension function δ_X on X. Then $f^!K$ is naturally equipped with a structure of a putative dualizing complex for the dimension function δ_Y on Y defined by $\delta_Y(y) = \delta_X(f(y))$ for all $y \in Y$ (cf.* ***XIV****-2.5.2).*

Let $y \in Y$. Let $x = f(y)$ and $\pi : y \to x$ be the finite morphism induced by f. We have a canonical isomorphism in $\mathrm{D}^+(y_{\text{ét}}, \Lambda)$:

$$\mathrm{R}\Gamma_y(f^!K) \simeq \pi^!\mathrm{R}\Gamma_x(K) \; .$$

The pinning at x gives an isomorphism $\mathrm{R}\Gamma_x(K) \simeq \Lambda(\delta_X(x))[2\delta_X(x)]$. To obtain the desired isomorphism $\mathrm{R}\Gamma_y(f^!K) \simeq \Lambda(\delta_Y(y))[2\delta_Y(y)]$, it is sufficient to define an isomorphism $\Lambda \xrightarrow{\sim} \pi^!\Lambda$: we use the Gysin morphism Cl_π.

Remarque 2.4.3.2. Let $g : Z \to Y$ be another quasi-finite morphism. *Via* the transitivity isomorphism $g^!f^! \simeq (f \circ g)^!$, the structure of a putative dualizing complex on $g^!(f^!K)$ obtained by applying this construction to f then to g is the same as that obtained by applying the construction directly to $f \circ g$: this follows immediately from the composition properties of Gysin morphisms.

Proposition 2.4.3.3. *Let $f : Y \to X$ be a finite surjective morphism and K a putative dualizing complex on X. Then K is a potential dualizing complex if and only if the putative dualizing complex $f^!K$ from proposition 2.4.3.1 is a potential dualizing complex.*

By virtue of propositions 2.3.2 and 2.3.3, we can assume that X and Y are integral strictly Henselian local schemes of dimension 1 and that the dimension functions take the values 0 and -1. Given remark 2.4.3.2, it then follows that it is sufficient to treat two cases :

(1) Y is the normalization of X;
(2) X and Y are traits.

We obtain the conclusion in case (1) by using the fact that the transition morphism for X and K is defined from that for Y and $f^\star K$ (cf. definition 1.3.1) and that if M/L is a purely inseparable finite field extension, the Gysin morphism $\mathrm{Cl}_\pi : \Lambda \to \pi^!\Lambda$ associated with the morphism $\pi : \operatorname{Spec}(M) \to \operatorname{Spec}(L)$ is identified with multiplication by the degree of M/L *via* the tautological isomorphisms $\pi^!\Lambda \simeq \pi^\star\Lambda \simeq \Lambda$.

The proof in case (2) will use the following general lemma :

Lemme 2.4.3.4. *Let $f : Y \to X$ be a quasi-finite morphism between regular excellent Noetherian $\mathbf{Z}\left[\frac{1}{n}\right]$-schemes of virtual relative dimension $-c$. We equip X with the dimension function $\delta_X = -\operatorname{codim}$ and Y with the dimension function δ_Y defined by composition with f as in proposition 2.4.3.1. Given the relation $\delta_Y(y) = -\operatorname{codim}_Y(y) - c$ for all $y \in Y$, proposition 2.4.1.1 equips $\Lambda(-c)[-2c]$ with a structure of a putative dualizing complex for the dimension function δ_Y on Y. Proposition 2.4.1.1 gives a structure of a putative dualizing complex on Λ on X, from which we deduce, by proposition 2.4.3.1, a structure of a putative dualizing complex on $f^!\Lambda$ on Y for the dimension function δ_Y. Then, the purity isomorphism $\mathrm{Cl}_f : \Lambda(-c)[-2c] \xrightarrow{\sim} f^!\Lambda$ is compatible with the pinnings of these two putative dualizing complexes.*

Let us study the pinnings at a point $y \in Y$. Let $x = f(y)$. We can assume that X and Y are local (strictly Henselian) with respective closed points x and y. We have a commutative diagram :

$$\begin{tikzcd} y \arrow[r, "j"] \arrow[d, "g"'] \arrow[dr, "h"] & Y \arrow[d, "f"] \\ x \arrow[r, "i"'] & X \end{tikzcd}$$

The schemes appearing on this diagram are affine and regular and the morphisms between them are of finite type. These morphisms are therefore locally of complete intersection (smoothable), we can apply to them the theory of Gysin morphisms (cf. **XVI**-2.5). The result of the lemma then follows immediately from their compatibility with composition, since it gives a commutative diagram in $\mathrm{D}^+(y_{\text{ét}}, \Lambda)$, where we have denoted by c' the codimension of y in Y:

$$\begin{tikzcd}
\Lambda \arrow[rr, "\mathrm{Cl}_g"] \arrow[d, "\mathrm{Cl}_j"'] \arrow[drr, "\mathrm{Cl}_h"] & & g^!\Lambda \arrow[d, "g^!(\mathrm{Cl}_i)"] \\
j^!\Lambda(c')[2c'] \arrow[rr, "j^!(\mathrm{Cl}_f)"] & & h^!\Lambda(c+c')[2c+2c']
\end{tikzcd}$$

Let us return to the proof of proposition 2.4.3.3. Assume that K is a potential dualizing complex. According to proposition 2.4.2.1 applied to X, K is canonically isomorphic to Λ (with the pinnings of proposition 2.4.1.1). We thus assume $K = \Lambda$. We have a purity isomorphism $\mathrm{Cl}_f : \Lambda \xrightarrow{\sim} f^!\Lambda$. According to lemma 2.4.3.4, this isomorphism is compatible with the pinnings of proposition 2.4.1.1 for Λ and those of proposition 2.4.3.1 for $f^!\Lambda$. According to proposition 2.4.2.1 (a) applied to Y, it follows that $f^!\Lambda$ is indeed a potential dualizing complex.

Conversely, assume that $f^!K$ is a potential dualizing complex. Proposition 2.4.2.1 (b) for X gives a canonical morphism $\Lambda \to K$ compatible with the pinnings at the generic point of X. The morphism $f^!\Lambda \to f^!K$ which is deduced from it is a morphism between two potential dualizing complexes compatible with the pinnings at the generic point of Y. According to proposition 2.4.2.1 (c), $f^!\Lambda \to f^!K$ is an isomorphism. The functor $f^!$ being conservative, it follows that the canonical morphism $\Lambda \to K$ is an isomorphism. We can therefore assume that $K = \Lambda$ (in a way compatible with the pinnings at the generic point). The pinning of K at the closed point x can only be of the form $\lambda \cdot \mathrm{Cl}_{x \subset X}^{-1} : \mathrm{R}\Gamma_x(\Lambda) \xrightarrow{\sim} \Lambda(-1)[-2]$ for $\lambda \in \Lambda^\times$. Given the above, it immediately follows that the putative dualizing complex $f^!K$ on Y is identified with Λ, pinned trivially at the generic point and by $\lambda \cdot \mathrm{Cl}_{y \subset Y}^{-1}$ at the closed point y of Y. Given proposition 2.4.2.1 (a), this putative dualizing complex on Y can obviously only be a potential dualizing complex if $\lambda = 1$. Thus, K is indeed a potential dualizing complex on X, since it is identified with Λ in a way compatible with the pinnings.

2.4.4. *A potential dualizing complex.*

Proposition 2.4.4.1. *Let X be an excellent regular scheme, equipped with the dimension function* $-\operatorname{codim}$*. The putative dualizing complex Λ from proposition 2.4.1.1 is a potential dualizing complex.*

Let $\overline{y} \to \overline{x}$ be an immediate specialization of geometric points of X. We want to show that the pinnings on Λ are compatible with the transition morphism $\mathrm{sp}^X_{\overline{y} \to \overline{x}}$. For this, we can assume that X is a local strictly Henselian scheme with closed point $\overline{x}$. Let C be the closure of the image of $\overline{y}$ in X. Let $i : C \to X$ be the closed immersion of C in X. Let $n : \tilde{C} \to C$ be the normalization of C. Showing the compatibility of the pinnings with the transition morphism associated with $\overline{y} \to \overline{x}$ amounts to showing that the putative dualizing complex $i^!\Lambda$ from proposition 2.4.3.1 is a potential dualizing complex, which, according to proposition 2.4.3.3, is equivalent to saying that $n^!i^!\Lambda$ is one. According to lemma 2.4.3.4, the putative dualizing complex $n^!i^!\Lambda$ is identified with the putative dualizing complex $\Lambda(-c)[-2c]$ obtained by twisting and shifting from the putative dualizing complex Λ from proposition 2.4.1.1 applied to the trait $\tilde{C}$. By virtue of proposition 2.4.2.1 (a), this is indeed a potential dualizing complex, which completes the proof.

3. General transition morphisms and cohomology class in maximal degree

3.1. Statements of the main theorems.

Théorème 3.1.1. *Let X be a strictly Henselian excellent normal local $\mathbf{Z}\left[\frac{1}{n}\right]$-scheme of dimension d and closed point x. Then, $\mathrm{H}^q_{x,\text{ét}}(X, \Lambda(d)) = 0$ for $q > 2d$ and we have an isomorphism $[x] : \Lambda \xrightarrow{\sim} \mathrm{H}^{2d}_{x,\text{ét}}(X, \Lambda(d))$ compatible with the transition morphisms associated with immediate specializations.*

Théorème 3.1.2. *Let X be an excellent Noetherian $\mathbf{Z}\left[\frac{1}{n}\right]$-scheme. For any specialization $\overline{y} \to \overline{x}$ of geometric points of X of codimension c, we can define, for any $K \in \mathsf{D}^+(X_{\text{ét}}, \Lambda)$ and $i \in \mathbf{Z}$, a transition morphism $\mathrm{sp}^X_{\overline{y}\to\overline{x}} \colon \mathrm{H}^i_{\overline{y}}(K) \to \mathrm{H}^{i+2c}_{\overline{x}}(K(c))$, compatible with the composition of specializations and induced by the definitions of section 1 for $c \leq 1$. Furthermore, these generalized transition morphisms satisfy a compatibility with finite morphisms stated in proposition 3.5.4.*

Remarque 3.1.3. Locally for the étale topology, a quasi-excellent scheme is excellent (cf. **XIV**-2.3.1 and **XIV**-2.2.6). It is therefore evident that in the statement of theorem 3.1.2, we can replace the hypothesis « excellent » by « quasi-excellent ».

3.2. Cohomological dimension. We state here two cohomological dimension results which are deduced from Gabber's affine Lefschetz theorem, cf. **XV**-1.2.2. See $\mathrm{XVIII}_{\mathrm{A}}$ and $\mathrm{XVIII}_{\mathrm{B}}$ for variants of these results without the excellence hypothesis.

Proposition 3.2.1. *Let ℓ be a prime number. Let X be a strictly Henselian excellent local $\mathbf{Z}\left[\frac{1}{\ell}\right]$-scheme with closed point x. Let $U = X - x$. Let $\mathscr{M}$ be a sheaf of $\mathbf{Z}/\ell\mathbf{Z}$-modules on U. Let $d \in \mathbf{N}$. Assume that if $u \in U$ is such that $\mathscr{M}_{\overline{u}} \neq 0$, then the dimension of the closure of u in X is $\leq d$. Then $\mathrm{H}^q_{\text{ét}}(U, \mathscr{M}) = 0$ for $q \geq 2d$.*

*In particular, if $d = \dim X$, we have $\mathrm{cd}_\ell\, U \leq 2d - 1$ (cf. [**SGA 4** x 1]) and for any ℓ-torsion sheaf $\mathscr{M}$ on $X_{\text{ét}}$, the groups $H^q_{x,\text{ét}}(X, \mathscr{M})$ are zero for $q > 2d$.*

By passing to the filtered colimit over the constructible subsheaves of $\mathscr{M}$, we can assume that $\mathscr{M}$ is constructible. Up to replacing X by the closure in X of the support of $\mathscr{M}$, we can then assume that $d = \dim X$. We are then reduced to showing that $\mathrm{cd}_\ell\, U \leq 2d - 1$. Since the (separated) scheme U can be covered by d affine open sets (cf. [**Serre, 1965**, § B.3, Chapter III]), and since an affine open set of X has cohomological dimension at most d (cf. **XV**-1.2.4), by appropriately using the Mayer-Vietoris exact sequences, we indeed obtain that $\mathrm{cd}_\ell\, U \leq 2d - 1$.

Proposition 3.2.2. *Let ℓ be a prime number. Let X be a strictly Henselian integral excellent local $\mathbf{Z}\left[\frac{1}{\ell}\right]$-scheme of dimension d, with generic point η. Then $\mathrm{cd}_\ell\, \eta \leq d$.*

The generic point η is identified with the projective limit of the projective system formed by the non-empty affine open sets of X. Since each of these affine open sets has ℓ-cohomological dimension at most d, the same holds for η.

3.3. Finite morphisms.

Proposition 3.3.1. *Let $f \colon X' \to X$ be a finite morphism between excellent Noetherian $\mathbf{Z}\left[\frac{1}{n}\right]$-schemes. Let $\overline{y'} \to \overline{x'}$ be an immediate specialization of geometric points of X' above an immediate specialization $\overline{y} \to \overline{x}$ of geometric points of X (cf. definition 2.3.1). For any $K \in \mathsf{D}^+(X_{\text{ét}}, \Lambda)$ and $i \in \mathbf{Z}$, the following diagram commutes :*

$$\begin{tikzcd}
\mathrm{H}^i_{\overline{y'}}(f^!K) \arrow[r, "\mathrm{sp}^{X'}_{\overline{y'}\to\overline{x'}}"] \arrow[d, "\mathrm{Cl}_{y'\to y}"', "\sim"] & \mathrm{H}^{i+2}_{\overline{x'}}(f^!K(1)) \arrow[d, "\sim"', "\mathrm{Cl}_{x'\to x}"] \\
\mathrm{H}^i_{\overline{y}}(K) \arrow[r, "\mathrm{sp}^X_{\overline{y}\to\overline{x}}"] & \mathrm{H}^{i+2}_{\overline{x}}(K(1))
\end{tikzcd}$$

We can assume that f is a dominant morphism between integral strictly Henselian local schemes of dimension 1. As in the proof of proposition 2.4.3.3, there are two cases to treat :

(1) X' is the normalization of X;
(2) X and X' are traits.

Case (1) being trivial, we focus on the case where X and X' are traits. To check the compatibility, we can obviously assume that $i = 0$. For any $K \in \mathsf{D}^+(X_{\text{ét}}, \Lambda)$, since the vertical arrows are isomorphisms, we can denote by $\delta_K \colon \mathrm{H}^0_{\overline{y}}(K) \to \mathrm{H}^2_{\overline{x}}(K(1))$ the difference of the arrows obtained by following the two possible paths. We need to show that for any $K \in \mathsf{D}^+(X_{\text{ét}}, \Lambda)$, we have $\delta_K = 0$.

The results of section 2 show that $\delta_\Lambda = 0$. The rest of the proof will consist in reducing to this case.

Since $\overline{y}$ is above the generic point of X, $\mathrm{H}^0_{\overline{y}}$ is identified with the fiber at $\overline{y}$ of the cohomology sheaf of K in degree 0. If we denote by $\tau_{\leq 0}K \to K$ the canonical morphism deduced from the canonical t-structure on $\mathrm{D}^+(X_{\text{ét}}, \Lambda)$ (v), we obtain a commutative square :

$$\begin{tikzcd}
\mathrm{H}^0_{\overline{y}}(\tau_{\leq 0}K) \arrow[r, "\delta_{\tau_{\leq 0}K}"] \arrow[d, "\sim"] & \mathrm{H}^2_{\overline{x}}(\tau_{\leq 0}K(1)) \arrow[d] \\
\mathrm{H}^0_{\overline{y}}(K) \arrow[r, "\delta_K"] & \mathrm{H}^2_{\overline{x}}(K(1))
\end{tikzcd}$$

It follows from this diagram that if $\delta_{\tau_{\leq 0}K} = 0$, then $\delta_K = 0$.

Let $\mathscr{H}^0K$ be the cohomology sheaf of K in degree zero. For cohomological dimension reasons (cf. proposition 3.2.1), the canonical morphism $\tau_{\leq 0}K \to \mathscr{H}^0K$ induces an isomorphism after application of the functor $\mathrm{H}^2_{\overline{x}}(-(1))$ (and also of the functor $\mathrm{H}^0_{\overline{y}}$). Consequently, by considering a commutative square of the preceding type, we now obtain that $\delta_{\mathscr{H}^0K} = 0$ is equivalent to $\delta_{\tau_{\leq 0}K} = 0$.

It follows from these remarks that to show that $\delta_K = 0$ for any $K \in \mathrm{D}^+(X_{\text{ét}}, \Lambda)$, we can assume that K is concentrated in degree 0.

We now assume that $K = \mathscr{M}$ where $\mathscr{M}$ is a sheaf of Λ-modules on X. Let $j \colon y \to X$ be the inclusion of the generic point of X. Using the description of the functor $\mathrm{H}^2_{\overline{x}}$ from proposition 1.2.2.1, we obtain that the canonical morphism $j_!j^\star\mathscr{M} \to \mathscr{M}$ induces isomorphisms after application of $\mathrm{H}^0_{\overline{y}}$ and $\mathrm{H}^2_{\overline{x}}(-(1))$: $\delta_K = 0$ is equivalent to $\delta_{j_!j^\star K} = 0$. The property, for a sheaf K on X, of being such that $\delta_K = 0$ thus depends only on its restriction to the generic point y.

For any sheaf of Λ-modules $\mathscr{L}$ on y, we denote $\delta_{\mathscr{L}} = \delta_{j_!\mathscr{L}}$. We need to show that $\delta_{\mathscr{L}} = 0$ for any sheaf of Λ-modules on y. Let $\mathscr{L} \to \mathscr{L}_{\text{constant}}$ be the largest constant quotient of $\mathscr{L}$: formally, the functor $\mathscr{L} \longmapsto \mathscr{L}_{\text{constant}}$ is the left adjoint functor to the inclusion functor of the category of constant sheaves of Λ-modules into the category of sheaves of Λ-modules on y. According to remark 1.2.4.8, it follows that the canonical morphism $\mathscr{L} \to \mathscr{L}_{\text{constant}}$ induces an isomorphism after application of $\mathrm{H}^2_{\overline{x}}(j_! - (1))$ (and a surjection after application of $\mathrm{H}^0_{\overline{y}}$). Hence $\delta_{\mathscr{L}} = 0$ if and only if $\delta_{\mathscr{L}_{\text{constant}}} = 0$. It follows that we can assume that $\mathscr{L}$ is a constant sheaf of Λ-modules. Furthermore, we can obviously assume that $\mathscr{L}$ is constructible. If $\mathscr{L} \to \mathscr{L}'$ is an epimorphism of constructible constant sheaves of Λ-modules, it immediately follows that $\delta_{\mathscr{L}} = 0$ implies $\delta_{\mathscr{L}'} = 0$. It follows that we can assume that $\mathscr{L} = \Lambda^r$ for some $r \in \mathbf{N}$, then, by additivity, that $r = 1$. In short, we have indeed reduced to the case of the constant sheaf Λ, which completes the proof of this proposition.

3.4. The case of dimension 2. Given the results established so far, theorems 3.1.1 and 3.1.2 can be considered as having been established in dimension 0 and 1. This subsection focuses on the case of dimension 2, which is the crucial step to pass to the general case.

We fix a normal excellent strictly Henselian local $\mathbf{Z}\left[\frac{1}{n}\right]$-scheme X of dimension 2. We denote by x the closed point of X and we set $U = X - x$. With the help of a resolution of singularities $X' \to X$, we will define in paragraph 3.4.1 a class in $\mathrm{H}^4_{x,\text{ét}}(X, \Lambda(2))$, then we will prove in paragraph 3.4.2 that it is independent of the resolution. Finally, we will establish in paragraph 3.4.3 a compatibility between this class, the classes defined for the strict localizations of X at points of codimension 1 and the transition morphisms associated with the corresponding immediate specializations.

3.4.1. *Construction of a class.* According to the main result of [**Lipman, 1978**], there exists a proper birational morphism $p \colon X' \to X$ with X' regular. The morphism p then automatically induces an isomorphism $p^{-1}(U) \xrightarrow{\sim} U$. We can further assume that $p^{-1}(x)_{\text{réd}}$ is a strict normal crossings divisor (whose irreducible components we denote by $D_1, \dots, D_n$).

(v) The author thinks it would have been more consistent to denote this truncation by $\tau^{\leq 0}$ to respect the convention that cohomological degrees are indicated as superscripts, but since tradition has consecrated the opposite usage, he reluctantly complies.

Indeed, we can first assume that the irreducible components $D_1, \dots, D_n$ have dimension 1 : if this is not the case, we can blow up the corresponding closed point (this can only happen if X is already regular and p is an isomorphism). In a second step, we can arrange for the irreducible components of $p^{-1}(x)_{\text{réd}}$ to be regular by iterating the process of blowing up the singular points (cf. [**Šafarevič, 1966**, page 38]). In a last step, we can force the crossings to become normal by blowing up the recalcitrant closed points ; since numerical invariants decrease strictly in this operation (cf. [**Šafarevič, 1966**, page 21]), this process terminates.

Proposition 3.4.1.1.

(a) *The boundary morphism* $\mathrm{H}^{q-1}_{\text{ét}}(U, \Lambda(2)) \to \mathrm{H}^{q}_{x,\text{ét}}(X, \Lambda(2))$ *is an isomorphism for* $q \geq 2$ *and these groups are zero for* $q \geq 5$ *;*

(b) *The evident morphism* $\mathrm{H}^{q}_{\text{ét}}(X', \Lambda(2)) \to \mathrm{H}^{q}_{\text{ét}}(p^{-1}(x), \Lambda(2))$ *is an isomorphism for any* $q \in \mathbf{Z}$ *and these groups are zero for* $q \geq 3$ *;*

(c) *The boundary morphism* $\mathrm{H}^{q-1}_{\text{ét}}(U, \Lambda(2)) \to \mathrm{H}^{q}_{p^{-1}(x),\text{ét}}(X', \Lambda(2))$ *is an isomorphism for* $q \geq 4$ *;*

(d) *The evident morphism* $\mathrm{H}^{q}_{x,\text{ét}}(X, \Lambda(2)) \to \mathrm{H}^{q}_{p^{-1}(x),\text{ét}}(X', \Lambda(2))$ *is an isomorphism for* $q \geq 4$ *and these groups are zero for* $q \geq 5$.

(a) is obtained by using the canonical exact sequence :

$$\mathrm{H}^{q-1}_{\text{ét}}(X, \Lambda(2)) \to \mathrm{H}^{q-1}_{\text{ét}}(U, \Lambda(2)) \to \mathrm{H}^{q}_{x,\text{ét}}(X, \Lambda(2)) \to \mathrm{H}^{q}_{\text{ét}}(X, \Lambda(2)) .$$

Indeed, since X is local and strictly Henselian, we have $\mathrm{H}^{i}_{\text{ét}}(X, \Lambda(2)) = 0$ for $i > 0$. We conclude by using the fact that $\mathrm{cd}_\ell\, U \leq 3$ (cf. proposition 3.2.1).

(b) results from the base change theorem for a proper morphism and the fact that $p^{-1}(x)$ is a proper curve and therefore has cohomological dimension 2.

(c) follows immediately from (b).

(d) results from (a), (c) and the commutativity of the following evident diagram :

$$\begin{tikzcd}
\mathrm{H}^{q-1}_{\text{ét}}(U, \Lambda(2)) \arrow[r] \arrow[d] & \mathrm{H}^{q}_{x,\text{ét}}(X, \Lambda(2)) \arrow[d] \\
\mathrm{H}^{q-1}_{\text{ét}}(U, \Lambda(2)) \arrow[r] & \mathrm{H}^{q}_{p^{-1}(x),\text{ét}}(X', \Lambda(2))
\end{tikzcd}$$

Proposition 3.4.1.2. *We have an exact sequence*

$$\bigoplus_{i<j} \mathrm{H}^{4}_{D_i \cap D_j,\text{ét}}(X', \Lambda(2)) \to \bigoplus_{i} \mathrm{H}^{4}_{D_i,\text{ét}}(X', \Lambda(2)) \to \mathrm{H}^{4}_{p^{-1}(x),\text{ét}}(X', \Lambda(2)) \to 0 ,$$

where the arrows are induced by enlargement of support morphisms, and their differences.

For any closed subset F of X', we denote by Λ_F the sheaf $i_{F\star}\Lambda$ where i_F is the immersion of F into X'. We have an evident short exact sequence of sheaves of Λ-modules on X' :

$$0 \to \Lambda_{p^{-1}(x)} \to \bigoplus_{i} \Lambda_{D_i} \to \bigoplus_{i<j} \Lambda_{D_i \cap D_j} \to 0 .$$

By applying $\mathrm{R}\,\mathbf{Hom}(-, \Lambda(2))$ to the distinguished triangle of $\mathrm{D}^+(X'_{\text{ét}}, \Lambda)$ associated with this short exact sequence, we can obtain the desired exact sequence, provided we can show that $\mathrm{H}^{5}_{D_i \cap D_j,\text{ét}}(X', \Lambda(2))$ is zero for $i < j$. According to the absolute purity theorem, this group is identified with $\mathrm{H}^{1}_{\text{ét}}(D_i \cap D_j, \Lambda)$, which is indeed zero since $D_i \cap D_j$ is a finite disjoint union of spectra of separably closed fields.

Proposition 3.4.1.3. *The exact sequence of proposition 3.4.1.2 can be rewritten as :*

$$\bigoplus_{i<j} \bigoplus_{y \in D_i \cap D_j} \Lambda \xrightarrow{\delta} \Lambda^n \xrightarrow{\varepsilon} \mathrm{H}^{4}_{p^{-1}(x),\text{ét}}(X', \Lambda(2)) \to 0 .$$

If we denote by $(\chi_y)_{i<j, y\in D_i\cap D_j}$ *the canonical basis of the left-hand group and by* $\chi_{D_1}, \dots, \chi_{D_n}$ *that of* Λ^n*, the differential* δ *satisfies the formula*

$$\delta(\chi_y) = [y : x] \cdot \left(\chi_{D_i} - \chi_{D_j}\right) .$$

Furthermore, $\varepsilon(\chi_{D_i}) \in \mathrm{H}^4_{p^{-1}(x),\text{ét}}(X', \Lambda(2))$ *is obtained by enlargement of support from the class* $\frac{1}{[z:x]}\mathrm{Cl}_{z\to X'} \in \mathrm{H}^4_{z,\text{ét}}(X', \Lambda(2))$ *for any closed point* z *of* D_i*.*

The absolute cohomological purity theorem gives isomorphisms

$$\mathrm{Cl}_{D_i\cap D_j\to X'} : \mathrm{H}^0_{\text{ét}}(D_i \cap D_j, \Lambda) \xrightarrow{\sim} \mathrm{H}^4_{D_i\cap D_j,\text{ét}}(X', \Lambda(2)) ,$$

which allows us to describe the left-hand group in the exact sequence of proposition 3.4.1.2. The same theorem also gives isomorphisms

$$\mathrm{Cl}_{D_i\to X'} : \mathrm{H}^2_{\text{ét}}(D_i, \Lambda(1)) \xrightarrow{\sim} \mathrm{H}^4_{D_i,\text{ét}}(X', \Lambda(2)) .$$

Furthermore, we have the trace morphism (relative to x) $\mathrm{Tr} : \mathrm{H}^2_{\text{ét}}(D_i, \Lambda(1)) \xrightarrow{\sim} \Lambda$. This is characterized by the fact that for any closed point z of D_i, the image of $\mathrm{Cl}_{z\to D_i} \in \mathrm{H}^2_{z,\text{ét}}(D_i, \Lambda(1))$ by the composite morphism

$$\mathrm{H}^2_{z,\text{ét}}(D_i, \Lambda(1)) \to \mathrm{H}^2_{\text{ét}}(D_i, \Lambda(1)) \xrightarrow{\mathrm{Tr}} \Lambda$$

is $[z : x]$ (which is invertible in Λ because $\kappa(y)/\kappa(x)$ is purely inseparable). The description of the morphisms δ and ε given in the statement is then immediately obtained from the composition properties of Gysin morphisms.

Proposition 3.4.1.4. *The following sequence is exact :*

$$\bigoplus_{i<j} \bigoplus_{y\in D_i\cap D_j} \Lambda \xrightarrow{\delta} \bigoplus_i \Lambda \xrightarrow{\Sigma} \Lambda \to 0 ,$$

where Σ *is defined by* $\Sigma(\chi_{D_i}) = 1$ *for all* i*.*

First, from the formula given for δ in proposition 3.4.1.3, it is clear that $\Sigma \circ \delta = 0$. Σ thus induces a morphism $\mathrm{Coker}(\delta) \to \Lambda$. Since the degrees $[y : x]$ which appear are invertible in Λ, we see that if i and j are such that $D_i \cap D_j$ is non-empty, then χ_{D_i} and χ_{D_j} have the same class in $\mathrm{Coker}(\delta)$. The fiber $p^{-1}(x)$ being connected (Main Theorem), we deduce that all the elements χ_{D_i} have the same class in $\mathrm{Coker}(\delta)$. There thus exists a morphism $\Lambda \to \mathrm{Coker}(\delta)$ sending 1 to χ_{D_i} for all i; this morphism is the inverse isomorphism of $\mathrm{Coker}(\delta) \to \Lambda$.

Corollaire 3.4.1.5. *The exact sequences of propositions 3.4.1.3 and 3.4.1.4 give rise to an isomorphism* $\Lambda \xrightarrow{\sim} \mathrm{H}^4_{p^{-1}(x),\text{ét}}(X', \Lambda(2))$*.*

Corollaire 3.4.1.6. Via *the canonical isomorphism* $\mathrm{H}^4_{x,\text{ét}}(X, \Lambda(2)) \xrightarrow{\sim} \mathrm{H}^4_{p^{-1}(x),\text{ét}}(X', \Lambda(2))$ *from proposition 3.4.1.1 (d), the isomorphism of the preceding corollary gives an isomorphism* $\Lambda \xrightarrow{\sim} \mathrm{H}^4_{x,\text{ét}}(X, \Lambda(2))$*.*

Définition 3.4.1.7. We denote by $[x]_{X'} \in \mathrm{H}^4_{x,\text{ét}}(X, \Lambda(2))$ the generator defined by the isomorphism of the preceding corollary.

3.4.2. *Independence of the resolution.* In this paragraph, we will show that if $q : X'' \to X$ is another resolution of the type considered in paragraph 3.4.1, then $[x]_{X'} = [x]_{X''}$. Up to introducing a desingularization of the irreducible component dominating X of the fiber product of X'' and X' over X, we can assume that one of the two desingularizations X' and X'' considered sits over the other. We thus assume for example that there exists a (unique) X-morphism $\pi : X'' \to X'$. The morphism π is projective and birational between two regular schemes of dimension 2, so it induces an isomorphism over an open set U' of X' such that the closed set $X' - U'$ has dimension 0. There thus certainly exists a closed point y' of $p^{-1}(y)$ such that the induced morphism is an isomorphism $\pi^{-1}(y') \to y'$. We denote by y'' the unique closed point of $\pi^{-1}(y')$. The compatibility of Gysin classes with base change (cf. **XVI**-2.3.2) implies that the class $\mathrm{Cl}_{y'\to X'}$ is sent to $\mathrm{Cl}_{y''\to X''}$ by the restriction morphism

$\pi^\star : \mathrm{H}^4_{y',\text{ét}}(X',\Lambda(2)) \to \mathrm{H}^4_{y'',\text{ét}}(X'',\Lambda(2))$. This compatibility still holds after enlargement of support to $p^{-1}(x)$ and to $q^{-1}(x)$. By considering the following composition :

$$\mathrm{H}^4_{x,\text{ét}}(X,\Lambda(2)) \xrightarrow{\sim} \mathrm{H}^4_{p^{-1}(x),\text{ét}}(X',\Lambda(2)) \xrightarrow{\sim} \mathrm{H}^4_{q^{-1}(x),\text{ét}}(X'',\Lambda(2))\ ,$$

we immediately obtain that $[y' : x] \cdot [x]_{X'} = [y'' : x] \cdot [x]_{X''}$, which allows us to conclude that $[x]_{X'} = [x]_{X''}$.

Remarque 3.4.2.1. By using arguments similar to the preceding ones, we can show that if $p : X' \to X$ is a projective birational morphism with X' regular, then, even without assuming that $p^{-1}(x)_{\text{réd}}$ is a strict normal crossings divisor, the map $\mathrm{H}^4_{x,\text{ét}}(X,\Lambda(2)) \to \mathrm{H}^4_{p^{-1}(x),\text{ét}}(X',\Lambda(2))$ is an isomorphism and the class $[x]$ is indeed sent to the element induced by $\frac{1}{[x':x]}\mathrm{Cl}_{x'\to X'}$ for any closed point x' of $p^{-1}(x)$.

3.4.3. *Compatibility with transition morphisms.* Let $\overline{y} \to X$ be a geometric point above a point y of codimension 1. In other words, we have an immediate specialization $\overline{y} \to x$ of geometric points of X. We will show that if $\overline{\eta} \to \overline{y}$ is an immediate specialization, then we have the equality

$$[x] = \mathrm{sp}^X_{\overline{y}\to x}(\mathrm{sp}^X_{\overline{\eta}\to\overline{x}}(1))$$

in $\mathrm{H}^4_{x,\text{ét}}(X,\Lambda(2))$, which will complete the proof of theorem 3.1.1 up to dimension 2.

We denote by C the closure of y in X. There exists a desingularization $X' \to X$ of the type considered in paragraph 3.4.1 such that the closure C' of y in X' is a trait. More precisely, the evident morphism $C' \to C$ identifies C' with the normalization of C. The morphism $C' \to C$ being a universal homeomorphism, we can denote by x' the closed point of C' (that of C is of course x) and $U' = X' - x'$. We have evident closed immersions $i : C \to X$, $i' : C' \to X'$, $k : y \to U$ and $k' : y \to U'$.

Lemme 3.4.3.1. *With the above notations, the following evident diagram is commutative* (*the arrows marked as being isomorphisms should be considered as bidirectional*) *:*

$$\begin{tikzcd}
 & \mathrm{H}^4_{x,\text{ét}}(X,\Lambda(2)) \arrow[r] \arrow[dl, "\sim"'] & \mathrm{H}^4_{x,\text{ét}}(C,i^!\Lambda(2)) \arrow[dr, "\sim"] & \\
\mathrm{H}^4_{p^{-1}(x),\text{ét}}(X',\Lambda(2)) \arrow[dr, "\sim"'] & & & \mathrm{H}^3_{\text{ét}}(y,k^!\Lambda(2)) \\
 & \mathrm{H}^4_{x',\text{ét}}(X',\Lambda(2)) \arrow[r] & \mathrm{H}^4_{x',\text{ét}}(C',i'^!\Lambda(2)) \arrow[ur] &
\end{tikzcd}$$

This is equivalent to the commutativity of the evident diagram :

$$\begin{tikzcd}
\mathrm{H}^3_{\text{ét}}(U,\Lambda(2)) \arrow[r, "\sim"] \arrow[d, "\sim"] & \mathrm{H}^4_{x,\text{ét}}(X,\Lambda(2)) \arrow[r] & \mathrm{H}^4_{x,\text{ét}}(C,i^!\Lambda(2)) \arrow[r] & \mathrm{H}^3_{\text{ét}}(y,k^!\Lambda(2)) \arrow[d] \\
\mathrm{H}^3_{\text{ét}}(U',\Lambda(2)) \arrow[r, "\sim"] & \mathrm{H}^4_{x',\text{ét}}(X',\Lambda(2)) \arrow[r] & \mathrm{H}^4_{x',\text{ét}}(C',i'^!\Lambda(2)) \arrow[r] & \mathrm{H}^3_{\text{ét}}(y,k'^!\Lambda(2))
\end{tikzcd}$$

We can identify the outer square of this diagram with the following one, where the horizontal arrows are the arrows of forgetting the support :

$$\begin{tikzcd}
\mathrm{H}^3_{\text{ét}}(U,\Lambda(2)) \arrow[r] \arrow[d, "\sim"] & \mathrm{H}^3_{y,\text{ét}}(U,\Lambda(2)) \arrow[d, "\sim"] \\
\mathrm{H}^3_{\text{ét}}(U',\Lambda(2)) \arrow[r] & \mathrm{H}^3_{y,\text{ét}}(U',\Lambda(2))
\end{tikzcd}$$

This diagram is of course commutative, which completes the proof of lemma 3.4.3.1.

We can extend the commutative diagram of the lemma to the right :

$$\begin{tikzcd}
\mathrm{H}^4_{x,\text{ét}}(C, i^!\Lambda(2)) \arrow[d, "\sim"'] & \mathrm{H}^2_{\overline{y}}(C, i^!\Lambda(1)) \arrow[l, "\mathrm{sp}^C_{\overline{y}\to x}"'] & & \\
\mathrm{H}^3_{\text{ét}}(y, k^!\Lambda(2)) \arrow[d, "\sim"] & & (\mathscr{H}^2 k^!\Lambda(1))_{\overline{y}} \arrow[ul, "\sim", "{[x':x]}"'] \arrow[dl, "\sim"', "1"] & \Lambda \arrow[l, "\mathrm{sp}^X_{\overline{\eta}\to\overline{y}}"', "\sim"] \\
\mathrm{H}^4_{x',\text{ét}}(C', i'^!\Lambda(2)) & \mathrm{H}^2_{\overline{y}}(C', i'^!\Lambda(1)) \arrow[l, "\mathrm{sp}^{C'}_{\overline{y}\to x'}"'] & &
\end{tikzcd}$$

From the group $(\mathscr{H}^2 k^!\Lambda(1))_{\overline{y}}$ start two canonical isomorphisms to cohomology groups with supports in C and C' ; here, we have multiplied them respectively by $[x' : x]$ and 1 so as to make the diagram commutative (cf. definition 1.3.1).

We can start from the element 1 in the group Λ on the far right and consider its image in the group $\mathrm{H}^4_{x,\text{ét}}(X, \Lambda(2))$ appearing in the diagram of the lemma. By following the bottom path, we obtain $[x' : x] \cdot [x]_{X'}$. By following the top path, we obtain $[x' : x] \cdot \mathrm{sp}^X_{\overline{y}\to x}(\mathrm{sp}^X_{\overline{\eta}\to\overline{y}}(1))$. We can thus conclude that we indeed have the equality

$$[x]_{X'} = \mathrm{sp}^X_{\overline{y}\to x}(\mathrm{sp}^X_{\overline{\eta}\to\overline{y}}(1))$$

in $\mathrm{H}^4_{x,\text{ét}}(X, \Lambda(2))$.

3.5. Transition morphisms in arbitrary codimension. We will now prove theorem 3.1.2 by relying on theorem 3.1.1, established for the moment up to dimension 2. This result will then allow us to establish theorem 3.1.1 in full generality.

Définition 3.5.1. Let X be an excellent Noetherian $\mathbf{Z}\left[\frac{1}{n}\right]$-scheme. Let $\overline{y} = \overline{x_0} \to \ldots \to \overline{x_n} = \overline{x}$ be a sequence of specializations of geometric points of X such that for any $0 \leq i < n$, the specialization $\overline{x}_i \to \overline{x}_{i+1}$ is of codimension 0 or 1. We denote by c the codimension of the specialization $\overline{y} \to \overline{x}$. For any $K \in \mathrm{D}^+(X_{\text{ét}}, \Lambda)$, we denote $\mathrm{sp}^X_{\overline{x}_0\to\ldots\to\overline{x}_n} : \mathrm{H}^p_{\overline{y}}(K) \to \mathrm{H}^{p+2c}_{\overline{x}}(K(c))$ the transition morphism obtained by the composition $\mathrm{sp}^X_{\overline{x}_{n-1}\to\overline{x}_n} \circ \ldots \circ \mathrm{sp}^X_{\overline{x}_0\to\overline{x}_1}$.

Définition 3.5.2. Let X be an excellent Noetherian $\mathbf{Z}\left[\frac{1}{n}\right]$-scheme. Let $\overline{y} \to \overline{x}$ be a specialization of geometric points of X of codimension c. Let $K \in \mathrm{D}^+(X_{\text{ét}}, \Lambda)$. We say that the property $(\mathrm{C})^c_{\overline{y}\to\overline{x},K}$ is satisfied if the morphism $\mathrm{sp}^X_{\overline{x_0}\to\ldots\to\overline{x_n}} : \mathrm{H}^0_{\overline{y}}(K) \to \mathrm{H}^{2c}_{\overline{x}}(K(c))$ does not depend on the choice of the factorization of $\overline{y} \to \overline{x}$ into $\overline{x_0} \to \ldots \to \overline{x_n}$.

Définition 3.5.3. We will say that the property $(\mathrm{C})^{\leq c}$ is satisfied if all the properties $(\mathrm{C})^{c'}_{\overline{y}\to\overline{x},K}$ considered in definition 3.5.2 are satisfied for $c' \leq c$. We will say that the property $(\mathrm{C})^{\leq c}_{\text{loc.}}$ is satisfied if the properties $(\mathrm{C})^{c'}_{\overline{\eta}\to x,K}$ are satisfied in the situation, called local, where the scheme X is assumed to be an integral strictly Henselian local scheme of dimension $\leq c$ with closed point x and where $\overline{\eta}$ is above the generic point of X. Finally, we will say that the property $(\mathrm{C})^{\leq c}_{\text{loc.,normal},\Lambda}$ is satisfied if we further assume that X is normal and that $K = \Lambda$.

We thus need to establish the property $(\mathrm{C})^{\leq c}$ for all $c \geq 0$. According to remark 1.4.2, we can consider that the property $(\mathrm{C})^{\leq 1}$ is known. Furthermore, note that theorem 3.1.1 asserts in particular $(\mathrm{C})^{\leq c}_{\text{loc.,normal},\Lambda}$ for all $c \geq 0$. The results of subsection 3.4 show that $(\mathrm{C})^{\leq 2}_{\text{loc.,normal},\Lambda}$ is satisfied.

Proposition 3.5.4. *Let $f : X' \to X$ be a finite morphism between excellent Noetherian $\mathbf{Z}\left[\frac{1}{n}\right]$-schemes. Let $\overline{y'} \to \overline{x'}$ be a specialization of geometric points of X' of codimension c above a specialization $\overline{y} \to \overline{x}$ of geometric points of X (cf. definition 2.3.1). For any factorization $\overline{x'_0} \to \ldots \overline{x'_n}$ of $\overline{y'} \to \overline{x'}$ into a sequence of specializations* *of codimension* 0 *or* 1*, if we denote by* $\overline{x_0} \to \dots \overline{x_n}$ *the factorization of* $\overline{y} \to \overline{x}$ *below the preceding one, for any* $K \in \mathsf{D}^+(X_{\text{ét}}, \Lambda)$ *and* $i \in \mathbf{Z}$*, the following diagram commutes :*

$$\begin{tikzcd}
\mathrm{H}^i_{\overline{y'}}(f^!K) \arrow[r, dash, "\mathrm{sp}^{X'}_{\overline{x'_0} \to \dots \to \overline{x'_n}}"] \arrow[d, dash, "\mathrm{Cl}_{y' \to y}"', "\sim"] & \mathrm{H}^{i+2c}_{\overline{x'}}(f^!K(c)) \arrow[d, dash, "\sim"', "\mathrm{Cl}_{x' \to x}"] \\
\mathrm{H}^i_{\overline{y}}(K) \arrow[r, dash, "\mathrm{sp}^{X}_{\overline{x_0} \to \dots \to \overline{x_n}}"] & \mathrm{H}^{i+2c}_{\overline{x}}(K(c))
\end{tikzcd}$$

Furthermore, we have the equivalence $(\mathrm{C})^c_{\overline{y} \to \overline{x}, K} \iff (\mathrm{C})^c_{\overline{y'} \to \overline{x'}, f^!K}$.

The commutativity of the diagram follows immediately from proposition 3.3.1. The stated equivalence results from the commutativity of the diagram and from the fact that it is essentially equivalent to give a factorization of the specialization $\overline{y'} \to \overline{x'}$ or to give one of $\overline{y} \to \overline{x}$ (cf. propositions 2.3.2 and 2.3.3).

Lemme 3.5.5. *For any integer* $c \geq 0$*, we have the equivalence* $(\mathrm{C})^{\leq c} \iff (\mathrm{C})^{\leq c}_{\text{loc.}}$.

Assume $(\mathrm{C})^{\leq c}_{\text{loc.}}$ and let us show $(\mathrm{C})^{\leq c}$. It is obviously sufficient to show $(\mathrm{C})^{c'}_{\overline{y} \to \overline{x}, K}$ for any $K \in \mathsf{D}^+(X_{\text{ét}}, \Lambda)$, with X a local strictly Henselian scheme with closed point $\overline{x}$ and a specialization $\overline{y} \to \overline{x}$ of codimension $c' \leq c$. We need to check that we can assume that X is integral and that $\overline{y}$ is above the generic point. For this, we introduce the closed immersion $i \colon Z \to X$ where $Z = \overline{\{y\}}$. Proposition 3.5.4 shows that $(\mathrm{C})^{c'}_{\overline{y} \to \overline{x}, i^!K}$ implies $(\mathrm{C})^{c'}_{\overline{y} \to \overline{x}, K}$. To conclude, it is sufficient to observe that $(\mathrm{C})^{c'}_{\overline{y} \to \overline{x}, i^!K}$ is a particular case of $(\mathrm{C})^{\leq c}_{\text{loc.}}$.

Lemme 3.5.6. *Let X be an integral strictly Henselian local excellent Noetherian* $\mathbf{Z}\left[\frac{1}{n}\right]$*-scheme of dimension* c*, with closed point* x *and generic point* η*. Let* $\overline{\eta}$ *be a geometric point above* η*. Let* $K \in \mathsf{D}^+(X_{\text{ét}}, \Lambda)$*.*

(a) $(\mathrm{C})^c_{\overline{\eta} \to x, \tau_{\leq 0}K} \Longrightarrow (\mathrm{C})^c_{\overline{\eta} \to x, K}$*;*
(b) $(\mathrm{C})^c_{\overline{\eta} \to x, \tau_{\leq 0}K} \iff (\mathrm{C})^c_{\overline{\eta} \to x, \mathscr{H}^0 K}$*;*
(c) *Let* $j \colon U \to X$ *be the inclusion of a dense open set and* $\mathscr{M}$ *and* $\mathscr{N}$ *be two sheaves of* Λ*-modules on* $X_{\text{ét}}$ *such that* $j^\star \mathscr{M} \simeq j^\star \mathscr{N}$*, then* $(\mathrm{C})^c_{\overline{\eta} \to x, \mathscr{M}} \iff (\mathrm{C})^c_{\overline{\eta} \to x, \mathscr{N}}$.

Implication (a) results from the fact that the canonical morphism $\tau_{\leq 0}K \to K$ induces an isomorphism $\mathrm{H}^0_{\overline{\eta}}(\tau_{\leq 0}K) \xrightarrow{\sim} \mathrm{H}^0_{\overline{\eta}}(K)$.

We then consider the canonical morphism $\tau_{\leq 0}K \to \mathscr{H}^0 K$. It obviously induces an isomorphism after application of $\mathrm{H}^0_{\overline{\eta}}$, but also after application of H^{2c}_x, for cohomological dimension reasons (cf. proposition 3.2.1). Equivalence (b) follows immediately.

To show (c), we can assume that $\mathscr{N} = j_! j^\star \mathscr{M}$. We then have a canonical monomorphism $j_! j^\star \mathscr{M} \to \mathscr{M}$, whose cokernel is $i_\star i^\star \mathscr{M}$, where $i \colon Z \to X$ denotes a complementary closed immersion. Proposition 3.2.1 then shows that $\mathrm{H}^q_{\text{ét}}(X - x, (i_\star i^\star \mathscr{M})_{|X-x}) = 0$ for $q \geq 2c - 2$. Thus, the morphism $j_! j^\star \mathscr{M} \to \mathscr{M}$ induces an isomorphism not only after application of $\mathrm{H}^0_{\overline{\eta}}$, but also of $\mathrm{H}^{2c}_{x,\text{ét}}$, which allows us to conclude.

Lemme 3.5.7. *For any integer* $c \geq 0$*, we have the implication* $(\mathrm{C})^{\leq c}_{\text{loc.,normal},\Lambda} \Longrightarrow (\mathrm{C})^{\leq c}_{\text{loc.}}$.

Let X be an integral strictly Henselian local excellent Noetherian $\mathbf{Z}\left[\frac{1}{n}\right]$-scheme of dimension $c' \leq c$, with closed point x and generic point η. Let $\overline{\eta}$ be a geometric point above η. According to lemma 3.5.6, to show $(\mathrm{C})^{c'}_{\overline{\eta} \to x, K}$ for any $K \in \mathsf{D}^+(X_{\text{ét}}, \Lambda)$, it is sufficient to show $(\mathrm{C})^{c'}_{\overline{\eta} \to x, \mathscr{M}}$ for any sheaf of Λ-modules $\mathscr{M}$ on X. We can obviously assume that $\mathscr{M}$ is constructible. But then, there exists a dense open set of X over which $\mathscr{M}$ is locally constant. There obviously exists a finite surjective morphism $f \colon \tilde{X} \to X$, with $\tilde{X}$ normal and integral, and a dense open set U of X such that f induces a finite étale covering

$\tilde{U} = f^{-1}(U) \to U$ and that $(f^\star\mathscr{M})_{|\tilde{U}}$ is isomorphic to a constant sheaf of value some Λ-module N. Let us choose a specialization $\overline{\eta} \to \tilde{x}$ above $\overline{\eta} \to x$. The property $(\mathrm{C})^{\leq c}_{\mathrm{loc.,normal},\Lambda}$ contains $(\mathrm{C})^{c'}_{\overline{\eta}\to\tilde{x},\Lambda}$ as a particular case. This last property in turn implies the property $(\mathrm{C})^{c'}_{\overline{\eta}\to\tilde{x},N}$. We observe that we have a canonical isomorphism $(f^!\mathscr{M})_{|\tilde{U}} \simeq N$. Assertion (c) of lemma 3.5.6 allows us to obtain the property $(\mathrm{C})^{c'}_{\overline{\eta}\to\tilde{x},\mathscr{H}^0(f^!\mathscr{M})}$, and assertions (a) and (b) to deduce $(\mathrm{C})^{c'}_{\overline{\eta}\to\tilde{x},f^!\mathscr{M}}$. Finally, proposition 3.5.4 allows us to verify $(\mathrm{C})^{c'}_{\overline{\eta}\to x,\mathscr{M}}$, which completes the proof of the lemma.

Lemme 3.5.8. *For any integer $c \geq 3$, we have the implication* $(\mathrm{C})^{\leq c-1} \Longrightarrow (\mathrm{C})^{\leq c}_{\mathrm{loc.,normal},\Lambda}$.

Let $X = \mathrm{Spec}(A)$ be a normal strictly Henselian local excellent Noetherian scheme of dimension c, with closed point x and generic point η. We choose a geometric point $\overline{\eta}$ above η. We denote by $s\colon\ \overline{\eta} \to x$ the canonical specialization. We will show the property $(\mathrm{C})^{c}_{s,\Lambda}$.

Let $\overline{z} \to X$ be a geometric point such that $z \neq x$ and $z \neq \eta$. We have a canonical specialization $t\colon\ \overline{z} \to x$. We can choose a specialization $u\colon\ \overline{\eta} \to \overline{z}$. We may not obtain a factorization of the specialization $\overline{\eta} \to x$ fixed above, but there certainly exists $\sigma \in \mathrm{Gal}(\overline{\eta}/\eta)$ making the following diagram of specializations of geometric points of X commute :

$$\begin{tikzcd} \overline{\eta} \arrow[r, "s"] & x \\ \overline{\eta} \arrow[u, "\sigma", "\sim"'] \arrow[r, "u"] & \overline{z} \arrow[u, "t"'] \end{tikzcd}$$

The specializations σ, u and t have codimension $\leq c-1$, so the associated transition morphisms are well-defined. We can consider the image of $1 \in \mathrm{H}^0_{\overline{\eta}}(X,\Lambda)$ by the composite of these transition morphisms :

$$\gamma_{\sigma,u,t} := (\mathrm{sp}^X_t \circ \mathrm{sp}^X_u \circ \mathrm{sp}^X_\sigma)(1) \in \mathrm{H}^{2c}_{x,\text{ét}}(X,\Lambda(c))\ .$$

Of course, we have $\mathrm{sp}^X_\sigma(1) = 1$. We thus simply denote by $\gamma_{u,t}$ the element $\gamma_{\sigma,u,t} = (\mathrm{sp}^X_t \circ \mathrm{sp}^X_u)(1)$. I claim that this class $\gamma_{u,t}$ depends only on the point z of X below which $\overline{z}$ lies. Indeed, let $\overline{z'}$ be another geometric point above z, and $t'\colon\ \overline{z'} \to x$ the canonically associated specialization. We choose a specialization $u'\colon\ \overline{\eta} \to \overline{z'}$. We can choose a z-isomorphism $\sigma'\colon\ \overline{z} \xrightarrow{\sim} \overline{z'}$. The scheme X being geometrically unibranch, we can easily show that there exists an η-isomorphism $\sigma''\colon\ \overline{\eta} \to \overline{\eta}$ inducing a commutative diagram of specializations of geometric points of X :

$$\begin{tikzcd} \overline{\eta} \arrow[r, "u"] \arrow[d, "\sigma''", "\sim"'] & \overline{z} \arrow[r, "t"] \arrow[d, "\sigma'", "\sim"'] & x \arrow[d, equal] \\ \overline{\eta} \arrow[r, "u'"] & \overline{z'} \arrow[r, "t'"] & x \end{tikzcd}$$

By using this time that σ'' acts trivially on $1 \in \mathrm{H}^0_{\overline{\eta}}(X,\Lambda)$, we show that $\gamma_{u,t} = \gamma_{u',t'}$. We can thus simply denote this class by γ_z.

We need to show that the class γ_z is independent of $z \in X - \{\eta, x\}$. By construction, if $z' \in X - \{\eta, x\}$ is such that there exists a (Zariski) specialization $z \to z'$, we have $\gamma_z = \gamma_{z'}$.

Since A is a normal excellent local ring of dimension ≥ 3, according to [**SGA 2** XIII 2.1] applied to the completion of A, if $f \in A - \{0\}$, the scheme $\mathrm{Spec}\,(A/(f)) - \{x\}$ is connected. It immediately follows that the classes γ_z for $z \in \mathrm{Spec}\,(A/(f)) - \{x\}$ are equal. Now, if z and z' are two elements of $X - \{\eta, x\}$, there obviously exists $f \in A - \{0\}$ such that z and z' belong to the hypersurface defined by f. We thus indeed obtain $\gamma_z = \gamma_{z'}$, which completes the proof of the lemma.

We are now in a position to prove theorem 3.1.2. We need to establish the property $(\mathrm{C})^{\leq c}$ for all $c \geq 0$. It is equivalent to the property $(\mathrm{C})^{\leq c}_{\mathrm{loc.}}$ according to lemma 3.5.5. According to lemma 3.5.7, it is also equivalent to the property $(\mathrm{C})^{\leq c}_{\mathrm{loc.,normal},\Lambda}$. Given these equivalences, lemma 3.5.8 shows that for

$c \geq 3$, $(\mathrm{C})^{\leq c-1}_{\mathrm{loc.,normal},\Lambda}$ implies $(\mathrm{C})^{\leq c}_{\mathrm{loc.,normal},\Lambda}$. An argument by induction thus allows us to conclude since the property $(\mathrm{C})^{\leq 2}_{\mathrm{loc.,normal},\Lambda}$ was established above.

3.6. End of the proof.

Proposition 3.6.1. *Let X be a normal Henselian local excellent Noetherian $\mathbf{Z}\left[\frac{1}{n}\right]$-scheme of dimension d, with closed point x and generic point η. Let $\overline{x}$ be a geometric point above x. Let $\overline{\eta}$ be a geometric point above η. Let $\overline{\eta} \to \overline{x}$ be a specialization. Then, the image of 1 by the transition morphism $\mathrm{sp}^X_{\overline{\eta}\to\overline{x}} : \mathrm{H}^0_{\overline{\eta}}(X,\Lambda) \to \mathrm{H}^{2d}_{\overline{x}}(X,\Lambda(d))$ is invariant under $\mathrm{Gal}(\overline{x}/x)$, and independent of the chosen specialization $\overline{\eta} \to \overline{x}$; we denote it by $[x]$. We thus obtain a morphism $\Lambda[-2d] \to \tau_{\geq 2d}\mathrm{R}\Gamma_x(\Lambda(d))$ in $\mathrm{D}(x_{\text{ét}},\Lambda)$.*

This follows immediately from theorem 3.1.2 and the fact that $1 \in \mathrm{H}^0_{\overline{\eta}}(X,\Lambda)$ is fixed by $\mathrm{Gal}(\overline{\eta}/\eta)$.

By construction, the classes $[x]$ are compatible with transition morphisms. To finish proving theorem 3.1.1, it remains to show that in the situation of the preceding proposition, the morphism $\Lambda[-2d] \to \tau_{\geq 2d}\mathrm{R}\Gamma_x(\Lambda(d))$ is an isomorphism, or equivalently, in the situation where X is strictly Henselian, that the morphism $\Lambda \to \mathrm{H}^{2d}_{x,\text{ét}}(X,\Lambda(d))$ induced by $[x]$ is an isomorphism. Indeed, cohomological dimension considerations (cf. proposition 3.2.1) then explain the vanishing of $\mathrm{H}^q_{x,\text{ét}}(X,\Lambda(d))$ for $q > 2d$.

Lemme 3.6.2. *Let $d \geq 3$. Assume the result of theorem 3.1.1 is known up to dimension $d-1$. Then, for any normal strictly Henselian local excellent Noetherian $\mathbf{Z}\left[\frac{1}{n}\right]$-scheme X of dimension d, with closed point x, the morphism $\Lambda \to \mathrm{H}^{2d}_{x,\text{ét}}(X,\Lambda(d))$ induced by $[x]$ is surjective.*

We use the coniveau spectral sequence calculating the cohomology of the open set $U = X - x$ with coefficients in $\Lambda(d)$:

$$E_1^{pq} = \bigoplus_{y\in U^p} \mathrm{H}^{p+q}_{y,\text{ét}}(X_{(y)},\Lambda(d)) \Longrightarrow \mathrm{H}^{p+q}_{\text{ét}}(U,\Lambda(d)) ,$$

where U^p denotes the set of points of codimension p in U. The étale topology being finer than the Nisnevich topology, in the expression for the E_1 term, we can replace the local schemes $X_{(y)}$ by their Henselizations $X^{\mathrm{h}}_{(y)}$ for all $y \in U^p$ with $0 \leq p \leq d-1$. We thus obtain a canonical isomorphism :

$$\mathrm{R}_y(X_{(y)},\Lambda(d)) \simeq \mathrm{R}(y,\mathrm{R}\Gamma_y(\Lambda(p))(d-p)) .$$

The structure of $\tau_{\geq 2p}\mathrm{R}\Gamma_y(\Lambda(p))$ is known by our limited knowledge of theorem 3.1.1 since $p \leq d-1$. According to proposition 3.2.2, we have an upper bound for the ℓ-cohomological dimension of y for any prime number ℓ dividing n : $\mathrm{cd}_\ell\, y \leq d-p$. By applying the composition of derived functors spectral sequence to the calculation of $\mathrm{R}_y(X_{(y)},\Lambda(d))$, we obtain on the one hand that $\mathrm{H}^i_{y,\text{ét}}(X_{(y)},\Lambda(d)) = 0$ for $i > d+p$, that is, $\mathrm{H}^{p+q}_{y,\text{ét}}(X_{(y)},\Lambda(d)) = 0$ for $q > d$, and on the other hand that $\mathrm{H}^{p+d}_{y,\text{ét}}(X_{(y)},\Lambda(d)) \simeq \mathrm{H}^{d-p}_{\text{ét}}(y,\Lambda(d-p))$.

We have thus shown that $E_1^{p,q} = 0$ for $q \geq d+1$, and furthermore, it is obvious that $E_1^{p,q} = 0$ if $p \geq d$. We immediately deduce that we have canonical isomorphisms $\mathrm{Coker}(E_1^{d-2,d} \to E_1^{d-1,d}) \xrightarrow{\sim} \mathrm{H}^{2d-1}_{\text{ét}}(U,\Lambda(d)) \xrightarrow{\sim} \mathrm{H}^{2d}_{x,\text{ét}}(X,\Lambda(d))$. In particular, the canonical morphism $E_1^{d-1,d} \to \mathrm{H}^{2d}_{x,\text{ét}}(X,\Lambda(d))$ is surjective. According to the calculation above, we know the structure of $E_1^{d-1,d}$:

$$E_1^{d-1,d} \simeq \bigoplus_{y\in U^{d-1}} \mathrm{H}^1_{\text{ét}}(y,\Lambda(1)) .$$

For any $y \in U^{d-1}$, we have an evident isomorphism $\mathrm{H}^1_{\text{ét}}(y,\Lambda(1)) \simeq \Lambda$ (induced by an immediate specialization $\overline{y} \to x$ of geometric points of X). It is not difficult to check that the image of the canonical morphism $\Lambda \simeq \mathrm{H}^1_{\text{ét}}(y,\Lambda(1)) \subset E_1^{d-1,d} \to \mathrm{H}^{2d}_{x,\text{ét}}(X,\Lambda(d))$ is the subgroup generated by $[x]$. Consequently, the image of the surjective morphism $E_1^{d-1,d} \to \mathrm{H}^{2d}_{x,\text{ét}}(X,\Lambda(d))$ is the subgroup generated by $[x]$, which completes the proof of the lemma.

Lemme 3.6.3. *Let $d \geq 3$. Assume the result of theorem 3.1.1 is known up to dimension $d-1$. Then, for any normal strictly Henselian local excellent Noetherian $\mathbf{Z}\left[\frac{1}{n}\right]$-scheme X of dimension d, with closed point x, the morphism $\Lambda \to \mathrm{H}^{2d}_{x,\text{ét}}(X, \Lambda(d))$ induced by $[x]$ is an isomorphism, in other words, the statement of theorem 3.1.1 holds up to dimension d.*

By virtue of lemma 3.6.2, it is now only a matter of determining the structure of the Λ-module $\mathrm{H}^{2d}_{x,\text{ét}}(X, \Lambda(d))$. Given the formal base change theorem (cf. [**Fujiwara, 1995**, corollary 6.6.4]), we can assume that X is complete. The structure theorems of complete Noetherian local rings (cf. [**ÉGA** 0_{IV} 19.8.8]) show that X is then isomorphic to a closed subscheme of a regular scheme. Let $i\colon X \to Y$ be a closed immersion of X into a regular excellent scheme Y. Given what we already know about potential dualizing complexes on regular schemes, there exists a potential dualizing complex for $(Y, -\operatorname{codim})$ thanks to proposition 2.4.4.1 and we can deduce the existence of a potential dualizing complex on X for the induced dimension function by using for example the construction of proposition 2.4.3.1. Up to shifting and twisting, if we denote by δ the dimension function on X such that $\delta(\eta) = 0$ where η is the generic point of X, we obtain that there exists a dualizing complex K for (X, δ). We will use the following general lemma :

Lemme 3.6.4. *Let X be an integral normal excellent Noetherian $\mathbf{Z}\left[\frac{1}{n}\right]$-scheme with generic point η. Assume X is equipped with the dimension function δ such that $\delta(\eta) = 0$. Let K be a potential dualizing complex for (X, δ). Then, the cohomology sheaves $\mathscr{H}^q K$ are zero for $q < 0$ and the pinning at η extends to an isomorphism $\mathscr{H}^0 K \simeq \Lambda$. In other words, we have a canonical isomorphism $\Lambda \xrightarrow{\sim} \tau_{\leq 0} K$.*

To obtain the result for X, it is sufficient to have it for its strict Henselizations. We can thus assume that X is strictly Henselian of dimension d and closed point x. We proceed by induction on d. If $d = 0$, the result is evident. We thus assume that $d \geq 1$ and that the result is known for the open set $U = X - x$. Let $j\colon U \to X$ and $i\colon x \to X$ be the evident immersions. We have a distinguished triangle in $\mathrm{D}^+(X_{\text{ét}}, \Lambda)$:

$$i_\star i^! K \to K \to \mathrm{R}j_\star j^\star K \to i_\star i^! K[1]\,.$$

Thanks to the pinning at x, it follows that $\mathscr{H}^q i^! K = 0$ for $q \leq 1$. The hypothesis on U shows that $\Lambda \xrightarrow{\sim} \tau_{\leq 0} j^\star K$. We deduce that $\mathscr{H}^q K = 0$ for $q < 0$ and that $\mathscr{H}^0 K \simeq j_\star \Lambda$. Since X is normal, the canonical morphism $\Lambda \to j_\star \Lambda$ is an isomorphism, which completes the proof of the lemma.

Let us return to the proof of lemma 3.6.3. We consider the canonical morphism $\Lambda \to K$ deduced from lemma 3.6.4. Let $\overline{\eta}$ be a geometric point above η. Let us choose a specialization $\overline{\eta} \to x$. We consider the following commutative diagram, where the vertical arrows are induced by $\Lambda \to K$ and the horizontal arrows by the transition morphisms associated with the specialization $\overline{\eta} \to x$:

$$\begin{tikzcd}
\mathrm{H}^0_{\overline{\eta}}(\Lambda) \arrow[r] \arrow[d] & \mathrm{H}^{2d}_{x,\text{ét}}(X, \Lambda(d)) \arrow[d] \\
\mathrm{H}^0_{\overline{\eta}}(K) \arrow[r] & \mathrm{H}^{2d}_{x,\text{ét}}(X, K(d))
\end{tikzcd}$$

The left morphism is obviously an isomorphism. The bottom one is too since K is a potential dualizing complex. The top morphism is therefore injective, but we already know it is surjective. The Λ-module $\mathrm{H}^{2d}_{x,\text{ét}}(X, \Lambda(d))$ is therefore isomorphic to Λ, which allows us to conclude.

4. Complements on potential dualizing complexes

4.1. Statements. Thanks to the results of section 3, we will be able to continue the study of certain properties of potential dualizing complexes. The main result of section 2 was the construction of a potential dualizing complex on regular schemes (cf. proposition 2.4.1.1). The two following propositions establish stability properties of potential dualizing complexes with respect to morphisms of finite type and regular morphisms [**ÉGA** IV 6.8.1].

Proposition 4.1.1. *Let $f\colon Y \to X$ be a regular morphism between excellent Noetherian $\mathbf{Z}\left[\frac{1}{n}\right]$-schemes. Assume X is equipped with a dimension function δ_X. We equip Y with the dimension function δ_Y defined by the equality $\delta_Y(y) = \delta_X(f(y)) - \mathrm{codim}_{f^{-1}(f(y))}(y)$ for all $y \in Y$ (cf. **XIV**-2.5.4).*

If K is a putative dualizing complex on X, then $f^\star K$ is naturally equipped with a structure of a putative dualizing complex on Y, and it is a potential dualizing complex if K is one.

Proposition 4.1.2. *Let $f\colon Y \to X$ be a compactifiable morphism of finite type between excellent Noetherian $\mathbf{Z}\left[\frac{1}{n}\right]$-schemes. Assume X is equipped with a dimension function δ_X. We equip Y with the dimension function δ_Y defined by the equality $\delta_Y(y) = \delta_X(f(y)) + \mathrm{deg.\,tr.}(y/f(y))$ for all $y \in Y$ (cf. **XIV**-2.5.2).*

If K is a putative dualizing complex on X, then $f^! K$ is naturally equipped with a structure of a putative dualizing complex on Y, and it is a potential dualizing complex if K is one.

Remarque 4.1.3. During the proofs, we will observe that the constructions of propositions 4.1.1 and 4.1.2 are compatible with the composition of morphisms : if $g\colon Z \to Y$ is another morphism of the type considered and K is a putative dualizing complex on X, the transitivity isomorphism $g^\star f^\star K \simeq (f\circ g)^\star K$ (resp. $g^! f^! K \simeq (f\circ g)^! K$) is compatible with the pinnings. Proposition 4.1.1 generalizes the construction of proposition 2.2.1 (case where f is étale).

We will see below that proposition 4.1.2 results easily from proposition 4.1.1. We will therefore focus more particularly on regular morphisms. The following subsection on the base change theorem for a regular morphism will allow us to define a structure of a putative dualizing complex on $f^\star K$ in the situation of proposition 4.1.1. In the case of a potential dualizing complex, it is the results of section 3 that will allow us to check that the pinnings on $f^\star K$ are compatible with specializations.

4.2. Base change for a regular morphism. In this section, we study some consequences of the base change theorem for a regular morphism **XIV**-2.5.3, whose statement we recall :

Proposition 4.2.1. *Let $f\colon Y \to X$ be a regular morphism between locally Noetherian $\mathbf{Z}\left[\frac{1}{n}\right]$-schemes. Let $g\colon X' \to X$ be a quasi-compact and quasi-separated morphism. We form the Cartesian square of schemes :*

$$\begin{tikzcd} Y' \arrow[r, "f'"] \arrow[d, "g'"] & X' \arrow[d, "g"] \\ Y \arrow[r, "f"] & X \end{tikzcd}$$

Then, for any $K \in \mathsf{D}^+(X'_{\text{ét}}, \Lambda)$, the base change morphism $f^\star \mathrm{R}g_\star K \to \mathrm{R}g'_\star f'^\star K$ is an isomorphism in $\mathsf{D}^+(Y_{\text{ét}}, \Lambda)$.

Proposition 4.2.2. *Let $f\colon Y \to X$ be a regular morphism between Noetherian $\mathbf{Z}\left[\frac{1}{n}\right]$-schemes. Let $K \in \mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(X_{\text{ét}}, \Lambda)$. Let $L \in \mathsf{D}^+(X_{\text{ét}}, \Lambda)$. Then, the evident morphism is an isomorphism in $\mathsf{D}^+(Y_{\text{ét}}, \Lambda)$:*

$$f^\star \mathrm{R}\,\mathbf{Hom}(K, L) \xrightarrow{\sim} \mathrm{R}\,\mathbf{Hom}(f^\star K, f^\star L)\ .$$

Before proving it, let us mention a corollary of this proposition (case where $K := i_\star \Lambda$ and $L := M$) :

Corollaire 4.2.3. *Let $f\colon Y \to X$ be a regular morphism between Noetherian $\mathbf{Z}\left[\frac{1}{n}\right]$-schemes. Let $i\colon Z \to X$ be a closed immersion. Let $i'\colon f^{-1}(Z) \to Y$ be the closed immersion deduced by base change, and $f'\colon f^{-1}(Z) \to Z$ be the projection. Then, for any $M \in \mathsf{D}^+(X_{\text{ét}}, \Lambda)$, the evident morphism $f'^\star i^! M \to i'^! f^\star M$ is an isomorphism in $\mathsf{D}^+(f^{-1}(Z)_{\text{ét}}, \Lambda)$.*

To prove proposition 4.2.2, let us start by observing that this corollary is a particular case of proposition 4.2.1 (apply it with g being the complementary open immersion of i).

Lemme 4.2.4. *Let $f: Y \to X$ be a regular morphism between Noetherian $\mathbf{Z}\left[\frac{1}{n}\right]$-schemes. Let $i: Z \to X$ be a closed immersion. We form the following Cartesian square :*

$$\begin{tikzcd} Z' \arrow[r, "i'"] \arrow[d, "f'"] & Y \arrow[d, "f"] \\ Z \arrow[r, "i"] & X \end{tikzcd}$$

Let $K \in \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(Z_{\text{ét}}, \Lambda)$. Assume that for any $L \in \mathrm{D}^{+}(Z_{\text{ét}}, \Lambda)$, the canonical morphism $f'^{\star}\mathrm{R}\,\mathbf{Hom}(K, L) \to \mathrm{R}\,\mathbf{Hom}(f'^{\star}K, f'^{\star}L)$ is an isomorphism. Then, for any $L \in \mathrm{D}^{+}(X_{\text{ét}}, \Lambda)$, the canonical morphism $f^{\star}\mathrm{R}\,\mathbf{Hom}(i_{\star}K, L) \to \mathrm{R}\,\mathbf{Hom}(f^{\star}i_{\star}K, f^{\star}L)$ is an isomorphism.

This lemma follows immediately from corollary 4.2.3.

Lemme 4.2.5. *Let $f: Y \to X$ be a regular morphism between Noetherian $\mathbf{Z}\left[\frac{1}{n}\right]$-schemes. Let $j: U \to X$ be an open immersion. We form the following Cartesian square :*

$$\begin{tikzcd} U' \arrow[r, "j'"] \arrow[d, "f'"] & Y \arrow[d, "f"] \\ U \arrow[r, "j"] & X \end{tikzcd}$$

Let $K \in \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(U_{\text{ét}}, \Lambda)$. Assume that for any $L \in \mathrm{D}^{+}(U_{\text{ét}}, \Lambda)$, the canonical morphism $f'^{\star}\mathrm{R}\,\mathbf{Hom}(K, L) \to \mathrm{R}\,\mathbf{Hom}(f'^{\star}K, f'^{\star}L)$ is an isomorphism. Then, for any $L \in \mathrm{D}^{+}(X_{\text{ét}}, \Lambda)$, the canonical morphism $f^{\star}\mathrm{R}\,\mathbf{Hom}(j_{!}K, L) \to \mathrm{R}\,\mathbf{Hom}(f^{\star}j_{!}K, f^{\star}L)$ is an isomorphism.

This lemma results from the isomorphism $f^{\star}\mathrm{R}j_{\star} \simeq \mathrm{R}j'_{\star}f'^{\star}$ which is a particular case of proposition 4.2.1.

Let us prove proposition 4.2.2. Lemmas 4.2.4 and 4.2.5 allow us to assume that K is a locally constant constructible sheaf of Λ-modules on X. By using that the result is trivial if f is étale, we are reduced to the case where K is a constant sheaf of value M where M is a finitely generated Λ-module. By introducing a projective resolution (possibly infinite) of M, we can reduce to the trivial case where M is a free of finite rank over Λ.

4.3. Proof of proposition 4.1.1.

4.3.1. *Structure of a putative dualizing complex on $f^{\star}K$.*

Lemme 4.3.1.1. *In the situation of proposition 4.1.1, if K is a putative dualizing complex on X, we can equip $f^{\star}K$ with a structure of a putative dualizing complex on Y.*

Let K be a putative dualizing complex on X. Let $x \in X$. Let us introduce the localization $X' = X_{(x)}$ of X at x and $Y' = Y \times_X X'$. Let $i: f^{-1}(x) \to Y'$ be the immersion of the fiber above x, $g: f^{-1}(x) \to x$ the projection and $j: Y' \to Y$ the canonical morphism. According to corollary 4.2.3, we have a canonical isomorphism $g^{\star}\mathrm{R}\Gamma_x(K) \simeq i^{!}j^{\star}f^{\star}K$. Defining pinnings on $f^{\star}K$ at the points of $f^{-1}(x)$ amounts to defining a structure of a putative dualizing complex on $i^{!}j^{\star}f^{\star}K$ for the dimension function $\delta_{Y|f^{-1}(x)}$. We have just seen that this object is identified with $g^{\star}\mathrm{R}\Gamma_x(K)$ which is itself identified with $\Lambda(\delta_X(x))[2\delta_X(x)]$ by virtue of the given pinning of K at x. The fiber $f^{-1}(x)$ being regular, the constant sheaf Λ is naturally equipped with a structure of a potential dualizing complex for the dimension function $-\operatorname{codim}$ (cf. proposition 2.4.4.1). Given the definition of δ_Y, we immediately deduce a structure of a potential dualizing complex on $\Lambda(\delta_X(x))[2\delta_X(x)] \in \mathrm{D}^{+}(f^{-1}(x)_{\text{ét}}, \Lambda)$ for the dimension function $\delta_{Y|f^{-1}(x)}$. We have thus obtained pinnings for $f^{\star}K$ at all points of $f^{-1}(x)$. Thanks to this fiber-by-fiber construction, we have defined a structure of a putative dualizing complex on $f^{\star}K$.

Lemme 4.3.1.2. *Let $f\colon Y \to X$ be a regular morphism between regular excellent Noetherian $\mathbf{Z}\left[\frac{1}{n}\right]$-schemes. We equip X and Y with the dimension functions $\delta_X = -\operatorname{codim}$ and $\delta_Y = -\operatorname{codim}$. We equip the constant sheaves Λ on X and Y with the structures of putative dualizing complexes defined in proposition 2.4.1.1; lemma 4.3.1.1 equips $f^\star\Lambda$ with a structure of a putative dualizing complex on Y for the dimension function δ_Y. Then, the canonical isomorphism $\Lambda \simeq f^\star\Lambda$ on Y is compatible with the pinnings.*

Let us check the compatibility of the canonical isomorphism $\Lambda \simeq f^\star\Lambda$ with the pinnings at a point y of Y. Up to localizing, we can assume that X is local with closed point $x = f(y)$ and that y is a closed point in the fiber $F = f^{-1}(y)$. Let $c = \operatorname{codim}_X x$ and $d = \operatorname{codim}_F y$. The pinning of $f^\star\Lambda$ at y is given by the product of the classes $f^\star(\mathrm{Cl}_{x\to X}) \in \mathrm{H}^{2c}_{F,\text{ét}}(Y,\Lambda(c))$ and $\mathrm{Cl}_{y\to F} \in \mathrm{H}^{2d}_{y,\text{ét}}(F,\Lambda(d))$, this product living in $\mathrm{H}^{2c+2d}_{y,\text{ét}}(Y,\Lambda(c+d))$. The compatibility of Gysin classes with base change (cf. **XVI**-2.3.2) implies that $f^\star(\mathrm{Cl}_{x\to X})$ is the Gysin class $\mathrm{Cl}_{F\to Y} \in \mathrm{H}^{2c}_{F,\text{ét}}(Y,\Lambda(c))$. Using the compatibility of Gysin classes with composition, the product considered above is $\mathrm{Cl}_{y\to Y}$, which is precisely the class that defines the pinning of Λ at y.

Remarque 4.3.1.3. Assume we have two composable regular morphisms $Z \xrightarrow{g} Y \xrightarrow{f} X$ between excellent Noetherian $\mathbf{Z}\left[\frac{1}{n}\right]$-schemes. If K is a putative dualizing complex on X, then applying the construction of lemma 4.3.1.1 to f, then to g, is the same as applying it directly to $f \circ g$, in other words the evident isomorphism $g^\star f^\star K \simeq (f\circ g)^\star K$ in $\mathrm{D}^+(Z_{\text{ét}},\Lambda)$ is compatible with the pinnings. Indeed, the constructions being performed fiber by fiber, we can assume that X is the spectrum of a field and, up to modifying the dimension function, that $K = \Lambda$; in particular, X, Y and Z are regular, the constant sheaves Λ on X, Y and Z are naturally equipped with structures of putative dualizing complexes (and even potential ones, cf. proposition 2.4.4.1) for the considered dimension functions, the required compatibility is obtained by applying lemma 4.3.1.2 to the morphisms f, g and $f\circ g$.

4.3.2. *Compatibility with specializations.* Assume now that K is a potential dualizing complex. By definition of potential dualizing complexes, we now need to show that the pinnings on $f^\star K$ are compatible with the transition morphisms associated with immediate specializations. According to theorem 3.1.2, we can give a meaning to this compatibility for any specialization $\overline{y'} \to \overline{y}$ of geometric points of Y, whatever its codimension; this is the object of the following definition :

Définition 4.3.2.1. Let L be a putative dualizing complex on an excellent Noetherian $\mathbf{Z}\left[\frac{1}{n}\right]$-scheme Y equipped with a dimension function δ. Let $\overline{y'} \to \overline{y}$ be a specialization of geometric points of Y. We say that the pinnings of L are **compatible with the specialization** $\overline{y'} \to \overline{y}$ if the following diagram is commutative :

$$\begin{tikzcd}
\mathrm{H}^{-2\delta(y')}_{\overline{y'}}(L(-\delta(y'))) \arrow[r, "\mathrm{sp}^X_{\overline{y'}\to\overline{y}}"] \arrow[dr, "\sim"'] & \mathrm{H}^{-2\delta(y)}_{\overline{y}}(L(-\delta(y))) \arrow[d, "\sim"] \\
& \Lambda
\end{tikzcd}$$

(The isomorphisms with Λ are induced by the pinnings at y and y'.)

It is evident that a putative dualizing complex is a potential dualizing complex (cf. definition 2.1.2) if and only if its pinnings are compatible with immediate specializations in the sense above and if this is the case, the construction of theorem 3.1.2 shows that they are compatible with all specializations.

Lemme 4.3.2.2. *Let us place ourselves in the situation of proposition 4.1.1. Let K be a potential dualizing complex on X. Let $\overline{y'} \to \overline{y}$ be a specialization of geometric points of Y above a specialization $\overline{x'} \to \overline{x}$ of geometric points of X. If $\overline{x'} \to \overline{x}$ has codimension 0 or 1, then the pinnings on $f^\star K$ are compatible with the specialization $\overline{y'} \to \overline{y}$.*

Given that the construction of the structure of a putative dualizing complex on $f^\star K$ was carried out fiber by fiber, we can assume that X is an integral strictly Henselian local scheme, with generic point x' and closed point x. To prove the lemma, it is sufficient to show that in this case $f^\star K$ is a potential dualizing complex. We can assume that $\delta_X(x') = 0$. The scheme X having dimension ≤ 1, if we denote by $n\colon \tilde{X} \to X$ the normalization of X, the scheme $\tilde{X}$ is regular. We form the following Cartesian square :

$$\begin{tikzcd} \tilde{Y} \arrow[r, "n'"] \arrow[d, "\tilde{f}"] & Y \arrow[d, "f"] \\ \tilde{X} \arrow[r, "n"] & X \end{tikzcd}$$

The morphisms n and n' are finite surjective and radicial. For them, we have the constructions $n^!$ and $n'^!$ on putative dualizing complexes (cf. proposition 2.4.3.1). For the regular morphisms f and $\tilde{f}$, we have the constructions $f^\star$ and $\tilde{f}^\star$. We can therefore consider the putative dualizing complexes $\tilde{f}^\star n^! K$ and $n'^! f^\star K$. According to the upcoming lemma 4.3.2.3, we have a canonical isomorphism of putative dualizing complexes $\tilde{f}^\star n^! K \simeq n'^! f^\star K$. Furthermore, proposition 2.4.3.3 shows that to show that $f^\star K$ is a potential dualizing complex, it is sufficient to show that $n'^! f^\star K$ is one. Since it is identified with $\tilde{f}^\star n^! K$ and since proposition 2.4.3.3 also tells us that $n^! K$ is a potential dualizing complex, we can finally replace f by $\tilde{f}$ and assume that X is regular of dimension ≤ 1. We can then assume that $K = \Lambda$, pinned as it should be. The complex $f^\star\Lambda$ is identified with Λ which we know can be equipped with a structure of a potential dualizing complex (cf. proposition 2.4.4.1). We need to show that the pinnings on $f^\star\Lambda$ and on Λ are compatible : this is the meaning of lemma 4.3.1.2.

Lemme 4.3.2.3. *Let us place ourselves in the situation of proposition 4.1.1. Let $g\colon X' \to X$ be a finite surjective radicial morphism. We form the Cartesian square :*

$$\begin{tikzcd} Y' \arrow[r, "g'"] \arrow[d, "f'"] & Y \arrow[d, "f"] \\ X' \arrow[r, "g"] & X \end{tikzcd}$$

Let K be a putative dualizing complex on X. Then, the evident isomorphism $f'^\star g^! K \simeq g'^! f^\star K$ in $\mathsf{D}^+(Y'_{\text{ét}}, \Lambda)$ is compatible with the pinnings.

The constructions considered being carried out fiber by fiber, we can assume that X and X' are the spectra of fields denoted respectively by E and E'. The morphisms f and f' being flat with geometrically regular fibers, the schemes Y and Y' are regular. In this situation, given proposition 2.4.3.3 and the construction of lemma 4.3.1.1, it is manifest that the putative dualizing complexes $f'^\star g^! K$ and $g'^! f^\star K$ are potential dualizing complexes. To show that the evident isomorphism $f'^\star g^! K \simeq g'^! f^\star K$ is compatible with the pinnings, it is therefore sufficient to do so at the maximal points of Y'. In short, we can assume that Y is also the spectrum of a field F. Since Y' is regular and homeomorphic to Y, the scheme Y' is in turn the spectrum of a field F'. Given the construction of proposition 2.4.3.1 involving the Gysin morphisms associated with the complete intersection morphisms $Y' \to Y$ and $X' \to X$, the comparison of the two structures of putative dualizing complexes considered on Y' reduces to the equality of degrees $[F' : F] = [E' : E]$, which follows immediately from the definition of Y' : $F' = E' \otimes_E F$.

To finish the proof of proposition 4.1.1, we still need to establish compatibilities between the pinnings on $f^\star K$ and the specializations of geometric points of Y. The following lemmas on specializations will be useful to us.

Lemme 4.3.2.4. *Let $f\colon Y \to X$ be a regular morphism between Noetherian schemes. Assume that f is a local morphism between strictly Henselian local schemes. Let $\overline{x} \to X$ be a geometric point of X. Then, $f^{-1}(\overline{x}) = Y_{\overline{x}} = Y \times_X \overline{x}$ is an integral scheme.*

Let $Y_x := Y \times_X x$. Let x'/x be a finite separable extension. More generally, let us denote $Y \times_x x' := Y_x \times_x x' \simeq Y \times_X x'$. If $n \geq 2$ is an integer invertible in x, by applying proposition 4.2.1 to the constant sheaf $\mathbf{Z}/n\mathbf{Z}$, to the morphism $x' \to X$ and to the regular base change $Y \to X$, we obtain an isomorphism $H^0_{\text{ét}}(x', \mathbf{Z}/n\mathbf{Z}) \xrightarrow{\sim} H^0_{\text{ét}}(Y_{x'}, \mathbf{Z}/n\mathbf{Z})$. We deduce that $Y_{x'}$ is connected. The morphism $f : Y \to X$ being regular, the scheme $Y_{x'}$ is regular and connected, hence integral. The scheme $Y_{\overline{x}}$ being a filtered projective limit of these integral affine schemes $Y_{x'}$, we obtain that $Y_{\overline{x}}$ is integral.

Lemme 4.3.2.5. *Let $f : Y \to X$ be a regular morphism between Noetherian schemes. Let $\overline{x'} \to \overline{x}$ be a specialization of geometric points of X. Let $\overline{\eta}$ be a geometric point of Y above $\overline{x}$ such that η has codimension 0 in its fiber for f. Then, up to an isomorphism that is not necessarily unique, there exists a unique specialization $\overline{\eta'} \to \overline{\eta}$ above $\overline{x'} \to \overline{x}$ such that η' has codimension 0 in its fiber. The codimension of $\overline{\eta'} \to \overline{\eta}$ is the same as that of $\overline{x'} \to \overline{x}$.*

We can assume that f is a local morphism between local strictly Henselian schemes Y and X with respective closed points $\overline{\eta}$ and $\overline{x}$. Establishing the lemma in this precise case amounts to showing that, up to isomorphism, the geometric fiber $Y_{\overline{x'}}$ has only one geometric point above a maximal point of $Y_{\overline{x'}}$, which results from lemma 4.3.2.4. It follows from [**ÉGA** iv 6.1.2] that the codimension of the specialization of $\overline{\eta'} \to \overline{\eta}$ is the same as that of $\overline{x'} \to \overline{x}$.

Lemme 4.3.2.6. *Let $f : Y \to X$ be a regular morphism between Noetherian schemes. Let $\overline{\eta'} \to \overline{\eta}$ be a specialization of geometric points of Y above a specialization $\overline{x'} \to \overline{x}$ of geometric points of X. Assume that η and η' have codimension 0 in their fiber for f. Let $\overline{x_0} \to \ldots \to \overline{x_n}$ be a factorization of the specialization $\overline{x'} \to \overline{x}$ into a sequence of immediate specializations. Then, we can decompose $\overline{\eta'} \to \overline{\eta}$ into a sequence of immediate specializations $\overline{\eta_0} \to \ldots \to \overline{\eta_n}$ above $\overline{x_0} \to \ldots \to \overline{x_n}$, each of the points η_i being of codimension 0 in its fiber for f.*

This follows immediately from lemma 4.3.2.5.

Lemme 4.3.2.7. *Let $f : Y \to X$ be a regular morphism between Noetherian schemes. Let $\overline{y'} \to \overline{y}$ be a specialization of geometric points of Y above $\overline{x'} \to \overline{x}$. Then, there exists a commutative diagram of specializations of geometric points of Y :*

$$\begin{tikzcd}
\overline{\eta'} \arrow[r] \arrow[d] & \overline{\eta} \arrow[d] \\
\overline{y'} \arrow[r] & \overline{y}
\end{tikzcd}$$

where the geometric points $\overline{\eta'}$ and $\overline{\eta}$ are respectively above $\overline{x'}$ and $\overline{x}$, and where η' and η have codimension 0 in their fiber for f.

We can assume that X and Y are local strictly Henselian schemes with respective closed points $\overline{y}$ and $\overline{x}$. We choose a specialization $\overline{\eta} \to \overline{y}$ of geometric points of the fiber $Y_{\overline{x}}$ with η of codimension 0 in this fiber. According to lemma 4.3.2.5, there exists a specialization $\overline{\eta'} \to \overline{\eta}$ of geometric points of Y above $\overline{x'} \to \overline{x}$, with η' of codimension 0 in its fiber for f. Furthermore, the geometric fiber $Y_{\overline{x'}}$ being integral, we can choose a geometric point $\overline{\eta''}$ of it above the generic point. The geometric point $\overline{y'}$ of Y being above $\overline{x'}$, it defines a geometric point of $Y_{\overline{x'}}$; we thus have a specialization $\overline{\eta''} \to \overline{y'}$ of geometric points above $\overline{x'}$. The geometric fiber $Y_{\overline{x'}}$ being integral, there exists an isomorphism $\overline{\eta'} \simeq \overline{\eta''}$, which gives the desired commutative diagram.

4.3.3. *End of the proof.* Let us finish the proof of proposition 4.1.1. We need to show that for any potential dualizing complex K on X, the pinnings on the putative dualizing complex $f^\star K$ are compatible with the specializations of geometric points of Y. Let $\overline{y'} \to \overline{y}$ be such a specialization. Lemma 4.3.2.7 applies and we obtain a commutative diagram of specializations as above. To show the compatibility for the specialization $\overline{y'} \to \overline{y}$, it is sufficient to obtain it for the three other specializations which appear in the commutative square. For the specializations $\overline{\eta'} \to \overline{y'}$ and $\overline{\eta} \to \overline{y}$, this follows immediately from the facts observed in the very construction of lemma 4.3.1.1, namely that each fiber of f is equipped with a potential dualizing complex. We can apply lemma 4.3.2.6 to obtain a factorization of $\overline{\eta'} \to \overline{\eta}$ into a sequence of immediate specializations $\overline{\eta_0} \to \dots \to \overline{\eta_n}$ above a composition of immediate specializations $\overline{x_0} \to \dots \to \overline{\eta_n}$ of geometric points of X. We can apply lemma 4.3.2.2 to the specializations $\overline{\eta_i} \to \overline{\eta_{i+1}}$ to obtain the desired compatibility for $\overline{\eta'} \to \overline{\eta}$, which completes the proof of proposition 4.1.1.

4.4. Proof of proposition 4.1.2.

Lemme 4.4.1. *In the situation of proposition 4.1.2, if K is a putative dualizing complex on X, we can equip $f^!K$ with a structure of a putative dualizing complex on Y.*

Defining the structure of a putative dualizing complex on $f^!K$ can be done fiber by fiber. We thus assume that $X = x$ is the spectrum of a field. Let $y \in Y$. Let U be a non-empty open set of regularity of the (reduced) closure $\overline{\{y\}}$. Let $j : U \to Y$ be the immersion of U in Y and $\pi : U \to x$ be the projection. We have canonical isomorphisms induced by the Gysin morphism Cl_π and the pinning of K at x :

$$\Lambda(\delta_Y(y))[2\delta_Y(y)] \xrightarrow{\sim} \pi^!\Lambda(\delta_X(x))[2\delta_X(x)] \xleftarrow{\sim} \pi^!\mathrm{R}\Gamma_x(K) \simeq j^!f^!K\,.$$

By passing to the generic point of U, we obtain the desired isomorphism : $\mathrm{R}\Gamma_y(f^!K) \simeq \Lambda(\delta_Y(y))[2\delta_Y(y)]$. It obviously does not depend on the open set U, which allows us to define the desired pinning of $f^!K$ at y.

We can check that the construction of this structure of a putative dualizing complex on $f^!K$ is compatible with the composition of morphisms of finite type (for the meaning of this assertion, cf. remark 4.1.3). Furthermore, it is evident that in the case where f is quasi-finite, the pinnings defined here are the same as those of proposition 2.4.3.1. Finally, in the case where f is smooth of relative dimension d, *via* the canonical isomorphism $f^\star K \simeq f^!K(d)[2d]$, the pinnings are compatible with those of proposition 4.1.1, taking into account the shift between the two dimension functions considered on Y.

Let us prove proposition 4.1.2. The question being of a local nature, we can assume that $f : X \to Y$ is a morphism of finite type between affine schemes. There thus exists a factorization $f = p \circ i$ with i a closed immersion and p a smooth morphism. Let K be a potential dualizing complex on X. The preceding remarks show that $p^!K$, then $i^!p^!K$ are potential dualizing complexes and that the latter is identified with $f^!K$. Consequently, $f^!K$ is a potential dualizing complex, which completes the proof.

5. Existence and uniqueness of potential dualizing complexes

5.1. Statement of the theorem. The objective of this section is to establish the following theorem :

Théorème 5.1.1. *Let X be an excellent Noetherian $\mathbf{Z}\left[\frac{1}{n}\right]$-scheme equipped with a dimension function δ. Then (X, δ) admits a potential dualizing complex K_X, unique up to unique isomorphism, and the evident morphism $\Lambda \to \tau_{\leq 0}\mathrm{R}\,\mathbf{Hom}(K_X, K_X)$ is an isomorphism in $\mathsf{D}(X_{\text{ét}}, \Lambda)$. Moreover, $K_X \in \mathrm{Perv}^{-2\delta}(X, \Lambda)$ (cf. subsection 5.2).*

5.2. Preliminaries on perverse sheaves. If X is a Noetherian scheme and $p : X \to \mathbf{Z} \cup \{+\infty\}$ is a perversity function (that is, for any specialization $\overline{y} \to \overline{x}$ of geometric points of X, we have $p(x) \geq p(y)$), Gabber defined in [**Gabber, 2004**] a t-structure $(\mathsf{D}^{\leq p}(X_{\text{ét}}, \Lambda), \mathsf{D}^{\geq p}(X_{\text{ét}}, \Lambda))$ on $\mathsf{D}^+(X_{\text{ét}}, \Lambda)$ such that for any $K \in \mathsf{D}^+(X_{\text{ét}}, \Lambda)$, we have :

$$K \in \mathsf{D}^{\leq p}(X_{\text{ét}}, \Lambda) \iff \forall x \in X, K_{|x} \in \mathsf{D}^{\leq p(x)}(x_{\text{ét}}, \Lambda)\,,$$

$$K \in \mathsf{D}^{\geq p}(X_{\text{ét}}, \Lambda) \iff \forall x \in X, \mathrm{R}\Gamma_x(K) \in \mathsf{D}^{\geq p(x)}(x_{\text{ét}}, \Lambda)\,,$$

where we have equipped each of the categories $\mathsf{D}^+(x_{\text{ét}}, \Lambda)$ with its canonical t-structure.

We denote by $\mathrm{Perv}^p(X, \Lambda) \subset \mathsf{D}^+(X_{\text{ét}}, \Lambda)$ the heart of this t-structure, the truncation functors being denoted by $\tau_{\leq p}$ and $\tau_{\geq p}$.

Définition 5.2.1. Let $f : Y \to X$ be a morphism of schemes. We say that f is a **pseudo-open immersion** if f induces a homeomorphism onto its image and the induced morphism $f^{-1}\mathscr{O}_X \to \mathscr{O}_Y$ is an isomorphism.

This class of morphisms is stable by composition; it contains open immersions and localizations.

PROPOSITION 5.2.2. *Let $p\colon X \to \mathbf{Z}\cup\{+\infty\}$ be a perversity function. Let $f\colon Y \to X$ be a pseudo-open immersion.*

(a) *The function $p\circ f\colon Y \to \mathbf{Z}\cup\{+\infty\}$ is a perversity function (still denoted by p) and for the t-structures defined by p, $f^\star\colon \mathsf{D}^+(X_{\text{ét}},\Lambda) \to \mathsf{D}^+(Y_{\text{ét}},\Lambda)$ is t-exact and $\mathrm{R}f_\star\colon \mathsf{D}^+(Y_{\text{ét}},\Lambda) \to \mathsf{D}^+(X_{\text{ét}},\Lambda)$ is left t-exact;*

(b) *The functor $f^\star$ induces an exact functor $f^\star\colon \mathrm{Perv}^p(X,\Lambda) \to \mathrm{Perv}^p(Y,\Lambda)$ which admits a right adjoint $f^p_\star\colon \mathrm{Perv}^p(Y,\Lambda) \to \mathrm{Perv}^p(X,\Lambda)$ defined by the formula $f^p_\star K = \tau_{\leq p}\mathrm{R}f_\star K$;*

(c) *The adjunction morphism $f^\star f^p_\star \to \mathrm{Id}_{\mathrm{Perv}^p(Y,\Lambda)}$ is an isomorphism and the functor $f^p_\star\colon \mathrm{Perv}^p(Y,\Lambda) \to \mathrm{Perv}^p(X,\Lambda)$ is fully faithful.*

(d) *If $g\colon Z \to Y$ is a pseudo-open immersion composable with f, we have a transitivity isomorphism $f^p_\star \circ g^p_\star \simeq (f\circ g)^p_\star$.*

The t-exactness of $f^\star$ is trivial. In particular, $f^\star$ is right t-exact; by adjunction, $\mathrm{R}f_\star$ is left t-exact. (b) follows immediately from (a). (c) also follows from it given that the adjunction morphism $f^\star\mathrm{R}f_\star \to \mathrm{Id}_{\mathrm{Perv}^p(Y,\Lambda)}$ is an isomorphism. The isomorphism of (d) is deduced by adjunction from the transitivity isomorphism of the inverse image functors.

5.3. Case of a normal scheme.

PROPOSITION 5.3.1. *The statement of theorem 5.1.1 is true if we further assume that the scheme X is normal. More precisely, let X be an irreducible normal excellent Noetherian $\mathbf{Z}\left[\frac{1}{n}\right]$-scheme with generic point η, equipped with a dimension function δ such that $\delta(\eta)=0$. We denote by $j\colon \eta \to X$ the inclusion of the generic point and we set $T = j^\varphi_\star\Lambda$ where $\varphi\colon X \to \mathbf{N}$ is the perversity function defined by the equality :*

$$\varphi(x) = \max(0, 2\dim \mathscr{O}_{X,x} - 2)$$

for any $x\in X$.

(a) *The adjunction morphism $\Lambda \to j^\varphi_\star j^\star\Lambda$ defines an evident morphism $\Lambda \to T$ such that for any geometric point $\overline{x} \to X$, the induced map*

$$\mathrm{H}^{-2\delta(x)}_{\overline{x}}(\Lambda) \to \mathrm{H}^{-2\delta(x)}_{\overline{x}}(T)$$

is an isomorphism and $\mathrm{H}^q_{\overline{x}}(T)=0$ if $q\neq -2\delta(x)$. Theorem 3.1.1 then provides a pinning of T at x. With these pinnings, T is a potential dualizing complex for (X,δ).

(b) *If K is a potential dualizing complex for (X,δ), then K belongs to $\mathrm{Perv}^\varphi(X,\Lambda)$ (and to $\mathrm{Perv}^{-2\delta}(X,\Lambda)$) and the morphism $K\to T$ which is immediately deduced from it is an isomorphism compatible with the pinnings.*

(c) *If K is a potential dualizing complex on X, then the evident morphism $\Lambda \to \tau_{\leq 0}\mathrm{R}\,\mathbf{Hom}(K,K)$ is an isomorphism.*

Let us establish (a). The essential point is to show that for any $x\in X$, the evident morphism $\mathrm{H}^{-2\delta(x)}_{\overline{x}}(\Lambda) \to \mathrm{H}^{-2\delta(x)}_{\overline{x}}(T)$ is an isomorphism and that $\mathrm{H}^q_{\overline{x}}(T)$ is zero if $q\neq -2\delta(x)$. The fact that T is a potential dualizing complex will then follow immediately from theorem 3.1.1. To establish the desired result, we can assume that X is a local strictly Henselian scheme with closed point x. By induction on $d=\dim X$, we can assume (a) is known on the open set $U = X - x$. If $d=0$, the result is evident. We thus assume $d\geq 1$, so that U contains the generic point of X. Let $L = T_{|U}$. According to proposition 5.2.2, if we denote by $g\colon U\to X$ the evident open immersion, we have a canonical isomorphism $T = g^\varphi_\star L$. We immediately deduce a canonical isomorphism $T = \tau_{\leq\varphi(x)}\mathrm{R}g_\star L$.

If $d=1$, $\varphi(\eta)=0$, so $T = g_\star\Lambda = \Lambda$. Proposition 2.4.2.1 allows us to conclude that T is indeed a potential dualizing complex with the pinnings considered here. We thus assume that $d\geq 2$. In this case, we have $\varphi(x) = 2d-2$ and $T = \tau_{\leq 2d-2}\mathrm{R}g_\star L$.

According to lemma 3.6.4, the structure of the cohomology objects $\mathscr{H}^q L$ for $q\leq 0$ is known. Let $i\colon x\to X$ be the immersion of the closed point of X. Let us use the canonical distinguished triangle :

$$i_\star i^! T \to T \to \mathrm{R}g_\star L \to i_\star i^! T[1]\ .$$

It results in a long exact sequence :

$$\ldots \to \mathrm{H}^{q-1}_{\text{ét}}(U, L) \to \mathrm{H}^{q}_{x,\text{ét}}(X, T) \to (\mathscr{H}^q T)_x \to \mathrm{H}^{q}_{\text{ét}}(U, L) \to \ldots$$

By construction, $(\mathscr{H}^q T)_x \xrightarrow{\sim} \mathrm{H}^q_{\text{ét}}(U, L)$ if $q \leq 2d-2$ and $(\mathscr{H}^q T)_x = 0$ if $q \geq 2d-1$. It follows that $\mathrm{H}^q_{x,\text{ét}}(X, T) = 0$ if $q \leq 2d-1$ and that we have isomorphisms $\mathrm{H}^{q-1}_{\text{ét}}(U, L) \xrightarrow{\sim} \mathrm{H}^q_{x,\text{ét}}(X, T)$ for $q \geq 2d$. It also follows that $\mathscr{H}^q T = 0$ if $q < 0$ and that we have a canonical isomorphism $\Lambda \xrightarrow{\sim} \mathscr{H}^0 T$. Thus, we have a canonical morphism $\Lambda \to T$ inducing an isomorphism $\Lambda \xrightarrow{\sim} \tau_{\leq 0} T$.

Lemme 5.3.2. *Let U be the complement of the closed point in a normal excellent strictly Henselian local scheme X of dimension $d \geq 2$. Let $M \in \mathsf{D}^{\leq \varphi}(U_{\text{ét}}, \Lambda)$. Assume there exists an isomorphism $\Lambda \xrightarrow{\sim} \tau_{\leq 0} M$. Then, for any $q \geq 2d$, the evident morphism is an isomorphism :*

$$\mathrm{H}^{q-1}_{\text{ét}}(U, \Lambda) \xrightarrow{\sim} \mathrm{H}^{q-1}_{\text{ét}}(U, M)\,.$$

Let M^+ be a cone of the evident morphism $\Lambda \to M$. It is sufficient to show that $\mathrm{H}^q_{\text{ét}}(U, M^+) = 0$ for $q \geq 2d-2$. The hypotheses imply that the cohomology objects of M^+ are zero outside the interval $[1, 2d-4]$. We can also observe that for any $1 \leq i \leq d-2$, if $y \in U$ is such that $(\mathscr{H}^{2i-1} M^+)_{\overline{y}}$ or $(\mathscr{H}^{2i} M^+)_{\overline{y}}$ is non-zero, then the closure (in X) of y has dimension $\leq d-i-1$. According to proposition 3.2.1 and the hypercohomology spectral sequence, we indeed obtain that $\mathrm{H}^q_{\text{ét}}(U, M^+) = 0$ if $q \geq 2d-2$.

We can apply lemma 5.3.2 with $M = L$. It then results from the above and from theorem 3.1.1 that $\mathrm{H}^q_{x,\text{ét}}(X, T) = 0$ if $q \neq 2d$, that we have a canonical isomorphism $\mathrm{H}^{2d}_{x,\text{ét}}(X, \Lambda(d)) \xrightarrow{\sim} \mathrm{H}^{2d}_{x,\text{ét}}(X, T(d))$, and that we can thus define pinnings on T which make it a potential dualizing complex on X.

Let us now show (b). Let K be a potential dualizing complex on X. It is evident that $K \in \mathsf{D}^{\geq -2\delta}(X_{\text{ét}}, \Lambda) \subset \mathsf{D}^{\geq \varphi}(X_{\text{ét}}, \Lambda)$. To show that $K \in \mathsf{D}^{\leq \varphi}(X_{\text{ét}}, \Lambda) \subset \mathsf{D}^{\leq -2\delta}(X_{\text{ét}}, \Lambda)$, we can assume that X is a strictly Henselian local scheme of dimension d. Let $i \colon x \to X$ be the inclusion of the closed point and $g \colon U \to X$ the inclusion of the complementary open set $X - x$. The case where $d = 0$ being trivial and that where $d = 1$ having been treated in proposition 2.4.2.1, we can assume that $d \geq 2$. By induction on d, we can assume (b) is known for the open set U. We denote $L = K_{|U} \in \mathsf{D}^{\leq \varphi}(U_{\text{ét}}, \Lambda)$.

Since K is a potential dualizing complex, the structure of the cohomology sheaves $\mathscr{H}^q K$ for $q \leq 0$ is known (cf. lemma 3.6.4). We can thus apply lemma 5.3.2 with $M = L$. Thus, the evident morphism is an isomorphism $\mathrm{H}^{2d-1}_{\text{ét}}(U, \Lambda) \xrightarrow{\sim} \mathrm{H}^{2d-1}_{\text{ét}}(U, L)$, and $\mathrm{H}^q_{\text{ét}}(U, L) = 0$ for $q \geq 2d$ (cf. proposition 3.2.1).

As for establishing (a), we use the canonical distinguished triangle :

$$i_\star i^! K \to K \to \mathrm{R}g_\star L \to i_\star i^! K[1]\,,$$

and the long exact sequence which is deduced from it :

$$\ldots \to \mathrm{H}^{q-1}_{\text{ét}}(U, L) \to \mathrm{H}^{q}_{x,\text{ét}}(X, K) \to (\mathscr{H}^q K)_x \to \mathrm{H}^{q}_{\text{ét}}(U, L) \to \ldots$$

The structure of $\mathrm{H}^q_{x,\text{ét}}(X, K)$ being known for any $q \in \mathbf{Z}$, it immediately follows that $(\mathscr{H}^q K)_x = 0$ for $q \geq 2d+1$ and that we have an exact sequence

$$0 \to (\mathscr{H}^{2d-1} K)_x \to \mathrm{H}^{2d-1}_{\text{ét}}(U, L) \to \mathrm{H}^{2d}_{x,\text{ét}}(X, K) \to (\mathscr{H}^{2d} K)_x \to 0\,.$$

To show that $K \in \mathsf{D}^{\leq \varphi}(X_{\text{ét}}, \Lambda)$, it thus remains to show that the canonical morphism $\mathrm{H}^{2d-1}_{\text{ét}}(U, L) \to \mathrm{H}^{2d}_{x,\text{ét}}(X, K)$ is an isomorphism. Let us choose a specialization $\overline{\eta} \to x$. We can consider the following commutative diagram, where the vertical arrows are induced by the canonical morphism $\Lambda \to K$:

$$\begin{tikzcd}
\mathrm{H}^0_{\overline{\eta}}(X, \Lambda) \arrow[r, "\mathrm{sp}^X_{\overline{\eta} \to x}"] \arrow[d, "\sim"] & \mathrm{H}^{2d}_{x,\text{ét}}(X, \Lambda(d)) \arrow[d] & \mathrm{H}^{2d-1}_{\text{ét}}(U, \Lambda(d)) \arrow[l, "\sim"'] \arrow[d, "\sim"] \\
\mathrm{H}^0_{\overline{\eta}}(X, K) \arrow[r, "\mathrm{sp}^X_{\overline{\eta} \to x}"] & \mathrm{H}^{2d}_{x,\text{ét}}(X, K(d)) & \mathrm{H}^{2d-1}_{\text{ét}}(U, L(d)) \arrow[l]
\end{tikzcd}$$

The arrows of the left column to the middle one are the transition morphisms introduced in theorem 3.1.2. Here, they are isomorphisms : for the top arrow, this results from theorem 3.1.1 and for the bottom arrow, from the fact that K is a potential dualizing complex. We deduce that the middle morphism $\mathrm{H}^{2d}_{x,\text{ét}}(X,\Lambda(d)) \to \mathrm{H}^{2d}_{x,\text{ét}}(X,K(d))$ is an isomorphism. Given the other known isomorphisms, it follows that the evident morphism $\mathrm{H}^{2d-1}_{\text{ét}}(U,L(d)) \to \mathrm{H}^{2d}_{x,\text{ét}}(X,K(d))$ is an isomorphism, which completes the proof that $K \in \mathrm{Perv}^{\varphi}(X,\Lambda)$.

We then have an adjunction morphism $K \to j^{\varphi}_{\star}j^{\star}K = T$ in $\mathrm{Perv}^{\varphi}(X,\Lambda)$. To show that it is an isomorphism, we can place ourselves in the preceding local situation, and do an induction on the dimension to be able to assume that the induced morphism $K_{|U} \to T_{|U}$ is an isomorphism. Let us choose a specialization $\overline{\eta} \to x$. We deduce from it a commutative diagram :

$$\begin{tikzcd} \mathrm{H}^0_{\overline{\eta}}(X,K) \arrow[r, "\mathrm{sp}^X_{\overline{\eta}\to x}"] \arrow[d] & \mathrm{H}^{2d}_{x,\text{ét}}(X,K(d)) \arrow[d] \\ \mathrm{H}^0_{\overline{\eta}}(X,T) \arrow[r, "\mathrm{sp}^X_{\overline{\eta}\to x}"] & \mathrm{H}^{2d}_{x,\text{ét}}(X,T(d)) \end{tikzcd}$$

The left arrow is obviously an isomorphism. The horizontal arrows are too since K and T are potential dualizing complexes. It thus follows that the induced morphism $i^!K \to i^!T$ is an isomorphism. It follows that $K \to T$ is an isomorphism (compatible with the pinnings at the generic point, thus with all pinnings), which completes the proof of (b).

Let us show (c). Let K be a potential dualizing complex on X. To show that the canonical morphism $\Lambda \to \tau_{\leq 0}\mathrm{R}\,\mathbf{Hom}(K,K)$ is an isomorphism, up to replacing X by connected étale schemes over it, it is sufficient to show that $\Lambda \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{D}^+(X_{\text{ét}},\Lambda)}(K,K)$ and that $\mathrm{Hom}_{\mathrm{D}^+(X_{\text{ét}},\Lambda)}(K,K[q]) = 0$ for $q < 0$. The vanishing of $\mathrm{Hom}_{\mathrm{D}^+(X_{\text{ét}},\Lambda)}(K,K[q])$ for $q < 0$ follows immediately from the fact that K belongs to the heart of the t-structure defined by φ. According to (b), we have a canonical isomorphism $K = j^{\varphi}_{\star}\Lambda$. The isomorphism $\Lambda \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{D}^+(X_{\text{ét}},\Lambda)}(K,K)$ then results from the fact that $j^{\varphi}_{\star} \colon \mathrm{Perv}^{\varphi}(\eta,\Lambda) \to \mathrm{Perv}^{\varphi}(X,\Lambda)$ is fully faithful (cf. proposition 5.2.2).

5.4. A gluing result. The following proposition is a gluing result which will allow us to pass from the normal case to the general case :

Proposition 5.4.1. *Let X be an excellent Noetherian $\mathbf{Z}\left[\frac{1}{n}\right]$-scheme equipped with a dimension function δ. Assume we are given a Cartesian square*

$$\begin{tikzcd} Y' \arrow[r, "i'"] \arrow[d, "p'"] & X' \arrow[d, "p"] \\ Y \arrow[r, "i"] & X \end{tikzcd}$$

where i is a closed immersion with open complement U and where p is a finite surjective morphism inducing an isomorphism $p^{-1}(U) \xrightarrow{\sim} U$. Let $q = p \circ i'$. Assume that the statement of theorem 5.1.1 is known for X', Y and Y' (relative to the dimension functions deduced from that on X by the process of proposition 2.4.3.1). Then, this statement is also true for X, and if we denote by K_X, $K_{X'}$, K_Y and $K_{Y'}$ the potential dualizing complexes of X, X', Y and Y' respectively, we have a canonical distinguished triangle in $\mathrm{D}^+(X_{\text{ét}},\Lambda)$:

$$q_{\star}K_{Y'} \to i_{\star}K_Y \oplus p_{\star}K_{X'} \to K_X \to q_{\star}K_{Y'}[1]\ .$$

First, let us assume that X admits a potential dualizing complex K_X and let us show that we can define a canonical distinguished triangle of the desired form. For this, let us consider the evident short exact sequence of sheaves on X :

$$0 \to \Lambda \overset{(+,+)}{\to} i_{\star}\Lambda \oplus p_{\star}\Lambda \overset{(+,-)}{\to} q_{\star}\Lambda \to 0\ .$$

By applying $\mathrm{R}\,\mathbf{Hom}(-,K_X)$ to the corresponding distinguished triangle, we obtain a distinguished triangle :

$$q_\star q^! K_X \overset{(+,-)}{\to} i_\star i^! K_X \oplus p_\star p^! K_X \overset{(+,+)}{\to} K_X \to q_\star q^! K_X[1]\ ,$$

which, given proposition 4.1.2 and the uniqueness property of potential dualizing complexes on X', Y and Y', can be rewritten as :

$$q_\star K_{Y'} \overset{(+,-)}{\to} i_\star K_Y \oplus p_\star K_{X'} \overset{(+,+)}{\to} K_X \to q_\star K_{Y'}[1]\ .$$

Returning to the hypotheses of the proposition, we will show that conversely, if we define K_X so as to have such a distinguished triangle (but *a priori* not in a canonical way), we indeed obtain a potential dualizing complex on X. We thus assume theorem 5.1.1 is known only for X', Y and Y' and we denote by $K_{X'}$, K_Y and $K_{Y'}$ the corresponding potential dualizing complexes. The uniqueness property for potential dualizing complexes on Y' gives canonical isomorphisms $K_{Y'} \simeq p'^! K_Y$ and $K_{Y'} \simeq i'^! K_{X'}$. By adjunction, we deduce canonical morphisms $p'_\star K_{Y'} \to K_Y$ and $i'_\star K_{Y'} \to K_{X'}$, then by applying respectively $i_\star$ and $p_\star$, we obtain canonical morphisms $q_\star K_{Y'} \to i_\star K_Y$ and $q_\star K_{Y'} \to p_\star K_{X'}$. We can consider their difference and construct a distinguished triangle in $\mathsf{D}^+(X_{\text{ét}},\Lambda)$:

$$q_\star K_{Y'} \overset{(+,-)}{\to} i_\star K_Y \oplus p_\star K_{X'} \to K_X \to q_\star K_{Y'}[1]\ .$$

We thus obtain an object $K_X \in \mathsf{D}^+(X_{\text{ét}},\Lambda)$ and two privileged morphisms $i_\star K_Y \to K_X$ and $p_\star K_{X'} \to K_X$.

Let $j\colon U \to X$ and $j'\colon U \to X'$ be the evident open immersions. By applying $j^\star$ to the distinguished triangle above, we can start by observing that the evident morphism $j'^\star K_{X'} \simeq j^\star p_\star K_{X'} \to j^\star K_X$ is an isomorphism. Consequently, we can equip K_X with pinnings at the points of the open set U in a way compatible with the structure of a potential dualizing complex obtained on $j'^\star K_{X'}$.

Let us consider the canonical morphism $q_\star K_{Y'} \to p_\star K_{X'}$. I claim that it induces an isomorphism after application of the functor $i^!$. Indeed, we have evident isomorphisms of functors $i^! q_\star i'^! \simeq i^! i_\star p'_\star i'^! \simeq p'_\star i'^! \simeq i^! p_\star$ and, given the canonical isomorphism $K_{Y'} \simeq i'^! K_{X'}$, we indeed obtain that the evident morphism $i^! q_\star K_{Y'} \to i^! p_\star K_{X'}$ is an isomorphism. By applying $i^!$ to the distinguished triangle defining K_X, it then follows that the canonical morphism $i_\star K_Y \to K_X$ induces an isomorphism $K_Y \xrightarrow{\sim} i^! K_X$ after application of $i^!$. This allows us to define pinnings for K_X at all points of Y.

Finally, we have obtained a structure of a putative dualizing complex on K_X. According to proposition 2.4.3.3, to show that K_X is a potential dualizing complex, it is sufficient to show that $p^! K_X$ is one. We will of course compare it with $K_{X'}$. By construction of K_X, we have a commutative diagram :

$$\begin{tikzcd} & i_\star K_Y \arrow[dr] & \\ q_\star K_{Y'} \arrow[ur] \arrow[dr] & & K_X \\ & p_\star K_{X'} \arrow[ur] & \end{tikzcd}$$

By adjunction, we obtain that two competing definitions of a morphism $K_{Y'} \to q^! K_X$ coincide :

$$\begin{tikzcd} & p'^! K_Y \arrow[r] & p'^! i^! K_X \arrow[dr] & \\ K_{Y'} \arrow[ur] \arrow[dr] & & & q^! K_X \\ & i'^! K_{X'} \arrow[r] & i'^! p^! K_X \arrow[ur] & \end{tikzcd}$$

We have already shown that the canonical morphism $K_Y \to i^! K_X$ was an isomorphism. Consequently, on the diagram above, all the arrows are isomorphisms. Thus, the evident morphism $K_{X'} \to p^! K_X$

induces an isomorphism not only after application of $j'^{\star}$, but also after application of $i'^{!}$. It follows that this morphism $K_{X'} \to p^! K_X$ is an isomorphism. Furthermore, on the diagram above, all objects are naturally equipped with a structure of a putative dualizing complex and all isomorphisms, except perhaps the bottom one, are compatible with the pinnings. This isomorphism $i'^! K_{X'} \to i'^! p^! K_X$ is therefore also compatible with the pinnings. Consequently, the isomorphism $K_{X'} \xrightarrow{\sim} p^! K_X$ is compatible with the pinnings not only on U but also on Y'. It follows that $p^! K_X$ is a potential dualizing complex; according to proposition 2.4.3.3, we can conclude that K_X is also a potential dualizing complex.

Furthermore, the hypothesis that the potential dualizing complexes on X', Y and Y' are perverse for the perversity function -2δ implies by construction that K_X is also perverse for -2δ.

To conclude, we need to show that if K and L are two potential dualizing complexes on X, we have a privileged isomorphism $\Lambda \xrightarrow{\sim} \tau_{\leq 0} \mathrm{R}\,\mathbf{Hom}(K, L)$ which gives rise to a morphism $\psi : K \to L$ which is an isomorphism of potential dualizing complexes. Indeed, this will show that if $\phi : K \to L$ is another isomorphism, then $\phi = \lambda \cdot \psi$ where $\lambda : X \to \Lambda^{\times}$ is a locally constant function. Requiring that ϕ be compatible with the pinnings implies that $\lambda = 1$, so we will indeed have a unique isomorphism $K \xrightarrow{\sim} L$ of potential dualizing complexes.

According to the uniqueness property of potential dualizing complexes on X', Y and Y', we have isomorphisms of potential dualizing complexes $p^! K \xrightarrow{\sim} p^! L$ and $i^! K \xrightarrow{\sim} i^! L$ inducing the same isomorphism $q^! K \xrightarrow{\sim} q^! L_Y$. We have constructed above a canonical distinguished triangle :

$$q_\star q^! K \overset{(+,-)}{\to} i_\star i^! K \oplus p_\star p^! K \overset{(+,+)}{\to} K \to q_\star q^! K[1] \ .$$

By applying $\mathrm{R}\,\mathbf{Hom}(-, L)$ to it, we obtain another distinguished triangle :

$$\mathrm{R}\,\mathbf{Hom}(K, L) \overset{(+,+)}{\to} i_\star \mathrm{R}\,\mathbf{Hom}(i^! K, i^! L) \oplus p_\star \mathrm{R}\,\mathbf{Hom}(p^! K, p^! L) \overset{(+,-)}{\to} q_\star \mathrm{R}\,\mathbf{Hom}(q^! K, q^! L) \overset{+}{\to}$$

The statement of theorem 5.1.1 for X', Y and Y' immediately implies that the cohomology objects $\mathscr{H}^q \mathrm{R}\,\mathbf{Hom}(K, L)$ are zero for $q < 0$, and, given the canonical exact sequence of sheaves :

$$0 \to \Lambda \overset{(+,+)}{\to} i_\star \Lambda \oplus p_\star \Lambda \overset{(+,-)}{\to} q_\star \Lambda \to 0 \ ,$$

that we have a privileged isomorphism $\Lambda \xrightarrow{\sim} \mathscr{H}^0 \mathrm{R}\,\mathbf{Hom}(K, L)$. The corresponding morphism $K \to L$ induces of course the unique isomorphisms of potential dualizing complexes $i^! K \xrightarrow{\sim} i^! L$ and $p^! K \xrightarrow{\sim} p^! L$. This morphism $K \to L$ is therefore compatible with the pinnings on Y and on U : it is an isomorphism of potential dualizing complexes. This completes the proof of proposition 5.4.1.

5.5. General case. Let us prove theorem 5.1.1 in the general case. To prove it for all excellent Noetherian $\mathbf{Z}\left[\frac{1}{n}\right]$-schemes equipped with a dimension function, by Noetherian induction, we can assume the result is known for schemes finite over a closed set with empty interior in X. We can assume that X is reduced. Let $p : X' \to X$ be the normalization of X. The morphism p is finite surjective and induces an isomorphism over the dense open set of normality U of the excellent scheme X. Let $Y = (X - U)_{\text{réd}}$ and form the following Cartesian square :

$$\begin{tikzcd} Y' \arrow[r, "i'"] \arrow[d, "p'"] & X' \arrow[d, "p"] \\ Y \arrow[r, "i"] & X \end{tikzcd}$$

Since X' is normal, proposition 5.3.1 shows that the statement of theorem 5.1.1 is known for X'. The Noetherian induction hypothesis shows that this is also the case for Y and Y'. Proposition 5.4.1 gives the desired conclusion for X.

6. The local duality theorem

6.1. Statement of the theorem.

Théorème 6.1.1. *Let X be an excellent Noetherian $\mathbf{Z}\left[\frac{1}{n}\right]$-scheme equipped with a dimension function δ. Let K be the potential dualizing complex of (X,δ) (cf. theorem 5.1.1). Then*

- *$K \in \mathsf{D}^{\mathsf{b}}_{\mathsf{ctf}}(X_{\text{ét}},\Lambda)$;*
- *K has finite quasi-injective dimension if and only if X has finite Krull dimension;*
- *the functor $\mathrm{D}_K = \mathrm{R}\,\mathbf{Hom}(-,K)$ preserves $\mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X_{\text{ét}},\Lambda)$;*
- *for any $M \in \mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X_{\text{ét}},\Lambda)$, the biduality morphism $M \to \mathrm{D}_K\,\mathrm{D}_K\,M$ is an isomorphism.*

In particular, if X has finite Krull dimension, K is a dualizing complex in the sense of [**SGA 5** I 1.7].

6.2. Constructibility, tor-dimension, quasi-injective dimension.

6.2.1. *Change of coefficients.*

Proposition 6.2.1.1. *Let $\Lambda = \mathbf{Z}/n\mathbf{Z}$. Let m be a divisor of n. Let $\Lambda' = \mathbf{Z}/m\mathbf{Z}$. The ring Λ' is a Λ-algebra. Let $K \in \mathsf{D}^+(X_{\text{ét}},\Lambda)$ be a potential dualizing complex on X relative to the coefficient ring Λ. Then $K' = \mathrm{R}\,\mathbf{Hom}_\Lambda(\Lambda',K) \in \mathsf{D}^+(X_{\text{ét}},\Lambda')$ is naturally equipped with a structure of a potential dualizing complex relative to the coefficient ring Λ'. Moreover, if $M \in \mathsf{D}^+(X_{\text{ét}},\Lambda')$, we have a canonical isomorphism in $\mathsf{D}^+(X_{\text{ét}},\Lambda)$:*

$$\mathrm{R}\,\mathbf{Hom}_\Lambda(M,K) \simeq \mathrm{R}\,\mathbf{Hom}_{\Lambda'}(M,K')\ .$$

This results easily from the commutation of the cohomology with supports functors with the functor $\mathrm{R}\,\mathbf{Hom}_\Lambda(\Lambda',-)$. (We also use a privileged isomorphism $\mathrm{R}\,\mathbf{Hom}_\Lambda(\Lambda',\Lambda) \simeq \Lambda'$: *via* this isomorphism, the canonical generator of Λ' corresponds to the morphism $\Lambda' \to \Lambda$ sending $1 \in \mathbf{Z}/m\mathbf{Z}$ to $[n/m] \in \mathbf{Z}/n\mathbf{Z}$.)

6.2.2. *Constructibility, tor-finiteness.*

Proposition 6.2.2.1. *Let K be a potential dualizing complex on an excellent Noetherian $\mathbf{Z}\left[\frac{1}{n}\right]$-scheme equipped with a dimension function δ. Then $K \in \mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X_{\text{ét}},\Lambda)$.*

We will use the following lemma :

Lemme 6.2.2.2. *Let X be an excellent Noetherian $\mathbf{Z}\left[\frac{1}{n}\right]$-scheme. Let $i\colon Z \to X$ be a closed immersion. Let $j\colon U \to X$ be the complementary open immersion. Let $K \in \mathsf{D}^+(X_{\text{ét}},\Lambda)$. The following conditions are equivalent :*

- *$K \in \mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X_{\text{ét}},\Lambda)$;*
- *$j^\star K \in \mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(U_{\text{ét}},\Lambda)$ and $i^!K \in \mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(Z_{\text{ét}},\Lambda)$.*

This lemma follows immediately from the non-trivial fact that the functor $\mathrm{R}j_\star$ sends $\mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(U_{\text{ét}},\Lambda)$ to $\mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X_{\text{ét}},\Lambda)$, cf. **XIII**-1.1.1.

Let us prove the proposition. We can assume X is reduced. Since X is excellent, X admits a dense regular open set. Let Z be the closed subscheme $(X-U)_{\text{réd}}$ and $i\colon Z \to X$ its closed immersion in X. By Noetherian induction, we can assume that the potential dualizing complex $i^!K$ of Z is in $\mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(Z_{\text{ét}},\Lambda)$. By virtue of the lemma, we are reduced to showing that $j^\star K \in \mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(U_{\text{ét}},\Lambda)$. In other words, we can assume that X is regular. We can further assume that X is connected. Let η be the generic point of X. According to proposition 2.4.4.1 and theorem 5.1.1, we have a canonical isomorphism $K \simeq \Lambda(\delta(\eta))[2\delta(\eta)]$. Thus, K indeed belongs to $\mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X_{\text{ét}},\Lambda)$, which completes the proof of the proposition.

Proposition 6.2.2.3. *Let X be an excellent Noetherian scheme equipped with a dimension function δ. The potential dualizing complex of (X,δ) belongs to $\mathsf{D}^{\mathsf{b}}_{\mathsf{ctf}}(X_{\text{ét}},\Lambda)$.*

We already know that the potential dualizing complex K of (X,δ) belongs to $\mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X_{\text{ét}},\Lambda)$. We thus need to obtain a tor-finiteness result for K. For this, we can assume that $\Lambda = \mathbf{Z}/\ell^\nu\mathbf{Z}$ where ℓ is a prime number and $\nu \geq 1$. According to proposition 6.2.1.1, $\mathrm{R}\,\mathbf{Hom}_\Lambda(\mathbf{Z}/\ell\mathbf{Z},K)$ is a potential dualizing complex relative to the coefficient ring $\mathbf{Z}/\ell\mathbf{Z}$; according to proposition 6.2.2.1, this object belongs to $\mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X_{\text{ét}},\mathbf{Z}/\ell\mathbf{Z})$, the criterion of the following lemma allows us to conclude that K belongs to $\mathsf{D}^{\mathsf{b}}_{\mathsf{tf}}(X_{\text{ét}},\Lambda)$.

Lemme 6.2.2.4. *Let X be a Noetherian scheme. Let ℓ be a prime number. Let $\nu \geq 1$. Let $\Lambda = \mathbf{Z}/\ell^\nu\mathbf{Z}$. Let $K \in \mathsf{D}^{\mathsf{b}}(X_{\text{ét}},\Lambda)$. The following conditions are equivalent :*

(i) $K \in \mathsf{D}^{\mathrm{b}}_{\mathrm{tf}}(X_{\text{ét}}, \Lambda)$;
(ii) $K \overset{\mathrm{L}}{\otimes}_{\Lambda} \mathbf{Z}/\ell\mathbf{Z} \in \mathsf{D}^{\mathrm{b}}(X_{\text{ét}}, \mathbf{Z}/\ell\mathbf{Z})$;
(iii) $\mathrm{R}\,\mathbf{Hom}_{\Lambda}(\mathbf{Z}/\ell\mathbf{Z}, K) \in \mathsf{D}^{\mathrm{b}}(X_{\text{ét}}, \mathbf{Z}/\ell\mathbf{Z})$.

In this case, we have a canonical isomorphism $\mathrm{R}\,\mathbf{Hom}_{\Lambda}(\mathbf{Z}/\ell\mathbf{Z}, K) \simeq K \overset{\mathrm{L}}{\otimes}_{\Lambda} \mathbf{Z}/\ell\mathbf{Z}$ *in* $\mathsf{D}^{\mathrm{b}}(X_{\text{ét}}, \mathbf{Z}/\ell\mathbf{Z})$.

The equivalence between (i) and (ii) is only indicated for the record. It is here mainly a question of showing that conditions (ii) and (iii) are equivalent. We represent K by a bounded complex. Let R be the following (acyclic) complex of Λ-modules :

$$\begin{array}{ccccccccccccccc} & -3 & & -2 & & -1 & & 0 & & 1 & & 2 & & 3 & \\ \cdots \longrightarrow & \Lambda & \xrightarrow{\ell} & \Lambda & \xrightarrow{\ell^{\nu-1}} & \Lambda & \xrightarrow{\ell} & \Lambda & \xrightarrow{\ell^{\nu-1}} & \Lambda & \xrightarrow{\ell} & \Lambda & \xrightarrow{\ell^{\nu-1}} & \Lambda & \xrightarrow{\ell} \cdots \end{array}$$

We denote by $R^{\leq 0}$ and $R^{>0}$ the brutal truncations of R, so that we have a short exact sequence of complexes :

$$0 \to R^{>0} \xrightarrow{i} R \xrightarrow{p} R^{\leq 0} \to 0 \,.$$

We deduce a short exact sequence of complexes of sheaves of Λ-modules on $X_{\text{ét}}$:

$$0 \to K \otimes_{\Lambda} R^{>0} \to K \otimes_{\Lambda} R \to K \otimes_{\Lambda} R^{\leq 0} \to 0 \,,$$

then a distinguished triangle in $\mathsf{D}(X_{\text{ét}}, \Lambda)$:

$$K \otimes_{\Lambda} R^{>0} \to K \otimes_{\Lambda} R \to K \otimes_{\Lambda} R^{\leq 0} \xrightarrow{\delta} (K \otimes_{\Lambda} R^{>0})[1] \,.$$

We have an evident quasi-isomorphism $R^{\leq 0} \xrightarrow{\sim} \mathbf{Z}/\ell\mathbf{Z}$ which allows us to make the identification $K \otimes_{\Lambda} R^{\leq 0} \simeq K \overset{\mathrm{L}}{\otimes}_{\Lambda} \mathbf{Z}/\ell\mathbf{Z}$ in $\mathsf{D}^{-}(X_{\text{ét}}, \Lambda)$. With the sign conventions of [**Conrad, 2000**, §1.3] (see also **XVI**-4.7.3), we have a canonical isomorphism $(K \otimes_{\Lambda} R^{>0})[1] \simeq K \otimes_{\Lambda} (R^{>0}[1])$ and another (which does not involve any signs) $\mathbf{Hom}(Q, \Lambda) = R^{>0}[1]$ where Q is the following complex of constant sheaves of Λ-modules :

$$\begin{array}{ccccccccccccccc} & -4 & & -3 & & -2 & & -1 & & 0 & & 1 & & 2 & \\ \cdots \longrightarrow & \Lambda & \xrightarrow{-\ell^{\nu-1}} & \Lambda & \xrightarrow{\ell} & \Lambda & \xrightarrow{-\ell^{\nu-1}} & \Lambda & \xrightarrow{\ell} & \Lambda & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow \cdots \end{array}$$

Proceeding as in **XVI**-4.7.2, we deduce an isomorphism $K \otimes_{\Lambda} (R^{>0}[1]) \to \mathbf{Hom}(Q, K)$. The evident quasi-isomorphism $Q \xrightarrow{\sim} \mathbf{Z}/\ell\mathbf{Z}$ induces an isomorphism $\mathbf{Hom}(Q, K) \simeq \mathrm{R}\,\mathbf{Hom}_{\Lambda}(\mathbf{Z}/\ell\mathbf{Z}, K)$ in $\mathsf{D}^{+}(X_{\text{ét}}, \Lambda)$. Given the above, we have obtained a canonical isomorphism $(K \otimes_{\Lambda} R^{>0})[1] \simeq \mathrm{R}\,\mathbf{Hom}_{\Lambda}(\mathbf{Z}/\ell\mathbf{Z}, K)$. If we denote by $\tilde{\delta} \colon K \overset{\mathrm{L}}{\otimes}_{\Lambda} \mathbf{Z}/\ell\mathbf{Z} \to \mathrm{R}\,\mathbf{Hom}_{\Lambda}(\mathbf{Z}/\ell\mathbf{Z}, K)$ the morphism in $\mathsf{D}(X_{\text{ét}}, \Lambda)$ induced by δ *via* the preceding isomorphisms, we have obtained a distinguished triangle in $\mathsf{D}(X_{\text{ét}}, \Lambda)$:

$$K \otimes_{\Lambda} R \to K \overset{\mathrm{L}}{\otimes}_{\Lambda} \mathbf{Z}/\ell\mathbf{Z} \xrightarrow{\tilde{\delta}} \mathrm{R}\,\mathbf{Hom}_{\Lambda}(\mathbf{Z}/\ell\mathbf{Z}, K) \to (K \otimes_{\Lambda} R)[1]$$

Since $R = R[2]$, we note that we have isomorphisms at the level of cohomology objects $\mathscr{H}^{q}(K \otimes_{\Lambda} R) \simeq \mathscr{H}^{q+2}(K \otimes_{\Lambda} R)$ for any $q \in \mathbf{Z}$. Consequently, the following conditions are equivalent :

(a) $K \otimes_{\Lambda} R$ is acyclic.
(b) $K \otimes_{\Lambda} R \in \mathsf{D}^{+}(X_{\text{ét}}, \Lambda)$;
(c) $K \otimes_{\Lambda} R \in \mathsf{D}^{-}(X_{\text{ét}}, \Lambda)$;

Thanks to the distinguished triangle constructed above, given the fact that $\mathrm{R}\,\mathbf{Hom}_{\Lambda}(\mathbf{Z}/\ell\mathbf{Z}, K) \in \mathsf{D}^{+}(X_{\text{ét}}, \Lambda)$, we obtain that (b) is equivalent to (ii). Similarly, that $K \overset{\mathrm{L}}{\otimes}_{\Lambda} \mathbf{Z}/\ell\mathbf{Z}$ belongs to $\mathsf{D}^{-}(X_{\text{ét}}, \Lambda)$ implies that (c) and (iii) are equivalent. Consequently, (ii) and (iii) are equivalent.

If we assume that these equivalent conditions are satisfied (that is, that $K \in \mathsf{D}^{\mathrm{b}}_{\mathrm{tf}}(X_{\text{ét}}, \Lambda)$), the morphism $\tilde{\delta} \colon K \overset{\mathrm{L}}{\otimes}_{\Lambda} \mathbf{Z}/\ell\mathbf{Z} \to \mathrm{R}\,\mathbf{Hom}_{\Lambda}(\mathbf{Z}/\ell\mathbf{Z}, K)$ is an isomorphism in $\mathsf{D}^{\mathrm{b}}(X_{\text{ét}}, \Lambda)$, which does not allow us to conclude immediately since we wish to obtain an isomorphism in the category $\mathsf{D}^{\mathrm{b}}(X_{\text{ét}}, \mathbf{Z}/\ell\mathbf{Z})$. For this, we can represent K by a bounded complex of flat sheaves of Λ-modules on $X_{\text{ét}}$. The objects $K \overset{\mathrm{L}}{\otimes}_{\Lambda} \mathbf{Z}/\ell\mathbf{Z}$

and $\mathrm{R}\,\mathbf{Hom}_\Lambda(\mathbf{Z}/\ell\mathbf{Z}, K)$ are then represented respectively by $K/\ell K$ and ${}_\ell K := \mathrm{Ker}(\ell : K \to K)$ and the sought quasi-isomorphism is the isomorphism induced by multiplication by $\ell^{\nu-1} : K/\ell K \xrightarrow{\sim} {}_\ell K$. (We can show that after application of the evident functor $\mathrm{D^b}(X_{\text{ét}}, \mathbf{Z}/\ell\mathbf{Z}) \to \mathrm{D^b}(X_{\text{ét}}, \Lambda)$, the isomorphism we have just defined becomes $\tilde{\delta}$.)

6.2.3. *Preservation of* $\mathrm{D^b_c}(X_{\text{ét}}, \Lambda)$.

Proposition 6.2.3.1. *Let X be an excellent Noetherian $\mathbf{Z}\left[\frac{1}{n}\right]$-scheme equipped with a dimension function δ. Let K_X be the potential dualizing complex of (X, δ). Then, the functor $\mathrm{D}_X = \mathrm{R}\,\mathbf{Hom}(-, K_X)$ preserves $\mathrm{D^b_c}(X_{\text{ét}}, \Lambda)$.*

Thanks to proposition 6.2.1.1, we can assume that the coefficient ring is $\mathbf{Z}/\ell\mathbf{Z}$, where ℓ is a prime number. We need to show that for any constructible sheaf of Λ-modules $\mathscr{M}$ on X, $\mathrm{D}_X \mathscr{M} \in \mathrm{D^b_c}(X_{\text{ét}}, \Lambda)$. According to proposition 6.2.2.1, if $\mathscr{M}$ is constant, we indeed have $\mathrm{D}_X \mathscr{M} \in \mathrm{D^b_c}(X_{\text{ét}}, \Lambda)$. If $p : Y \to X$ is an étale morphism, we have an evident isomorphism $p^\star \mathrm{D}_X \mathscr{M} \simeq \mathrm{D}_Y p^\star \mathscr{M}$ for any constructible sheaf $\mathscr{M}$ on X. If $\mathscr{M}$ is locally constant, by introducing a surjective étale morphism p such that $p^\star\mathscr{M}$ is constant, we obtain that $\mathrm{D}_X \mathscr{M} \in \mathrm{D^b_c}(X_{\text{ét}}, \Lambda)$ if $\mathscr{M}$ is locally constant.

We then reason by Noetherian induction. For any constructible sheaf $\mathscr{M}$, there exists a dense open set U of X on which $\mathscr{M}$ is locally constant. We denote by $j : U \to X$ the corresponding open immersion and by $i : Z \to X$ a complementary closed immersion. From the above, we have $j^\star \mathrm{D}_X \mathscr{M} = \mathrm{D}_U \mathscr{M}_{|U} \in \mathrm{D^b_c}(U_{\text{ét}}, \Lambda)$. Furthermore, $i^! \mathrm{D}_X \mathscr{M} \simeq \mathrm{D}_Z i^\star \mathscr{M}$. By the Noetherian induction hypothesis, we obtain that $i^! \mathrm{D}_X \mathscr{M}$ belongs to $\mathrm{D^b_c}(Z_{\text{ét}}, \Lambda)$. Lemma 6.2.2.2 allows us to conclude that $\mathrm{D}_X \mathscr{M}$ belongs to $\mathrm{D^b_c}(X_{\text{ét}}, \Lambda)$.

The stability of $\mathrm{D^b_c}(X_{\text{ét}}, \Lambda)$ under D_X allows us to state the following important result :

Proposition 6.2.3.2. *Let $p : X' \to X$ be a regular morphism between excellent Noetherian $\mathbf{Z}\left[\frac{1}{n}\right]$-schemes. Assume X is equipped with a dimension function δ_X and equip X' with the dimension function $\delta_{X'}$ defined in proposition 4.1.1. Then, for any $L \in \mathrm{D^b_c}(X_{\text{ét}}, \Lambda)$, we have a canonical isomorphism $p^\star \mathrm{D}_X L \xrightarrow{\sim} \mathrm{D}_{X'} p^\star L$ and the biduality morphism $p^\star L \to \mathrm{D}^2_{X'} p^\star L$ is identified with the image by $p^\star$ of the biduality morphism $L \to \mathrm{D}^2_X L$.*

This follows immediately from propositions 4.1.1, 4.2.2, 6.2.3.1 and the result of §12.3.2.

6.2.4. *Quasi-injective dimension.*

Proposition 6.2.4.1. *Let X be an excellent Noetherian $\mathbf{Z}\left[\frac{1}{n}\right]$-scheme equipped with a dimension function δ. Let K_X be the potential dualizing complex of (X, δ). The quasi-injective dimension of K_X is $-2\inf_{x\in X} \delta(x)$. In particular, it is finite if and only if X has finite Krull dimension.*

Let us start by giving a lower bound for the quasi-injective dimension of K_X. Let $x \in X$. We denote by $i : Z \to X$ the inclusion of the integral subscheme of X with generic point x. We have a canonical isomorphism $i^! K_X \simeq i^\star \mathrm{R}\,\mathbf{Hom}(\Lambda_Z, K_X)$. The complex $K_Z = i^! K_X$ is a potential dualizing complex for $(Z, \delta_{|Z})$. Consequently $(K_Z)_{\overline{x}} \simeq \Lambda(\delta(x))[2\delta(x)]$. It follows that the quasi-injective dimension of K_X is at least $-2\delta(x)$. We thus obtain the lower bound

$$-2 \inf_{x\in X} \delta(x) \leq \dim.\mathrm{q.inj.}\, K_X\ .$$

Let us show that this inequality is in fact an equality if X has finite Krull dimension. We can proceed by induction on the dimension of X. We can furthermore assume that X is local strictly Henselian (and reduced) with closed point x. Let $\mathscr{M}$ be a constructible sheaf of Λ-modules on X. We need to show that $\mathrm{D}_X \mathscr{M} \in \mathrm{D}^{\leq -2\delta(x)}(X_{\text{ét}}, \Lambda)$. There exists a dense affine open set U on which $\mathscr{M}$ is locally constant. The scheme X being reduced and excellent, up to shrinking X, we can assume that U is regular. Let $j : U \to X$ be the immersion of U. Let $U_1, \dots, U_n$ be the connected components of U, and $\eta_1, \dots, \eta_n$ be the generic points of these components. The scheme U being regular, we know the structure of the potential dualizing complex K_U : for any $1 \leq i \leq n$, we have a canonical isomorphism $K_{U_i} \simeq \Lambda(\delta(\eta_i))[2\delta(\eta_i)]$. In particular, $K_{U_i} \in \mathrm{D}^{\leq -2\delta(\eta_i)}(U_{i_{\text{ét}}}, \Lambda)$. The sheaf $\mathscr{M}_{|U_i}$ being locally constant, we obtain that $\mathrm{D}_{U_i} \mathscr{M}_{|U_i} \in \mathrm{D}^{\leq -2\delta(\eta_i)}(U_{i_{\text{ét}}}, \Lambda)$. According to the affine Lefschetz theorem (cf. **XV**-1.2.2) applied to the affine open immersions $j_i : U_i \to X$, it then follows that $\mathrm{R}j_{i_\star} \mathrm{D}_{U_i} \mathscr{M}_{|U_i}$

belongs to $\mathrm{D}^{\leq \dim \overline{\{\eta_i\}} - 2\delta(\eta_i)}(X_{\text{ét}}, \Lambda)$. Since we have $\dim \overline{\{\eta_i\}} - 2\delta(\eta_i) \leq -2\delta(x)$, it follows that $\mathrm{R}j_\star \, \mathrm{D}_U \mathscr{M}_{|U}$ belongs to $\mathrm{D}^{\leq -2\delta(x)}(X_{\text{ét}}, \Lambda)$.

Let $i\colon Z \to X$ be a closed immersion complementary to j. Thanks to the induction on the dimension, we know that $\mathrm{D}_Z \, i^\star \mathscr{M} \in \mathrm{D}^{\leq -2\delta(x)}(Z_{\text{ét}}, \Lambda)$. Using the canonical distinguished triangle

$$i_\star \, \mathrm{D}_Z \, i^\star \mathscr{M} \to \mathrm{D}_X \mathscr{M} \to \mathrm{R}j_\star \, \mathrm{D}_U \mathscr{M}_{|U} \to i_\star \, \mathrm{D}_Z \, i^\star \mathscr{M}[1] \; ,$$

we indeed obtain that $\mathrm{D}_X \mathscr{M} \in \mathrm{D}^{\leq -2\delta(x)}(X_{\text{ét}}, \Lambda)$, which completes the proof of the proposition.

6.3. The theorem in non-positive degree.

Proposition 6.3.1. *Let X be an excellent Noetherian $\mathbf{Z}\left[\frac{1}{n}\right]$-scheme equipped with a dimension function δ. Let $\mathscr{M}$ be a constructible sheaf of Λ-modules on X. Then, the canonical morphism is an isomorphism in $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X_{\text{ét}}, \Lambda)$.*

$$\mathscr{M} \xrightarrow{\sim} \tau_{\leq 0} \, \mathrm{D}_X \, \mathrm{D}_X \mathscr{M} \; .$$

During this proof, we will say that a constructible sheaf of Λ-modules $\mathscr{M}$ on X is **weakly reflexive** if the canonical morphism $\mathscr{M} \to \tau_{\leq 0} \, \mathrm{D}_X \, \mathrm{D}_X \mathscr{M}$ of the proposition is an isomorphism.

According to theorem 5.1.1, we know that Λ is weakly reflexive. If $g\colon Z \to X$ is a closed immersion and $\mathscr{N}$ is a constructible sheaf of Λ-modules on Z, it is clear that $\mathscr{N}$ is weakly reflexive on Z if and only if $g_\star \mathscr{N}$ is weakly reflexive on X. More generally, if $f\colon Y \to X$ is a finite morphism and $\mathscr{N}$ is a constructible sheaf of Λ-modules on Y, then $\mathscr{N}$ is weakly reflexive if and only if $f_\star \mathscr{N}$ is (see [**SGA 5** I 1.13] and §12.4.4). Let us also note that an appropriate use of the five lemma shows that if we have a short exact sequence $0 \to \mathscr{M}' \to \mathscr{M} \to \mathscr{M}'' \to 0$ of constructible sheaves of Λ-modules, and if $\mathscr{M}''$ is weakly reflexive, then $\mathscr{M}$ is weakly reflexive if and only if $\mathscr{M}'$ is weakly reflexive.

Thanks to the stability by extension stated above and to proposition 6.2.1.1, we can assume that $\Lambda = \mathbf{Z}/\ell\mathbf{Z}$ where ℓ is a prime number. From the preceding remarks, it follows that if $f\colon Y \to X$ is finite and U is an open set of Y, then $f_\star \Lambda_U$ is weakly reflexive. The class of weakly reflexive constructible sheaves of Λ-modules on X being stable by direct factors and extensions, we can conclude by using the dévissage of constructible sheaves from [**SGA 4** IX 5.8].

6.4. The argument of [SGA 4½ [Th. finitude] 4.3].

Définition 6.4.1. Let X be a $\mathbf{Z}\left[\frac{1}{n}\right]$-scheme Noetherian excellent equipped with a dimension function δ. Let $\mathscr{M}$ be a constructible sheaf of Λ-modules on X. We say that $\mathscr{M}$ is **reflexive** if the biduality morphism $\mathscr{M} \to \mathrm{D}_X \, \mathrm{D}_X \mathscr{M}$ is an isomorphism. We will say that the biduality morphism is an isomorphism for X if every constructible sheaf of Λ-modules on X is reflexive.

Proposition 6.4.2. *Let $d \geq 0$. If the biduality morphism is an isomorphism for the $\mathbf{Z}\left[\frac{1}{n}\right]$-schemes excellent Noetherian of dimension $\leq d$, then it is also for the schemes of finite type over such schemes.*

Remarque 6.4.3. In [**SGA 4½** [Th. finitude] 4.3], such a biduality isomorphism is constructed for schemes of finite type over a regular $\mathbf{Z}\left[\frac{1}{n}\right]$-scheme of dimension at most one (but not necessarily excellent). Apart from the hypotheses of excellence and regularity of the base scheme, it is essentially the case $d = 1$ of the proposition. The proof that follows resumes and generalizes that of [**SGA 4½** [Th. finitude] 4.3].

By induction on d, one can assume that the biduality isomorphism is an isomorphism for schemes of finite type (and their strict henselizations) over $\mathbf{Z}\left[\frac{1}{n}\right]$-schemes Noetherian excellent of dimension $< d$. To show that the biduality isomorphism is an isomorphism for every scheme Y of finite type over X, it suffices obviously to do so for $Y = (\mathbf{P}^1)^k \times X$ for every $k \in \mathbf{N}$.

We show by induction on k that for every $\mathbf{Z}\left[\frac{1}{n}\right]$-scheme Noetherian excellent X of dimension at most d (equipped with a dimension function δ), the biduality morphism is an isomorphism for $(\mathbf{P}^1)^k \times X$. The hypothesis of the proposition settles the case $k = 0$. Suppose the property established up to rank $k-1$, with $k \geq 1$. Let X be a Noetherian excellent scheme of dimension d equipped with a dimension function δ. We show that the biduality morphism is an isomorphism for $(\mathbf{P}^1)^k \times X$. Thanks to the proposition 6.2.3.2, one can assume that X is local strictly henselian, with closed point x. Let $\mathscr{M}$ be a constructible sheaf on $(\mathbf{P}^1)^k \times X$. Let C be a cone of the biduality morphism $\mathscr{M} \to \mathrm{D}\,\mathrm{D}\,\mathscr{M}$. We already know that $C \in \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}((\mathbf{P}^1)^k \times X_{\text{ét}}, \Lambda)$ (cf. proposition 6.2.3.1). We will first show that the cohomology sheaves of C are skyscraper, i.e., supported on closed points. The induction hypothesis on d shows that the support of C is contained in the closed subscheme $(\mathbf{P}^1)^k \times x$. Set $Y = (\mathbf{P}^1 \times X)_{(y)}$ where y is the generic point of $\mathbf{P}^1 \times x \subset \mathbf{P}^1 \times X$. We consider the n canonical projections $(\mathbf{P}^1)^k \times X \to \mathbf{P}^1 \times X$ and their base change $(\mathbf{P}^1)^{k-1} \times Y \to Y$ over Y. The scheme Y being of dimension d and the proposition 6.2.3.2 showing in particular that "duality commutes with localizations", the induction hypothesis for $n-1$ implies that if $z \in (\mathbf{P}^1)^k \times X$ is such that $C_{\overline{z}} \neq 0$, then the images of z by the k canonical projections $(\mathbf{P}^1)^k \times X \to \mathbf{P}^1 \times X$ are closed points (since they are over x and different from the generic point y of $\mathbf{P}^1 \times x$). Consequently, such a point z is a closed point of $(\mathbf{P}^1)^k \times x$. In short, the cohomology sheaves of C are supported on closed points. It follows that if we denote $\pi\colon (\mathbf{P}^1)^k \times X \to X$ the canonical morphism, then to show that $C \simeq 0$, it suffices to show that $\mathrm{R}\pi_\star C \simeq 0$. According to [**SGA 4½** [Th. finitude] 4.4], $\mathrm{R}\pi_\star C$ identifies with the cone of the biduality morphism $\mathrm{R}\pi_\star \mathscr{M} \to \mathrm{D}_X\,\mathrm{D}_X\,\mathrm{R}\pi_\star \mathscr{M}$, which is an isomorphism by hypothesis. Consequently, $C \simeq 0$, which completes the proof of the proposition.

6.5. End of the proof. We prove the theorem 6.1.1. Given the previous results, it remains only to show that the biduality morphism is an isomorphism for every $\mathbf{Z}\left[\frac{1}{n}\right]$-scheme Noetherian excellent X equipped with a dimension function δ. As it suffices to obtain the conclusion for the strict henselizations of X, one can assume that X is of finite Krull dimension d. We will reason by induction on d.

Définition 6.5.1. Let $\mathscr{M}$ be a constructible sheaf of Λ-modules on a $\mathbf{Z}\left[\frac{1}{n}\right]$-scheme Noetherian excellent X equipped with a dimension function δ. For every $q \geq 1$, we say that $\mathscr{M}$ satisfies the property $(\mathrm{D})_q$ if the cohomology sheaf $\mathscr{H}^q(\mathrm{D}^2_X\,\mathscr{M})$ is zero.

According to the proposition 6.3.1, $\mathscr{M}$ is reflexive if and only if it satisfies the property $(\mathrm{D})_q$ for every $q \geq 1$.

The dimension function δ serves to formulate the property $(\mathrm{D})_q$. However, it does not depend on it. Indeed, if δ and δ' are two dimension functions on X (connected), there exists a relative integer k such that $\delta' = \delta + k$. The potential dualizing complex $K_{X,\delta'}$ identifies canonically with $K_{X,\delta}(k)[2k]$: the biduality functors $\mathrm{D}^2_{X,\delta}$ and $\mathrm{D}^2_{X,\delta'}$ are canonically isomorphic. There is therefore no need to mention the dimension function in the notation D^2_X, and the properties $(\mathrm{D})_q$ defined relative to δ and δ' are equivalent.

Moreover, the properties $(\mathrm{D})_q$ are clearly local for the étale topology. As Noetherian excellent schemes admit locally for the étale topology dimension functions, one can give them sense even in the absence of a global dimension function. By gluing, one can even give sense to the cohomology sheaves $\mathscr{H}^q(\mathrm{D}^2_X\,\mathscr{M})$.

Let $d \geq 0$. We assume that the biduality isomorphism is an isomorphism for every $\mathbf{Z}\left[\frac{1}{n}\right]$-scheme Noetherian excellent of dimension at most $d-1$ equipped with a dimension function.

We will show by induction on $q \geq 1$ that every constructible sheaf of Λ-modules $\mathscr{M}$ on a Noetherian excellent scheme X of dimension $\leq d$ satisfies the property $(\mathrm{D})_q$.

Let $q \geq 1$. We assume that for every $1 \leq q' < q$, every sheaf of Λ-modules on a Noetherian excellent scheme X of dimension $\leq d$ satisfies the property $(\mathrm{D})_{q'}$.

Lemme 6.5.2. *The integers d and q having been fixed as above, the property $(\mathrm{D})_q$ for the constructible sheaves of Λ-modules on the Noetherian excellent schemes of dimension at most d is stable under extensions and subobjects.*

Indeed, if we have a short exact sequence $0 \to \mathscr{M}' \to \mathscr{M} \to \mathscr{M}'' \to 0$ of constructible sheaves of Λ-modules on such a scheme X, the induction hypothesis if $q \geq 2$ or the proposition 6.3.1 if $q = 1$

implies that we have an exact sequence of sheaves :

$$0 \to \mathscr{H}^q(\mathrm{D}^2_X \mathscr{M}') \to \mathscr{H}^q(\mathrm{D}^2_X \mathscr{M}) \to \mathscr{H}^q(\mathrm{D}^2_X \mathscr{M}'')$$

The property $(\mathrm{D})_q$ is therefore stable under extensions and subobjects.

Lemme 6.5.3. *Let $p\colon Y \to X$ be a finite morphism between Noetherian excellent schemes. Let $\mathscr{M}$ be a constructible sheaf of Λ-modules on Y. Then, $\mathscr{M}$ satisfies the property $(\mathrm{D})_q$ if and only if $p_\star\mathscr{M}$ does.*

This follows immediately from the canonical isomorphism $p_\star\mathscr{H}^q(\mathrm{D}^2_Y \mathscr{M}) \simeq \mathscr{H}^q(\mathrm{D}^2_X\, p_\star\mathscr{M})$ (see [**SGA 5** I 1.12 (a)]) and the conservativity of the functor $p_\star$.

Lemme 6.5.4. *Let X be a Noetherian excellent scheme. Let $\mathscr{C}$ be a strictly full subcategory of the category* $\mathrm{Cons}(X, \Lambda)$ *of constructible sheaves of Λ-modules on X, stable under direct factors and extensions. We assume that for every finite morphism $p\colon Y \to X$, every open immersion $j\colon U \to Y$ with Y normal integral and every prime number ℓ dividing n, we have $p_\star j_!\mathbf{Z}/\ell\mathbf{Z} \in \mathscr{C}$. Then, $\mathscr{C} = \mathrm{Cons}(X, \Lambda)$.*

This is an easy variant of [**SGA 4** IX 5.8].

Lemme 6.5.5. *The integers d and q having been fixed as above, if the property $(\mathrm{D})_q$ is satisfied by the constant sheaf Λ on the Noetherian excellent normal (strictly henselian) schemes of dimension at most d, then the property $(\mathrm{D})_q$ is satisfied by every constructible sheaf of Λ-modules on a Noetherian excellent scheme of dimension at most d.*

According to the lemma 6.5.4, it suffices to establish the property $(\mathrm{D})_q$ for a sheaf of the form $p_\star j_!\mathbf{Z}/\ell\mathbf{Z}$ with ℓ a prime divisor of n, p a finite morphism and j an open immersion between normal integral schemes. According to the lemma 6.5.3, it suffices to establish the property $(\mathrm{D})_q$ for $j_!\mathbf{Z}/\ell\mathbf{Z}$ with $j\colon U \to Y$ an open immersion, with Y normal integral. According to the stability under subobject and extensions of the property $(\mathrm{D})_q$ (cf. lemma 6.5.2), it suffices to treat the case of the sheaf $j_!\Lambda$, which is itself a sub-sheaf of the constant sheaf Λ on the normal scheme Y, which completes the proof of the lemma.

We are thus reduced to showing the property $(\mathrm{D})_q$ for the constant sheaf Λ on the Noetherian excellent normal schemes X of dimension d. One can assume X local strictly henselian with closed point x and generic point η. If $d \leq 1$, X is regular, and then, if we choose the dimension function δ on X such that $\delta(\eta) = 0$, the associated potential dualizing complex K_X on X is the constant sheaf Λ, and then it is evident that Λ satisfies the property $(\mathrm{D})_q$ since we have tautologically $\mathrm{D}_X\,\mathrm{D}_X\,\Lambda \simeq \Lambda$. One can therefore assume that $d \geq 2$. By applying the following lemma to the completion $\widehat{X} \to X$, one sees that one can assume that X is complete :

Lemme 6.5.6. *Let $q \geq 1$. Let $p\colon X' \to X$ be a regular morphism between $\mathbf{Z}\left[\frac{1}{n}\right]$-schemes Noetherian excellent. Let $\mathscr{M}$ be a constructible sheaf of Λ-modules on X. We assume that p is surjective. Then, the sheaf $\mathscr{M}$ satisfies the property $(\mathrm{D})_q$ if and only if $p^\star\mathscr{M}$ does.*

This follows immediately from the proposition 6.2.3.2.

It remains to show that if X is a local strictly henselian Noetherian normal complete scheme of dimension $d \geq 2$, then the constant sheaf Λ on X satisfies the property $(\mathrm{D})_q$. According to the partial algebraization theorem (cf. **V**-3.1.3 and **V**-2.1.2), there exists a finite surjective morphism $p\colon X' \to X$ such that

- the scheme X' is normal ;
- there exists a local Noetherian complete scheme Y of dimension $< d$, a finite type morphism $Z \to Y$, a geometric point $\overline{z} \to Z$ and an isomorphism $X' \simeq \widehat{Z_{(\overline{z})}}$.

The scheme Y is Noetherian excellent and of dimension $< d$. The induction hypothesis on d and the proposition 6.4.2 imply that the biduality morphism is an isomorphism for $Z_{(\overline{z})}$. In particular, the constant sheaf Λ on $Z_{(\overline{z})}$ is reflexive. Applied to the completion morphism $X' \to Z_{(\overline{z})}$, the proposition 6.2.3.2 shows that the constant sheaf Λ on X' is reflexive. In particular, the constant sheaf Λ on X' satisfies the property $(\mathrm{D})_q$. According to the lemma 6.5.3, one can deduce that the sheaf $p_\star\Lambda$ on X satisfies the property $(\mathrm{D})_q$. The finite morphism p being surjective, the canonical morphism $\Lambda \to p_\star\Lambda$ is a monomorphism of sheaves. The property $(\mathrm{D})_q$ being stable under subobjects (cf. lemma 6.5.2), the constant sheaf Λ on X indeed satisfies the property $(\mathrm{D})_q$, which completes the proof of the local duality theorem.

7. General coefficient rings

7.1. Statements.

Définition 7.1.1. Let X be a Noetherian scheme. Let A be a commutative Noetherian ring. We call **dualizing complex** on $\mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X_{\text{ét}}, A)$ (resp. $\mathsf{D}^{\mathsf{b}}_{\mathsf{ctf}}(X_{\text{ét}}, A)$) an object $K \in \mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X_{\text{ét}}, A)$ (resp. $\mathsf{D}^{\mathsf{b}}_{\mathsf{ctf}}(X_{\text{ét}}, A)$) such that the functor $\mathrm{D}_K = \mathrm{R}\,\mathbf{Hom}(-, K)$ preserves $\mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X_{\text{ét}}, A)$ (resp. $\mathsf{D}^{\mathsf{b}}_{\mathsf{ctf}}(X_{\text{ét}}, A)$) and that for every $M \in \mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X_{\text{ét}}, A)$ (resp. $M \in \mathsf{D}^{\mathsf{b}}_{\mathsf{ctf}}(X_{\text{ét}}, A)$), the biduality morphism $M \to \mathrm{D}^2_K M$ is an isomorphism.

This section aims to establish the two following theorems :

Théorème 7.1.2. *Let A be a Λ-algebra Noetherian. Let X be a Noetherian scheme. If they exist, the dualizing complexes on $\mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X_{\text{ét}}, A)$ are unique up to tensor product with invertible objects. Let $R \in \mathsf{D}(A)$ be a strongly pointwise dualizing complex in the sense of* [**Conrad, 2000**, page 120] (vi). *Let K be a dualizing complex on $\mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X_{\text{ét}}, \Lambda)$. Then, $R \overset{\mathrm{L}}{\otimes}_\Lambda K$ is a dualizing complex on $\mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X_{\text{ét}}, A)$.*

Théorème 7.1.3. *Let A be a Λ-algebra Noetherian. Let X be a Noetherian scheme. If they exist, the dualizing complexes on $\mathsf{D}^{\mathsf{b}}_{\mathsf{ctf}}(X_{\text{ét}}, A)$ are unique up to tensor product with invertible objects. Let K be a dualizing complex on $\mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X_{\text{ét}}, \Lambda)$. Then, $A \overset{\mathrm{L}}{\otimes}_\Lambda K$ is a dualizing complex on $\mathsf{D}^{\mathsf{b}}_{\mathsf{ctf}}(X_{\text{ét}}, A)$.*

7.2. Local systems.

Définition 7.2.1. Let X be a Noetherian scheme. A **local system** (of sets) on X is a sheaf of sets on $X_{\text{ét}}$ isomorphic to an inductive limit filtered of sheaves represented by finite étale coverings of X. A **finite local system** is a local system represented by a finite étale covering.

Proposition 7.2.2. *Let X be a Noetherian scheme. The category of local systems on X is equivalent to the category* $\mathrm{Ind}(\mathrm{Rev}(X))$ *of ind-objects in the category* $\mathrm{Rev}(X)$ *of finite étale coverings of X.*

The functor that associates to a finite étale covering $Y \to X$ the sheaf of sets on $X_{\text{ét}}$ represented by Y is fully faithful. By using [**SGA 4** I 8.7.5 a)], one deduces, by passing to the inductive limit, a fully faithful functor from the category $\mathrm{Ind}(\mathrm{Rev}(X))$ to that of sheaves of sets on $X_{\text{ét}}$. By definition, the essential image of this functor is the category of local systems.

Proposition 7.2.3. *Let X be a connected Noetherian scheme. Let $\overline{x}$ be a geometric point of X. The functor that associates to a local system $\mathscr{F}$ the fiber $\mathscr{F}_{\overline{x}}$ is naturally equipped with an action of $\pi_1(X, \overline{x})$ and defines an equivalence between the category of local systems on X and the category $\pi_1(X, \overline{x}) - \mathsf{Ens}$ of sets on which the profinite group $\pi_1(X, \overline{x})$ acts continuously. In other words, the category of local systems of sets on X identifies with the category of sheaves of sets on the classifying topos of the profinite group $\pi_1(X, \overline{x})$.*

Let Ensf be the category of finite sets. According to [**SGA 1** V 7], the functor $\mathrm{Rev}(X) \to \mathsf{Ensf}$ that associates to Y the underlying set of the scheme $Y_{\overline{x}}$ enriches with an action of the group $\pi_1(X, \overline{x})$ to define an equivalence of categories $\mathrm{Rev}(X) \xrightarrow{\sim} \pi_1(X, \overline{x}) - \mathsf{Ensf}$ where $\pi_1(X, \overline{x}) - \mathsf{Ensf}$ is the category of finite (discrete) sets equipped with a continuous action of the profinite group $\pi_1(X, \overline{x})$. By passing this

(vi) We recall that this means here that R belongs to $\mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(A)$ and that for every $x \in \mathrm{Spec}(A)$, $R_{(x)} \in \mathsf{D}(A_{(x)})$ is a dualizing complex for $A_{(x)}$ in the sense of [**Hartshorne, 1966**, page 258], which means that $R_{(x)}$ is of finite injective dimension and that the functor $\mathrm{RHom}(-, R_{(x)})$ induces an involution of $\mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(A_{(x)})$. According to, [**Conrad, 2000**, lemma 3.1.5], it is equivalent to ask that the functor $\mathrm{RHom}_A(-, R)$ induces an involution of $\mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(A)$.

equivalence to ind-objects, one obtains an equivalence $\mathrm{Ind}(\mathrm{Rev}(X)) \xrightarrow{\sim} \pi_1(X,\overline{x}) - \mathsf{Ens}$, which allows to conclude according to the proposition 7.2.2.

From the definition of local systems of sets, one can define local systems of abelian groups, of torsion abelian groups, of modules, etc.

Proposition 7.2.4.

(a) *For every Noetherian scheme X, the category of local systems of sets (resp. of abelian groups) on X admits inductive limits and finite projective limits and the inclusion functor of the category of local systems of sets (resp. of abelian groups) on X into the category of sheaves of sets (resp. of abelian groups) on $X_{\text{ét}}$ commutes with them.*

(b) *Let $p\colon Y \to X$ be a morphism between Noetherian schemes. If $\mathscr{F}$ is a local system on X, then $p^\star\mathscr{F}$ is a local system on Y; the converse is true if p is a surjective finite étale covering.*

(c) *If $\mathscr{G}$ is a local system on Y and p a finite étale covering, then $p_\star\mathscr{G}$ is a local system on X.*

By applying the proposition 7.2.3 to the connected components of X and to all their geometric points, one obtains directly (a). The first part of (b) is trivial. To show the second part of (b), begin by establishing (c). It suffices for that to show that if $\mathscr{G}$ is represented by an étale covering of Y, then $p_\star\mathscr{G}$ is represented by an étale covering of X (which one can assume connected) : by faithfully flat descent, one reduces to the trivial case where Y is a disjoint union of copies of X. Show the last part of (b). Assume that p is a surjective finite étale covering and that $\mathscr{F}$ a sheaf of sets on $X_{\text{ét}}$ such that $p^\star\mathscr{F}$ is a local system. We have an evident isomorphism between $\mathscr{F}$ and the equalizer of the two evident morphisms $p_\star p^\star\mathscr{F} \to p_\star p^\star p_\star p^\star\mathscr{F}$ deduced from the pair of adjoint functors $(p^\star, p_\star)$. According to the other stability properties of local systems, $p_\star p^\star\mathscr{F}$ and $p_\star p^\star p_\star p^\star\mathscr{F}$ are local systems and the equalizer of two morphisms between local systems is again a local system according to (a). One thus obtains that $\mathscr{F}$ is a local system, which completes the proof of (b).

Proposition 7.2.5. *Let X be a Noetherian scheme. The abelian category of local systems of torsion abelian groups on X is stable under extension in the category of sheaves of abelian groups on $X_{\text{ét}}$. More precisely, if $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ is a short exact sequence of sheaves of abelian groups on X such that $\mathscr{F}'$ and $\mathscr{F}''$ are local systems and that $\mathscr{F}'$ is torsion, then $\mathscr{F}$ is a local system of abelian groups.*

This proposition follows immediately from the following lemma :

Lemme 7.2.6. *Let X be a Noetherian scheme. Let $\mathscr{G}$ be a sheaf of torsion abelian groups acting freely on a sheaf of sets $\mathscr{T}$. We denote $p\colon \mathscr{T} \to \mathscr{T}/\mathscr{G}$ the quotient morphism. If $\mathscr{G}$ and $\mathscr{T}/\mathscr{G}$ are local systems, then $\mathscr{T}$ also.*

One can assume that X is connected. Let $\mathscr{Y} = \mathscr{T}/\mathscr{G}$. One can write $\mathscr{Y}$ as union of finite sub-local systems $\mathscr{Y}'$. For each of these $\mathscr{Y}'$, one can consider $p^{-1}(\mathscr{Y}')$: it is a sheaf of sets on $X_{\text{ét}}$ on which $\mathscr{G}$ acts freely with $\mathscr{Y}'$ for quotient. The sheaf $\mathscr{T}$ being union of the sub-sheaves $p^{-1}(\mathscr{Y}')$ associated, to show that $\mathscr{T}$ is a local system, it suffices to show that for every finite sub-local system $\mathscr{Y}'$ of $\mathscr{Y}$, $p^{-1}(\mathscr{Y}')$ is a local system. In short, one can assume that $\mathscr{Y}$ is a finite local system.

We thus assume that $\mathscr{Y}$ is represented by a finite étale covering $q\colon Y \to X$. We would like to show that $\mathscr{T}$ is a local system. For that, admit provisionally that this fact is known in the particular case where $Y = X$ (which amounts to asking that $\mathscr{T}$ be a $\mathscr{G}$-torsor over $X_{\text{ét}}$) ; we will study this case below. Introduce $r\colon Z \to X$ a Galois étale covering trivializing q, i.e., $Y \times_X Z = \coprod_{i\in I} Z_i$ where $Z_i \to Z$ is an isomorphism for every $i \in I$. By inverse image, $r^\star\mathscr{G}$ is a local system on Z acting freely on $r^\star\mathscr{T}$ and the quotient $r^\star(\mathscr{T}/\mathscr{G})$ identifies with $\coprod_{i\in I} \bullet$ where $\bullet$ is the final object of the category of sheaves on $Z_{\text{ét}}$. For every $i \in I$, denote $\mathscr{T}_i \subset r^\star\mathscr{T}$ the inverse image by $r^\star\mathscr{T} \to \coprod_{i\in I} \bullet$ of the copy of $\bullet$ corresponding to i. The sheaf $\mathscr{T}_i$ on $Z_{\text{ét}}$ is therefore a $r^\star\mathscr{G}$-torsor over $Z_{\text{ét}}$. According to what we have provisionally admitted, the sheaves $\mathscr{T}_i$ are local systems on $Z_{\text{ét}}$, therefore $r^\star\mathscr{T} = \coprod_{i\in I} \mathscr{T}_i$ is also a local system on $Z_{\text{ét}}$. According to the proposition 7.2.4 (b), $\mathscr{T}$ is a local system on X.

We have reduced to the situation where $\mathscr{T}/\mathscr{G}$ is the final object of the category of sheaves on $X_{\text{ét}}$, i.e., that $\mathscr{T}$ is a torsor under $\mathscr{G}$. If $\mathscr{G}$ is a finite local system, then $\mathscr{T}$ is representable by a finite étale covering and is therefore a local system ; we will reduce to this case.

The isomorphism class of the $\mathscr{G}$-torsor $\mathscr{T}$ is defined by an element in the set $\mathrm{H}^1_{\text{ét}}(X,\mathscr{G})$. As $\mathrm{H}^1_{\text{ét}}(X,-)$ commutes with filtered inductive limits, there exists a sub-local system of finite abelian groups $\mathscr{G}'$ of $\mathscr{G}$ (assumed torsion), a $\mathscr{G}'$-torsor $\mathscr{T}'$ and a $\mathscr{G}$-isomorphism $\mathscr{T} \simeq \mathscr{G} \otimes_{\mathscr{G}'} \mathscr{T}'$ where we have denoted $\otimes$ the functor of extension of the structure group (cf. [**Giraud, 1971**, proposition 1.3.6, Chapter III]). The extension of the structure group commuting with filtered inductive limits, $\mathscr{T}$ identifies with the inductive limit of the $\mathscr{G}'' \otimes_{\mathscr{G}'} \mathscr{T}'$ for $\mathscr{G}''$ ranging over the ordered set of sub-local systems of finite abelian groups of $\mathscr{G}$ containing $\mathscr{G}'$. According to what precedes, $\mathscr{G}'' \otimes_{\mathscr{G}'} \mathscr{T}'$ is a local system of sets on X. By passing to the inductive limit, $\mathscr{T}$ is indeed a local system.

The result of the following exercise shows that the "torsion" hypothesis is indeed necessary in the proposition 7.2.5, and that moreover, a sheaf that is a local system locally for the étale topology is not necessarily a local system.

Exercice 7.2.7. Let A be the sub-ring of $\mathbf{C}[t]$ formed of polynomials f such that $f(0) = f(1)$: the scheme $C = \mathrm{Spec}(A)$ corresponding is obtained by identifying 0 and 1 in the complex affine line $\mathbf{A}^1_{\mathbf{C}}$.

- Show that C is isomorphic to the plane cubic of equation $x^3 - y^2 + xy = 0$ in the complex affine plane $\mathrm{Spec}(\mathbf{C}[x,y])$ (send x and y respectively to $t(t-1)$ and $t^2(t-1)$).
- Show that C admits a unique singular point O.
- Show that the evident morphism $p : \mathbf{A}^1_{\mathbf{C}} \to C$ is the normalization of C and that the closed reduced subscheme $p^{-1}(O)$ of $\mathbf{A}^1_{\mathbf{C}}$ is $\{0,1\}$.
- Construct an isomorphism $\mathrm{H}^1_{\text{ét}}(C,\mathbf{Z}) \simeq \mathbf{Z}$.
- Show that there exists a sheaf of abelian groups $\mathscr{F}$ on $C_{\text{ét}}$ such that :
  (i) $\mathscr{F}$ be extension of two local systems, and inserts more precisely into a short exact sequence $0 \to \mathbf{Z} \to \mathscr{F} \to \mathbf{Z} \to 0$;
  (i′) Locally for the étale topology, $\mathscr{F}$ be a local system;
  (ii) $\mathscr{F}$ not be a local system.

7.3. Galois partitions.

Définition 7.3.1. A **Galois partition** of a Noetherian scheme X consists of a finite partition of X by reduced connected (non-empty) locally closed subschemes $(S_i)_{i\in I}$ and a Galois étale covering $S'_i \to S_i$ for every $i \in I$.

Définition 7.3.2. Let $p : Y \to X$ be a finite Galois étale covering between Noetherian schemes. Let A be a Λ-algebra. We say that a local system of A-modules $\mathscr{F}$ on X is **rendered ind-unipotent by** Y, if for a geometric point $\overline{y}$ of Y (and thus for all) over a geometric point $\overline{x}$ of X, the discrete $\pi_1(X,\overline{x})$-module $\mathscr{F}_{\overline{x}}$ is ind-unipotent for the distinguished subgroup $\pi_1(Y,\overline{y})$ (cf. subsection 11.3), in other words that $\mathscr{F}$ is filtered inductive limit of sheaves locally constant successive extensions of sheaves whose inverse image by p is a constant sheaf.

Définition 7.3.3. Let X be a Noetherian scheme equipped with a Galois partition $\mathscr{P} = (S'_i \to S_i)_{i\in I}$. Let A be a Λ-algebra. We say of a sheaf of A-modules on X that it is **weakly constructible with respect to** $\mathscr{P}$ if for every $i \in I$, its restriction to S_i is a local system rendered ind-unipotent by S'_i. We denote $\mathrm{FCons}^{\mathscr{P}}(X,A)$ the full sub-category of the category of sheaves of A-modules on X formed of the sheaves weakly constructible for $\mathscr{P}$. If $\mathscr{P}'$ is a second Galois partition, we say that $\mathscr{P}'$ **refines** $\mathscr{P}$ if we have the inclusion $\mathrm{FCons}^{\mathscr{P}}(X,\Lambda) \subset \mathrm{FCons}^{\mathscr{P}'}(X,\Lambda)$ (and thus also $\mathrm{FCons}^{\mathscr{P}}(X,A) \subset \mathrm{FCons}^{\mathscr{P}'}(X,A)$ for every Λ-algebra A).

Proposition 7.3.4. *Let X be a Noetherian scheme equipped with a Galois partition. Let A be a Λ-algebra. The category* $\mathrm{FCons}^{\mathscr{P}}(X,A)$ *is abelian and admits inductive limits ; its inclusion functor into the category of sheaves of A-modules on X is exact and commutes with inductive limits.* $\mathrm{FCons}^{\mathscr{P}}(X,A)$ *is stable under extensions in the category of sheaves of A-modules on X.*

This follows immediately from the properties of ind-unipotent modules for a subgroup (cf. proposition 11.3.4) and the general properties of local systems (cf. subsection 7.2).

Définition 7.3.5. We say of a Galois partition $\mathscr{P} = (S'_i \to S_i)_{i\in I}$ on a Noetherian scheme X that it is **directed** if we have equipped I with a total order such that, either I is empty, or, if we denote i_0 the smallest element of I, S_{i_0} is open and, recursively, $(S'_i \to S_i)_{i\in I-\{i_0\}}$ is a directed Galois partition of the reduced closed subscheme $X - S_{i_0}$. We say that a Galois partition is **dirigible** if there exists a total order on the index set that makes it a directed Galois partition.

Proposition 7.3.6. *Every Galois partition of a Noetherian scheme is refined by a dirigible Galois partition.*

Lemme 7.3.7. *Let $X' \to X$ be a Galois étale covering. Let $(S_i \to X)_{i\in I}$ be a partition of X by a finite number of reduced connected subschemes. We denote S'_i a connected component of the fiber product $S_i \times_X X'$. Then, $(S'_i \to S_i)_{i\in I}$ is a Galois partition of X that refines the Galois partition $(X' \to X)$.*

This follows immediately from Galois theory.

We prove the proposition 7.3.6 by Noetherian induction on X. Let $\mathscr{P} = (S'_i \to S_i)_{i\in I}$ be a Galois partition of a non-empty Noetherian scheme X. First, show that, up to refining $\mathscr{P}$, one can assume that there exists an index $i_0 \in I$ such that S_{i_0} is an open. Indeed, if one chooses an $i_0 \in I$ such that S_{i_0} contains a maximal point of X, S_{i_0} contains a non-empty open U of S_{i_0}. Let $V_1, \dots, V_n$ be the connected components of the reduced closed subscheme $S_{i_0} - U$ of S_{i_0}. According to the lemma, there exists a Galois partition $\mathscr{Q} = (U' \to U, V'_1 \to V_1, \dots, V'_n \to V_n)$ of S_{i_0} that refines the Galois partition $(S'_{i_0} \to S_{i_0})$ of S_{i_0}. Up to replacing the initial Galois partition of X by its refinement $\mathscr{Q} \cup \mathscr{P}'$ with $\mathscr{P}' = (S'_i \to S_i)_{i\in I-\{i_0\}}$, one can effectively assume that S_{i_0} is open.

One can apply the Noetherian induction hypothesis to the Galois partition $\mathscr{P}'$ of $X - S_{i_0}$ to obtain a refinement $\mathscr{P}''$ indexed by a certain totally ordered set J that makes $\mathscr{P}''$ a directed Galois partition of $X - S_{i_0}$. The Galois partition $(S'_{i_0} \to S_{i_0}) \cup \mathscr{P}''$ refines $\mathscr{P}$, and if one extends the order on J to an order on the disjoint union $\{i_0\} \amalg J$ in such a way as to make i_0 the smallest element, one has obtained a directed partition.

Exercice 7.3.8. Show that if $\mathscr{P}$ and $\mathscr{P}'$ are two Galois partitions of a Noetherian scheme X, there exists a Galois partition (dirigible) refining both $\mathscr{P}$ and $\mathscr{P}'$.

7.4. Devisages. The goal of this subsection is to establish the following result :

Proposition 7.4.1. *Let A be a Λ-algebra Noetherian. Let X be a Noetherian scheme. Let $\mathscr{T}$ be a strictly full triangulated subcategory of $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X_{\text{ét}}, A)$ stable under direct factors. We assume that for every prime number ℓ dividing n and every finite type $A/\ell A$-module N, the functor $N \overset{\mathrm{L}}{\otimes}_{\mathbf{Z}/\ell\mathbf{Z}} - : \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X_{\text{ét}}, \mathbf{Z}/\ell\mathbf{Z}) \to \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X_{\text{ét}}, A)$ takes values in $\mathscr{T}$. Then, $\mathscr{T} = \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X_{\text{ét}}, A)$.*

The proof of this proposition is postponed to the end of this subsection.

Proposition 7.4.2. *Let A be a Λ-algebra. Let $\mathscr{P} = (S'_i \to S_i)_{i\in I}$ with $I = \{1, \dots, N\}$ a directed Galois partition of a Noetherian scheme X. Let $k_i : S_i \to X$ be the canonical immersion for every $i \in I$. For every $i \in I$, the functor $k_{i!} : \mathrm{FCons}^{(S'_i \to S_i)}(S_i, A) \to \mathrm{FCons}^{\mathscr{P}}(X, A)$ is fully faithful. Every object $\mathscr{F}$ of $\mathrm{FCons}^{\mathscr{P}}(X, A)$ admits an increasing (functorial) filtration $(\mathrm{Fil}_n \mathscr{F})_{n\in\mathbf{Z}}$ such that $\mathrm{Fil}_0 \mathscr{F} = 0$, $\mathrm{Fil}_N \mathscr{F} = \mathscr{F}$ and that for every $1 \le i \le N$, the quotient $\mathrm{Fil}_i \mathscr{F} / \mathrm{Fil}_{i-1} \mathscr{F}$ is in $k_{i!}\, \mathrm{FCons}^{S'_i \to S_i}(S_i, A)$.*

This is trivial.

Proposition 7.4.3. *Let A be a Λ-algebra. Let $\mathscr{P} = (S'_i \to S_i)_{i\in I}$ be a dirigible Galois partition of a Noetherian scheme X. Let $k_i : S_i \to X$ be the canonical immersion for every $i \in I$. If A is Noetherian, then $\mathrm{FCons}^{\mathscr{P}}(X, A)$ is a locally Noetherian category (cf.* [**Gabriel, 1962**, pages 325–326]*). If A is Artinian, $\mathrm{FCons}^{\mathscr{P}}(X, A)$ is locally finite and admits a finite number of simple objects; more precisely, if we denote $k_i : S_i \to X$ the canonical inclusions, W_i a finite set representative of the simple objects of the category of $A[\mathrm{Gal}(S'_i/S_i)]$-modules* (via *the choice of a geometric point of S'_i, we identify these objects to local systems on S_i trivialized by S'_i), then the objects $k_{i!}\mathscr{F}$ for $i \in I$ and $\mathscr{F} \in W_i$ form a representative set of the simple objects of $\mathrm{FCons}^{\mathscr{P}}(X, A)$.*

Assume A Noetherian. Show that $\mathrm{FCons}^{\mathscr{P}}(X,A)$ is locally Noetherian. We already know that this abelian category admits filtered inductive limits and that these are exact. It remains to show that every object of $\mathrm{FCons}^{\mathscr{P}}(X,A)$ is inductive limit of Noetherian objects (or more precisely, but this amounts to the same, "union" of its Noetherian subobjects). If the index set I of $\mathscr{P}$ is empty, it is trivial. Otherwise, one can choose a total order on I that makes $\mathscr{P}$ a directed Galois partition, and denote i_0 the smallest element of I. Let $j: S_{i_0} \to X$ be the corresponding (open) immersion and $k: X - S_{i_0} \to X$ the immersion of the complementary reduced closed subscheme. For every object $\mathscr{F} \in \mathrm{FCons}^{\mathscr{P}}(X,A)$, we have a short exact sequence :

$$0 \to j_! j^\star \mathscr{F} \to \mathscr{F} \to k_\star k^\star \mathscr{F} \to 0 .$$

By induction on the cardinality of I, one can assume that $k^\star \mathscr{F}$ is "union" of its Noetherian subobjects in $\mathrm{FCons}^{\mathscr{P}'}(X - S_{i_0}, A)$ with $\mathscr{P}' = \mathscr{P} - (S'_{i_0} \to S_{i_0})$. Of course, an object $\mathscr{R} \in \mathrm{FCons}^{\mathscr{P}'}(X - S_{i_0}, A)$ is Noetherian if and only if $k_\star \mathscr{R}$ is in $\mathrm{FCons}^{\mathscr{P}}(X,A)$. One thus disposes of an inductive system $(\mathscr{H}_b)_{b \in B}$ indexed by a filtered ordered set B of Noetherian subobjects of $k_\star k^\star \mathscr{F}$ such that $\mathscr{F}$ be the union of the subobjects $\pi^{-1}(\mathscr{H}_b)$ of $\mathscr{F}$ for $b \in B$. If each of the $\pi^{-1}(\mathscr{H}_b)$ is union of its Noetherian subobjects, then $\mathscr{F}$ also. This allows to assume that $k^\star \mathscr{F}$ is Noetherian. Concerning $j^\star \mathscr{F}$, by using that the category of discrete $A[G]$-modules (with G profinite group) is locally Noetherian, one obtains that the object $j^\star \mathscr{F}$ of $\mathrm{FCons}^{S'_{i_0} \to S_{i_0}}(S_{i_0}, A)$ is union of its Noetherian subobjects ; one deduces immediately that $j_! j^\star \mathscr{F}$ is also union of its Noetherian subobjects in $\mathrm{FCons}^{\mathscr{P}}(X,A)$. To conclude that $\mathscr{F}$ is union of its Noetherian subobjects, one uses the following lemma :

Lemme 7.4.4. *Let A be a Λ-algebra Noetherian. Let $\mathscr{P}$ be a dirigible Galois partition of a Noetherian scheme X. Let $0 \to \mathscr{H} \to \mathscr{F} \to \mathscr{G} \to 0$ be a short exact sequence in $\mathrm{FCons}^{\mathscr{P}}(X,A)$. We assume that $\mathscr{G}$ is a Noetherian object of $\mathrm{FCons}^{\mathscr{P}}(X,A)$ and that $\mathscr{H}$ is union of its Noetherian subobjects. Then, $\mathscr{F}$ is also union of its Noetherian subobjects.*

It is evident that $\mathscr{G}$ is a constructible sheaf of A-modules on X. According to [**SGA 4** IX 2.7.3], the functor $\mathrm{Ext}^1(\mathscr{G}, -)$ from the category of sheaves of A-modules on X to that of A-modules commutes with filtered inductive limits. The given exact sequence defining an element in $\mathrm{Ext}^1(\mathscr{G}, \mathscr{H})$ and $\mathscr{H}$ writing as a filtered inductive limit of its Noetherian subobjects, there exists a Noetherian subobject $\mathscr{H}'$ of $\mathscr{H}$, a short exact sequence $0 \to \mathscr{H}' \to \mathscr{F}' \to \mathscr{G} \to 0$ and a commutative diagram of the following form, where the left square is cocartesian :

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{H}' \arrow[r] \arrow[d] & \mathscr{F}' \arrow[r] \arrow[d] & \mathscr{G} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathscr{H} \arrow[r] & \mathscr{F} \arrow[r] & \mathscr{G} \arrow[r] & 0
\end{tikzcd} ,$$

In fact, one can replace $\mathscr{H}'$ by any subobject (Noetherian) $\mathscr{H}''$ of $\mathscr{H}$ containing $\mathscr{H}'$, and $\mathscr{F}$ identifies with the union of the subobjects $\mathscr{F}''$ thus defined. To conclude, it suffices to show that such a $\mathscr{F}''$ is a Noetherian object of $\mathrm{FCons}^{\mathscr{P}}(X,A)$, which is evident since it is extension of two Noetherian objects $\mathscr{H}''$ and $\mathscr{G}$.

Finish the proof of the proposition 7.4.3. Assume A Artinian. It is a matter of finding a finite set of simple objects from which all Noetherian objects are obtained by successive extensions. Given the devissage of the proposition 7.4.2, one can assume that $\mathscr{P}$ consists of a single Galois covering $X' \to X$. Choose a geometric point $\overline{x}'$ of X' over a geometric point $\overline{x}$ of X. Denote $G = \pi_1(X, \overline{x})$ and $H = \pi_1(X', \overline{x'})$. The profinite group H identifies with an open distinguished subgroup of G. Denote $K = G/H$ the finite quotient group. The ring $A[K]$ is obviously left Artinian, one can denote W a finite representative set of simple objects ; considered as discrete $A[G]$-modules, the elements of W are still simple.

The category $\mathrm{FCons}^{\mathscr{P}}(X,A)$ identifies with the category of discrete G-modules ind-unipotent for H. Every object of this category writes as a union of finite type subobjects unipotent for H, and such subobjects devisse themselves into successive extensions of objects on which H acts trivially, these latter identifying with finite type $A[K]$-modules, they devisse into successive extensions of elements of W.

Définition 7.4.5. Let A be a Λ-algebra Noetherian. Let X be a Noetherian scheme. Let $\mathscr{P}$ be a Galois partition. We denote $\mathrm{Cons}^{\mathscr{P}}(X, A)$ the full sub-category of $\mathrm{FCons}^{\mathscr{P}}(X, A)$ formed of the constructible sheaves of A-modules (which amounts here to saying that the fibers are finite type A-modules). If $\mathscr{P}$ is dirigible, it is of course the abelian sub-category of Noetherian objects of $\mathrm{FCons}^{\mathscr{P}}(X, A)$.

Proposition 7.4.6. *Let A be a Λ-algebra Noetherian. Let X be a Noetherian scheme. Let $\mathscr{P}$ be a dirigible Galois partition of X. Let $\mathscr{F}$ be an object of $\mathrm{Cons}^{\mathscr{P}}(X, A)$. There exists a finite filtration of $\mathscr{F}$ in $\mathrm{Cons}^{\mathscr{P}}(X, A)$ whose successive quotients are direct factors of objects of the form $M \otimes_{\mathbf{Z}/\ell\mathbf{Z}} \mathscr{F}_0$ where ℓ is a prime number dividing n, M a finite type $A/\ell A$-module and $\mathscr{F}_0$ an object of $\mathrm{Cons}^{\mathscr{P}}(X, \mathbf{Z}/\ell\mathbf{Z})$.*

According to the proposition 7.4.2, one can assume that $\mathscr{P}$ consists of a single Galois covering $X' \to X$. By resuming the notations used in the proof of the Artinian case of the proposition 7.4.3, one can identify $\mathscr{F}$ with a discrete $A[G]$-module unipotent for the closed distinguished subgroup H. This unipotence property allows to assume that H acts trivially, so that one ends up with an action of the finite group $K = G/H = \mathrm{Gal}(X'/X)$ (in short, one can assume that $\mathscr{F}$ is a local system of A-modules trivialized by X'). The following lemma applied to the group algebra $B = \Lambda[K]$ allows to conclude.

Lemme 7.4.7. *Let A be a Λ-algebra Noetherian. Let B be a finite Λ-algebra not necessarily commutative. Every finite type (left) $A \otimes_\Lambda B$-module admits a finite filtration whose successive quotients be direct factors of $A \otimes_\Lambda B$-modules of the form $N \otimes_{\mathbf{F}_\ell} L$ where ℓ is a prime number dividing n, N a finite type $A/\ell A$-module and L a simple $B/\ell B$-module.*

First, as $A \otimes_\Lambda B$ is obviously left Noetherian, one can proceed to a Noetherian induction; it suffices therefore to show that every non-zero $A \otimes_\Lambda B$-module admits a non-zero sub-module direct factor of a module of the form $N \otimes_{\mathbf{F}_\ell} L$ with N a finite type $A/\ell A$-module, L a simple $B/\ell B$-module and ℓ a prime number dividing n. This allows to assume that B is a semi-simple ring. Indeed, for every non-zero B-module M, the annihilator of the Jacobson radical $\mathscr{N}$ of B in M is a non-zero sub-B-module of M (cf. [**Lam, 1991**, § 4]); if M is a non-zero $A \otimes_\Lambda B$-module, the annihilator of $\mathscr{N}$ in M identifies therefore with a non-zero $A \otimes_\Lambda (B/\mathscr{N})$-module and the ring $B/\mathscr{N}$ is indeed semi-simple.

Second, the statement of the lemma is true for a product $B = B_1 \times \cdots \times B_k$ of rings if and only if it is true for each of the B_i and the statement is also invariant under Morita equivalence (cf. [**Lam, 1999**, § 18]) since one can formulate it intrinsically in terms of the categories of B-modules. Given the Artin-Wedderburn theorem on the structure of semi-simple rings (cf. [**Lam, 1991**, 3.5]), one can therefore assume that B is a finite field, *a priori* non-commutative, but effectively commutative by virtue of Wedderburn's theorem (cf. [**Lam, 1991**, 13.1]).

Third, the statement is true in the particular case to which one has reduced above. Let $B = L$ a finite extension of $\mathbf{F}_\ell$, for a certain prime number ℓ dividing n. One can assume that ℓ annihilates A. Let M be an $A \otimes_{\mathbf{F}_\ell} L$-module. The extension $L/\mathbf{F}_\ell$ is Galois, denote G its Galois group. The map $\gamma\colon L \otimes_{\mathbf{F}_\ell} L \to \prod_{\sigma \in G} L$ defined by $\sigma(a \otimes b) = (\sigma(a)b)_{\sigma \in G}$ is a bijection according to Galois theory and it realizes an isomorphism of (L, L)-bimodules $\gamma\colon L \otimes_{\mathbf{F}_\ell} L \xrightarrow{\sim} \prod_{\sigma \in G} {}_\sigma L_{\mathrm{Id}}$ where we have indicated in index the morphisms $L \to L$ that define the left and right module structures on the different factors. As $M \otimes_{L\,\mathrm{Id}} L_{\mathrm{Id}}$ identifies tautologically with M as $A \otimes_{\mathbf{F}_\ell} L$-module, one deduces that the $A \otimes_{\mathbf{F}_\ell} L$-module M identifies as one wanted with a direct factor of the $A \otimes_{\mathbf{F}_\ell} L$-module $M \otimes_L (L \otimes_{\mathbf{F}_\ell} L) \simeq M \otimes_{\mathbf{F}_\ell} L$ (where L acts by multiplication on the right factor).

We are now in a position to prove the proposition 7.4.1. Let $\mathscr{T}$ be such a triangulated subcategory of $\mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X_{\text{ét}}, A)$. Let $\mathscr{F}$ be a constructible sheaf of A-modules on X. It is a matter of showing that $\mathscr{F}$ belongs to $\mathscr{T}$. There obviously exists a Galois partition $\mathscr{P}$ such that $\mathscr{F}$ belongs to $\mathrm{Cons}^{\mathscr{P}}(X, A)$. According to the proposition 7.3.6, one can assume that $\mathscr{P}$ is dirigible. One can then apply the proposition 7.4.6 to conclude that $\mathscr{F}$ belongs to $\mathscr{T}$.

7.5. Dualizing complexes on $\mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X_{\text{ét}}, A)$.

7.5.1. *Uniqueness.*

PROPOSITION 7.5.1.1. *Let X be a Noetherian scheme. Let A be a Noetherian ring. If K and K' are two dualizing complexes on $\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(X_{\text{ét}}, A)$ (resp. $\mathsf{D}^{\mathrm{b}}_{\mathrm{ctf}}(X_{\text{ét}}, A)$), then there exists an invertible object $L \in \mathsf{D}^{\mathrm{b}}_{\mathrm{ctf}}(X_{\text{ét}}, A)$ (cf. proposition 9.2 for more precision) such that K' is isomorphic to $L \overset{\mathrm{L}}{\otimes}_A K$.*

LEMME 7.5.1.2. *Let X be a Noetherian scheme. Let A be a Noetherian ring. We assume that K is a dualizing complex on $\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(X_{\text{ét}}, A)$. For every $F \in \mathsf{D}^{-}_{\mathrm{c}}(X_{\text{ét}}, A)$, if $\mathrm{D}_K F \in \mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(X_{\text{ét}}, A)$, then $F \in \mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(X_{\text{ét}}, A)$.*

Begin by showing that one can assume that $\mathrm{D}_K F = 0$. For every $F \in \mathsf{D}(X_{\text{ét}}, A)$, denote $\varepsilon_F : F \to \mathrm{D}_K \mathrm{D}_K F$ the biduality morphism. The composed morphism

$$\mathrm{D}_K F \xrightarrow{\varepsilon_{\mathrm{D}_K F}} \mathrm{D}_K \mathrm{D}_K \mathrm{D}_K F \xrightarrow{\mathrm{D}_K(\varepsilon_F)} \mathrm{D}_K F$$

is the identity of $\mathrm{D}_K F$. As $\mathrm{D}_K F$ is in $\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(X_{\text{ét}}, A)$, the fact that K is dualizing implies that $\varepsilon_{\mathrm{D}_K F}$ is an isomorphism. Consequently, $\mathrm{D}_K(\varepsilon_F)$ is an isomorphism. Up to replacing F by a cone of ε_F, one can assume that $\mathrm{D}_K F = 0$.

By contradiction, assume that F is non-zero. There then exists a non-zero morphism $p : F \to F'$ with $F' \in \mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(X_{\text{ét}}, A)$ (for example, the canonical morphism $F \to \tau_{\geq n} F$ for a well-chosen integer n). Consider the commutative square :

$$\begin{tikzcd}
F \arrow[r, "p"] \arrow[d, "\varepsilon_F"] & F' \arrow[d, "\varepsilon_{F'}", "\sim"'] \\
\mathrm{D}^2_K F \arrow[r, "\mathrm{D}^2_K(p)"] & \mathrm{D}^2_K F'
\end{tikzcd}$$

On one side, $\mathrm{D}^2_K F$ is zero, so $\varepsilon_{F'} \circ p = 0$, but on the other, $\varepsilon_{F'}$ is an isomorphism, hence $p = 0$, which leads to a contradiction.

Establish the proposition 7.5.1.1 in the case of dualizing complexes on $\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(X_{\text{ét}}, A)$, the proof that follows will also hold for $\mathsf{D}^{\mathrm{b}}_{\mathrm{ctf}}(X_{\text{ét}}, A)$ (with the caveat that it will no longer be necessary to resort to the lemma above). We set $Y = \mathrm{D}_K K' \in \mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(X_{\text{ét}}, A)$. As K is dualizing, we also have a privileged isomorphism $K' = \mathrm{D}_K Y$. For every $Z \in \mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(X_{\text{ét}}, A)$, we have a functorial isomorphism $\mathrm{D}_{K'} Z \simeq \mathrm{D}_K(Z \overset{\mathrm{L}}{\otimes}_A Y)$ in $\mathsf{D}(X_{\text{ét}}, A)$. Thanks to the lemma, one deduces that for every $Z \in \mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(X_{\text{ét}}, A)$, one has $Z \overset{\mathrm{L}}{\otimes}_A Y \in \mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(X_{\text{ét}}, A)$. We thus have a commutative triangle of categories and functors (up to isomorphism of functors) :

$$\begin{tikzcd}
\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(X_{\text{ét}}, A) \arrow[r, "- \overset{\mathrm{L}}{\otimes}_A Y"] \arrow[dr, "\mathrm{D}_{K'}"'] & \mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(X_{\text{ét}}, A) \arrow[d, "\mathrm{D}_K"] \\
& (\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(X_{\text{ét}}, A))^{\mathrm{opp}}
\end{tikzcd}$$

As D_K and $\mathrm{D}_{K'}$ are equivalences, the functor $- \overset{\mathrm{L}}{\otimes}_A Y$ also. In particular, Y is an invertible complex. Denote Y' the inverse of Y. We have an isomorphism of functors $\mathrm{R}\,\mathbf{Hom}(Y, -) \simeq Y' \overset{\mathrm{L}}{\otimes}_A -$ (if a functor is an equivalence, its right adjoint is a quasi-inverse). As $K' = \mathrm{D}_K Y$, one can deduce that $K' \simeq Y' \overset{\mathrm{L}}{\otimes}_A K$.

7.5.2. *Reduction to the case $\Lambda = \mathbf{Z}/\ell\mathbf{Z}$.*

PROPOSITION 7.5.2.1. *Let A be a Noetherian ring. Let $R \in \mathsf{D}(A)$ be a strongly pointwise dualizing complex. Let J be an ideal of A. We set $A' = A/J$ and $R' = \mathrm{RHom}_A(A', R) \in \mathsf{D}(A')$. Then, R' is a strongly pointwise dualizing complex. If we denote* D *(resp.* D'*) the functor* $\mathrm{RHom}_A(-, R)$ *(resp.* $\mathrm{RHom}_{A'}(-, R')$*) on $\mathsf{D}(A)$ (resp. $\mathsf{D}(A')$) and* oub $: \mathsf{D}(A') \to \mathsf{D}(A)$ *the "restriction of scalars" functor, we have a canonical isomorphism :*

$$\mathrm{oub} \circ \mathrm{D}' \simeq \mathrm{D} \circ \mathrm{oub}\ .$$

By passing to the left adjoints of these functors, this follows from the results of §8 :

Proposition 7.5.2.2. *Let X be a Noetherian scheme. Let $K \in \mathsf{D}^{\mathsf{b}}_{\mathsf{ctf}}(X_{\text{ét}}, \Lambda)$. Let A be a Noetherian Λ-algebra. Let J be an ideal of A. Let $R \in \mathsf{D}(A)$ be a strongly pointwise dualizing complex. We set $A' = A/J$ and $R' = \mathrm{RHom}_A(A', R) \in \mathsf{D}(A')$. We denote $K_R = K \overset{\mathrm{L}}{\otimes}_\Lambda R \in \mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X_{\text{ét}}, A)$ and $K_{R'} = K \overset{\mathrm{L}}{\otimes}_\Lambda R' \in \mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X_{\text{ét}}, A')$. We denote $\mathrm{oub} : \mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X_{\text{ét}}, A') \to \mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X_{\text{ét}}, A)$ the evident conservative functor. Then, for every $M \in \mathsf{D}(X_{\text{ét}}, A')$, we have a canonical isomorphism in $\mathsf{D}(X_{\text{ét}}, A)$:*

$$\mathrm{oub}(\mathrm{R}\,\mathbf{Hom}_{A'}(M, K_{R'})) \simeq \mathrm{R}\,\mathbf{Hom}_A(\mathrm{oub}(M), K_R)\,.$$

Moreover, if K_R is a dualizing complex on $\mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X_{\text{ét}}, A)$, then $K_{R'}$ is one on $\mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X_{\text{ét}}, A')$ and the converse is true if J is nilpotent.

The other assertions being easy consequences, it is a matter of showing that we have a canonical isomorphism $\mathrm{R}\,\mathbf{Hom}_A(A', K_R) \simeq \mathrm{oub}(K_{R'})$ in $\mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X_{\text{ét}}, A)$, which follows from proposition 10.1.2.

Corollaire 7.5.2.3. *To prove the theorem 7.1.2, one can assume that $\Lambda = \mathbf{Z}/\ell\mathbf{Z}$ where ℓ is a prime number.*

It being understood that the uniqueness property of dualizing complexes has already been obtained (cf. proposition 7.5.1.1), it is evident that to prove the theorem 7.1.2, one can assume that $\Lambda = \mathbf{Z}/\ell^\nu\mathbf{Z}$ where ℓ is a prime number and $\nu \geq 1$. Set $A' = A/\ell A$. Denote $R \in \mathsf{D}(A)$ a strongly pointwise dualizing complex. According to the proposition 7.5.2.1, the complex $R' = \mathrm{RHom}_A(A', R) \in \mathsf{D}(A')$ is one for A'. Applying first the proposition 7.5.2.2 to the case where $\Lambda \to A$ is $\Lambda \to \mathbf{Z}/\ell\mathbf{Z}$, we obtain that $K'' = \mathrm{RHom}_\Lambda(\mathbf{Z}/\ell\mathbf{Z}, K)$ is a dualizing complex on $\mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X_{\text{ét}}, \mathbf{Z}/\ell\mathbf{Z})$. According to the lemma 6.2.2.4, we also have an isomorphism $K'' \simeq K \overset{\mathrm{L}}{\otimes}_\Lambda \mathbf{Z}/\ell\mathbf{Z}$ in $\mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X_{\text{ét}}, \mathbf{Z}/\ell\mathbf{Z})$. Apply the theorem 7.1.2 in the case of the $\mathbf{Z}/\ell\mathbf{Z}$-algebra A' : one obtains that $K'' \overset{\mathrm{L}}{\otimes}_{\mathbf{Z}/\ell\mathbf{Z}} R'$ is a dualizing complex on $\mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X_{\text{ét}}, A')$. This object $K'' \overset{\mathrm{L}}{\otimes}_{\mathbf{Z}/\ell\mathbf{Z}} R'$ also identifies with $K \overset{\mathrm{L}}{\otimes}_\Lambda R' = K_{R'}$. According to the proposition 7.5.2.2, it follows that $K \overset{\mathrm{L}}{\otimes}_\Lambda R = K_R$ is a dualizing complex on $\mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X_{\text{ét}}, A)$, which completes the proof of this corollary.

7.5.3. *Proof of the theorem 7.1.2.* The uniqueness statement of dualizing complexes on $\mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X_{\text{ét}}, A)$ has already been obtained, cf. proposition 7.5.1.1. For the existence statement, according to the corollary 7.5.2.3, one can assume that $\Lambda = \mathbf{Z}/\ell\mathbf{Z}$ with ℓ a prime number. We take K a dualizing complex on $\mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X_{\text{ét}}, \mathbf{Z}/\ell\mathbf{Z})$. Let A be a $\mathbf{Z}/\ell\mathbf{Z}$-algebra Noetherian and $R \in \mathsf{D}(A)$ a strongly pointwise dualizing complex. Denote $\mathrm{D}_X = \mathrm{R}\,\mathbf{Hom}_\Lambda(-, K)$ the duality functor on $\mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X_{\text{ét}}, \Lambda)$ induced by K and D_A that induced by R on $\mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(A)$. Denote $\mathrm{D}_{X,A}$ the functor $\mathrm{R}\,\mathbf{Hom}_A(-, K_R)$ on $\mathsf{D}(X_{\text{ét}}, A)$ where $K_R = K \overset{\mathrm{L}}{\otimes}_A R$. The proposition 10.1.3 (Λ is a field) shows that we have a canonical isomorphism, for every $N \in \mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(A)$ and $\mathscr{F} \in \mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X_{\text{ét}}, \mathbf{Z}/\ell\mathbf{Z})$:

$$\mathrm{D}_{X,A}(N \overset{\mathrm{L}}{\otimes}_{\mathbf{Z}/\ell\mathbf{Z}} \mathscr{F}) \simeq (\mathrm{D}_A\, N) \overset{\mathrm{L}}{\otimes}_{\mathbf{Z}/\ell\mathbf{Z}} (\mathrm{D}_X\, \mathscr{F})\,.$$

By hypothesis, $\mathrm{D}_A\, N \in \mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(A)$ and $\mathrm{D}_X\, \mathscr{F} \in \mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X_{\text{ét}}, \mathbf{Z}/\ell\mathbf{Z})$, which allows to deduce that $\mathrm{D}_{X,A}(N \overset{\mathrm{L}}{\otimes}_{\mathbf{Z}/\ell\mathbf{Z}} \mathscr{F})$ belongs to $\mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X_{\text{ét}}, A)$, then that $\mathrm{D}_{X,A}$ preserves $\mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X_{\text{ét}}, A)$ thanks to the devissage of the proposition 7.4.1. With the same notations, the biduality morphisms $N \to \mathrm{D}^2_A\, N$ and $\mathscr{F} \to \mathrm{D}^2_X\, \mathscr{F}$ are isomorphisms, the biduality morphism $N \overset{\mathrm{L}}{\otimes}_{\mathbf{Z}/\ell\mathbf{Z}} \mathscr{F} \to \mathrm{D}^2_{X,A}(N \overset{\mathrm{L}}{\otimes}_{\mathbf{Z}/\ell\mathbf{Z}} \mathscr{F})$ is therefore one too ; the same devissage allows to conclude that $\mathrm{D}_{X,A}$ defines an involution of $\mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X_{\text{ét}}, A)$, i.e., that K_R is a dualizing complex on $\mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X_{\text{ét}}, A)$, which completes the proof of the theorem 7.1.2.

7.6. Dualizing complexes on $\mathsf{D}^{\mathsf{b}}_{\mathsf{ctf}}(X_{\text{ét}}, A)$. The uniqueness assertion of dualizing complexes on $\mathsf{D}^{\mathsf{b}}_{\mathsf{ctf}}(X_{\text{ét}}, A)$ has already been stated in the proposition 7.5.1.1. The essential part of this subsection aims to establish the following theorem, from which we will deduce in a few lines the théorème 7.1.3 :

Théorème 7.6.1. *Let X be a Noetherian scheme. We assume that there exists a dualizing complex on $\mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X_{\text{ét}}, \Lambda)$. Let A be a Noetherian Λ-algebra. For all K and L objects of $\mathsf{D}^{\mathsf{b}}_{\mathsf{ctf}}(X_{\text{ét}}, A)$, the object $\mathrm{R}\,\mathbf{Hom}_A(K, L)$ belongs to $\mathsf{D}^{\mathsf{b}}_{\mathsf{ctf}}(X_{\text{ét}}, A)$ and for every $M \in \mathsf{D}^+(A)$, the following canonical morphism is an isomorphism :*

$$M \overset{\mathrm{L}}{\otimes}_A \mathrm{R}\,\mathbf{Hom}_A(K, L) \xrightarrow{\sim} \mathrm{R}\,\mathbf{Hom}_A(K, M \overset{\mathrm{L}}{\otimes}_A L)\,.$$

If A' is an A-algebra, that K and L are objects of $\mathsf{D}^{\mathrm{b}}_{\mathrm{ctf}}(X_{\text{ét}}, A)$, then we have a canonical isomorphism in $\mathsf{D}^{\mathrm{b}}(X_{\text{ét}}, A')$:

$$A' \overset{\mathrm{L}}{\otimes}_A \mathrm{R}\,\mathbf{Hom}_A(K, L) \xrightarrow{\sim} \mathrm{R}\,\mathbf{Hom}_{A'}(A' \overset{\mathrm{L}}{\otimes}_A K, A' \overset{\mathrm{L}}{\otimes}_A L) .$$

Show that one can deduce the théorème 7.1.3 from this theorem 7.6.1 and the théorème 7.1.2 which has already been established. Begin with a lemma :

Lemme 7.6.2. *Let A be a Noetherian ring. Let $K \in \mathsf{D}^-_{\mathrm{c}}(A)$. Then K is zero if and only if for every maximal ideal $\mathfrak{m}$ of A, the object $(A/\mathfrak{m}) \overset{\mathrm{L}}{\otimes}_A K$ is zero.*

We assume that K is not zero. We want to show that there exists a maximal ideal $\mathfrak{m}$ of A such that $(A/\mathfrak{m}) \overset{\mathrm{L}}{\otimes}_A K$ is not zero. Let q be the largest integer such that $\mathrm{H}^q(K)$ is non-zero. One can assume that K is a complex formed of projective finite type A-modules and zero in degrees strictly greater than q. By construction of the derived tensor product, we have an isomorphism $\mathrm{H}^q((A/\mathfrak{m}) \overset{\mathrm{L}}{\otimes}_A K) \xrightarrow{\sim} \mathrm{H}^q(K)/\mathfrak{m}\mathrm{H}^q(K)$ for every maximal ideal $\mathfrak{m}$ of A. To conclude, it suffices therefore to show that there exists a maximal ideal $\mathfrak{m}$ such that $\mathrm{H}^q(K) \neq \mathfrak{m}\mathrm{H}^q(K)$ or still, according to Nakayama's lemma, that $\mathrm{H}^q(K) \otimes_A A_{\mathfrak{m}} \neq 0$. The support of $\mathrm{H}^q(K)$ is a non-empty closed subscheme of $\mathrm{Spec}(A)$ (cf. [**ÉGA** 0_{I} 1.7]), it contains a closed point that one identifies with a maximal ideal $\mathfrak{m}$ of A, and this maximal ideal satisfies the desired condition.

Prove the théorème 7.1.3 assuming known the theorem 7.6.1. Let K be a dualizing complex on $\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(X_{\text{ét}}, \Lambda)$ and A a Noetherian Λ-algebra. Thanks to the lemma 6.2.2.4, it follows that K belongs to $\mathsf{D}^{\mathrm{b}}_{\mathrm{ctf}}(X_{\text{ét}}, \Lambda)$ and therefore that $K_A = A \overset{\mathrm{L}}{\otimes}_\Lambda K$ belongs to $\mathsf{D}^{\mathrm{b}}_{\mathrm{ctf}}(X_{\text{ét}}, A)$. According to the théorème 7.6.1, the functor $\mathrm{D}_A = \mathrm{R}\,\mathbf{Hom}_A(-, K_A)$ preserves $\mathsf{D}^{\mathrm{b}}_{\mathrm{ctf}}(X_{\text{ét}}, A)$. It remains to show that the biduality morphism $L \to \mathrm{D}^2_A\, L$ is an isomorphism for every $L \in \mathsf{D}^{\mathrm{b}}_{\mathrm{ctf}}(X_{\text{ét}}, A)$. According to the lemma, it suffices to show that after derived tensor product with $A/\mathfrak{m}$, the morphism $L \to \mathrm{D}^2_A\, L$ induces an isomorphism. According to the théorème 7.6.1, the considered duality functor commutes with change of ring, thus, after tensor product with $A/\mathfrak{m}$, thanks to the théorème 12.2.5, one obtains the biduality morphism for $A/\mathfrak{m} \overset{\mathrm{L}}{\otimes}_A L$ in $\mathsf{D}^{\mathrm{b}}_{\mathrm{ctf}}(X_{\text{ét}}, A/\mathfrak{m})$. In short, one can assume that the ring A is a field. In this case, one can conclude by using the théorème 7.1.2.

Proposition 7.6.3. *Let X be a Noetherian scheme. We assume that there exists a dualizing complex on $\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(X_{\text{ét}}, \Lambda)$. Then, for every immersion j of an open U of X, the functor $\mathrm{R}j_\star$ sends $\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(U_{\text{ét}}, \Lambda)$ into $\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(X_{\text{ét}}, \Lambda)$.*

Let K be a dualizing complex on $\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(X_{\text{ét}}, \Lambda)$. For evident reasons, $j^\star K$ is a dualizing complex on $\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(U_{\text{ét}}, \Lambda)$. We denote D_X (resp. D_U) the dualities induced by K and $j^\star K$ on the triangulated categories $\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(X_{\text{ét}}, \Lambda)$ (resp. $\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(U_{\text{ét}}, \Lambda)$). We have a canonical isomorphism $\mathrm{D}_X \circ j_! \simeq \mathrm{R}j_\star\, \mathrm{D}_U$. One deduces that for every $M \in \mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(U_{\text{ét}}, \Lambda)$, $\mathrm{R}j_\star M \simeq \mathrm{D}_X\, j_!\, \mathrm{D}_U\, M$, which allows to conclude that $\mathrm{R}j_\star M$ belongs to $\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(X_{\text{ét}}, \Lambda)$.

Définition 7.6.4. If X is a Noetherian scheme, A a Noetherian Λ-algebra and $\mathscr{P}$ a dirigible Galois partition of X, we denote $\mathsf{D}(X_{\text{ét}}, A)^{\mathscr{P}}$ the triangulated subcategory of $\mathsf{D}(X_{\text{ét}}, A)$ whose cohomology objects are in $\mathrm{FCons}^{\mathscr{P}}(X, A)$. We define similarly the variants $\mathsf{D}^{\mathrm{b}}(X_{\text{ét}}, A)^{\mathscr{P}}$, $\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(X_{\text{ét}}, A)^{\mathscr{P}}$, etc.

Proposition 7.6.5. *Let $j : U \to X$ be an open immersion between Noetherian schemes. We assume that $\mathrm{R}j_\star$ applies $\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(U_{\text{ét}}, \Lambda)$ into $\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(X_{\text{ét}}, \Lambda)$. For every dirigible Galois partition $\mathscr{P}$ of U, there exists a Galois partition $\mathscr{P}'$ of X and an integer c such that $\mathrm{R}j_\star$ sends $\mathsf{D}^{\mathrm{b}}(U_{\text{ét}}, \Lambda)^{\mathscr{P}}$ into $\mathsf{D}^{\mathrm{b}}(X_{\text{ét}}, \Lambda)^{\mathscr{P}'}$ and that for every $q > c$ and $\mathscr{F} \in \mathrm{FCons}^{\mathscr{P}}(U, \Lambda)$, we have $\mathrm{R}^q j_\star \mathscr{F} = 0$.*

We know that the category $\mathrm{FCons}^{\mathscr{P}}(U, \Lambda)$ is locally finite (cf. [**Gabriel, 1962**, pages 356]) and even admits a finite number of simple objects (and these are constructible sheaves), cf. proposition 7.4.3. As $\mathrm{R}j_\star$ applies $\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(U_{\text{ét}}, \Lambda)$ into $\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(X_{\text{ét}}, \Lambda)$, one can choose an integer c and a Galois partition $\mathscr{P}'$ of X such that for every simple object $\mathscr{F}$ of $\mathrm{FCons}^{\mathscr{P}}(U, \Lambda)$, $\mathrm{R}j_\star \mathscr{F}$ belongs to $\mathsf{D}^{\mathrm{b}}(X_{\text{ét}}, \Lambda)^{\mathscr{P}'}$ and has cohomology objects zero in degrees strictly greater than c. This result extends by devissage to the Noetherian objects of $\mathrm{FCons}^{\mathscr{P}}(U,\Lambda)$ then to this entire category due to the commutation of the functors $\mathrm{R}^q j_\star$ with filtered inductive limits.

Corollaire 7.6.6. *Let $i\colon Z \to X$ be a closed immersion between Noetherian schemes. We assume that $i^!$ applies $\mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X_{\text{ét}},\Lambda)$ into $\mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(Z_{\text{ét}},\Lambda)$. For every dirigible Galois partition $\mathscr{P}$ on X, there exists a Galois partition $\mathscr{P}'$ on Z and an integer c such that $i^!$ sends $\mathsf{D}^{\mathsf{b}}(X_{\text{ét}},\Lambda)^{\mathscr{P}}$ into $\mathsf{D}^{\mathsf{b}}(Z_{\text{ét}},\Lambda)^{\mathscr{P}'}$ and that for every $q > c$ and $\mathscr{F} \in \mathrm{FCons}^{\mathscr{P}}(X,\Lambda)$, we have $\mathscr{H}^q(i^!\mathscr{F}) = 0$.*

If we denote $j\colon U \to X$ the complementary open immersion, the hypothesis on $i^!$ stated here is equivalent to that required on $\mathrm{R}j_\star$ in the proposition 7.6.5. Up to refining $\mathscr{P}$, one can assume that the constituents of $\mathscr{P}$ are either over U, or over Z. One can thus write $\mathscr{P} = \mathscr{P}_U \cup \mathscr{P}_Z$ where $\mathscr{P}_U$ and $\mathscr{P}_Z$ are Galois partitions of U and Z respectively. One applies the proposition 7.6.5 to $\mathscr{P}_U$. One obtains a Galois partition $\mathscr{P}''$ of X such that $\mathrm{R}j_\star$ applies $\mathsf{D}^{\mathsf{b}}(U_{\text{ét}},\Lambda)^{\mathscr{P}_U}$ into $\mathsf{D}^{\mathsf{b}}(X_{\text{ét}},\Lambda)^{\mathscr{P}''}$ and a natural integer c' such that for every $\mathscr{F} \in \mathrm{FCons}^{\mathscr{P}_U}(U,\Lambda)$, we have $\mathrm{R}^q j_\star\mathscr{F} = 0$ for $q > c'$. Up to refining $\mathscr{P}''$, one can assume that $\mathscr{P}'' = \mathscr{P}''_Z \cup \mathscr{P}''_U$ as above. Up to refining $\mathscr{P}''_Z$, one can assume that this Galois partition of Z refines $\mathscr{P}_Z$. By using the distinguished triangle

$$i^!K \to i^\star K \to i^\star \mathrm{R}j_\star j^\star K \to i^!K[1]$$

for every $K \in \mathsf{D}^+(X_{\text{ét}},\Lambda)$, one obtains immediately that $\mathscr{P}' = \mathscr{P}''_Z$ and $c = c'+1$ work.

Proposition 7.6.7. *Let $j\colon U \to X$ be an open immersion between Noetherian schemes. We assume that $\mathrm{R}j_\star$ sends $\mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(U_{\text{ét}},\Lambda)$ into $\mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X_{\text{ét}},\Lambda)$. Then, for every Noetherian Λ-algebra A, $\mathrm{R}j_\star$ sends $\mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(U_{\text{ét}},A)$ (resp. $\mathsf{D}^{\mathsf{b}}_{\mathsf{ctf}}(U_{\text{ét}},A)$) into $\mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X_{\text{ét}},A)$ (resp. $\mathsf{D}^{\mathsf{b}}_{\mathsf{ctf}}(X_{\text{ét}},A)$). Moreover, for every $Y \in \mathsf{D}^{\mathsf{b}}_{\mathsf{ctf}}(U_{\text{ét}},A)$ and $M \in \mathsf{D}^+(A)$, the canonical morphism $M \overset{\mathrm{L}}{\otimes}_A \mathrm{R}j_\star Y \to \mathrm{R}j_\star(M \overset{\mathrm{L}}{\otimes}_A Y)$ is an isomorphism. Furthermore, if $i\colon Z \to X$ is a closed immersion complementary to j, then $i^!$ sends $\mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X_{\text{ét}},A)$ (resp. $\mathsf{D}^{\mathsf{b}}_{\mathsf{ctf}}(X_{\text{ét}},A)$) into $\mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(Z_{\text{ét}},A)$ (resp. $\mathsf{D}^{\mathsf{b}}_{\mathsf{ctf}}(Z_{\text{ét}},A)$), and for every $M \in \mathsf{D}^+(A)$ and $Y \in \mathsf{D}^{\mathsf{b}}_{\mathsf{ctf}}(X_{\text{ét}},A)$, the canonical morphism $M \overset{\mathrm{L}}{\otimes}_A i^!Y \to i^!(M \overset{\mathrm{L}}{\otimes}_A Y)$ is an isomorphism.*

The statement on $i^!$ follows immediately from that on $\mathrm{R}j_\star$, one concentrates therefore on the latter. To show that $\mathrm{R}j_\star$ sends $\mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(U_{\text{ét}},A)$ into $\mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X_{\text{ét}},A)$, one can assume by an evident devissage that $\Lambda = \mathbf{Z}/\ell\mathbf{Z}$, where ℓ is a prime number. Consequently, for every Λ-module N and every object $Y \in \mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(U_{\text{ét}},\Lambda)$, we have an isomorphism $\mathrm{R}j_\star(N \overset{\mathrm{L}}{\otimes}_\Lambda Y) \simeq N \overset{\mathrm{L}}{\otimes}_\Lambda \mathrm{R}j_\star Y$ (cf. proposition 10.2.1). If N has a structure of finite type A-module, as we know that $\mathrm{R}j_\star Y$ belongs to $\mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X_{\text{ét}},\Lambda)$, one can deduce that $\mathrm{R}j_\star(N \overset{\mathrm{L}}{\otimes}_\Lambda Y)$ belongs to $\mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X_{\text{ét}},A)$. According to the devissage of the proposition 7.4.1, it follows that $\mathrm{R}j_\star$ sends $\mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(U_{\text{ét}},A)$ into $\mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X_{\text{ét}},A)$.

Show now that $\mathrm{R}j_\star$ sends $\mathsf{D}^{\mathsf{b}}_{\mathsf{ctf}}(U_{\text{ét}},A)$ into $\mathsf{D}^{\mathsf{b}}_{\mathsf{ctf}}(X_{\text{ét}},A)$. Given the previous result, it suffices to show that if $\mathscr{F}$ is a flat and constructible sheaf of A-modules on U, then $\mathrm{R}j_\star\mathscr{F} \in \mathsf{D}^{\mathsf{b}}_{\mathsf{tf}}(X_{\text{ét}},A)$, i.e., that $M \overset{\mathrm{L}}{\otimes}_A \mathrm{R}j_\star\mathscr{F}$ is bounded independently of the A-module M. According to the proposition 10.2.1, it suffices to show that $\mathrm{R}j_\star(\mathscr{F} \otimes_A M)$ is bounded independently of the A-module M, and if this is the case, the formula of universal coefficients stated here will be satisfied. There exists a dirigible Galois partition $\mathscr{P}$ of U such that $\mathscr{F}$ belongs to $\mathrm{FCons}^{\mathscr{P}}(U,A)$. For every A-module M, $\mathscr{F} \otimes_A M$ is an object of $\mathrm{FCons}^{\mathscr{P}}(U,A)$, thus, it suffices to show that there exists an integer c such that for every object $\mathscr{G}$ of $\mathrm{FCons}^{\mathscr{P}}(U,A)$, we have $\mathrm{R}^q j_\star\mathscr{G} = 0$ for $q > c$, which follows from the proposition 7.6.5.

Définition 7.6.8. Let X be a Noetherian scheme. Let A be a Noetherian Λ-algebra. Let $K \in \mathsf{D}^{\mathsf{b}}_{\mathsf{ctf}}(X_{\text{ét}},A)$. We say that K satisfies the **condition (B)** if for every Galois partition $\mathscr{P}$ of X, there exists an integer c and a Galois partition $\mathscr{P}'$ of X such that for every $L \in \mathsf{D}^{\mathsf{b}}(X_{\text{ét}},A)^{\mathscr{P}}$, $\mathrm{R}\,\mathbf{Hom}_A(K,L)$ belongs to $\mathsf{D}^{\mathsf{b}}(X_{\text{ét}},A)^{\mathscr{P}'}$, that if we assume that $\mathscr{H}^qL = 0$ for $q > 0$, then $\mathscr{H}^q\mathrm{R}\,\mathbf{Hom}_A(K,L) = 0$ for $q > c$ and finally, that if L belongs to $\mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X_{\text{ét}},A)^{\mathscr{P}}$, then $\mathrm{R}\,\mathbf{Hom}_A(K,L)$ belongs to $\mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X_{\text{ét}},A)^{\mathscr{P}'}$.

Proposition 7.6.9. *Let X be a Noetherian scheme such that there exists a dualizing complex on $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X_{\text{ét}},\Lambda)$. Then, every object $K\in \mathrm{D}^{\mathrm{b}}_{\mathrm{ctf}}(X_{\text{ét}},A)$ satisfies condition (B).*

Lemme 7.6.10. *Let X be a Noetherian scheme such that there exists a dualizing complex on $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X_{\text{ét}},\Lambda)$. Let $i\colon Z\to X$ be a closed immersion. Let $j\colon U\to X$ be the complementary open immersion. Let $K\in\mathrm{D}^{\mathrm{b}}_{\mathrm{ctf}}(X_{\text{ét}},A)$. We assume that $i^\star K$ and $j^\star K$ satisfy condition (B). Then, K satisfies condition (B).*

For every $L\in\mathrm{D}(X_{\text{ét}},A)$, we have a distinguished triangle in $\mathrm{D}(X_{\text{ét}},A)$:

$$i_\star\mathrm{R}\,\mathbf{Hom}_A(i^\star K,i^! L)\to\mathrm{R}\,\mathbf{Hom}_A(K,L)\to\mathrm{R}j_\star\mathrm{R}\,\mathbf{Hom}_A(j^\star K,j^\star L)\to+$$

Thanks to the result of the exercise 7.3.8 and given the proposition 7.6.3, one can combine on one hand the result on $i^!$ of the corollary 7.6.6 and condition (B) for $i^\star K$ and on the other hand the proposition 7.6.5 concerning $\mathrm{R}j_\star$ and condition (B) for $j^\star K$ to obtain that K satisfies condition (B).

Prove the proposition 7.6.9. Condition (B) defines a triangulated subcategory of $\mathrm{D}^{\mathrm{b}}_{\mathrm{ctf}}(X_{\text{ét}},A)$. To show the proposition, it suffices to show that if K is a flat and constructible sheaf of A-modules, then K satisfies condition (B). The existence of a dualizing complex being a condition preserved by passing to a subscheme, the previous lemma provides a means to devisse the situation to reduce to the case where K is locally constant. One is reduced to the following lemma :

Lemme 7.6.11. *Let X be a Noetherian scheme. Let $\mathscr{F}$ be a constructible sheaf of A-modules, flat and locally constant. Then, $\mathscr{F}$ satisfies property (B).*

First, for every $L\in\mathrm{D}^{\mathrm{b}}(X_{\text{ét}},A)$, if $\mathscr{H}^qL=0$ for $q>0$, then for every $q>0$, $\mathscr{H}^q(\mathrm{R}\,\mathbf{Hom}_A(\mathscr{F},L))\simeq\mathbf{Hom}_A(\mathscr{F},\mathscr{H}^qL)=0$, and if $L\in\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X_{\text{ét}},A)$, then $\mathrm{R}\,\mathbf{Hom}_A(\mathscr{F},L)\in\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X_{\text{ét}},A)$. It remains therefore to show that if $\mathscr{P}$ is a Galois partition of X, there exists a Galois partition $\mathscr{P}'$ of X such that for every $L\in\mathrm{D}^{\mathrm{b}}(X_{\text{ét}},A)^{\mathscr{P}}$, then $\mathrm{R}\,\mathbf{Hom}_A(\mathscr{F},L)$ belongs to $\mathrm{D}^{\mathrm{b}}(X_{\text{ét}},A)^{\mathscr{P}'}$. One can assume that $\mathscr{P}$ is constituted of a single Galois étale covering $X'\to X$. One chooses a Galois étale covering $X''\to X$ such that the inverse image of $\mathscr{F}$ on X'' is a constant sheaf, then a Galois covering $X'''\to X$ covering X' and X''. One sees immediately that the Galois partition $\mathscr{P}'=(X'''\to X)$ of X works.

Prove the theorem 7.6.1. Let $K\in\mathrm{D}^{\mathrm{b}}_{\mathrm{ctf}}(X_{\text{ét}},A)$. According to the proposition 7.6.9, K satisfies condition (B). Let $L\in\mathrm{D}^{\mathrm{b}}_{\mathrm{ctf}}(X_{\text{ét}},A)$. There exists a Galois partition $\mathscr{P}$ of X such that L belongs to $\mathrm{D}^{\mathrm{b}}(X_{\text{ét}},A)^{\mathscr{P}}$. For every A-module M, the object $M\overset{\mathrm{L}}{\otimes}_A L$ still belongs to this category (and is bounded independently of M). It results from condition (B) of K that $\mathrm{R}\,\mathbf{Hom}_A(K,L)$ belongs to $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X_{\text{ét}},A)$ and that there exists a Galois partition $\mathscr{P}'$ such that $\mathrm{R}\,\mathbf{Hom}_A(K,M\overset{\mathrm{L}}{\otimes}_A L)$ is an object of $\mathrm{D}^{\mathrm{b}}(X_{\text{ét}},A)^{\mathscr{P}'}$ bounded independently of the A-module M. The proposition 10.1.1 allows to deduce that $\mathrm{R}\,\mathbf{Hom}_A(K,L)$ belongs to $\mathrm{D}^{\mathrm{b}}_{\mathrm{tf}}(X_{\text{ét}},A)$ and that for every $M\in\mathrm{D}^+(A)$, the canonical morphism $M\overset{\mathrm{L}}{\otimes}_A\mathrm{R}\,\mathbf{Hom}_A(K,L)\to\mathrm{R}\,\mathbf{Hom}_A(K,M\overset{\mathrm{L}}{\otimes}_A L)$ is an isomorphism. One deduces immediately from this formula the compatibility with the change of ring $A\to A'$ for every A-algebra A'.

7.7. Elimination of the Noetherian hypothesis on A.

Définition 7.7.1. Let A be a commutative ring. Let X be a Noetherian scheme. We say of a complex $K\in\mathrm{D}(X_{\text{ét}},A)$ that it is *c*-**perfect** if there exists a finite partition $(U_i)_{i\in I}$ of X by reduced subschemes such that for every $i\in I$, $K_{|U_i}\in\mathrm{D}(U_{i_{\text{ét}}},A)$ is a perfect complex (cf. [**SGA 6** I 4.8]). We denote $\mathrm{D}^{\mathrm{b}}_{\mathrm{c-parf}}(X_{\text{ét}},A)$ the triangulated subcategory of $\mathrm{D}(X_{\text{ét}},A)$ formed of the c-perfect complexes.

Of course, for every ring morphism $A\to A'$, the functor $A'\overset{\mathrm{L}}{\otimes}_A -\colon\mathrm{D}^-(X_{\text{ét}},A)\to\mathrm{D}^-(X_{\text{ét}},A')$ induces a functor $\mathrm{D}^{\mathrm{b}}_{\mathrm{c-parf}}(X_{\text{ét}},A)\to\mathrm{D}^{\mathrm{b}}_{\mathrm{c-parf}}(X_{\text{ét}},A')$. Moreover, if A is a Noetherian ring, $\mathrm{D}^{\mathrm{b}}_{\mathrm{c-parf}}(X_{\text{ét}},A)=\mathrm{D}^{\mathrm{b}}_{\mathrm{ctf}}(X_{\text{ét}},A)$.

Théorème 7.7.2. *Let A be a commutative Λ-algebra. Let X be a Noetherian scheme. If they exist, the dualizing complexes on $\mathrm{D}^{\mathrm{b}}_{\mathrm{c-parf}}(X_{\text{ét}},A)$ are unique up to tensor product with invertible objects. Let K be a dualizing complex on $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X_{\text{ét}},\Lambda)$. Then, $A\overset{\mathrm{L}}{\otimes}_{\Lambda}K$ is a dualizing complex on $\mathrm{D}^{\mathrm{b}}_{\mathrm{c-parf}}(X_{\text{ét}},A)$. Moreover, the bifunctor $\mathrm{R}\,\mathbf{Hom}_A$ preserves $\mathrm{D}^{\mathrm{b}}_{\mathrm{c-parf}}(X_{\text{ét}},A)$ and commutes with every change of ring $A\to A'$.*

The theorem 7.6.1 states that if B is a Noetherian Λ-algebra, that K and L are two objects of $\mathrm{D}^{\mathrm{b}}_{\mathrm{c-parf}}(X_{\text{ét}},B)$, then for every B-algebra A, we have a canonical isomorphism

$$\mathrm{R}\,\mathbf{Hom}_A(A\overset{\mathrm{L}}{\otimes}_B K, A\overset{\mathrm{L}}{\otimes}_B L)\simeq A\overset{\mathrm{L}}{\otimes}_B \mathrm{R}\,\mathbf{Hom}_B(K,L)\ .$$

As the right object belongs to $\mathrm{D}^{\mathrm{b}}_{\mathrm{c-parf}}(X_{\text{ét}},A)$, it is an isomorphism in $\mathrm{D}^{\mathrm{b}}_{\mathrm{c-parf}}(X_{\text{ét}},A)$. In short, given the theorem 7.6.1 and the theorem 7.1.3 (whose uniqueness statement of dualizing complexes also holds for $\mathrm{D}^{\mathrm{b}}_{\mathrm{c-parf}}(X_{\text{ét}},A)$ with the same proof, cf. proposition 7.5.1.1), the above theorem is reduced to the following lemma :

Lemme 7.7.3. *Let A be a commutative ring. Let X be a Noetherian scheme. For every object K of $\mathrm{D}^{\mathrm{b}}_{\mathrm{c-parf}}(X_{\text{ét}},A)$, there exists a Noetherian sub-ring (even of finite type over $\mathbf{Z}$) B of A and $K'\in\mathrm{D}^{\mathrm{b}}_{\mathrm{c-parf}}(X_{\text{ét}},B)$ such that the objects K and $A\overset{\mathrm{L}}{\otimes}_B K'$ of $\mathrm{D}^{\mathrm{b}}_{\mathrm{c-parf}}(X_{\text{ét}},A)$ are isomorphic.*

As it requires a more careful examination of the notion of c-perfection, we postpone the proof of this lemma to the end of this subsection.

Lemme 7.7.4. *Let A be a commutative ring. Let $\mathscr{F}$ be a sheaf of A-modules on $X_{\text{ét}}$. The following conditions are equivalent :*

(i) *There exists a partition $(U_i)_{i\in I}$ of X by reduced subschemes such that for every $i\in I$, $\mathscr{F}_{|U_i}$ is locally constant and that for every geometric point $\overline{x}$ of X, the A-module $\mathscr{F}_{\overline{x}}$ is projective of finite type;*
(ii) *The sheaf of A-modules $\mathscr{F}$ is constructible*[(vii)] *and for every geometric point $\overline{x}$ of X, the A-module $\mathscr{F}_{\overline{x}}$ is projective of finite type;*
(iii) *The sheaf of A-modules $\mathscr{F}$ is flat and constructible.*

By definition of constructible sheaves, we obviously have the equivalence (i) $\Longleftrightarrow$ (ii). If $\mathscr{F}$ is constructible, the fibers $\mathscr{F}_{\overline{x}}$ are A-modules of finite presentation, it is then equivalent to require that these modules are flat or projective of finite type, which shows the equivalence (ii) $\Longleftrightarrow$ (iii).

We denote $\mathscr{C}$ the fibered category over the site $X_{\text{ét}}$ that to $U\in X_{\text{ét}}$ associates the category $\mathscr{C}_X$ of sheaves of A-modules on $U_{\text{ét}}$ and $\mathscr{C}_c$ the sub-$X_{\text{ét}}$-category of $\mathscr{C}$ formed of flat and constructible sheaves of A-modules. We will use the terminology of [**SGA 6** I 1.2] and [**SGA 6** I 2]. It is evident that an object of $\mathscr{C}$ that is locally in $\mathscr{C}_c$ is in $\mathscr{C}_c$ and that $\mathscr{C}_c$ is stable under kernel of epimorphism. According to [**SGA 4** IX 2.7], a sheaf of A-modules on $X_{\text{ét}}$ is constructible if and only if it is isomorphic to the cokernel of a morphism $A_V\to A_U$ for U and V two étale X-schemes of finite presentation. The sheaves A_U for U étale and of finite presentation over X are therefore obviously flat and constructible. It follows from these results that an object of $\mathscr{C}_X$ is of finite $\mathscr{C}_c$-type if and only if it is generated by a finite number of sections (which amounts to asking that it be of finite $\mathscr{C}_{cX}$-type) and that an object of $\mathscr{C}_X$ is of finite $\mathscr{C}_c$-presentation if and only if it is constructible (which still amounts to asking that it be of $\mathscr{C}_{cX}$-finite presentation). It is moreover evident that $\mathscr{C}_c$ is quasi-relevable in $\mathscr{C}$ and even that $\mathscr{C}_{cX}$ is quasi-relevable in $\mathscr{C}_X$. The categories $\mathscr{C}_c$ and $\mathscr{C}$ therefore satisfy the hypotheses of [**SGA 6** I 2.0] and of [**SGA 6** II 1.1] (but in general not of [**SGA 6** I 4.0]). One thus disposes of a notion of pseudo-coherent complex (relative to $\mathscr{C}_c$) and this can be defined globally (i.e., relative to $\mathscr{C}_{cX}$) :

Définition 7.7.5. Let A be a ring. Let X be a Noetherian scheme. Let $K\in C(X_{\text{ét}},A)$. We say that K is **strictly c-pseudo-coherent** (resp. **strictly c-perfect**) if K is an upper bounded (resp. bounded) complex formed of flat and constructible sheaves of A-modules.

[(vii)] We use the definition given in [**SGA 4** IX 2.3] even if A is not Noetherian, and not the definition suggested in note at that place; a constructible sheaf for this other definition is what we call a c-pseudo-coherent sheaf (cf. definition 7.7.6).

Définition 7.7.6. Let A be a ring. Let X be a Noetherian scheme. Let $K \in \mathsf{D}(X_{\text{ét}}, A)$. We say that K is **c-pseudo-coherent** if it is isomorphic to the image in $\mathsf{D}(X_{\text{ét}}, A)$ of a strictly c-pseudo-coherent complex[(viii)]. The c-pseudo-coherent objects form an anonymous triangulated sub-category of $\mathsf{D}(X_{\text{ét}}, A)$.

Remarque 7.7.7. The $X_{\text{ét}}$-category $\mathscr{C}$ also contains the $X_{\text{ét}}$-category $\mathscr{C}_0$ of constant sheaves with value an A-module projective of finite type. Of course, $\mathscr{C}_0$ is contained in $\mathscr{C}_c$. Thus, the notions of strict pseudo-coherence (resp. strict perfection) defined relative to $\mathscr{C}_0$ imply the corresponding notions relative to $\mathscr{C}_c$.

Proposition 7.7.8. *Let A be a ring. Let X be a Noetherian scheme. Let $K \in \mathsf{D}(X_{\text{ét}}, A)$. Then, the following conditions are equivalent :*

(i) *K is c-perfect ;*
(ii) *K is c-pseudo-coherent and of finite tor-dimension ;*
(iii) *K is isomorphic to the image in $\mathsf{D}(X_{\text{ét}}, A)$ of a strictly c-perfect complex.*

Lemme 7.7.9. *Let A be a ring. Let X be a Noetherian scheme. Let $(U_i)_{i\in I}$ be a finite partition of X by reduced subschemes. Let $K \in \mathsf{D}(X_{\text{ét}}, A)$. We assume that for every $i \in I$, the restriction of K to U_i is c-pseudo-coherent. Then, K is c-pseudo-coherent.*

By the usual arguments, one reduces to the case where the partition of X is constituted of an open U and a closed Z. We denote $i\colon Z \to X$ and $j\colon U \to X$ the corresponding immersions. We dispose of a distinguished triangle in $\mathsf{D}(X_{\text{ét}}, A)$:

$$j_! j^\star K \to K \to i_\star i^\star K \to j_! j^\star K[1]$$

We know that $j^\star K$ and $i^\star K$ are c-pseudo-coherent. It is evident that $j_!$ and $i_\star$ preserve the notion of flat and constructible sheaf ; at the level of triangulated categories, these functors therefore preserve the notion of c-pseudo-coherence. The objects $j_! j^\star K$ and $i_\star i^\star K$ are c-pseudo-coherent ; it follows that K is also c-pseudo-coherent.

Lemme 7.7.10. *Let A be a ring. Let X be a Noetherian scheme. Let $K \in \mathsf{D}(X_{\text{ét}}, A)$. If K is c-perfect, then K is c-pseudo-coherent.*

According to the lemma 7.7.9, up to passing to a finite covering by suitable locally closed subschemes, one can assume that K is perfect. Consequently, K is pseudo-coherent (understood relative to the $X_{\text{ét}}$-category $\mathscr{C}_0$) ; *a fortiori*, K is c-pseudo-coherent.

Prove the proposition 7.7.8. The implication (iii) $\Longrightarrow$ (i) is evident. The implication (i) $\Longrightarrow$ (ii) follows essentially from the previous lemma ; it remains however to verify that if K is c-perfect, then it is of finite tor-dimension. Assume therefore that K is c-perfect. There exists a finite covering $(U_i)_{i\in I}$ of X by locally closed subschemes such that $K_{|U_i}$ is perfect for every $i \in I$. To obtain (ii) for K, it suffices to show that the perfect complex $K_{|U_i}$ is of finite tor-dimension, which follows immediately from [**SGA 6** I 5.8.1]. It remains to establish the implication (ii) $\Longrightarrow$ (iii), the most interesting for us. Let $K \in \mathsf{D}(X_{\text{ét}}, A)$ be a c-pseudo-coherent complex and of finite tor-dimension. By definition of c-pseudo-coherence, one can replace if necessary K by an upper bounded complex formed of flat and constructible sheaves of A-modules. For every $a \in \mathbf{Z}$, one can consider the canonical truncation $\tau_{\leq a}K$ of K, if we denote Z^a the kernel of $K^a \to K^{a+1}$, it is the following sub-complex of K :

$$\dots \to K^{a-2} \to K^{a-1} \to Z^a \to 0 \to \dots \ .$$

As K is of finite tor-dimension, there exists an integer $a \in \mathbf{Z}$ such that for every A-module M, $\tau_{\leq a}(K \otimes_A M)$ is acyclic. By applying this with $M = A$, one obtains a flat resolution of Z^a :

$$\dots \to K^{a-3} \to K^{a-2} \to K^{a-1} \to Z^a \to 0\ .$$

Next, one obtains immediately that for any A-module M, this sequence remains exact after passing to the tensor product with M. Thus, for every $i > 0$, $\mathbf{Tor}_i^A(Z^a, M) = 0$ for every A-module M, which implies

(viii) Due to the existence of “global resolutions”, the global definition given here is equivalent to the local definition of [**SGA 6** I 2.3].

that Z^a is a flat sheaf of A-modules. Moreover, Z^a is the cokernel of the morphism $K^{a-2} \to K^{a-1}$, so Z^a is a constructible sheaf. Thus, Z^a is flat and constructible. The complex K is quasi-isomorphic to the strictly c-perfect complex

$$\dots \to 0 \to Z^a \to K^a \to K^{a+1} \to \dots ,$$

thus K satisfies condition (iii).

We are now in a position to prove the lemma 7.7.3. Given the proposition 7.7.8, it is a matter of showing that if $K \in C(X_{\text{ét}}, A)$ is a strictly c-perfect complex, then A contains a sub-ring A' of finite type over $\mathbf{Z}$ such that there exists a strictly c-perfect complex $K' \in C(X_{\text{ét}}, A')$ and an isomorphism $K \simeq A \otimes_{A'} K'$. This follows immediately from the following lemma :

Lemme 7.7.11. *Let $(A_\alpha)_{\alpha \in I}$ be an inductive system of commutative rings indexed by a filtered ordered set (non-empty) I. We denote A the inductive limit. Let X be a Noetherian scheme. Then, the data of a constructible sheaf of A-modules (resp. flat and constructible) on X is equivalent to the data of a constructible sheaf of A_α-modules (resp. flat and constructible) on X for α large enough.*

In accordance with the great principles of [**ÉGA** iv 8], this means on one hand that if F is a constructible sheaf of A-modules on X, there exists $\alpha \in I$ and F_α a constructible sheaf of A_α-modules on X such that F is isomorphic to $A \otimes_{A_\alpha} F_\alpha$ and on the other hand that if $\alpha \in I$ and that F_α and G_α are two constructible sheaves of A_α-modules on X, if we denote $F_\beta = A_\beta \otimes_{A_\alpha} F_\alpha$ and $G_\beta = A_\beta \otimes_{A_\alpha} G_\alpha$ for every $\beta \geq \alpha$ and $F = A \otimes_{A_\alpha} F_\alpha$ and $G = A \otimes_{A_\alpha} G_\alpha$, then the canonical map

$$\operatorname*{colim}_{\beta \geq \alpha} \operatorname{Hom}_{A_\beta}(F_\beta, G_\beta) \to \operatorname{Hom}_A(F, G)$$

is an isomorphism. Moreover, if $\alpha \in I$ and that F_α is a constructible sheaf of A_α-modules on X, then $F = A \otimes_{A_\alpha} F_\alpha$ is flat if and only if for $\beta \geq \alpha$ large enough, $F_\beta = A_\beta \otimes_{A_\alpha} F_\alpha$ is flat.

The statement in the non-flat case follows from the description of constructible sheaves of B-modules (for every commutative ring B) as cokernel of an arrow $B_V \to B_U$ where U and V are étale and of finite presentation over X, and from the fact that the functor $\mathrm{H}^0_{\text{ét}}(V, -)$ from the category of sheaves of abelian groups on $X_{\text{ét}}$ to that of abelian groups commutes with filtered inductive limits [**SGA 4** vii 3.3].

It remains to show that if F_α is a constructible sheaf of A_α-modules such that, with the above notations, F is A-flat, then for $\beta \geq \alpha$ large enough, F_β is A_β-flat. By using a decomposition of X into union of locally closed connected (non-empty) over which F_α is locally constant trivialized by an étale covering (non-empty), one can assume that X is connected (non-empty) and that F_α is locally constant and trivialized by an étale covering (non-empty) $Y \to X$. To verify the flatness of F_β, it suffices to obtain it for *one* fiber $(F_\beta)_{\overline{x}}$; one is thus reduced to the following lemma :

Lemme 7.7.12. *Let $(A_\alpha)_{\alpha \in I}$ be an inductive system of commutative rings indexed by a filtered ordered set (non-empty) I. We denote A the inductive limit. Let $\alpha \in I$, let M_α be an A_α-module of finite presentation. We assume that $M = A \otimes_{A_\alpha} M_\alpha$ is A-flat. Then, there exists $\beta \geq \alpha$ such that $M_\beta = A_\beta \otimes_{A_\alpha} M_\alpha$ is A_β-flat.*

The considered modules being of finite presentation, the module M (resp. M_β) is flat if and only if it is direct factor of a free module of finite type. One can conclude by using appropriately [**ÉGA** iv 8.5.2].

8. Tensor products of unbounded complexes

8.1. K-flatness.

Définition 8.1.1. Let $(\mathscr{T}, \mathscr{A})$ be an annelid topos in commutative rings. We denote $C(\mathscr{T}, \mathscr{A})$ (resp. $K(\mathscr{T}, \mathscr{A})$, $\mathrm{D}(\mathscr{T}, \mathscr{A})$) the category of complexes of $\mathscr{A}$-Modules (resp. the corresponding homotopy category, the associated derived category). We dispose of a bifunctor $\otimes_{\mathscr{A}}$ on $C(\mathscr{T}, \mathscr{A})$ inducing a bifunctor on $K(\mathscr{T}, \mathscr{A})$[(ix)]. Let $K \in C(\mathscr{T}, \mathscr{A})$.We say that K is **K-flat** if the triangulated functor $K \otimes_{\mathscr{A}} - : K(\mathscr{T}, \mathscr{A}) \to K(\mathscr{T}, \mathscr{A})$ preserves quasi-isomorphisms, in other words that for every acyclic complex L of $C(\mathscr{T}, \mathscr{A})$, the complex $K \otimes_{\mathscr{A}} L$ is acyclic (cf. [**Spaltenstein, 1988**, definition 5.1]).

[(ix)] The reader can consult the sign conventions of **XVI**-4.5

The full subcategory of $C(\mathscr{T},\mathscr{A})$ formed of K-flat complexes is stable under filtered inductive limits, direct sums and direct factors. Moreover, the corresponding full subcategory of $K(\mathscr{T},\mathscr{A})$ is a triangulated subcategory of $K(\mathscr{T},\mathscr{A})$.

Proposition 8.1.2. *Let $(\mathscr{T},\mathscr{A})$ be an annelid topos in commutative rings. Let K be an upper bounded complex formed of flat $\mathscr{A}$-modules (cf. [***SGA 4*** v 1]). Then, K is K-flat.*

By using the stupid truncation functors and the stability under filtered inductive limits of K-flatness, one reduces to the case where K is bounded. As K-flatness defines a triangulated subcategory of $K(\mathscr{T},\mathscr{A})$, one can proceed to a devissage using still the stupid truncations to reduce to the case where $K^q = 0$ for $q \neq 0$. One is then reduced to showing that if F is a flat $\mathscr{A}$-Module and $L \in C(\mathscr{T},\mathscr{A})$ an acyclic complex, then $F \otimes_{\mathscr{A}} L$ is acyclic, which follows immediately from the definition of flatness.

8.2. *K*-flat resolutions.

8.2.1. *Definition of the derived tensor product.*

Théorème 8.2.1.1. *Let $(\mathscr{T},\mathscr{A})$ be an annelid topos in commutative rings. There exists a functor $\rho\colon C(\mathscr{T},\mathscr{A}) \to C(\mathscr{T},\mathscr{A})$ and a natural transformation $\rho K \to K$ for $K \in C(\mathscr{T},\mathscr{A})$ such that :*

- *for every $K \in C(\mathscr{T},\mathscr{A})$, ρK is K-flat ;*
- *for every $K \in C(\mathscr{T},\mathscr{A})$, the morphism $\rho K \to K$ is a quasi-isomorphism ;*
- *the functor ρ commutes with filtered inductive limits.*

This theorem will be proved below. From it we deduce immediately the trivial proposition following, which constitutes our definition of the tensor product on $\mathrm{D}(\mathscr{T},\mathscr{A})$:

Proposition 8.2.1.2. *Let $(\mathscr{T},\mathscr{A})$ be an annelid topos in commutative rings. The total left derived functor of $\otimes_{\mathscr{A}}\colon C(\mathscr{T},\mathscr{A}) \times C(\mathscr{T},\mathscr{A}) \to C(\mathscr{T},\mathscr{A})$ exists. More precisely, for all K and L in $C(\mathscr{T},\mathscr{A})$, we denote $K \overset{\mathrm{L}}{\otimes}_{\mathscr{A}} L = (\rho K) \otimes_{\mathscr{A}} (\rho L) \in C(\mathscr{T},\mathscr{A})$; this bifunctor $\overset{\mathrm{L}}{\otimes}_{\mathscr{A}}$ commutes with filtered inductive limits in each argument and, preserving quasi-isomorphisms, it induces a bifunctor of the same name $\mathrm{D}(\mathscr{T},\mathscr{A})\times\mathrm{D}(\mathscr{T},\mathscr{A}) \to \mathrm{D}(\mathscr{T},\mathscr{A})$; the evident natural transformation $\overset{\mathrm{L}}{\otimes}_{\mathscr{A}} \to \otimes_{\mathscr{A}}$ makes $\overset{\mathrm{L}}{\otimes}_{\mathscr{A}}$ the total left derived functor of $\otimes_{\mathscr{A}}$ (cf. [**Goerss & Jardine, 1999**, remark 7.4, Chapter II] for a definition of total derived functors in terms of Kan extensions). Moreover, if K and L are two objects of $C(\mathscr{T},\mathscr{A})$ of which at least one is K-flat, then the canonical morphism $K \overset{\mathrm{L}}{\otimes}_{\mathscr{A}} L \to K \otimes_{\mathscr{A}} L$ is an isomorphism in $\mathrm{D}(\mathscr{T},\mathscr{A})$.*

8.2.2. *Modules over a ring.* We place ourselves here in the particular case where the topos $\mathscr{T}$ is punctual. One can identify the sheaves of $\mathscr{A}$-modules with A-modules for a ring A. The following lemma demonstrates the theorem 8.2.1.1 in this particular case.

Lemme 8.2.2.1. *For every commutative ring A, one can define a functor ρ_A and a natural transformation $\rho_A \to \mathrm{Id}$ of functors from the category $C(A)$ of complexes of A-modules to itself such that ρ_A commutes with filtered inductive limits, preserves monomorphisms, that for every $K \in C(A)$, the morphism of complexes $\rho_A(K) \to K$ is a quasi-isomorphism, that for every relative integer n, $\rho_A(K)^n$ is a free A-module, and that $\rho_A(K)$ is the filtered inductive limit of bounded sub-complexes formed of free A-modules (in particular, $\rho_A(K)$ is K-flat).*

One can define such K-flat resolutions ρ_A for every commutative ring A in such a way that if $A \to A'$ is a ring morphism, we have a functorial morphism $\rho_A(K) \to \rho_{A'}(K)$ in $C(A)$ for $K \in C(A')$, this morphism satisfying an evident compatibility to the composition of ring morphisms.

We denote G the left adjoint functor of the forgetful functor oub from the category of A-modules to that of pointed sets. We set $F = G \circ \mathrm{oub}$. If M is an A-module, FM is the quotient of the free A-module with basis the set M by the rank 1 free sub-module generated by the zero of M. We set $(F'M)_0 := FM$ and denote Z_0 the kernel of the adjunction morphism $(F'M)_0 = FM \to M$ which is an epimorphism. Next, evidently, for every natural integer $n \geq 1$, one can define recursively an object $(F'M)_n = FZ_{n-1}$, a morphism $d_n\colon (F'M)_n \to (F'M)_{n-1}$ and the kernel $Z_n = \mathrm{Ker}(d_n)$. It is evident that one thus defines a complex $F'M$ concentrated in negative or zero degrees, equipped with an augmentation $F'M \to M$ which is a quasi-isomorphism. As F preserves monomorphisms, one sees that F' also preserves monomorphisms.

Let $K \in C(A)$. The functor F' defined above is not additive (unless $A = 0$), but it is such that $F'(0) = 0$. Thus, if one applies term by term the functor F' to the objects K_n for every $n \in \mathbf{Z}$, one obtains a double complex $((F'K^p)_{-q})_{(p,q)\in\mathbf{Z}}$ in the category of A-modules. We denote $\rho_A(K)$ the associated simple complex (defined in terms of sums), cf. **XVI**-4.4. One disposes of course of an augmentation morphism $\rho_A(K) \to K$. It is evident that ρ_A commutes with filtered inductive limits, preserves monomorphisms and that for every integer n, $\rho_A(K)^n$ is a free A-module. If K is upper bounded, the fact that for every $n \in \mathbf{Z}$, the morphism $\rho_A(K^n) \to K^n$ is a quasi-isomorphism implies, by passing to the simple complex, that $\rho_A(K) \to K$ is a quasi-isomorphism. As every complex of A-modules can be written as a filtered inductive limit of upper bounded sub-complexes, it follows that for every $K \in C(A)$, the morphism $\rho_A(K) \to K$ is a quasi-isomorphism.

If K is upper bounded $\rho_A(K)$ is upper bounded and formed of free A-modules. By using the stupid truncations, one obtains that $\rho_A(K)$ is filtered inductive limit of bounded sub-complexes formed of free A-modules. In the general case, K is filtered inductive limit of upper bounded sub-complexes. By using that ρ_A preserves monomorphisms and commutes with filtered inductive limits, one obtains that $\rho_A(K)$ is filtered inductive limit of bounded sub-complexes formed of free A-modules.

The last assertion concerning the change of ring being evident, one can consider that the lemma has been proved.

8.2.3. *Presheaves of modules.* We now assume that $\mathscr{T}$ is the topos of presheaves on a small category $\mathscr{C}$. The ring sheaf $\mathscr{A}$ is a presheaf of commutative rings on $\mathscr{C}$.

Let $K \in C(\mathscr{T}, \mathscr{A})$. For every object U of $\mathscr{C}$, $K(U)$ identifies with an object of $C(\mathscr{A}(U))$. One applies the construction of the lemma 8.2.2.1 to the ring $\mathscr{A}(U)$. We set $(\rho K)(U) = \rho_{\mathscr{A}(U)}(K(U)) \in C(\mathscr{A}(U))$. If $V \to U$ is a morphism in $\mathscr{C}$, one defines an $\mathscr{A}(U)$-morphism $(\rho K)(U) \to (\rho K)(V)$ in the following way :

$$\rho_{\mathscr{A}(U)}(K(U)) \to \rho_{\mathscr{A}(U)}(K(V)) \to \rho_{\mathscr{A}(V)}(K(V))$$

where the left morphism is induced by the structure of complex of presheaves of $\mathscr{A}$-modules on K and the right arrow by the compatibility of the construction of the lemma 8.2.2.1 to the change of ring. According to this lemma, these transition morphisms define a presheaf structure on ρK. Thus, one has defined an object $\rho K \in C(\mathscr{T}, \mathscr{A})$ and it is equipped with a functorial morphism $\rho K \to K$.

By construction, one can verify the presumed virtues of ρ term by term ; thus, this functor ρ allows to establish the théorème 8.2.1.1 in the case where the topos is a topos of presheaves.

8.2.4. *Sheaves of modules.*

Proposition 8.2.4.1. *Let $\mathscr{T}$ be the topos of sheaves on a site whose underlying category is denoted $\mathscr{C}$. We denote $\mathscr{T}'$ the topos of presheaves of sets on $\mathscr{C}$. Let $\mathscr{A}'$ be a presheaf of commutative rings on $\mathscr{C}$. We denote $\mathscr{A}$ the sheaf of rings $a\mathscr{A}'$ on $\mathscr{T}$ associated to $\mathscr{A}'$. If $K \in C(\mathscr{T}', \mathscr{A}')$ is K-flat, then $aK \in C(\mathscr{T}, \mathscr{A})$ is K-flat.*

Lemme 8.2.4.2. *We keep the notations of the proposition 8.2.4.1. Let $K \in C(\mathscr{T}', \mathscr{A}')$ be an object such that aK is zero in $\mathrm{D}(\mathscr{T}, \mathscr{A})$. Then, for every $L \in C(\mathscr{T}', \mathscr{A}')$, $a(K \overset{L}{\otimes}_{\mathscr{A}'} L)$ (where the derived tensor product over $\mathscr{A}'$ is that defined above in the case of presheaves) is zero in $\mathrm{D}(\mathscr{T}, \mathscr{A})$.*

If M is a flat $\mathscr{A}'$-Module, the $\mathscr{A}$-Module aM is flat according to [**SGA 4** v 1.7.1][(x)]. The isomorphism $a(K \otimes_{\mathscr{A}'} M) \simeq aK \otimes_{\mathscr{A}} aM$ then allows to obtain that $a(K \otimes_{\mathscr{A}'} M)$ is acyclic. By devissage, if M is a bounded complex formed of flat $\mathscr{A}'$-Modules, then $a(K \otimes_{\mathscr{A}'} M)$ is acyclic. For every $L \in C(\mathscr{T}', \mathscr{A}')$, as $\rho_{\mathscr{A}'}L$ is filtered inductive limit of complexes bounded formed of flat $\mathscr{A}'$-Modules, it follows that $a(K \otimes_{\mathscr{A}'} \rho_{\mathscr{A}'}L)$ is acyclic. As $\rho_{\mathscr{A}'}L$ is K-flat (over $\mathscr{A}'$), the quasi-isomorphism $\rho_{\mathscr{A}'}K \to K$ induces a quasi-isomorphism $\rho_{\mathscr{A}'}K \otimes_{\mathscr{A}'} \rho_{\mathscr{A}'}L \to K \otimes_{\mathscr{A}'} \rho_{\mathscr{A}'}L$. By passing to the associated sheaves, one deduces isomorphisms in $\mathrm{D}(\mathscr{T}, \mathscr{A})$: $a(K \overset{L}{\otimes}_{\mathscr{A}'} L) \simeq a(K \otimes_{\mathscr{A}'} \rho_{\mathscr{A}'}L) \simeq 0$.

Prove the proposition 8.2.4.1. We denote $\mathscr{Z}$ the triangulated subcategory of $\mathrm{D}(\mathscr{T}', \mathscr{A}')$ formed of the complexes that are annihilated by the associated sheaf functor $\mathrm{D}(\mathscr{T}', \mathscr{A}') \to \mathrm{D}(\mathscr{T}, \mathscr{A})$. The induced functor $\mathrm{D}(\mathscr{T}', \mathscr{A}')/\mathscr{Z} \to \mathrm{D}(\mathscr{T}, \mathscr{A})$ is obviously an equivalence of triangulated categories.

(x) We leave to the reader the care to find an alternative proof that is more direct and that avoids resorting to local inductive limits [**SGA 4** v 8] in the case where the topos $\mathscr{T}$ would not admit a conservative family of points.

According to the lemma 8.2.4.2, the bifunctor $\overset{\mathrm{L}}{\otimes}_{\mathscr{A}'}$ on $\mathsf{D}(\mathscr{T}', \mathscr{A}')$ passes to the quotient by $\mathscr{Z}$ to define a bifunctor on $\mathsf{D}(\mathscr{T}, \mathscr{A})$. The proposition follows immediately. Indeed, let $K \in C(\mathscr{T}', \mathscr{A}')$ K-flat, let $L \in C(\mathscr{T}, \mathscr{A})$ such that L is zero in $\mathsf{D}(\mathscr{T}, \mathscr{A})$. One can identify L with an object L' of $C(\mathscr{T}', \mathscr{A}')$ and this object L' belongs to the triangulated subcategory $\mathscr{Z}$ of $\mathsf{D}(\mathscr{T}', \mathscr{A}')$. The lemma shows that $K \overset{\mathrm{L}}{\otimes}_{\mathscr{A}'} L'$ belongs to $\mathscr{Z}$, in other words, K being K-flat, that $K \otimes_{\mathscr{A}'} L'$ belongs to $\mathscr{Z}$, i.e., that the complex of sheaves $aK \otimes_{\mathscr{A}} L$ is acyclic, which shows that $a(K \otimes_{\mathscr{A}'} L') = aK \in C(\mathscr{T}, \mathscr{A})$ is K-flat.

Corollaire 8.2.4.3. *With the notations of the proposition 8.2.4.1, the functor $\rho_{\mathscr{A}}$ that associates to a complex $K \in C(\mathscr{T}, \mathscr{A})$ the $a\rho_{\mathscr{A}'}K'$ where K' is the $\mathscr{A}'$-Module defined by K is a K-flat resolution functor on $C(\mathscr{T}, \mathscr{A})$ satisfying the conditions of the théorème 8.2.1.1. Thus, one disposes of a bifunctor $\overset{\mathrm{L}}{\otimes}_{\mathscr{A}}$ on $\mathsf{D}(\mathscr{T}, \mathscr{A})$ and of a bifunctorial isomorphism*

$$aK \overset{\mathrm{L}}{\otimes}_{\mathscr{A}} aL \simeq a(K \overset{\mathrm{L}}{\otimes}_{\mathscr{A}'} L)$$

in $\mathsf{D}(\mathscr{T}, \mathscr{A})$ for all K and L in $\mathsf{D}(\mathscr{T}', \mathscr{A}')$.

With the statement of this corollary ends the proof of the théorème 8.2.1.1.

8.3. Complements.

8.3.1. *Internal homomorphisms.*

Définition 8.3.1.1. Let $(\mathscr{T}, \mathscr{A})$ be an annelid topos in commutative rings. We denote $\mathbf{Hom}_{\mathscr{A}}$ the right adjoint functor of the tensor product functor $\otimes_{\mathscr{A}}$ on $C(\mathscr{T}, \mathscr{A})$, i.e., that for X, K and L objects of $C(\mathscr{T}, \mathscr{A})$, we have a canonical isomorphism of abelian groups :

$$\mathrm{Hom}_{C(\mathscr{T},\mathscr{A})}(X \otimes_{\mathscr{A}} K, L) \simeq \mathrm{Hom}_{C(\mathscr{T},\mathscr{A})}(X, \mathbf{Hom}_{\mathscr{A}}(K, L)) .$$

We denote $\mathrm{Hom}^{\bullet}_{\mathscr{A}}(K, L) := \Gamma(\mathscr{T}, \mathbf{Hom}_{\mathscr{A}}(K, L))$ the complex of abelian groups obtained by applying the global sections functor $\Gamma(\mathscr{T}, -)$ to the complex of sheaves $\mathbf{Hom}_{\mathscr{A}}(K, L)$; thus, $H^0(\mathrm{Hom}^{\bullet}_{\mathscr{A}}(K, L)) \simeq \mathrm{Hom}_{K(\mathscr{T},\mathscr{A})}(K, L)$.

We recall that an object $L \in C(\mathscr{T}, \mathscr{A})$ is ***K*-injective** if for every acyclic complex $K \in C(\mathscr{T}, \mathscr{A})$, the complex of abelian groups $\mathrm{Hom}^{\bullet}_{\mathscr{A}}(K, L)$ is acyclic. We also recall that there exist K-injective resolution functors[(xi)].

The following proposition, indicated for memory, is essentially trivial :

Proposition 8.3.1.2. *Let $(\mathscr{T}, \mathscr{A})$ be an annelid topos in commutative rings. The functor $\mathbf{Hom}_{\mathscr{A}}$ on $C(\mathscr{T}, \mathscr{A})$ admits a total right derived functor $\mathrm{R}\,\mathbf{Hom}_{\mathscr{A}} : \mathsf{D}(\mathscr{T}, \mathscr{A})^{\mathrm{opp}} \times \mathsf{D}(\mathscr{T}, \mathscr{A}) \to \mathsf{D}(\mathscr{T}, \mathscr{A})$, right adjoint to $\overset{\mathrm{L}}{\otimes}_{\mathscr{A}}$. Moreover, if K and L are objects of $C(\mathscr{T}, \mathscr{A})$, with L K-injective, then the canonical morphism*

$$\mathbf{Hom}_{\mathscr{A}}(K, L) \to \mathrm{R}\,\mathbf{Hom}_{\mathscr{A}}(K, L)$$

is an isomorphism in $\mathsf{D}(\mathscr{T}, \mathscr{A})$; if we assume furthermore that K is K-flat, then $\mathbf{Hom}_{\mathscr{A}}(K, L)$ is K-injective. Finally, if K, L and M are three objects of $\mathsf{D}(\mathscr{T}, \mathscr{A})$, we have a functorial isomorphism in $\mathsf{D}(\mathscr{T}, \mathscr{A})$ « dear to Cartan » :

$$\mathrm{R}\,\mathbf{Hom}_{\mathscr{A}}(K \overset{\mathrm{L}}{\otimes}_{\mathscr{A}} L, M) \simeq \mathrm{R}\,\mathbf{Hom}_{\mathscr{A}}(K, \mathrm{R}\,\mathbf{Hom}_{\mathscr{A}}(L, M)) .$$

(xi) This result is stated in [**Spaltenstein, 1988**, § 4] in the framework of annelid topological spaces, but the proof can be extended to the case of annelid topoi. The principle of the proof is similar to that used by Grothendieck to show the existence of sufficiently many injectives in the categories of sheaves in [**Grothendieck, 1957**, § I.10]; in the context of homotopical algebra, this argument is known under the name of "small object argument". One can find a proof for the case that interests us in [**Hovey, 2001**].

8.3.2. *Compatibility with inverse images.*

Proposition 8.3.2.1. *Let $(\mathscr{T},\mathscr{A})$ be an annelid topos in commutative rings. For every $K \in C(\mathscr{T},\mathscr{A})$, the K-flat resolution $\rho_A K \in C(\mathscr{T},\mathscr{A})$ is such that for every annelid topos morphism in commutative rings $u\colon (\mathscr{T}',\mathscr{A}') \to (\mathscr{T},\mathscr{A})$, the complex $u^\star\rho_{\mathscr{A}}K \in C(\mathscr{T}',\mathscr{A}')$ is K-flat.*

As $u^\star$ and $\rho_{\mathscr{A}}$ commute with filtered inductive limits, one can assume that K is a bounded complex. The complex $\rho_{\mathscr{A}}K$ is then upper bounded. According to the proposition 8.1.2, it suffices for concluding to show that $u^\star\rho_A K$ is formed of flat $\mathscr{A}$-Modules. If we denote $F_{\mathscr{A}}$ (resp. $F_{\mathscr{A}'}$) the left adjoint functor of the forgetful functor of the category of $\mathscr{A}$-Modules (resp. $\mathscr{A}'$-Modules) to that of sheaves of pointed sets on $\mathscr{T}$ (resp. $\mathscr{T}'$), for every $n \in \mathbf{Z}$, there exists a sheaf of pointed sets X^n on $\mathscr{T}$ such that $(\rho_A K)^n \simeq F_{\mathscr{A}}X^n$ and thus $(u^\star\rho_A K)^n \simeq u^\star F_{\mathscr{A}}X^n \simeq F_{\mathscr{A}'}u^{-1}X^n$, where u^{-1} is the inverse image functor associated to u at the level of sheaves of pointed sets. This allows to conclude that $u^\star\rho_A K$ is formed of flat $\mathscr{A}'$-Modules.

Proposition 8.3.2.2. *Let $u\colon (\mathscr{T}',\mathscr{A}') \to (\mathscr{T},\mathscr{A})$ be an annelid topos morphism in commutative rings. The functor $u^\star \circ \rho_{\mathscr{A}}\colon C(\mathscr{T}',\mathscr{A}') \to C(\mathscr{T},\mathscr{A})$ preserves quasi-isomorphisms and induces a functor that we denote $\mathrm{L}u^\star\colon \mathsf{D}(\mathscr{T},\mathscr{A}) \to \mathsf{D}(\mathscr{T}',\mathscr{A}')$ which is the total left derived functor of $u^\star\colon C(\mathscr{T},\mathscr{A}) \to C(\mathscr{T}',\mathscr{A}')$.*

It is necessary to show that if $K_1 \to K_2$ is a quasi-isomorphism in $C(\mathscr{T},\mathscr{A})$, then $u^\star\rho_{\mathscr{A}}K_1 \to u^\star\rho_{\mathscr{A}}K_2$ is a quasi-isomorphism. Denote $u^{-1}\mathscr{A}$ the inverse image sheaf of rings of $\mathscr{A}$ and $u'\colon (\mathscr{T}',u^{-1}\mathscr{A}) \to (\mathscr{T},\mathscr{A})$ the evident annelid topos morphism. By applying the proposition 8.3.2.1 to u', it follows that $u^{-1}\rho_{\mathscr{A}}K_1$ and $u^{-1}\rho_{\mathscr{A}}K_2$ are K-flat complexes of $u^{-1}\mathscr{A}$-Modules. Denote $C \in C(\mathscr{T},\mathscr{A})$ the cone of $\rho_{\mathscr{A}}K_1 \to \rho_{\mathscr{A}}K_2$. The complex $u^{-1}C$ identifying with the cone of $u^{-1}\rho_{\mathscr{A}}K_1 \to u^{-1}\rho_{\mathscr{A}}K_2$, it follows that $u^{-1}C$ is K-flat. As u^{-1} is exact and C acyclic, it follows that $u^{-1}C$ is acyclic. The K-flatness of $u^{-1}C$ leads to quasi-isomorphisms $u^\star C = \mathscr{A}' \otimes_{u^{-1}\mathscr{A}} u^{-1}C \simeq \rho_{u^{-1}\mathscr{A}}\mathscr{A}' \otimes_{u^{-1}\mathscr{A}} u^{-1}C$. The acyclicity of $u^{-1}C$ and the K-flatness of $\rho_{u^{-1}\mathscr{A}}\mathscr{A}'$ implies next that $\rho_{u^{-1}\mathscr{A}}\mathscr{A}' \otimes_{u^{-1}\mathscr{A}} u^{-1}C$ is acyclic. Finally, $u^\star C$ is acyclic and as $u^\star C$ identifies with the cone of the morphism $u^\star\rho_{\mathscr{A}}K_1 \to u^\star\rho_{\mathscr{A}}K_2$, this morphism of complexes is a quasi-isomorphism.

Corollaire 8.3.2.3. *Let $u\colon (\mathscr{T},\mathscr{A}) \to (\mathscr{T}',\mathscr{A}')$ be an annelid topos morphism in commutative rings. For all K and L objects of $\mathsf{D}(\mathscr{T}',\mathscr{A}')$, we have a canonical isomorphism*

$$\mathrm{L}u^\star K \overset{\mathrm{L}}{\otimes}_{\mathscr{A}} \mathrm{L}u^\star L \simeq \mathrm{L}u^\star(K \overset{\mathrm{L}}{\otimes}_{\mathscr{A}'} L)$$

in $\mathsf{D}(\mathscr{T},\mathscr{A})$.

Corollaire 8.3.2.4. *If u and v are composable annelid topos morphisms in commutative rings, then we have a canonical isomorphism of functors $\mathrm{L}v^\star \circ \mathrm{L}u^\star \xrightarrow{\sim} \mathrm{L}(u \circ v)^\star$.*

Remarque 8.3.2.5. It is possible to establish a more precise statement than the proposition 8.3.2.1, namely that for every annelid topos morphism in commutative rings $u\colon (\mathscr{T}',\mathscr{A}') \to (\mathscr{T},\mathscr{A})$, if $K \in C(\mathscr{T},\mathscr{A})$ is K-flat, then $u^\star K$ is K-flat. Given the proposition 8.3.2.1, one reduces to showing that if $K \in C(\mathscr{T},\mathscr{A})$ is K-flat and acyclic, then $(u^\star K) \otimes_{\mathscr{A}'} L$ is acyclic for every $\mathscr{A}'$-Module L and this can be demonstrated thanks to the method of local inductive limits used in [**SGA 4** v 8.2.9] (or by using a conservative family of fiber functors if the considered topoi possess one). The conclusion of this remark is that to compute $\mathrm{L}u^\star K$ it suffices to apply $u^\star$ to an arbitrary K-flat resolution of K.

9. Invertible complexes

Proposition 9.1. *Let A be a commutative ring. Let $X \in \mathsf{D}(A)$. Let $Y \in \mathsf{D}(A)$. Let $X \overset{\mathrm{L}}{\otimes}_A Y \simeq A$ be an isomorphism in $\mathsf{D}(A)$. Then, there exists a locally constant function $k\colon \operatorname{Spec}(A) \to \mathbf{Z}$, an invertible A-module L and isomorphisms $X \simeq L[k]$ and $Y \simeq L^\vee[-k]$ (the shift functor $[k]$ being defined evidently).*

This result appears in [**Hartshorne, 1966**, lemma 3.3, Chapter V] under additional hypotheses saying that A is Noetherian and that X and Y are in $\mathsf{D}^-_{\mathrm{c}}(A)$. This version implies evidently another where one asks A to be Noetherian and X and Y to be perfect complexes. It is then easy to suppress the Noetherian hypothesis (cf. [**ÉGA** iv 8]). In short, to complete the proof, it suffices to show that in the conditions of the above proposition, the complex X (and thus Y by symmetry of roles) is a perfect complex. We say of an object X of $\mathsf{D}(A)$ that it is of **finite presentation** if the functor $\mathrm{Hom}_{\mathsf{D}(A)}(X,-)$ from $\mathsf{D}(A)$ to the category of abelian groups commutes with direct sums (infinite). One can show that $X \in \mathsf{D}(A)$ is a perfect complex if and only if it is of finite presentation [xii]. From the isomorphism $X \overset{\mathrm{L}}{\otimes}_A Y \simeq A$, one draws an isomorphism of functors $\mathrm{R}\,\mathbf{Hom}(X,-) \simeq Y \overset{\mathrm{L}}{\otimes}_A - : \mathsf{D}(A) \to \mathsf{D}(A)$. As the functor $Y \overset{\mathrm{L}}{\otimes}_A -$ commutes obviously with direct sums, this is also the case for the functor $\mathrm{Hom}_{\mathsf{D}(A)}(X,-)$, which shows that X is of finite presentation : it is a perfect complex.

Proposition 9.2. *Let $(\mathscr{T},\mathscr{A})$ be an annelid topos in commutative rings. Let $X \in \mathsf{D}(\mathscr{T},\mathscr{A})$. Let $Y \in \mathsf{D}(\mathscr{T},\mathscr{A})$. Let $X \overset{\mathrm{L}}{\otimes}_{\mathscr{A}} Y \simeq \mathscr{A}$ be an isomorphism in $\mathsf{D}(\mathscr{T},\mathscr{A})$. Then, the final object of $\mathscr{T}$ is covered by objects U such that $X_{|U}$ and $Y_{|U}$ can be induced by perfect complexes X' and Y' of $\mathsf{D}(\mathscr{A}(U))$ and that we have an isomorphism $X' \overset{\mathrm{L}}{\otimes}_{\mathscr{A}(U)} Y' \simeq \mathscr{A}(U)$ compatible with the given isomorphism ; the form of the complexes X' and Y' is therefore known thanks to the proposition 9.1.*

Prove the proposition 9.2. One can assume that $\mathscr{T}$ is the topos of sheaves on a site $\mathscr{C}$. Denote $\mathscr{T}'$ the topos of presheaves on the underlying category to the site $\mathscr{C}$, also denoted $\mathscr{C}$, and $\mathscr{A}'$ the presheaf of rings on $\mathscr{C}$ defined by $\mathscr{A}$. One can identify $X \overset{\mathrm{L}}{\otimes}_{\mathscr{A}} Y$ with the sheaf associated to $\rho_{\mathscr{A}'}X \otimes_{\mathscr{A}'} \rho_{\mathscr{A}'}Y$ where $\rho_{\mathscr{A}'}$ is the K-flat resolution functor defined in the paragraph 8.2.3. Thus, the given isomorphism $\mathscr{A} \xrightarrow{\sim} X \overset{\mathrm{L}}{\otimes}_{\mathscr{A}} Y$ can be represented locally by a 0-cocycle s of $(\rho_{\mathscr{A}'}X \otimes_{\mathscr{A}'} \rho_{\mathscr{A}'}Y)(U) = \rho_{\mathscr{A}(U)}(X(U)) \otimes_{\mathscr{A}(U)} \rho_{\mathscr{A}(U)}(Y(U))$. Let U be an object of $\mathscr{C}$ on which such a description is possible. By construction of $\rho_{\mathscr{A}(U)}$, it follows that there exist bounded sub-complexes formed of free finite type $\mathscr{A}(U)$-modules X' of $\rho_{\mathscr{A}(U)}(X(U))$ and Y' of $\rho_{\mathscr{A}(U)}(X(U))$ such that $X' \otimes_{\mathscr{A}(U)} Y'$ contains s. Denote $\mathscr{T}_{|U}$ the topos of sheaves on $\mathscr{T}$ over U (an underlying site is given by the category $\mathscr{C}/U$). One disposes of an evident annelid topos morphism π from $(\mathscr{T}_{|U},\mathscr{A}_{|U})$ to the punctual topos equipped with the ring $\mathscr{A}(U)$. Set $X'' = \pi^\star X'$ and $Y'' = \pi^\star Y'$: these are strictly perfect complexes on $\mathscr{T}_{|U}$. One disposes of evident morphisms $X'' \to X_{|U}$ and $Y'' \to Y_{|U}$ in $\mathsf{D}(\mathscr{T}_{|U},\mathscr{A}_{|U})$ and according to what precedes, we have a morphism $\mathscr{A}_{|U} \to X'' \otimes_{\mathscr{A}_{|U}} Y''$ factorizing the isomorphism $\mathscr{A}_{|U} \simeq X_{|U} \overset{\mathrm{L}}{\otimes}_{\mathscr{A}_{|U}} Y_{|U}$. In short, $\mathscr{A}_{|U} \simeq X_{|U} \overset{\mathrm{L}}{\otimes}_{\mathscr{A}_{|U}} Y_{|U}$ is a direct factor of $X'' \otimes_{\mathscr{A}_{|U}} Y''$. Moreover, the isomorphism $\mathscr{A}_{|U} \xrightarrow{\sim} X_{|U} \overset{\mathrm{L}}{\otimes}_{\mathscr{A}_{|U}} Y_{|U}$ also factorizes through $X'' \otimes_{\mathscr{A}_{|U}} Y_{|U}$, so $\mathscr{A}_{|U}$ is a direct factor of $X'' \otimes_{\mathscr{A}_{|U}} Y_{|U}$ in $\mathsf{D}(\mathscr{T},\mathscr{A})$; by tensoring this fact with $X_{|U}$, one obtains that $X_{|U}$ is a direct factor of X''. Let $p : X'' \to X''$ be a projector whose image is isomorphic to $X_{|U}$. As X' and Y' are (strictly) perfect, up to replacing U by a family of objects covering it, one can assume that p comes from a projector $\tilde{p}$ on X' in $\mathsf{D}(\mathscr{A}(U))$. The image of $\tilde{p}$ is a perfect complex $\tilde{X}$ in $\mathsf{D}(\mathscr{A}(U))$ inducing $X_{|U}$. Similarly, one can assume that $Y_{|U}$ is induced by a perfect complex of $\mathscr{A}(U)$-modules $\tilde{Y}$. Up to refining the covering of U, one can therefore assume that the isomorphism $\mathscr{A}_{|U} \xrightarrow{\sim} X_{|U} \otimes_{\mathscr{A}_{|U}} Y_{|U}$ is induced by a morphism $\tilde{s} : \mathscr{A}(U) \to \tilde{X} \otimes_{\mathscr{A}(U)} \tilde{Y}$. Denote C a cone of $\tilde{s}$. The complex C is perfect and satisfies $\pi^\star C \simeq 0$. It follows that up to refining the covering of U, one can assume that C is acyclic, i.e., that $\tilde{s}$ is an isomorphism.

10. Universal coefficients

10.1. Statements for $\mathrm{R}\,\mathbf{Hom}$.

[xii] It is a good exercise. However, one can also obtain this criterion by using general principles. The object A of $\mathsf{D}(A)$ is a finite presentation generator (*i.e.* the cohomological functor $\mathrm{Hom}_{\mathsf{D}(A)}(A,-)$ commutes with direct sums and is conservative) ; the triangulated subcategory of $\mathsf{D}(A)$ formed of finite presentation objects is therefore the pseudo-abelian envelope of the triangulated subcategory generated by A, i.e., the subcategory of perfect complexes : combine [**Neeman, 2001**, proposition 8.4.1], [**Neeman, 2001**, lemma 4.4.5] and [**Neeman, 2001**, remark 4.2.6].

Proposition 10.1.1. *Let Z be a Noetherian scheme. Let A be a commutative Noetherian ring. Let $X \in \mathsf{D}^-_{\mathrm{c}}(Z_{\text{ét}}, A)$. Let $Y \in \mathsf{D}^{\mathrm{b}}_{\mathrm{tf}}(Z_{\text{ét}}, A)$. We assume either that A is a field or that $\mathrm{R}\,\mathbf{Hom}(X, M \overset{\mathrm{L}}{\otimes}_A Y)$ is bounded independently of the A-module M. Then, for every $M \in \mathsf{D}^+(A)$, the canonical morphism*

$$M \overset{\mathrm{L}}{\otimes}_A \mathrm{R}\,\mathbf{Hom}(X, Y) \to \mathrm{R}\,\mathbf{Hom}(X, M \overset{\mathrm{L}}{\otimes}_A Y)$$

is an isomorphism. Moreover, if we are in the case where $\mathrm{R}\,\mathbf{Hom}(X, M \overset{\mathrm{L}}{\otimes}_A Y)$ is bounded independently of the A-module M, then $\mathrm{R}\,\mathbf{Hom}(X, Y) \in \mathsf{D}^{\mathrm{b}}_{\mathrm{tf}}(Z_{\text{ét}}, A)$.

We denote $u_M\colon M \overset{\mathrm{L}}{\otimes}_A \mathrm{R}\,\mathbf{Hom}(X, Y) \to \mathrm{R}\,\mathbf{Hom}(X, M \overset{\mathrm{L}}{\otimes}_A Y)$ the canonical morphism, for every $M \in \mathsf{D}^+(Z_{\text{ét}}, A)$. The hypothesis that $X \in \mathsf{D}^-_{\mathrm{c}}(Z_{\text{ét}}, A)$ implies that the functors $\mathbf{Ext}^q(X, -) = \mathscr{H}^q \mathrm{R}\,\mathbf{Hom}(X, -)$ commute with filtered inductive limits of sheaves of A-modules on Z. In particular, the functor $\mathrm{R}\,\mathbf{Hom}(X, -)$ from the category $\mathsf{D}^+(Z_{\text{ét}}, A)$ to itself commutes with representable direct sums. One deduces that u_M is an isomorphism if M is a free A-module. Thus, if M is a bounded complex of free A-modules, then u_M is an isomorphism; if A is a field, one can conclude that u_M is an isomorphism for every $M \in \mathsf{D}^+(A)$.

We now place ourselves in the case where $\mathrm{R}\,\mathbf{Hom}(X, M \overset{\mathrm{L}}{\otimes}_A Y)$ is bounded independently of the A-module M. Let $M \in \mathsf{D}^{\mathrm{b}}(A)$. Show that u_M is an isomorphism. For every relative integer q, there exists a morphism $P \to M$ with P a bounded complex of free A-modules such that if we denote C a cone of this morphism, then $C \leq q$ (such inequalities are to be understood relative to the canonical t-structure). We consider the commutative square :

$$\begin{tikzcd}
P \overset{\mathrm{L}}{\otimes}_A \mathrm{R}\,\mathbf{Hom}(X, Y) \arrow[r, "u_P", "\sim"'] \arrow[d] & \mathrm{R}\,\mathbf{Hom}(X, P \overset{\mathrm{L}}{\otimes}_A Y) \arrow[d] \\
M \overset{\mathrm{L}}{\otimes}_A \mathrm{R}\,\mathbf{Hom}(X, Y) \arrow[r, "u_M"] & \mathrm{R}\,\mathbf{Hom}(X, M \overset{\mathrm{L}}{\otimes}_A Y)
\end{tikzcd}$$

Denote N the smallest natural integer such that $\mathrm{R}\,\mathbf{Hom}(X, V \overset{\mathrm{L}}{\otimes}_A Y)$ is $\leq N$ for every A-module V. By devissage, one obtains that the cones of the vertical arrows of the above diagram are $\leq q + N$. The two vertical arrows therefore induce isomorphisms on the cohomology objects $\mathscr{H}^i$ for $i \geq q+N+2$. The top arrow being an isomorphism, the bottom arrow also induces isomorphisms on the $\mathscr{H}^i$ for $i \geq q+N+2$. This fact being verified for every $q \in \mathbf{Z}$, the bottom morphism is indeed an isomorphism.

One deduces immediately from what precedes that $\mathrm{R}\,\mathbf{Hom}(X, Y) \in \mathsf{D}^{\mathrm{b}}_{\mathrm{tf}}(Z_{\text{ét}}, A)$. We also know that $Y \in \mathsf{D}^{\mathrm{b}}_{\mathrm{tf}}(Z_{\text{ét}}, A)$ and that $X \in \mathsf{D}^-(Z_{\text{ét}}, A)$. Consequently, there exists a relative integer N' such that if $M \in \mathsf{D}^{\geq c}(A)$ for a certain integer c, then the source and target of u_M are $\geq c + N'$. By reasoning as above, one deduces from the fact that u_M is an isomorphism for every $M \in \mathsf{D}^{\mathrm{b}}(A)$ that this result holds in fact for every $M \in \mathsf{D}^+(A)$.

Proposition 10.1.2. *Let Z be a scheme. Let A be a commutative ring. Let $Y \in \mathsf{D}^{\mathrm{b}}_{\mathrm{tf}}(Z_{\text{ét}}, A)$. Let M be a pseudo-coherent complex in $\mathsf{D}(A)$*[(xiii)]*. Let $N \in \mathsf{D}^+(A)$. Then, the canonical morphism*

$$\mathrm{RHom}(M, N) \overset{\mathrm{L}}{\otimes}_A Y \to \mathrm{R}\,\mathbf{Hom}(M, N \overset{\mathrm{L}}{\otimes}_A Y)$$

is an isomorphism in $\mathsf{D}^+(Z_{\text{ét}}, A)$. (We denoted RHom *the internal Hom in* $\mathsf{D}(A)$.)

We fix $N \in \mathsf{D}^+(A)$. For every $M \in \mathsf{D}(A)$, denote v_M the canonical morphism

$$\mathrm{RHom}(M, N) \overset{\mathrm{L}}{\otimes}_A Y \to \mathrm{R}\,\mathbf{Hom}(M, N \overset{\mathrm{L}}{\otimes}_A Y)\,.$$

(xiii) We recall that this means that for every $n \in \mathbf{Z}$, there exists a morphism of complexes $P \to M$ inducing isomorphisms $H^q P \to H^q M$ for $q \geq n$ where P is a strictly perfect complex of A-modules (i.e., that P is bounded and formed of projective finite type A-modules).

Of course, if $M \in \mathsf{D}(A)$ is a perfect complex, v_M is an isomorphism. For every $M \in \mathsf{D}(A)$ pseudo-coherent and every relative integer q, there exists a morphism $P \to M$ with P perfect whose cone is $\leq q$. To be able to deduce that v_M is an isomorphism for every $M \in \mathsf{D}(A)$ from the particular case where M is assumed perfect, it suffices therefore to show that there exists a relative integer c such that for every integer q and every $M \in \mathsf{D}^{\leq q}(A)$, the source and target of v_M are $\geq -q + c$. Denote a an integer such that $N \geq a$ and b an integer such that for every A-module V, $V \overset{\mathrm{L}}{\otimes}_A Y \geq b$. One verifies immediately that $c = a + b$ works, which completes the proof of the proposition.

Proposition 10.1.3. *Let Z be a Noetherian scheme. Let A be a commutative Noetherian ring. Let $X \in \mathsf{D}^-_{\mathrm{c}}(Z_{\text{ét}}, A)$. Let $Y \in \mathsf{D}^{\mathrm{b}}_{\mathrm{tf}}(Z_{\text{ét}}, A)$. We assume either that A is a field or that $\mathrm{R}\,\mathbf{Hom}(X, N \overset{\mathrm{L}}{\otimes}_A Y)$ is bounded independently of the A-module N. Let B be an A-algebra. Then, for every pseudo-coherent complex $M \in \mathsf{D}(B)$ and for every $N \in \mathsf{D}^+(B)$, the canonical morphism*

$$\mathrm{RHom}_B(M, N) \overset{\mathrm{L}}{\otimes}_A \mathrm{R}\,\mathbf{Hom}_A(X, Y) \to \mathrm{R}\,\mathbf{Hom}_B(M \overset{\mathrm{L}}{\otimes}_A X, N \overset{\mathrm{L}}{\otimes}_A Y)$$

is an isomorphism in $\mathsf{D}(Z_{\text{ét}}, B)$.

The hypotheses make that $\mathrm{RHom}_B(M, N)$ belongs to $\mathsf{D}^+(B)$. If A is not a field, one can apply the result of the proposition 10.1.2 by replacing respectively A, M, N and Y by B, M, N and $B \overset{\mathrm{L}}{\otimes}_A \mathrm{R}\,\mathbf{Hom}_A(X, Y)$ (one will note that $\mathrm{R}\,\mathbf{Hom}_A(X, Y) \in \mathsf{D}^{\mathrm{b}}_{\mathrm{tf}}(Z_{\text{ét}}, A)$ according to the proposition 10.1.1). One obtains thus a canonical isomorphism

$$\mathrm{RHom}_B(M, N) \overset{\mathrm{L}}{\otimes}_A \mathrm{R}\,\mathbf{Hom}_A(X, Y) \xrightarrow{\sim} \mathrm{R}\,\mathbf{Hom}_B(M, N \overset{\mathrm{L}}{\otimes}_A \mathrm{R}\,\mathbf{Hom}_A(X, Y)) .$$

If A is a field, one establishes this isomorphism by a similar argument by reducing to the case where M is a perfect complex of B-modules. By applying now the proposition 10.1.1, one obtains a new isomorphism :

$$\mathrm{R}\,\mathbf{Hom}_B(M, N \overset{\mathrm{L}}{\otimes}_A \mathrm{R}\,\mathbf{Hom}_A(X, Y)) \xrightarrow{\sim} \mathrm{R}\,\mathbf{Hom}_B(M, \mathrm{R}\,\mathbf{Hom}_A(X, N \overset{\mathrm{L}}{\otimes}_A Y)) .$$

Finally, one uses the isomorphism dear to Cartan :

$$\mathrm{R}\,\mathbf{Hom}_B(M, \mathrm{R}\,\mathbf{Hom}_A(X, N \overset{\mathrm{L}}{\otimes}_A Y)) \simeq \mathrm{R}\,\mathbf{Hom}_B(M \overset{\mathrm{L}}{\otimes}_A X, N \overset{\mathrm{L}}{\otimes}_A Y) .$$

The wanted isomorphism is the isomorphism obtained by composing the different canonical isomorphisms above.

10.2. Consequences for $\mathrm{R}j_\star$ and $i^!$.

Proposition 10.2.1. *Let $j : U \to X$ be an open immersion between Noetherian schemes. Let A be a commutative Noetherian ring. Let $Y \in \mathsf{D}^{\mathrm{b}}_{\mathrm{tf}}(U_{\text{ét}}, A)$. We assume either that A is a field or that $\mathrm{R}j_\star(M \overset{\mathrm{L}}{\otimes}_A Y)$ is bounded independently of the A-module M. Then, for every $M \in \mathsf{D}^+(A)$, the canonical morphism*

$$M \overset{\mathrm{L}}{\otimes}_A \mathrm{R}j_\star Y \to \mathrm{R}j_\star(M \overset{\mathrm{L}}{\otimes}_A Y)$$

is an isomorphism in $\mathsf{D}(X_{\text{ét}}, A)$. If $\mathrm{R}j_\star(M \overset{\mathrm{L}}{\otimes}_A Y)$ is bounded independently of the A-module M, then $\mathrm{R}j_\star Y$ belongs to $\mathsf{D}^{\mathrm{b}}_{\mathrm{tf}}(X_{\text{ét}}, A)$.

This follows from the proposition 10.1.1, given the formula $\mathrm{R}j_\star Y \simeq \mathrm{R}\,\mathbf{Hom}(j_!A, j_!Y)$ for every $Y \in \mathsf{D}(U_{\text{ét}}, A)$.

Proposition 10.2.2. *Let $i : Z \to X$ be a closed immersion between Noetherian schemes. Let A be a commutative Noetherian ring. Let $Y \in \mathsf{D}^{\mathrm{b}}_{\mathrm{tf}}(X_{\text{ét}}, A)$. We assume either that A is a field or that $i^!(M \overset{\mathrm{L}}{\otimes}_A Y)$ is bounded independently of the A-module M. Then, for every $M \in \mathsf{D}^+(A)$, the canonical morphism*

$$M \overset{\mathrm{L}}{\otimes}_A i^! Y \to i^!(M \overset{\mathrm{L}}{\otimes}_A Y)$$

is an isomorphism in $\mathsf{D}(Z_{\text{ét}}, A)$*. If* $i^!(M \overset{\mathrm{L}}{\otimes}_A Y)$ *is bounded independently of the* A*-module* M*, then* $i^! Y$ *belongs to* $\mathsf{D}^{\mathsf{b}}_{\mathsf{tf}}(Z_{\text{ét}}, A)$*.*

This time, one uses the formula $i^! Y \simeq i^\star \mathrm{R}\,\mathbf{Hom}(i_\star A, Y)$.

11. Ind-unipotent modules

11.1. Definitions. Let G be a topological group. Let A be a ring. We will call here $A[G]$-module a left $A[G]$-module in the usual sense when G is a group that is not equipped with a topology. A discrete $A[G]$-module is an $A[G]$-module in which the stabilizer of every element is open. An $A[G]$-module is said **trivial** if the group G acts trivially on it ; such an $A[G]$-module is discrete. We denote I_G the augmentation ideal of $A[G]$; an $A[G]$-module is trivial if and only if it is annihilated by I_G.

The category of discrete $A[G]$-modules is a Grothendieck abelian category ; the inclusion functor of this category into that of $A[G]$-modules is exact, commutes with inductive limits, but in general not with projective limits. The category of discrete $A[G]$-modules is stable under sub-quotient (but in general not under extensions) in that of $A[G]$-modules.

Définition 11.1.1. A discrete $A[G]$-module is **unipotent** if it admits a finite filtration whose successive quotients are trivial $A[G]$-modules. The **order of unipotence** of an unipotent $A[G]$-module is the smallest length of such a filtration ; thus, a non-zero trivial $A[G]$-module is unipotent of order 1. A discrete $A[G]$-module is **ind-unipotent** if all its finite type sub-$A[G]$-modules are unipotent.

Définition 11.1.2. Let M be a discrete $A[G]$-module. One defines recursively an increasing filtration $(\operatorname{Fil}_n M)_{n \in \mathbf{Z}}$ of M by sub $A[G]$-modules, so that $\operatorname{Fil}_0 M = 0$ and that for every integer $n \in \mathbf{N}$, we have a canonical isomorphism $\operatorname{Fil}_{n+1} M / \operatorname{Fil}_n M \simeq \mathrm{H}^0(G, M/\operatorname{Fil}_n M) \subset M/\operatorname{Fil}_n M$.

The filtration $\operatorname{Fil}_\bullet M$ is obviously functorial in M in the sense that if $f : M \to M'$ is a morphism of $A[G]$-modules, then $f(\operatorname{Fil}_n M) \subset \operatorname{Fil}_n(M')$ for every $n \in \mathbf{Z}$.

11.2. Properties.

Proposition 11.2.1. *The notions of unipotence and ind-unipotence of discrete* $A[G]$*-modules enjoy the following properties :*

(i) *The unipotent character of discrete* $A[G]$*-modules is stable under sub-quotients and extensions ; the order of unipotence decreases by passing to a sub-quotient and is sub-additive with respect to extensions ;*
(ii) *The ind-unipotent character of discrete* $A[G]$*-modules is stable under sub-quotients, and, if* G *is profinite, under extensions ;*
(iii) *The category of discrete unipotent* $A[G]$*-modules (resp. ind-unipotent) is abelian, the inclusion functor into that of discrete* $A[G]$*-modules is exact ;*
(iv) *A discrete* $A[G]$*-module* M *is unipotent if and only if there exists a natural integer* n *such that* $\operatorname{Fil}_n M = M$*; in this case, the order of unipotence of* M *is the smallest of these integers* n*;*
(v) *The unipotent (resp. ind-unipotent) character of a discrete* $A[G]$*-module does not depend on the coefficient ring* A *and this notion is not altered either if one considers* G *as a discrete group ;*
(vi) *For every natural integer* n *and every discrete* $A[G]$*-module,* $\operatorname{Fil}_n M$ *is the annihilator of* I_G^n *in* M*;*
(vii) *For every natural integer* n*, a discrete* $A[G]$*-module is unipotent of order at most* n *if and only if it is annihilated by the ideal* I_G^n*. A discrete* $A[G]$*-module* M *is ind-unipotent if and only if every element of* M *is annihilated by a power of* I_G*, i.e., that* $M = \bigcup_{n \in \mathbf{N}} \operatorname{Fil}_n M$*;*
(viii) *A unipotent discrete* $A[G]$*-module is ind-unipotent ;*
(ix) *A finite type discrete* $A[G]$*-module is unipotent if and only if it is ind-unipotent ;*
(x) *A direct sum (resp. an inductive limit) of ind-unipotent discrete* $A[G]$*-modules is an ind-unipotent discrete* $A[G]$*-module.*
(xi) *The ind-unipotent discrete* $A[G]$*-modules are exactly the* $A[G]$*-modules filtered inductive limits of unipotent discrete* $A[G]$*-modules.*

Prove (i). If $(M, F_\bullet M)$ is a filtered discrete $A[G]$-module, every sub-object M' (resp. sub-quotient M'') of M is naturally equipped with a filtration $F_\bullet M'$ (resp. $F_\bullet M''$) such that for every $n \in \mathbf{Z}$, the induced morphism at the level of graded pieces $\mathrm{Gr}_n M' \to \mathrm{Gr}_n M$ (resp. $\mathrm{Gr}_n M \to \mathrm{Gr}_n M''$) is injective (resp. surjective) ; as every sub-quotient of a trivial $A[G]$-module is trivial, it follows immediately that the unipotent character of an $A[G]$-module is stable under sub-quotient and that the order of unipotence decreases in this operation. It is evident from the definition that the unipotent character of discrete $A[G]$-modules is stable under extensions, and that the order of unipotence is sub-additive with respect to them. (viii) is a trivial consequence of (i).

Prove property (iv). Let M be a discrete $A[G]$-module. If $\mathrm{Fil}_n M = M$, the filtration $(\mathrm{Fil}_i M)_{0 \leq i \leq n}$ is a finite filtration of M of length n whose successive quotients are trivial ; thus, M is unipotent of order at most n. Conversely, let M be an $A[G]$-module equipped with a filtration $(F_i M)_{i \geq 0}$ such that $F_0 M = 0$ and whose successive quotients are trivial. An evident induction on $i \in \mathbf{N}$ shows that we have an inclusion $F_i M \subset \mathrm{Fil}_i M$, so that if a natural integer n is such that $F_n M = M$, then $\mathrm{Fil}_n M = M$.

In the unipotent case, property (v) is a consequence of property (iv) : the filtration $\mathrm{Fil}_\bullet M$ is the same whether one considers M as $A[G]$-module or as $\mathbf{Z}[G]$-module, and whether one considers G as an authentic topological group or as a discrete group. When it will not be pertinent, we will therefore not systematically specify in the sequel that the $A[G]$-modules are discrete.

Concerning property (vi), it is easy to show by induction on $n \in \mathbf{N}$ that $\mathrm{Fil}_n M$ is the annihilator of I_G^n in M.

Consider (vii). One deduces trivially from (iv) and (vi) that a discrete $A[G]$-module is unipotent of order at most n if and only if it is annihilated by I_G^n. Prove the other part of (vii). Let M be an ind-unipotent discrete $A[G]$-module. For every $m \in M$, the sub-$A[G]$-module of M generated by m is unipotent, according to what one has just shown, this module is annihilated by a power of I_G, in particular, m is annihilated by I_G^n for a certain natural integer n. Conversely, assume that every element of M is annihilated by a power of I_G. If one considers a finite type sub-$A[G]$-module N of M, by applying the hypothesis to the elements of a finite set of generators of N, one obtains that N is annihilated by I_G^n for a certain natural integer n, and thus that N is unipotent. This completes the proof of (vii). We thus know that a discrete $A[G]$-module M is ind-unipotent if and only if $M = \bigcup_{n \in \mathbf{N}} \mathrm{Fil}_n M$. As the filtration $\mathrm{Fil}_n M$ does not depend on the topology of G nor on the coefficient ring A, one can obtain (v). Property (ix) is an immediate consequence of (vii). The stability under direct sums and sub-quotients of the ind-unipotent character follows also, which establishes (x).

Assume G profinite and prove the stability under extensions of ind-unipotent discrete $A[G]$-modules, which will complete the proof of (ii). According to (v), one can assume that $A = \mathbf{Z}$. One gives oneself a short exact sequence $0 \to M' \to M \to M'' \to 0$ of discrete $\mathbf{Z}[G]$-modules with M' and M'' ind-unipotent. Let N be a finite type sub-$\mathbf{Z}[G]$-module of M. It is a matter of showing that N is unipotent. One can consider the image N'' of N in M'', and N' the kernel of the projection $N \to N''$. The $\mathbf{Z}[G]$-module N' (resp. N'') is a sub-$\mathbf{Z}[G]$-module of M' (resp. of M''). By passing to sub-objects, N' and N'' are ind-unipotent. The group G being profinite and N a discrete $\mathbf{Z}[G]$-module, N is a finitely generated abelian group ; the abelian groups N' and N'' which are sub-quotients of it are also finitely generated. According to (ix), N' and N'' are unipotent ; according to (i), N is unipotent, which completes the proof of (ii).

(iii) follows immediately from (i) and (ii). Finally, (xi) follows from (vii), (viii) and (x).

Remarque 11.2.2. If $H \to G$ is a morphism of topological groups and M an unipotent (resp. ind-unipotent) $A[G]$-module, then, as $A[H]$-module, M is unipotent (resp. ind-unipotent).

11.3. Ind-unipotent modules for a distinguished subgroup. Let A be a ring. Let G be a topological group. Let H be a distinguished (closed) subgroup of G. If M is a discrete $A[G]$-module, one can wonder if as $A[H]$-module, M is unipotent (resp. ind-unipotent).

In this situation, we denote $\mathrm{Fil}_\bullet M$ the filtration of the definition 11.1.2 for the group H. The fact that H is distinguished in G allows to obtain immediately that this filtration $\mathrm{Fil}_\bullet M$ is constituted of sub-$A[G]$-modules of M. From this remark and the proposition 11.2.1 (iv), one draws :

Proposition 11.3.1. *Let M be an $A[G]$-module. The following conditions are equivalent :*

- *as $A[H]$-module, M is unipotent ;*
- *there exists a finite filtration of M by sub-$A[G]$-modules such that H acts trivially on the successive quotients.*

Proposition 11.3.2. *Let M be an $A[G]$-module. The following conditions are equivalent :*

- *as $A[H]$-module, M is ind-unipotent ;*
- *every finite type sub-$A[G]$-module of M is unipotent for H.*

It is a matter of showing that if a finite type sub-$A[H]$-module N of M is unipotent, then the finite type sub-$A[G]$-module of M generated by N is unipotent for H. To show this, it suffices to establish the following lemma :

Lemme 11.3.3. *Let M be an unipotent $A[H]$-module. As $A[H]$-module, $A[G] \otimes_{A[H]} M$ is unipotent.*

We denote $\rho : H \to \mathrm{Aut}_A(M)$ the morphism defining the action of H on M. For every $g \in G$, we denote $c_g : H \to H$ the map $h \longmapsto ghg^{-1}$ and M^g the H-module defined by the morphism $\rho \circ c_g : H \to \mathrm{Aut}_A(M)$. The $A[H]$-module $A[G] \otimes_{A[H]} M$ identifies with $\oplus_{i \in I} M^{g_i}$ where $(g_i)_{i \in I}$ is a set of representatives of G/H. As M is unipotent, it is evident that the $A[H]$-modules M^{g_i} are unipotent and that they all have the same order of unipotence. One deduces easily that $A[G] \otimes_{A[G]} M$ is unipotent for H.

Now that the definitions of unipotence and ind-unipotence for the distinguished subgroup H are clarified, one can state the following proposition :

Proposition 11.3.4. *The statements of the proposition 11.2.1 remain true if one replaces "unipotent" by "unipotent for H", "ind-unipotent" by "ind-unipotent for H" and I_G by I_H, and that, as above, one defines the filtration* $\mathrm{Fil}_\bullet$ *relative to the group H.*

Given the clarifications made above, this is trivial.

12. The biduality morphism

The biduality morphism $L \to D_K D_K L$ studied in this exposé was defined in **XVI**-4.7.7. We will here establish its compatibilities with respect to the functors $f^\star$ and $\mathrm{R}f_\star$. These compatibilities result from [**Calmès & Hornbostel, 2009**], but, by developing it here somewhat, we would like to show all the relevance of the argument given by Deligne in [**SGA 4½** [Dualité] 1.2].

12.1. Couplings. Let $\mathscr{C}$ be a category equipped with a bifunctor that we will denote $\otimes : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$. We assume given a symmetry isomorphism $s_{X,Y} : X \otimes Y \xrightarrow{\sim} Y \otimes X$ functorial in X and Y such that $s_{Y,X}^{-1} = s_{X,Y}$: we will say that $\otimes$ is a symmetric bifunctor. We also assume that "$\otimes$ admits a right adjoint **Hom**", i.e., a bifunctor $\mathbf{Hom} : \mathscr{C}^{\mathrm{opp}} \times \mathscr{C} \to \mathscr{C}$ equipped with trifunctorial bijections said "dear to Cartan" :

$$\mathrm{Hom}_{\mathscr{C}}(X, \mathbf{Hom}(Y, Z)) \simeq \mathrm{Hom}_{\mathscr{C}}(X \otimes Y, Z) .$$

Définition 12.1.1. A **coupling** in $\mathscr{C}$ consists of the data of three objects M_1, M_2, K and of a morphism $M_1 \otimes M_2 \xrightarrow{a} K$. We say that a is a coupling between M_1 and M_2 with values in K. If we dispose of morphisms $M_1' \to M_1$, $M_2' \to M_2$ and $K \to K'$, the coupling a induces by functoriality a coupling $a' : M_1' \otimes M_2' \to K'$.

Proposition 12.1.2. *The data of a coupling $a : M_1 \otimes M_2 \to K$ is equivalent by adjunction to the data of a morphism $b : M_1 \to \mathbf{Hom}(M_2, K)$: we say that b is the* ***adjoint description*** *of the coupling a. If we dispose of morphisms $M_1' \to M_1$, $M_2' \to M_2$ and $K \to K'$, by using the bifunctoriality of* **Hom** *and the composition with $M_1' \to M_1$, one obtains a morphism $b' : M_1' \to \mathbf{Hom}(M_2', K')$ which is none other than the adjoint description of the coupling $a' : M_1' \otimes M_2' \to K'$ defined by functoriality in the definition 12.1.1.*

This proposition merely reformulates the adjunction between $\otimes$ and **Hom** and the trifunctoriality of the isomorphism "dear to Cartan".

Définition 12.1.3. If $a\colon M_1 \otimes M_2 \to K$ is a coupling between M_1 and M_2 with values in K, one defines the coupling a^τ between M_2 and M_1 with values in K deduced by symmetry from a as being the composed morphism $M_2 \otimes M_1 \underset{\sim}{\overset{s_{M_2,M_1}}{\to}} M_1 \otimes M_2 \overset{a}{\to} K$. (This construction is evidently compatible with the trifunctoriality highlighted on the couplings.)

Définition 12.1.4. Let $K \in \mathscr{C}$. We denote $D_K := \mathbf{Hom}(-,K)$. The identity $D_K M \overset{\mathrm{Id}}{\to} D_K M = \mathbf{Hom}(M,K)$ is the adjoint description (cf. definition 12.1.1) of a coupling $\mathrm{ev}\colon D_K M \otimes M \to K$ (it is the "evaluation morphism") which induces by symmetry (cf. definition 12.1.3) a coupling $M \otimes D_K M \to K$ whose adjoint description is a morphism that we denote $\beta_M\colon M \to D_K D_K M$ and that we will call **biduality morphism** (of M).

12.2. Functoriality. Assume that $\mathscr{C}$ and $\mathscr{D}$ are two categories equipped with symmetric bifunctors denoted $\otimes$ and that these admit right adjoints **Hom** as in the §12.1. Let $F\colon \mathscr{C} \to \mathscr{D}$ be a functor equipped with a morphism $FX \otimes FY \overset{\varphi_{X,Y}}{\to} F(X \otimes Y)$ bifunctorial in the objects X and Y of $\mathscr{C}$. We make the hypothesis that F is symmetric, i.e., that for all X and Y, the following diagram commutes :

$$\begin{tikzcd}
FX \otimes FY \arrow[r, "\varphi_{X,Y}"] \arrow[d, "s_{FX,FY}"', "\sim"] & F(X \otimes Y) \arrow[d, "F(s_{X,Y})"', "\sim"] \\
FY \otimes FX \arrow[r, "\varphi_{Y,X}"] & F(Y \otimes X)
\end{tikzcd}$$

Définition 12.2.1. Let $a\colon M_1 \otimes M_2 \to K$ be a coupling in $\mathscr{C}$. The coupling $F[a]\colon FM_1 \otimes FM_2 \to FK$ in $\mathscr{D}$ deduced from a by applying F is the following composed morphism :

$$FM_1 \otimes FM_2 \overset{\varphi_{M_1,M_2}}{\to} F(M_1 \otimes M_2) \overset{F(a)}{\to} FK\,.$$

(This construction is evidently compatible with the functoriality in the three variables M_1, M_2 and K.)

The key of Deligne's argument in [**SGA 4½** [Dualité] 1.2] lies in the following proposition :

Proposition 12.2.2. *The construction of definition 12.2.1 commutes with symmetry.*

If $a\colon M_1 \otimes M_2 \to K$ is a coupling in $\mathscr{C}$, one can use the symmetry in $\mathscr{C}$ then apply F to obtain a coupling $FM_2 \otimes FM_1 \to FK$ in $\mathscr{D}$. One can also begin by applying F then use the symmetry in $\mathscr{D}$. The statement of the proposition means that the two couplings thus obtained are equal. This compatibility is nothing other than the symmetry hypothesis made on the functor F.

Définition 12.2.3. Assume that M and K are objects of $\mathscr{C}$. One disposes of a canonical coupling $\mathrm{ev}\colon \mathbf{Hom}(M,K) \otimes M \to K$ whose adjoint description is the identity of $\mathbf{Hom}(M,K)$ (cf. definition 12.1.4). The definition 12.2.1 gives rise to a coupling $F[\mathrm{ev}]\colon F\mathbf{Hom}(M,K) \otimes FM \to FK$ in $\mathscr{D}$ whose adjoint description is a morphism that we will denote $\mathbf{F}\colon F\mathbf{Hom}(M,K) \to \mathbf{Hom}(FM,FK)$. (This morphism is evidently functorial in the two variables M and K.)

Proposition 12.2.4. *Assume that $b\colon M_1 \to \mathbf{Hom}(M_2,K)$ is the adjoint description of a coupling $a\colon M_1 \otimes M_2 \to K$ in $\mathscr{C}$. Then, the coupling $F[a]\colon FM_1 \otimes FM_2 \to FK$ deduced from a by applying F admits for adjoint description the following composed morphism that we will denote $F[b]$:*

$$F[b]\colon FM_1 \overset{F(b)}{\to} F\mathbf{Hom}(M_2,K) \overset{\mathbf{F}}{\to} \mathbf{Hom}(FM_2,FK)\,.$$

By construction of $\mathbf{F}$, this is true if b is the identity and the general case follows from it by using the functoriality with respect to M_1.

Théorème 12.2.5. *Let $K \in \mathscr{C}$. We denote $D_K := \mathbf{Hom}(-,K)$ and $D_{FK} := \mathbf{Hom}(-,FK)$. For every $X \in \mathscr{C}$, we denote $\mathbf{F}_X : FD_KX \to D_{FK}FX$ the morphism $\mathbf{F} : F\mathbf{Hom}(X,K) \to \mathbf{Hom}(FX,FK)$ of the definition 12.2.3. Then, for every $M \in \mathscr{C}$, the following diagram commutes :*

$$\begin{tikzcd}
FM \arrow[r, "F(\beta_M)"] \arrow[d, "\beta_{FM}"'] & FD_KD_KM \arrow[d, "\mathbf{F}_{D_KM}"] \\
D_{FK}D_{FK}FM \arrow[r, "D_{FK}(\mathbf{F}_M)"] & D_{FK}FD_KM
\end{tikzcd}$$

Consider the canonical coupling $D_KM \otimes M \xrightarrow{a} K$ whose adjoint description is the identity of D_KM. By symmetry, a induces a coupling $a^\tau : M \otimes D_KM \to K$ whose adjoint description is the biduality morphism $\beta_M : M \to D_KD_KM$. One has defined a coupling $F[a^\tau] : FM \otimes FD_KM \to FK$ whose adjoint description is therefore $F[\beta_M]$ according to proposition 12.2.4 :

$$F[\beta_M] : FM \xrightarrow{F(\beta_M)} FD_KD_KM \xrightarrow{\mathbf{F}_{D_KM}} D_KFD_KM\,.$$

Moreover, since $a : D_KM \otimes M \to K$ is the canonical coupling, the morphism $F[a] : FD_KM \otimes FM \to FK$ admits for adjoint description the morphism $\mathbf{F}_M : FD_KM \to D_{FK}FM$ (cf. definition 12.2.3). If we denote $\alpha : D_{FK}FM \otimes FM \to FK$ the canonical coupling given by the evaluation morphism, this means that one has a commutative diagram :

$$\begin{tikzcd}
FD_KM \otimes FM \arrow[r, "F[a]"] \arrow[d, "\mathbf{F}_M \otimes \mathrm{Id}"'] & FK \arrow[d] \\
D_{FK}FM \otimes FM \arrow[r, "\alpha"] & FK
\end{tikzcd}$$

By symmetry, one deduces a commutative diagram :

$$\begin{tikzcd}
FM \otimes FD_KM \arrow[r, "F[a]^\tau"] \arrow[d, "\mathrm{Id} \otimes \mathbf{F}_M"'] & FK \arrow[d] \\
FM \otimes D_{FK}FM \arrow[r, "\alpha^\tau"] & FK
\end{tikzcd}$$

The adjoint description of α^τ being by definition the biduality morphism β_{FM} of FM, one deduces that the adjoint description of $F[a]^\tau$ is equal to the following composition :

$$FM \xrightarrow{\beta_{FM}} D_{FK}D_{FK}FM \xrightarrow{D_{FK}(\mathbf{F}_M)} D_{FK}FD_KM\,.$$

The proposition 12.2.2 states that $F[a]^\tau$ and $F[a^\tau]$ are the same couplings $FM \otimes FD_KM \to FK$. Their adjoint descriptions (described above) are therefore equal, which completes the proof of the theorem 12.2.5.

The theorem 12.2.5 admits the following generalization :

Théorème 12.2.6. *Let $K \in \mathscr{C}$. Let $K' \in \mathscr{D}$. We assume given a morphism $FK \xrightarrow{\varepsilon} K'$. We denote $D_K := \mathbf{Hom}(-,K)$ and $D_{K'} := \mathbf{Hom}(-,K')$. For every $X \in \mathscr{C}$, we denote $\mathbf{F}_{\varepsilon X} : FD_KX \to D_{K'}FX$ the composed morphism $FD_KX = F\mathbf{Hom}(X,K) \xrightarrow{\mathbf{F}} \mathbf{Hom}(FX,FK) \xrightarrow{\varepsilon_\star} \mathbf{Hom}(FX,K') = D_{K'}FX$. Then, for every $M \in \mathscr{C}$, the following diagram commutes :*

$$\begin{tikzcd}
FM \arrow[r, "F(\beta_M)"] \arrow[d, "\beta_{FM}"'] & FD_KD_KM \arrow[d, "\mathbf{F}_{\varepsilon D_KM}"] \\
D_{K'}D_{K'}FM \arrow[r, "D_{FK}(\mathbf{F}_{\varepsilon M})"] & D_{K'}FD_KM
\end{tikzcd}$$

If $a : M_1 \otimes M_2 \to K$ is a coupling in $\mathscr{C}$, one can define $F_\varepsilon[a] : FM_1 \otimes FM_2 \to K'$ as being the coupling obtained by composition

$$FM_1 \otimes FM_2 \xrightarrow{F[a]} FK \xrightarrow{\varepsilon} K' .$$

Just like the construction $F[a]$, this one commutes with symmetry, and the théorème 12.2.6 is proved like the théorème 12.2.5 by replacing $F[-]$ by $F_\varepsilon[-]$.

12.3. Application to $f^\star$.

Proposition 12.3.1. *Let $f : (\mathscr{T}', \mathscr{A}') \to (\mathscr{T}, \mathscr{A})$ be an annelid topos morphism in commutative rings. Denote $\mathrm{L}f^\star : \mathrm{D}(\mathscr{T}, \mathscr{A}) \to \mathrm{D}(\mathscr{T}', \mathscr{A}')$ the functor defined in §8.3.2. Let $K \in \mathrm{D}(\mathscr{T}, \mathscr{A})$. We denote $D_K := \mathrm{R}\,\mathbf{Hom}_{\mathscr{A}}(-, K)$ and $D_{\mathrm{L}f^\star K} := \mathrm{R}\,\mathbf{Hom}_{\mathscr{A}}(-, \mathrm{L}f^\star K)$. Then, for every $M \in \mathrm{D}(\mathscr{T}, \mathscr{A})$, the following diagram is commutative in $D(\mathscr{T}', \mathscr{A}')$:*

$$\begin{tikzcd}
\mathrm{L}f^\star M \arrow[r, "\mathrm{L}f^\star(\beta_M)"] \arrow[d, "\beta_{\mathrm{L}f^\star M}"'] & \mathrm{L}f^\star D_K D_K M \arrow[d, "\mathbf{Lf}^\star_{D_K M}"] \\
D_{\mathrm{L}f^\star K} D_{\mathrm{L}f^\star K} \mathrm{L}f^\star M & D_{\mathrm{L}f^\star K} \mathrm{L}f^\star D_K M \arrow[l, "D_{\mathrm{L}f^\star K}(\mathbf{Lf}^\star_M)"']
\end{tikzcd}$$

This follows from the theorem 12.2.5 applied to the functor $\mathrm{L}f^\star : \mathrm{D}(\mathscr{T}, \mathscr{A}) \to \mathrm{D}(\mathscr{T}', \mathscr{A}')$.

12.3.2. If $f : X' \to X$ is a morphism of schemes, one can apply this proposition to the topos morphism $X'_{\text{ét}} \xrightarrow{f} X_{\text{ét}}$ (these topoi being equipped with a constant sheaf of rings). In certain favorable cases (cf. proposition 6.2.3.2), the morphisms $f^\star D_K M \to D_{f^\star K} f^\star M$ are isomorphisms, in which case the proposition 12.3.1 states that for certain objects M and K in $\mathrm{D}(X_{\text{ét}}, \Lambda)$, the image by $f^\star$ of the biduality morphism for M identifies with the biduality morphism for $f^\star M$.

12.4. The Künneth morphism.

Proposition 12.4.1. *Assume that $\mathscr{C}$ and $\mathscr{D}$ are categories equipped with symmetric bifunctors $\otimes$ admitting a right adjoint* **Hom**. *Let $G : \mathscr{D} \to \mathscr{C}$ be a symmetric functor. We assume that for all X and Y in $\mathscr{D}$, the morphism $\varphi_{X,Y} : GX \otimes GY \to G(X \otimes Y)$ is an isomorphism. Assume that G admits a right adjoint $F : \mathscr{C} \to \mathscr{D}$. Then, F is naturally equipped with a symmetric natural transformation $FX \otimes FY \to F(X \otimes Y)$ for all X and Y in $\mathscr{C}$.*

If X and Y are objects of $\mathscr{C}$, the adjunction between G and F defines morphisms $GFX \to X$ and $GFY \to Y$, which allows to define a morphism $GFX \otimes GFY \to X \otimes Y$ whose source identifies with $G(FX \otimes FY)$ thanks to our hypothesis. The morphism $G(FX \otimes FY) \to X \otimes Y$ thus deduced corresponds by adjunction to the wanted morphism $FX \otimes FY \to F(X \otimes Y)$.

12.4.2. If $f : (\mathscr{T}', \mathscr{A}') \to (\mathscr{T}, \mathscr{A})$ is an annelid topos morphism in commutative rings, by applying the proposition 12.4.1 to $\mathrm{L}f^\star$, one obtains the Künneth morphism $\mathrm{R}f_\star X \overset{\mathrm{L}}{\otimes}_{\mathscr{A}} \mathrm{R}f_\star Y \to \mathrm{R}f_\star(X \overset{\mathrm{L}}{\otimes}_{\mathscr{A}'} Y)$.

Corollaire 12.4.3. *Let $f : (\mathscr{T}', \mathscr{A}') \to (\mathscr{T}, \mathscr{A})$ be an annelid topos morphism in commutative rings. Let $K' \in \mathrm{D}(\mathscr{T}, \mathscr{A})$. Let $K \in \mathrm{D}(\mathscr{T}, \mathscr{A})$. We assume given a morphism $\mathrm{R}f_\star K' \xrightarrow{\varepsilon} K$ in $\mathrm{D}(\mathscr{T}, \mathscr{A})$. We denote $D_K := \mathrm{R}\,\mathbf{Hom}_{\mathscr{A}}(-, K)$ and $D_{K'} := \mathrm{R}\,\mathbf{Hom}_{\mathscr{A}'}(-, K')$. We denote $\mathbf{f}_\varepsilon : \mathrm{R}f_\star D_{K'} \to D_K \mathrm{R}f_\star$ the canonical natural transformation. Then, the following diagram is commutative for every $M' \in \mathrm{D}(\mathscr{T}', \mathscr{A}')$:*

$$\begin{tikzcd}
\mathrm{R}f_\star M' \arrow[r, "\mathrm{R}f_\star(\beta_{M'})"] \arrow[d, "\beta_{\mathrm{R}f_\star M'}"'] & \mathrm{R}f_\star D_{K'} D_{K'} M' \arrow[d, "\mathbf{f}_\varepsilon"] \\
D_K D_K \mathrm{R}f_\star M' & D_K \mathrm{R}f_\star D_{K'} M' \arrow[l, "D_K(\mathbf{f}_\varepsilon)"']
\end{tikzcd}$$

12.4.4. If $f : X' \to X$ is a proper morphism between $\mathbf{Z}\left[\frac{1}{n}\right]$-Noetherian schemes and $K \in \mathsf{D}^+(X_{\text{ét}}, \Lambda)$, one can set $K' := f^!K$ and denote $\varepsilon : \mathrm{R}f_\star K' \to K$ the adjunction morphism. One can apply the corollary 12.4.3 to this situation. The morphism $\mathbf{f}_\varepsilon : \mathrm{R}f_\star \mathrm{R}\,\mathbf{Hom}(M', f^!K) \to \mathrm{R}\,\mathbf{Hom}(\mathrm{R}f_\star M', K)$ is an isomorphism if $M' \in \mathsf{D}^-(X_{\text{ét}}, \Lambda)$ thanks to the adjunction formula [**SGA 4** xviii 3.1.10]. If one knows that M' and $D_{K'}M'$ are in $\mathsf{D}^-(X_{\text{ét}}, \Lambda)$ (for example, if one knows that $M' \in \mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(X'_{\text{ét}}, \Lambda)$ and that $D_{K'}$ preserves $\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(X'_{\text{ét}}, \Lambda)$), the corollary states that the image by $\mathrm{R}f_\star$ of the biduality morphism $\beta_{M'} : M' \to D_{K'}D_{K'}M'$ identifies with the biduality morphism $\beta_{\mathrm{R}f_\star M'} : \mathrm{R}f_\star M' \to D_K D_K \mathrm{R}f_\star M'$.

EXPOSÉ $\mathrm{XVIII_A}$

Cohomological dimension: First results

Luc Illusie

In this exposé we establish Gabber's bound on cohomological dimension stated in the introduction (in the comments on the proof of the finiteness theorem).

1. Bound in the strictly local case and applications

Theorem 1.1. *Let X be a strictly local, noetherian scheme of dimension $d > 0$, and let ℓ be a prime number invertible on X. Then, for any open subset U of X, we have*

$$\mathrm{cd}_\ell(U) \leq 2d - 1. \tag{1.1.1}$$

Recall that, for a scheme S, $\mathrm{cd}_\ell(S)$ (ℓ-**cohomological dimension** of S) denotes the infimum of the integers n such that for all ℓ-torsion abelian sheaves F on S, and all $i > n$, $\mathrm{H}^i(S, F) = 0$.

Corollary 1.2. *Let $X = \operatorname{Spec} A$ be as in 1.1, and assume A is a domain, with fraction field K. Then*

$$\mathrm{cd}_\ell(K) \leq 2d - 1. \tag{1.2.1}$$

Indeed, it suffices to show that if F is a finitely generated $\mathbf{F}_\ell$-module over $\eta = \operatorname{Spec} K$, then $\mathrm{H}^i(\eta, F) = 0$ for $i > 2d-1$. But η is a filtering projective limit of affine open subsets U_α of X, F is induced from a locally constant constructible $\mathbf{F}_\ell$-sheaf F_{α_0} on U_{α_0}, and $\mathrm{H}^i(\eta, F) = \varinjlim \mathrm{H}^i(U_\alpha, F_\alpha)$, where $F_\alpha = F_{\alpha_0}|U_\alpha$ for $\alpha \geq \alpha_0$ ([**SGA 4** VII 5.7]).

Remark 1.3. (a) The proof shows that, given X as in 1.1, with X integral, then, if (1.1.1) holds for any affine open subset U, (1.2.1) holds, too.

(b) If X is an integral noetherian scheme of dimension d, with generic point $\operatorname{Spec} K$, and ℓ is a prime number invertible on X, then $\mathrm{cd}_\ell(K) \geq d$ ([**SGA 4** X 2.5]). Gabber can prove that under the assumptions of 1.2 one has $\mathrm{cd}_\ell(K) = d$ (see $\mathrm{XVIII_B}$).

Corollary 1.4. *Let Y be a noetherian scheme of finite dimension, $f : X \to Y$ a morphism of finite type, and ℓ a prime number invertible on Y. Then*

$$\mathrm{cd}_\ell(\mathrm{R}f_\star) < \infty,$$

i.e. there exists an integer N such that for any ℓ-torsion abelian sheaf F on X, $\mathrm{R}^q f_\star F = 0$ for $q > N$.

Proof of 1.4. We may assume Y affine. Covering X by finitely many open affine subsets U_i ($0 \leq i \leq n$), and using the alternating Čech spectral sequence

$$E_1^{pq} = \bigoplus \mathrm{R}^q(f|U_{i_0\cdots i_p})_\star(F|U_{i_0\cdots i_p}) \Rightarrow \mathrm{R}^{p+q} f_\star F,$$

where $U_{i_0\cdots i_p} = U_{i_0} \cap \cdots \cap U_{i_p}$, we may assume f separated. Repeating the procedure, we may assume X affine. Choose an immersion $X \to \mathbf{P}^n_Y$, and replace $\mathbf{P}^n_Y$ by the scheme-theoretic closure of X. We get a commutative diagram

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[d, "f"'] & \overline{X} \arrow[dl, "g"] \\ Y & \end{tikzcd}$$

with j open and g projective of relative dimension $\leq n$. By the proper base change theorem we have $\mathrm{cd}_\ell(\mathrm{R}g_\star) \leq 2n$. By the Leray spectral sequence $\mathrm{R}^p g_\star \mathrm{R}^q j_\star F \Rightarrow \mathrm{R}^{p+q} f_\star F$ it thus suffices to prove 1.4 for j, in other words, we may assume that f is an open immersion. Let d be the dimension of Y. Let y be a geometric point of Y, and let $U = Y_{(y)} \times_Y X$ be the corresponding open subset of the strictly local scheme $Y_{(y)}$ (of dimension $\leq d$, that we may assume to be > 0). Then

$$\mathrm{R}^i f_\star(F)_y = \mathrm{H}^i(U, F)$$

(where we still denote by F its inverse image on U). The conclusion follows from 1.1.

Remarks 1.5. (a) Under the assumptions of 1.1, if X is quasi-excellent and U is affine, then by Gabber's affine Lefschetz theorem (**XV**-1.2.4) we have $\mathrm{cd}_\ell(U) \leq d$. More generally, see **XVII**-3.2.1 for a proof of 1.1 for X quasi-excellent.

(b) Gabber can show that, under the assumptions of 1.1, one has $\mathrm{cd}_\ell(U) \geq d$ if U is not empty and does not contain the closed point and that for each n such that $d \leq n \leq 2d-1$, there exists a pair (X, U) as in 1.1, with U affine, such that $\mathrm{cd}_\ell(U) = n$ (by (a), for $n > d$, X is not quasi-excellent). These results are proved in XVIII$_\text{B}$.

2. Proof of the main result

Lemma 2.1. *Let X be as in 1.1, and let x be the closed point of X. Then (1.1.1) holds for $U = X - \{x\}$.*

Proof. It suffices to show that for any constructible $\mathbf{F}_\ell$-sheaf F on U, $\mathrm{H}^i(U, F) = 0$ for $i \geq 2d$. Let $\widehat{X}$ be the completion of X at $\{x\}$ and set $\widehat{U} := \widehat{X} \times_X U = \widehat{X} - \{x\}$. Let $\widehat{F}$ be the inverse image of F on $\widehat{U}$. By Gabber's formal base change theorem ([**Fujiwara, 1995**, 6.6.4]), the natural map

$$\mathrm{H}^i(U, F) \to \mathrm{H}^i(\widehat{U}, \widehat{F})$$

is an isomorphism for all i. Therefore we may assume X complete, and in particular, excellent. Let $(f_1, \cdots, f_d)$ be a system of parameters of X, and let $U_i = X_{f_i}$, so that $U = \bigcup_{1 \leq i \leq d} U_i$. Consider the (alternate) Čech spectral sequence

$$E_1^{pq} = \bigoplus \mathrm{H}^q(U_{i_0 \cdots i_p}, F) \Rightarrow \mathrm{H}^{p+q}(U, F),$$

with $U_{i_0 \cdots i_p} = U_{i_0} \cap \cdots \cap U_{i_p}$ as above. By definition, $E_1^{pq} = 0$ for $p \geq d$. On the other hand, as X is excellent, by Gabber's affine Lefschetz theorem (**XV**-1.2.4), we have $E_1^{pq} = 0$ for $q \geq d+1$. Therefore $E_1^{pq} = 0$ for $p + q \geq 2d$, hence $\mathrm{H}^i(U, F) = 0$ for $i \geq 2d$. □

Lemma 2.2. *Let X be a noetherian scheme of finite dimension, Y a closed subset, ℓ a prime number invertible on X. Then, for any ℓ-torsion sheaf F on X,*

$$\mathrm{H}^i_Y(X, F) = 0$$

for

$$i > \sup_{x \in Y}(\mathrm{cd}_\ell(k(x)) + 2 \dim \mathscr{O}_{X,x}).$$

In particular,

$$\mathrm{cd}_\ell(X) \leq \sup_{x \in X}(\mathrm{cd}_\ell(k(x)) + 2 \dim \mathscr{O}_{X,x}).$$

Proof. For $p \geq 0$, let Φ^p be the set of closed subsets of Y of codimension $\geq p$ in X. We have $\Phi^p = \varnothing$ for $p > \dim(X)$. Consider the (biregular) coniveau spectral sequence of the filtration (Φ^p) (cf. [**Grothendieck, 1968**, 10.1]),

$$E_1^{pq} = \mathrm{H}^{p+q}_{\Phi^p/\Phi^{p+1}}(X, F) \Rightarrow \mathrm{H}^{p+q}_Y(X, F). \tag{2.2.1}$$

We have

$$E_1^{pq} = \bigoplus_{x \in Y^{(p)}} \mathrm{H}^{p+q}_{\{x\}}(X_x, F|X_x),$$

where $Y^{(p)}$ denotes the set of points of Y of codimension p in X, and $X_x = \operatorname{Spec} \mathscr{O}_{X,x}$. For $x \in Y^{(p)}$ (i.e. $\dim \mathscr{O}_{X,x} = p$), let $\overline{x}$ be a geometric point above x. Consider the diagram

$$\begin{tikzcd} \{\overline{x}\} \arrow[r, "i_{\overline{x}}"] \arrow[d] & X_{(\overline{x})} \arrow[d] & \overline{U} \arrow[l, "\overline{j}"'] \arrow[d] \\ \{x\} \arrow[r, "i_x"] & X_x & U \arrow[l, "j"'] \end{tikzcd},$$

where $U = X_x - \{x\}$, $\overline{U} = X_{(\overline{x})} - \{\overline{x}\}$. We have

$$R\Gamma_{\{x\}}(X_x, F|X_x) = R\Gamma(\{x\}, \mathrm{R}i_x^!(F|X_x)).$$

The stalk of $\mathrm{R}i_x^!(F|X_x)$ at $\overline{x}$ is

$$\mathrm{R}i_x^!(F|X_x)_{\overline{x}} = \mathrm{R}i_{\overline{x}}^!(F|X_{(\overline{x})}),$$

as $(\mathrm{R}j_\star(F|U))_{\overline{x}} = R\overline{j}_\star(F|\overline{U})_{\overline{x}}$. We thus have a spectral sequence

$$E_2^{rs} = \mathrm{H}^r(k(x), \mathrm{R}^s i_{\overline{x}}^!(F|X_{(\overline{x})})) \Rightarrow \mathrm{H}^{r+s}_{\{x\}}(X_x, F|X_x), \tag{2.2.2}$$

It suffices to show that, in the initial term of (2.2.1),

$$\mathrm{H}^{p+q}_{\{x\}}(X_x, F|X_x) = 0 \tag{2.2.3}$$

for $p + q > \mathrm{cd}_\ell(k(x)) + 2p$. If $p = 0$, then $\mathrm{R}i_{\overline{x}}^!(F|X_{(\overline{x})}) = F_{\overline{x}}$, and, in (2.2.2), $E_2^{rs} = 0$ for $s > 0$, $E_2^{r0} = 0$ for $r > \mathrm{cd}_\ell(k(x))$, so (2.2.3) is true in this case. Assume $p > 0$. We have

$$\mathrm{R}^s i_{\overline{x}}^! F = \mathrm{H}^{s-1}(\overline{U}, F|\overline{U}) \tag{2.2.4}$$

for $s \geq 2$, where, as above, $\overline{U} = X_{(\overline{x})} - \{\overline{x}\}$. By 2.1, $\mathrm{H}^{s-1}(\overline{U}, F|\overline{U}) = 0$ for $s - 1 \geq 2p$, hence, by (2.2.4), $\mathrm{R}^s i_{\overline{x}}^!(F|X_{(\overline{x})}) = 0$ and $E_2^{rs} = 0$ for $s \geq 2p + 1$. If $r + s \geq \mathrm{cd}_\ell(k(x)) + 2p + 1$ and $s \leq 2p$, then $r > \mathrm{cd}_\ell(k(x))$, hence $E_2^{rs} = 0$ as well. Therefore, by (2.2.2), (2.2.3) holds, which finishes the proof. □

Proof of 1.1. We prove 1.1 by induction on d. For $n \geq 0$ consider the assertion

G_n : *For every strictly local, noetherian scheme X of dimension n, all open subsets U of X and any prime number ℓ invertible on X, we have* $\mathrm{cd}_\ell(U) \leq \sup(0, 2n - 1)$.

Let $d > 0$. Assume G_n holds for $n < d$, and let us prove G_d. Let X be as in 1.1. If $(X_i)_{1 \leq i \leq r}$ are the reduced irreducible components of X and $U_i = U \times_X X_i$, we have $\mathrm{cd}_\ell(U) \leq \sup(\mathrm{cd}_\ell(U_i))$, hence we may assume X integral. Let x be the closed point of X, and $U = X - \{x\}$ the punctured spectrum. Let $j : V \to U$ be a nonempty open subset of U, and F be a constructible $\mathbf{F}_\ell$-sheaf on V. As $F = j^\star j_! F$, by 2.1 it suffices to show that, for any constructible $\mathbf{F}_\ell$-sheaf L on U, the restriction map

$$\mathrm{H}^i(U, L) \to \mathrm{H}^i(V, j^\star L) \tag{$*$}$$

is an isomorphism for $i \geq 2d$. Let $Y = U - V$. Consider the exact sequence

$$\mathrm{H}^i_Y(U, L) \to \mathrm{H}^i(U, L) \to \mathrm{H}^i(V, j^\star L) \to \mathrm{H}^{i+1}_Y(U, L).$$

By 2.2, we have $\mathrm{H}^i_Y(U, L) = 0$ for $i > \sup_{y \in Y}(\mathrm{cd}_\ell(k(y)) + 2 \dim \mathscr{O}_{X,y})$. For $y \in Y$, denote by Z the closed, integral subscheme of X defined by the closure of $\{y\}$ *in* X. As X is integral and V nonempty, Z is a strictly local scheme of dimension $n < d$, with generic point y. By 1.3 (a) and G_n (inductive assumption), we have $\mathrm{cd}_\ell(k(y)) \leq 2n - 1$. We have $2n - 1 + 2 \dim \mathscr{O}_{X,y} \leq 2d - 1$. Hence, for $i \geq 2d$, $\mathrm{H}^i_Y(U, L) = \mathrm{H}^{i+1}_Y(U, L) = 0$, and $(*)$ is an isomorphism, which finishes the proof.

REMARK 2.3. Gabber has an alternate proof of 1.1, based on the theory of Zariski-Riemann spaces. By 2.2, it suffices to show 1.2. Here is a sketch, pasted from an e-mail of Gabber to Illusie of 2007, Aug. 15 :

"For $Y \to X$ proper birational with special fiber Y_0, consider $i : Y_0 \to Y$ and $j : \eta \to Y$, η the generic point. We have by proper base change a spectral sequence

$$\mathrm{H}^p(Y_0, i^\star \mathrm{R}^q j_\star F) \to \mathrm{H}^{p+q}(\eta, F)$$

for F an ℓ-torsion Galois module. We take the direct limit and get a spectral sequence involving cohomologies on the étale topos of ZRS_0 defined as the limit of étale topoi of Y_0 or viewing ZRS_0 as a locally ringed topos and applying a universal construction in the book of M. Hakim. The limit of the $\mathrm{R}^q j_\star F$ is $\mathrm{R}^q(\eta \to ZRS)_\star F$. By a classical result of Abhyankar, also proved in Appendix 2 of the book of Zariski-Samuel Vol. II, if R is a noetherian local domain of dimension d and V a valuation ring of $\mathrm{Frac}(R)$ dominating R, the sum of the rational rank and the residue transcendence degree is at most d. For a strictly henselian valuation ring V with residue characteristic exponent p and value group Γ, the absolute Galois group of $\mathrm{Frac}(V)$ is an extension of the tame part (product for ℓ prime not equal to p of $\mathrm{Hom}(\Gamma, \mathbf{Z}_\ell(1))$) by a p-group, so the ℓ-cohomological dimension is the dimension of Γ tensored with the prime field $\mathbf{F}_\ell$, which is at most the dimension of Γ tensored with the rationals. If A is an ℓ-torsion sheaf on the étale topos of ZRS_0, let $\delta(A)$ be the sup of transcendence degrees of points where the stalk is non-zero. I claim that $\mathrm{H}^n(ZRS_0, A)$ vanishes for $n > 2\delta(A)$. One reduces it to the finite type case (passage to the limit [**SGA 4** VI 8.7.4]) using that the δ of the direct image of A to Y_0 is at most $\delta(A)$. In Y_0 the transcendence degrees over the closed point of X are at most $d-1$ by the dimension inequality. Summing up, for the limit spectral sequence the q-th direct image sheaf restricted to the special fiber has δ at most $\min(d-1, d-q)$, giving vanishing for certain $E_2^{p,q}$ and the result."

EXPOSÉ XVIII_{B}

Cohomological dimension : refinements and complements

Fabrice Orgogozo

Our first objective is to demonstrate that for every prime number ℓ there exists an *affine* open subset of a regular strictly Henselian Noetherian scheme of dimension 2 whose ℓ-cohomological dimension is equal to 3. Besides cohomological ingredients — purity, Gysin morphism and comparison to completion —, we use a construction whose principle is due to Nagata : using "formal dilatations", we construct a regular strictly Henselian Noetherian scheme X of dimension 2, with a regular completion $\widehat{X}$ of dimension 2, and an *irreducible* curve C in X becoming the ℓ-th multiple of a regular divisor in $\widehat{X}$. This construction is then extended to the more delicate case of higher dimension. From there, we easily construct schemes whose existence was announced in the first part (**XVIII_{A}**-1.5). To verify that their cohomological dimension is indeed the expected one, we appeal to a fairly general upper bound established without any excellence hypothesis. Finally, we conclude with a lower bound for the cohomological dimension of an open subset (not necessarily affine) of the punctured spectrum of an integral strictly local Noetherian scheme.

1. Preliminaries

1.1. Formal dilatations. Let R be a ring, π a non-zero divisor element and f an element of the π-adic completion $\widehat{R}$ of R. For every $n \geq 0$, choose an $f_n \in R$ such that $f \equiv f_n$ modulo π^n.

Définition 1.1.1. We denote by $\mathrm{Dil}_{\pi}^{f} R$ the R-subalgebra of $R[\pi^{-1}, F]$, where F is an indeterminate, colimit of the R-algebras $R[\frac{F-f_n}{\pi^n}]$.

We will also denote by F the image of this variable in $\mathrm{Dil}_{\pi}^{f} R$.

Remarque 1.1.2. Note that the R-algebras considered are all isomorphic to a polynomial algebra in one variable over R.

1.1.3. We immediately verify the following facts :

(i) the construction does not depend on the choice of f_n, and depends on the element π only through the ideal it generates ;

(ii) the morphisms $R \to \mathrm{Dil}_{\pi}^{f} R$ and $\mathrm{Dil}_{\pi}^{f} R \to \widehat{R}$, $\frac{F-f_n}{\pi^n} \mapsto \frac{f-f_n}{\pi^n}$, induce *isomorphisms* on the π-adic completion.

1.2. Flatness and Noetheriality.

1.2.1. Recall that if a morphism $A \to B$ is faithfully flat, A is Noetherian if B is ([**ÉGA** IV□ 2.2.14]). To verify flatness, it is sometimes convenient to use the following criterion ([**Raynaud & Gruson, 1971**, II.1.4.2.1]).

Proposition 1.2.2. *Let M be an R-module, and $\pi \in R$. Assume that π is not a zero divisor neither in R nor in M. Then, M is* flat *over R if and only if M/π is flat over R/π and $M[\pi^{-1}]$ is flat over $R[\pi^{-1}]$.*

Remarque 1.2.3. To prove the Noetheriality of the rings considered below, one could also use Cohen's criterion recalled in **XIX**-3.2, by verifying in particular that ideals of height 1 are principal.

1.3. Blow-ups.

1.3.1. Let A be a local Noetherian ring, with maximal ideal $\mathfrak{m}$. Following [**Bourbaki**, **AC**, IX, appendice, § 2], we denote by $A]t[$ and call **elementary blow-up** of A the localization of the polynomial ring $A[t]$ at the prime ideal $\mathfrak{m}A[t]$. It is a local Noetherian ring. (It is denoted $A(t)$, by analogy with rational functions, in [**Nagata, 1962**, p. 17—18]; see also [**Matsumura, 1980b**, p. 138].) More generally, one can consider an arbitrary set of variables $t_e, e \in E$, and define the ring $A]t_e, e \in E[$, localization of $A[t_e, e \in E]$ at the prime ideal generated by $\mathfrak{m}$. Recall the following fact ([**Bourbaki**, **AC**, IX, appendice, prop. 2 et corollaire]).

Proposition 1.3.2. *The ring $A]t_e, e \in E[$ is local Noetherian with the same dimension as A.*

1.3.3. Note that the case of a finite number of variables is very elementary and the general case results from lemma [**ÉGA** 0_{III} 10.3.1.3], reproduced in **XIX**-3.1, by passing to the (co)limit. For another proof, see also [**Bourbaki**, **AC**, III, § 5, exercice 7].

1.3.4. Observe also that if the residue field of A is κ, that of $A]t_e, e \in E[$ is canonically isomorphic to its pure transcendental extension $\kappa(t_e, e \in E)$. Furthermore, it is shown that if F is a subset of E, the morphism $A]t_e, e \in F[\to A]t_e, e \in E[$ is *faithfully flat*. (See [**Bourbaki**, **AC**, IX, appendice, prop. 2] for a proof in the case where $F = \varnothing$, to which we are immediately reduced.)

2. Nagata's construction in dimension 2, cohomological application

2.1. Dilatation relative to a transcendental series.

2.1.1. Let W be a discrete valuation ring, with maximal ideal $\mathfrak{m}_W = (\pi)$, residue field k, fraction field K and completion $\widehat{W}$. We denote by $\widehat{K}$ the fraction field of $\widehat{W}$. Suppose there exists an element $\varphi \in \widehat{W}$ *transcendental* over K. This is the case if W is countable or, for example, when $W = k[t]_{(t)}$ in which case $\varphi = \sum_n t^{n!}$ works. By translation, we can assume that φ belongs to the maximal ideal of $\widehat{W}$.

2.1.2. Fix an integer $\ell \geq 1$ and consider the element $f = (y - \varphi)^\ell$ of the π-adic completion $\widehat{W[y]}$ of $W[y]$. Note that this completion injects into the total completion $\widehat{W}[[y]]$ and that f belongs to the subring $\widehat{W}[y]$ of $\widehat{W[y]}$ and $\widehat{W}[[y]]$. Consequently, the canonical morphism $\mathrm{Dil}_\pi^f W[y] \to \widehat{W[y]}$ factors into a morphism $\mathrm{Dil}_\pi^f W[y] \to \widehat{W}[y]$.

Proposition 2.1.3. *The morphism $\mathrm{Dil}_\pi^f W[y] \to \widehat{W}[y]$ is* flat.

Démonstration. Let's simplify notation by denoting $\mathcal{D}$ the source of this morphism. According to the flatness criterion recalled above, it suffices to show that the morphism $\mathcal{D}[\pi^{-1}] \to \widehat{W}[y][\pi^{-1}]$ is flat because $\mathcal{D} \to \widehat{W}[y]$ visibly induces an isomorphism modulo π. Now, when π is inverted, the $W[y]$-algebras whose colimit $\mathcal{D}$ is by definition are all isomorphic to a polynomial ring $K[y, F]$ where, recall, K is the fraction field $W[\pi^{-1}]$ of W. Consequently, we need to show that the morphism $K[y, F] \to \widehat{W}[y][\pi^{-1}]$, composed of the morphisms

$$\begin{cases} K[y, F] \to K[y, F'] \\ F \mapsto F'^{\ell} \end{cases} \qquad\qquad \begin{cases} K[y, F'] \to \widehat{K}[y] \\ F' \mapsto y - \varphi \end{cases}$$

is flat. The flatness of the first is obvious. For the second, we reduce by translation and base change to show that the morphism $K[F'] \to \widehat{K}$, $F' \mapsto \varphi$, is flat. It factors into the composition of the field of fractions $K[F'] \to K(F')$ with the *injection* $K(F') \hookrightarrow \widehat{K}$ deduced from φ. Each of these morphisms is flat. □

Remarque 2.1.4. The construction of the dilatation ring $\mathrm{Dil}_\pi^f W[y]$ is inspired by that of [**Nagata, 1958**], which considers the case $\ell = 2$. (See also [**Nagata, 1962**, appendix, E4.1] and [**Heinzer et al., 1997**].)

2.2. The divisor $C = V(F)$.

2.2.1. We keep the preceding notations. Let A be the localization of the dilatation ring $\mathcal{D}$ at the prime ideal inverse image of the ideal $(\pi, y) \subset \widehat{W}[y]$. It is a Noetherian ring by faithful flatness, with residue field k.

Lemme 2.2.2. *The ring A satisfies the following properties :*

(i) *it is regular and the sequence (π, F) is regular ;*

(ii) *its quotient A/F is integral ;*

(iii) *the schematic intersection of the closed subset $V(F)$ with the punctured spectrum of A is a regular scheme.*

Démonstration. (i) It suffices to establish both statements for $\mathcal{D}$. To do this, we can complete π-adically (cf. for example [**Bourbaki**, **AC**, X, § 4, nº 2, cor. 3]). The completion of $\mathcal{D}$ is isomorphic to $\widehat{W[y]}$ so that the regularity of the ring and of the sequence (π, F) — that is, the injectivity of multiplication by π, and by F modulo π — are evident. (ii) If we invert π, the ring A becomes a localization of the polynomial ring $K[y, F]$. The restriction of the divisor to this open set is integral. Multiplication by π in the quotient A/F being injective according to (i), the integrality of A/F results from that of $A[\pi^{-1}]/F$. The quotient is non-zero because $F \in \mathfrak{m}_A$. (iii) On the open complement of $V(\pi)$, the element F is an indeterminate so the result is clear. On the other hand, the intersection of the complement of $V(y)$ with the divisor is contained in the complement of $V(\pi)$ because the equation is — in the π-adic completion — of the form $(y-\varphi)^\ell$, with $\varphi \in (\pi)$. This is sufficient to conclude. □

2.2.3. Note the following fact, trivial but crucial : by construction, the divisor $V(F)$ becomes $V((y-\varphi)^\ell)$ in the completion $\widehat{A}$ of A relative to its maximal ideal.

2.3. Henselization. To simplify notation, we now assume the field k to be separably closed.

2.3.1. Let A^{hs} be the Henselization of A at its closed point, $\widehat{A}$ the completion of A (as well as of A^{hs}), and let $X = \mathrm{Spec}(A^{\mathrm{hs}})$ and $\widehat{X} = \mathrm{Spec}(\widehat{A})$ be their respective spectra, as well as $\star$ and $\widehat{\star}$ the closed points. Like $\mathrm{Spec}(A)$, the scheme X is integral. Furthermore, the divisor C with equation $F = 0$ in X is reduced, this property being also preserved by Henselization.

2.3.2. Let's verify that the divisor C is *irreducible*. According to Elkik's comparison theorem, the morphism $\pi_0(\widehat{C} - \widehat{\star}) \to \pi_0(C - \star)$ is a bijection. Now, $\widehat{C} - \widehat{\star}$ is connected : in a regular local ring B, the spectrum $\mathrm{Spec}(B/g^\ell)$ is irreducible for every $g \in \mathfrak{m}_B - \mathfrak{m}_B^2$. Thus, the open set $C - \star$ of C is connected and, finally, C is irreducible.

2.4. Cohomological application. *We now assume the integer ℓ to be invertible on X.*

2.4.1. To simplify, let G be the element $y - \varphi$ of $\widehat{A}$, so that — by construction — we have the equality $F = G^\ell$ in $\widehat{A}$. Let U be the affine open set $X - C$ of the strictly local scheme X. Let j be the open immersion of U into the punctured spectrum $X - \star$ and i the closed immersion $C - \star \hookrightarrow X - \star$.

2.4.2. The triangle

$$i_\star i^! \to \mathrm{Id} \to \mathrm{R}j_\star j^\star$$

on $X - \star$ induces the exact sequence

$$\mathrm{H}^3_{C-\star}(X-\star, \mathbf{Z}/\ell\mathbf{Z}) \to \mathrm{H}^3(X-\star, \mathbf{Z}/\ell\mathbf{Z}) \to \mathrm{H}^3(U, \mathbf{Z}/\ell\mathbf{Z}) \to \mathrm{H}^4_{C-\star}(X-\star, \mathbf{Z}/\ell\mathbf{Z}) = \mathrm{H}^2(C-\star, i^!\mathbf{Z}/\ell\mathbf{Z}[2]).$$

By purity (**XVI**-3.1.1), the local cohomology sheaf $i^!\mathbf{Z}/\ell\mathbf{Z}$ is concentrated in degree 2. Now the cohomology group $\mathrm{H}^2(C-\star, \mathbf{Z}/\ell\mathbf{Z})$ is zero : the cohomology of the fraction field of a strictly local integral ring B of dimension 1 is zero in degree ≥ 2. (We reduce to the well-known case of a discrete valuation ring by observing that the normalization of B in its fraction field is a Noetherian, Dedekind, and strictly local ring as a local colimit of strictly local rings.) It follows that the restriction map $\mathrm{H}^3(X-\star, \mathbf{Z}/\ell\mathbf{Z}) \to \mathrm{H}^3(U, \mathbf{Z}/\ell\mathbf{Z})$ is surjective ; we will see that it is an isomorphism.

2.4.3. As recalled above, the morphism $\mathrm{H}^3_{C-\star}(X-\star, \mathbf{Z}/\ell\mathbf{Z}) \to \mathrm{H}^3(X-\star, \mathbf{Z}/\ell\mathbf{Z})$ is identified, by purity, with the Gysin morphism $\mathrm{Gys}(f) : \mathrm{H}^1(C-\star, \mathbf{Z}/\ell\mathbf{Z}) \to \mathrm{H}^3(X-\star, \mathbf{Z}/\ell\mathbf{Z})$. It results from the commutativity of the diagram

$$\begin{tikzcd}
\mathrm{H}^1(C-\star, \mathbf{Z}/\ell\mathbf{Z}) \arrow[r, "\mathrm{Gys}(F)"] \arrow[d] & \mathrm{H}^3(X-\star, \mathbf{Z}/\ell\mathbf{Z}) \arrow[d] \\
\mathrm{H}^1(\widehat{C}-\star, \mathbf{Z}/\ell\mathbf{Z}) \arrow[r, "\mathrm{Gys}(F) = \mathrm{Gys}(G^\ell)"] & \mathrm{H}^3(\widehat{X}-\star, \mathbf{Z}/\ell\mathbf{Z}),
\end{tikzcd}$$

of the equality $\mathrm{Gys}(F) = \ell \cdot \mathrm{Gys}(G)$ and finally from the fact that the vertical arrows are isomorphisms (comparison to completion, [**Fujiwara, 1995**, 6.6.4]) that the morphism $\mathrm{Gys}(F)$ is zero. (The commutativity of the diagram results for example from definition **XVI**-2.3.1 and **XVI**-2.2.3.1.) Thus, the restriction morphism induces an isomorphism

$$\mathrm{H}^3(X - \bigstar, \mathbf{Z}/\ell\mathbf{Z}) \xrightarrow{\sim} \mathrm{H}^3(U, \mathbf{Z}/\ell\mathbf{Z}).$$

Now the left term is non-zero, again by purity. The affine scheme U is therefore of cohomological dimension > 2. Q.E.D.

3. Gabber's formal series, cohomological application

We extend the previous construction to arbitrary dimension ≥ 2.

3.1. A formal series and its decomposition.

3.1.1. Let A be a commutative ring. Recall that the A-linear map

$$\sum_{i=1}^{n} A[x_{j\neq i}][[x_i]] \to A[[x_1, \ldots, x_n]]$$

sum of canonical injections is *surjective* : if $G \in A[[x_1, \ldots, x_n]]$, one can for example group for each $i \in [1, n]$ the terms $ax_1^{\beta_1} \cdots x_n^{\beta_n} \in A[[x_1, \ldots, x_n]]$ of G for which $\beta_i = \max_{j\in[1,n]} \beta_j$ and $\beta_1, \ldots, \beta_{i-1} < \beta_i$. (This last condition is only there to define i unambiguously ; any other choice would suffice.) The sum g_i of these terms belongs to $A[x_{j\neq i}][[x_i]]$, and $G = g_1 + \cdots + g_n$.

3.1.2. We now fix two non-zero integers n and ℓ and consider

$$S = \big(y + \sum_{i=1}^{n} \sum_{\alpha=1}^{\infty} t_{i\alpha} x_i^{\alpha}\big)^{\ell} \in \mathbf{Z}[y, t_{i\in[1,n],\alpha\geq 1}][[x_1, \ldots, x_n]].$$

It follows from the preceding observation that this series can be written in the form

$$y^{\ell} + f_1 + \cdots + f_n$$

where each f_i is a formal series in x_i, with polynomial coefficients in the other variables.

3.1.3. In order for the flatness proposition below to be true, we proceed in a slightly different way to define the formal series $f_i \in \mathbf{Z}[y, t_{j\alpha}, x_{k\neq i}][[x_i]]$ such that $S - y^{\ell} = \sum_{i=1}^{n} f_i$. Let's write

$$S = y^{\ell} + \sum_{\alpha=1}^{\infty} (\sum_{i=1}^{n} t_{i\alpha} x_i^{\alpha})^{\ell} + \big(\text{element of degree } < \ell \text{ in the } t_{j\beta}\big).$$

Let $i \in [1, n]$ and $\alpha \geq 1$ be indices. We consider the terms $ax_1^{\beta_1} \cdots x_n^{\beta_n}$ of $(\sum_{i=1}^{n} t_{i\alpha} x_i^{\alpha})^{\ell}$ for which i is the largest index such that $\beta_i \neq 0$, i.e., the terms of $(\sum_{i=1}^{n} t_{i\alpha} x_i^{\alpha})^{\ell}$ which are in $\mathbf{Z}[t_{1\alpha}, \ldots, t_{n\alpha}, x_1, \ldots, x_i]$ but not in $\mathbf{Z}[t_{1\alpha}, \ldots, t_{n\alpha}, x_1, \ldots, x_{i-1}]$. For a fixed i, the sum over α of these terms is an element $f_{i,=\ell}$ of $\mathbf{Z}[t_{j\beta}, x_{k\neq i}][[x_i]]$. By construction, we have the equality $\sum_{\alpha=1}^{\infty} (\sum_{i=1}^{n} t_{i\alpha} x_i^{\alpha})^{\ell} = \sum_{i=1}^{n} f_{i,=\ell}$. Finally, we decompose the remaining term, $S - y^{\ell} - \sum_{i=1}^{n} f_{i,=\ell}$, into a sum $\sum_{i=1}^{n} f_{i,<\ell}$ where each $f_{i,<\ell}$ belongs to $\mathbf{Z}[y, t_{j\beta}, x_{k\neq i}][[x_i]]$, by proceeding for example as in 3.1.1. We then set $f_i = f_{i,=\ell} + f_{i,<\ell}$; each of these elements has total degree in the $t_{j\beta}$ less than or equal to ℓ, and therefore also belongs to $\mathbf{Z}[y, t_{j,\beta\neq\alpha}][[x_1, \ldots, x_n]][t_{1\alpha}, \ldots, t_{n\alpha}]$ for each $\alpha \geq 1$.

Proposition 3.1.4. *Fix $\alpha \geq 1$. Let $T_i = t_{i\alpha}$ for each $i \in [1, n]$ and R_α be the ring $\mathbf{Z}[y, t_{j,\beta\neq\alpha}][[x_1, \ldots, x_n]]$. The morphism*

$$R_\alpha[F_1, \ldots, F_n] \to R_\alpha[T_1, \ldots, T_n]$$

$$F_i \mapsto f_i$$

is free, *hence* flat, *over the open set $x_1 \cdots x_n \neq 0$ of* $\mathrm{Spec}(R)$.

Démonstration. By construction, each f_i is a sum $f_{i,=\ell}+f_{i,<\ell}$, where $f_{i,=\ell}$ (resp. $f_{i,<\ell}$) is a polynomial in $R_\alpha[T_1,\ldots,T_n]$ of total degree equal (resp. strictly less than) to ℓ. Furthermore, $f_{i,=\ell}$ is, as a polynomial in T_i, of the form $x_i^{\alpha\ell}T_i^\ell+\sum_{m<\ell}c_m T_i^m$ where the c_m belong to $R_\alpha[T_{j<i}]$. Let's equip the monomials of $R_\alpha[T_1,\ldots,T_n]$ with the graded lexicographical order as follows : $T_1^{d_1}\cdots T_n^{d_n}\le T_1^{d'_1}\cdots T_d^{d'_n}$ if and only if $\sum d_i<\sum d'_i$ or $\sum d_i=\sum d'_i$ and $d_i<d'_i$ for the largest i such that $d_i\neq d'_i$. It is immediately clear that the *leading term* $\mathrm{in}_\le(g)$ for this order of a polynomial $g=f_1^{q_1}\cdots f_n^{q_n}$ in the f_i is $T_1^{q_1\ell}\cdots T_n^{q_n\ell}$, up to multiplication by a monomial in the x_i. It follows immediately that by inverting the x_i, the ring $R_\alpha[T_1,\ldots,T_n]$ is free over $R_\alpha[F_1,\ldots,F_n]$ with basis the monomials $T_1^{r_1}\cdots T_d^{r_d}$, with $0\le r_i<\ell$. □

3.1.5. Let A be a regular local Noetherian ring of dimension n for which we denote $x_1,\ldots,x_n$ a regular system of parameters. One can think for example of the localization $k[x_1,\ldots,x_n]_{(x_1,\ldots,x_n)}$ of a polynomial ring over a field. We consider the blow-up $A]\underline{t}[$ defined in 1.3, where $\underline{t}$ is the set of variables $\{t_{i\alpha}: i\in[1,\ldots,n],\alpha\in\mathbf{N}_{\ge1}\}$.

3.2. Construction of a pathological regular local ring. Let $i\in[1,\ldots,n]$. Let f_i still denote the image of the formal series with integer coefficients considered in 3.1.3 in the x_i-adic completion of $A]\underline{t}[\,[y]$, and $\mathcal{D}_i$ the dilatation ring $\mathrm{Dil}_{x_i}^{f_i}A]\underline{t}[\,[y]$. The tensor product $\mathcal{P}$ of these $A]\underline{t}[\,[y]$-algebras naturally maps to the completion $\widehat{A]\underline{t}[}\,[[y]]$, where the first completion is done with respect to the maximal ideal $(x_1,\ldots,x_n)$ of $A]\underline{t}[$. Let $\mathcal{D}$ be the localization of $\mathcal{P}$ at the image of the maximal ideal $(x_1,\ldots,x_n,y)$ of $\mathrm{Spec}(\widehat{A]\underline{t}[}\,[[y]])$.

Proposition 3.2.1. *The morphism $\mathcal{D}\to\widehat{A]\underline{t}[}\,[[y]]$ is faithfully* flat.

It follows that the ring $\mathcal{D}$ is local *Noetherian*, regular.

Remarque 3.2.2. Note that it is clear that $\mathcal{D}$ is "quasi-regular". Indeed, the graded ring of $\mathcal{P}$ with respect to the ideal $(x_1,\ldots,x_n,y)$ is a symmetric algebra : for each integer r, the morphism $A]\underline{t}[\,[y]\to\mathcal{P}$ induces an isomorphism modulo $(x_1,\ldots,x_n)^r$.

Démonstration. It suffices to show that the morphism $\mathcal{P}\to\widehat{A]\underline{t}[}\,[[y]]$ is flat. According to the flatness criterion recalled previously (1.2.2), it suffices to show flatness on the open set $x_1\cdots x_n\neq0$. Indeed, the case where only some x_i are zero reduces to this particular case : by tensoring by $A]\underline{t}[\,[y]/x_i$ the morphism $\mathcal{P}\to\widehat{A]\underline{t}[}\,[[y]]$, we obtain an arrow of the same type defined by the ring A/x_i of dimension $n-1$ and series that coincide with the evaluation at $x_i=0$ of $f_1,\ldots,f_{i-1},f_{i+1},\ldots,f_n$. For each i, $\mathcal{D}_i[x_i^{-1}]$ is a polynomial algebra $A]\underline{t}[\,[y,F_i][x_i^{-1}]$ so that the morphism whose flatness we want to prove is

$$A]\underline{t}[\,[y,F_1,\ldots,F_n]\,[\frac{1}{x_1\cdots x_n}]\to\widehat{A]\underline{t}[}\,[[y]]\,[\frac{1}{x_1\cdots x_n}],$$

$$F_i\mapsto f_i.$$

It suffices to show that for each *finite* subset $\mathcal{T}$ of the variables $\underline{t}$, the morphism

$$A]t\in\mathcal{T}[\,[y,F_1,\ldots,F_n]\,[\frac{1}{x_1\cdots x_n}]\to\widehat{A]t\in\mathcal{T}[}\,[[y]]\,[\frac{1}{x_1\cdots x_n}]$$

is flat. By possibly enlarging such a set $\mathcal{T}$, we reduce to the case where $\mathcal{T}$ is *cofinite*, with complement of variables $t_{1\alpha},\ldots,t_{n\alpha}$ for any index $\alpha\ge1$. Setting then $R_\alpha=A]t_{i,\beta\neq\alpha}[$ and $R'=A]\underline{t}[$, it suffices to show that the morphism $R_\alpha[y,F_1,\ldots,F_n]\to R'[[y]]$ is flat over the open set $x_1\cdots x_n\neq0$. A final unravelling brings us back to showing the flatness of the morphism $R_\alpha[y,F_1,\ldots,F_n]\to R_\alpha[y,t_{1\alpha},\ldots,t_{n\alpha}]$, over the same open set. This last point results from proposition 3.1.4. □

Proposition 3.2.3. *The divisor $C=V(y^\ell+F_1+\cdots+F_n)$ of* $\mathrm{Spec}(\mathcal{D})$ *is* regular *outside the closed point.*

In this statement, F_i abusively denotes the image in $\mathcal{D}$ of the element of $\mathcal{D}_i$ corresponding to f_i (cf. 1.1).

Démonstration. It suffices to show that for each strict subset E of $[1,n]$, the schematic intersection of C with the subscheme $X_E = \{x_i = 0, i \in E\,; x_i \neq 0, i \notin E\}$ of $\mathrm{Spec}(\mathcal{D})$ is a regular divisor of X_E. If $E = \varnothing$, this results from the fact that the scheme $X_{\varnothing}$ is the localization of a polynomial algebra in $y, F_1, \ldots, F_n$. The general case reduces easily to this particular case. (Note that if $x_i = 0$, then $F_i = 0$.) □

Proposition 3.2.4. *The inverse image of C in a strict localization of* $\mathrm{Spec}(\mathcal{D})$ *is* irreducible.

Démonstration. Same argument as in dimension 2. □

Corollaire 3.2.5. *For every integer $d \geq 1$, there exists a Noetherian strictly local regular scheme of dimension d possessing an affine open set of ℓ-cohomological dimension $2d-1$.*

Démonstration. The same proof as in dimension 2 allows us to bound the cohomological dimension from below by $2d-1$. According to **XVIII$_{\mathbf{A}}$**-1.1.1, this is an equality. □

4. Cohomological dimension : upper bound for a "generic Milnor fiber"

4.1. Statement.

Théorème 4.1.1. *Let $R \to R'$ be a local essentially of finite type morphism of integral strictly local Noetherian rings. Let K denote the fraction field of R. Then, for every prime number ℓ invertible on R, we have the upper bound*

$$\mathrm{cd}_\ell(R' \otimes_R K) \leq \dim(R'),$$

where the left term denotes the ℓ-cohomological dimension étale of the spectrum of the ring $R' \otimes_R K$ and the right term denotes the Krull dimension of R'.

4.1.2. In this statement, the finiteness hypothesis on f means that R' is a colimit of R-algebras of finite type with étale transition morphisms.

Corollaire 4.1.3. *Let R be an integral strictly local Noetherian ring with fraction field K and let ℓ be a prime number invertible on R. Then, we have the upper bound*

$$\mathrm{cd}_\ell(K) \leq \dim(R).$$

Remarque 4.1.4. Conversely, it follows by passing to the limit from the results of § 6 *infra* that, under the hypotheses of the corollary, if U is a non-empty strict open subset of $\mathrm{Spec}(R)$, then $\mathrm{cd}_\ell(U) \geq \dim(R)$ and that, when $R' \otimes_R K \neq 0$, the upper bound of theorem 4.1.1 is an equality, except in the trivial case $R' \xrightarrow{\sim} R' \otimes_R K$. Another way to proceed would be to use a variant of the method (also due to O. Gabber) exposed in [**Gabber & Orgogozo, 2008**, § 6.1] and based on a "quadratic trick". Let's recall finally that the "limit" lower bound $\mathrm{cd}_\ell(K) \geq \dim(R)$ is elementary : one proceeds by successive specializations in codimension 1 (see [**SGA 4** x 2.4]).

4.2. Proof. We proceed by induction on $d' = \dim(R')$ and reduce to the excellent case.

4.2.1. Let $X = \mathrm{Spec}(R)$, $Y = \mathrm{Spec}(R')$ and respectively $X^\star$ and $Y^\star$ the punctured spectra. Consider the open set $Y_\star = Y \times_X X^\star$ of $Y^\star$, $V = \mathrm{Spec}(R' \otimes_R K)$ the generic fiber of $Y \to X$ and finally j the morphism $V \hookrightarrow Y^\star$. It follows from the induction hypothesis that for each sheaf $\mathcal{F}$ of $\mathbf{Z}/\ell\mathbf{Z}$-modules on V, the complex $\Phi_{\mathcal{F}} = \mathrm{R}j_\star\mathcal{F}$ belongs to $\mathsf{D}^{\leq \mathrm{cod}}(Y^\star)$, where cod is the perversity function $y \mapsto \dim \mathcal{O}_{Y,y}$. (This is still true with $y \mapsto d' - \dim \overline{\{y\}}$.) We want to show that $\mathrm{H}^r(V, \mathcal{F}) = \mathrm{H}^r(Y^\star, \Phi_{\mathcal{F}})$ is zero for $r > d'$. Fix such an r and a class $c \in \mathrm{H}^r(Y^\star, \Phi_{\mathcal{F}})$.

4.2.2. We assume $d \geq 2$, and choose a system of parameters $x_1, \ldots, x_d$ for the strictly local ring R. Let $Z = Y\}t_1, \ldots, t_{d-1}\{$ be the "étale blow-up", strict Henselization of $\mathbf{A}_Y^{d-1} = Y[t_1, \ldots, t_{d-1}]$ at a geometric generic point of the special fiber over Y. The "hyperplane" $H = V(t_1x_1 + \cdots + t_{d-1}x_{d-1} + x_d)$ of Z is of codimension 1, essentially smooth over $Y_\star$. Consider the distinguished triangle

$$\mathrm{R}\Gamma_{H_\star}(Z_\star, \Phi_{\mathcal{F}}) \to \mathrm{R}\Gamma(Z_\star, \Phi_{\mathcal{F}}) \to \mathrm{R}\Gamma(Z_\star - H_\star, \Phi_{\mathcal{F}}) \xrightarrow{+1},$$

where we denote $Z_\star$ the fiber product $Z \times_Y Y_\star$ and, abusively, $\Phi_{\mathcal{F}}$ its various inverse images. Let i be the closed immersion $H_\star \hookrightarrow Z_\star$, where $H_\star = H \times_Z Z_\star$. The morphism $\mathbf{Z}/\ell\mathbf{Z} \to i^!\mathbf{Z}/\ell\mathbf{Z}(1)[2]$ of complexes on $H_\star$ is an isomorphism by relative purity. It is shown by unravelling that it is the same for the arrow $\Phi_{\mathcal{F}|H_\star} \to i^!\Phi_{\mathcal{F}|Z_\star}(1)[2]$ obtained by tensoring from the previous one. We use here the fact that the restriction of $\Phi_{\mathcal{F}}$ to $Z_\star$ comes from the base $Y_\star$. We deduce the piece of exact sequence :

$$\mathrm{H}^{r-2}(H_\star, \Phi_{\mathcal{F}})(-1) \to \mathrm{H}^r(Z_\star, \Phi_{\mathcal{F}}) \to \mathrm{H}^r(Z_\star - H_\star, \Phi_{\mathcal{F}}).$$

Note that $Z_\star - H_\star = Z - H$ because H contains the special fiber of $Z \to X$. Let $\widehat{Y}$ be the completion ($\mathfrak{m}_{R'}$-adic) of Y and $\widetilde{Z}$ a strict Henselization of the fiber product $Z \times_Y \widehat{Y}$. Note that the morphism $\widetilde{Z} \to Z$ is a local morphism between strictly local schemes inducing an isomorphism on the completion along the special fiber over Y. It therefore follows from Fujiwara-Gabber's comparison theorem ([**Fujiwara, 1995**, 6.6.4]) that the morphism $\mathrm{H}^r(Z_\star, \Phi_{\mathcal{F}}) \to \mathrm{H}^r(\widetilde{Z}_\star, \Phi_{\mathcal{F}})$ is an *isomorphism* for each r. It is the same for $\mathrm{H}^{r-2}(H_\star, \Phi_{\mathcal{F}}) \to \mathrm{H}^{r-2}(\widetilde{H}_\star, \Phi_{\mathcal{F}})$, where $\widetilde{H}_\star = H_\star \times_{Z_\star} \widetilde{Z}_\star$. The scheme $\widetilde{Z}_\star - \widetilde{H}_\star$ is an affine open set, which coincides with $\widetilde{Z} - \widetilde{H}$, of a strictly local scheme essentially of finite type over the complete Noetherian local scheme $\widehat{Y}$. Since the membership of $\Phi_{\mathcal{F}}$ in $\mathsf{D}^{\leq \mathrm{cod}}$ is preserved by completion, it follows from the affine Lefschetz theorem (**XV**-1.2.2), in the excellent case, that the cohomology group $\mathrm{H}^r(\widetilde{Z}_\star - \widetilde{H}_\star, \Phi_{\mathcal{F}})$ is zero for each $r > \dim(Z) = \dim(Y) = d'$. Consequently, the morphism $\mathrm{H}^r(\widetilde{Z}_\star, \Phi) \to \mathrm{H}^r(\widetilde{Z}_\star - \widetilde{H}_\star, \Phi)$ is zero for the same r. From this, the aforementioned comparison theorems and the compatibility of the Gysin morphism with completion, it formally follows that any class $c \in \mathrm{H}^r(Z_\star, \Phi_{\mathcal{F}})$ comes from a class in $\mathrm{H}^{r-2}(H_\star, \Phi_{\mathcal{F}})(-1)$ and is therefore killed by restriction to $Z_\star - H_\star$.

4.2.3. There therefore exists an étale neighborhood $e : W \to \mathbf{A}^{d-1}_Y$ whose image meets the special fiber over Y such that the class $c \in \mathrm{H}^r(Y^\star, \Phi_{\mathcal{F}})$ is killed by restriction to $W - H_W$. Let k' be the residue field of Y, and k that of X. The set k^{d-1} is dense in $\mathbf{A}^{d-1}_{k'}$, since k is infinite. It follows that there exists a section $\sigma : Y \to \mathbf{A}^{d-1}_Y$, corresponding to specializations of the t_i to values in R, such that W_σ has a non-empty special fiber over Y. Since the scheme Y is strictly local, we lift this section to $Y \to W_\sigma \to W$. The cohomology class c is therefore zero on $Y - H_Y$, where H_Y is now a hypersurface of equation $x_d + t_1x_1 + \cdots + t_{d-1}x_{d-1}$ with coefficients t_i in R. This affine open set $Y - H_Y$ contains the generic fiber $V = Y \otimes_R K$ because the element $x_d + t_1x_1 + \cdots + t_{d-1}x_{d-1} \in R$ is non-zero, the x_i forming a system of parameters of R. Finally, the restriction of $c \in \mathrm{H}^r(Y^\star, \Phi_{\mathcal{F}})$ to $\mathrm{H}^r(V, \Phi_{\mathcal{F}}) = \mathrm{H}^r(V, \mathcal{F})$, which is the class we started with, is zero. Q.E.D.

5. Upper bound : improvement

5.1. Statement.

5.1.1. Let $f : Y \to X$ be a morphism between non-empty sober topological spaces. We denote

$$\dim.\mathrm{cat}(f) = \sup\{n \in \mathbf{N} : \exists y_0 \rightsquigarrow y_1 \rightsquigarrow \cdots \rightsquigarrow y_n, f(y_0) \neq f(y_1) \neq \cdots \neq f(y_n)\} \in \mathbf{N} \cup \{\infty\}$$

the **catenary dimension** of f, where each $y_i \rightsquigarrow y_{i+1}$ is a specialization.

5.1.2. By construction, $\dim.\mathrm{cat}(f : Y \to X)$ is bounded above by the dimensions of X and Y with equality for example when f is the identity. More generally, when f is a *generizing* morphism ([**ÉGA** I′ 3.9.2]) — as is the case for a flat morphism of schemes — the catenary dimension coincides with the dimension of the image.

Remarque 5.1.3. If f is a dominant essentially of finite type morphism (i.e., Zariski-locally as in 4.1.2) between integral Noetherian schemes, it can be shown that the catenary dimension of f is the dimension of the image of a flattening of f.

Théorème 5.1.4. *Let $f : Y \to X$ be an essentially of finite type morphism between strictly local Noetherian schemes and let V be an affine open subset of Y. Then, for every prime number ℓ invertible on X, we have the upper bound*

$$\mathrm{cd}_\ell(V) \leq \dim(Y) + \max\big(0, \dim.\mathrm{cat}(f) - 1\big).$$

Corollaire 5.1.5. *Let $f : Y \to X$ be an essentially of finite type morphism between strictly local Noetherian schemes, where $\dim(X) \geq 1$, and let V be an affine open subset of Y. Then, for every prime number ℓ invertible on X, we have the upper bound*

$$\mathrm{cd}_\ell(V) \leq \dim(Y) + \dim(X) - 1.$$

Corollaire 5.1.6. *Let $d \geq 1$ be an integer and let n be an integer in the closed interval $[d, 2d-1]$. There exists a strictly local Noetherian scheme X, regular of dimension d, and an affine open set U of this scheme such that for every prime number ℓ invertible on X we have the equality*

$$\mathrm{cd}_\ell(U) = n.$$

Proof of corollary 5.1.6. It suffices to show that for every integer $d \geq 1$, and every integer $r \geq 0$, there exists a strictly local regular Noetherian scheme Y of dimension $d+r$ and an affine open set V of Y of ℓ-cohomological dimension equal to $2d+r-1$. Let X and f be as in 3.2.5 : the affine open set $U = X[f^{-1}]$ is of dimension d, ℓ-cohomological dimension $\delta = 2d-1$. More precisely, it follows from the proof that there exists a non-zero class in $\mathrm{H}^\delta(U, \mathbf{Z}/\ell\mathbf{Z})$. Now consider $Y = X[T]_{(0)}$ a strict Henselization of the affine line over X at the origin of the special fiber, $g = fT \in \Gamma(Y, \mathcal{O}_Y)$ and V the affine open set $Y[g^{-1}]$. We have $\dim(Y) = d+1$. By cohomological purity, it is immediately verified that the cohomology group $\mathrm{H}^{d+1}(V, \mathbf{Z}/\ell\mathbf{Z})$ is also non-zero. By induction, we obtain a pair (Y, V) as above such that $\mathrm{cd}_\ell(V) \geq 2d+r-1$. According to the previous corollary, we also have the upper bound $\mathrm{cd}_\ell(V) \leq 2d+r-1$, hence the equality. $\square$

5.2. Proof. We proceed by induction on the catenary dimension of f.

5.2.1. $\dim.\mathrm{cat}(f) = 0$. This equality occurs if and only if Y is contained in the special fiber. The theorem is therefore known in this case : we are over a field thus in an excellent situation.

5.2.2. $\dim.\mathrm{cat}(f) = 1$. We can assume the schemes X and Y to be reduced. By induction on the dimension of Y, we can also assume Y to be irreducible : if Y is the union of two strict closed subsets Y_1 and Y_2, consider for example the morphism $\pi : Y_1 \amalg Y_2 \to Y$ and the exact sequence $0 \to \mathcal{F} \to \pi^\star\mathcal{F} \to i_\star\mathcal{H} \to 0$, where i is the closed immersion $Y_1 \cap Y_2 \hookrightarrow Y$ and $\mathcal{H}$ a sheaf on this intersection. By possibly replacing X by the closure of the image of f, we can also assume the base to be integral and f dominant. Let η (resp. s) be the generic (resp. closed) point of X and η' (resp. s') the generic (resp. closed) point of Y. Since every point y of Y is inserted into a sequence of specializations $\eta' \rightsquigarrow y \rightsquigarrow s'$, with image $\eta \rightsquigarrow f(y) \rightsquigarrow s$, it follows from the hypothesis $\dim.\mathrm{cat}(f) = 1$ that $f(y) = \eta$ or $f(y) = s$. Let $\widehat{X}$ be the completion of X and $\widetilde{Y}$ a strict Henselization of the fiber product $Y \times_X \widehat{X}$. This is a strictly local scheme of dimension $\dim(Y)$ and *excellent* because essentially of finite type over the complete Noetherian local scheme — hence excellent — $\widehat{X}$. We denote $\widetilde{V}$ the open set $V \times_Y \widetilde{Y}$. It follows from [**SGA 4½** [Th. finitude] 1.9] that over η, and thus over $X - \{s\}$, the formation of direct images by $j : V \hookrightarrow Y$ commutes with base change $\widetilde{Y} \to Y$. In other words, if $V' = V \cup (Y - Y_s)$ and j' denotes the intermediate immersion $V \hookrightarrow V'$, the formation of $\mathrm{R}j'_\star$ commutes with $\widetilde{Y} \to Y$. It is the same for $j'' : V' \to Y$ according to the comparison theorem to completion of Fujiwara-Gabber ([**Fujiwara, 1995**, 6.6.4]). We use here the fact that if F is a closed subset of Y included in the special fiber then the completions of Y along F and that of the Henselization of $\widetilde{Y}$ (at the point corresponding to the closed point of Y) are naturally isomorphic. Finally, the functor $\mathrm{R}\Gamma(V) = \mathrm{R}\Gamma(Y) \circ \mathrm{R}j_\star$ is identified with the functor $\mathrm{R}\Gamma(\widetilde{Y}) \circ \mathrm{R}\tilde{j}_\star = \mathrm{R}\Gamma(\widetilde{V})$, applied to the inverse image. Thus we have the inequality $\mathrm{cd}_\ell(V) \leq \mathrm{cd}_\ell(\widetilde{V})$. The scheme $\widetilde{Y}$ is quasi-excellent, the right term is therefore subject to the affine Lefschetz theorem. The inequality $\mathrm{cd}_\ell(V) \leq \dim(Y)$ then results from the equality $\dim(Y) = \dim(\widetilde{Y})$.

Remarque 5.2.3. When $\dim(X) = 1$, we saw in **XIII**-2.3 that the theorem can also be proven by normalization.

5.2.4. $\dim.\mathrm{cat}(f) > 1$. Let j again denote the open immersion $V \hookrightarrow Y$. By restriction to the generic fiber, we have an isomorphism $\mathrm{R}\Gamma(V_\eta, \mathcal{F}) = \mathrm{R}\Gamma(Y_\eta, (Rj_\star\mathcal{F})_{|Y_\eta})$. If y is a geometric point of Y localized at Y_η, the fiber $(\mathrm{R}^q j_\star\mathcal{F})_y$ is zero as soon as $q > \dim(\mathcal{O}_{Y_\eta, y})$. This follows by passing to the limit from Artin's theorem for affine schemes of finite type over a field and from the trivial fact that $Y_{(y)} \to Y$ factors through Y_η. Let $q \geq 0$ be an integer, $\mathcal{G}$ a constructible subsheaf of $(\mathrm{R}^q j_\star\mathcal{F})_{|Y_\eta}$ and S the closure in Y of its support. From the foregoing, we have the upper bound $\mathrm{codim}(S_\eta, Y_\eta) \geq q$. It follows that $\mathrm{codim}(S, Y) \geq q$, and this without any catenarity hypothesis on the schemes. From the spectral sequence of composition of functors and theorem 4.1.1, we deduce that the cohomology group $\mathrm{H}^n(V_\eta, \mathcal{F})$ is zero when $n > \dim(Y)$.

Consider the adjunction map $\mathcal{F} \to k_\star k^\star \mathcal{F}$, where k is the immersion $V_\eta \hookrightarrow V$, and $\mathcal{K}$ its kernel. By construction, the restriction of $\mathcal{K}$ to V_η is zero. The dimension of the closure of the support of $\mathcal{K}$ is therefore at most $\dim(Y)-1$. It therefore follows from the induction hypothesis that the desired vanishing result is known for $\mathcal{K}$. Proceeding in the same way for the cokernel of the previous adjunction, we reduce to proving the vanishing of the group $\mathrm{H}^p(V, R^0 k_\star k^\star \mathcal{F})$ for $p \geq \dim(Y) + \mathrm{dim.cat}(f)(> \dim(Y))$. Given the vanishing result previously established for $\mathrm{R}\Gamma(V_\eta, \mathcal{F}) = \mathrm{R}\Gamma(V, Rk_\star k^\star \mathcal{F})$ and Leray's spectral sequence $E_2^{p,q} = \mathrm{H}^p(V, \mathrm{R}^q k_\star k^\star \mathcal{F}) \Rightarrow \mathrm{H}^{p+q}(V_\eta, \mathcal{F})$, it suffices to show that for each such q, the groups $\mathrm{H}^{p-q-1}(V, \mathrm{R}^q k_\star k^\star \mathcal{F})$ are zero for $q > 0$. Fix q. Let y be a geometric point of Y such that the fiber of $\mathrm{R}^q k_\star k^\star \mathcal{F}$ at y is non-zero and x the point image of y in X. The scheme $\eta \times_X X_{(x)}$ decomposes into a coproduct of spectra of fields η_α; similarly, the fiber product $Y_{(y)} \times_X \eta$, whose cohomology we consider, is isomorphic to the coproduct of the $Y_{(y)} \times_{X_{(x)}} \eta_\alpha$. According to *op. cit.* (4.1.1), these last have cohomology only in degree $q \leq \dim Y_{(y)} \leq \dim(Y) - \dim \overline{\{y\}}$. It follows that the dimension of the support of each of the constructible subsheaves of $\mathrm{R}^q k_\star k^\star \mathcal{F}$ is at most $\dim(Y) - q$. Moreover, the catenary dimension of the morphism f restricted to such a support is at most $\mathrm{dim.cat}(f) - 1$. It therefore follows from the induction hypothesis that the groups $\mathrm{H}^{p-q-1}(V, \mathrm{R}^q k_\star k^\star \mathcal{F})$ are zero when $p - q - 1 \geq (\dim(Y) - q) + (\mathrm{dim.cat.}(f) - 1)$. Q.E.D.

6. Cohomological dimension of an open subset of the punctured spectrum : lower bound

Théorème 6.1. *Let X be an integral strictly local Noetherian scheme of dimension d and Ω a non-empty open subset of the punctured spectrum. Then, for every prime number ℓ invertible on X, we have*

$$\mathrm{cd}_\ell(\Omega) \geq d.$$

The first proof occupies the following two paragraphs.

6.2. Local combinatorial construction.

6.2.1. *Notations.* Let X be a regular strictly local Noetherian scheme of dimension $d \geq 2$ and let $t_1, \dots, t_{d-1}, t_d$ be a regular system of parameters. For reasons that will appear later, we also denote π the element t_d. For each $1 \leq i \leq d-1$, we denote H_i the regular divisor $V(t_i)$; for $i = d$, we set $H_d = V(t_1 + \cdots + t_{d-1} - \pi)$. Finally, we denote U the open affine $X[\pi^{-1}]$, k the open immersion $U \hookrightarrow X$ and j the open immersion $U - \bigcup_{i=1}^d H_i \hookrightarrow U$. We fix a prime number ℓ invertible on X and set $\Lambda = \mathbf{Z}/\ell\mathbf{Z}$.

6.2.2. Let P be a subset of $\{1, \dots, d\}$. Let H_P denote the intersection $\bigcap_{p \in P} H_p$, and let H'_P denote the intersection $H_P \cap U$ — open in H_P and closed in U —, and k_P the open immersion $H'_P \hookrightarrow H_P$. For each integer q, the cohomology group $\mathrm{H}^q(U, \Lambda_{H'_P})$ is isomorphic to the group $\mathrm{H}^q(H'_P, \Lambda)$. Since $H_P - H'_P$ is the regular divisor defined by π in H_P, it follows from cohomological purity that $\mathrm{H}^q(U, \Lambda_{H'_P})$ is zero for $q > 1$, isomorphic to Λ for $q = 0$ and of rank 1, generated by the Kummer class of π for $q = 1$. This holds also for $P = \varnothing$, with the obvious convention that $H_\varnothing = X$ and $H'_\varnothing = U$.

6.2.3. Consider now the quasi-isomorphism

$$j_! \Lambda \xrightarrow{\sim} \big(\Lambda \to \bigoplus_{1 \leq i \leq d} \Lambda_{H'_i} \to \cdots \to \bigoplus_{|P| = d-1} \Lambda_{H'_P} \to 0\big)$$

of sheaves of Λ-modules on U, where the first term of the right complex is placed in degree 0. (We use here the fact that $H'_P = \varnothing$ if $|P| = d$ because H_P is then the closed point of X.) The differentials are sums, with signs, of restriction maps. To this resolution is associated — via the "stupid" filtration — the spectral sequence

$$E_1^{p,q} = \mathrm{H}^q(U, \bigoplus_{|P| = p} \Lambda_{H'_P}) \Rightarrow \mathrm{H}^{p+q}(U, j_! \Lambda).$$

According to the observations in the previous paragraph, every class in $E_1^{d-1,1} = \bigoplus_{|P|=d-1} \mathrm{H}^1(U, \Lambda_{H'_P})$ not belonging to the image of $E_1^{d-2,1}$ survives in the abutment $\mathrm{H}^d(U, j_! \Lambda)$.

6.2.4. Just as the sheaf $j_!\Lambda$ is isomorphic to the tensor product

$$(U - H'_1 \hookrightarrow U)_!\Lambda \otimes \cdots \otimes (U - H'_d \hookrightarrow U)_!\Lambda),$$

the complex quasi-isomorphic to $j_!\Lambda$ above is isomorphic to the tensor product of the complexes $(\Lambda \to \Lambda_{H'_i})$, $1 \leq i \leq d$, where the arrow is the unit of the adjunction, isomorphic respectively to $(U - H'_i \hookrightarrow U)_!\Lambda$. From this observation, combined with (6.2.2), we deduce that the complex $E_1^{\bullet,1}$ is quasi-isomorphic to the naive truncation $\sigma_{\leq d-1}\big((\Lambda \xrightarrow{\mathrm{Id}} \Lambda)^{\otimes d}\big)$ obtained by replacing the d-th term, isomorphic to Λ, by zero. It is well-known that this ("Koszul") tensor product complex is acyclic (before truncation), cf. for example [**ÉGA** III$_1$ § 1.1]. (The exactness also results from the quasi-isomorphism above, applied to other closed sets.) In particular, the image of the differential $E_1^{d-2,1} \to E_1^{d-1,1}$ is naturally the kernel of a non-zero linear form (explicit) on $E_1^{d-1,1}$ and is therefore not $E_1^{d-1,1}$ entirely. There therefore exist direct sums of Kummer classes of π that survive in $\mathrm{H}^d(U, j_!\Lambda)$.

6.3. Blow-up and partial normalization.

6.3.1. Now let $X = \mathrm{Spec}(R)$ be an integral strictly local Noetherian scheme of dimension $d \geq 2$, with closed point x, and let Ω be a non-empty strict open subset of X. We will show that after blow-up and "partial normalization" the open set Ω is — locally and "modulo nilpotents" — a regular scheme of the type of scheme U considered above. This allows us to produce a non-zero cohomology class of degree d on Ω.

6.3.2. Let $Y = \mathrm{Écl}_x(X)$. Let j denote the open immersion of $Y - Y_x$ into Y. We denote by $\widehat{X}$ the completion of the local scheme X, by Y' the fiber product $Y \times_X \widehat{X}$ and by $Y'_{\mathrm{réd}}$ the reduction of Y'. Note that the scheme Y' is *excellent* because $\widehat{X}$ is.

6.3.3. Let $\mathcal{O}_Y^{\oslash}$ be the normalization of $\mathcal{O}_Y$ in $j_\star\mathcal{O}_{Y-Y_x}$. We similarly define $\mathcal{O}_{Y'}^{\oslash}$ and $\mathcal{O}_{Y'_{\mathrm{réd}}}^{\oslash}$. The $\mathcal{O}_Y$-algebra $\mathcal{O}_Y^{\oslash}$ is a (filtered) colimit of its finite $\mathcal{O}_Y$-subalgebras $\mathcal{A}_\lambda$.

Proposition 6.3.4. *The functor sending a $\mathcal{O}_Y$-subalgebra $\mathcal{B}$ of $j_\star\mathcal{O}_{Y-Y_x}$ to the image of $(Y' \to Y)^\star\mathcal{B}$ by the adjunction $(Y' \to Y)^\star j_\star\mathcal{O}_{Y-Y_x} \to j'_\star\mathcal{O}_{Y'-Y_x}$ induces a* bijection *between the finite $\mathcal{O}_Y$-subalgebras of $j_\star\mathcal{O}_{Y-Y_x}$ and the finite $\mathcal{O}_{Y'}$-subalgebras of $j'_\star\mathcal{O}_{Y'-Y_x}$. Moreover, the algebras $\mathcal{O}_Y^{\oslash}$ and $\mathcal{O}_{Y'}^{\oslash}$ correspond to each other via this functor.*

Démonstration. We reduce to the case where $Y = \mathrm{Spec}(A)$ and $Y_x = V(t)$. It suffices to show that if A is an integral R-algebra and $A' = A \otimes_R \widehat{R}$ then the morphism $A[t^{-1}]/A \to A'[t^{-1}]/A'$ is an isomorphism and that the normalizations correspond. Note that the rings A and A' have the same t-adic completion. The first point then follows from the fact that if M is an A-module whose every element is killed by a power of t, we have $M \xrightarrow{\sim} M \otimes_A A'$. Finally, let $(f'/t^r)^d + a'_1(f'/t^r)^{d-1} + \cdots + a'_d = 0$ be an integral dependence relation where f' and the a'_i belong to A'. Let N be a sufficiently large integer. Write $f' = f + t^N g'$, $a'_i = a_i + t^N b'_i$ where f and the a_i belong to A. The previous relation becomes $(f/t^r)^d + a_1(f/t^r)^{d-1} + \cdots + a_d \in A' \cap A[t^{-1}] = A$. It follows that the element $f'/t^r = f/t^r + t^{N-r}g'$ is, modulo an element of A', in the image of $A^{\oslash}$. □

6.3.5. By excellence, the algebra $\mathcal{O}_{Y'_{\mathrm{réd}}}^{\oslash}$ is finite over $\mathcal{O}_{Y'_{\mathrm{réd}}}$. The ring $\mathcal{O}_{Y'}^{\oslash}$ is its "inverse image" by the natural surjection. From these observations and the previous proposition, we deduce that there exists an index λ such that, if $Z = \mathrm{Spec}(\mathcal{A}_\lambda)$ and $Z' = Z_{\widehat{X}}$, then $Z'_{\mathrm{réd}}$ is integrally closed in $Z'_{\mathrm{réd}}$ deprived of the inverse image of Y_x. Note that Z and Y are isomorphic outside Y_x.

6.3.6. Let E be a component of dimension $d-1$ of Y_x and let e be a maximal point of $E'_{\mathrm{réd}}$ in $Z'_{\mathrm{réd}}$. The localization at e is a discrete valuation ring : it is a local reduced ring of dimension 1, integrally closed in the complement of the closed point. By excellence of Z' there exists a dense open subset of $E'_{\mathrm{réd}}$ along which $Z'_{\mathrm{réd}}$ is regular. We can also assume that $E'_{\mathrm{réd}}$ is regular on this open set. (For this last point it suffices to note that $E'_{\mathrm{réd}}$ is of finite type over a field.) Let $U'_{\mathrm{réd}}$ be an open subset of $Z'_{\mathrm{réd}}$ inducing the open subset of $E'_{\mathrm{réd}}$ above and U an open subset of Z inducing the corresponding open subset of E. (The morphism $Z' \to Z$ is an isomorphism on E.) We have $U' \subset U \times_Z Z'$. Below, we allow ourselves to shrink the open sets U and U', provided they contain all maximal points of E. We further assume that $U \cap Y_x = U \cap E$.

6.3.7. Let t denote an equation of E in U and π an equation of $E'_{\text{réd}}$ in $U'_{\text{réd}}$ so that there exists a unit u and an integer e such that we have the equality $t = u \times \pi^e$ on $U'_{\text{réd}}$. The existence of a lifting shows that we can assume the equation π to be defined on U'. Let's check that we can also assume π to be defined on U. Since schemes Z' and Z have the same t-adic completion, it suffices to observe that if a is a function on U', we have the equality of ideals $(\pi) = (\pi + at^2)$, at least when $1 + ua\pi^{2e-1} \in \mathbf{G}_m(U')$, which we can assume by restricting U'.

6.3.8. Let Ω be a non-empty open set of $X - \{x\}$ (cf. 6.3.1). We also denote Ω its inverse images in Y and Z; they are isomorphic to it. On an open neighborhood of the maximal points of E, the open set Ω coincides with the complement of E. Generically on E, we therefore have $\Omega = Z[\pi^{-1}]$. Let z be a closed point of Z belonging to such an open set as well as to the open set U. Let $t_1, \dots, t_{d-1}$ be functions of $\mathcal{O}_{Z'_{\text{réd}},z}$ constituting, with π, a regular system of parameters. We can assume that they extend to $U'_{\text{réd}}$. Using again the fact that the morphism $Z' \to Z$ is an isomorphism over E, we can also assume that they come from U, possibly changing them modulo π. For each $i < d$, consider the schematic closure $H_i \subset Z$ of the hypersurface $V(t_i)$ in U and H_d the schematic closure of $V(t_1 + \dots + t_{d-1} - \pi)$. We denote $H'_i = H_i \cap \Omega$.

6.3.9. *Strategy.* We will construct a non-zero class in the cohomology group $\mathrm{H}^d(\Omega, j_!\Lambda_{\Omega - \bigcup_1^d H'_i})$, where j is the open immersion $\Omega - \bigcup_1^d H'_i \hookrightarrow \Omega$ and $\Lambda = \mathbf{Z}/n\mathbf{Z}$ with n invertible on X. *Locally*, these cohomology groups are invariant under passage to the completion of the base X (and of course to the reduced scheme) so that we will be able to use the calculations of 6.2. However, care must be taken here that the intersection $\bigcap_1^d H'_i$ is not necessarily empty, unlike the previously studied local case : the analogue of complex 6.2.3 therefore has one more term (in degree d). Nevertheless, we will lift to Ω a "local" class of degree d — i.e., from the scheme $\Omega \times_Z Z_{(\overline{z})}$ (or rather the analogue on $Z'_{\text{réd}}$) — with coefficients in $j_!\Lambda$.

6.3.10. Consider again the spectral sequence from 6.2.3 :

$$E_1^{p,q} = \mathrm{H}^q(\Omega, \Lambda_p) \Rightarrow \mathrm{H}^{p+q}(\Omega, j_!\Lambda),$$

where Λ_p denotes the direct sum of the $\Lambda_{H'_P}$ with $|P| = p$. For each $P \subset [1, d]$ of cardinality $d - 1$, the intersection H_P of the corresponding hypersurfaces of Z is proper over X. By construction, it is also *quasi-finite* in the neighborhood of the closed point z. It follows from the proper base change theorem for the set of connected components (or Zariski's main theorem) — according to which its decomposition into connected components is read on the special fiber — that each H_P decomposes into the coproduct of a local scheme finite over X and a scheme not meeting z. Thus, $\mathrm{H}^1(\Omega, \Lambda_{d-1})$ has a direct factor isomorphic to $\mathrm{H}^1(Z_{(z)}[\pi^{-1}], \Lambda_{d-1})$ and, by comparison to completion, to $\mathrm{H}^1(Z'_{\text{réd}(z)}[\pi^{-1}], \Lambda_{d-1})$. Moreover, the differential $E_1^{d-1,1} \to E_1^{d,1}$ sends this factor to 0 in $\mathrm{H}^1(\Omega, \Lambda_d)$ because the intersection of $H_{[1,d]}$ with Ω is empty in the neighborhood of z. Indeed, if $t_1 = t_2 = \dots = t_{d-1} = t_1 + \dots + t_{d-1} - \pi = 0$ then $\pi = 0$; on the other hand, in the neighborhood of z, $\Omega = Z[\pi^{-1}]$. Thus, any cohomology class of the "local" direct factor induces a class in $\mathrm{H}^d(\Omega, j_!\Lambda)$ which lifts the corresponding class in $\mathrm{H}^d(Z'_{\text{réd}(z)}[\pi^{-1}], j_!\Lambda)$. We have seen (6.2.4) that such non-zero classes exist. Consequently, $\mathrm{H}^d(\Omega, j_!\Lambda) \neq 0$ and finally $\mathrm{cd}_n(\Omega) \geq d$. Q.E.D.

6.4. Joins of Henselian rings and cohomological dimension.

6.4.1. *Joins.* Let's begin by stating a variant of the main result of [**Artin, 1971**], to which we reduce.

Proposition 6.4.1.1. *Let A be a ring, $\mathfrak{p}$ and $\mathfrak{q}$ two prime ideals and C a component of the tensor product $A^{\mathrm{h}}_{\mathfrak{p}} \otimes_A A^{h}_{\mathfrak{q}}$. The ring C is local Henselian. Moreover, if neither $A^{\mathrm{h}}_{\mathfrak{p}} \to C$ nor $A^{\mathrm{h}}_{\mathfrak{q}} \to C$ are* local, *it is* strictly *Henselian.*

Here, $A^{\mathrm{h}}_{\mathfrak{p}}$ denotes the Henselization of A at $\mathfrak{p}$. By "connected component", we mean — as in *op. cit.*, § 3 — the ring of functions on a connected component of the spectrum. Note that the first statement is only given for memory (*op. cit.*, 3.4 (i)) ; the second generalizes *loc. cit.* (ii) because if $A^{\mathrm{h}}_{\mathfrak{p}} \to C$ is local, we have the inclusion $\mathfrak{p} \subset \mathfrak{q}$ (and similarly for $\mathfrak{q}$, *mutatis mutandis*).

Corollaire 6.4.1.2. *Let A be a ring and $\mathfrak{p}$, $\mathfrak{q}$ two prime ideals. If $\mathfrak{p} \not\subset \mathfrak{q}$, every connected component of the tensor product $A^{\mathrm{h}}_{\mathfrak{p}} \otimes_A A^{\mathrm{hs}}_{\mathfrak{q}}$ is strictly Henselian.*

Here, $A^{\mathrm{hs}}_{\mathfrak{q}}$ denotes a strict Henselization of A at $\mathfrak{q}$.

Proof of the corollary. Let C' be a component of the tensor product $A^{\mathrm{h}}_{\mathfrak{p}} \otimes_A A^{\mathrm{hs}}_{\mathfrak{q}}$ and C the corresponding component of $A^{\mathrm{h}}_{\mathfrak{p}} \otimes_A A^{h}_{\mathfrak{q}}$. The ring C is local Henselian; the morphism $C \to C'$ being ind-finite étale, the ring C' is also local Henselian. By hypothesis, the morphism $A^{\mathrm{h}}_{\mathfrak{p}} \to C$ is not local. Two cases arise. If $A^{\mathrm{h}}_{\mathfrak{q}} \to C$ is not local either, the ring C is (local) *strictly* Henselian and so is C'. If $A^{\mathrm{h}}_{\mathfrak{q}} \to C$ is local, the ind-étale morphism $A^{\mathrm{hs}}_{\mathfrak{q}} \to C'$ is also local. The residue fields of $A^{\mathrm{hs}}_{\mathfrak{q}}$ and C' are therefore isomorphic. In particular, the Henselian ring C' is *strictly* Henselian. □

Proof of the proposition. Write A as a quotient of an integral normal ring A' and let $\mathfrak{p}$ and $\mathfrak{q}$ still denote the prime ideals of A' corresponding to those of A. For any connected component C' of $A'^{\mathrm{h}}_{\mathfrak{p}} \otimes_{A'} A'^{\mathrm{h}}_{\mathfrak{q}}$ — necessarily local Henselian —, the quotient $C' \otimes_{A'} A$ is either zero or local Henselian. The connected components of $A^{\mathrm{h}}_{\mathfrak{p}} \otimes_A A^{\mathrm{h}}_{\mathfrak{q}}$ are exactly the local Henselian quotients thus obtained. This allows us to assume A to be integral normal. (We could also assume A to be of finite type over $\mathbf{Z}$; cf. *op. cit.*, proof of theorem 3.4.) As in *op. cit.*, 2.1, we can choose an algebraic closure $\overline{K}$ of $K = \mathrm{Frac}(A)$ and embed our rings in a diagram

$$
\begin{array}{ccccccc}
 & & & \overline{K} & & & \\
 & & & | & & & \\
 & & & C & & & \\
 & & \diagup & \cup & \diagdown & & \\
\overline{A}_1 & \subset & A^{\mathrm{h}}_{\mathfrak{p}} & \overline{A} & A^{\mathrm{h}}_{\mathfrak{q}} & \supset & \overline{A}_2 \\
 & & \diagdown & & \diagup & & \\
 & & & A & & &
\end{array}
$$

where C is the subring of $\overline{K}$ generated by $A^{\mathrm{h}}_{\mathfrak{p}}$ and $A^{\mathrm{h}}_{\mathfrak{q}}$ — this is the "join", denoted $[A^{\mathrm{h}}_{\mathfrak{p}}, A^{\mathrm{h}}_{\mathfrak{q}}]$, of *op. cit.* —, and $\overline{A}_1$, $\overline{A}_2$ and $\overline{A}$ are the integral closures of A in the fraction fields of, respectively, $A^{\mathrm{h}}_{\mathfrak{p}}$, $A^{\mathrm{h}}_{\mathfrak{q}}$ and C. Note that the ring C is normal : it is a component of the ind-étale A-algebra $A^{\mathrm{h}}_{\mathfrak{p}} \otimes_A A^{\mathrm{h}}_{\mathfrak{q}}$. On the other hand, the ring $A^{\mathrm{h}}_{\mathfrak{p}}$ is a (Zariski) localization of $\overline{A}_1$ at a prime ideal; this prime ideal admits — by the Hensel property — a unique lift to a prime ideal $\overline{\mathfrak{p}}$ of C. Similarly for $\mathfrak{q}$. The ring C is also the join of the Henselian rings $\overline{A}_{\overline{\mathfrak{p}}}$ and $\overline{A}_{\overline{\mathfrak{q}}}$. Recall that, by hypothesis, neither of the morphisms $A^{\mathrm{h}}_{\mathfrak{p}} \to C$ and $A^{\mathrm{h}}_{\mathfrak{q}} \to C$ is local; the same is true for $\overline{A}_{\overline{\mathfrak{p}}} \to C$ and $\overline{A}_{\overline{\mathfrak{q}}} \to C$ and, in particular, $\overline{\mathfrak{p}} \not\subset \overline{\mathfrak{q}}$ and $\overline{\mathfrak{q}} \not\subset \overline{\mathfrak{p}}$. These reductions being made, the fact that C is strictly Henselian results from [**Artin, 1971**, theorems 2.2 and 2.5]. □

6.4.2. *Comparison to the cohomology of a Henselian discretely valued field.* Let's now see a cohomological consequence of the previous proposition.

Let X be a scheme such that every pair of points belongs to an affine open set. Let $x \in X$ such that $\mathcal{O}_{X,x}$ is (local) integral Noetherian of dimension 1 and finally $\Omega \subset X$ an open set such that $\Omega \cap \mathrm{Spec}(\mathcal{O}_{X,x})$ is the generic point $\mathrm{Spec}(K)$ of $\mathrm{Spec}(\mathcal{O}_{X,x})$. Let K^{h} denote the total ring of fractions $\mathcal{O}^{\mathrm{h}}_{X,x} \otimes_{\mathcal{O}_{X,x}} K$ of the Henselization $\mathcal{O}^{\mathrm{h}}_{X,x}$ and ε the morphism $\mathrm{Spec}(K^{\mathrm{h}}) \to \Omega$.

Proposition 6.4.2.1. *For every integer $j > 0$, we have $\mathrm{R}^j\varepsilon_\star = 0$. Thus, for every étale abelian sheaf $\mathcal{F}$ on* $\mathrm{Spec}(K^{\mathrm{h}})$, *and every integer $i \geq 0$, the morphism*

$$\mathrm{H}^i(\Omega, \varepsilon_\star\mathcal{F}) \to \mathrm{H}^i(\mathrm{Spec}\, K^{\mathrm{h}}, \mathcal{F})$$

is an isomorphism. *In particular, for every prime number ℓ, we have the lower bound* $\mathrm{cd}_\ell(\Omega) \geq \mathrm{cd}_\ell(K^{\mathrm{h}})$.

Démonstration. Only the vanishing result needs to be proven; it follows immediately from the previous proposition by passing to the fibers at geometric points of U. □

It is not difficult to deduce an alternative proof of theorem 6.1 (see also 4.1.4) which we recall here in the form of a corollary.

Corollaire 6.4.2.2. *Let R be an integral strictly local Noetherian ring with fraction field K and let ℓ be a prime number invertible on R. For any non-empty strict open set Ω of* $\mathrm{Spec}(R)$*, we have the lower bound*

$$\mathrm{cd}_\ell(\Omega) \geq \dim(R).$$

Démonstration. Let $X = \mathrm{Spec}(R)$ and consider, as in § 6.3, the blow-up Y of X at its closed point. Let y be the generic point of an irreducible component of dimension $\dim(X) - 1$ of the special fiber. According to proposition 6.4.2.1 applied to point y of Y and to open set Ω (seen in Y), we have the lower bound $\mathrm{cd}_\ell(\Omega) \geq \mathrm{cd}_\ell(K^{\mathrm{h}})$, where K^{h} denotes the total ring of fractions of $\mathcal{O}^{\mathrm{h}}_{Y,y}$. According to the Krull-Akizuki theorem, the normalization of $\mathcal{O}_{Y,y}$ in K^{h} is a finite product of Henselian discrete valuation rings, with residue fields finite over $\kappa(y)$. It then follows from the formula $\mathrm{cd}_\ell(\mathrm{Frac}\,A) = 1 + \mathrm{cd}_\ell(\kappa)$ — where A is a Henselian trait with residue field κ of characteristic $\neq \ell$ ([**SGA 4** x 2.2.(i)]) —, and from the formula $\mathrm{cd}_\ell(\kappa) = \mathrm{deg.\,tr.}(\kappa/k)$ — where κ is an extension of finite type of a separably closed field k of characteristic $\neq \ell$ ([**SGA 4** x 2.1]) —, that the ℓ-cohomological dimension of K^{h} is $\dim(R)$. Q.E.D. □

Remarques 6.4.2.3. The same argument allows for a new, simpler proof of the results in [**Gabber & Orgogozo, 2008**, § 6.2].

Finally, note that 6.4.2.2 can be extended to any non-empty *subset* Ω of the punctured scheme $X^\star$ that is stable by generization : such a set is an intersection of open sets and one can define its étale topos, by passing to the limit ([**SGA 4** vi § 8]) or via the étale topos of an annelated topos [**Hakim, 1972**, chap. IV, § 5].

EXPOSÉ XIX

A Counterexample

Yves Laszlo

1. Introduction

This lecture is dedicated to constructing, following Gabber ([**Gabber, 2001**]), an example of an open immersion $j : U \to X$ of Noetherian schemes such that $\mathrm{R}^1 j_\star \mathbf{Z}/2\mathbf{Z}$ is not constructible. This shows that the quasi-excellence hypothesis in Gabber's constructibility theorem (**XIII**-1.1.1) is essential. From a geometric point of view, the construction is interesting : U is the complement of a divisor D in a regular surface X but has infinitely many ordinary double points ; in particular, its regular locus is not open, which prevents it from being quasi-excellent. This divisor is an example of a divisor in a regular surface that is locally with normal crossings (in the sense of de Jong) but not globally (5.5).

2. The Construction

If K is a field and $\underline{x} = (x_1, ..., x_n)$, we denote by $K\{\underline{x}\}$ the henselization at the origin of the polynomial ring $K[\underline{x}]$. We choose an infinite perfect field k, at most countable, such that k^*/k^{*2} is infinite. For example, we can take k to be a number field.

Remarque 2.1. For any finite extension L/k, the group L^*/L^{*2} is infinite. Indeed, according to Kummer theory, the kernel of

$$k^*/k^{*2} \to L^*/L^{*2}$$

parameterizes the intermediate quadratic extensions of L/k, of which there are a finite number since L/k is separable.

We denote by $\bar{k}$ an algebraic closure of k. In what follows, we denote $\Lambda = \mathbf{Z}/2\mathbf{Z}$.

We start by considering the plane $\mathbf{A}^2 = \mathrm{Spec}(k[x, y])$ minus the irreducible curves that do not intersect the line $\Delta = \mathrm{Spec}(k[x])$ with equation $y = 0$. These curves are exactly the irreducible curves with equation $u(1 + yg(x, y)), u \in k^*$. We therefore set

$$A_0 = (1 + yk[x, y])^{-1}k[x, y].$$

The localization morphism $k[x, y] \to A_0$ identifies $\mathrm{Spec}(A_0)$ with the desired subset of the plane $\mathbf{A}^2 = \mathrm{Spec}(k[x, y])$. The points of $\mathrm{Spec}(A_0)$ are of three types

- The generic point of $\mathbf{A}^2$;
- The generic points of the irreducible curves in the plane that meet Δ ;
- The points of $\mathrm{Spec}(A_0)$ that are closed in $\mathbf{A}^2$ (which are the closed points of Δ, as we will see).

Note that a generic point of a curve C that intersects Δ specializes in $\mathrm{Spec}(A_0)$ to any closed point of $C \cap \Delta$ and thus is not closed in $\mathrm{Spec}(A_0)$.

Furthermore, a point in $\mathrm{Specmax}(A_0)$ is thus defined by $(\bar{x}, \bar{y}) \in \bar{k}$. If $\bar{y}$ is nonzero, being algebraic over k, its inverse is in $k[\bar{y}]$, which implies $(\bar{x}, \bar{y}) \notin \mathrm{Spec}(A_0)(\bar{k})$. The closed immersion

$$\mathrm{Spec}(k[x]) = \mathrm{Spec}(A_0/yA_0) \hookrightarrow \mathrm{Spec}(A_0)$$

thus induces a homeomorphism $\mathrm{Specmax}(k[x]) \xrightarrow{\sim} \mathrm{Specmax}(A_0)$.

If $\xi \in \mathrm{Specmax}(A_0)$, we denote by $\pi_\xi \in k[x]$ the monic generator of the polynomials vanishing at ξ, and we choose a root of π_ξ in $\bar{k}$ defining a geometric point $\bar{\xi}$ over ξ. We see it as an element of A_0

via the tautological embedding $k[x] \hookrightarrow A_0$. The pair (π_ξ, y) is a system of local coordinates of A_0 at ξ, *i.e.*, we have an isomorphism [(i)]

$$k(\xi)\{\pi_\xi, y\} \xrightarrow{\sim} A^{\mathrm{h}}_{0,\xi}. \tag{2.1.1}$$

Since k is countable, $\operatorname{Specmax}(k[x])$ is countable. We denote by $[i], i \geq 0$ the sequence of its points, which can also be seen as the sequence of maximal ideals of A_0. We then denote by $P(i)$ the image of $P \in k[X]$ in $k([i])$.

Let us start with an elementary Bertini-type lemma.

Lemme 2.2. *Let $P, Q \in k[X]$. Assume $P' \neq 0$ or $Q' \neq 0$ and* $\operatorname{GCD}(P, Q) = 1$. *Then, for all but a finite number of $t \in k$, $P + tQ$ is separable.*

Démonstration. Since $\operatorname{GCD}(P, Q) = 1$, the linear system (P, Q) is base-point free and defines a morphism

$$\mathbf{A}^1_k \xrightarrow{(P:Q)} \mathbf{P}^1_k.$$

The point $(T : 1) \in \mathbf{P}^1_k(k(T))$ is generic, so that the geometric fiber

$$F_\eta \subset \mathbf{A}^1_{\overline{k(T)}}$$

has the equation

$$P(X) - TQ(X) = 0.$$

The polynomial in T

$$P(X) - TQ(X)$$

is primitive ($\operatorname{GCD}(P, Q) = 1$) and of degree 1. It is therefore irreducible in $k[X, T] = k[T][X]$ and thus in $k(T)[X]$. Furthermore, $P'(X) - TQ'(X)$ is not zero. Otherwise, P' would be zero and so would Q', which is not the case. Thus, the irreducible polynomial $P(X) - TQ(X) \in k(T)[X]$ is coprime to its derivative, which ensures the smoothness of F_η. We conclude by the theorem of generic smoothness. □

Lemme 2.3. *There exist sequences $\xi_n \in$* $\operatorname{Specmax}(k[X])$, *$g_n \in k[X]$ such that*

(i) *the g_n are pairwise coprime;*
(ii) *ξ_n is a zero of multiplicity 2 of g_n;*
(iii) *the other zeros of g_n are simple;*
(iv) *for all $i \leq n$, $g_n(i) \neq 0$ and*

$$(g_n(i) \bmod k([i])^{*2}) \notin \mathbf{F}_2\langle (g_j(i) \bmod k([i])^{*2}), j < n \mid g_j(i) \neq 0\rangle \subset k([i])^*/k([i])^{*2}.$$

Démonstration. Assume ξ_i, $g_i, i < n$ have been constructed (an empty condition if $n = 0$). Choose $\xi_n \in \operatorname{Specmax}(k[X])$ different from the zeros of $g_m, m < n$ and from the $[i], i \leq n$.

For all $i \leq n$, choose a polynomial P_i such that

$$P_i(i) \neq 0 \text{ and } (P_i(i) \bmod k([i])^{*2}) \notin \mathbf{F}_2\langle g_j(i) \mid g_j(i) \neq 0, j < n\rangle$$

which is possible by (2.1). Set $P_i = 1$ if $i > n$. Let $V \subset \operatorname{Specmax}(k[X])$ be the set of zeros of the $g_m, m < n$.

Then choose $\tilde{g}_n$ such that

$$\tilde{g}_n \equiv \pi^2_{\xi_n} \bmod \pi^3_{\xi_n} \text{ and } \tilde{g}_n \equiv P_i \bmod [i] \text{ if } i \leq n \text{ or if } [i] \in V.$$

By construction, $(\tilde{g}_n, \xi_n)$ satisfies all the required properties except the third one. Let

$$P = \tilde{g}_n \pi^{-2}_{\xi_n} \text{ and } Q = \pi_{\xi_n} \prod_{j<n} \tilde{g}_j \prod_{j \leq n} \pi_{[j]}.$$

Since Q vanishes to order 1 at ξ_n, its derivative is not identically zero, so that by (2.2) we can choose $t \in k$ such that

$$P + tQ$$

[(i)] We should rather say that the morphism $k[X, Y]_{(0,0)} \to A_{0,\xi}$ that sends X to π_ξ and Y to y induces a unique isomorphism $k(\xi)\{\pi_\xi, y\} \xrightarrow{\sim} A^{\mathrm{h}}_{0,\xi}$.

is separable. By construction,

$$(g_n = \pi_{\xi_n}^2(P + tQ),\ \xi_n)$$

satisfies the properties required in the lemma. □

We then define

$$A_{n+1} = A_n[z_n]/(z_n^2 - y - g_n) \text{ and } A = \operatorname{colim} A_n.$$

Let $n \in \mathbf{N}$.

By construction, $\mathrm{Spec}(A_n)$ is an integral regular scheme of dimension 2, finite over $\mathrm{Spec}(A_0)$. Since the extension of integral domains $A_n \hookrightarrow A$ is integral, we can choose a geometric point $\overline{\xi_{n,\infty}}$ of $\mathrm{Spec}(A)$ over ξ_n. It thus defines geometric points $\overline{\xi_n}$ over ξ_n.

By construction, the inclusion $A_{n+1} \hookrightarrow A$ defines an isomorphism (cf. note (i))

$$\bar{k}\{\pi_{\xi_n}, y, z_n\}/(z_n^2 - y - g_n) \xrightarrow{\sim} A^{\mathrm{hs}}_{\overline{\xi_{n,\infty}}}$$

compatible with (2.1.1), *i.e.*, such that the diagram

$$\begin{tikzcd}
\overline{k}\{\pi_{\xi_n}, y, z_n\}/(z_n^2 - y - g_n) \arrow[r, "\sim"] & A^{\mathrm{hs}}_{\overline{\xi_{n,\infty}}} \\
\overline{k}\{\pi_{\xi_n}, y\} \arrow[u] \arrow[r, "\sim"] & A^{\mathrm{hs}}_{0,\overline{\xi_n}} \arrow[u]
\end{tikzcd}$$

commutes (recall

that strict henselization commutes with filtered direct limits, cf. [**ÉGA** IV$_4$ 18.8.18]).

Lemme 2.4. *The divisor $D = V(y)$ of the surface* Spec(A) *is integral.*

Démonstration. Let us show that this is already true for the divisors D_n of $\mathrm{Spec}(A_n)$. The fiber of $D_n \to \Delta$ over $[0]$ is defined by the equations

$$z_i^2 = g_i(0),\ i \leq n$$

in $\mathbf{A}^n_{k[0]}$. It is the spectrum of a field because the $g_i(0), i \leq n$ are non-zero and linearly independent mod $k([0])^{*2}$, which allows us to invoke Kummer theory. If we now had two components in D_n, they would project onto Δ (by properness and flatness) and so the fiber over $[0]$ would not be reduced. □

With these preparations, we can state the main result.

Proposition 2.5. *Let j be the open immersion* $\mathrm{Spec}(A[1/y]) \hookrightarrow \mathrm{Spec}(A)$ *and let η be the generic point of $D = V(y)$.*

(i) *A is Noetherian.*
(ii) *For all n, the dimension (over Λ) of* $(\mathrm{R}^1 j_\star\Lambda)_{\overline{\xi_{n,\infty}}}$ *is* 2, *whereas the dimension of* $(\mathrm{R}^1 j_\star\Lambda)_{\bar{\eta}}$ *is* 1.
(iii) *In particular,* $\mathrm{R}^1 j_\star\Lambda$ *is not constructible.*

Remarque 2.6. Note that the divisor (hence integral) $D = V(y)$ of the regular surface $\mathrm{Spec}(A)$ has each $\xi_{n,\infty}$ as an (ordinary) double point. It is therefore not quasi-excellent since its regular locus (or normal locus, which is the same thing here) is not open. We then obtain a counterexample to constructibility with a regular (though admittedly not excellent) ambient scheme!

Point (iii) follows immediately from points (i) and (ii). The rest of the lecture is dedicated to proving points (i) and (ii), the only points remaining to be shown.

3. Noetherianity of A

We will adapt (cf. proposition 3.4) to the situation (by using it) the usual criterion for the Noetherianity of direct limits, which we recall :

Théorème 3.1 ([**ÉGA** 0_{III} 10.3.1.3]). *Let $(A_i, \mathfrak{m}_i)$ be a filtered direct system of Noetherian local rings. We assume that all the A_i are Noetherian and that the transition morphisms are local and flat. Then, if for all $i \leq j$, we have $\mathfrak{m}_i A_j = \mathfrak{m}_j$, then* $\operatorname{colim} A_i$ *is Noetherian.*

We will use without further mention Cohen's criterion for Noetherianity ([**Nagata, 1962**, 3.4]) :

Proposition 3.2 (Cohen). *A ring is Noetherian if and only if every prime ideal is finitely generated.*

Let $A_i, i \geq 0$ be a direct system of rings and $A_\infty = \operatorname{colim} A_i$. We assume

- the morphisms $A_i \to A_{i+1}$ are finite and injective ;
- each A_i is Noetherian (or, what amounts to the same thing, that A_0 is Noetherian).

In particular, $\operatorname{Spec}(A_{i+1}) \to \operatorname{Spec}(A_i)$ is finite and surjective and $\operatorname{Spec}(A_\infty) \to \operatorname{Spec}(A_i)$ is integral and surjective for all i, which we will use without further caution. Their fibers have dimension zero. For $\mathfrak{p} \in \operatorname{Spec}(A_0)$, we denote by $\tilde{\star}_{\mathfrak{p}}$ the property

Property $\tilde{\star}_{\mathfrak{p}}$: *There exists an i such that for all $j \geq i$ and all $\mathfrak{q} \in \operatorname{Spec}(A_j)$ over $\mathfrak{p}$, the ideal $\mathfrak{q}A_\infty$ is prime.*

Proposition 3.3. *A_∞ is Noetherian if and only if every prime ideal $\mathfrak{p}$ of A_0 satisfies property $\tilde{\star}_{\mathfrak{p}}$.*

Démonstration. We denote $f : \operatorname{Spec}(A_\infty) \to \operatorname{Spec}(A_0)$. Sufficiency. Let $\mathfrak{q}_\infty \in \operatorname{Spec}(A_\infty)$ and let $\mathfrak{p}$ be its image in $\operatorname{Spec}(A_0)$. Let us show that $\mathfrak{q}_\infty$ is finitely generated. Choose i as in $\tilde{\star}_{\mathfrak{p}}$ and let $\mathfrak{q} = \mathfrak{q}_\infty \cap A_i$. We have on the one hand $\mathfrak{q}A_\infty \subset \mathfrak{q}_\infty$ and, on the other hand

$$\mathfrak{q}_\infty \cap A_i \subset \mathfrak{q}A_\infty \subset \mathfrak{q}_\infty$$

which ensures the equality $\mathfrak{p} = \mathfrak{q}_\infty \cap A_0 = \mathfrak{q}A_\infty \cap A_0$, so that $\mathfrak{q}A_\infty$ specializes to $\mathfrak{q}_\infty$ in $f^{-1}(\mathfrak{p})$ which has dimension 0. We thus have $\mathfrak{q}_\infty = \mathfrak{q}A_\infty$, which proves that $\mathfrak{q}_\infty$ is finitely generated like $\mathfrak{q}$, and we invoke 3.2.

Necessity. Assume A_∞ is Noetherian and let $\mathfrak{p} \in \operatorname{Spec}(A_0)$. The fiber

$$f^{-1}(\mathfrak{p}) = \operatorname{Spec}(A_\infty \otimes_{A_0} \kappa(\mathfrak{p}))$$

is Noetherian of dimension zero, hence of finite cardinality. Since A_∞ is Noetherian, we can thus assume that all prime ideals of $f^{-1}(\mathfrak{p})$ are generated by elements of A_i for a suitable i. Let then $\mathfrak{q} \in \operatorname{Spec}(A_j)$, $j \geq i$ be over $\mathfrak{p}$. Let $\mathfrak{q}' \in \operatorname{Spec}(A_\infty)$ be over $\mathfrak{q}$. Since $\mathfrak{q}'$ is generated by $\mathfrak{q}' \cap A_i$, it is generated by $\mathfrak{q}' \cap A_j = \mathfrak{q}$, so that $\mathfrak{q}A_\infty = \mathfrak{q}'$, which is therefore prime. □

Proposition 3.4. *We keep the hypotheses and notations of 3.3. If, moreover, the extensions A_{i+1}/A_i are* flat, *then A_∞ is Noetherian if and only if every* maximal *ideal $\mathfrak{m}$ of A_0 satisfies property $\tilde{\star}_{\mathfrak{m}}$.*

Démonstration. The necessity follows from 3.3. It is therefore sufficient to prove the sufficiency. Assume then that every *maximal* ideal $\mathfrak{m}$ of A_0 satisfies property $\tilde{\star}_{\mathfrak{m}}$. Let then $\mathfrak{p} \in \operatorname{Spec}(A_0)$ and let us show that $\mathfrak{p}$ satisfies $\tilde{\star}_{\mathfrak{p}}$.

Lemme 3.5. *Under the conditions of the proposition, property $\tilde{\star}_{\mathfrak{p}}$ is equivalent to the property*
Property $\star_{\mathfrak{p}}$: There exists an i such that for all $l \geq j \geq i$ and all $\mathfrak{q} \in \operatorname{Spec}(A_j)$ over $\mathfrak{p}$, the ideal $\mathfrak{q}A_l$ is prime.

Démonstration. Assume $\star_{\mathfrak{p}}$ holds. Let then $\mathfrak{q} \in \operatorname{Spec}(A_j)$ over $\mathfrak{p}$. We already deduce $1 \notin \mathfrak{q}A_\infty$. Moreover, if $xy \in \mathfrak{q}A_\infty$, there exists $l \geq j$ such that $x, y \in A_l$. By choosing a larger l if necessary, we can also assume $xy \in \mathfrak{q}A_l$ and thus for example $x \in \mathfrak{q}A_l \subset \mathfrak{q}A_\infty$. We thus have $\star_{\mathfrak{p}} \Rightarrow \tilde{\star}_{\mathfrak{p}}$ (without the flatness hypothesis). The other implication follows directly from the equality $\mathfrak{q}A_\infty \cap A_l = \mathfrak{q}A_l$ (faithful flatness). □

The key is to note that the condition $\star_{\mathfrak{p}}$ depends only on the schematic fibers of $f_i : \operatorname{Spec}(A_i) \to \operatorname{Spec}(A_0)$ and is therefore *invariant under localization*, which will allow us to reduce to the local case to apply (3.1). Let us be more precise.

Lemme 3.6. *Let $\mathfrak{p} \in \mathrm{Spec}(A_0)$. The following two properties are equivalent.*

- *Property $\star_{\mathfrak{p}}$ is satisfied.*
- *There exists an i such that for all $l \geq j \geq i$ the induced morphism*

$$\phi : f_l^{-1}(\mathfrak{p}) \to f_j^{-1}(\mathfrak{p})$$

between the schematic fibers is bijective with reduced fibers.

Démonstration. Assume $\star_{\mathfrak{p}}$ holds and choose $i \leq j \leq l$ as in $\star_{\mathfrak{p}}$. Let $\mathfrak{q} \in f_j^{-1}(\mathfrak{p})$ and set $A = A_j/\mathfrak{q}$ and $B = A_l/\mathfrak{q}A_l$. The schematic fiber $\phi^{-1}(\mathfrak{q})$ is the $k(\mathfrak{q}) = \mathrm{Frac}(A)$-scheme $\mathrm{Spec}(B \otimes_A \mathrm{Frac}(A))$. Since $\mathfrak{q}A_l$ is prime, B is an integral domain and so is $B \otimes_A \mathrm{Frac}(A)$ (tensoring by $\mathrm{Frac}(A)$ is a localization). Since A_l/A_j is finite, the extension B/A is finite, so that $B \otimes_A \mathrm{Frac}(A)$ is both finite-dimensional over $\mathrm{Frac}(A)$ and an integral domain, hence it is a field, which implies the second condition.

Conversely, assume that $\phi^{-1}(\mathfrak{q})$ is the spectrum of a field. In other words, we assume that $B \otimes_A \mathrm{Frac}(A)$ is an integral domain and we want to show that B is an integral domain. But we have

Sous-lemme 3.7. *Let $\phi : A \to B$ be a flat morphism of rings. Assume A is an integral domain. Then, B is an integral domain if and only if the generic fiber $B \otimes_A \mathrm{Frac}(A)$ is an integral domain.*

Démonstration. By tensoring the inclusion $A \hookrightarrow \mathrm{Frac}(A)$ by B, we obtain (by flatness) that the tautological morphism

$$B \to B \otimes_A \mathrm{Frac}(A) = (A - \{0\})^{-1}B$$

is injective.

Assume B is an integral domain. Since B is non-zero, the same is true for $(A - \{0\})^{-1}B$. Moreover, if, with obvious notations, $(b/a)(b'/a') = 0$ in $(A - \{0\})^{-1}B$, there exists $\alpha \in A - \{0\}$ such that $\alpha bb' = 0$ (in B). Thus, αb or b' is zero, and so $b/1$ or $b'/1$ is zero in $(A - \{0\})^{-1}B$.

Conversely, if $B \otimes_A \mathrm{Frac}(A)$ is an integral domain, so is B as a subring. □

□

Let us finish the proof of proposition 3.4 by reducing to the local case. Choose a maximal ideal $\mathfrak{m}$ in A_0 containing $\mathfrak{p}$. According to (3.5), $\mathfrak{m}$ satisfies property $\star_{\mathfrak{m}}$. Then choose i as in $\star_{\mathfrak{m}}$. Let $\mathfrak{m}_i \in \mathrm{Specmax}(A_i)$ be over $\mathfrak{m}$ ($\mathrm{Spec}(A_i) \to \mathrm{Spec}(A_0)$ is finite and surjective).

By construction, the ideal $\mathfrak{m}_iA_j$ is prime for all $j \geq i$. But we have $\mathfrak{m}_iA_j \cap A_i = \mathfrak{m}_i$ (A_j/A_i is faithfully flat) so that $\mathfrak{m}_iA_j$ is maximal (A_j/A_i is finite) and defines by localization the maximal ideal of $A_{j,\mathfrak{m}_i}$. We thus have $A_{j,\mathfrak{m}_i} = A_{j,\mathfrak{m}_iA_j}$ so that we can apply (3.4) and deduce that

$$A_{\mathfrak{m}_i} = \mathrm{colim}_{j \geq i} A_{j,\mathfrak{m}_i}$$

is a Noetherian ring. In other words, the localizations $A_{\mathfrak{m}_i}$ of $A_{\mathfrak{m}}$ at $\mathfrak{m}_i \in \mathrm{Specmax}(A_{i,\mathfrak{m}})$ are Noetherian. But $A_{i,\mathfrak{m}}$ is semilocal (because it is finite over $A_{0,\mathfrak{m}}$, which is local), so $A_{\mathfrak{m}}$ is Noetherian (exercise). Let us then set

$$A'_j = A_{j,\mathfrak{m}} \text{ and } \mathfrak{p}' = \mathfrak{p}A_{0,\mathfrak{m}}$$

According to (3.3), (3.5) and (3.6), up to changing i, for all $l \geq j \geq i$, the morphism between the schematic fibers $\phi' : f_l^{-1}(\mathfrak{p}') \to f_j^{-1}(\mathfrak{p}')$ is bijective with reduced fibers. Since Φ' is identified with Φ, by (3.3), (3.5) and (3.6) we deduce that A is Noetherian, which completes the proof of 3.4. □

In the situation of proposition 2.5, the maximal ideals $\mathfrak{m}$ of A_0 satisfy $\star_{\mathfrak{m}}$ by construction (this is where the condition of linear independence of the $g_n(i) \bmod k([i])^{*2}$ is fully used), which completes the proof of point (i) of *loc. cit.*

4. Study of the Double Points

It remains to prove point (ii) of proposition 2.5.

The restriction of the closed immersion $D = V(y) \hookrightarrow \mathrm{Spec}(A)$ to $\mathrm{Spec}(\mathscr{O}_\eta)$ being an immersion of a regular divisor into a regular scheme, the purity theorem (**XVI**-3.1.4) ensures that the dimension of $(\mathrm{R}^1 j_\star\Lambda)_{\bar\eta}$ is 1.

To simplify the notation, we set $x = \pi_{\xi_n}, z_n = z, g = g_n$ and $R = \bar{k}\{x, y, z\}/(z^2 - y - g)$, and we recall the expression

$$g = x^2 u(x)$$

where $u(x) \in \bar{k}\{x\}$ can be assumed to satisfy $u(0) = 1$. There exists a unique square root $\sqrt{u(x)} \in \bar{k}\{x\}$ of $u(x)$ such that $\sqrt{u(x)}(0) = 1$, which defines a local coordinate $X = x\sqrt{u(x)}$ of $\bar{k}\{x\}$ (recall that the characteristic of k is different from 2). In these new coordinates X, y, z of $\bar{k}\{x, y, z\}$, we have

$$z^2 - y - g = z^2 - y - X^2$$

and we invoke (**XVI**-3.1.4) again to conclude [(ii)]. This completes the proof of proposition 2.5.

5. *D* is locally but not globally a normal crossings divisor

Let us start with a definition. In this section D denotes an effective divisor on a regular scheme X and $j : U = X - D \hookrightarrow X$ is the open immersion of the complement.

Définition 5.1. We keep the preceding notations.

- We say that D is a **locally normal crossings divisor** (abbreviated as *locally ncd*) if for all $x \in D$, the Zariski localization $\mathrm{Spec}(\mathscr{O}_{D,x})$ is a normal crossings divisor of $\mathrm{Spec}(\mathscr{O}_{X,x})$.
- Assume D is locally ncd. We denote by $\varepsilon(x)$, $x \in D$ the number of analytic branches of the strict henselization $D_{(\bar{x})}$, where $\bar{x}$ is a geometric point over x, and by $\zeta(x)$ its number of irreducible components. The function $\varepsilon : x \mapsto \varepsilon(x)$ (resp. $\zeta : x \mapsto \zeta(x)$) is called the analytic counting function (resp. Zariski counting function).

With the preceding notations, if $\bar{x}$ is a geometric point over $x \in D$ with D a locally normal crossings divisor, the strict henselization $D_{(\bar{x})}$ is a strict normal crossings divisor of $X_{(\bar{x})}$. We then have the following characterization :

Lemme 5.2. *With the preceding notations, assume moreover that D is locally ncd and $\Lambda = \mathbf{Z}/\ell\mathbf{Z}$ with ℓ a prime number invertible on X. Then, the following propositions are equivalent.*

- $\mathrm{R}^1 j_\star\Lambda$ *is constructible ;*
- $\mathrm{R}^p j_\star\Lambda$ *is constructible for all* p *;*
- *the analytic counting function* ε *is constructible.*

Démonstration. According to the purity theorem (**XVI**-3.1.4), the fiber $(\mathrm{R}j_\star\Lambda)_{\bar{x}}$ is the exterior algebra over

$$(\mathrm{R}^1 j_\star\Lambda)_{\bar{x}} = \Lambda^{\varepsilon(x)}.$$

The lemma follows immediately from the characterization of constructible sheaves with finite fibers ([**SGA 4** IX prop. 2.13 (iii)]). ☐

The interest of this lemma lies in the following proposition.

Proposition 5.3. *With the preceding notations, assume moreover that D is locally ncd. Then, ε is constructible if and only if D is a normal crossings divisor.*

(ii) One can avoid resorting to the general purity theorem by using the Kummer exact sequence to restrict to calculating the $H^1(-, \Lambda)$ of the complement of a strict normal crossings divisor in the spectrum of a regular local ring.

Démonstration. The constructibility of ε if D has normal crossings follows directly from the definitions (cf. [**de Jong, 1996**]) . Assume then that ε is constructible and let us show that D has normal crossings. Let $\bar{x}$ be a geometric point over $x \in D$. Since $D_{(\bar{x})}$ is a strict normal crossings divisor, there exists an étale neighborhood $\pi : X' \to X$ of $\bar{x}$ in X such that the divisor $D' = \pi^{-1}(D)$ is the sum of divisors D'_i that are regular at x' (the image of $\bar{x}$ in X') and that intersect transversally at x'. The analytic counting function ε' of D' is the sum of the analytic counting functions ε'_i. Since ε' only depends on the strict henselization, we thus have

$$\varepsilon' = \varepsilon \circ \pi = \sum \varepsilon'_i.$$

In particular, ε' is constructible like ε. The Zariski counting function ζ' of D' is certainly constructible, so that the difference $\varepsilon' - \zeta'$ is as well. By hypothesis, $\varepsilon' - \zeta'$ vanishes on Spec $\mathscr{O}_{X',x'}$, hence on the set of generizations of x'. Since it is constructible, it is zero on an open (Zariski) neighborhood U' of x'. Since $\varepsilon'_i \geq \zeta'_i$, we have $\varepsilon'_i = \zeta'_i$ on U', so that, up to shrinking U', each divisor D_i is regular on U'. By restricting to the strict localization at each point of U', on which we know that D' is a normal crossings divisor, we obtain that the D_i intersect transversally, so that the restriction of D' to U' is a strict normal crossings divisor. □

Remarque 5.4. The preceding argument applied to ζ ensures that if the Zariski localization of D at every point is a strict normal crossings divisor then D is a strict normal crossings divisor.

With the notations of proposition 2.5, we have thus obtained the following result.

Corollaire 5.5. *The divisor D of the regular surface* Spec A *is locally with normal crossings but not globally.*

TALK XX

Rigidity

Yves Laszlo et Alban Moreau

1. Introduction

The goal of this talk is to prove the two technical results 2.1.1 (comparison of torsors over the complementary open set $\operatorname{Spec} A - V(I)$ defined by a Henselian pair that is not necessarily Noetherian and the corresponding open set of $\operatorname{Spec} \widehat{A}$ where A denotes the I-adic completion of A) and 5.3.1 (rigidity of ramification). They will allow us in the next talk to prove the following finiteness statement (**XXI**-1.4) :

Théorème. *Let A be a strictly local ring of dimension 2. We assume that A is normal, excellent, and we denote by $X' = \operatorname{Spec}(A) - \{\mathfrak{m}_A\}$ its punctured spectrum. Then, for any finite group G, the set $\mathrm{H}^1(X', G)$ is finite.*

This result is the key to proving the following general finiteness result (**XXI**-1.2) :

Théorème. *Let $f : Y \to X$ be a morphism of finite type between quasi-excellent schemes. Let $\mathbb{L}$ be a set of prime numbers invertible on X. For any constructible sheaf of groups F on $Y_{\text{ét}}$ with $\mathbb{L}$-torsion, the sheaf $\mathrm{R}^1 f_\star(F)$ on $X_{\text{ét}}$ is constructible.*

By ultrafilter techniques, dear to model theorists, we are led to study étale coverings of punctured spectra of *non-Noetherian* rings, which explains why we are forced to prove the technical statements outside of any Noetherian framework.

Remarque. Let X be a scheme. We will consider stacks in groupoids $\mathscr{C}$ on $X_{\text{ét}}$ (we will simply say stacks). In general, the fibered category $\mathscr{C}$ is not split, so that if x, y are two objects of $\mathscr{C}(S)$ where $S \to X$ is étale, some precautions are needed to speak of the sheaf $\underline{\mathrm{Hom}}(x, y)$ on $S_{\text{ét}}$. Precisely, following [**Giraud, 1971**, I.2.6.3.1], we consider the equivalence of fibered categories $\mathscr{C} \to \mathfrak{L}\mathscr{C}$ between $\mathscr{C}$ and the free category $\mathfrak{L}\mathscr{C}$ generated by $\mathscr{C}$, which free category is split. We then define

$$\underline{\mathrm{Hom}}(x, y)(S') = \mathrm{Hom}_{\mathfrak{L}\mathscr{C}(S')}(\mathfrak{L}x', \mathfrak{L}y')$$

where $\mathfrak{L}x', \mathfrak{L}y'$ are the inverse images by the étale morphism $S' \to S$ of $\mathfrak{L}x, \mathfrak{L}y$ in $\mathfrak{L}\mathscr{C}(S')$. Of course ([**Giraud, 1971**, I.2.6.3.2 (1)]), $\mathfrak{L}$ induces a bijection

$$\mathrm{Hom}_{\mathscr{C}(S)}(x, y) \xrightarrow{\sim} \mathrm{H}^0(S, \underline{\mathrm{Hom}}(x, y)).$$

These remarks justify that we can if necessary assume without loss of generality that the stacks we consider are split.

2. Rigidity lemma

Let (A, I) be a Henselian pair (**V**-1.2.1 or [**ÉGA** IV$_4$ 18.5.5]) not necessarily Noetherian, *with I of finite type*[(i)]. Let U be an open subset of $X = \operatorname{Spec}(A)$ containing $\operatorname{Spec}(A) - V(I)$. We denote by $\widehat{A}$ the I-adic completion[(ii)] of A and by $\widehat{U}$ the inverse image of U by the completion morphism $\pi : \widehat{X} = \operatorname{Spec}(\widehat{A}) \to X$. We assume for simplicity that U is quasi-compact (cf. 2.1.4).

(i) This hypothesis will be used to compare the I-adic graded rings of A and of its completion $\widehat{A}$ ([**Bourbaki**, **AC**, III, §2, n$^{\text{o}}$ 12])

(ii) We will simply say completion for separated completion.

2.1. Statements. Recall from [**SGA 4** IX 1.5] that a sheaf of groups $\mathscr{F}$ on X is ind-finite if for any étale open subset $u\,:\,U \to X$ with U quasi-compact, the group $\mathscr{F}(u)$ is a filtered inductive limit of its finite subgroups. We then say that a stack in groupoids $\mathscr{C}$ on X is ind-finite if for any étale open subset $u\,:\,U \to X$ with U quasi-compact and any $x_u \in \mathscr{C}(u)$, the sheaf of groups $\pi_1(\mathscr{C}, x_u) = \mathrm{Aut}_{\mathscr{C}}(x_u)$ is ind-finite.

The goal of this section is to prove the following rigidity theorem.

Théorème 2.1.1 (Gabber's rigidity theorem). *Let $\mathscr{F}$ be a sheaf of sets on $U_{\text{ét}}$. Then we have*

(i) *the natural arrow $\mathrm{H}^0(U, \mathscr{F}) \to \mathrm{H}^0(\widehat{U}, \pi^\star \mathscr{F})$ is bijective;*

(ii) *if $\mathscr{F}$ is moreover an ind-finite sheaf of groups, the natural arrow $\mathrm{H}^1(U, \mathscr{F}) \to \mathrm{H}^1(\widehat{U}, \pi^\star \mathscr{F})$ is bijective.*

The two statements of the preceding theorem are a consequence of the following theorem, which is apparently stronger, the stacky form of the rigidity theorem[(iii)].

Théorème 2.1.2 (Gabber's rigidity theorem, stacky form). *Let $\mathscr{C}$ be an ind-finite stack in groupoids on $U_{\text{ét}}$. Then, the natural arrow $\gamma(\mathscr{C})\,:\,\Gamma(U, \mathscr{C}) \to \Gamma(\widehat{U}, \pi^\star \mathscr{C})$ is an equivalence.*

Remarque 2.1.3. In fact, the rigidity theorem 2.1.1 is *a priori* equivalent to the stacky version 2.1.2. This is what emerges, for example, from the statement 6.3.2. But, formally, we do not need to prove this at this stage.

Remarque 2.1.4. The preceding results are also valid when U is not necessarily quasi-compact. This follows from the fact that the category of sections of a stack on U is equivalent to the 2-projective limit of the sections over the quasi-compact open subsets of U containing $\mathrm{Spec}(A) - V(I)$. The quasi-compactness hypothesis is used in a blow-up argument below (cf. 2.4.2).

2.2. Reduction to the constant case. The result is as follows

Proposition 2.2.1. *Suppose that for any U as above,*

(i) *for any finite set F, the arrow $\mathrm{H}^0(U, F) \to \mathrm{H}^0(\widehat{U}, F)$ is bijective. Then, 2.1.1 (i) is true, that is, the rigidity theorem 2.1.2 is true for discrete stacks.*

(ii) *for any finite group G, the arrow $\mathrm{Tors}(U, G) \to \mathrm{Tors}(\widehat{U}, G)$ is an equivalence and 2.1.1 (i) is true. Then the rigidity theorem 2.1.2 is true.*

Démonstration. According to [**SGA 4** XII prop. 6.5], to prove 2.1.1 (i) (resp. 2.1.2), it suffices to prove that for any finite $U' \to U$ and any finite set F (resp. finite group G), the arrow

$$\mathrm{H}^0(U', F) \to \mathrm{H}^0(\widehat{U'}, F) \text{ (resp. } \mathrm{Tors}(U', G) \to \mathrm{Tors}(\widehat{U'}, G)) \tag{2.2.1.1}$$

is bijective (resp. an equivalence) where $\widehat{U'} = \widehat{U} \times_U U'$.

Lemme 2.2.2. *There exists an affine scheme $\mathrm{Spec}(B)$ and a Cartesian diagram*

$$\begin{tikzcd} U' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathrm{Spec}(B) \arrow[d] \\ U \arrow[r] & \mathrm{Spec}(A) \end{tikzcd}$$

where B is finite over A. The morphism $U' \to \mathrm{Spec}(B)$ is identified with the open immersion $U_B \hookrightarrow \mathrm{Spec}(B)$. Moreover, U_B contains $\mathrm{Spec}(B) - V(IB)$.

Démonstration. Since $U' \to U$ is finite, it is projective ([**ÉGA** II 6.1.11]). Since U is quasi-compact, the open immersion $U \hookrightarrow X$ is quasi-affine ([**ÉGA** II 5.1.1]), hence quasi-projective, so that the composition $f\,:\,U' \to U \to X$ is quasi-projective ([**ÉGA** II 5.3.4]). Since $X = \mathrm{Spec}(A)$ is affine, $\mathscr{O}_X$ is certainly ample (cf. the definition or [**ÉGA** II 5.1.2]). The hypotheses of Zariski's main theorem ([**ÉGA** IV$_3$ 8.12.8]) are therefore satisfied. There thus exists a finite $X' \to X$ such that f factors as $U' \hookrightarrow X' \to X$

(iii) Discrete (ind-finite) stacks in groupoids are identified with sheaves of sets : we will sometimes say a *discrete stack*.

where $U' \hookrightarrow X'$ is an open immersion and $X' \to X$ is finite. The schematic closure of U' in X' is closed in X' : it can therefore be written as $\mathrm{Spec}(B)$ where B is finite over A. We have thus a commutative diagram

$$\begin{tikzcd} U' \arrow[r] \arrow[dr] & U_B \arrow[r] \arrow[d] & \mathrm{Spec}(B) \arrow[d] \\ & U \arrow[r] & \mathrm{Spec}(A) \end{tikzcd}$$

where the non-horizontal arrows are finite. The arrow $U' \to U_B$ is therefore proper. Since it is also an open immersion with dense image, it is an isomorphism. Since the open set U contains $\mathrm{Spec}(A) - V(I)$, we deduce that $U' = U_B$ contains $\mathrm{Spec}(B) - V(IB) = (\mathrm{Spec}(A) - V(I))_B$. □

According to the lemma, the arrow (2.2.1.1) is identified with

(2.2.2.1) $$\mathrm{H}^0(U_B, F) \to \mathrm{H}^0(\widehat{U}_B, F) \text{ (resp. } \mathrm{Tors}(U_B, G) \to \mathrm{Tors}(\widehat{U}_B, G))$$

(where $?_B$ is the extension of scalars of the A-scheme $?$ to $\mathrm{Spec}(B)$). It is therefore a matter of showing that (2.2.2.1) is bijective (resp. an equivalence).

By definition, we have

$$\widehat{U}_B = \pi_C^{-1}(U)$$

where π_C is the natural projection

$$\pi_C : \mathrm{Spec}(C) \to \mathrm{Spec}(A), \text{ with } C = \widehat{A} \otimes_A B.$$

In the Noetherian case, C is the IB-adic completion $\widehat{B}$ of B, which proves the proposition in this case — apply hypothesis 2.2.1 (i) to constant $\mathscr{F}$ with value F on U_B —. In the general case, the arrow $C \to \widehat{B}$ is not in general an isomorphism.

Lemme 2.2.3. *With the preceding notations, we have*

(i) *Let (A_n, I_n) be a projective system of Henselian pairs. The pair $(A_\infty, I_\infty) = (\varprojlim A_n, \varprojlim I_n)$ is Henselian.*
(ii) *The I-adic completion $\hat{A}$ of A is $\hat{I}$-Henselian.*
(iii) *The pairs (B, IB) and (C, IC) are Henselian and have the same I-adic completion.*

Démonstration. Let P be a polynomial in $A_\infty[x]$ and $\bar{a} \in A_\infty/I_\infty$ be a simple root (that is, such that $P'(\bar{a})$ is invertible in A_∞/I_∞). The image $\bar{a}_n$ of $\bar{a}$ in A_n/I_n is a simple root of P. It therefore lifts uniquely to a root $a_n \in A_n$ of P by Hensel's lemma. Since I_{n+1} maps into I_n, by uniqueness of the lifts, the image of a_{n+1} in A_n is equal to a_n, so that the sequence $a = (a_n) \in A_\infty$ is the sought-after lift of $\bar{a}$, which proves (i) according to [**Crépeaux, 1967**, Prop. 1].

Since $A \to A/I^n$ is notoriously integral, the pairs $(A/I^n, IA/I^n)$ are Henselian, so that (ii) follows from (i).

By associativity of the tensor product, the natural morphism $B/I^nB \to C/I^nC$ is identified with the tensor product by B of the natural morphism $A/I^nA \to \widehat{A}/I^n\widehat{A}$. Since the latter is an isomorphism ([**Bourbaki**, **AC**, III, § 2, nº 12, prop. 15 et cor. 2]), B and C have the same I-adic completion. (iii) then follows from (ii) because a pair that is finite over a Henselian one is Henselian. □

We thus have $\widehat{(U_B)} = \widehat{(U_C)}$. According to the preceding lemma, under the hypotheses of 2.2.1 (i) (resp. (ii)), the natural arrow

$$\mathrm{H}^0(U_B, F) \to \mathrm{H}^0(\widehat{(U_B)}, F) = \mathrm{H}^0(\widehat{(U_C)}, F) \leftarrow \mathrm{H}^0(U_C, F) = \mathrm{H}^0(\widehat{U}_B, F)$$

(resp.

$$\mathrm{Tors}(U_B, G) \to \mathrm{Tors}(\widehat{(U_B)}, G) = \mathrm{Tors}(\widehat{(U_C)}, G) \leftarrow \mathrm{Tors}(U_C, G) = \mathrm{Tors}(\widehat{U}_B, G))$$

is then a bijection (resp. equivalence), which is what we wanted. □

2.3. Reduction to the strictly Henselian case. Let us summarize the notations in the following Cartesian diagram

$$\begin{tikzcd} \widehat{U} \arrow[r, "\pi"] \arrow[d, "\hat{j}"'] \arrow[dr, phantom, "\square"] & U \arrow[d, "j"] \\ \widehat{X} \arrow[r, "\pi"] & X \end{tikzcd}$$

with U quasi-compact containing $\mathrm{Spec}(A) - V(I)$. Let us show the following result.

Proposition 2.3.1. *Suppose that for any U as above,*

(i) *for any finite set F, the arrow $\mathrm{H}^0(U, F) \to \mathrm{H}^0(\widehat{U}, F)$ is bijective if A is moreover strictly local. Then, 2.1.1 (i) is true (whether or not A is strictly local).*
(ii) *for any finite group G, the arrow $\mathrm{Tors}(U, G) \to \mathrm{Tors}(\widehat{U}, G)$ is an equivalence if A is moreover strictly Henselian and 2.1.1 (i) is true. Then, 2.1.2 is true (whether or not A is strictly local).*

Démonstration. Let us start with a lemma.

Lemme 2.3.2. *Suppose that for any U as above,*

(i) *for any finite set F, the arrow $\mathrm{H}^0(U, F) \to \mathrm{H}^0(\widehat{U}, F)$ is bijective if A is moreover strictly local. Then, the base change arrow*

$$\gamma : \pi^\star j_\star F \to \hat{j}_\star \pi^\star F = \hat{j}_\star F$$

is an isomorphism (whether or not A is strictly local).
(ii) *for any finite group G, the arrow $\mathrm{Tors}(U, G) \to \mathrm{Tors}(\widehat{U}, G)$ is an equivalence if A is moreover strictly Henselian and 2.1.1 (i) is true. Then, the base change arrow*

$$\gamma : \pi^\star j_\star \underline{\mathrm{Tors}}(U, G) \to \hat{j}_\star \pi^\star \underline{\mathrm{Tors}}(U, G) = \hat{j}_\star \underline{\mathrm{Tors}}(\widehat{U}, G),$$

*where the equality follows from [**Giraud, 1971**, III.2.1.5.7], is an equivalence (whether or not A is strictly local).*

Démonstration. The formulas $j^\star j_\star = \mathrm{Id}$ and $\hat{j}^\star \hat{j}_\star = \mathrm{Id}$ ensure that we have

$$\hat{j}^\star \pi^\star j_\star = \pi^\star j^\star j_\star = \pi^\star = \hat{j}^\star \hat{j}_\star \pi^\star$$

so that the inverse image on $\widehat{U}$ of the base change arrow

$$\pi^\star j_\star \mathscr{C} \to \hat{j}_\star \pi^\star \mathscr{C} \tag{2.3.2.1}$$

is an equivalence for any stack in groupoids $\mathscr{C}$.

Let $\hat{x}$ be a geometric point of $\widehat{X}$ with image the geometric point $x = \pi \circ \hat{x}$ of X and let us show that the fiber of the base change arrow (2.3.2.1) at $\hat{x}$ is an equivalence. From the preceding, we can assume $\hat{x} \notin \widehat{U}$. In particular, $x \in V(I)$.

Let A^{hs} (resp. $X_{(x)}$) be the strict Henselization of A (resp. X) at x and $\widehat{A}^{\mathrm{hs}}$ (resp. $\widehat{X}_{(\hat{x})}$) that of $\widehat{A}$ (resp. $\widehat{X}$) at $\hat{x}$. We have a commutative diagram where the arrows are the functoriality, completion, or strict Henselization arrows

$$\begin{tikzcd} \widehat{\widehat{X}_{(\hat{x})}} \arrow[r] \arrow[d] & \widehat{X}_{(\hat{x})} \arrow[r] \arrow[d] & \widehat{X} \arrow[d] \\ \widehat{X}_{(\hat{x})} \arrow[r] & X_{(x)} \arrow[r] & X \end{tikzcd}$$

We then denote by

$$\begin{tikzcd}
\widehat{\widehat{U}_{(\hat{x})}} \arrow[r] \arrow[d] & \widehat{U}_{(\hat{x})} \arrow[r] \arrow[d] & \widehat{U} \arrow[d] \\
\widehat{U}_{(\hat{x})} \arrow[r] & U_{(x)} \arrow[r] & U
\end{tikzcd}$$

the inverse image of the diagram by the open immersion $U \to X$. In particular, $U_{(x)}$ (resp. $\widehat{U}_{(\hat{x})}$) denotes the inverse image of the strict Henselization $X_{(x)}$ (resp. $\widehat{X}_{(\hat{x})}$) of X (resp. $\widehat{X}$) at x (resp. $\hat{x}$) by j (resp. $\hat{j}$). Since U is quasi-compact, the same is true for the open sets $U_{(x)}, \widehat{U}_{(\hat{x})}$ of $X_{(x)}, \widehat{X}_{(\hat{x})}$.

The morphisms $j, \hat{j}$ being coherent, in case (i), the fiber $\gamma_{\hat{x}}$ is identified with the natural arrow

$$\mathrm{H}^0(U_{(x)}, F) \to \mathrm{H}^0(\widehat{U}_{(\hat{x})}, F)$$

while in case (ii) it is identified with

$$\mathrm{Tors}(U_{(x)}, G) \to \mathrm{Tors}(\widehat{U}_{(\hat{x})}, G).$$

We deduce that the natural arrows

$$\mathrm{H}^0(U_{(x)}, F) \to \mathrm{H}^0(\widehat{U_{(x)}}, F) \text{ and } \mathrm{H}^0(\widehat{U}_{(\hat{x})}, F) \to \mathrm{H}^0(\widehat{\widehat{U}_{(\hat{x})}}, F)$$

are bijective in case (i) and that the arrows

$$\mathrm{Tors}(U_{(x)}, G) \to \mathrm{Tors}(\widehat{U_{(x)}}, G) \text{ and } \mathrm{Tors}(\widehat{U}_{(\hat{x})}, G) \to \mathrm{Tors}(\widehat{\widehat{U}_{(\hat{x})}}, G)$$

are equivalences in case (ii). It is therefore sufficient to see that the natural arrow

$$\widehat{\widehat{U}_{(\hat{x})}} \to \widehat{U_{(x)}} \tag{2.3.2.2}$$

is an isomorphism, or equivalently that

$$A^{\mathrm{hs}} \text{ and } \widehat{A}^{\mathrm{hs}} \text{ have the same } I\text{-completion.}$$

Since the local ring A^{hs} is Henselian, it is a fortiori I-Henselian (**V**-1.2.1). Using (2.2.3), we see that the I-completion $\widehat{A^{\mathrm{hs}}}$ is Henselian. Since its residue field is that of A^{hs}, it is strictly Henselian. The arrow $\widehat{A} \to \widehat{A^{\mathrm{hs}}}$ thus induces an arrow $\widehat{A}^{\mathrm{hs}} \to \widehat{A^{\mathrm{hs}}}$ and therefore, by I-completion, an arrow

$$\widehat{\widehat{A}^{\mathrm{hs}}} \to \widehat{A^{\mathrm{hs}}}. \tag{$*$}$$

Furthermore, the completion arrow $A \to \widehat{A}$ induces by strict Henselization and then completion an arrow

$$\widehat{A^{\mathrm{hs}}} \to \widehat{\widehat{A}^{\mathrm{hs}}}. \tag{$**$}$$

The arrows $(*)$ and $(**)$ are inverse to each other, hence the lemma. □

We have the commutative diagram with a Cartesian square

$$\begin{tikzcd}
\widehat{U} \arrow[r, "\pi"] \arrow[d, "\hat{j}"'] \arrow[dr, phantom, "\square"] & U \arrow[d, "j"] \\
\widehat{X} \arrow[r, "\pi"] & X \\
X_I = \mathrm{Spec}(A/I) \arrow[u] \arrow[ur] &
\end{tikzcd}$$

As we have observed, the pairs (A, I) and $(\widehat{A}, I\widehat{A})$ are Henselian. The arrow $\mathrm{H}^0(\widehat{X}, \mathscr{C}) \to \mathrm{H}^0(X_I, \mathscr{C}_{|X_I})$ is therefore an equivalence for any ind-finite stack $\mathscr{C}$ on $X_{\text{ét}}$ according to [**Gabber, 1994**, theorem 1′].

We deduce on the one hand

$$\mathrm{H}^0(U, F) = \mathrm{H}^0(X, j_\star F) = \mathrm{H}^0(X_I, (j_\star F)_{|X_I})$$

and, on the other hand

$$\mathrm{H}^0(\widehat{U}, F) = \mathrm{H}^0(\widehat{X}, \hat{j}_\star F) \overset{2.3.2}{=} \mathrm{H}^0(\widehat{X}, \pi^\star j_\star F) = \mathrm{H}^0(X_I, (\pi^\star j_\star F)_{|X_I})$$

the latter being none other than $\mathrm{H}^0(X_I, (j_\star F)_{|X_I})$ (of course the induced isomorphism

$$\mathrm{H}^0(U, F) \xrightarrow{\sim} \mathrm{H}^0(\widehat{U}, F)$$

is the restriction).

Similarly, we have

$$\mathrm{H}^0(U, \underline{\mathrm{Tors}}(U, G)) = \mathrm{H}^0(X, j_\star \underline{\mathrm{Tors}}(U, G)) = \mathrm{H}^0(X_I, j_\star \underline{\mathrm{Tors}}(U, G)_{|X_I})$$

and, on the other hand

$$\begin{aligned} \mathrm{H}^0(\widehat{U}, \underline{\mathrm{Tors}}(\widehat{U}, G)) &= \mathrm{H}^0(\widehat{X}, \hat{j}_\star \underline{\mathrm{Tors}}(\widehat{U}, G)) \\ &\overset{2.3.2}{=} \mathrm{H}^0(\widehat{X}, \pi^\star j_\star \underline{\mathrm{Tors}}(U, G)) \\ &= \mathrm{H}^0(X_I, \pi^\star j_\star \underline{\mathrm{Tors}}(U, G)_{|X_I}) \end{aligned}$$

the latter being none other than $\mathrm{H}^0(X_I, j_\star \underline{\mathrm{Tors}}(U, G)_{|X_I})$), the equivalence of course inducing

$$\mathrm{H}^0(U, \underline{\mathrm{Tors}}(U, G)) \xrightarrow{\sim} \mathrm{H}^0(\widehat{U}, \underline{\mathrm{Tors}}(\widehat{U}, G)).$$

It remains to invoke 2.2.1. □

2.4. End of the proof of 2.1.2. According to 2.3.1, to prove 2.1.2, it suffices to prove the following statement

PROPOSITION 2.4.1. *Suppose A is strictly Henselian (and $I \subset rad(A)$) and let U be as above.*

(i) *for any finite set F, the arrow $\mathrm{H}^0(U, F) \to \mathrm{H}^0(\widehat{U}, F)$ is bijective.*
(ii) *for any finite group G, the arrow $\mathrm{Tors}(U, G) \to \mathrm{Tors}(\widehat{U}, G)$ is an equivalence.*

The formula $\pi^\star \underline{\mathrm{Tors}}(U, G) = \underline{\mathrm{Tors}}(\widehat{U}, G)$ ([**Giraud, 1971**], III.2.1.5.7) allows us to rewrite 2.4.1 in the following form

PROPOSITION 2.4.2. *Suppose A is strictly Henselian (and $I \subset rad(A)$) and let U be as above. Let $\mathscr{C}$ denote the discrete stack F_U or $\underline{\mathrm{Tors}}(U, G)$. Then, the arrow $\mathrm{H}^0(U, \mathscr{C}) \to \mathrm{H}^0(\widehat{U}, \pi^\star \mathscr{C})$ is an equivalence.*

Démonstration. We will reduce by blow-up to the case where the ideal J defining the complement of U is principal.

For any ideal $\tilde{I}$ of a ring $\tilde{A}$, we denote by

$$\mathrm{Écl}_{\tilde{I}}(\tilde{A}) = \mathrm{Proj}(\bigoplus_{n \geq 0} \tilde{I}^n)$$

the blow-up of $\tilde{I}$ in $\mathrm{Spec}(\tilde{A})$. If $\tilde{I}$ is of finite type, the structural morphism $e : \mathrm{Écl}_{\tilde{I}}(\tilde{A}) \to \mathrm{Spec}(\tilde{A})$ is projective, in particular proper.

We thus assume that A is strictly Henselian with residue field k and $\mathscr{F} = F_U$ as above. We have already observed that $\widehat{A}$ is also strictly Henselian. It follows in particular that the set of global sections of any étale sheaf on X or $\widehat{X}$ is identified with its special fiber, which we will use without further precaution.

Since U is quasi-compact, there exists an ideal J of finite type such that $U = \mathrm{Spec}(A) - V(J)$. Since U contains $\mathrm{Spec}\, A - V(I)$ and since I is of finite type, we can assume $I \subset J$. Let

$$Y = \mathrm{Écl}_J(A) \text{ and } Y' = \mathrm{Écl}_J(\widehat{A}).$$

(We should have written $\mathrm{Écl}_{J\widehat{A}}(\widehat{A})$ for $\mathrm{Écl}_J(\widehat{A})$). For coherence reasons, we will simply denote by X' the scheme $\widehat{X} = \mathrm{Spec}(\widehat{A})$ (resp. U' its restriction $\widehat{U} = \pi^{-1}(U)$ to U).

Sous-lemme 2.4.3. *Let n, m be integers ≥ 0. The completion morphism defines isomorphisms*

$$A/I^mJ^n \simeq \widehat{A}/I^mJ^n\widehat{A} \text{ and } A/J^n \simeq \widehat{A}/J^n\widehat{A}$$

inducing an isomorphism

$$J^n/I^mJ^n \simeq J^n\widehat{A}/I^mJ^n\widehat{A}.$$

Démonstration. Since I is of finite type, the completion morphism induces isomorphisms

$$A/I^{m+n} \simeq \widehat{A}/I^{m+n}\widehat{A} \text{ and } A/I^n \simeq \widehat{A}/I^n\widehat{A}$$

according to [**Bourbaki**, **AC**, III, § 2, nº 12, cor. 2 of prop. 16]. But since J contains I, we have

$$I^{m+n} \subset I^mJ^n \text{ and } I^n \subset J^n,$$

so that the base changes

$$A/I^{m+n} \to A/I^mJ^n \text{ and } A/I^n \to A/J^n$$

then give isomorphisms

$$A/I^mJ^n \simeq \widehat{A}/I^mJ^n\widehat{A} \text{ and } A/J^n \simeq \widehat{A}/J^n\widehat{A}$$

which give 2.4.3. □

The natural arrow $Y' \to Y$ is therefore an isomorphism over $\mathrm{Spec}(A/I) \subset X$ because it is induced by the graded morphism

$$\bigoplus J^n/IJ^n \to J^n\widehat{A}/IJ^n\widehat{A}$$

which is an isomorphism. We will identify these restrictions hereafter. In particular, the morphism $p_s : Y'_s \to Y_s$ between special fibers (that is, over the closed point of $s \in \mathrm{Spec}(A/I) \subset X$) is an isomorphism, by which we will identify them. Let us look at the commutative solid

$$\begin{tikzcd}
\widehat{U} = U' \arrow[r, "j'"] \arrow[dd, "p"] & Y' \arrow[dd, "p"] \arrow[drr] & Y_s \arrow[l, "i'"'] \arrow[dd] \arrow[drr, "e"] & & \\
& {} \arrow[r, phantom, "\square"] & {} & \widehat{X} = X' \arrow[r] \arrow[dd, "\pi"] & \mathrm{Spec}(k) \arrow[dd] \\
U \arrow[r, "j"] & Y \arrow[drr] & Y_s \arrow[l, "i"'] \arrow[drr, "e"] & & \\
& & & X \arrow[r] & \mathrm{Spec}(k)
\end{tikzcd}$$

Let us admit for a moment the following result.

Lemme 2.4.4. *Let $\mathscr{C} = F_U$ (resp. $\mathscr{C} = \underline{\mathrm{Tors}}(U, G)$). Then, the base change arrow*

$$\gamma : p^\star j_\star \mathscr{C} \to j'_\star p^\star \mathscr{C}$$

is bijective (resp. an equivalence).

Let us then deduce the sought-after equivalence

$$\mathrm{H}^0(U, \mathscr{C}) \xrightarrow{\sim} \mathrm{H}^0(U', \mathscr{C}) = \mathrm{H}^0(U', p^\star\mathscr{C})$$

thanks to Artin-Grothendieck's proper base change theorem ([**Giraud, 1971**] in the Noetherian case and theorem 7.1 in the general case) applied to the lower and upper faces of the preceding diagram. Indeed, we have an essentially commutative diagram where all the arrows are the natural arrows (obtained by adjunction)

$$\begin{tikzcd}
\mathrm{H}^0(U,\mathscr{C}) \arrow[r] \arrow[d, "\alpha"] & \mathrm{H}^0(Y, j_\star\mathscr{C}) \arrow[r, "b"] \arrow[d, "a"] & \mathrm{H}^0(Y_s, i^\star j_\star\mathscr{C}) \arrow[r] \arrow[d] & \mathrm{H}^0(Y_s, i'^\star p^\star j_\star\mathscr{C}) \\
\mathrm{H}^0(U', p^\star\mathscr{C}) \arrow[r] & \mathrm{H}^0(Y', j'_\star p^\star\mathscr{C}) \arrow[r, "d"] & \mathrm{H}^0(Y_s, i'^\star j'_\star p^\star\mathscr{C}) \arrow[ur, "c"'] &
\end{tikzcd}$$

The arrows b, d are bijective (resp. equivalences) thanks to the proper base change theorem (7.1) while c is a bijection (resp. an equivalence) thanks to 2.4.4. It follows that a and α are bijections (resp. equivalences).

Proof of Lemma 2.4.4. Let x' be a geometric point of Y' with image x in Y. We can assume $x' \in V(J\mathscr{O}_{Y'})$. Let B be the (strict) Henselization of Y at x' and B' that of Y' at x. We must study the arrow

$$(\star) \qquad \mathrm{H}^0(\mathrm{Spec}(B) - V(JB), \mathscr{C}) \to \mathrm{H}^0(\mathrm{Spec}(B') - V(JB'), \mathscr{C})$$

Let us observe that by definition of the blow-up, JB (resp. JB') is a principal ideal generated by a non-zero-divisor and non-invertible element $t \in B$ (resp. $t' \in B'$) (local equation of the exceptional divisor). Furthermore, the pairs (B, JB) and (B', JB') are Henselian because B, B' are local Henselian rings (exercise). The isomorphisms

$$J^n A/J^{n+m}A \xrightarrow{\sim} J^n\widehat{A}/J^{n+m}\widehat{A}, n, m \geq 0$$

ensure that B and B' have the same J-adic completion $\widehat{B} = \widehat{B'}$.

We then use the generalizations of Elkik's results [**Elkik, 1973**] — and thus of Ferrand-Raynaud for π_0 — to the non-Noetherian principal case from [**Gabber & Ramero, 2003**]. Precisely, Theorem 5.4.37 *loc. cit.* applied to the discrete $B[t^{-1}]$-groupoid $F_B = \mathrm{Spec}(B[t^{-1}]) \times F$ ensures that we have

$$\mathrm{H}^0(\mathrm{Spec}(B[t^{-1}]), F) = \pi_0(F_B) = \pi_0(F_{\widehat{B}}) = \mathrm{H}^0(\mathrm{Spec}(\widehat{B}[t^{-1}]), F)$$

and similarly by replacing B, t with B', t'. Since B and B' have the same J-adic completion, we thus have

$$\mathrm{H}^0(\mathrm{Spec}(B[t^{-1}]), F) = \mathrm{H}^0(\mathrm{Spec}(B'[t'^{-1}]), F),$$

which is what we wanted. In the case $\mathscr{C} = \underline{\mathrm{Tors}}(U, G)$, we deduce from the discrete case that $(*)$ is fully faithful. Let then $\hat{P}$ be a Galois covering with group G over $\widehat{U} = \mathrm{Spec}(\widehat{B}) - V(J\widehat{B})$. According to Theorem 5.4.53 of [**Gabber & Ramero, 2003**], it comes from a (unique) covering P of U. The full faithfulness of $(\star)$ ensures that the automorphism group of P is G. To say that P is Galois with group G is to say that the canonical arrow

$$\phi : P \times G \to P \times_U P$$

is an isomorphism. We can see this arrow as a morphism of étale coverings of U. After inverse image over $\widehat{U}$, it is identified with the analogous arrow

$$\hat{P} \times G \to \hat{P} \times_{\widehat{U}} \hat{P}$$

which is an isomorphism (of étale coverings of $\hat{P}$ hence of étale coverings of $\widehat{U}$) by hypothesis. The full faithfulness of $(\star)$ ensures that ϕ is an isomorphism, so that P is indeed Galois with group G. We have thus obtained that the natural functor between the categories of Galois G-coverings over U and $\widehat{U}$ are equivalent. The same is therefore true for the functor between the categories of Galois G-coverings over U' and $\widehat{U'}$. We conclude by remembering the equality $\widehat{U} = \widehat{U'}$. □

□

Remarque 2.4.5. Theorem 2.1.2 immediately implies that the base change arrow

$$\pi^\star j_\star\mathscr{C} \to \hat{j}_\star\pi^\star\mathscr{C}$$

is an equivalence. Indeed, we have already seen this on $\widehat{U}$ (2.3.2.1). If $\hat{x} \notin \widehat{U}$, we have already observed in the proof of 2.3.2 that $U_{(x)}$ and $\widehat{U}_{(\hat{x})}$ have the same I-adic completion, so that two applications of 2.1.2 ensure that the fiber of

$$\pi^\star j_\star\mathscr{C} \to \hat{j}_\star\pi^\star\mathscr{C}$$

at $\hat{x}$ is an equivalence.

3. Rigidity of ramification

3.1. The c_2 condition. Recall ([**ÉGA** IV$_4$ 18.6.7]) that the Henselization A^{h} of a semi-local ring A is the product of the Henselizations of the localizations of A at its maximal ideals. For any Noetherian ring, we denote by A^{nor} its normalization, namely the integral closure of A in the total ring $K(A)$ of fractions of $A_{\mathrm{réd}}$. Since $K(A)$ is the product of the $K(A/\mathfrak{p})$ where $\mathfrak{p}$ ranges over the maximal points of $\mathrm{Spec}(A)$, the normalization of A is the product of the normalizations of the $A/\mathfrak{p}$. Although A^{nor} is in general not Noetherian ([**Nagata, 1962**, example 5 of the appendix]), its reduced fibers over A are finite ([**Nagata, 1962**, V.33.10]). In particular, if A is local Noetherian, A^{nor} is semi-local, so its Henselization is well-defined. We then have (compare with [**Nagata, 1962**, 43.20 and exercise 43.21])

LEMME 3.2. *Let A be a local Noetherian ring.*

(i) *The canonical arrow $A^{\mathrm{h}} \to (A^{\mathrm{nor}})^{\mathrm{h}}$ induces an isomorphism $(A^{\mathrm{h}})^{\mathrm{nor}} \xrightarrow{\sim} (A^{\mathrm{nor}})^{\mathrm{h}}$.*

(ii) *This bijection induces a canonical bijection $\mathfrak{p} \mapsto \mathfrak{p}^\star$ between the maximal points $\mathfrak{p}$ of $\mathrm{Spec}(A^{\mathrm{h}})$ and the closed points $\mathfrak{p}^\star$ of $\mathrm{Spec}(A^{\mathrm{nor}})$ such that the integral domains $(A^{\mathrm{h}}/\mathfrak{p})^{\mathrm{nor}}$ and $(A^{\mathrm{nor}}_{\mathfrak{p}^\star})^{\mathrm{h}}$ are (canonically) isomorphic.*

Démonstration. According to [**ÉGA** IV$_4$ 18.6.8], the canonical morphism $A^{\mathrm{nor}} \otimes_A A^{\mathrm{h}} \to (A^{\mathrm{nor}})^{\mathrm{h}}$ is an isomorphism. The canonical morphism $A \to A^{\mathrm{h}}$ being ind-étale, it is normal. According to [**ÉGA** IV$_2$ 6.14.4], the canonical morphism $A^{\mathrm{h}} \to A^{\mathrm{nor}} \otimes_A A^{\mathrm{h}}$ identifies $A^{\mathrm{nor}} \otimes_A A^{\mathrm{h}}$ with the integral closure of A^{h} in $A^{\mathrm{h}} \otimes_A K(A)$. If now, $A \to B$ is étale, the fiber at the maximal point $\mathfrak{p} \in \mathrm{Spec}(A)$ is identified with $\mathrm{Spec}(K(B))$. By passing to the limit, we deduce the equality $A^{\mathrm{h}} \otimes_A K(A) = K(A^{\mathrm{h}})$ so that $A^{\mathrm{nor}} \otimes_A A^{\mathrm{h}}$ is identified with the integral closure of A^{h} in $A^{\mathrm{h}} \otimes_A K(A) = K(A^{\mathrm{h}})$ and thus $(A^{\mathrm{h}})^{\mathrm{nor}} \xrightarrow{\sim} A^{\mathrm{nor}} \otimes_A A^{\mathrm{h}}$. The composition

$$(A^{\mathrm{h}})^{\mathrm{nor}} \xrightarrow{\sim} A^{\mathrm{nor}} \otimes_A A^{\mathrm{h}} \xrightarrow{\sim} (A^{\mathrm{nor}})^{\mathrm{h}}$$

is the announced isomorphism. For the second point, we observe on the one hand that the spectrum of the normalization of A^{h} is the disjoint sum of the normalizations of its irreducible components

$$\mathrm{Spec}((A^{\mathrm{h}})^{\mathrm{nor}}) = \coprod_{\mathfrak{p} \text{ maximal point}} \mathrm{Spec}((A^{\mathrm{h}}/\mathfrak{p})^{\mathrm{nor}}), \tag{3.2.1}$$

each closed set $\mathrm{Spec}((A^{\mathrm{h}}/\mathfrak{p})^{\mathrm{nor}})$ being integral (since it is local and normal) so that 3.2.1 is the decomposition into irreducible components of $\mathrm{Spec}((A^{\mathrm{h}})^{\mathrm{nor}})$. On the other hand, by definition of the Henselization of a semi-local ring, we have

$$\mathrm{Spec}((A^{\mathrm{nor}})^{\mathrm{h}}) = \coprod_{\mathfrak{p}^\star \text{ closed point}} \mathrm{Spec}((A^{\mathrm{nor}}_{\mathfrak{p}^\star})^{\mathrm{h}}). \tag{3.2.2}$$

Now, $(A^{\mathrm{nor}}_{\mathfrak{p}^\star})^{\mathrm{h}}$ is local and normal (like $A^{\mathrm{nor}}_{\mathfrak{p}^\star}$), hence integral, proving that 3.2.2 is the decomposition into irreducible components of $\mathrm{Spec}((A^{\mathrm{nor}})^{\mathrm{h}})$. The lemma follows. □

PROPOSITION 3.3. *Let Z be a closed subscheme of a Noetherian scheme X. The following conditions are equivalent :*

(i) *Let $p : X^{\mathrm{nor}} \to X$ be the normalization morphism. Then, $p^{-1}(Z)$ is of codimension ≥ 2 in X^{nor}.*

(ii) *For any $z \in Z$, all irreducible components of $\mathrm{Spec}(\mathscr{O}^{\mathrm{h}}_{X,z})$ are of dimension ≥ 2.*

(ii$_{\mathrm{bis}}$) *For any $z \in Z$, all irreducible components of $\mathrm{Spec}(\mathscr{O}^{\mathrm{hs}}_{X,z})$ are of dimension ≥ 2.*

(iii) *For any $z \in Z$, all irreducible components of $\mathrm{Spec}(\widehat{\mathscr{O}_{X,z}})$ are of dimension ≥ 2.*

Démonstration. Let us denote $A = \mathscr{O}_{X,z}$ for $z \in Z$. First note that the morphism $A^{\mathrm{h}} \to A^{\mathrm{hs}}$ is injective, integral and faithfully flat. This proves that the morphism $h : \mathrm{Spec}(A^{\mathrm{hs}}) \to \mathrm{Spec}(A)$ satisfies $\dim(\overline{h(x)}) = \dim(\overline{x})$ and induces a surjection at the level of maximal points, which proves the equivalence of (ii) and (ii$_{\mathrm{bis}}$).

An integral domain and its normalization, as well as a local ring and its Henselization, have the same dimension. Keeping the notations of 3.2, we thus have

$$\dim A^{\mathrm{h}}/\mathfrak{p} = \dim A^{\mathrm{nor}}_{\mathfrak{p}^\star}.$$

Now, saying $\operatorname{codim} p^{-1}(Z) \geq 2$ is saying $\dim A^{\mathrm{nor}}_{\mathfrak{p}^\star} \geq 2$ when $\mathfrak{p}^\star$ ranges over the closed points of

$$\operatorname{Spec}(A^{\mathrm{nor}}) = p^{-1}(\operatorname{Spec}(\mathscr{O}_{X,z}))$$

as z ranges over Z. This is therefore equivalent to saying that all irreducible components $\operatorname{Spec}(A^{\mathrm{h}}/\mathfrak{p})$ of $\operatorname{Spec}(A^{\mathrm{h}})$ are of dimension ≥ 2, proving the equivalence of (i) and (ii).

To show the equivalence of (i) and (iii), we can assume that $X = \operatorname{Spec}(A)$ is local Henselian and that Z is reduced to its closed point.

Let us first prove that (iii) implies (ii). Let Y be an irreducible component of X. The completion morphism $c : \widehat{X} \to X$ being faithfully flat, $\widehat{Y} = c^{-1}(Y)$ is a union of irreducible components of $\widehat{X}$ so that we have $\dim(\widehat{Y}) \geq 2$. Since Y is local Noetherian, we have $\dim(Y) = \dim(\widehat{Y}) \geq 2$.

Let us prove the converse. Up to restricting to an irreducible (reduced) component, we can assume X is integral of dimension ≥ 2. Let $\hat{x}$ (resp. x) be the closed point of $\widehat{X}$ (resp. X) (it is not an irreducible component of $\widehat{X}$ which is of dimension ≥ 2). If one of the components of $\widehat{X}$ were of dimension ≤ 1, it would be of dimension 1 (because $\{\hat{x}\}$ is not a component) and thus its generic point would be an isolated point of $\widehat{X} - \{\hat{x}\}$ so that $\widehat{X} - \{\hat{x}\}$ would be disconnected (being of dimension ≥ 2). Now, according to [**Ferrand & Raynaud, 1970**, corollary 4.4], the arrow

$$\pi_0(\widehat{X} - \{\hat{x}\}) = \pi_0(c^{-1}(X - \{x\})) \to \pi_0(X - \{x\})$$

is bijective. Now, since X is integral of dimension ≥ 2, the open set $X - \{x\}$ is integral, hence connected. □

Définition 3.4. With the notations of 3.3, if Z satisfies the equivalent conditions of 3.3, we say that Z is c_2 in X.

Remarque 3.5. If X is integral and excellent, Z is c_2 if and only if $X - Z$ contains all points of codimension ≤ 1. Indeed, since the normalization morphism is finite and X is universally catenary, we have $\dim \mathscr{O}_{X^{\mathrm{nor}},z^{\mathrm{nor}}} = \dim \mathscr{O}_{X,p(z^{\mathrm{nor}})}$ for all $z^{\mathrm{nor}} \in p^{-1}(Z)$ (cf. [**ÉGA** IV$_2$ 5.6.10]).

Proposition 3.6. *Let $f : X' \to X$ be a flat morphism of Noetherian schemes and Z a closed subset of X. Then, if Z is c_2 in X, its inverse image $Z' = f^{-1}(Z)$ is c_2 in X'. In particular, the condition c_2 is invariant under Zariski or étale localization.*

Démonstration. Let $z' \in Z'$ with image $z = f(z') \in Z$. We thus assume (3.3) that all components of $A = \widehat{\mathscr{O}_{X,z}}$ are of dimension ≥ 2 and we want to prove that all components of $B = \widehat{\mathscr{O}_{X',z'}}$ are of dimension ≥ 2. We can therefore assume that f is a local morphism of Noetherian, local, and complete schemes. Since f is flat, every component of X' dominates a component X_0 of X and is a component of $f^{-1}(X_0)$. We can therefore assume X is integral of dimension > 1, with closed point z. According to [**SGA 2** VIII 2.3], the A-module $\mathscr{O}(X - z)$ is of finite type. Since B is flat over A, we deduce that $B \otimes_A \mathscr{O}(X - z) = \mathscr{O}(X' - f^{-1}(z))$ is of finite type over B. Since B is Noetherian, the B-submodule $\mathscr{O}(X' - z')$ of $\mathscr{O}(X' - f^{-1}(z))$ is of finite type.

Suppose that a component of X' is of dimension 1. Let η be the generic point of such a component and then define X'_0 as the schematic closure $\operatorname{Spec}(\mathscr{O}_{X',\eta})$ in X'. The complement $X'_0 - z'$ would then be reduced to η which would be isolated in $X' - z'$. Thus $\mathscr{O}(X'_0 - z')$ would be a B-submodule of $\mathscr{O}(X' - z')$, hence of finite type (B is Noetherian). A fortiori, $\mathscr{O}(X'_0 - z')$ would be of finite type as an $\mathscr{O}(X'_0)$-module, which contradicts [**SGA 2** VIII 2.3] since X'_0 is of dimension 1. □

4. Rigidity of ramification theorem I : weak form

We will start by proving a variant of the smooth base change theorem which is crucial in the proof of the rigidity theorem 4.2.1.

4.1. Variant of the smooth base change theorem. Let G be a finite group. We will prove a variant of the smooth base change theorem [**SGA 4** XVI 1.2] for sheaves of G-torsors without any hypothesis on the order of G, but by restricting to the case of open immersions (for a slightly different proof, see [**Gabber & Ramero, 2013**, 10.2.2]).

THÉORÈME 4.1.1. *Consider a Cartesian diagram*

$$\begin{tikzcd} U' \arrow[r, hook, "j'"] \arrow[d] \arrow[dr, phantom, "\square"] & X' \arrow[d, "p"] \\ U \arrow[r, hook, "j"] & X \end{tikzcd}$$

Suppose X is excellent normal, $p : X' \to X$ is smooth, and $j : U \to X$ is an open immersion such that U contains all points of codimension ≤ 1. Then, the base change morphism $\Phi : p^\star j_\star \underline{\mathrm{Tors}}(U, G) \to j'_\star \underline{\mathrm{Tors}}(U', G)$ is an equivalence.

Démonstration. According to the smooth base change theorem for sheaves of sets [**SGA 4** XVI 1.2], Φ is fully faithful. It suffices to prove essential surjectivity. Let x' be a geometric point of X' with image $x = p(x')$. Passing to the fibers, we are reduced to proving that the inverse image arrow of torsors

$$(*) \qquad \mathrm{H}^1(U_{(x)}, G) \to \mathrm{H}^1(U'_{(x')}, G)$$

is bijective, with moreover x' *closed* in its fiber [**SGA 4** VIII 3.13 b)]. Strict Henselization preserves normality and codimension (flatness). The permanence properties of excellent rings (cf. **I**-8) thus ensure that we can assume $X = \mathrm{Spec}(A), X' = \mathrm{Spec}(A')$ with $A = \mathscr{O}^{\mathrm{hs}}_{X,x}, A' = \mathscr{O}^{\mathrm{hs}}_{X',x'}$ strictly local, normal and excellent.

It may be that the residue extension $k(x')/k(x)$ is purely inseparable. As in the proof of the usual local acyclicity theorem ([**SGA 4** XV 2.1]), to reduce to the separable case, hence to degree 1, we consider a finite extension $A \subset B$ such that the residue extension contains $k(x')/k(x)$ (one can for example consider a *swelling* of A'/A in the sense of Bourbaki). We can assume B is integral and normal and then consider

$$\pi : \ Y = \mathrm{Spec}(B) \to X = \mathrm{Spec}(A)$$

as well as $Y' = Y \times_X X'$ and V the inverse image of U in Y. The pair (Y, V) satisfies the same properties as (X, U). The tautological morphism $\underline{\mathrm{Tors}}(U, G) \to \pi_* \underline{\mathrm{Tors}}(\pi^{-1}(U), G)$ is faithful. We can then invoke 6.2.1 to reduce the proof of $(*)$ to the analogous statement on (Y, V), in other words we can assume $k(x) = k(x')$.

Since p is smooth, the choice of local coordinates $t_1, \cdots, t_n$ of X' at x' defines an A-isomorphism $A\{t_1, \cdots, t_n\} \xrightarrow{\sim} A'$ where as usual $A\{t_1, \cdots, t_n\}$ denotes the strict Henselization of $A[t_1, \cdots, t_n]$ at the origin. An obvious recurrence allows us to assume $n = 1$. We have reduced to the situation

$$\begin{tikzcd} U' \arrow[r, hook] \arrow[d, "p"] \arrow[dr, phantom, "\square"] & X' \arrow[d, "p"'] \\ U \arrow[r, hook, "j"] \arrow[u, bend left, "\sigma"] & X \arrow[u, bend right, "\sigma"'] \end{tikzcd}$$

with A strictly local, normal and excellent, and σ the section of p defined by the closed immersion with equation $t = 0$. Since X, X' are local and normal, they are integral. The non-empty open sets of X, X' are therefore integral and thus connected. The composition

$$\pi_1(U) \xrightarrow{\sigma_\star} \pi_1(U') \xrightarrow{p_\star} \pi_1(U)$$

being the identity, it suffices to prove that $\sigma_\star$ is surjective. Let then V' be a connected étale covering of U'. We must prove that its restriction $V \to U$ to the closed subset $U \overset{\sigma}{\hookrightarrow} U'$ of equation $t = 0$ is connected.

Since X' is excellent, the integral closure Y' of X' in V' is finite over X', normal and integral (like X'). Since X' is Henselian, the same is true for Y', which is therefore a disjoint union of its local components.

Since Y' is integral, Y' is local. Let $D \subset Y'$ be the Cartier divisor with equation $t = 0$: D is connected, since it is closed in a local scheme.

We thus have a commutative diagram with Cartesian squares and where the vertical arrows are finite (and dominant).

$$\begin{tikzcd}
V' \arrow[r, hook] \arrow[d] & Y' \arrow[d] & D \arrow[l, hook'] \arrow[d] & V \arrow[l, hook'] \arrow[d] \\
U' \arrow[r, hook] & X' & X \arrow[l, hook', "\sigma"'] & U \arrow[l, hook']
\end{tikzcd}$$

Let x' be a point of $D - V$, with image x in $X - U$. Since $D \to X$ is finite, we have $\dim \overline{\{x'\}} = \dim \overline{\{x\}}$ and $\dim(D) = \dim(X)$. Since X, D are catenary (they are even excellent) and D is equidimensional, we deduce the equality $\dim \mathscr{O}_{D,x'} = \dim \mathscr{O}_{X,x}$ which ensures that the open set V in D contains all points of codimension 1 in D (just as U contains all points of codimension 1 in X). According to lemma **XXI**-4.2.1 applied to the connected Cartier divisor of the normal, excellent scheme Y', the scheme V is connected. □

4.2. Statement and reductions.

Théorème 4.2.1 (Rigidity of ramification). *Let X, X' be Noetherian schemes, $Z \subset X$ a closed subscheme, $U \overset{\tilde{j}}{\hookrightarrow} X$ the complementary open set and $X' \xrightarrow{\pi} X$ a flat morphism. Let us denote by $U' \overset{\tilde{j}'}{\hookrightarrow} X'$ the open immersion $U' = \pi^{-1}(U) \hookrightarrow X'$. We assume that π is regular over Z. Let $\mathscr{C}$ be a stack in groupoids on $U_{\text{ét}}$. Then, the base change arrow*

$$\phi(\mathscr{C}) : \pi^{\star} \tilde{j}_{\star} \mathscr{C} \to \tilde{j}'_{\star} \pi^{\star} \mathscr{C}$$

is an equivalence in the following two cases :

(i) *$\mathscr{C}$ is discrete (that is, $\mathscr{C}$ is equivalent to a sheaf of sets).*
(ii) *Z is c_2 and $\mathscr{C} = \underline{\text{Tors}}(U, G)$ with G a finite (ordinary) group.*

By considering the fibers, we can assume that π is a local morphism of strictly local schemes (the condition c_2 depending only on the strict Henselizations at the points of Z).

Let x, x' be the respective closed points of X, X'. By induction on the dimension of X', we can assume that $\phi(\mathscr{C})_{\bar{y}'}$ is an equivalence at any geometric point $\bar{y}'$ of $X' - \{x'\}$ and it suffices to prove that $\phi(\mathscr{C})_{x'}$ is an equivalence. We can moreover assume $x \in Z$ (otherwise $U = X$ and we are done). By hypothesis, the special fiber $F = \pi^{-1}(x)$ of π is geometrically regular.

We have a commutative diagram with Cartesian "squares" (with somewhat abusive notations)

$$\begin{tikzcd}
U' \arrow[r] \arrow[d, hook, "j'"'] \arrow[dd, bend right=60, "\tilde{j}'"'] & U \arrow[d, hook, "j"] \arrow[dd, bend left=60, "\tilde{j}"] \\
X' - F \arrow[r] \arrow[d, hook] & X - \{x\} \arrow[d, hook] \\
X' \arrow[r, "\pi"] & X
\end{tikzcd}$$

By induction hypothesis, the base change arrow associated with the upper square is an equivalence, so that we have an equivalence $\pi^{\star} j_{\star} \mathscr{C} \xrightarrow{\sim} j'_{\star} \pi^{\star} \mathscr{C}$ on $X' - F$. Since X, X' are strictly Henselian, the base change arrow $\phi(j_{\star}\mathscr{C})_{x'}$

$$\begin{array}{rcl}
\mathrm{H}^0(X, \tilde{j}_{\star}\mathscr{C}) & = & \mathrm{H}^0(X - \{x\}, j_{\star}\mathscr{C}) \\
& \xrightarrow{\pi^{\star}} & \mathrm{H}^0(X' - F, \pi^{\star} j_{\star}\mathscr{C}) \\
& = & \mathrm{H}^0(X' - F, j'_{\star}\pi^{\star}\mathscr{C}) \\
& = & \mathrm{H}^0(X', \tilde{j}'_{\star}\pi^{\star}\mathscr{C})
\end{array}$$

is identified with the inverse image arrow

$$\pi^\star : \mathrm{H}^0(X - \{x\}, j_\star\mathscr{C}) \to \mathrm{H}^0(X' - F, \pi^\star j_\star \mathscr{C}). \tag{4.2.1.1}$$

Let us denote by $\widehat{X}$ the completion *scheme* of X along its closed point and by $\widehat{X'}$ the completion of X' along F. For any S-space $\mathscr{E}$ on $S_{\text{ét}}$ with $S = X, X'$, we denote by $\widehat{\mathscr{E}}$ its inverse image on $\hat{S}$. We have a *commutative* diagram

$$\begin{tikzcd} \widehat{X'} \arrow[r, "\gamma'"] \arrow[d, "\hat{\pi}"] & X' \arrow[d, "\pi"] \\ \widehat{X} \arrow[r, "\gamma"] & X \end{tikzcd}$$

where γ, γ' are the completion morphisms, hence are flat, and $\hat{\pi}$ is flat since π is a local morphism of Noetherian schemes. Its special fiber is still F so it is geometrically regular. Thus, $\hat{\pi}$ is formally smooth ([**ÉGA** IV$_4$ 19.7.1]) and therefore regular ([**André, 1974**]) since $\widehat{X}$ is local Noetherian complete, hence excellent. According to 3.6, $\widehat{Z} = \widehat{X} - \widehat{U}$ is still c_2 (in case (ii)). According to Gabber's rigidity theorem (2.1.2) applied to the Henselian pairs (X, x) and (X', F), it suffices, to prove that the functor 4.2.1.1 is an equivalence, to prove that the functor

$$\hat{\pi}^\star : \mathrm{H}^0(\widehat{X} - \{x\}, \widehat{j_\star\mathscr{C}}) \to \mathrm{H}^0(\widehat{X'} - F, \hat{\pi}^\star \widehat{j_\star\mathscr{C}}) \tag{4.2.1.2}$$

is an equivalence.

If $\mathscr{C}$ is a sheaf of sets, we proceed as in (**XIV**-2.5.3) to show the bijectivity of (4.2.1.2) and complete the proof of theorem 4.2.1 in the discrete case.

In case (ii), let us show a lemma.

Lemme 4.2.2. *We can assume that π is an essentially smooth morphism of strictly local and excellent schemes.*

Démonstration. But the base change morphism

$$\iota : \gamma^\star j_\star \mathscr{C} = \widehat{j_\star\mathscr{C}} \to \hat{j}_\star \gamma^\star \mathscr{C} = \hat{j}_\star \widehat{\mathscr{C}}$$

is faithful. Indeed, by considering the sheaves of morphisms, it suffices to show that the base change morphism

$$\gamma^\star j_\star \mathscr{F} \to \hat{j}_\star \gamma^\star \mathscr{F}$$

is injective for any sheaf of sets on U. The fiber of this morphism at a geometric point $\hat{\xi}$ of $\hat{X}$ with image ξ in X is identified with the inverse image morphism

$$\Gamma : \mathrm{H}^0(U \times_X X_{(\xi)}, \mathscr{F}) \to \mathrm{H}^0(\hat{U} \times_{\hat{X}} \hat{X}_{(\hat{\xi})}, \gamma_\xi^\star \mathscr{F})$$

by the canonical morphism

$$\gamma_\xi : \hat{U} \times_{\hat{X}} \hat{X}_{(\hat{\xi})} \to U \times_X X_{(\xi)}.$$

But the flatness of γ ensures that γ_ξ is surjective and thus Γ is injective.

On the other hand

$$\widehat{\mathscr{C}} = \underline{\mathrm{Tors}}(\widehat{U}, G)$$

according to [**Giraud, 1971**, III.2.1.5.7]. In case (ii), to prove that (4.2.1.1) is an equivalence, we can therefore assume according to (6.2.1) that X is complete, hence excellent, and π is a regular local morphism.

According to Popescu's theorem ([**Swan, 1998**]), the regular morphism π is a filtered projective limit of essentially smooth local morphisms $\pi_i : X'_i \to X$. Note that the X'_i are strictly local and excellent like X. Since the X'_i are coherent, the global section functor commutes with the projective limit in the sense of [**SGA 4** VII 5.7] so that it suffices to prove the theorem for the π_i. □

4.3. Proof of 4.2.1. We thus assume that π is an essentially smooth local morphism of excellent schemes and $\mathscr{C} = \underline{\mathrm{Tors}}(U, G)$. To conclude the proof of theorem 4.2.1, we must prove the following variant of Gabber's smooth base change theorem (4.1.1).

Proposition 4.3.1. *Consider a Cartesian diagram*

$$\begin{tikzcd} U' \arrow[r, "j'"] \arrow[d] \arrow[dr, phantom, "\square"] & X' \arrow[d, "\pi"] \\ U \arrow[r, "j"] & X \end{tikzcd}$$

where π is an essentially smooth local morphism of strictly local excellent schemes. We assume that the complementary closed set $Z = X - U$ is non-empty and c_2 (that is, under these hypotheses, that U contains the points of codimension 1 (3.5)). Then, the morphism

$$\pi^\star : \mathrm{H}^0(X - \{x\}, j_\star \underline{\mathrm{Tors}}(U, G)) \to \mathrm{H}^0(X' - \pi^{-1}\{x\}, \pi^\star j_\star \underline{\mathrm{Tors}}(U, G))$$

is an equivalence, where x is the closed point of X.

Démonstration. The normalization morphism $p : X^{\mathrm{nor}} \to X$ being integral, its image is closed. Since p is (set-theoretically) dominant, p is surjective. Since p is surjective, the functor

$$j_\star \underline{\mathrm{Tors}}(U, G) \to j_\star p_\star p^\star \underline{\mathrm{Tors}}(U, G) \overset{[\textbf{Giraud, 1971}, \text{ III.2.1.5.7}]}{=} p_\star j^{\mathrm{nor}}_\star \underline{\mathrm{Tors}}(U^{\mathrm{nor}}, G)$$

is faithful[(iv)]. According to 6.2.1 and theorem 4.2.1 (i), it suffices to prove that the arrow

(4.3.1.1) $$\pi^\star : \mathrm{H}^0(X - \{x\}, p_\star j^{\mathrm{nor}}_\star \underline{\mathrm{Tors}}(U^{\mathrm{nor}}, G)) \to \mathrm{H}^0(X' - \pi^{-1}\{x\}, \pi^\star p_\star j^{\mathrm{nor}}_\star \underline{\mathrm{Tors}}(U^{\mathrm{nor}}, G))$$

is an equivalence.

Consider the Cartesian diagram

$$\begin{tikzcd} X'^{\mathrm{nor}} \arrow[r, "\pi^{\mathrm{nor}}"] \arrow[d, "p'"] \arrow[dr, phantom, "\square"] & X^{\mathrm{nor}} \arrow[d, "p"] \\ X' \arrow[r, "\pi"] & X \end{tikzcd}$$

Since p is finite (hence proper), we have $\pi^\star p_\star = p'_\star \pi^{\mathrm{nor}\star}$ so that (4.3.1.1) is identified with the inverse image arrow

(4.3.1.2)
$$\pi^{\mathrm{nor}\star} : \mathrm{H}^0(X^{\mathrm{nor}} - \{x\}^{\mathrm{nor}}, j^{\mathrm{nor}}_\star \underline{\mathrm{Tors}}(U^{\mathrm{nor}}, G)) \to \mathrm{H}^0(X'^{\mathrm{nor}} - (\pi^{\mathrm{nor}})^{-1}\{x\}^{\mathrm{nor}}, \pi^{\mathrm{nor}\star} j^{\mathrm{nor}}_\star \underline{\mathrm{Tors}}(U^{\mathrm{nor}}, G)).$$

Note that, since the c_2 condition depends only on the normalization, the complement Z^{nor} of U^{nor} is still c_2 in X^{nor}, and U^{nor} contains all points of codimension 1. According to Gabber's smooth base change theorem 4.1.1, the base change arrow

$$\pi^{\mathrm{nor}\star} j^{\mathrm{nor}}_\star \underline{\mathrm{Tors}}(U^{\mathrm{nor}}, G) \to j'^{\mathrm{nor}}_\star \underline{\mathrm{Tors}}(U'^{\mathrm{nor}}, G)$$

is an equivalence, so that (4.3.1.2) is identified with the inverse image

$$\pi^{\mathrm{nor}\star} : \mathrm{Tors}(U^{\mathrm{nor}}, G) \to \mathrm{Tors}(U'^{\mathrm{nor}}, G).$$

It then suffices to note that the proof of theorem 4.3.1 ensures that $\pi^{\mathrm{nor}\star}$ is an equivalence. $\square$

(iv) We denote by $\mathscr{E} \mapsto \mathscr{E}^{\mathrm{nor}}$ the inverse image functor by p.

4.4. Comparison with completion : case of abelian coefficients in the not necessarily Noetherian case.

The following paragraph is a *sketch* of a proof of the analogue of Theorem 4.2.1 for abelian coefficients. The case of Noetherian schemes is treated in [**Fujiwara, 1995**]. We reproduce here faithfully a letter from Ofer Gabber to the editors (June 20, 2012).

Let $(A, I) \to (A', I')$ be a map of henselian pairs with I finitely generated, $I' = IA'$, $\widehat{A} \xrightarrow{\sim} \widehat{A'}$ (I-adic completions). $X = \mathrm{Spec}(A)$, $X' = \mathrm{Spec}(A')$, $\pi : X' \to X$, $U = X - V(I)$, $U' = X' - V(I')$, $j : U \to X$, $j' : U' \to X'$.

CTC : For every torsion abelian sheaf F on U, the base change arrow $\pi^\star \mathrm{R}^q j_\star F \to \mathrm{R}^q j'_\star \pi^\star F$ is an isomorphism for all q.

Analogue of 4.2.1 (notations as there) : If F is a sheaf of $\mathbf{Z}/n\mathbf{Z}$-modules on U where $n > 0$ is invertible on X, then $\pi^\star \mathrm{R}^q j_\star F \to \mathrm{R}^q j'_\star \pi^\star F$ are isomorphisms.

This is reduced to CTC by the same argument.

Sketch of proof of CTC using Zariski-Riemann spaces : For comparing stalks we may assume A, A' strictly henselian and I a proper ideal, and we want

$$(*) \qquad \mathrm{H}^q(U, F) \xrightarrow{\sim} \mathrm{H}^q(U', F).$$

We call a finitely generated ideal $J \subset A$ containing a power of I *admissible*. We consider the admissible blow-ups $\mathrm{Bl}_J(X)$ which form a cofiltered category using X-scheme morphisms. In general there can be more than one X-morphism between two admissible blow-ups but if we restrict ourselves to J's with $V(J) = V(I)$ (set theoretically) (so that U is schematically dense in the blow-up), there is at most one. Define $J \leq J'$ iff there is an X-morphism $\mathrm{Bl}_{J'}(X) \to \mathrm{Bl}_J(X)$. This is a filtered preorder. When $V(J) \subset V(J')$, $J \leq J'$ is equivalent to the condition that for some $n > 0$ and ideal K, $J'^n = JK$. Thus we have an isomorphism of the preordered set of admissible J's of full support in A and the corresponding set for A'. Let $ZRS_I(X) = \varprojlim \mathrm{Bl}_J(X)$ (a locally ringed space). For the closed point s of X we can consider the special fiber $ZRS_I(X)_s$ and its étale topos, which for our purposes may be defined as the projective limit of the étale topoi $(\mathrm{Bl}_J(X)_s)_{\text{ét}}$ as in [**SGA 4**]. It has enough points by Deligne's theorem. The points are given by "geometric points" of $ZRS_I(X)_s$ (*i.e.* a point and a choice of a separable closure of the residue field). For every admissible J we have

$$j_J : U \hookrightarrow \mathrm{Bl}_J(X)$$

giving a spectral sequence (using proper base change)

$$(**) \qquad \mathrm{H}^p(\mathrm{Bl}_J(X)_s, \mathrm{R}^q j_{J\star} F) \Rightarrow \mathrm{H}^{p+q}(U, F).$$

We pass to the limit using the general theory of [**SGA 4** VI]. We get a spectral sequence $(**)_{\lim}$ involving cohomology on $(ZRS_J(X)_s)_{\text{ét}}$. Since the latter topos is the same for X', to show $(*)$ we use the morphism of the limit spectral sequence to reduce to stalks of the limits of the $\mathrm{R}^q j_{J\star}$ sheaves.

Using the study in [**Fujiwara, 1995**] of the local rings of ZRS's and their henselizations, one reduces $(*)$ to the case of local rings at geometric points of the special fibers of ZRS's. Thus we are reduced to the case A, A' are henselian and I-valuative. Say $I = (\varphi)$. Then $A[\varphi^{-1}]$ is a henselian local ring with maximal ideal corresponding to $P = \bigcap I^n$, and A/P is a henselian valuation ring whose valuation topology is the φ-adic one. In this case to prove $(*)$ one reduces to the corresponding statement for $\mathrm{Frac}(A/P) \to \mathrm{Frac}(A'/P')$. In fact for $K \to K'$ a dense embedding of henselian valued fields, if we choose separable closures K_{sep}, K'_{sep} and a map between them we have $\mathrm{Gal}(K'_{\mathrm{sep}}/K') \xrightarrow{\sim} \mathrm{Gal}(K_{\mathrm{sep}}/K)$, using forms of Krasner's lemma (cf. [**Bourbaki**, **AC**, VI, § 8, exercices 12, 14 a]).

Note: For admissible J, $\mathrm{Bl}_J(X') \to \mathrm{Bl}_J(X)$ gives an isomorphism on I-adic completions (as in the discussion in the proof of 2.4.4) as for every m the map

$$\bigoplus_n J^n \to \bigoplus_n J^n A'$$

is an isomorphism mod I^m.

5. Rigidity of ramification II : strong form

5.1. Immediate étale generizations. We denote by $X_{(x)}, X^{\mathrm{h}}_{(x)}$ and $\widehat{X}_{(x)}$ the localizations, Henselizations, and completions, respectively, of X at x. We denote by $\overline{\{y\}}$ the closure of y in X equipped with its reduced structure. Henselization and completion commute with closed immersions, so that $\overline{\{y\}}^{\mathrm{h}}, \widehat{\overline{\{y\}}}$ respectively coincide with the inverse image of $\overline{\{y\}}$ by the morphisms of Henselization and completion, respectively. Recall (**XIV**-2.1.2) that a generization $y \in X$ of a point x of a scheme X is an *immediate étale generization* of x if the strict Henselization at $\bar{x}$ of the closure of y has an irreducible component of dimension 1.

Lemme 5.1.1. *Let y be a generization of x. Let us denote by $c : \widehat{X}_{(x)} \to X_{(x)}$ the completion morphism. Then, y is an immediate étale generization of x if and only if one of the maximal points of $c^{-1}(y)$ is an immediate étale generization of the closed point of $\widehat{X}_{(x)}$.*

Démonstration. Let us denote for simplicity $Y = \overline{\{y\}}$. Let us first observe that one of the three schemes $Y_{(x)}, Y^{\mathrm{h}}_{(x)}$ and $\widehat{Y}_{(x)}$ has a maximal point of dimension zero if and only if each is reduced (set-theoretically) to its closed point. We can therefore exclude this case. The morphism $Y_{(\bar{x})} \to Y^{\mathrm{h}}_{(x)}$ is faithfully flat and integral. Thus, the strict Henselization has a maximal point of dimension 1[(v)] if and only if the Henselization $Y^{\mathrm{h}}_{(x)}$ has a maximal point of dimension 1. According to (3.3), $Y^{\mathrm{h}}_{(x)}$ has a maximal point of dimension 1 if and only if $\widehat{Y}_{(x)}$ has a maximal point of dimension 1. By flatness of $\widehat{Y}_{(x)} \to Y$, it necessarily maps to y, the generic point of Y. □

We can characterize immediate étale generizations nicely.

Lemme 5.1.2. *Let $f : X_{(\bar{x})} \to X_{(x)}$ be the strict Henselization morphism. The immediate étale generizations of x are the images $y = f(y')$ of the $y' \in X_{(\bar{x})}$ such that* $\dim \overline{\{y'\}} = 1$.

Démonstration. Let $y' \in X_{(\bar{x})}$ be such that $\dim \overline{\{y'\}} = 1$. The image $y = f(y')$ is a strict generization of x (because for example the fibers of f are discrete). For this same reason, $\overline{\{y'\}}$ is a component of $f^{-1}(\overline{\{y\}}) = \overline{\{y\}}_{(\bar{x})}$. Conversely, if y is an immediate étale generization of x, the generic point y' of a component of dimension 1 of $\overline{\{y\}}_{(\bar{x})}$ maps to y (flatness of f) and its closure is of dimension 1. □

Exemple 5.1.3. Let us take the example of the pinch point from [**ÉGA** IV$_2$ 5.6.11]. Keeping the notations of *loc. cit.*, the pinched ring C is local Noetherian of dimension 2 and its normalization has two maximal ideals of height 1, 2 respectively. According to 3.2, the Henselization of C has two irreducible components of dimension 1 and 2 with generic points c, c'. As in the proof of **XIV**-2.1.9, this ensures the existence of $\bar{c}$ (above c) in the strict Henselization of C whose closure is of dimension 1 and thus that the generic point of Spec(C) is an immediate étale generization of its closed point.

5.2. Associated pairs and condition $(*)$. Let us start with a definition.

Définition 5.2.1. Let x be a point of a scheme X. Let us choose a separable closure of $k(x)$ defining a geometric point $\bar{x}$ of X.

(i) Let G be a group scheme over X. We define the local sections of G with support in $\bar{x}$ by the formula
$$\mathrm{H}^0_{\bar{x}}(G) = \mathrm{Ker}(\mathrm{H}^0(X_{(\bar{x})}, G) \to \mathrm{H}^0(X_{(\bar{x})} - \{\bar{x}\}, G)).$$
(ii) Let $\mathscr{C}$ be a stack (in groupoids) on $X_{\text{ét}}$ and p a prime number. We say that (x, p) is **associated** with $\mathscr{C}$ and we write $(x, p) \in \mathrm{Ass}(\mathscr{C})$ if there exists $\sigma \in \mathscr{C}_{\bar{x}}$ such that $\mathrm{H}^0_{\bar{x}}(\underline{\mathrm{Aut}}(\sigma))$ has p-torsion.
(iii) Let $\mathscr{C}$ be an ind-finite stack (in groupoids) on an open subset U of X. We say that $\mathscr{C}$ satisfies condition $(*)$ if for any $x \in X - U$ of characteristic $p > 0$, there is no immediate étale generization y of x such that $(y, p) \in \mathrm{Ass}(\mathscr{C})$.

(v) One should rather say a maximal point whose closure is of dimension 1.

Note that the condition of (x, p) being associated does not depend on the choice of the separable closure of $k(x)$.

EXEMPLE 5.2.2. Suppose X is normal and G is a finite group. Let U be an open subset of X. Then, (x, p) is associated with $\mathscr{C} = \underline{\mathrm{Tors}}(U, G)$ if and only if $p|\,\mathrm{card}(G)$ and x is a maximal point of U. Indeed, the unique object of $\mathscr{C}_{\bar{x}}$ is the trivial torsor σ and $\underline{\mathrm{Aut}}(\sigma) = G$. Now, $X_{(\bar{x})} - \{\bar{x}\}$ is connected (resp. empty) if x is non-maximal (resp. maximal). Thus, we have $\mathrm{H}^0_{\bar{x}}(\underline{\mathrm{Aut}}(\sigma)) = \{1\}$ (resp. $\mathrm{H}^0_{\bar{x}}(\underline{\mathrm{Aut}}(\sigma)) = G$). We deduce that $\mathscr{C}$ satisfies $(*)$ if and only if U contains all points of codimension 1 whose characteristic divides the order of G.

LEMME 5.2.3. *Let $f : X \to Y$ be a flat morphism of Noetherian schemes, $x \in X$ with image $y = f(x)$ in Y and $\mathscr{C}$ a stack in groupoids on Y. Then, $(x, p) \in \mathrm{Ass}(f^\star\mathscr{C})$ if and only if $(y, p) \in \mathrm{Ass}(\mathscr{C})$ and $x \in \mathrm{Max}(f^{-1}(y))$.*

Démonstration. Let us choose a geometric point $\bar{x}$ above x, which defines $\bar{y}$ above of y.

Suppose $(x, p) \in \mathrm{Ass}(f^\star\mathscr{C})$. Since the arrow $(f^\star\mathscr{C})_{\bar{x}} \to \mathscr{C}_{\bar{y}}$ is an equivalence, there exists $\sigma \in \mathscr{C}_{\bar{y}}$ and $g \in \mathrm{Aut}(\sigma)$ such that $f^\star g$ is of order p and has support $\{\bar{x}\}$. Let F be the strict Henselization of $f^{-1}(y)$ at $\bar{x}$. It is also the fiber of the Henselization $\varphi : X_{(\bar{x})} \to Y_{(\bar{y})}$ of f above $\bar{y}$. If F were not reduced to $\bar{x}$, one of the points of F would not be in the support of $f^\star g$ so that $f^\star g$ would be the identity at this point. But $f^\star g$ is constant on $F = \varphi^{-1}(\bar{y})$ so that $f^\star g$ would also be the identity at $\bar{x} \in F$, which is not the case. Therefore, F is reduced to $\bar{x}$ so that $\dim \mathscr{O}_{f^{-1}(y),x} = 0$ (since a local ring has the same dimension as its strict Henselization) and $x \in \mathrm{Max}(f^{-1}(y))$. Moreover, g is trivial on $\varphi(X_{(\bar{x})} - \{\bar{x}\}) = Y_{(\bar{y})} - \{\bar{y}\}$ (faithful flatness of φ) which ensures $(y, p) \in \mathrm{Ass}(\mathscr{C})$.

Conversely, suppose $(y, p) \in \mathrm{Ass}(\mathscr{C})$ and $x \in \mathrm{Max}(f^{-1}(y))$. We thus have an automorphism g of order p of $\sigma \in \mathscr{C}_{\bar{y}}$ with support $\{\bar{y}\}$. The support of $\varphi^\star g$ is the fiber $\varphi^{-1}(\bar{y}) = f^{-1}(y)_{(\bar{x})}$. Since x is maximal in $f^{-1}(y)$, we deduce (dimension) that the local scheme $\varphi^{-1}(\bar{y})$ is of dimension zero, hence reduced to $\bar{x}$, which is what we wanted. □

COROLLAIRE 5.2.4. *Let (X, x) be a Henselian local Noetherian scheme, $U = X - \{x\}$ the complementary open set of the closed point, and $c : \widehat{X} \to X$ the completion morphism. Then, the stack in groupoids $\mathscr{C}$ on U satisfies $(*)$ if and only if $\widehat{\mathscr{C}} = c^\star\mathscr{C}$ satisfies $(*)$.*

Démonstration. We still denote by x the closed point of $\widehat{X}$ and we choose a geometric point $\bar{x}$ above x.

Suppose that $\widehat{\mathscr{C}}$ satisfies $(*)$. Let $(y, p = \mathrm{car}(x)) \in \mathrm{Ass}(\mathscr{C})$ and let us denote by Y the closure of y in X. We must show that all components of $Y_{(\bar{x})}$ are of dimension ≥ 2, or equivalently (3.3) that all components of Y are of dimension ≥ 2. But this is indeed the case because, according to Lemma 5.2.3, we have $(\hat{y}, p) \in \mathrm{Ass}(\widehat{\mathscr{C}})$ for any maximal point $\hat{y}$ of $\widehat{Y}$.

Conversely, suppose that $\mathscr{C}$ satisfies $(*)$. So let $(\hat{y}, p = \mathrm{car}(x)) \in \mathrm{Ass}(\widehat{\mathscr{C}})$ and let $y = c(\hat{y})$. According to Lemma 5.2.3, $(y, p) \in \mathrm{Ass}(\mathscr{C})$ and $\hat{y}$ is maximal in $c^{-1}(y) = \widehat{Y}$ where $Y = \overline{\{y\}}$. Then, all components of $Y_{(\bar{x})}$ (hence of $\widehat{Y}$ according to (3.3)) are of dimension > 1. In particular, $\dim \overline{\{\hat{y}\}} > 1$, which is what we wanted. □

5.3. The rigidity of ramification theorem.

THÉORÈME 5.3.1 (Rigidity of ramification II). *Let $\pi : X' \to X$ be a flat morphism of Noetherian schemes, regular over a closed subscheme $Z \subset X$. Let $j : U = X - Z \hookrightarrow X$ be the open immersion of the complement of Z and $\mathscr{C}$ be an ind-finite stack on U satisfying condition $(*)$. Then, the base change arrow $\pi^\star j_\star\mathscr{C} \to j'_\star\pi'^\star\mathscr{C}$ is an equivalence.*

Démonstration. According to the rigidity of ramification theorem I (4.2.1), the theorem is true in the discrete case, so that $\pi^\star j_\star\mathscr{C} \to j'_\star\pi'^\star\mathscr{C}$ is always fully faithful. As in the proof of 4.2.1, we can assume X, X' are strictly local with closed points x, x' and π is a local morphism. By induction on the dimension of X, we can assume that the base change by π is an equivalence for the immersion $U \hookrightarrow X - \{x\}$, so that we can assume $U = X - \{x\}$. As in the proof of 4.2.1 and by using the invariance under completion of condition $(*)$ (5.2.4), we can further assume that X is complete and π is an essentially smooth and local morphism, and it is a matter of proving that the arrow

$$\pi^{\star} : \mathrm{H}^0(X - \{x\}, \mathscr{C}) \to \mathrm{H}^0(X' - \pi^{-1}\{x\}, \mathscr{C}')$$

is essentially surjective.

So let σ' be an object of $\mathrm{H}^0(X' - \pi^{-1}\{x\}, \mathscr{C}')$. Since condition $(*)$ is stable by passage to (maximal) subgerbes, we can, as in the proof of 6.2.2, by considering the maximal subgerbe of $\mathscr{C}'$ generated by σ', further assume that $\mathscr{C}$ is a gerbe. Since $\mathscr{C}$ is ind-finite, we can assume that $\mathscr{C}$ is *constructible* (8.3.3).

For any maximal point $y \in U$, let us denote by i_y the canonical morphism

$$i_y : \mathrm{Spec}(k(y)) \to U = X - \{x\}.$$

The fiber category of $i_{y\star} i_y^{\star} \mathscr{C}$ over an étale open set $V \to U$ is identified with the rational sections of $\mathscr{C}$ defined in a neighborhood of the maximal points of V above of y. Let

$$\Psi : \mathscr{C} \to \mathscr{D} := \prod_{y \in \mathrm{Max}(U)} i_{y\star} i_y^{\star} \mathscr{C}$$

be the morphism deduced from the adjunction morphisms.

For any section $\tau \in \mathrm{H}^0(U, \mathscr{D})$ (seen as a morphism of U-spaces $\tau : U \to \mathscr{D}$), the associated stack of liftings $K(\tau) = U \times_{\mathscr{D}} \mathscr{C}$ is a stack in groupoids. The canonical morphism

$$K(\tau) \to \mathscr{C}$$

is faithful ([**Giraud, 1971**, IV.2.5.2]). Since $\mathscr{C}$ is constructible and satisfies $(*)$, the same holds for the maximal subgerbes of $K(\tau)$.

It then suffices (exercise) to check that the base change arrow is essentially surjective for

1) the gerbes $\mathscr{G} = i_{y\star} i_y^{\star} \mathscr{C}$;
2) the maximal subgerbes of the stack of liftings $K(\tau) = U \times_{\mathscr{D}} \mathscr{C}$ associated with $\tau \in \mathrm{H}^0(U, \mathscr{D})$.

5.3.2. *First case : base change for $\mathscr{G} = i_{y\star} i_y^{\star} \mathscr{C}$.* So let us assume $\mathscr{G} = i_{y\star} i_y^{\star} \mathscr{C}$. Up to changing X, U to $\overline{\{y\}}, U \cap \overline{\{y\}}$, we can assume that X is irreducible with generic point y.

If the dimension of X is 1, we have $U = \{y\}$ and $\mathscr{G}_{\bar{y}} = \underline{\mathrm{Tors}}(\mathrm{Spec}(\overline{k(y)}), G)$ with $p = \mathrm{car}(y)$ not dividing the order of G (cf. the argument in Example 5.2.2). We then invoke the base change by a usual smooth morphism ([**Giraud, 1971**, VII.2.1.2]).

We thus assume that the dimension of X is > 1.

Let us choose a separable closure $k(y) \hookrightarrow k_y$ and denote by $j_y : \mathrm{Spec}(k_y) \to U$ the canonical morphism. We have

$$j_y^{\star} \mathscr{C} = \underline{\mathrm{Tors}}(\mathrm{Spec}(k_y), G)$$

where G is a finite constant group. Since

$$i_{y\star} i_y^{\star} \mathscr{C} \to j_{y_\star} j_y^{\star} \mathscr{C}$$

is faithful, we can (6.2.1) replace $\mathscr{G}$ by $j_{y_\star} j_y^{\star} \mathscr{C} = j_{y_\star} \underline{\mathrm{Tors}}(\mathrm{Spec}(k_y), G)$.

LEMME 5.3.3. *We have $\mathrm{R}^1 j_{y\star} G = \{*\}$ and $j_{y_\star} \underline{\mathrm{Tors}}(\mathrm{Spec}(k_y), G) = \underline{\mathrm{Tors}}(U, j_{y_\star} G)$.*

Démonstration. The second equality follows from the first and from the formula ([**Giraud, 1971**, V.3.1.5])

$$\pi_0(j_{y_\star} \underline{\mathrm{Tors}}(\mathrm{Spec}(k_y), G)) = \mathrm{R}^1 j_{y\star} G.$$

Let $\tilde{A}$ be the strict Henselization of $X = \mathrm{Spec}(A)$ at a geometric point $\bar{\xi}$ of X. It is a filtered inductive limit of algebras A_i of finite type which are generically étale. We deduce that $j_y^{-1}(\mathrm{Spec}(\tilde{A}))$ is the spectrum of the filtered inductive limit of the étale algebras $B_i = k_y \otimes_{k(y)} A_i$ which are therefore split since k_y is separably closed. Thus, the schemes considered being coherent, we have

$$(\mathrm{R}^1 j_{y\star} G)_{\bar{\xi}} = \mathrm{H}^1(j_y^{-1}(\mathrm{Spec}(\tilde{A})), G) = \varinjlim \mathrm{H}^1(\mathrm{Spec}(B_i), G) = \{*\}.$$

□

In terms of Galois modules, $j_{y\star}G$ is the (continuous) induced representation $\mathrm{Hom}_c(\Gamma, G)$ where Γ is the profinite group $\mathrm{Gal}(k_y/k(y))$. By writing $\Gamma = \operatorname{colim} \mathrm{Gal}(K_\alpha/k(y))$ where $(K_\alpha/k(y))_\alpha$ is the inductive system of finite Galois subextensions of $k_y/k(y)$, we find

$$j_{y\star}G = \lim j_{\alpha\star}G$$

where $j_\alpha : \mathrm{Spec}(K_\alpha) \to \mathrm{Spec}(k(y)) \to U$ is the canonical morphism. Since U, U' are Noetherian, hence coherent, we have

$$\mathrm{H}^0(U, \underline{\mathrm{Tors}}(U, \operatorname{colim} j_{\alpha\star}G)) = \operatorname{colim} \mathrm{H}^0(U, \underline{\mathrm{Tors}}(U, j_{\alpha\star}G))$$

and

$$\begin{aligned} \mathrm{H}^0(U', \pi'^{\star}\, \underline{\mathrm{Tors}}(U, \operatorname{colim} j_{\alpha\star}G)) &= \mathrm{H}^0(U', \underline{\mathrm{Tors}}(U', \pi'^{\star} \operatorname{colim} j_{\alpha\star}G)) \\ &= \operatorname{colim} \mathrm{H}^0(U, \underline{\mathrm{Tors}}(U', \pi'^{\star} j_{\alpha\star}G)) \end{aligned}$$

so that for case 1) we are reduced to studying the base change for the gerbe $\mathscr{G}_\alpha = \underline{\mathrm{Tors}}(U, j_{\alpha\star}G)$.

Let $p : W \to X$ be the normalization of X in $\mathrm{Spec}(K_\alpha) \to X$: it is a finite morphism (because X is excellent) and surjective, so that W is semi-local and Henselian (like X). We deduce that W is the disjoint union of its Henselizations at the closed points. Since W is integral, W is strictly local : we denote by w its closed point. Moreover, W is normal, thus geometrically unibranch, so that $j_{\alpha\star}G = p_\star G_{|W-w}$. Since $\mathrm{R}^1 p_\star G$ is trivial (p is finite), we deduce the equality

$$\mathscr{G}_\alpha = p_\star\, \underline{\mathrm{Tors}}(W - \{w\}, G)$$

as in the proof of Lemma 6.3.3 *infra*.

By using the proper base change for p, we are reduced to proving that the arrow

$$\mathrm{Tors}(W - \{w\}, G) \to \mathrm{Tors}(W' - \pi^{-1}\{w\}, G)$$

is an equivalence.

Recall that we have assumed that the dimension of X (or W, it is the same) is > 1. In this case, $\{w\}$ is c_2 in W and we invoke the variant of Gabber's smooth base change theorem (4.3.1).

5.3.4. *Second case : base change for a maximal subgerbe K of the stack of liftings $K = K(\tau)$.* The gerbe K satisfies $(*)$ like $K(\tau)$. Since $\Psi_{\bar{y}}$ is an equivalence for any maximal point $y \in U$, we deduce that $K(\tau)_{\bar{y}}$ is punctual and thus that $K_{\bar{y}}$ is the trivial gerbe at all these points. By induction hypothesis, to complete the proof, it suffices to prove the following lemma.

Lemme 5.3.5. *There exists a nowhere dense closed immersion $i : F \to X$ such that $K = i_\star i^\star K$.*

Démonstration. It suffices to prove that for any maximal y, there exists a Zariski open set containing y on which K is trivial. By construction, there exists an étale neighborhood $V \to U$ of y and $\sigma \in K(V)$. Since $\underline{\mathrm{Aut}}(\sigma)$ is a constructible sheaf on $V_{\text{ét}}$, the isomorphism

$$\{\mathrm{Id}\}_{\bar{y}} \xrightarrow{\sim} \underline{\mathrm{Aut}}(\sigma)_{\bar{y}}$$

comes from an isomorphism

$$\{\mathrm{Id}\}_W \xrightarrow{\sim} \underline{\mathrm{Aut}}(\sigma)_W$$

on an étale neighborhood $W \to V \to U$ of y. Up to localizing, we can assume that $W \to U$ is a Galois covering of its image. The section σ descends to U and has no automorphism by construction. Thus, the restriction to U of K is a neutral gerbe with trivial automorphism group, hence is trivial. □

□

6. Appendix 1 : Stacky sorites

6.1. In the situation of Theorem 2.1.2, we already know (2.1.1 (i)) that the functor $\mathrm{H}^0(U, \mathscr{C}) \to \mathrm{H}^0(\widehat{U}, \pi^\star\mathscr{C})$ is fully faithful whether $\mathscr{C}$ is ind-finite or not.

Let $\mathscr{C}$ be an ind-finite stack on $Y_{\text{ét}}$. We are looking for conditions ensuring that the hypothesis

Hypothèse 6.1.1. Let $f : X \to Y$ be a morphism of schemes. We assume that for any sheaf of sets $\mathscr{F}$ on $Y_{\text{ét}}$, the arrow $\mathrm{H}^0(Y, \mathscr{F}) \to \mathrm{H}^0(X, f^\star\mathscr{F})$ is bijective.

implies that the conclusion

Conclusion 6.1.2. The arrow $\phi : \mathrm{H}^0(Y, \mathscr{C}) \to \mathrm{H}^0(X, f^\star\mathscr{C})$ is an equivalence of categories.

is true, in other words, ensuring that the assertion

Assertion 6.1.3. We have the implication 6.1.1 $\Rightarrow$ 6.1.2.

is true. We already know that 6.1.1 implies that ϕ is fully faithful (cf. 2.1).

6.2. First reductions. Let us start with a formal lemma :

Lemme 6.2.1. *Let $f : X \to Y$ be a morphism of schemes and $\mathscr{C}_1 \to \mathscr{C}_2$ a morphism of stacks on Y which we assume to be faithful. If 6.1.3 is true for $\mathscr{C}_2$, then 6.1.3 is true for $\mathscr{C}_1$.*

Démonstration. We have already observed (2.1) that ϕ is fully faithful. So let

$$c_1^X \in \mathrm{H}^0(X, f^\star\mathscr{C}_1) = \mathrm{Hom}_X(X, f^\star\mathscr{C}_1)$$

for which we are looking for a preimage in $\mathrm{H}^0(Y, \mathscr{C}_1)$. Its image

$$c_2^X \in \mathrm{H}^0(X, f^\star\mathscr{C}_2)$$

has a preimage (up to isomorphism)

$$c_2^Y \in \mathrm{H}^0(Y, \mathscr{C}_2).$$

The pair $(c_1^X, c_2^X = f^\star c_2^Y)$ defines a section of the stack of liftings

$$K(f^\star c_2^X) = X \times_{f^\star\mathscr{C}_1} f^\star\mathscr{C}_2.$$

But (associativity of the fibered product) $K(f^\star c_2^Y)$ is identified with

$$f^\star K(c_2^Y) = f^\star(Y \times_{\mathscr{C}_1} \mathscr{C}_2).$$

Now, $K(c_2^Y)$ is a sheaf of sets because $\mathscr{C}_1 \to \mathscr{C}_2$ is faithful. Therefore, $(c_1^X, c_2^X = f^\star c_2^Y) \in \mathrm{H}^0(X, f^\star K(c_2^Y))$ has a unique preimage of the form (c_1^Y, c_2^Y) and c_1^Y is indeed the sought-after preimage. ☐

Lemme 6.2.2. *If 6.1.3 is true for any gerbe (resp. any ind-finite gerbe), then 6.1.3 is true for any stack (resp. any ind-finite stack).*

Démonstration. Let $t \in \mathrm{H}^0(X, f^\star\mathscr{C})$ and $\gamma_t \subset f^\star\mathscr{C}$ be the maximal subgerbe generated by t in $f^\star\mathscr{C}$ ([**Giraud, 1971**, III.2.1.3.2])[(vi)]. This defines a section $\tau \in \pi_0(f^\star\mathscr{C})$ of the sheaf of sets $\pi_0(f^\star\mathscr{C})$ of maximal subgerbes of $f^\star\mathscr{C}$ (*loc. cit.*, 2.1.4). According to *loc. cit.*, 2.1.5, the natural arrow

$$\pi_0(f^\star\mathscr{C}) \to f^\star\pi_0(\mathscr{C})$$

is bijective. But, by hypothesis, the arrow

$$\mathrm{H}^0(Y, \pi_0(\mathscr{C})) \to \mathrm{H}^0(X, f^\star\pi_0(\mathscr{C})) = \mathrm{H}^0(X, \pi_0(f^\star\mathscr{C}))$$

is bijective, so that there exists a (unique) (maximal) subgerbe $\gamma \subset \mathscr{C}$ such that $f^\star\gamma = \gamma_t$, which will be ind-finite if $\mathscr{C}$ is. The image in $\mathrm{H}^0(Y, \mathscr{C})$ of the preimage of $t \in \mathrm{H}^0(X, f^\star\gamma)$ in $\mathrm{H}^0(Y, \gamma)$ is the sought-after preimage. ☐

(vi) In *loc. cit.*, $\pi_0(\mathscr{C})$ is denoted by $\mathrm{Ger}(\mathscr{C})$, which is not the current standard notation.

6.3. Reduction to the case of a stack of torsors under a finite constant group. Let us admit for a moment the following result, a generalization to the case of stacks of Godement's flasque resolution.

Lemme 6.3.1 (Effacement Lemma). *Let γ be an ind-finite gerbe on a coherent scheme X. There exists an ind-finite sheaf of groups $\mathscr{G}$ on X and a faithful functor $\gamma \hookrightarrow \underline{\mathrm{Tors}}(X, \mathscr{G})$.*

We can then prove the following criterion :

Proposition 6.3.2. *Let $f : X \to Y$ be a morphism of coherent schemes. We assume that for any sheaf of sets $\mathscr{F}$ on Y, the arrow $\mathrm{H}^0(Y, \mathscr{F}) \to \mathrm{H}^0(X, f^\star\mathscr{F})$ is bijective (6.1.1). We further assume that for any finite morphism $p : Y' \to Y$ inducing $f' : X' = X \times_Y Y' \to Y'$ and any finite constant group G, the arrow $\mathrm{Tors}(Y', G) \to \mathrm{Tors}(X', G)$ is an equivalence. Then, for any ind-finite stack $\mathscr{C}$ on Y, the arrow $\mathrm{H}^0(Y, \mathscr{C}) \to \mathrm{H}^0(X, f^\star\mathscr{C})$ is an equivalence.*

Démonstration. Only essential surjectivity is a problem. The effacement lemma, 6.2.1 and 6.2.2 allow us to assume that $\mathscr{C} = \underline{\mathrm{Tors}}(Y, \mathscr{G})$ where $\mathscr{G}$ is an ind-finite group on Y. Since X, Y are coherent, non-abelian cohomology commutes with filtered inductive limits [**SGA 4** vii rem. 5.14]. Since $\mathscr{G}$ is ind-finite, it is a filtered inductive limit of constructible sheaves of groups [**SGA 4** ix 2.7.2] : we can therefore assume $\mathscr{G}$ is constructible. Since Y is coherent, there exists (*loc. cit.*, 2.14) a finite family of finite morphisms $p_i : Y_i \to Y$ and finite constant groups G_i such that $\mathscr{G}$ embeds into the product $\prod p_{i\star}G_i$. We thus have a faithful morphism

$$\underline{\mathrm{Tors}}(Y, \mathscr{G}) \hookrightarrow \underline{\mathrm{Tors}}(Y, \prod p_{i\star}G_i) = \prod \underline{\mathrm{Tors}}(Y, p_{i\star}G_i)$$

thanks to [**Giraud, 1971**, III.2.4.4].

Using 6.2.1 again, we can assume

$$\mathscr{C} = \underline{\mathrm{Tors}}(Y, p_\star G)$$

with G a finite constant group and $p : Y' \to Y$ finite.

Lemme 6.3.3. *We have $\underline{\mathrm{Tors}}(Y, p_\star G) = p_\star\underline{\mathrm{Tors}}(Y', G)$.*

Démonstration. Since p is finite, $\mathrm{R}^1p_\star G$ is trivial. But $\pi_0(p_\star\underline{\mathrm{Tors}}(Y', G)) = \mathrm{R}^1p_\star G$ ([**Giraud, 1971**, V.3.1.9.1]) so that $p_\star\underline{\mathrm{Tors}}(Y', G)$ is a gerbe, visibly neutral, and therefore must be $\underline{\mathrm{Tors}}(Y, p_\star G)$. □

The proper base change theorem for sheaves (trivial in this case) ensures that we have $f^\star p_\star G = p'_\star f'^\star G = p'_\star G$. The arrow

$$\mathrm{H}^0(Y, p_\star\underline{\mathrm{Tors}}(Y', G)) \to \mathrm{H}^0(X, f^\star p_\star\underline{\mathrm{Tors}}(Y', G))$$

is then identified with the natural arrow

$$\begin{array}{rcll} \mathrm{Tors}(Y', G) & = & \mathrm{H}^0(Y, p_\star\underline{\mathrm{Tors}}(Y', G)) & \\ & \to & \mathrm{H}^0(X, f^\star p_\star\underline{\mathrm{Tors}}(Y', G)) & \\ & = & \mathrm{H}^0(X, \underline{\mathrm{Tors}}(X, f^\star p_\star G)) & \text{(by 6.3.3 and [\textbf{Giraud, 1971}, III.2.1.5.7])} \\ & = & \mathrm{H}^0(X, \underline{\mathrm{Tors}}(X, p'_\star G)) & \\ & = & \mathrm{H}^0(X, p'_\star\underline{\mathrm{Tors}}(X', G)) & \\ & = & \mathrm{H}^0(X', \underline{\mathrm{Tors}}(X', G)) & \\ & = & \mathrm{Tors}(X', G) & \end{array}$$

which is bijective by hypothesis. □

6.4. Proof of the effacement lemma. Let X be a coherent scheme.

Lemme 6.4.1. *There exists an affine scheme X', a quasi-compact and surjective morphism $f : X' \to X$ such that for any $x' \in X'$, the residue field $k(x')$ is the algebraic closure of the residue field $k(f(x))$.*

Démonstration. Since X is quasi-compact, we can cover X by a finite number of affine open sets X_i. The morphism $X_i \to X$ is surjective and quasi-compact (X is quasi-separated). Since all residue extensions are isomorphisms, we can therefore assume $X = \mathrm{Spec}(A)$ is affine, up to replacing X with $\coprod X_i$.

Sous-lemme 6.4.2. *Let I be the set of monic (non-constant) polynomials in $A[X]$ and*

$$T'(A) = A[X_P, P \in I]/(P(X_P))$$

and

$$f : X' = \operatorname{Spec} T'(A) \to X = \operatorname{Spec} A.$$

The morphism f is surjective and, for any $\xi \in X'$, the residue field $k(\xi)$ is the algebraic closure of $k(f(\xi))$.

Démonstration. Let $x \in X$ and $\overline{k(x)}$ be an algebraic closure of $k(x)$. The schematic fiber $f^{-1}(x)$ is the spectrum of

$$B = k(x)[X_P, P \in I]/(\tilde{P}(X_P))$$

where $\tilde{P}$ denotes the image of $P \in I$ by the localization of coefficients morphism

$$A[X] \to k(x)[X].$$

The choice of roots $x_P \in \overline{k(x)}$ for all polynomials P in I defines a point of $f^{-1}(x)$, ensuring the surjectivity of f.

Let then $\xi \in f^{-1}(x)$: it is a closed point because f is integral and $k(\xi)$ is algebraic over $k(x)$. Let Q be a monic polynomial in $k(x)$ of degree $d > 0$. There exists $a \in A$ with non-zero image $a(x)$ in $k(x)$, a monic polynomial $P \in I$ and an integer $n > 0$ such that

$$a^{nd} Q(X) = P(a^n X).$$

We deduce that the image X_P/a^n in $k(\xi)$ is a root of Q. Since $k(\xi)$ is algebraic over $k(x)$, this ensures that $k(\xi)$ is an algebraic closure of $k(x)$. (This is an (easy) exercise in Galois theory or [**Bourbaki**, **A**, V, § 10, exercise 20].) □

Since f is quasi-compact because it is affine, the lemma is proven. □

Secondly, let us recall the construction of the constructible topology on X (cf. [**ÉGA** IV$_1$ 1.9.13]).

Remarque 6.4.3. For our purpose, we will in fact only use it for the affine scheme X'.

We construct the topological space X^{cons} whose underlying set coincides with the underlying set $|X|$ of X but whose open (resp. closed) sets are the ind (resp. pro)-constructible parts, namely the unions (resp. intersections) of constructible parts. Since X is coherent, X is a compact, totally disconnected topological space ([**ÉGA** IV$_1$ 1.9.15]). Moreover, the coherence of X implies that the constructible parts are then finite unions of intersections $U \cap (X - V)$ with U, V quasi-compact open sets ([**ÉGA** III$_1$ 9.1.3] and [**ÉGA** IV$_1$ 1.2.7]). The complement of $U \cap (X - V)$ being $(X - U) \cup V$, it is therefore also open in X^{cons}, so that X has a basis of compact open sets.

The identity of $|X|$ induces a continuous map $X^{\text{cons}} \to X$ since a closed set is pro-constructible. If $X = \operatorname{Spec}(A)$ is affine, X^{cons} is naturally homeomorphic to the spectrum of a certain A-algebra $T(A)$ for a certain endofunctor T of the category of rings ([**Olivier, 1968**], proposition 5). This homeomorphism is compatible with localization so that these schematic structures glue, endowing X^{cons} with a natural structure of an X-scheme relatively affine compatible with the continuous (identity !) map $X^{\text{cons}} \to X$. If $x \in |X|$, we have $\mathscr{O}_{X^{\text{cons}},x} = k(x)$.

We then define

$$f : X^c \to X$$

as the composition

$$f : X^c = (X')^{\text{cons}} \to X' \to X.$$

By construction, X^c is compact – in particular coherent – (reduced), totally disconnected and has a basis of open-compact neighborhoods (which are therefore open-closed since X^c is compact hence topologically separated). Its residue fields are algebraically closed and f is quasi-compact[(vii)] and surjective (as a composition of quasi-compact morphisms).

[(vii)]We will apply this construction here to an open (quasi-compact) subset of an affine scheme, thus to a separated scheme so that f will even be affine in this case.

Recall that the empty sheaf on a topological space is the sheaf associated with the presheaf of constant value $\varnothing$. The set of its sections on any non-empty open set is $\varnothing$ and is reduced to a point on the empty open set.

LEMME 6.4.4. *Any étale morphism $f : Y \to X^c$ is Zariski locally trivial. In particular, the canonical morphism of topoi $(\varepsilon^{-1}, \varepsilon_\star) : X^c_{\text{ét}} \to X^c_{\text{Zar}}$ is an equivalence. Any sheaf on X^c that has sections locally has global sections. Moreover, any torsor on X^c is trivial and any gerbe is neutral.*

Démonstration. Let $y \in Y$ with image $x \in X^c$. Since f is quasi-finite and $k(x)$ is algebraically closed, the inclusion $k(x) \hookrightarrow k(y)$ is an equality. The composite morphism

$$\mathrm{Spec}(\mathscr{O}_x) = \mathrm{Spec}(k(x)) = \mathrm{Spec}(k(y)) \to Y$$

extends to a neighborhood of x as a local section of f, which proves the first point.

Let $\mathscr{F}$ be a sheaf (Zariski or étale, it's the same thing) on X^c. Since X^c is compact, we can find a finite covering by compact open sets U_i on which $\mathscr{F}$ has a section. We show by induction on the number of open sets that we can refine this covering into a finite covering V_j by disjoint compact open sets. Since for any j there exists an i such that $V_j \subset U_i$, the sheaf $\mathscr{F}$ has local sections on each V_j. These open sets being disjoint, these sections glue to a global section. The rest follows because any torsor (resp. any gerbe) on X^c has sections locally. $\square$

The proof of the effacement lemma is then easy. Let γ be an ind-finite gerbe on X. The adjunction functor

$$\gamma \to f_\star f^\star \gamma$$

is faithful because f is surjective. The gerbe $f^\star\gamma$ ([**Giraud, 1971**, III.2.1.5.6]) is neutral and ind-finite (6.4.4) so that it is equivalent to $\underline{\mathrm{Tors}}(X^c, \mathscr{G}^c)$ for a suitable ind-finite sheaf of groups $\mathscr{G}^c$. Furthermore, as we have already seen ([**Giraud, 1971**, V.3.1.9.1]), the sheaf of maximal subgerbes $\pi_0(f_\star \underline{\mathrm{Tors}}(X^c, \mathscr{G}^c))$ is identified with $\mathrm{R}^1 f_\star \mathscr{G}^c$.

LEMME 6.4.5. *The sheaf $\mathrm{R}^1 f_\star \mathscr{G}^c$ is trivial (constant punctual).*

Démonstration. Let $U \to X$ be an étale morphism with U quasi-compact and quasi-separated. As in the proof of 6.4.4, $X^c_U \to X^c$ is étale and thus a local isomorphism for the Zariski topology because the residue fields of X^c are algebraically closed. We deduce that the étale and Zariski topoi of X^c_U are equivalent. Any étale $\mathscr{G}^c$-torsor on X^c_U thus comes from a Zariski torsor. Since $X^c_U \to X^c$ is a local isomorphism, X^c_U has a basis of open-closed sets and is therefore separated. Since X^c_U is quasi-compact (since f is quasi-compact and U is quasi-compact like X), the underlying topological space of X^c_U is moreover compact. We deduce as in 6.4.4 that any torsor on X^c_U is trivial $\mathrm{H}^1(X^c_U, \mathscr{G}^c) = \{*\}$. Passing to the limit (we do not use the coherence of f here), we find that the fibers of $\mathrm{R}^1 f_\star \mathscr{G}_c$ are trivial. $\square$

According to the lemma, $\pi_0(f_\star \underline{\mathrm{Tors}}(X^c, G^c))$ is the punctual sheaf, which ensures that $f_\star \underline{\mathrm{Tors}}(X^c, G^c)$ is a gerbe. Since it has a section, it is neutral with group $G = f_\star G^c$ and is therefore identified with $\underline{\mathrm{Tors}}(X, G)$. But f is quasi-compact so that G is ind-finite like G^c [**SGA 4** IX 1.6]. The proof of the effacement lemma is complete.

7. Appendix 2 : Artin-Grothendieck's proper base change theorem for ind-finite stacks on non-Noetherian schemes

We will prove the following statement

THÉORÈME 7.1. *Consider a Cartesian diagram*

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

with f proper. Then, for any ind-finite stack $\mathscr{C}$ on X, the base change arrow $g^\star f_\star \mathscr{C} \to f'_\star g'^\star \mathscr{C}$ is an equivalence.

Note that this theorem is known in the discrete case [**SGA 4** XII 5.1] as well as in the case where Y is locally Noetherian ([**Giraud, 1971**, VII.2.2.2]). The following proof is an adaptation of the proof of the latter statement.

Démonstration. According to [**Giraud, 1971**, VII.2.2.5], it suffices to prove the following statement : let X be proper over a local Henselian S and $i : X_0 \hookrightarrow X$ be the immersion of the closed fiber. Then, for any ind-finite stack $\mathscr{C}$ on X, the arrow

$$\gamma : \mathrm{H}^0(X, \mathscr{C}) \to \mathrm{H}^0(X_0, i^\star \mathscr{C})$$

is an equivalence. Note that X/S being proper, it is coherent, so X is coherent like S [**SGA 4** VI 2.5]. Thus, i is a coherent morphism of coherent schemes. If $\mathscr{C}$ is discrete, the theorem is an immediate consequence of the proper base change theorem for sheaves of sets [**SGA 4** XII 5.1 (i)]. We deduce that γ is fully faithful. According to 6.3.2, it suffices to show that for any finite morphism $X' \to X$ (inducing a closed immersion $X'_0 \hookrightarrow X'$) and any finite group G, the arrow

$$\gamma : \mathrm{Tors}(X', G) \to \mathrm{Tors}(X'_0, G) \tag{7.1.1}$$

is an equivalence (and in fact is essentially surjective since we already know that it is fully faithful). We then apply [**SGA 4** XII 5.5 (ii)] to the proper morphism $X' \to S$ to conclude. □

8. Appendix 3 : Sorites on gerbes

We show that any ind-finite gerbe on a Noetherian X is a filtered inductive limit of its constructible subgerbes (compare with [**ÉGA** IV$_3$ IX. 2.2 and 2.9]).

8.1. Image of a morphism of stacks.

8.1.1. Let $\varphi : \mathscr{C} \to \mathscr{C}'$ be a (Cartesian) morphism of stacks on $X_{\text{ét}}$. We define the image $\mathfrak{I}(\varphi) = \varphi(\mathscr{C})$ as the category having for objects those of $\mathscr{C}$ and such that $\mathrm{Hom}_{\varphi(\mathscr{C})}(g_1, g_2) = \mathrm{Hom}_{\mathscr{C}'}(\varphi(g_1), \varphi(g_2))$, the fibered category structure being deduced from the (compatible) structures of $\mathscr{C}$ and $\mathscr{C}'$. Note that $\mathfrak{I}(\varphi)$ is naturally equivalent to the full subcategory $\mathscr{C}'_\varphi$ of $\mathscr{C}'$ whose objects are the images of the objects of $\mathscr{C}$. We will identify $\varphi(\mathscr{C})$ and $\mathscr{C}'_\varphi$. We will denote by $\widetilde{\mathfrak{I}}(\varphi)$ the stack associated with the prestack $\mathfrak{I}(\varphi)$. This is the full subcategory of $\mathscr{C}'_\varphi$ of objects locally isomorphic to the image of elements of $\mathscr{C}$.

LEMME 8.1.2. *Let $\bar{x}$ be a geometric point of X and $f : W \to X$ be a morphism of schemes. Then,*

(i) *the natural arrow $\mathfrak{I}(\varphi)_{\bar{x}} \to \mathfrak{I}(\varphi_{\bar{x}})$ is an equivalence ;*

(ii) *we have an equivalence of stacks, defined up to unique isomorphism, $f^\star \widetilde{\mathfrak{I}}(\varphi) \xrightarrow{\sim} \widetilde{\mathfrak{I}}(f^\star \varphi)$.*

Démonstration. The objects of $\mathfrak{I}(\varphi)_{\bar{x}}$ and $\mathfrak{I}(\varphi_{\bar{x}})$ coincide with those of $\mathscr{C}_{\bar{x}}$ and the natural arrow is simply the identity. Let us construct the inverse of the arrow. So let $a, b \in \mathcal{C}'_{\bar{x}}$ and $\psi \in \mathrm{Hom}(\varphi(a), \varphi(b))$ which comes from $\Psi_S \in \mathrm{Hom}_S(\varphi(\alpha), \varphi(\beta))$. But Ψ_S can be seen as an arrow of $\mathfrak{I}(\varphi)(S)$: we take its germ at $\bar{x}$ to define the inverse (which does not depend on the choices). We check that this defines the sought-after inverse.

Let us move on to the second point and define the arrow. By adjunction, we must define a (Cartesian) arrow

$$\widetilde{\mathfrak{I}}(\varphi) \to f_\star \widetilde{\mathfrak{I}}(f^\star \varphi). \tag{8.1.2.1}$$

According to the 2-universal property of the associated stack, it suffices to define a Cartesian morphism of prestacks

$$\mathfrak{I}(\varphi) \to f_\star \mathfrak{I}(f^\star \varphi).$$

Let $S \to X$ be étale. The objects of the left-hand member are the objects of $\mathscr{C}(S)$ while those of the right-hand member are those of $(f^\star \mathscr{C})(f^{-1}(S)) = f_\star f^\star \mathscr{C}(S)$. The adjunction arrow $\mathscr{C} \to f_\star f^\star \mathscr{C}$ then allows us to define the sought-after arrow $x \mapsto f^\star(x)$ at the level of objects. So let x, y be objects of $\mathscr{C}(S)$ and

$$g \in \mathrm{Hom}_S(\varphi(x), \varphi(y)) = \mathrm{Hom}_{\mathfrak{I}(\varphi)(S)}(x, y)$$

This is thus a section on S of $\underline{\mathrm{Hom}}(\varphi(x),\varphi(y))$ which provides (by inverse image) a section on $f^{-1}(S)$ of $\underline{\mathrm{Hom}}(f^\star\varphi(x), f^\star\varphi(y))$ [**Giraud, 1971**, II.3.2.8.1 (4)], thus an arrow of

$$\mathrm{Hom}_{f^{-1}(S)}(f^\star\varphi(x), f^\star\varphi(y)) = \mathrm{Hom}_{f^{-1}(S)}((\varphi(f^\star x),\varphi(f^\star y)) = \mathrm{Hom}_{f_\star\mathfrak{I}(f^\star\varphi)(S)}(x,y).$$

The functor thus defined is visibly Cartesian (like φ). The first point ensures that the fibers of this functor, and thus also those of the corresponding functor (8.1.2.1), are equivalences, which completes the proof of the lemma. $\square$

8.2. Free groupoids.

8.2.1. Let $\Gamma = E \overset{s}{\underset{b}{\rightrightarrows}} V$ be a directed graph (E is the set of edges, V the set of vertices, b, s the "target, source" maps). We associate (see [**Berger, 1995**]) the free groupoid $L(\Gamma)$ which can be described as follows. Let $E^\pm$ be the set $E^\pm = \{e^\pm, e \in E\}$, the disjoint union of two copies of E : its objects are the vertices and the morphisms between $v, v' \in V$ are the (reduced) words, possibly empty,

$$e_1^\pm \cdots e_n^\pm \text{ with } s(e_i^\pm) = b(e_{i+1}^\pm)(i = 1, \cdots, n-1), s(e_n^\pm) = v, b(e_1^\pm) = v'.$$

Remarque 8.2.2. It is well known that $L(\Gamma)$ is the fundamental groupoid $\Pi_1(\Gamma_{\mathbf{R}})$ of the geometric realization $\Gamma_{\mathbf{R}}$ of Γ.

8.2.3. By construction, the functors from $L(\Gamma)$ to a groupoid G are naturally identified with the families

$$(g_v) \in \mathrm{Ob}(G)^V, (\gamma_e) \in \mathrm{Fl}(G)^E \text{ such that } \gamma_e \in \mathrm{Hom}_G(g_{s(e)}, g_{b(e)}).$$

If one prefers, L is the left adjoint of the forgetful functor Groupoids → Graphs.

8.2.4. The construction globalizes in the following way. Consider a diagram of étale X-schemes

$$\Gamma_X : \begin{tikzcd} E \arrow[rr, shift left, "s"] \arrow[rr, shift right, "b"'] \arrow[dr] & & V \arrow[dl] \\ & X & \end{tikzcd}$$

By functoriality of the construction L, we define a (split) fibered category in groupoids $L(\Gamma_X)$ on $X_{\text{ét}}$ by the formula $S \mapsto L(\Gamma_X(S))$. By construction, the fibers of $L(\Gamma_X)$ are non-empty if and only if $V \to X$ is surjective. The local sections are locally isomorphic if and only if for any geometric point $\bar{x} \to X$, the graph $E_{\bar{x}} \rightrightarrows V_{\bar{x}}$ is connected and non-empty. By construction, we have two tautological sections

$$g \in \mathrm{Ob}L(\Gamma_X)(V), \gamma \in \mathrm{Hom}_{L(\Gamma_X)(E)}(s^\star g, b^\star g)$$

defined by the identity of V and of E respectively. Let $\mathscr{G}$ be a groupoid on $X_{\text{ét}}$. We then have the following adjunction property : the arrow which to a *Cartesian* functor

$$\varphi : L(\Gamma_X) \to \mathscr{G}$$

associates

$$\varphi(g) \in \mathscr{G}(V), \varphi(\gamma) \in \mathrm{Hom}_{\mathscr{G}(E)}(s^\star\varphi(g), b^\star\varphi(g))$$

is an equivalence.

8.3. Constructibility of subgerbes. Consider a hypercovering (viii) of X, namely a diagram of étale (of finite type) X-schemes

$$H_X : \begin{tikzcd} E \arrow[rr, shift left, "s"] \arrow[rr, shift right, "b"'] \arrow[dr] & & V \arrow[dl] \\ & X & \end{tikzcd}$$

(viii) The terminology is abusive : the diagonal section $V \to E$ is missing to have a (truncated) hypercovering.

where the arrows $V \to X$ and $(s, b) : E \to V \times_X V$ are surjective. For any geometric point $\bar{x} \to X$, the graph $H_{\bar{x}}$ is connected so that $L(H_X)$ is a gerbe. Let

$$\tilde{g} \in \mathscr{G}(V), \tilde{\gamma} \in \mathrm{Hom}_{\mathscr{G}(E)}(s^\star \tilde{g}, b^\star \tilde{g})$$

define a Cartesian morphism $\varphi : L(H_X) \to \mathscr{G}$.

Définition 8.3.1. A gerbe $\mathscr{G}$ on $X_{\text{ét}}$ is said to be **constructible** if for any étale open set $S \to X$, any local section $\sigma \in \mathscr{G}(S)$, the sheaf of groups $\underline{\mathrm{Aut}}(\sigma)$ on $S_{\text{ét}}$ is constructible.

Lemme 8.3.2. *With the preceding notations, suppose $\mathscr{G}$ is ind-finite. Then, the image $I = \widetilde{\mathfrak{I}}(\varphi)$ is constructible.*

Démonstration. Since the formation of the image and of L commute with the inverse image, we can proceed by Noetherian induction. It is therefore sufficient to prove that I is constructible on a non-empty open subset of X assumed to be integral. Since constructibility can be tested after any surjective base change locally of finite presentation, [**SGA 4** IX 2.8], we can assume that V, E are étale *coverings* of X that are completely decomposed, in other words that H_X is a finite constant graph Γ.

So let $\sigma \in I(S)$. Since Γ is constant, σ is locally isomorphic to $n = \mathrm{card}(V)$ sections $\sigma_i \in \mathscr{G}(X)$ that are pairwise isomorphic. In particular, each sheaf $\underline{\mathrm{Aut}}(\sigma_i)$ is generated by a finite number of sections coming from a finite generating family of $\pi_1(\Gamma_{\mathbf{R}}, i)$ and is contained in an ind-finite sheaf of groups. This ensures its constructibility (cf. the proof of [**SGA 4** IX 2.9 (iii)]). □

Proposition 8.3.3. *Let $\pi : X \to Y$ be a morphism of Noetherian schemes, $\mathscr{G}$ an ind-finite gerbe on $Y_{\text{ét}}$ and $\sigma \in \mathrm{H}^0(X, \pi^\star \mathscr{G})$. There exists a constructible subgerbe $\mathscr{G}_1$ of $\mathscr{G}$ such that $\sigma \in \mathrm{H}^0(X, \pi^\star \mathscr{G}_1) \subset \mathrm{H}^0(X, \pi^\star \mathscr{G})$*(ix).

Démonstration. The formula $(\pi^\star \mathscr{G})_{\bar{x}} = \mathscr{G}_{\pi(\bar{x})}$ ensures that locally σ is isomorphic to the inverse image of a local section of $\mathscr{G}$. Since Y is quasi-compact, we can find an étale $V \to Y$ (surjective of finite type) and $\tau \in \mathscr{G}(V)$ such that $\pi^\star \tau$ and σ are locally isomorphic (for the étale topology) on $X_V := X \times_Y V$ where we have still denoted by π the second projection $X_V \to V$. Let us thus choose an étale

$$e : X' \to X_V$$

(surjective of finite type) and an isomorphism

$$e^* \pi^* \tau \xrightarrow{\sim} e^* p^* \sigma \tag{8.3.3.1}$$

where p denotes the first projection $X_V \to X$. Consider the commutative diagram

$$\begin{tikzcd} X' \times_X X' \arrow[r, "h"] \arrow[d, shift right, "pr_1"'] \arrow[d, shift left, "pr_2"] & V \times_Y V \arrow[d, shift right, "pr_1"'] \arrow[d, shift left, "pr_2"] \\ X' \arrow[r, "\pi \circ e"] & V \end{tikzcd}$$

By definition, $p \circ e \circ pr_1 = p \circ e \circ pr_2$ so that (8.3.3.1) defines an isomorphism

$$h^* pr_1^* \tau = pr_1^* e^* \pi^* \tau \xrightarrow{\sim} pr_2^* e^* \pi^* \tau = h^* pr_2^* \tau \tag{8.3.3.2}$$

and thus ([**Giraud, 1971**, II.3.2.8]) a global section of the sheaf $h^* \underline{\mathbf{Isom}}(pr_1^* \tau, pr_2^* \tau)$. As previously, there then exists an étale $(s, b) : E \to V \times_Y V$ (surjective of finite type) and an isomorphism $\gamma : s^\star \tau \to b^\star \tau$ inducing (8.3.3.2) locally on $X' \times_X X'$. We then check that the image

$$\mathscr{G}_1 = \widetilde{\mathfrak{I}}(L(V, E) \xrightarrow{(\tau, \gamma)} \mathscr{G})$$

of the adjunction morphism defined by τ and γ (cf. previous paragraph) is suitable. □

(ix) More precisely, σ is in the essential image of $\mathrm{H}^0(X, \pi^\star \mathscr{G}_1)$ in $\mathrm{H}^0(X, \pi^\star \mathscr{G})$.

Remarque 8.3.4. We circumvent here the absence of sorites on inductive limits. The statement should be in two parts : first that an ind-finite gerbe on a Noetherian scheme is a filtered inductive limit of its constructible subgerbes, which is essentially the content of the preceding lemma, then that on a coherent scheme, the global sections functor commutes with filtered inductive limits.

EXPOSÉ XXI

The finiteness theorem for non-abelian coefficients

Frédéric Déglise

In memory of my uncle Olivier.

1. Introduction

The purpose of this exposition is to demonstrate the following theorems, which generalize Artin's theorem (cf. [**SGA 4** XIV 1.1]) in the set-theoretic and non-abelian case :

Théorème 1.1. *Let $f : Y \to X$ be a finite type morphism between Noetherian schemes. For every constructible sheaf F on $Y_{\text{ét}}$, the sheaf $f_\star F$ is constructible.*

Théorème 1.2. *Let $f : Y \to X$ be a finite type morphism between quasi-excellent schemes. Let $\mathbb{L}$ be a set of prime numbers invertible on X. For every constructible sheaf of groups F on $Y_{\text{ét}}$ of $\mathbb{L}$-torsion, the sheaf $\mathrm{R}^1 f_\star(F)$ on $X_{\text{ét}}$ is constructible.*

Théorème 1.3. *Let X be an excellent scheme, $Z \subset X$ a closed subset such that for every irreducible component X' of X, $\operatorname{codim}_{X'}(Z \cap X') \geq 2$. Let $j : U \to X$ be the open immersion of the complement of Z. For every finite group G, the sheaf $\mathrm{R}^1 j_\star(G_U)$ is constructible.*

Théorème 1.4. *Let A be a strictly local ring of dimension 2. Assume that A is normal, excellent, and let $X' = \operatorname{Spec}(A) - \{\mathfrak{m}_A\}$ be its punctured spectrum. Then, for every finite group G, the set $\mathrm{H}^1(X', G)$ is finite.*

Theorem 1.1 is proved in Section 2. This theorem is used by the following ones in the case where X is quasi-excellent. This case is much simpler, as we highlight in the proof.

Theorem 1.2 is reduced — in three steps — to Theorem 1.3 in Section 3. However, the attentive reader will note that this latter theorem is not a simple special case because it is not necessary to make any hypothesis on the cardinality of the group G.

Theorem 1.3 is reduced to Theorem 1.4 in Section 4. This reduction appears in 4.3 and uses two lemmas that were established earlier (Lemmas 3.3.2 and 4.2.2).

The last theorem is indeed a special case of 1.3. However, we have chosen to highlight it in this introduction both as an important result and as a key point. It is proved in Section 5 following a proof by contradiction that uses the ultrafilter method (see 5.2.1 for recollections).

Notations and conventions.

- When a topology on a scheme is implicit, it is the étale topology.
- Given a set D (resp. a group G), we will sometimes denote D (resp. G) for the induced constant étale sheaf on a scheme X when X is clear from the context. If one wants to specify X, we denote this sheaf D_X (resp. G_X), following common practice.
- When we talk about the normalization of a scheme X, it refers to the canonical morphism

$$X' = \coprod_{i \in I} X'_i \to X$$

where I denotes the set of irreducible components of X and X'_i denotes the normalized scheme of the irreducible component of X corresponding to i, endowed with its reduced subscheme structure. We also say that X' is the normalized scheme associated with X.

2. Direct image of constructible set sheaves

In the case where X is quasi-excellent, the proof is an application of already known results (cf. [**SGA 4** IX]). We begin by presenting the proof in this case, then in the general case. However, the reduction step presented in the following section is valid in both cases.

2.1. Reduction of the theorem. We begin by reducing Theorem 1.1 to the following assertion :

($\mathscr{P}$) Let D be a finite set and $j : U \to X$ an open immersion between Noetherian schemes. Then, the sheaf of sets $j_\star(D_U)$ is constructible.

Consider the hypotheses of Theorem 1.1. According to [**SGA 4** IX 2.14], we can find a monomorphism

$$F \to \prod_{i=1}^{n} \pi_{i\star}(C_i) = Q$$

for finite morphisms $\pi_i : Y_i \to Y$ and finite constant sheaves C_i on Y_i. Since a subsheaf of a constructible sheaf is constructible ([**SGA 4** IX 2.9 (ii)]), it suffices to show that $f_\star(Q)$ is constructible. We are therefore reduced to the case of $(f\pi_i)_\star(C_i)$ for each i, which shows that we can assume $F = D_Y$ for a finite set D.

Note that in this case, the theorem is local on Y. Indeed, if we are given a finite type étale covering $\pi : W \to Y$, the adjunction morphism

$$D_Y \to \pi_\star\pi^\star(D_Y) = \pi_\star(D_W)$$

is a monomorphism. Applying $f_\star$ to it, we deduce a monomorphism

$$f_\star(D_Y) \to (f\pi)_\star(D_W).$$

It therefore suffices to show that the right-hand side is constructible ([**SGA 4** IX 2.9 (ii)] again). In particular, we can therefore assume that Y is affine.

Then, f is separated of finite type. We can therefore consider a factorization

$$Y \xrightarrow{j} \bar{X} \xrightarrow{\bar{f}} X$$

of f such that j is an open immersion and $\bar{f}$ is a proper morphism. The result is known for $\bar{f}$ (cf. [**SGA 4** XIV 1.1]) so we are reduced to the case of the open immersion j, that is, to assertion ($\mathscr{P}$).

Remarque 2.1.1. (i) If we assume that X is quasi-excellent, the scheme $\bar{X}$ that appears in the above reduction is still quasi-excellent since $\bar{f}$ is of finite type.
(ii) In this reduction, we have seen that ($\mathscr{P}$) is local on U.

2.2. Case where X is quasi-excellent. Note the following easy lemma :

Lemme 2.2.1. *Consider a Cartesian square of Noetherian schemes*

$$\begin{tikzcd} U' \arrow[r, "j'"] \arrow[d, "q"'] & X' \arrow[d, "p"] \\ U \arrow[r, "j"] & X \end{tikzcd}$$

such that j is an open immersion and p a finite surjective morphism. Then, for every finite set D, if $j'_\star(D_{U'})$ is constructible, $j_\star(D_U)$ is constructible.

Démonstration. By hypothesis, q is surjective. We deduce that the adjunction morphism

$$D_U \to q_\star q^\star(D_U) = q_\star(D_{U'})$$

is a monomorphism. Applying $j_\star$, we deduce a monomorphism

$$j_\star(D_U) \to p_\star(j'_\star(D_{U'})).$$

Since p is finite, $p_\star$ preserves constructibility according to [**SGA 4** IX 2.14 (i)]. The lemma follows since a subsheaf of a constructible sheaf of sets is constructible ([**SGA 4** IX 2.9 (ii)]). ☐

2.2.2. Before moving on to the proof in the general case, let's note that the proof of Theorem 1.1 in the case where X is quasi-excellent is simpler. Thanks to the previous remark, we reduce to assertion $(\mathscr{P})$ in the case where X is quasi-excellent. Thus, the normalization $p : X' \to X$ of X is finite. So, the previous lemma applied to the obvious Cartesian square reduces us to the case where X is normal.

We are only interested in the case where X and U are non-empty connected. Then, according to [**SGA 4** IX 2.14.1], $j_\star(D_U) = D_X$, which concludes the proof.

2.3. General case.

2.3.1. Consider the hypotheses of assertion $(\mathscr{P})$. This is local in X and it is therefore sufficient to treat the case where X is affine, the spectrum of a Noetherian ring A. Using Lemma 2.2.1 — by taking for X' the disjoint union of the irreducible components of X — we can also assume that A is integral. We have already seen that $(\mathscr{P})$ is also local in U (point (ii) of Remark 2.1.1). We can therefore reduce to the case where $U = \operatorname{Spec}(A_f)$ for a non-zero element $f \in A$.

To prove $(\mathscr{P})$, we will demonstrate by Noetherian induction on the irreducible closed subsets Z of X the slightly stronger property :

$(*_Z)$ For every finite morphism $Z' \to Z$, Z' integral, every finite set D and every open immersion $l : V' \to Z'$, the sheaf $l_\star(D_{V'})$ is constructible.

From the foregoing, we are reduced to proving $(\mathscr{P})$ for $X = \operatorname{Spec}(A)$ integral and $U = \operatorname{Spec}(A_f)$, $f \neq 0$, by assuming the following induction property :

$(\mathscr{H})$ For every proper closed subset Z of X, every finite morphism $Z' \to Z$, every finite set D and every open immersion $l : V' \to Z'$, the sheaf $l_\star(D_{V'})$ is constructible.

2.3.2. Let A' be the integral closure of A in its fraction field. We set $X' = \operatorname{Spec}(A')$, $Z = \operatorname{Spec}(A/(f))$ and we consider the diagram formed by Cartesian squares :

$$\begin{tikzcd}
 & & Z'' \arrow[d, "h"] \arrow[dd, bend left, "\pi"] \\
U' \arrow[r, "j'"] \arrow[d, "q"] & X' \arrow[d, "p"] & Z' \arrow[l, "i'"'] \arrow[d, "r"] \\
U \arrow[r, "j"] & X & Z \arrow[l, "i"']
\end{tikzcd}$$

such that i and j are the obvious immersions, p (resp. h) is the normalization of X (resp. Z').

2.3.3. *First step (generic fibers of p).* According to [**Nagata, 1962**, 3.3.10], the ring A' is a Krull ring. It follows that Z' has only a finite number of generic points $z'_1, \dots, z'_n$. We set $z_r = p(z'_r)$. Although p is not necessarily finite, $p^{-1}(z_s)$ is finite and the residual extension $\kappa(z'_s)/\kappa(z_s)$ is finite (see [**Nagata, 1962**, Chap. V, 33.10]). According to Lemma 2.2.1, we can always replace A by a finite extension $A \subset B \subset A'$. The hypothesis $(\mathscr{H})$ is indeed still satisfied for $Y = \operatorname{Spec}(B)$. Thus, we can assume that the following conditions are satisfied :

(h1) For every index s, $p^{-1}(\{z_s\}) = \{z'_s\}$.

(h2) For every index s, $\kappa(z'_s)/\kappa(z_s)$ is trivial.

Let A_s (resp. A'_s) denote the localized ring of X at z_s (resp. X' at z'_s). Then, A'_s is a discrete valuation ring. From the fact that the induced extension A'_s/A_s is integral, we deduce that A_s is of dimension 1, which implies that z_s is a maximal point of the divisor Z of X. Since q is surjective, we deduce from (h1) that $z_1, \dots, z_n$ is the set of generic points of Z.

2.3.4. *Second step (restriction to an open subset of Z).* Since trivially $j^\star j_\star(D_U) = D_U$, it suffices to show that $i^\star j_\star(D_U)$ is constructible. Note the following facts :

(i) q surjective : $D_U \to q_\star q^\star(D_U) = q_\star(D_{U'})$ is a monomorphism.

(ii) X' normal, j' dominant : $j'_\star(D_{U'}) = D_{X'}$ (see [**SGA 4** IX 2.14.1]).

(iii) h surjective : $D_{Z'} \to h_\star h^\star(D_{Z'}) = h_\star(D_{Z''})$ is a monomorphism.

We deduce from (i) and (ii) a monomorphism

$$j_\star(D_U) \to j_\star q_\star(D_{U'}) = p_\star(D_{X'}).$$

Note that p is pro-finite. The proper base change theorem extends to this case, which gives the relation : $i^\star p_\star = r_\star i'^\star$. If we apply $i^\star$ to the previous monomorphism, we deduce from this relation and from (iii) a composite monomorphism :

$$\sigma : i^\star j_\star(D_U) \to i^\star p_\star(D_{X'}) = r_\star(D_{Z'}) \to \pi_\star(D_{Z''}).$$

Now consider a dense open immersion $l : V \to Z$ as well as the following Cartesian square :

$$\begin{tikzcd} Z'' \arrow[r, "l'"] \arrow[d, "\pi"'] & V'' \arrow[d, "\pi_V"] \\ Z \arrow[r, "l"] & V. \end{tikzcd}$$

We deduce a commutative diagram of sheaves of sets :

$$\begin{tikzcd} i^\star j_\star(D_U) \arrow[r, "\alpha"] \arrow[d, "\sigma"'] & l_\star l^\star i^\star j_\star(D_U) \arrow[d, "l_\star l^\star \sigma"] \\ \pi_\star(D_{Z''}) \arrow[r, "\beta"] & l_\star l^\star \pi_\star(D_{Z''}). \end{tikzcd}$$

However, the morphism β — induced by the unit of adjunction $(l^\star, l_\star)$ — is an isomorphism. Indeed, since l is an open immersion, $l^\star \pi_\star = \pi_{V\star} l'^\star$ by base change. We thus obtain the identification : $l_\star l^\star \pi_\star(D_{Z''}) = \pi_\star l'_\star(D_{V''})$. Moreover, through this identification, the morphism β is the image[(i)] by $\pi_\star$ of the unit of adjunction

$$D_{Z''} \to l'_\star l'^\star(D_{Z''}) = l'_\star(D_{V''}).$$

The latter is an isomorphism since Z'' is normal and l' is dense (see [**SGA 4** IX 2.14.1]). Since σ is a monomorphism, we deduce that α is a monomorphism.

However, according to $(\mathscr{H})$, the sheaf of sets $l_\star(D_V)$ is constructible. To conclude, it therefore suffices (according to [**SGA 4** IX 2.9 (ii)]) to find a dense open set V of Z such that

$$j_\star(D_U)|_V = D_V. \tag{2.3.4.1}$$

2.3.5. *Third step (embedded components of Z).* Consider the union T of the embedded components of Z in X. Then, the open set $V = Z - T$ satisfies relation (2.3.4.1).

We must show that for every geometric point $\bar{x}$ of V, the fiber of the canonical morphism $D_V \to j_\star(D_U)|_V$ at point $\bar{x}$ is an isomorphism. In other words, we must show that the canonical morphism

$$\pi_0(U \times_X X_{(\bar{x})}) \to \pi_0(X_{(\bar{x})})$$

is an isomorphism. By induction on the dimension of $X_{(\bar{x})}$, we reduce to showing that the following morphism

$$\pi_0(X_{(\bar{x})} - \{\bar{x}\}) \to \pi_0(X_{(\bar{x})}) \tag{2.3.5.1}$$

is an isomorphism.

Let x be the point of $X = \operatorname{Spec} A$ corresponding to $\bar{x}$. If x is not a maximal point, since V has no embedded component,

$$\operatorname{Prof}_x(A/(f)) \geq 1 \Rightarrow \operatorname{Prof}_x(A) \geq 2.$$

According to Hartshorne's theorem ([**SGA 2** III 3.6]), the morphism (2.3.5.1) is therefore an isomorphism.

Suppose that x is a maximal point. According to (h1), there exists an index i such that $x = z_i$. However, according to (h2), the birational integral morphism

$$X'_{(z'_i)} = \operatorname{Spec}(A'_i) \to \operatorname{Spec}(A_i) = X_{(z_i)}$$

(i) This can be easily checked by returning to the definition of the base change morphism using the adjunctions $(l^\star, l_\star)$ and $(l'^\star, l'_\star)$ and by using that the following composition of unit/counit morphisms is the identity :

$$l_\star \to l_\star l^\star l_\star \to l_\star.$$

is radical. It is therefore a universal homeomorphism. We deduce that the morphism $X'_{(\bar{z}'_i)} \to X_{(\bar{z}_i)}$ is still a homeomorphism, where $\bar{z}'_i$ is the geometric point corresponding to the separable closure of $\kappa(z_i)$ defined by $\bar{z}_i$. Thus, (2.3.5.1) is an isomorphism since the corresponding property is true for the normal scheme $X'_{(\bar{z}'_i)}$. This concludes the proof.

3. Derived direct image of constructible group sheaves

3.1. Reduction to the case of a constant sheaf.

Lemme 3.1.1. *Let $f : Y \to X$ be a finite type morphism between Noetherian schemes and $u : F \to F'$ a monomorphism of constructible group sheaves on $Y_{\text{ét}}$. Then,* $\mathrm{R}^1 f_\star(F')$ *constructible implies* $\mathrm{R}^1 f_\star(F)$ *constructible.*

Démonstration. Let $C = F'/F$ be viewed as a pointed sheaf, constructible on $Y_{\text{ét}}$ by hypothesis. Consider the exact sequence of pointed sheaves (cf. [**SGA 4** XII 3.1])

$$f_\star(F') \to f_\star(C) \to \mathrm{R}^1 f_\star(F) \xrightarrow{v} \mathrm{R}^1 f_\star(F').$$

Assuming that $\mathrm{R}^1 f_\star(F')$ is constructible, we can find a generating family of local sections $(e_1, ..., e_n)$ of $\mathrm{R}^1 f_\star(F')$ where e_i is defined on an étale finite type X-scheme V_i. Let Φ_i be the fiber sheaf of v at e_i, defined by the Cartesian diagram of sheaves (of sets) on $X_{\text{ét}}$:

$$\begin{tikzcd} \Phi_i \arrow[r, "\nu_i"] \arrow[d] & V_i \arrow[d, "e_i"] \\ \mathrm{R}^1 f_\star(F) \arrow[r, "v"] & \mathrm{R}^1 f_\star(F'). \end{tikzcd}$$

where $e_i \in \Gamma(V_i, \mathrm{R}^1 f_\star(F'))$ is seen as a morphism of the étale topos of X. To show that $\mathrm{R}^1 f_\star(F)$ is constructible, it suffices to show that it is generated by a finite family of local sections on quasi-compact objects. It therefore suffices to show that for each i, Φ_i, viewed as an étale sheaf on V_i thanks to the morphism ν_i, is generated by a finite number of local sections. We are therefore reduced to showing that the étale sheaf Φ_i on V_i is constructible.

To show this, we assume $X = V_i$ to simplify notation and we use the interpretation of Φ_i in terms of twisted F-objects. The sheaf Φ_i is a vacuum extension of a sheaf with non-empty fibers on an open subset U of X. Let $\bar{x}$ be a geometric point of U. There exists an étale neighborhood V of $\bar{x}$ in X and a section e of Φ_i over V/X. By restricting the neighborhood V, e comes from an F-torsor P on $V \times_X Y$. We can then twist by P the sheaves F, F' and C and obtain an exact sequence on V :

$$f_\star(F'^P) \to f_\star(C^P) \to \mathrm{R}^1 f_\star(F^P) \xrightarrow{v^P} \mathrm{R}^1 f_\star(F'^P).$$

We know that $\Phi_i|_V$ is identified with the kernel of v^P. We deduce $\Phi_i|_V \simeq f_\star(F'^P)\backslash f_\star(C^P)$. However, this sheaf is constructible according to Theorem 1.1. □

Consider the hypotheses of Theorem 1.2. According to [**SGA 4** IX 2.14], we can find a monomorphism of group sheaves

$$F \to F' = \prod_{i=1}^{r} \pi_{i\star}(G_i)$$

for finite morphisms $\pi_i : Y_i \to Y$ and finite groups G_i for $i = 1, ..., r$. Note that according to the proof of *loc. cit.*, we can assume that the groups G_i are of $\mathbb{L}$-torsion, since F is of $\mathbb{L}$-torsion. According to the previous lemma, we are reduced to the case of F'. Since $\pi_{i\star}$ is exact, we are therefore reduced to the case of the morphism $f \circ \pi_i$ and the constant sheaf on Y_i of group G_i for each index i.

3.2. Reduction to the case of an open immersion. The key lemma in this reduction step is the following :

Lemme 3.2.1. *Let G be a finite group. Consider a commutative diagram*

$$\begin{tikzcd} & Y' \arrow[dr, "g"] & \\ Y \arrow[ur, "h"] \arrow[rr, "f"] & & X \end{tikzcd}$$

of finite type morphisms between Noetherian schemes. The following assertions are true :

(i) *If h is surjective,*
$\mathrm{R}^1 f_\star(G_Y)$ *constructible implies* $\mathrm{R}^1 g_\star(G_{Y'})$ *constructible.*

(ii) *If g is proper,*
$\mathrm{R}^1 h_\star(G_Y)$ *constructible implies* $\mathrm{R}^1 f_\star(G_Y)$ *constructible.*

Démonstration. Recall that we have the exact sequence of sheaves of sets on $X_{\text{ét}}$ (cf. [**SGA 4** XII 3.2]) :

$$* \to \mathrm{R}^1 g_\star(h_\star G_Y) \xrightarrow{u} \mathrm{R}^1 f_\star(G_Y) \xrightarrow{v} g_\star \mathrm{R}^1 h_\star(G_Y).$$

Consider assertion (i). Since h is surjective, the adjunction morphism

$$G_{Y'} \to h_\star h^\star G_{Y'} = h_\star G_Y$$

is a monomorphism. Applying Lemma 3.1.1, it suffices to show that $\mathrm{R}^1 g_\star(h_\star G_Y)$ is constructible. We can then conclude since the morphism $u : \mathrm{R}^1 g_\star(h_\star G_Y) \to \mathrm{R}^1 f_\star(G_Y)$ is a monomorphism.

Now consider assertion (ii). Since g is proper, the proper base change theorem [**SGA 4** XIV 1.1] combined with Theorem 1.1 shows that the source of u is constructible. By hypothesis and a new application of Theorem 1.1, the target of v is constructible.

It then suffices to reason as in the proof of 3.1.1 on the fibers of the morphism v associated with a finite family of local sections of $g_\star \mathrm{R}^1 h_\star(G_Y)$ which is generating. Each of its fibers is a vacuum extension of a sheaf locally isomorphic to the twisted sheaf $\mathrm{R}^1 g_\star((h_\star G_Y^P))$ for one of its local sections represented by a torsor P. Since this sheaf is always constructible, we can conclude. □

Now consider the hypotheses of Theorem 1.2, in the case $F = G_Y$. Since Y is Noetherian, there exists a Zariski covering $\pi : W \to Y$ such that W is affine. According to assertion (i) of the above lemma, it suffices to prove the theorem for $f \circ \pi$. We can therefore assume that Y is affine.

The morphism $f : Y \to X$ is then quasi-projective. We can therefore consider a factorization $Y \xrightarrow{j} Y' \xrightarrow{g} X$ of f such that g is projective and j is an open immersion. According to assertion (ii) of the above lemma, we are reduced to the case of the open immersion j.

3.3. Reduction to Theorem 1.3 (i.e., codimension 2).

3.3.1. Thanks to the two preceding reduction steps, we are reduced to the case of a dense open immersion $j : U \to X$, X quasi-excellent, and a constant sheaf on U of group G. In this reduction step, we consider the codimension of the complement Z of U in X.

To show that $\mathrm{R}^1 j_\star(G_U)$ is constructible, we can reason locally on X. We can therefore assume that X is Noetherian. Consider the normalization $p : X' \to X$ of X as well as the Cartesian square :

$$\begin{tikzcd} U' \arrow[r, "j'"] \arrow[d, "q"'] & X' \arrow[d, "p"] \\ U \arrow[r, "j"] & X. \end{tikzcd}$$

Since q is surjective, the adjunction morphism $G_U \to q_\star q^\star(G_U) = q_\star G_{U'}$ is a monomorphism. It therefore suffices to show that $\mathrm{R}^1 j_\star(q_\star G_{U'})$ is constructible. Since p and q are finite, $\mathrm{R}^1 j_\star(q_\star G_{U'}) = p_\star \mathrm{R}^1 j'_\star(G_{U'})$. We can therefore assume that X is normal.

Thus, X is a disjoint union of its irreducible components, and we can therefore assume it to be integral. Let K be its function field. For a finite extension L/K, we can consider the normalized scheme X' of X in L, as well as the following Cartesian diagram :

$$\begin{tikzcd} U' \arrow[r, "j'"] \arrow[d, "q"'] & X' \arrow[d, "p"] \\ U \arrow[r, "j"] & X. \end{tikzcd}$$

By adjunction, we obtain a canonical morphism

$$\phi_{U/X}^{(L)} : \mathrm{R}^1 j_\star(G_U) \to p_\star \mathrm{R}^1 j'_\star(G_{U'})$$

induced by the morphism of stacks on U which associates to a G-covering of an étale scheme V/U its pullback on $V' = V \times_U U'$.

Lemme 3.3.2. *Consider the preceding hypotheses and notations. Then, the following conditions are equivalent :*

(i) $\mathrm{R}^1 j_\star(G_U)$ *is constructible.*
(ii) *There exists a finite separable extension L/K such that $\phi_{U/X}^{(L)}$ is trivial.*

Démonstration. $(ii) \Rightarrow (i)$: Let L/K be a finite extension such that $\phi_{U/X}^{(L)}$ is trivial. With the preceding notations, we set $C = q_\star(G_{U'})/G_U$, a sheaf on $U_{\text{ét}}$ pointed in the obvious way. We can then form the exact sequence of pointed sheaves (cf. [**SGA 4** xii 3.1]) :

$$j_\star q_\star(G_{U'}) \to j_\star(C) \to \mathrm{R}^1 j_\star(G_U) \xrightarrow{(1)} \mathrm{R}^1 j_\star\left(q_\star(G_{U'})\right).$$

Note that, since p is finite, $\mathrm{R}^1 j_\star\left(q_\star(G_{U'})\right) = p_\star \mathrm{R}^1 j'_\star(G_{U'})$ and the morphism (1) is identified with the morphism $\phi_{U/X}^{(L)}$. By hypothesis, the above exact sequence therefore implies that $\mathrm{R}^1 j_\star(G) = j_\star(C)/j_\star q_\star(G_{U'})$ which concludes the proof according to Theorem 1.1.
$(i) \Rightarrow (ii)$: Consider a generating family $(e_1, ..., e_n)$ of sections of $\mathrm{R}^1 j_\star(G_U)$. Since the scheme X is assumed Noetherian, we can assume that e_i is defined on an étale finite type X-scheme V_i. By possibly replacing V_i by a finite type étale covering, we can further assume that section e_i corresponds to a covering $P_i \to V_i \times_X U$. Since $V_i \times_X U$ is quasi-compact, the scheme P_i is quasi-compact : let L_{ij}/K be the finite family of field extensions corresponding to the generic points of P_i. We consider the normal closure L of an extension of K composed of the L_{ij}/K for all indices (i, j). By definition, $\phi_{U/X}^{(L)}(e_i) = *$ for every integer i and the result follows since $(e_1, ..., e_n)$ is generating. □

Remarque 3.3.3. Consider the notations preceding the lemma. In terms of G-coverings, the triviality of the morphism $\phi_{U/X}^{(L)}$ is interpreted as follows :

(i) For every geometric point $\bar{s}$ of Z and for every G-covering $\pi : P \to X_{(\bar{s})} - Z_{(\bar{s})}$, the covering $\phi_{U/X,\bar{s}}^{(L)}(P)$ is trivial.
(ii) For every étale scheme V/X and every G-covering $\pi : P \to V - Z_V$, there exists an étale covering $W/V \times_X X'$ such that $\pi|_{W - Z_W}$ extends to W (the extension is then unique (up to unique isomorphism) since X' is normal).

In the case of open immersions, we can strengthen Lemma 3.2.1 as follows :

Lemme 3.3.4. *Consider a commutative diagram*

$$\begin{tikzcd} & V \arrow[dr, "k"] & \\ U \arrow[ur, "h"] \arrow[rr, "j"] & & X \end{tikzcd}$$

of open immersions such that $U \neq \varnothing$, X is integral, Noetherian, normal, quasi-excellent. Then, the following assertions are true :

(i) *If* $\mathrm{R}^1 j_\star(G)$ *is constructible, then* $\mathrm{R}^1 k_\star(G)$ *is constructible.*

(ii) *If* $\mathrm{R}^1 h_\star(G)$ *is constructible and for every finite surjective morphism* $X' \to X$ *where* X' *is the normalization of* X *in an extension finite of its function field,* $k' = k \times_X X'$, *the sheaf* $\mathrm{R}^1 k'_\star(G)$ *is constructible, then* $\mathrm{R}^1 j_\star(G)$ *is constructible.*

Démonstration. Note that the hypothesis on U and X implies that $h_\star(G_U) = G_V$ (cf. [**SGA 4** IX 2.14.1]).

Assertion (i) thus simply results from the fact that the canonical morphism $\mathrm{R}^1 k_\star(G_V) \to \mathrm{R}^1 j_\star(G_U)$ is always a monomorphism.

Consider the hypotheses of assertion (ii). Let K be the function field of X. According to the previous lemma applied to h, we can find a finite extension L/K such that $\phi^{(L)}_{U/V} = *$. Let X' be the normalization of X in L/K, $k' : V' \to X'$ the pullback of k on X'. Applying the previous lemma to k', we can find a finite extension E/L such that $\phi^{(E)}_{V'/X'} = *$.

We denote X'' the normalization of X in E/K, $X'' \xrightarrow{p'} X' \xrightarrow{p} X$ the canonical morphisms. We denote h', j', k' (resp. h'', j'', k'') the respective pullbacks of h, j, k on X' (resp. X''). We can then conclude thanks to the commutative diagram following :

$$\begin{tikzcd}
 & \mathrm{R}^1 j_\star(G) \arrow[r] \arrow[dl, dotted] \arrow[d, "\phi^{(L)}_{U/X}"] & k_\star \mathrm{R}^1 h_\star(G) \arrow[d, "k_\star\phi^{(L)}_{U/V}"] \\
p_\star \mathrm{R}^1 k'_\star(G) \arrow[r] \arrow[d, "p_\star\phi^{(E)}_{V'/X'}"'] & p_\star \mathrm{R}^1 j'_\star(G) \arrow[r] \arrow[d, "p_\star\phi^{(E)}_{U'/X'}"] & (pk')_\star \mathrm{R}^1 h'_\star(G) \\
(pp')_\star \mathrm{R}^1 k''_\star(G) \arrow[r] & (pp')_\star \mathrm{R}^1 j''_\star(G). &
\end{tikzcd}$$

The dotted arrow exists due to the exactness of the middle horizontal sequence and of $\phi^{(L)}_{U/V} = *$. We conclude since $\phi^{(E)}_{U/X} = (p_\star \phi^{(E)}_{U'/X'}) \circ \phi^{(L)}_{U/X}$. □

Lemme 3.3.5. *Let* X *be a regular scheme and* Z *a regular closed subscheme of* X. *Let* $j : U \to X$ *be the complementary open immersion. Let* n *be the order of* G. *Then, if* n *is invertible on* X, $\mathrm{R}^1 j_\star(G_U)$ *is constructible.*

Démonstration. The assertion is Zariski local on X. We can therefore assume that X is affine irreducible and Z is irreducible.

If Z is of codimension greater than 2 in X, it follows from Zariski-Nagata's purity theorem (cf. [**SGA 1** X 3.1, 3.3]) that $\mathrm{R}^1 j_\star(G) = *$ which concludes.

In codimension 1, we can assume that Z admits a regular parameter $f \in A$. Consider the scheme

$$X' = \operatorname{Spec}\bigl(A[t]/(t^n - f)\bigr).$$

Abhyankar's absolute lemma (cf. [**SGA 1** XIII 5.2]) then shows precisely that for every geometric point $\bar{y}$ of Z, for every covering $E \xrightarrow{\pi} U \times_X X_{(\bar{y})}$ principal Galois of group G, the covering $\pi \times_X X' : E' \to U \times_X X' \times_X X_{(\bar{y})}$ extends to $X' \times_X X_{(\bar{y})}$. It is therefore trivial and Lemma 3.3.2 accompanied by Remark 3.3.3 allows us to conclude. □

Let's return to the general case of an open immersion $j : U \to X$ with complementary closed subset Z, X being assumed integral, Noetherian, normal, and quasi-excellent. Assume that the order of G is invertible on X.

Let T be the union of the singular loci of X and Z. Let $V = X - T$ and $W = X - (Z \cup T)$ and consider the corresponding open immersions :

$$\begin{tikzcd}
 & V \arrow[dr, "k"] & \\
W \arrow[ur, "h"] \arrow[rr, "\gamma"] \arrow[dr, "l"'] & & X \\
 & U \arrow[ur, "j"'] &
\end{tikzcd}$$

According to the previous lemma, $\mathrm{R}^1h_\star(G)$ is constructible. According to Lemma 3.3.4, we are therefore reduced to proving that for every finite surjective morphism $X' \to X$ such that X' is the normalization of X in its function field, $k' = k \times_X X'$, the sheaf $\mathrm{R}^1k'_\star(G)$ is constructible. This last assertion is indeed implied by Theorem 1.3.

4. Codimension 2 case without torsion hypothesis

4.1. Resolution of singularities.

Lemme 4.1.1. *Let X be a normal, connected and excellent scheme, $Z \subset X$ a closed subset of codimension greater than 2 and $j : U \to X$ the complementary open immersion.*

Then, there exists a closed subset $T \subset Z$ of codimension greater than 3 in X such that for all geometric points $\bar{s}$ and $\bar{t}$ of $Z - T$ and all specialization $\eta : X_{(\bar{t})} \to X_{(\bar{s})}$, the kernel of the specialization morphism

$$\eta^\star : \mathrm{R}^1j_\star(G)_{\bar{s}} \to \mathrm{R}^1j_\star(G)_{\bar{t}}$$

is trivial.

Démonstration. In this proof, we consider Z endowed with its reduced structure as a subscheme of X.

If we denote by X_{sing} the singular locus of X, the subset

$$T_0 = \overline{(X_{\mathrm{sing}} - Z)} \cap Z$$

is of codimension greater than 3 in X. We can therefore assume that X_{sing} is included in Z. Since X is excellent, Z is also excellent. So by possibly removing a nowhere dense closed subset of Z, we can assume that Z is regular. We then reduce to the case where Z is furthermore integral and of codimension 2.

According to [**Lipman, 1978**], we can resolve the singularity of X at the maximal point of Z by a sequence of blow-ups and normalizations. So, by possibly removing a nowhere dense closed subset of Z again, we can assume that there exists a diagram formed by Cartesian squares

$$\begin{tikzcd} Z' \arrow[r] \arrow[d] & X' \arrow[d, "p"'] & U' \arrow[l, "j'"'] \arrow[d, "q"] \arrow[dl, "f"'] \\ Z \arrow[r] & X & U \arrow[l, "j"] \end{tikzcd}$$

such that X' is regular, Z' is a divisor in X', every irreducible component of Z' dominates Z, p is proper and q is an isomorphism.

We therefore deduce $\mathrm{R}^1j_\star(G) = \mathrm{R}^1f_\star(G)$. Since X' is regular and U' dominant in X', we obtain $j'_\star(G_{U'}) = G_{X'}$ (cf. [**SGA 4** IX 2.14.1]). We thus obtain a canonical monomorphism $\rho : \mathrm{R}^1p_\star(G) \to \mathrm{R}^1f_\star(G)$.

Considering the notations of the lemma (where we assumed $T = \varnothing$), we thus obtain a commutative diagram of pointed sets

$$\begin{tikzcd} \mathrm{R}^1p_\star(G)_{\bar{s}} \arrow[r, "\rho_{\bar{s}}"] \arrow[d, "(1)"'] & \mathrm{R}^1f_\star(G)_{\bar{s}} \arrow[d, "\eta^\star"] \\ \mathrm{R}^1p_\star(G)_{\bar{t}} \arrow[r, "\rho_{\bar{t}}"] & \mathrm{R}^1f_\star(G)_{\bar{t}}. \end{tikzcd}$$

According to the constructibility theorem applied to p (cf. [**SGA 4** XIV 1.1]), the sheaf of groups $\mathrm{R}^1p_\star(G)$ is constructible. From the fact that p is an isomorphism above U, we obtain that by removing a proper closed subset of Z, we can even assume that $\mathrm{R}^1p_\star(G)$ is locally constant. It therefore follows from [**SGA 4** IX 2.13 (i)] that the morphism (1) is injective. From the foregoing, $\rho_{\bar{t}}$ is also injective. We thus obtain that the composition $\eta^\star \circ \rho_{\bar{s}}$ is a monomorphism. It therefore suffices to verify that the kernel of $\eta^\star$ is included in the image of $\rho_{\bar{s}}$.

By pulling back the situation to $X_{(\bar{s})}$, we can assume that $X = X_{(\bar{s})}$ to simplify the notation — note that Z remains integral because, being assumed regular, it is geometrically unibranch. We are given a principal G-covering $P \to U'$ which is trivial on $U' \times_X X_{(\bar{t})}$. Let $\bar{P}$ be the normal closure of X' in P/U'. We will show that $\bar{P}/X'$ is étale and thus gives an antecedent to the class of P/U' by the map ρ_s as expected.

By construction $\bar{P} \times_{X'} U' = P$, so $\bar{P}/X'$ is unramified over U'. It therefore suffices to show that $\bar{P}/X'$ is unramified over Z'. According to Zariski-Nagata's theorem (cf. [**SGA 1** x 3.1]), it suffices to show that $\bar{P}/X'$ is unramified at every point of codimension 1 of the regular scheme X'. From the foregoing, it suffices to treat the maximal points z' of Z'. However, such a point maps to the maximal point of Z by construction. From the fact that η is a morphism of specialization, it follows that there exists a point y' of $Z' \times_X X_{(\bar{t})}$ that maps to z'. From the fact that $P \times_X X_{(\bar{t})}$ is trivial on $U' \times_X X_{(\bar{t})}$, it follows that $\bar{P} \times_X X_{(\bar{t})}$ is unramified over y'. We deduce that $\bar{P}/X'$ is unramified over z' which concludes the proof. □

4.2. A "Lefschetz-like" argument. For this argument, we will use the following lemma, which is an application of results related to "the Lefschetz method" from [**SGA 2** x §2].

Lemme 4.2.1. *Let X be a normal excellent connected scheme, D an effective connected Cartier divisor in X, and $Z \subset D$ a closed subset of codimension greater than 2. Then, $D - Z$ is connected.*

Démonstration. It suffices to show that for every point s of Z, the scheme $(D - Z) \times_X X_{(s)}$ is connected. By induction on $\dim(\mathcal{O}_{X,s})$, it suffices to show that $(D \times_X X_{(s)} - \{s\})$ is connected. We can therefore assume that X is local, of dimension greater than 3 and that Z is its closed point. Consider the completion $\hat{X}$ of the local scheme X. Since X is excellent, $\hat{X}$ is still normal. Since the morphism $\hat{X} \to X$ is surjective, it suffices to show that $(D - Z) \times_X \hat{X}$ is connected. We can therefore further assume that X is complete.

Then, the punctured spectrum $X' = X - Z$ is normal, connected and equidimensional of dimension greater than 2. It follows from Serre's normality criterion (cf. [**Matsumura, 1989**, 23.8]) that for every closed point x of X',

$$\operatorname{prof}(\mathscr{O}_{X',x}) \geq 2.$$

Thus, according to [**SGA 2** x 2.1] (see also more directly [**SGA 2** ix 1.4]), we obtain a canonical isomorphism

$$\Gamma(X', \mathscr{O}) \simeq \Gamma\left(\widehat{X'}^D, \mathscr{O}\right)$$

where $\widehat{X'}^D$ denotes the formal completion of X' along D, which concludes the proof. □

Lemme 4.2.2. *Let X be a normal excellent scheme, D an effective Cartier divisor in X and $Z \subset D$ a non-empty closed subset of codimension greater than 2. Let $U = X - Z$, $V = D - Z$.*

Consider the Cartesian square formed by the obvious immersions

$$\begin{tikzcd} V \arrow[r, "j'"] \arrow[d, "i_U"'] & D \arrow[d, "i"] \\ U \arrow[r, "j"] & X. \end{tikzcd}$$

Then, the associated base change morphism

$$i^\star \mathrm{R}^1 j_\star(G) \to \mathrm{R}^1 j'_\star(G)$$

is a monomorphism.

Démonstration. We can assume that X and D are strictly local Henselian. We must show that the restriction morphism

$$\mathrm{H}^1(U, G) \xrightarrow{i_U^\star} \mathrm{H}^1(V, G)$$

is injective.

Note that the previous lemma already shows us that V is connected. Let P and P' be two G-torsors on U that coincide on V. Consider the sheaf $L = \underline{\mathrm{Isom}}_G(P, P')$ of G-isomorphisms from P to P' on $U_{\text{ét}}$. We must show that it admits a section on U.

Since L is locally constant constructible, it is representable by a finite étale U-scheme denoted U'. Let $V' = U' \times_U V$. By hypothesis, V'/V admits a section.

We will show that for every connected component U'_0 of U' such that $U'_0 \times_U V/V$ admits a section, there exists a section of U'_0/U, which will suffice to conclude.

By possibly replacing U' by U'_0, we can assume for this purpose that U' is connected non-empty. The function fields of U' and U define a finite separable extension L/K. Let X' be the normalization of X in L/K. We still set $D' = X' \times_X D$ and $Z' = X' \times_X Z$.

Note that X' is normal excellent and connected. Thus, the previous lemma implies that $V' = D' - Z'$ is connected. By hypothesis, the étale V-scheme V' admits a section, so $V' = V$. It follows that the étale covering U'/U is of degree 1 over V, so $U' = U$. □

4.3. Reduction of Theorem 1.3 to Theorem 1.4. We can assume that X is affine reduced. For a fixed excellent ring A, we prove by Noetherian induction on the closed subsets Z of $\mathrm{Spec}(A)$ that the result holds for finite reduced Z-schemes. For this, it suffices to prove the result for X affine reduced, assuming the result holds for every finite scheme over a proper closed subscheme of an irreducible component of X.

First, let's show that we can assume that X is normal. Consider the normalization $p : X' \to X$ of X (sum of the normalizations of the irreducible components of X) as well as the Cartesian square :

$$\begin{tikzcd} U' \arrow[r, "j'"] \arrow[d, "q"] & X' \arrow[d, "p"] \\ U \arrow[r, "j"] & X. \end{tikzcd}$$

Since q is surjective, the adjunction morphism $G_U \to q_\star q^\star(G_U) = q_\star G_{U'}$ is a monomorphism. It therefore suffices to show that $\mathrm{R}^1 j_\star(q_\star G_{U'})$ is constructible. Since p and q are finite, $\mathrm{R}^1 j_\star(q_\star G_{U'}) = p_\star \mathrm{R}^1 j'_\star(G_{U'})$. Note that $Z \times_X X'$ is of codimension 2 in every irreducible component of X' (since X is universally catenary). We are therefore reduced to the case where X is normal and we can furthermore assume it to be affine and integral.

The case where X is of dimension 2 results from Theorem 1.4. We can therefore assume that $\dim(X) > 2$.

If $\mathrm{codim}(Z) > 2$, we can find a principal divisor $D \xrightarrow{i} X$ that contains Z (it suffices to take as parameter for D a non-zero element of the integral ring $\mathscr{O}_X(X)$ vanishing on Z). Let $j' : D - Z \to D$ be the induced morphism. According to Lemma 4.2.2, we obtain a monomorphism $i^\star \mathrm{R}^1 j_\star(G) \to \mathrm{R}^1 j'_\star(G)$. From the fact that $\mathrm{R}^1 j_\star(G) = i_\star i^\star \mathrm{R}^1 j_\star(G)$, it suffices to use that $\mathrm{R}^1 j'_\star(G)$ is constructible by induction hypothesis.

Let's place ourselves in the critical case where $\mathrm{codim}(Z) = 2$.

According to Theorem 1.4, the result is known for the semi-localized scheme of X at the generic points of Z which are of codimension 2 in X. There therefore exists, according to Lemma 3.3.2, a finite extension L of the function field K of X such that for every maximal point η of Z of codimension 2 in X, $\phi^{(L)}_{U/X,\bar{\eta}}$ is trivial.

Consider X' the normalization of X in L/K and $p : X' \to X$ its projection. We set $j' = j \times_X X'$ and $Z' = Z \times_X X'$. According to Lemma 4.1.1, there exists a closed subset $T' \subset Z'$ of codimension greater than 3 in X' such that the kernels of the specialization maps of $\mathrm{R}^1 j'_\star(G)$ at points of $Z' - T'$ are trivial.

Let T be the union of the irreducible components of Z of codimension greater than 3 and of the closed subset $p(T')$ in Z. Consider a geometric point $\bar{s}$ of $Z - T$. There exists a maximal geometric point $\bar{t}$ of $Z - T$ and a specialization $\eta : X_{(\bar{t})} \to X_{(\bar{s})}$. Considering the fibers of $\phi^{(L)}_{U/X}$, we obtain the following diagram :

$$\begin{tikzcd}
\mathrm{R}^1 j_\star(G)_{\bar{s}} \arrow[r, "\phi^{(L)}_{U/X,\bar{s}}"] \arrow[d] & {[p_\star \mathrm{R}^1 j'_\star(G)]_{\bar{s}}} \arrow[d, "\eta^\star"] \\
\mathrm{R}^1 j_\star(G)_{\bar{t}} \arrow[r, "\phi^{(L)}_{U/X,\bar{t}}"] & {[p_\star \mathrm{R}^1 j'_\star(G)]_{\bar{t}}}.
\end{tikzcd}$$

According to the choice of L/K, the composition $\eta^\star \circ \phi^{(L)}_{U/X,\bar{s}}$ is trivial. Furthermore, since p is finite, $\eta^\star$ has a trivial kernel. We deduce that $\phi^{(L)}_{U/X,\bar{s}}$ is trivial. According to Lemma 3.3.2, $\mathrm{R}^1 h_\star(G)$ is constructible for the open immersion $h : X - Z \to X - T$. According to Lemma 3.3.4, we are therefore reduced to showing that for every finite surjective morphism $X' \to X$, $\mathrm{R}^1 k'_\star(G)$ is constructible for the open immersion $k' : X' - T' \to X'$.

However, we can find a principal divisor $D \xrightarrow{i} X'$ that contains T'. We set $k'' = k' \times_{X'} D$.

According to Lemma 4.2.2, we obtain a monomorphism

$$i^\star \mathrm{R}^1 k'_\star(G) \to \mathrm{R}^1 k''_\star(G).$$

We can therefore again conclude according to the induction hypothesis applied to k'' and from the fact that $\mathrm{R}^1 k'_\star(G) = i_\star i^\star \mathrm{R}^1 k'_\star(G)$.

5. Principal coverings of a strictly local punctured surface

5.1. Setup. According to Elkik's algebraization theorem (cf. [**Elkik, 1973**], Theorem 5, also recalled in 1.2.6), we can assume that A is complete. Let $X = \mathrm{Spec}(A)$ and $X' = \mathrm{Spec}(A) - \{\mathfrak{m}_A\}$. We denote K the fraction field of A.

We begin by showing that we can assume there exists a regular subring $R \subset A$ such that the extension A/R is finite and generically étale.

According to Cohen's structure theorems (cf. [**Bourbaki**, **AC**, IX, § 2, nº 5, th. 3] if A is of mixed characteristic and [**Bourbaki**, **AC**, IX, § 3, nº 4, th. 2] if A contains a field), there exists a subring $R \subset A$ such that A/R is finite and R is a formal power series ring over a Cohen ring or over a field. The ring R is therefore in particular regular. Let E be the fraction field of R and E' the separable closure of E in K. Let A_0 denote the normal closure of R in E'/E. Then, the morphism $X_0 = \mathrm{Spec}(A_0) \to \mathrm{Spec}(A)$ is finite radical and surjective. According to the topological invariance of the étale site (cf. [**SGA 4** VIII 1.1]), we can replace A by A_0 which is finite generically étale over R, as expected.

First, let's remark the following fact :

Lemme 5.1.1. *Let R be a regular local ring of dimension 2 and A a finite dominant R-algebra such that A is a local and normal ring. Let m be the generic degree of $R \to A$. Then A is a free R-module of rank m.*

Démonstration. Since A is finite dominant over R, $\dim(A) = \dim(R) = 2$. Moreover, since A is local normal of dimension 2, it follows from Serre's criterion that $\mathrm{prof}(A) = 2$ (cf. [**Matsumura, 1989**, ex. 17.3]). So A is a Cohen-Macaulay ring. The lemma then results from [**ÉGA** 0_{IV} 17.3.5 (ii)]. □

Disregarding the group G, we fix an integer $n > 0$ and show that the set of isomorphism classes of étale coverings of X' of degree n is finite. In the rest of this proof, the étale coverings considered will be assumed connected.

We reason by contradiction. Consider a sequence $(P'_i \to X')_{i \in \mathbf{N}}$ of étale coverings of degree n such that for any $i \neq j$, P'_i is not X'-isomorphic to P'_j.

Let K be the fraction field of A. For every integer i, P'_i/X' corresponds to a finite separable extension L_i/K. Let B_i denote the integral closure of A in L_i, $P_i = \mathrm{Spec}(B_i)$. Note furthermore that according to the previous lemma, A/R is free of rank m and B_i/R is free of rank nm.

5.1.2. *Discriminant questions.* Recall for the purposes of the following proof the following considerations :

DÉFINITION 5.1.3. Let B/A be a finite free algebra of rank n. Let $\mathscr{B} = (e_i)_{1\leq i\leq n}$ be a basis of B/A. The determinant of the matrix $\big(\mathrm{Tr}_{B/A}(e_ie_j)\big)_{1\leq i,j\leq n}$ is called the **discriminant of** B/A **relative to** $\mathscr{B}$. Its class in the multiplicative monoid $A/(A^\times)^2$ is independent of $\mathscr{B}$. We denote it by $\mathrm{disc}_{B/A}$.

By abuse, we will consider the class $\mathrm{disc}_{B/A}$ as an element of A. Recall that B/A is étale if and only if $\mathrm{disc}_{B/A}$ is invertible in A. Subsequently, we will need the following formula (cf. [**Ramero, 2005**, 2.1.4]) : Let B/A and C/B be two finite free algebras. Let n be the rank of C/B. Then,

$$\mathrm{disc}_{C/A} = \mathrm{disc}^n_{B/A} \cdot N_{B/A}(\mathrm{disc}_{C/B}).$$

Returning to the situation of the previous subsection, we consider an ideal $\mathfrak{p}$ of height 1 of R. Let $A_\mathfrak{p}$ (resp. $B_{i,\mathfrak{p}}$) be the semi-localized ring of A (resp. B_i) corresponding to the fiber above $\mathfrak{p}$.

Note that $A_\mathfrak{p}$ is normal of dimension 1. Furthermore, since by hypothesis B_i/A is étale finite of degree n over the punctured spectrum of A, the extension of local rings $B_{i,\mathfrak{p}}/A_\mathfrak{p}$ is free of rank n. According to the formula recalled previously,

$$\mathrm{disc}_{B_{i,\mathfrak{p}}/R_\mathfrak{p}} = \mathrm{disc}^n_{A_\mathfrak{p}/R_\mathfrak{p}} \cdot N_{A_\mathfrak{p}/R_\mathfrak{p}}(\mathrm{disc}_{B_{i,\mathfrak{p}}/A_\mathfrak{p}}).$$

However, A/R (resp. B_i/R) is generically étale and $B_{i,\mathfrak{p}}/A_\mathfrak{p}$ is étale. We deduce from the previous relation that the element $(\mathrm{disc}_{B_i/R})(\mathrm{disc}^n_{A/R})^{-1}$ of Frac(R)$^\times$ belongs to $R_\mathfrak{p}^\times$. Since this is valid for every $\mathfrak{p}$ and R is normal, we deduce :

$$\frac{\mathrm{disc}_{B_i/R}}{\mathrm{disc}^n_{A/R}} \in R^\times \tag{5.1.3.1}$$

5.2. Key lemma. The technique for finding a contradiction to the situation we arrived at after the setup 5.1 relies on the use of ultrafilters. In the following section, we recall this theory and prove the results that will be useful to us. The essential point of the proof then reduces to proving the *key lemma* 5.2.8 as we show in paragraph 5.2.9. The proof of this lemma is given in the last section, 5.2.10.

5.2.1. *Ultraproducts.*

DÉFINITION 5.2.2. Let I be a set and $P(I)$ the power set of I. An **ultrafilter** $\mathscr{F}$ on I is given by a set of subsets of I satisfying the following properties :

(i) $\forall F \in \mathscr{F},\ \forall G \in P(I),\ F \subset G \Rightarrow\ G \in \mathscr{F}$.
(ii) $\forall F, G \in \mathscr{F},\ F \cap G \in \mathscr{F}$.
(iii) $\forall F \in P(I), F \in \mathscr{F}$ or $I - F \in \mathscr{F}$.
(iv) $\varnothing \notin \mathscr{F}$.

In the rest of the proof, we will understand an ultrafilter $\mathscr{F}$ as an ordered set such that for all $F, G \in \mathscr{F}$,

$$F \leq G \Leftrightarrow F \supset G.$$

Note that it then follows from the definition that $\mathscr{F}$ is a filtered set.

EXEMPLE 5.2.3. Let a be an element of I. Then, the set of subsets of I containing a is an ultrafilter $\mathscr{F}$ on I. In this case, we say that $\mathscr{F}$ is **principal**.

According to Zorn's lemma, there exist non-principal ultrafilters on an infinite set I.

DÉFINITION 5.2.4. Let $\mathscr{F}$ be an ultrafilter on a set I and $\mathscr{C}$ a category admitting filtered inductive limits and products.

Let $(X_i)_{i\in I}$ be a family of objects of $\mathscr{C}$. The inductive system $(\prod_{i\in F} X_i)_{F\in\mathscr{F}}$ is filtered. We define **the ultraproduct of** $(X_i)_{i\in I}$ **following** $\mathscr{F}$ as the inductive limit of this system :

$$\prod_{i\in I/\mathscr{F}} X_i = \mathrm{colim}_{F\in\mathscr{F}}\left(\prod_{i\in F} X_i\right).$$

If $(X_i)_i$ is the constant family with value an object X, we denote $X^{I/\mathscr{F}}$ its ultraproduct, called **the ultrapower of** X **following** $\mathscr{F}$. We always have the diagonal map $X \to X^{I/\mathscr{F}}$.

We will note in particular that an element x of the ultraproduct $\prod_{i\in I/\mathscr{F}} X_i$ is represented by a sequence $(x_i)_{i\in F}$ for an element $F \in \mathscr{F}$. Furthermore, given another element $y = (y_j)_{j\in G}$ of this ultraproduct, $x = y$ if and only if there exists $H \in \mathscr{F}$ such that $H \subset F \cap G$ satisfying $\forall i \in H$, $x_i = y_i$.

We will use this notion in the case of rings or modules and we will particularly use the following lemma :

Lemme 5.2.5. *Let I be a set and $\mathscr{F}$ an ultrafilter on I.*
Consider a family $(A_i)_{i\in I}$ of rings. Let $A_\infty = \prod_{i\in I/\mathscr{F}} A_i$.

(i) *If for all $i \in I$, A_i is integral (resp. a field, a separably closed field), it is the same for A_∞.*
(ii) *If for all $i \in I$, A_i is local (resp. local Henselian) with maximal ideal $\mathfrak{m}_i$, A_∞ is local (resp. local Henselian) with maximal ideal $\prod_{i\in I/\mathscr{F}} \mathfrak{m}_i$.*

Consider a family of algebras $(B_i/A_i)_{i\in I}$, B_∞/A_∞ its ultraproduct following $\mathscr{F}$:

(iii) *If for all $i \in I$, B_i/A_i is a local extension of local rings (resp. free of rank m), it is the same for B_∞/A_∞.*

Consider a ring A, and $A_\infty = A^{I/\mathscr{F}}$ its ultrapower.

(iv) *If M is a finitely presented A-module,*
$M \otimes_A A^{I/\mathscr{F}} = M^{I/\mathscr{F}}$.
(v) *If A is coherent, the diagonal map $A \to A^{I/\mathscr{F}}$ is flat.*

Démonstration. According to the characterization of elements of an ultraproduct recalled before the statement of the lemma, an element $x \in A_\infty$ is invertible (resp. zero) if and only if there exists $F \in \mathscr{F}$ such that x is represented by a family $(x_i)_{i\in F}$ such that for all index $i \in F$, x_i is invertible (resp. zero) in A_i.
(i) in the first two respective cases follows easily. To show that the ultraproduct $k_\infty = A_\infty$ of separably closed fields $k_i = A_i$ is still separably closed, consider a separable polynomial P with coefficients in k_∞ of degree $d > 0$. By reasoning about the coefficients of the polynomial P, we can assume that P is represented by a family of polynomials $(P_i)_{i\in F}$ for an element $F \in \mathscr{F}$ and polynomials P_i with coefficients in k_i. By possibly restricting F, we can assume that for all $i \in F$, P_i is separable of degree d. It therefore has a root x_i in k_i. It is then immediate that the family $(x_i)_{i\in F}$ represents an element x of k_∞ such that $P(x) = 0$.
(ii) : we first treat the non-respé assertion. Let $\mathfrak{m}_\infty = \prod_{i\in I/\mathscr{F}} \mathfrak{m}_i$. It is clearly an ideal of A_∞. Let x be an element of $A_\infty - \mathfrak{m}_\infty$, represented by a family $(x_i)_{i\in F}$. The hypothesis on x is translated as follows :

$$\nexists H \in \mathscr{F}, H \subset F \mid \forall i \in H, x_i \in \mathfrak{m}_i.$$

Let $G = \{i \in F \mid x_i \notin \mathfrak{m}_i\}$. Then, by hypothesis, $G \notin \mathscr{F}$. So $H = F - G$ belongs to $\mathscr{F}$ because $\mathscr{F}$ is an ultrafilter. Since for all $i \in H$, $x_i \notin \mathfrak{m}_i$, x_i is invertible in the local ring A_i. We deduce that x is invertible according to the characterization of invertible elements recalled at the beginning of the proof.

To show the respé assertion, we reason as in the case of the ultraproduct of separably closed fields treated above.
(iii) concerning the first assertion follows easily from (ii). The respé assertion is obvious.
(iv) follows easily from the elementary fact that the functor $M \otimes_A -$ commutes with products if M is finitely presented.
(v) follows from the fact that when A is coherent, a product of flat A-modules is flat (cf. [**Chase, 1960**]). □

Remarque 5.2.6. From a conceptual point of view, this lemma can be seen as a corollary of Łoś's theorem (cf. [**Bell & Slomson, 1969**, Chap. 5, Section 2]).

5.2.7. *Statement of the key lemma.* Let's return to the situation at hand. Let $\mathscr{F}$ be a non-principal ultrafilter on $\mathbf{N}$. Let B_∞ (resp. A_∞, R_∞) be the local ring obtained by ultraproduct following $\mathscr{F}$ of $(B_i)_{i\in\mathbf{N}}$ (resp. A, R). Let $\widehat{B_\infty}$, $\widehat{A_\infty}$, $\widehat{R_\infty}$ be their respective completions. We thus obtain the following towers of local rings :

$$\begin{tikzcd}
B_i \arrow[d, dash] & B_\infty \arrow[r, dash] \arrow[d, dash, "\varphi"'] & \widehat{B_\infty} \arrow[d, dash, "\hat{\varphi}"'] \\
A \arrow[r, dash] \arrow[d, dash] & A_\infty \arrow[r, dash] \arrow[d, dash] & \widehat{A_\infty} \arrow[d, dash] \\
R \arrow[r, dash] & R_\infty \arrow[r, dash] & \widehat{R_\infty}.
\end{tikzcd}$$

According to point (iii) of Lemma 5.2.5, A_∞ and B_∞ are free R_∞-algebras of finite rank. If k_∞ denotes the residue field of R_∞, we deduce that $A_\infty \otimes_{R_\infty} k_\infty$ and $B_\infty \otimes_{R_\infty} k_\infty$ are finite local k_∞-algebras : their maximal ideals are therefore nilpotent. It follows that the completion of the local ring A_∞ (resp. B_∞) coincides with its completion with respect to the maximal ideal of R_∞. Since A_∞ (resp. B_∞) is a free R_∞-module, we deduce :

$$\widehat{A_\infty} = A_\infty \otimes_{R_\infty} \widehat{R_\infty}, \ \ \widehat{B_\infty} = B_\infty \otimes_{R_\infty} \widehat{R_\infty}.$$

Considering the spectrum of the local rings on the first two lines, we obtain the following commutative diagram :

$$\begin{tikzcd}
P_i \arrow[d, dash] & P_\infty \arrow[r, dash] \arrow[d, dash, "p"] & \hat{P}_\infty \arrow[d, dash, "\hat{p}"] \\
X \arrow[r, dash] & X_\infty \arrow[r, dash] & \hat{X}_\infty
\end{tikzcd}$$

and the preceding discussion shows that the square that appears in this diagram is Cartesian. Considering again the punctured spectrum of the local rings A, A_∞ and $\widehat{A_\infty}$, we obtain by base change the diagram following :

$$\begin{tikzcd}
P_i' \arrow[d, dash] & P_\infty' \arrow[r, dash] \arrow[d, dash, "p'"] & \hat{P}_\infty' \arrow[d, dash, "\hat{p}'"] \\
X' \arrow[r, dash] & X_\infty' \arrow[r, dash] & \hat{X}_\infty'.
\end{tikzcd}$$

Note that the arrows in the bottom line exist because, the maximal ideal $\mathfrak{M}_A$ being of finite type, one obtains easily for the respective maximal ideals of A_∞ and $\widehat{A_\infty}$:

$$\mathfrak{M}_{A_\infty} = \mathfrak{M}_A \cdot A_\infty, \ \ \mathfrak{M}_{\widehat{A_\infty}} = \mathfrak{M}_A \cdot \widehat{A_\infty}.$$

Lemme 5.2.8. (1) *The scheme $\hat{X}_\infty$ is Noetherian.*

(2) *The morphism $\hat{X}_\infty \to X$ is flat with regular geometric fibers.*

(3) *The morphism $\hat{X}_\infty \to X_\infty$ is a closed immersion whose image contains all the closed points of X_∞'.*

(4) *The finite morphism $\hat{p} : \hat{P}_\infty \to \hat{X}_\infty$ is étale over $\hat{X}_\infty'$.*

(5) *The finite morphism $p : P_\infty \to X_\infty$ is étale over X_∞'.*

5.2.9. *Consequences of the key lemma.* Let's show why the previous lemma allows us to complete the proof by contradiction.

According to point (ii) of Lemma 5.2.5, A_∞ is a Henselian local ring. We can therefore apply variant c_2 of the smooth base change theorem (cf. **XX**-4.2.1, case (ii)) to the morphism $\hat{X}_\infty \to X$ and to the open set $\hat{X}_\infty'$. We deduce that there exists an étale covering $Q \to X'$ such that $\hat{P}_\infty' \simeq Q \times_{X'} \hat{X}_\infty'$.

According to Gabber's rigidity theorem (cf. 2.1.1) applied to A_∞, for every finite group G, the morphism

$$\mathrm{H}^1(X_\infty', G) \to \mathrm{H}^1(\hat{X}_\infty', G)$$

is an isomorphism. We therefore deduce an isomorphism :

$$P_\infty' \simeq Q \times_{X'} X_\infty'. \tag{5.2.9.1}$$

For every element F of the ultrafilter $\mathscr{F}$, we set : $P_F = \mathrm{Spec}(\prod_{i\in F} B_i), X_F = \mathrm{Spec}(\prod_{i\in F} A), Y_F = \mathrm{Spec}(\prod_{i\in F} R$, and we denote $p_F : P_F \to X_F, q_F : X_F \to Y_F$ the canonical arrows. By definition, the morphism $p : P_\infty \to X_\infty$ is the projective limit following $F \in \mathscr{F}$ of the morphisms p_F. We denote $p'_F : P'_F \to X'_F$ the pullback of p above X'.

Since for all $i \in F$, B_i/R (resp. A/R) is finite and free of rank nm (resp. m), we easily obtain that $q_F \circ p_F$ (resp. q_F) is finite free of rank nm (resp. m). In particular, p_F is finite of finite presentation. We can then apply [**ÉGA** IV$_3$ 8.8.2] to the families of X'_F-schemes P'_F and $Q \times_{X'} X'_F$ indexed by the elements of the ultrafilter $\mathscr{F}$ and we obtain that the isomorphism (5.2.9.1) lifts for a particular element $F \in \mathscr{F}$ to an isomorphism of the form :

$$P'_F \simeq Q \times_{X'} X'_F.$$

Since the ultrafilter $\mathscr{F}$ is non-principal, F contains at least two distinct elements i and j. The previous isomorphism therefore implies $P'_i \simeq Q \simeq P'_j$ which constitutes the announced contradiction.

5.2.10. *Proof of the key lemma.* Let's prove each assertion of Lemma 5.2.8 to conclude :

Assertion (1) : Note that the maximal ideal of the local ring A_∞ is of finite type according to point (ii) of Lemma 5.2.5. Thus, the completion $\widehat{A_\infty}$ is Noetherian according to [**ÉGA** 0$_{\text{I}}$ 7.2.7, 7.2.8]. The same holds for $\widehat{R_\infty}$ and $\widehat{B_\infty}$.

Assertion (2) : To show that $\hat{X}_\infty \to X_\infty$ is flat, it suffices, according to the flatness criterion by fibers (cf. [**ÉGA** IV$_3$ 11.3.10]), to show that for every integer $l > 0$, the morphism $A/\mathfrak{m}_A^l \to \widehat{A_\infty}/\mathfrak{m}^l_{\widehat{A_\infty}}$ is flat. However, $\widehat{A_\infty}/\mathfrak{m}^l_{\widehat{A_\infty}} = (A/\mathfrak{m}_A^l)^{I/\mathscr{F}}$ and the previous morphism is the diagonal map. We can therefore conclude by using property (v) of Lemma 5.2.5 and the fact that $A/\mathfrak{m}_A^l$ is coherent.
Note that the residue extension $\kappa_A^{I/\mathscr{F}}/\kappa_A$ of A_∞/A is separable. Indeed, for any finite extension L/κ_A, $L \otimes_{\kappa_A} \kappa_A^{I/\mathscr{F}} = L^{I/\mathscr{F}}$ is a field according to point (i) of Lemma 5.2.5. We deduce that the geometric fibers of the morphism $\hat{X}_\infty \to X$ are regular by application of the *formal smoothness localization theorem* of André (cf. [**André, 1974**]).

Assertion (3) : We begin by showing that the morphism $\hat{X}_\infty \to X$ is a closed immersion. It is a matter of showing that $\pi : A_\infty \to \widehat{A_\infty}$ is surjective. However, as we have already seen, $\widehat{A_\infty}$ is the completion (ii) of A_∞ for the $\mathfrak{M}_A$-adic topology. However, $\mathfrak{M}_A$ being of finite type, we obtain that A_∞ is complete (but not necessarily separated) for the $\mathfrak{M}_A$-adic topology — this results from the fact that it is trivially true for products A^F for an element $F \in \mathscr{F}$. The surjectivity of π follows.

To demonstrate the second part of assertion (3), it suffices to apply the following lemma to the maximal ideal of A_∞.

Lemme 5.2.11. *Let A be a ring and I a finitely generated ideal such that $I \subset \mathrm{rad}(A)$. Let $\hat{A}$ be the I-adic completion of A.*
Then the induced morphism $\mathrm{Spec}(\hat{A}) - V(I\hat{A}) \to \mathrm{Spec}(A) - V(I)$ *is surjective on the closed points of the target.*

Let $f : \mathrm{Spec}(\hat{A}) \to \mathrm{Spec}(A)$ be the canonical morphism. To prove the lemma, we must show that for every closed $Z \subset \mathrm{Spec}(A)$,

$$Z \subset V(I) \Leftrightarrow f^{-1}(Z) \subset V(I\hat{A}).$$

Since I is finitely generated, this is equivalent to showing that for every ideal $J \subset A$,

$$\exists n > 0 \mid I^n \subset J \Leftrightarrow \exists n > 0 \mid I^n\hat{A} \subset J\hat{A}.$$

However, this follows easily from Nakayama's lemma.

Assertion (4) : First note that according to [**ÉGA** IV$_2$ 6.14.1], the ring $\widehat{A_\infty}$ is normal. Indeed, A is normal and the morphism $A \to \widehat{A_\infty}$ is normal since it is flat with regular geometric fibers according to Lemma 5.2.8. However, $\widehat{B_\infty}/\widehat{R_\infty}$ is free (of rank nm), hence torsion-free. Since $\widehat{A_\infty}/\widehat{R_\infty}$ is integral, we deduce that $\widehat{B_\infty}/\widehat{A_\infty}$ is torsion-free. Thus, $\widehat{B_\infty}/\widehat{A_\infty}$ is flat in codimension ≤ 1.

(ii) Recall that completion means complete-separated.

However, from relations (5.1.3.1) for all $i \in \mathbf{N}$, we deduce

$$\frac{\mathrm{disc}_{\widehat{B_\infty}/\widehat{R_\infty}}}{\mathrm{disc}^n_{\widehat{A_\infty}/\widehat{R_\infty}}} \in (\widehat{R_\infty})^\times.$$

If $\mathfrak{p}$ is a prime ideal of height ≤ 1 of $\widehat{R_\infty}$, the extension $(\widehat{B_\infty})_\mathfrak{p}/(\widehat{A_\infty})_\mathfrak{p}$ is torsion-free, hence free. The previous relation shows that $\mathrm{disc}_{(\widehat{B_\infty})_\mathfrak{p}/(\widehat{A_\infty})_\mathfrak{p}}$ is invertible which proves that $(\widehat{B_\infty})_\mathfrak{p}/(\widehat{A_\infty})_\mathfrak{p}$ is étale, which proves (4).

Assertion (5) : According to the flatness criterion by fibers (cf. [**ÉGA** IV$_3$ 11.3.10] applied to $P_\infty \to X_\infty \to \mathrm{Spec}(R_\infty)$), for every point x of the punctured spectrum of R_∞, the localized morphism $(P_\infty)_{(x)} \to (X_\infty)_{(x)}$ is flat. As in point (4), relation (5.1.3.1) allows us to show that $(P_\infty)_{(x)}/(X_\infty)_{(x)}$ is étale. Property (5) therefore results from point (3).

APPENDIX A

Fac-similé : Orsay

Transparents de l'exposé d'Ofer Gabber, fait le 27 juin 2005 à la conférence en l'honneur de Luc Illusie ([**Gabber, 2005a**]).

A finiteness theorem for non abelian H^1 of excellent schemes

Ofer Gabber

(IHES, CNRS)

1. Statements

Theorem 1.0 Let $f : X \to Y$ be a morphism of finite type of noetherian schemes and F a constructible sheaf of sets on X_{et}.
Then $f_* F$ is constructible.

The proof is by reducing to the case f proper (SGA 4) and f an open immersion (easier when normalizations are finite)

Theorem 1.1 Let $\mathbb{L}$ be a finite set of primes, $f : X \to Y$ a morphism of finite type between quasi-excellent schemes on which (every $p \in \mathbb{L}$) is invertible. Let F be a constructible $\mathbb{L}$-torsion sheaf of groups on X.
Then $R^1 f_* F$ is constructible.

Also stack version.

Proved here using ultraproducts.

Theorem 1.2. $X, Y, \mathbb{L}, f$ as in 1.1. Y qc. If F is a constructible $\mathbb{L}$-torsion abelian sheaf on X, then the sheaves $R^i f_* F$ are constructible and 0 for $i \gg 0$.

(Planned for Deligne's conference. Using alterations)

2. Reductions.

Th. 1.1 known for proper f, so enough to consider open immersions. Reduce to Y normal affine, $X = Y - Z$ (Z reduced) and F constant.

Enough to find $Y' \to Y$, normalization of Y in a finite extension of $R(Y)$ which kills $R^1 f_* F$.

This exists outside $\mathrm{Sing}(Y) \cup \mathrm{Sing}(Z)$ (Abh. Lemma). Reduce to

<u>Th. 2.1.</u> Let Y be a normal excellent scheme, Z a closed subscheme of cod ≥ 2, $j: Y - Z \longrightarrow Y$, G a finite group. Then $R^1 j_* G$ is constructible.

(The order of G not necessarily inv. on Y)

This will be reduced to the following case:

Th. 2.2. Let A be a strictly henselian excellent normal local ring of dimension 2. For every finite group G

$$H^1(\mathrm{Spec}(A) - \{m\}, G) \quad \text{is finite.}$$

3. Lefschetz

Recall (SGA 2 XIII §2)

Th. 3.0. A cnlr $f \in \mathfrak{m}$, f nonzero divisor, $X = \mathrm{Spec}(A)$ $X' = X - \{m\}$, depth $\mathcal{O}_{X',x} \geq 2$ for closed points of X'. Then

$\Gamma(X', \mathcal{O}) \xrightarrow{\sim} \Gamma(\hat{X}', \mathcal{O})$ so

$\pi_0(X' \cap V(f)) \xrightarrow{\sim} \pi_0(X')$.

Cor. 1. A excellent normal nlr of dim ≥ 3, $f \in A$ nonzero nonunit, then the punctured spectrum of $\underset{= Y}{\mathrm{Spec}(A/fA)}$ is connected.

Y is connected in cod. 1, i.e. if $Z \subset Y$ is of cod ≥ 2 then $Y - Z$ is connected.

<u>Cor. 2.</u> Let X be normal excellent scheme
$D \subset X$ effective Cartier divisor
$Z \subset D$ closed, $\operatorname{cod}_X Z \geq 3$

$$\begin{array}{ccc} D - Z & \xrightarrow{j'} & D \\ \downarrow & & \downarrow \\ X - Z & \xrightarrow{j} & X \end{array}$$

If L is a locally constant constr sheaf on $X - Z$

$$(j_* L)|_D \xrightarrow{\sim} j'_* (L|_{D-Z}).$$

$(R^1 j_* G)|_D$ injects into $R^1 j'_* G$ for every finite group G.

4. Use of desingularization of 2 dimensional schemes.

<u>Lemma 4.1</u>. In the situation of Th. 2.1 if Z_i is an irreducible component of Z of codim 2 in Y then there is an open dense $U_i \subset Z_i$ s.t. the sheaf of pointed sets $F = R^1 j_* G$ on U_i has specialization maps $F_s \to F_t$ (for $t \to s$ map of geom. pts $\overset{\text{topos pts}}{\shortparallel}$ of U_i) with trivial kernel.

Proof. W.M.A. $Z = Z_i$ and that there is $p: Y' \to Y$ proper, birational, isomorphism outside Z, Y' regular, $p^{-1}(Z)_{red}$ is a normal crossings divisor with irr components dominating Z and $p^{-1}(Z) \to Z$ is (univ) locally 0-acyclic.
(Connected Milnor fibres)

5. Proof of Th. 2.1 assuming Th. 2.2.

WMA $Y = \mathrm{Spec}(A)$, A excellent normal domain and that the result is known

for

normalizations of $Y' \subsetneq Y$, Y' irr, in

finite extensions of $R(Y')$.

Using section 4 there is

$\mathrm{Spec}(A') \underset{\text{finite}}{\twoheadrightarrow} \mathrm{Spec}(A)$ which kills

$R^1 j_* G$ outside a cod ≥ 3 locus,

so reduce to Z of cod ≥ 3 and

use section 3.

6. Discriminants.

Lemma 6.1. Let $A = (A_{ij})$ be an $n \times n$ matrix of $m \times m$ matrices over a commutative ring k, with commuting A_{ij}. Let A be the resulting block $nm \times nm$ matrix.

Then $\det A = \det \det A$.

[Hint: replace A_{11} by $A_{11} + t$, invertible over $k' \supset k[t]$.]

Lemma 6.2. Let $A \subset B \subset C$ be comm. rings with C a free B-module with basis f_j $(1 \le j \le n)$ and B a free A-module with basis e_i $(1 \le i \le m)$. (so $(e_i f_j)$ is an A basis of C). We use these bases to define discriminants, e.g. $\operatorname{disc}_{B/A} = \det \operatorname{Tr}_{B/A}(e_i e_j)$.

Then $\operatorname{disc}_{C/A} = (\operatorname{disc}_{B/A})^n \operatorname{Norm}_{B/A}(\operatorname{disc}_{C/B})$.

7. Rigidity.

Th. 7.1. Let (A, I) be a henselian pair with I finitely generated, $U \subset \mathrm{Spec}(A)$ a quasi-compact open $U \supset \mathrm{Spec}(A) - V(I)$, $\hat{A}$ the I-adic completion of A, $\hat{U} \subset \mathrm{Spec}(\hat{A})$ the inverse image of U. Let F be a sheaf of sets (resp. an ind-finite sheaf of groups) on U. Then $H^0(U, F) \xrightarrow{\sim} H^0(\hat{U}, F)$ (resp. $H^1(U, F) \xrightarrow{\sim} H^1(\hat{U}, F)$).

version for stacks.

Can reduce to case F constant, which holds by Elkik's approximation when A is noetherian and also when I is principal and $U = \mathrm{Spec}(A) - V(I)$. In general blow up a f.g. ideal defining the complement of U and use proper base change for stacks (Giraud + extension to non noeth. case) and affine base change to reduce to the principal ideal case.

<u>Th. 7.2.</u> (close to SGA4 XV)

$$\begin{tikzcd} p: X' \arrow[r] & X \\ U' \arrow[u, hook, "j'"] & U \arrow[u, hook, "j"'] \end{tikzcd}$$

p smooth, X normal excellent, $U' = p^{-1}(U)$

U open containing all points of cod ≤ 1.

Then $p^* R^1 j_* G \xrightarrow{\sim} R^1 j'_* G$ (G finite).

Lemma 7.3. Let Z be a nowhere dense closed subscheme of a noetherian scheme X. Then the following conditions are equivalent

(1) Let $p: X' \longrightarrow X_{red}$ be the normalization. Then $p^{-1}(Z)$ is of codim ≥ 2 in X'.

(2) for every $y \in Z$, the irreducible components of $\operatorname{Spec}(\mathcal{O}^h_{X,y})$ are of dim ≥ 2

(3) same for $\hat{\mathcal{O}}_{X,y}$.

If this holds we say that Z is $c2$ in X.

Prop. 7.4. Let $A \longrightarrow B$ be a local homomorphism of nlr, formally smooth for m-adic topologies. Then There is a direct system of local essentially smooth A-algebras B_i with

$$\hat{B} \simeq (\varinjlim B_i)^\wedge$$

$$B_i \longrightarrow B_j \text{ flat} \quad m_i B_j = m_j .$$

<u>Th. 7.5.</u> Let Z be C2 in X

$p : X' \longrightarrow X$ a flat morphism of noetherian schemes, the fibres of p above points of Z are geometrically regular.

Conclusion as in 7.2.

8. Separable projection.

Lemma 8.1. Let R be a complete noetherian local domain ~~ring~~. Then R has a regular subring R_0, R finite over R_0 with $\mathrm{Frac}(R)$ separable over $\mathrm{Frac}(R_0)$.

Only problem in equal characteristic $p > 0$.
Let k be the residue field of R and $\{b_i\}_{i \in I}$ a p-basis of k. There is a bijection

$$\{\text{coefficient fields of } R\} \longleftrightarrow \{\text{liftings of } b_i\}.$$

Fix a coefficient field.
By the proof of Nagata's Jacobian criterion there is a finite subset $J \subset I$ s.t. if $k' = k^p(b_i,\ i \notin J)$ then

$$\Omega = \Omega^1_{R/\overline{k' R^p}}$$

has generic rank $\dim(R) + \mathrm{card}(J)$.
Then change the liftings of b_i $i \in J$ s.t. their differentials are linearly independent in Ω. With this coefficient field

one can take $k' = k$.

Let $t_1, \dots, t_d$ be a system of parameters for R and let $f_1, \dots, f_d \in R$ be s.t. df_i form a basis of $\Omega \otimes_R \mathrm{frac}(R)$.

WMA $f_i \in m$.

Let $t_i' = t_i^p (1 + f_i)$.

Can take

$$R_0 = k[[t_1', \dots, t_d']].$$

9. Ultraproducts

Let I be a set. There is a bijection

$$\{\text{ultrafilters on } I\} \longleftrightarrow \mathrm{Spec}\left((\mathbb{Z}/2\mathbb{Z})^I \right)$$

$$F \longmapsto p = \{ \chi_A \mid A \notin F \}.$$

If R_i $(i \in I)$ are rings then the ultraproduct $\prod_{/F} (R_i)$ is $\prod_i (R_i) / \sim$ where

$$(r_i) \sim (s_i) \Leftrightarrow \{ i \mid r_i = s_i \} \in F.$$

In the commutative case there is a map of topological spaces

$f : \mathrm{Spec}(\prod_i R_i) \longrightarrow \mathrm{Spec}((\mathbb{Z}/2)^I)$

defined by "$f^* \chi_A = \chi_A$".

The fibres of f (with the restriction of the structure sheaf) are Spec of the ultraproduct.

R_i all fields, domains, local rings,
$\Rightarrow$ same for $\pi_{/F}(R_i) = R_\infty$

(*) If R_i are local rings whose maximal ideals are generated by n elements then R_∞ has the same property, so its completion is $\underbrace{\text{a}}_{\text{complete}}$ noetherian local ring.

- If furthermore F is not ω complete R_∞ maps onto $\hat{R}_\infty$
- If$_{(*)}$, F is ω complete and R_i are noetherian

then R_∞ is noetherian.

<u>Lemma</u>. If $R_i \to S_i$ $(i \in I)$ are finite maps of noetherian local rings, $\exists n$ s.t. $\forall i$ the maximal ideal of R_i and S_i have n generators as an R_i-module, F is an ultrafilter on I, $R_\infty \to S_\infty$ the corresponding map on ultraproducts then $R_\infty \to S_\infty$ is finite and $\ker(R_\infty \to \hat{R}_\infty)$ generates $\ker(S_\infty \to \hat{S}_\infty)$.

The proof uses that in all cases $R_{\infty, sep} = \mathrm{Im}(R_\infty \to \hat{R}_\infty)$ is noetherian.

We will use this only for complete local rings, in which case it is easily seen that R_∞ maps onto $\hat{R}_\infty$.

10. Proof of Th. 2.2.

We may assume A is complete.
View A as a finite generically étale extension of a 2 dimensional regular complete local ring R. A/R finite free rk m.

If the assertion is false there is $n>0$ and connected pairwise non isomorphic finite étale maps $E_i \longrightarrow \mathrm{Spec}(A)-\{m_A\}$ of degree n, ($i\in\mathbb{N}$). Let B_i be the normalization of A in E_i. B_i is complete normal local ring and E_i is the punctured spectrum of B_i. B_i is finite free rk nm as an R-module.

$$\mathrm{disc}_R\, B_i \;=\; (\mathrm{disc}_R\, A)^n \quad \text{up to unit.}$$

Let F be an ultrafilter on $\mathbb{N}$. Consider the ultraproducts

$$R_\infty \longrightarrow A_\infty \longrightarrow B_\infty$$

$R_\infty \to A_\infty$ free rk m
$R_\infty \to B_\infty$ free rk nm
B_∞ a finitely presented A_∞-module
information on $\operatorname{disc}_R B$ retained
Same for $R_\infty^\wedge \longrightarrow A_\infty^\wedge \longrightarrow B_\infty^\wedge$.

Note $A \longrightarrow A_\infty^\wedge$ is flat : mod m_A^ℓ this reduces to the fact that over a coherent ring infinite products of flat modules are flat.
The residue field of $A_\infty^\wedge$ is a regular (in particular separable) extension of the residue field of A. By "localization de la lissité formelle" the fibre rings of $A \to A_\infty^\wedge$ are geometrically regular. Hence $A_\infty^\wedge$ is normal.
$B_\infty^\wedge$ is torsion free over $R_\infty^\wedge$, hence over $A_\infty^\wedge$.
Hence $B_\infty^\wedge / A_\infty^\wedge$ is finite flat over the punctured spectrum, necessarily of rank n.
By the discriminant information we get that

$\hat{B}_\infty / \hat{A}_\infty$ is finite étale on the punctured spectrum. Hence $\forall\, \xi \in \mathrm{Spec}(\hat{A_\infty}) - V(m_A)$, the criterion of flatness by fibres gives that $(B_\infty)_\xi$ is flat over $(A_\infty)_\xi$, hence finite étale. But every point of $\mathrm{Spec}(A_\infty) - V(m_A)$ is a generization of a point of $\mathrm{Spec}(\hat{A_\infty}) - V(m_A)$.

[If $I \subset \mathrm{Rad}(A)$ is a f.g. ideal in a ring A and $\hat{A}$ the I-adic completion of A then all closed points of $\mathrm{Spec}(A) - V(I)$ are in the image of $\mathrm{Spec}(\hat{A}) \to \mathrm{Spec}(A)$.]

This proves

__Theorem.__ $p : \mathrm{Spec}(\prod_i B_i) \longrightarrow \mathrm{Spec}(\prod_i A)$ is finite étale of deg n on $V(m_A)^c$.

Let F be a non principal ultrafilter on $\mathbb{N}$. By the rigidity facts the restriction of p to $\mathrm{Spec}(A_\infty) - V(m_A)$ comes from a finite etale cover of $\mathrm{Spec}(A) - V(m_A)$. By passage to the limit $\exists\, T \in F$ s.t. same holds for $p|_{\mathrm{Spec}(\prod_{i \in T} A) - V(m_A)}$. contradiction.

APPENDIX B

Fac-similé : Princeton

Transparents de l'exposé d'Ofer Gabber, fait le 17 octobre 2005 à la conférence en l'honneur de Pierre Deligne ([**Gabber, 2005b**]).

Finiteness theorems for étale cohomology of excellent schemes

Ofer Gabber

IHES, CNRS

The goal is to prove

<u>Theorem 0.1</u> Let $f : X \to Y$ be a morphism of finite type between quasi-excellent noetherian $\mathbb{Z}[\frac{1}{n}]$ schemes and F a constructible sheaf of $\mathbb{Z}/n$-modules on X_{et}. Then the $R^i f_* F$ are constructible and 0 for $i \gg 0$.

We also prove other expected properties (affine cohomological dimension, existence of dualizing complexes). We use a weak form of resolution of singularities.

Epp's theorem
Popescu's theorem
Absolute cohomological purity

We give a proof of ACP without using AKTEC (Thomason)

In SGA4 XIX Artin proves Th. 0.1 for equichar. excellent schemes assuming resolution. One idea (partial algebraization) comes from the proof of affine coh. dim. in loc. cit.

1. Weak local uniformization

1.1 Weak version

Let X be a quasi excellent scheme, $Z \subset X$ nowhere dense closed subset. Then there are

$$\begin{array}{lccc} \text{regular} & X_i & \xrightarrow{\text{gen. finite}} & X \\ & \cup & \square & \cup \\ \text{N.C.D.} & Z_i & \longrightarrow & Z \end{array}$$

covering family for the h-topology.

For a scheme S, the h-topology on the category of schemes locally of finite pres. over S is the topology generated by proper surjective maps and Zariski open coverings.

1.2 Suppose X is integral (qc, sep) $K = R(X)$

$$ZRS(X) = \{\text{valuation rings of } K \text{ dominating some } \mathcal{O}_{X,x}\} = \varprojlim_{X' \to X \text{ proper birational}} X'$$

$$ZRS_{\overline{K}}(X) = \varprojlim X'$$

X' integral, projective gen. finite over X

$R(X) \underbrace{\subset}_{\text{normal}} R(X') \subset \overline{K}$, Galois group acts on X'.

Weak version $\Longleftrightarrow$ every $v \in ZRS_{\bar{k}}(X)$ dominates a good local model

$\Longleftrightarrow$ $\exists$ proj. equiv. model X' and an open cover U_i of X' s.t. $U_i \longrightarrow X$ factorizes through a good local model.

1.3 For $v \in ZRS_{\bar{k}}(X)$ define the inertia and decomposition group. These are $\varprojlim$ of inertia and dec. groups for p.e. models. The latter are upper semi continuous for the Zariski resp. constructible topology.

Strong form of Theorem: $\forall\ V \in ZRS_{\bar{k}}$ l invertible on X S an l-sylow of D = dec. gp of V, V^S dominates a good local model.

$\Longleftrightarrow$ $\exists$ p.e. X', $U_i \subset X'$ open, $H_i \subset Gal(X'/X)$ acts on U_i, s.t. U_i/H_i dominates a good local model and

$\forall x \in X' \ \exists i \ (x \in U_i$ and $H_i \supset \ell$ - Sylow of $D(x)$).

$\Longleftrightarrow$ there is the following

$$\begin{tikzcd}
 & \Omega'_i \arrow[dl, "\text{finite flat}", "(\deg, \ell) = 1"'] \arrow[rr] & & Y_i \arrow[dd, "f_i"] \\
\Omega_i \arrow[dr, "\text{Nisnevich cover}"'] & & & \\
 & X' \arrow[rr, "\text{proper birational}"'] & & X
\end{tikzcd}$$

regular with $f_i^{-1}(Z)$ N.C.D.

Similarly when the condition is restricted to valuations center at $x \in X$.

Thm for $(X, x) \Longleftrightarrow$ Thm for (X^h, x)

$\Longleftrightarrow$ Thm for $(X^\wedge, x)$

X^h = Spec of henselization of local ring

$X^\wedge$ = 〃 completion 〃 .

2. Approximation.

Let R be an excellent henselian local ring. Given an algebro geometric datum of finite presentation over $\hat{R}$ it comes from a datum over a f.g. subalgebra R_1 of $\hat{R}$. By Popescu's thm ($\hat{R} = \varinjlim$ smooth R algebras) $R_1 \longrightarrow \hat{R}$ can be approximated by $R_1 \longrightarrow R$.
Want to preserve properties of schemes and morphisms.

2.1 Let $I \subset A$ be an ideal, M, M' A-modules. An (I, c) isomorphism $M \underset{(I,c)}{\simeq} M'$ is an isomorphism

$$\bigoplus_{n \in \mathbb{Z}} I^n M / I^{n+c} M \longrightarrow \bigoplus_{n \in \mathbb{Z}} I^n M' / I^{n+c} M'$$

over $\bigoplus_{n \in \mathbb{Z}} I^n$.

Similarly for algebras.

2.2 (Artin-Rees) A noetherian, $(*)\ M' \xrightarrow{f} M \xrightarrow{g} M''$ an exact sequence of f.g. A-modules, I an ideal. Then for $n \gg 0$ if $M'_1 \xrightarrow{f_1} M_1 \xrightarrow{g_1} M''_1$ is (I,n)-isomorphic to $(*)$ & $g_1 f_1 = 0$ then it is exact and all kernels, cokernels and images are $(I, n-c)$-isomorphic to those of $(*)$.

Given M with a resolution $F_2 \xrightarrow[\partial]{} F_1 \xrightarrow[\partial]{} F_0 \to M \to 0$ by f.g. free-modules, if we approximate the matrix entries of ∂ s.t. $\partial\partial = 0$ holds we get an approximation of M. Conversely, approx. of M $\Rightarrow$ approx. of truncated resolutions.

2.3. A noetherian excellent $B = A/J$, $B' = A/J'$ is (I,n) close to B. Then B reduced near $V(I)$ $\Rightarrow B'$ reduced near $V(I)$. Same for normal. The assertion for "reduced" is shown by blowing up I. When $I = (f)$ use

Lemma. B reduced Japanese, $h \in B$ non zero divisor
$\exists n \quad (x^2 \in h^n B \Rightarrow x \in hB)$.

For the above type of approximation, $\forall x \in \mathrm{Spec}(A/_{I+J})$
$\dim B_x = \dim B'_x$, B_x regular $\Leftrightarrow B'_x$ regular.

3. Log regular schemes and quotients.

A locally noetherian fs log scheme X is log regular iff $\forall x \in X$ the completion (of the strict hensel.) is isomorphic to $k[[x_1, \dots, x_n]][[M]]$ (M a sharp toric monoid) in the equal-characteristic case, to $I[[x_1, \dots, x_n]][[M]]/(f)$ ($f \mapsto p \in I$, I cohen ring) in the mixed characteristic case.

A canonical desingularization procedure for excellent schemes of char. 0 gives a canonical desingularization for toric varieties over $\mathbb{Z}$ and for log regular schemes.

A log regular scheme is normal and the log structure is determined by the locus of triviality of the log structure

$j : U \hookrightarrow X$ (dense) $\quad M_X = \mathcal{O}_X \cap j_* \mathcal{O}_U^*$.

(étale top)

We say $(X, X-U)$ is log regular.

3.1 Let X be a separated scheme equipped with an action of a finite group G. We say the action is tame iff $\forall x \in X$ the order of the inertia group G_x is invertible in $k(x)$.

3.2 Lemma. Let G be a finite group acting on a noetherian ring A with $|G|^{-1} \in A$. Then A^G is noetherian and A is finite over A^G

3.3. Let G be a finite group acting generically freely on a sep. log regular scheme (X, Z). We say the action is very tame at $x \in X$ iff G_x is of order prime to the char. exp. $k(x)$, G_x acts trivially on $\bar{M}_x$ and G_x acts trivially on the connected stratum containing x.

[connected component of $\{x \in X \mid \mathrm{rk}\, \overline{M}_{\bar{x}} = i\}$]

Then G_x is abelian, $\exists$ structure

thm for completion, condition is open.

If action is very tame $(X/G, Z/G)$ is

log regular and action is free on $X - Z$.

$X - Z \longrightarrow (X-Z)/G$ is a G torsor tamely

ramified at maximal points of Z/G.

Conversely, if (X', Z') is log regular and

$V \longrightarrow X' - Z'$ is a G-torsor tamely

ramified in cod. 1 then by FK purity it

extends to a finite Kummer log étale map

$X \longrightarrow X'$, X the normalization of X' in V

and G acts very tamely on (X, Z).

3.4 Thm. Let G act generically freely

and tamely on a sep, qc log regular (X, Z).

Then there is a projective birational

map $X' \overset{p}{\dashrightarrow} X$, s.t.

if $Z' = p^{-1}(Z \cup \text{locus of non free action})$

Then (X', Z') is log regular, G acts very tamely on X'.

<u>sketch of proof</u>. Use canonical desing. of X.
WMA X regular, Z N.C.D.
To ensure G_x act trivially on $\bar{M}_x$ need to blow up k-uple intersections of components of Z
$k \geq 2$. (Blow up N-uple intersections, then proper transform of $(N-1)$-uple intersections etc.)
(étale locally WMA simple NCD)

In a similar way blow up along X^H $\forall H \neq \{1\}$ and increase Z. This leads to a situation with abelian inertia groups and free action on $X - Z$. Étale locally can increase Z s.t. action is very tame. This depends on choosing eigenfunctions and is not unique. This gives a log regular structure on X/G and the canonical desingularization of X/G is shown independent of local choices by lifting to char. 0, $Y \xrightarrow{p} X/G$. Show $(Y, p^{-1}(Z/G))$ is log regular. Normalize Y in $X - Z$.

3.5. If in 3.4 (X, Z) is log smooth over a base S with a trivial G action then (X', Z') and its qt by G are log smooth over S.

3.6. Let (X, Z) be log regular $X' \xrightarrow{f} X$ a nodal curve smooth over $X - Z$, $D \subset X'$ divisor in smooth locus étale over X. Then $(X', D \cup f^{-1}(Z))$ is log regular

4. Absolute cohomological purity.

Recall (Azumino) That for a regular immersion $Z \subset X$ of cod. c have a global fundamental class in $H^{2c}_Z(X_{et}, \Lambda(c))$ $\Lambda = \mathbb{Z}/n$, n invertible on X.

All schemes below are assumed to have an ample line bundle.

$f : X \to Y$ relative complete intersection $\Leftrightarrow$ factorizable $X \underset{\text{reg. imm.}}{\hookrightarrow} M \xrightarrow[\text{lisse}]{} Y$ $\quad \mathrm{cod}(f) = \mathrm{cod}_M X - \dim(M/Y)$.

Get a gysin map $\Lambda_X \to f^! \Lambda_Y (c) [2c]$
$c = \mathrm{cod}(f)$

satisfying transitivity.

$\mathrm{Tr} : Rf_! \Lambda_X \to \Lambda_Y (c) [2c]$.

For f flat coincides with SGA4 XVIII 2.9.

4.1. Suppose

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & S & \end{tikzcd}$$

rel c.i., $\mathrm{cod}(f) = 0$

Gysin for f gives $f^* F \to f^! F \quad \forall F \in D^+$.

Get $f_! f^* F \to F$. For f proper

$F \to f_* f^* F \to F$. This is shown to

be multiplication by $\deg(f) = \chi_{\mathrm{rk}}(Rf_* \mathcal{O}_X)$.

Let $K_X = p^! \Lambda_S (c) [2c] \quad c = \mathrm{cod}(p)$, $K_Y = \ldots$

$f^* (\Lambda_Y \to K_Y) \to (\Lambda_X \to K_X) \to f^! (\Lambda_Y \to K_Y)$.

commutes.

For f finite get $(\Lambda_Y \to K_Y) \rightleftarrows f_* (\Lambda_X \to K_X)$

$\circlearrowleft$ = multip by $\deg(f)$ on 2 terms.

Hence if f is finite of constant generic degree prime to n and $\Lambda_X \xrightarrow{\sim} K_X$ then $\Lambda_Y \xrightarrow{\sim} K_Y$.

The problem of ACP in the mixed characteristic case is reduced to the case of schemes of finite type over a trait S (complete) and for such X, X is punctually pure iff $\Lambda_X \xrightarrow{\sim} K_X$.

By de Jong $\exists$ Galois alteration

$$\begin{tikzcd} G \circlearrowright X' \arrow[r] \arrow[d] & X \arrow[d] \\ G \circlearrowright T \arrow[r] & S \end{tikzcd}$$

$R(X')$ normal $/R(X)$, $Gal = G$.

s.t. X' has a G-invariant log regular structure, log lisse over T with canonical log structure. (By Vidal we can even take $X' \to T$ projective regular with semistable reduction.)

$n = \ell^i$, $S_\ell \subset G$ ℓ-sylow.

can modify X' s.t. S_ℓ action in very tame and $Y = X'/S_\ell$ regular, and log lisse over T/S_ℓ.

Then X'/S_ℓ is étale locally of the form

$\mathrm{Spec}\ \mathcal{O}_{T/S_\ell}[x_1, \ldots, x_N]/(\prod x_i^{e_i} - \pi)$ unif. of T/S_ℓ.

Such schemes are p.p. by [Rap. Zink], [Ill]

NB. punctual purity for such schemes is equivalent to $R^* j_* \Lambda = \Lambda^* R^1 j_* \Lambda$, j inclusion of general fiber. This is reduced to the case all $e_i = 1$.

Since $Y \longrightarrow X$ is of generic degree prime to ℓ, Y p.p $\Rightarrow$ X p.p.

5. Applying Gpp.

Th. 5. 1. Let R be a complete normal n.l.r. (of dim. ≥ 2), I_i ($i \in$ finite set) ideals of R.

Then $\exists$ $R' \supset R$ finite extension s.t. R' is normal and is the completion of a ring (R'') ess. of finite type over a regular complete n.l.r. of dimension $\dim(R) - 1$, and $I_i R'$ come from R''.

In the equal characteristic case can take $R' = R$ and holds for $\dim R = 1$.

Th. 5. 2. Let $T \longrightarrow R$ be an extension of complete DVR's. When the residue char. is $p > 0$ assume that the residue field k_T is perfect and that the maximal perfect subfield of k_R is algebraic over k_T. Then $\exists$ finite extn $T \subset T'$ s.t. $(T' \underset{T}{\otimes} R)^{\nu}_{red}$ has reduced special fibre over T'.

(Trivial in char. 0)

5.3. Consider the situation of Th. 5.1 in the case of mixed characteristic $(0, p)$. Let k_0 be the maximal perfect subfield of k_R.

$W = W(k_0) \longrightarrow R$

$\exists\ W \longrightarrow W'$ finite, $R' = (R \otimes W')^{\nu}$ reduced fibre over max. ideal of W'. For every connected component of R' choose a coefficient field k of the special fibre $\overline{R}'$ over which $\overline{R}'$ is analytically separable:

$$\exists\ \underset{\text{gen. étale}}{k[[t_1, \ldots, t_{d-1}]]} \xrightarrow{\text{finite}} \overline{R}'$$

Extend k' to a Cohen ring I mapping to R'. $I[[t_1, \quad , t_{d-1}]] \longrightarrow R'$ finite, changing coordinates WMA it is étale at pt $(\pi, t_1, \quad, t_{d-2})$, hence étale outside $V(f)$ for $f \in I[[t_1, \quad , t_{d-2}]][t_{d-1}]$ monic. Using Elkik's approximation descend to $I[[t_1, \ldots, t_{d-2}]]\{t_{d-1}\}$ (henselian power series).

5.4. Prove weak form (1.1) by induction on the dimension.

6. First proof of Th. 0.1.

This proof gives that each $R^i f_* F$ is constructible. For schemes of finite Krull dimension have bounds (cde open in Spec (R), R str. local $\leq 2 \dim(R) - 1$), [Hu].

6.1. Let $X_. \xrightarrow[\varepsilon]{} X$ be an h-hypercover.

Then $F \xrightarrow{\sim} R\varepsilon_* \varepsilon^* F \quad \forall\ F \in D^+(X, \mathbb{Z}/n)$.

If
$$\begin{tikzcd} X_. \arrow[r] \arrow[d, "f_."'] & X \arrow[d, "f"] \\ Y_. \arrow[r, "\varepsilon"] & Y \end{tikzcd}$$
cartesian, rows h-hypercov.

$$R f_* F = R\varepsilon_* (R f_{.*} F).$$

Analyzing the proof of this get also that this holds for pull-backs by an arbitrary $T \to Y$.

Reduce 0.1 to open immersions j, can choose hypercover s.t. $\forall\ i \leq N$ Y_i is regular and $X_i \subset Y_i$ is for every connected component empty or complement of N.C.D.

Use generic constructibility (SGA $4\frac{1}{2}$).

7. Second proof of Th. 0.1.

(P_c) For open immersion $U \underset{j}{\hookrightarrow} X$, $F \in D^b_c(U)$,

$\exists$ $T \subset X$ of Cod. $> c$ s.t. $Rj_* F|_{X-T} \in D^b_c$.

prove this inductively.

$c = 0$ trivial

$c = 1$ reduce to X normal and constant coefficients. Use ACP.

$\forall c\ (P_c) \Rightarrow$ Theorem easy

Assume (P_{c-1}) Let T_α be an irr. comp. of Cod. c of T. We allow to restrict to neighborhoods of the generic points of T.

Consider the diagram of page 3. Let $T' \subset X'$ be the locus coming from (P_{c-1}) for X'. We may assume $T' = \coprod T'_{\alpha\beta}$ $T'_{\alpha\beta}$ irreducible finite over T_α. $\forall \beta\ \exists i$ $\Omega_i \longrightarrow X'$ is an isomorphism over $T'_{\alpha\beta}$.

Given $\quad Z \subset Y \xrightarrow{p} X$

$$p^{-1}(U) \xrightarrow[j_Y]{} Y$$

Denote

$$Rp_*\left((Rj_{Y*}F)|Z\right) = \varphi(Z, Y)$$

$$\begin{tikzcd}
T''_\beta \subset \Omega'_\beta \arrow[d] \arrow[rr] & & Y_\beta \supset T'''_\beta \arrow[d] \\
T'_\beta \subset \Omega_\beta \arrow[r] & X' \supset T'_\beta \arrow[r] & X \supset T
\end{tikzcd}$$

Using inductive assumption

$$\varphi(T, X) \longrightarrow \bigoplus_\beta \varphi(T'_\beta, X')$$

has cone in D^b_c.

$$\varphi(T'_\beta, X') \xrightarrow{\sim} \varphi(T'_\beta, \Omega_\beta) \longrightarrow \varphi(T''_\beta, \Omega'_\beta)$$

has left inverse

$\varphi(T, X)$.

$\varphi(T, X) \longrightarrow \varphi(T''_\beta, \Omega'_\beta)$ factorizes through an object in D^b_c since on Y_β have good open immersion (N.C.D)

8. Affine morphisms.

$f: X \longrightarrow Y$ affine f.t., assume Y universally catenarian and $\exists$ dimension function δ on Y ($\exists$ étale locally on a quasi excellent scheme), i.e $\delta(y') = \delta(y)+1$ if y is an immediate specialization of y'.

Want to prove $\delta(R^i f_* F) \leq \delta(F) - i$

(cf. SGA4 XIX).

Reduce to

<u>Th. 8.1</u> $Y = \mathrm{Spec}(R)$ strictly local excellent

dim = d, $f \in R$, then $H^i(\underset{U}{\mathrm{Spec}\, R[f^{-1}]}, \mathbb{Z}/n) = 0$ $\forall\, i > d$.

Let $Y_\cdot \longrightarrow Y$ be a truncated h hypercover as in first proof of finiteness.

$U_\cdot \longrightarrow U$ $U_\cdot \subset \bar{U}_\cdot$ have log structure

$H^i(U) = H^i($ closed fibre of $\bar{U}_\cdot$ with Kummer étale topology $)$

By ~~formally~~ smooth base change w.m.a. R complete, quotient of $I[[x_1, \dots, x_m]]$.

Approximate to a quotient of $I\{x_1, \dots, x_m\}$.

9. Top cohomology of the punctured spectrum

9.1 Transition maps.

Let $\bar{y} \to \bar{x}$ be an immediate specialization of geometric points of a quasi excellent scheme X. So $\bar{y} \to X(\bar{x})$ is centered at y corresponding to a 1-dimensional $C \subset X(\bar{x})$, C^ν is a trait.

$F \in D^+(X, \mathbb{Z}/n)$.

$$H^i_{\bar{y}}(F) \longrightarrow H^1(\mathrm{Gal}(\bar{y}/y), H^i_{\bar{y}} F)(1) \longrightarrow H^{i+1}_y F(1) \qquad \text{(sign!)}$$

$$\longrightarrow H^{i+1}(X(\bar{x}) - \{\bar{x}\}, F(1)) \longrightarrow H^{i+2}_{\bar{y}}(F)(1)$$

divided by the degree of the (inseparable) residue field extension for $C^\nu \to C$.

9.2. Thm. For every strictly local normal excellent X $d = \dim(X)$, n invertible on X have an isomorphism

$$H^{2d}_s(X, \mathbb{Z}/n(d)) \longrightarrow \mathbb{Z}/n$$

compatible with transition maps
compatible with trace for finite $X' \to X$ (up to residue extension).

For $\dim(X) = 2$ let $X' \xrightarrow[p]{} X$ be a desingularization $p^{-1}(\xi)_{red}$ $\overset{(simple)}{NCD}$ $\bigcup_i D_i$

$j : U = X - \{\xi\} \hookrightarrow X'$.

$$d_2 : H^0(X'_\xi, R^2 j_* \mathbb{Z}/n) \longrightarrow H^2(X'_\xi, R^1 j_* \mathbb{Z}/n)$$
$$\mathbin{/\!/} \bigoplus \Lambda_{D_i}(-1)$$

can compute d_2,

$\mathrm{Coker}(d_2) = H^3(U)$.

For $\dim(X) > 2$ let X' be the normalization of $bl_q(X)$, q an $\overset{m-}{}$primary ideal.

$$E_2^{pq} = H^p(X'_\xi, R^q j_* \Lambda)$$

concentrated in $0 \le p \le 2d-2$, $0 \le q \le d$

$p \le 2(d-q)$ using §8.

can avoid using this by showing that the limit of E_2^{pq} over all X' vanishes outside this range.

$$H^{2d-4}(X'_\xi, R^2 j_* \Lambda) \to H^{2d-2}(X'_\xi, R^1 j_* \Lambda) \to H^{2d-1}(U, \Lambda) \to 0.$$

Using the information from the 2 dimensional case show the image of d_2 identifies the contributions of the irreducible components of $X'_{\bar{s}}$.

Check compatibility with transition map for curves $C \subset X$ whose proper transform is regular and meets $(X'_{\bar{s}})_{red} = E$ transversally at a regular point.

For X regular every C becomes transversal on some X'.

10. Dualizing complexes.

10.1 Given any specialization $\bar{y} \longrightarrow \bar{x}$ define $H^i_{\bar{y}}(F) \longrightarrow H^{i+2c}_{\bar{x}}(F(c))$ by decomposing to immediate specializations.

It is independent of the choice. (scheme q. exc.)

Assume X universally catenarian with a dimension function δ.

Definition 10.2. A candidate dualizing (c.d.) complex on X is $K \in D^+(X, \Lambda)$ equipped with $R\Gamma_{\bar{x}}(K) \simeq \Lambda(\delta(x))[2\delta(x)]$ compatible with transition maps.

10.3. If $Y \xrightarrow[f]{} X$ is of finite type and K is c.d. on X then $f^! K$ is c.d. on Y w.r.t. $\delta(y) = \delta(f(y)) + \text{tr.deg.}(y/f(y))$.

10.4. If $Y \xrightarrow[f]{} X$ is flat with geometrically regular fibres and K is c.d. on X, then $f^* K$ is c.d. on Y w.r.t.
$$\delta(y) = \delta(f(y)) - \dim(\mathcal{O}_{f^{-1}(f(y)), y}).$$

<u>Th. 10.5.</u> A c.d. complex exists and is unique up to a unique isomorphism.

$\Lambda_X \xrightarrow{\sim} \text{Iso}\, R\underline{\mathrm{Hom}}(K, K)$.

Remark. K is (-2δ) - perverse

for X normal irreducible with generic point

$j : \eta \longrightarrow X$, $\delta(\eta) = 0$, $K = \tau_{\leq \varphi} Rj_* \Lambda$

$\varphi(x) = \max(0, 2\dim(\mathcal{O}_{X,x}) - 2)$.

In the proof of 10.5 we may assume it is known for schemes finite over proper closed subschemes of X.

$$\begin{tikzcd} Y' \arrow[r, hook, "i'"] \arrow[d, "p'"'] & X' = X^{\nu}_{red} \arrow[d, "p"] \\ Y \arrow[r, hook, "i"] & X \end{tikzcd}$$

$X - Y$ = normal locus of X_{red}.

Have $K_{Y'} = p'^{!} K_Y = i'^{!} K_{X'}$.

$$p'_* K_{Y'} \longrightarrow K_Y \oplus p_* K_{X'}$$

show cone of this is c.d. for X.

10.6 $K \in D^b_{ctf}(X)$, compatible with change of Λ, $D_K : D^b_c \longrightarrow D^b_c$. For a constructible sheaf F show $F \xrightarrow{\sim} \tau_{\leq 0} D_K D_K F$ by reducing to the case of constant sheaves.

10.7 show $\mathcal{H}^r(\mathrm{Cone}(F \to D_K D_K F)) = 0$

by induction on $(\dim X, r)$.

The proof of [Th. Finitude] extends to

(biduality for excellent schemes of dim $\leq d$)

$\Rightarrow$ (biduality for schemes of finite type over excellent schemes of dim $\leq d$).

Enough to embed F in a sheaf for which biduality is known. Use 5.1.

Bibliography

[Abbes & Gros, 2011a] Abbes, A. & Gros, M. (2011a). Sur la correspondance de Simpson p-adique. I : étude locale. arXiv:1102.5466. 152

[Abbes & Gros, 2011b] Abbes, A. & Gros, M. (2011b). Topos co-évanescents et généralisations. arXiv:1107.2380. 145, 152, 155, 156

[Abramovich & de Jong, 1997] Abramovich, D. & de Jong, A. J. (1997). Smoothness, semistability, and toroidal geometry. *J. Algebraic Geom.*, 6, 789–801. 132

[Abramovich et al., 2013] Abramovich, D., Denef, J., & Karu, K. (2013). Weak toroidalization over non-closed fields. *Manuscripta math.*, 142(1-2), 257–271. 139

[Abramovich & Karu, 2000] Abramovich, D. & Karu, K. (2000). Weak semistable reduction in characteristic 0. *Invent. math.*, 139(2), 241–273. vii, 137

[André, 1974] André, M. (1974). Localisation de la lissité formelle. *Manuscripta math.*, 13, 297–307. 345, 376

[Artin, 1968] Artin, M. (1968). On the solutions of analytic equations. *Invent. math.*, 5, 277–291. 9

[Artin, 1969] Artin, M. (1969). Algebraic approximation of structures over complete local rings. *Publications mathématiques de l'IHÉS*, 36, 23–58. 9

[Artin, 1971] Artin, M. (1971). On the joins of Hensel rings. *Adv. Math.*, 7, 282–296. 321, 322

[Artin et al., 1973] Artin, M., Grothendieck, A., & Verdier, J.-L. (1972-1973). *Théorie des topos et cohomologie étale des schémas* (*SGA 4*). Springer-Verlag. Séminaire de géométrie algébrique du Bois-Marie 1963–1964, dirigé par M. Artin, A. Grothendieck, et J.-L. Verdier. Lecture Notes in Mathematics numéros 269 (tome 1), 270 (tome 2), et 305 (tome 3). 432

[Ayoub, 2007] Ayoub, J. (2007). *Les six opérations de Grothendieck et le formalisme des cycles évanescents dans le monde motivique*, volume 314-315 des *Astérisque*. Société mathématique de France. 158, 217

[Bell & Slomson, 1969] Bell, J. L. & Slomson, A. B. (1969). *Models and ultraproducts: An introduction*. North-Holland Publishing Co. 374

[Berger, 1995] Berger, C. (1995). Un groupoïde simplicial comme modèle de l'espace des chemins. *Bull. Soc. math. France*, 123(1), 1–32. 357

[Berthelot et al., 1971] Berthelot, P., Grothendieck, A., & Illusie, L. (1971). *Théorie des intersections et théorème de Riemann-Roch*, volume 225 des *Lecture Notes in Mathematics*. Springer-Verlag. Séminaire de géometrie algébrique du Bois-Marie 1966–1967 (SGA 6), avec la collaboration de D. Ferrand, J.-P. Jouanolou, O. Jussila, S. Kleiman, M. Raynaud et J.-P. Serre. 432

[Bhatt & Scholze, 2013] Bhatt, B. & Scholze, P. (2013). The pro-étale topology for schemes. arXiv:1309.1198. 186

[Bierstone & Milman, 1997] Bierstone, E. & Milman, P. (1997). Canonical desingularization in characteristic zero by blowing up the maximum strata of a local invariant. *Invent. math.*, 128(2), 207–302. 74

[Bierstone et al., 2011] Bierstone, E., Milman, P., & Temkin, M. (2011). **Q**-universal desingularization. *Asian Journal of Mathematics*, 15, 229–250. 74

[Bosch et al., 1995] Bosch, S., Lütkebohmert, W., & Raynaud, M. (1995). Formal and rigid geometry IV. The reduced fibre theorem. *Invent. math.*, 119(2), 361–398. 138

[Bravo et al., 2005] Bravo, A. M., Encinas, S., & Villamayor U., O. (2005). A simplified proof of desingularization and applications. *Rev. Mat. Iberoamericana*, 21(2), 349–458. 74

[Brodmann & Rotthaus, 1980] Brodmann, M. & Rotthaus, C. (1980). Über den regulären Ort in ausgezeichneten Ringen. *Math. Z.*, 175(1), 81–85. 8

[Calmès & Hornbostel, 2009] Calmès, B. & Hornbostel, J. (2009). Tensor-triangulated categories and dualities. *Theory Appl. Categ.*, 22, nº 6, 136–200. 302

[Chase, 1960] Chase, S. U. (1960). Direct products of modules. *Trans. Amer. Math. Soc.*, 97, 457–473. 374

[Conrad, 1999] Conrad, B. (1999). Irreducible components of rigid spaces. *Ann. Inst. Fourier*, 49(2), 473–541. 9

[Conrad, 2000] Conrad, B. (2000). *Grothendieck duality and base change*, volume 1750 des *Lecture Notes in Mathematics*. Springer-Verlag. 231, 233, 236, 237, 274, 279

[Conrad, 2007] Conrad, B. (2007). Deligne's notes on Nagata compactification. *J. Ramanujan Math. Soc.*, 22, 205–257. 68, 122

[Conrad & de Jong, 2002] Conrad, B. & de Jong, A. J. (2002). Approximation of versal deformations. *J. Algebra*, 255(2), 489–515. 23, 27

[Cossart et al., 2009] Cossart, V., Jannsen, U., & Saito, S. (2009). Canonical embedded and non-embedded resolution of singularities for excellent two-dimensional schemes. arXiv:0905.2191v2. 130

[Crépeaux, 1967] Crépeaux, E. (1967). Une caractérisation des couples henséliens. *Enseignement math.*, 13, 273–279. 46, 335 [de Jong, 1996] de Jong, A. J. (1996). Smoothness, semistability and alterations. *Publications mathématiques de l'IHÉS*, 83, 51–93. 53, 63, 66, 93, 125, 133, 331

[de Jong, 1997] de Jong, A. J. (1997). Families of curves and alterations. *Ann. Inst. Fourier*, 47(2), 599–621. vi, 66, 105, 109, 122, 129

[Deitmar, 2005] Deitmar, A. (2005). Schemes over $\mathbf{F}_1$. In *Number fields and function fields—two parallel worlds*, volume 239 des *Progress in Mathematics* (pp. 87–100). Birkhäuser. 77, 80

[Deligne, 1974] Deligne, P. (1974). Théorie de Hodge. III. *Publications mathématiques de l'IHÉS*, 44, 5–77. 162, 164, 180

[Deligne, 1977] Deligne, P. (1977). *Cohomologie étale*, volume 569 des *Lecture Notes in Mathematics*. Springer-Verlag. Avec la collaboration de J.-F. Boutot, A. Grothendieck, L. Illusie et J.-L. Verdier. 432

[Deligne, 1995] Deligne, P. (1995). Lettre à L. Illusie du 31 mars 1995. 152, 155

[Deligne et al., 1972] Deligne, P., Grothendieck, A., & Nicholas, K. (1972). *Groupes de monodromie en géométrie algébrique*. Springer-Verlag. Séminaire de géométrie algébrique du Bois-Marie 1967–1969 (SGA 7). Lecture Notes in Mathematics, vol. 288 (A. Grothendieck) et 340 (P. Deligne et N. Katz). 432

[Ekedahl,] Ekedahl, T. On the adic formalism. In *The Grothendieck Festschrift, II*. 186, 187, 188

[Ekedahl, 1984] Ekedahl, T. (1984). On the multiplicative properties of the de Rham-Witt complex. I. *Ark. Mat.*, 22(2), 185–239. 187

[Elkik, 1973] Elkik, R. (1973). Solutions d'équations à coefficients dans un anneau hensélien. *Ann. sci. École norm. sup.*, 6, 553–603. 46, 340, 372

[Epp, 1973] Epp, H. (1973). Eliminating wild ramification. *Invent. math.*, 19, 235–249. 40

[Faltings, 1978] Faltings, G. (1978). Ein einfacher Beweis, daß geometrische Regularität formal Glattheit impliziert. *Arch. Math.*, 30(3), 284–285. 5

[Faltings, 2002] Faltings, G. (2002). Almost étale extensions. In P. Berthelot, J.-M. Fontaine, L. Illusie, K. Kato, & M. Rapoport (éds), *Cohomologies p-adiques et applications arithmétiques*, volume 279 des *Astérisque* (pp. 185–270). Société mathématique de France. 152

[Faltings, 2005] Faltings, G. (2005). A p-adic Simpson correspondence. *Adv. Math.*, 198, 847–862. 152

[Fargues, 2009] Fargues, L. (2009). Filtration de monodromie et cycles évanescents formels. *Invent. math.*, 177(2), 281–305. 186, 187

[Ferrand, 1972] Ferrand, D. (1972). Monomorphismes et morphismes absolument plats. *Bull. Soc. math. France*, 100, 97–128. 8

[Ferrand & Raynaud, 1970] Ferrand, D. & Raynaud, M. (1970). Fibres formelles d'un anneau local noethérien. *Ann. sci. École norm. sup.*, 3, 295–311. 11, 342

[Fujiwara, 1995] Fujiwara, K. (1995). Theory of tubular neighborhood in étale topology. *Duke Math. J.*, 80(1), 15–57. viii, 181, 200, 261, 308, 314, 317, 318, 347

[Fujiwara, 2002] Fujiwara, K. (2002). A proof of the absolute purity conjecture (after Gabber). In *Algebraic geometry 2000, Azumino*, volume 36 des *Advanced Studies in Pure Mathematics* (pp. 153–183). 日本数学会. v, viii, 200, 205, 208, 213, 215, 222, 223, 231

[Fujiwara & Kato, 1995] Fujiwara, K. & Kato, K. (1995). Logarithmic étale topology theory. Manuscrit non publié. 107

[Fulton, 1993] Fulton, W. (1993). *Introduction to toric varieties*, volume 131 des *Annals of Mathematics Studies*. Princeton University Press. 82

[Fulton, 1998] Fulton, W. (1998). *Intersection theory*, volume 2 des *Ergebnisse der Mathematik und ihrer Grenzgebiete*. Springer-Verlag, seconde édition. 208, 212

[Gabber, 1992] Gabber, O. (1992). K-theory of Henselian local rings and Henselian pairs. In *Algebraic K-theory, commutative algebra, and algebraic geometry* (*Santa Margherita Ligure, 1989*), volume 126 des *Contemp. Math.* (pp. 59–70). Amer. Math. Soc. 46

[Gabber, 1994] Gabber, O. (1994). Affine analog of the proper base change theorem. *Israel J. Math.*, 87(1-3), 325–335. 337

[Gabber, 2001] Gabber, O. (2001). Lettre à Fabrice Orgogozo, 28 décembre 2001. 325

[Gabber, 2004] Gabber, O. (2004). Notes on some t-structures. In A. Adolphson, F. Baldassarri, P. Berthelot, N. Katz, & F. Loeser (éds), *Geometric aspects of Dwork theory, vol. II* (pp. 711–734). Walter de Gruyter. 239, 267

[Gabber, 2005a] Gabber, O. (2005a). A finiteness theorem for non abelian H^1 of excellent schemes. Notes d'un exposé donné à l'occasion de la conférence en l'honneur de Luc Illusie, Orsay. (Voir annexe A). v, vii, 35, 379

[Gabber, 2005b] Gabber, O. (2005b). Finiteness theorems for étale cohomology of excellent schemes. Conférence en l'honneur de Pierre Deligne à l'occasion de son soixante-et-unième anniversaire, Institute for Advanced Study, Princeton. (Voir annexe B). v, viii, ix, 21, 205, 239, 399

[Gabber, 2007] Gabber, O. (2007). Lettre à Yves Laszlo du 23 mai 2007. 8

[Gabber & Orgogozo, 2008] Gabber, O. & Orgogozo, F. (2008). Sur la p-dimension des corps. *Invent. math.*, 174(1), 47–80. 316, 323

[Gabber & Ramero, 2003] Gabber, O. & Ramero, L. (2003). *Almost ring theory*, volume 1800 des *Lecture Notes in Mathematics*. Springer-Verlag. 241, 340

[Gabber & Ramero, 2013] Gabber, O. & Ramero, L. (2013). Foundations for almost ring theory. arXiv:math/0409584v8. 51, 66, 108, 112, 181, 343

[Gabriel, 1962] Gabriel, P. (1962). Des catégories abéliennes. *Bull. Soc. math. France*, 90, 323–448. 282, 287

[Giraud, 1971] Giraud, J. (1971). *Cohomologie non abélienne*, volume 179 des *Grundlehren der mathematischen Wissenschaften*. Springer-Verlag. 150, 163, 281, 333, 336, 338, 339, 345, 346, 350, 352, 353, 355, 356, 357, 358

[Giraud, 1972] Giraud, J. (1972). Classifying topos. In *Toposes, Algebraic Geometry and Logic*, volume 274 des *Lecture Notes in Mathematics* (pp. 43–56). Springer-Verlag. 151

[Goerss & Jardine, 1999] Goerss, P. G. & Jardine, J. F. (1999). *Simplicial homotopy theory*, volume 174 des *Progress in Mathematics*. Birkhäuser. 293

[Goodwillie & Lichtenbaum, 2001] Goodwillie, T. G. & Lichtenbaum, S. (2001). A cohomological bound for the h-topology. *Amer. J. Math.*, 123(3), 425–443. v, 21, 180

[Greco, 1976] Greco, S. (1976). Two theorems on excellent rings. *Nagoya Math. J.*, 60, 139–149. 8

[Greco, 1982] Greco, S. (1982). A note on universally catenary rings. *Nagoya Math. J.*, 87, 95–100. 9

[Greco & Valabrega, 1974] Greco, S. & Valabrega, P. (1974). On the excellent property for strictly convergent power series over a complete non Archimedean valued field. *Atti Accad. Sci. Torino Cl. Sci. Fis. Mat. Natur.*, 108(3-4), 529–538. 9

[Grothendieck, 1957] Grothendieck, A. (1957). Sur quelques points d'algèbre homologique. *Tohoku Math. J.*, 9, 119–221. 232, 295

[Grothendieck, 1958a] Grothendieck, A. (1958a). La théorie des classes de Chern. *Bull. Soc. math. France*, 86, 137–154. 205

[Grothendieck, 1958b] Grothendieck, A. (1958b). Sur quelques propriétés fondamentales en théorie des intersections. In *Anneaux de Chow et applications*. Séminaire Claude Chevalley, deuxième année. 206

[Grothendieck, 1967] Grothendieck, A. (1960-1967). Éléments de géométrie algébrique (ÉGA). *Publications mathématiques de l'IHÉS*. Numéros 4 (I) ; 8 (II) ; 11,17 (III) ; 20, 24 et 28 (IV), rédigés avec la collaboration de J. Dieudonné. 432

[Grothendieck, 1968] Grothendieck, A. (1968). Le groupe de Brauer III : exemples et compléments. In J. Giraud, A. Grothendieck, S. L. Kleiman, M. Raynaud, & J. Tate (éds), *Dix exposés sur la cohomologie des schémas*, volume 3 des *Advanced studies in Pure Mathematics* (pp. 88–188). Masson et Cie, North-Holland. 308

[Grothendieck, 1977] Grothendieck, A. (1977). *Cohomologie ℓ-adique et fonctions L*, volume 589 des *Lecture Notes in Mathematics*. Springer-Verlag. Séminaire de géometrie algébrique du Bois-Marie 1965–1966 (SGA 5). Avec la collaboration de I. Bucur, C. Houzel, L. Illusie, J.-P. Jouanolou et J.-P. Serre. 432

[Grothendieck, 2003a] Grothendieck, A. (2003a). *Cohomologie locale des faisceaux cohérents et théorèmes de Lefschetz locaux et globaux*, volume 4 des *Documents mathématiques*. Société mathématique de France. Séminaire de géométrie algébrique du Bois-Marie 1962 (SGA 2). Augmenté d'un exposé de Mme Michèle Raynaud. 432

[Grothendieck, 2003b] Grothendieck, A. (2003b). *Revêtements étales et groupe fondamental*, volume 3 des *Documents mathématiques*. Société mathématique de France. Séminaire de géométrie algébrique du Bois-Marie 1960–1961 (SGA 1). Augmenté de deux exposés de Mme Michèle Raynaud. Réédition du LNM nº 224. 432

[Grothendieck & Dieudonné, 1971] Grothendieck, A. & Dieudonné, J. (1971). *Éléments de géométrie algébrique I*. Springer-Verlag. 432

[Gruson, 1972] Gruson, L. (1972). Une propriété des couples henséliens. In *Colloque d'algèbre commutative (Rennes, 1972), exp. nº10*. Université de Rennes. 46

[Hakim, 1972] Hakim, M. (1972). *Topos annelés et schémas relatifs*, volume 64 des *Ergebnisse der Mathematik und ihrer Grenzgebiete*. Springer-Verlag. 323

[Hartshorne, 1966] Hartshorne, R. (1966). *Residues and duality*, volume 20 des *Lecture Notes in Mathematics*. Springer-Verlag. 279, 296

[Heinzer et al., 1997] Heinzer, W., Rotthaus, C., & Wiegand, S. (1997). Noetherian rings between a semilocal domain and its completion. *J. Algebra*, 198(2), 627–655. 312

[Heinzer et al., 2004] Heinzer, W., Rotthaus, C., & Wiegand, S. (2004). Catenary local rings with geometrically normal formal fibers. In *Algebra, arithmetic and geometry with applications (West Lafayette, 2000)* (pp. 497–510). Springer-Verlag. 10

[Hironaka, 1964] Hironaka, H. (1964). Resolution of singularities of an algebraic variety over a field of characteristic zero. I, II. *Ann. Math.*, 79, 109–203, 205–326. 129

[Hovey, 2001] Hovey, M. (2001). Model category structures on chain complexes of sheaves. *Trans. Amer. Math. Soc.*, 353(6), 2441–2457. 295

[Illusie, 1972] Illusie, L. (1971-1972). *Complexe cotangent et déformations,* volume 239 & 283. Springer-Verlag. Lecture Notes in Mathematics. 76, 162

[Illusie, 1983] Illusie, L. (1983). Finiteness, duality, and Künneth theorems in the cohomology of the de Rham-Witt complex. In *Algebraic geometry (Tokyo/Kyoto, 1982)*, volume 1016 des *Lecture Notes in Mathematics* (pp. 20–72). Springer-Verlag. 187

[Illusie, 2002] Illusie, L. (2002). An overview of the work of K. Fujiwara, K. Kato, and C. Nakayama on logarithmic étale cohomology. In P. Berthelot, J.-M. Fontaine, L. Illusie, K. Kato, & M. Rapoport (éds), *Cohomologies p-adiques et applications arithmétiques*, volume 279 des *Astérisque* (pp. 271–322). Société mathématique de France. 53, 54, 121

[Illusie, 2003] Illusie, L. (2003). Perversité et variation. *Manuscripta math.*, 112(3), 271–295. viii, 181, 199, 203

[Illusie, 2004] Illusie, L. (2004). On the semistable reduction and the calculation of nearby cycles. In A. Adolphson, F. Baldassarri, P. Berthelot, N. Katz, & F. Loeser (éds), *Geometric aspects of Dwork theory, vol. II* (pp. 785–803). Walter de Gruyter. 229

[Illusie et al., 2005] Illusie, L., Kato, K., & Nakayama, C. (2005). Quasi-unipotent logarithmic Riemann-Hilbert correspondences. *J. Math. Sci. Univ. Tokyo*, 12, 1–66. 136, 137, 138, 139

[Illusie et al., 2013] Illusie, L., Nakayama, C., & Tsuji, T. (2013). On log flat descent. *Proc. Japan Academy*, 89, 1–5. 54

[Kato, 1988] Kato, K. (1988). Logarithmic structures of Fontaine-Illusie. In J.-i. Igusa (éd.), *Algebraic analysis, geometry and number* (pp. 191–224). The John Hopkins University Press. 51, 58, 110, 111, 134, 136, 137, 229

[Kato, 1991] Kato, K. (1991). Logarithmic structures of Fontaine-Illusie II. Logarithmic flat topology. Manuscrit non publié. 54

[Kato, 1994] Kato, K. (1994). Toric singularities. *Amer. J. Math.*, 116(5), 1073–1099. vi, 51, 52, 66, 77, 78, 80, 81, 86, 87, 88, 107, 109, 114, 116, 135, 222, 229, 231

[Katz & Laumon, 1985] Katz, N. M. & Laumon, G. (1985). Transformation de Fourier et majoration de sommes exponentielles. *Publications mathématiques de l'IHÉS*, 62, 361–418. 186

[Kelly, 2012] Kelly, S. (2012). *Triangulated categories of motives in positive characteristic*. Thèse, Australian National University. arXiv:1305.5349v1. 17

[Kempf et al., 1973] Kempf, G., Knudsen, F. F., Mumford, D., & Saint-Donat, B. (1973). *Toroidal embeddings I*, volume 339 des *Lecture Notes in Mathematics*. Springer-Verlag. vi

[Kiehl, 1969] Kiehl, R. (1969). Ausgezeichnete Ringe in der nichtarchimedischen analytischen Geometrie. *J. reine angew. Math.*, 234, 89–98. 9

[Kollár, 2007] Kollár, J. (2007). *Lectures on resolution of singularities*, volume 166 des *Annals of Mathematics Studies*. Princeton University Press. 74

[Kurke et al., 1975] Kurke, H., Pfister, G., & Roczen, M. (1975). *Henselsche Ringe und algebraische Geometrie*. Deutscher Verlag der Wissenschaften. Mathematische Monographien. 45, 46

[Lam, 1991] Lam, T. Y. (1991). *A first course in noncommutative rings*, volume 131 des *Graduate Texts in Mathematics*. Springer-Verlag. 284

[Lam, 1999] Lam, T. Y. (1999). *Lectures on modules and rings*, volume 189 des *Graduate Texts in Mathematics*. Springer-Verlag. 284

[Laumon, 1983] Laumon, G. (1983). Vanishing cycles over a base of dimension ≥ 1. In M. Raynaud & T. Shioda (éds), *Algebraic Geometry, Proceedings, Tokyo/Kyoto 1982*, volume 1016 des *Lecture Notes in Mathematics* (pp. 143–150). Springer-Verlag. vii, 141, 152, 153

[Lipman, 1978] Lipman, J. (1978). Desingularization of two-dimensional schemes. *Ann. Math.*, 107(1), 151–207. 253, 369

[Lipman & Hashimoto, 2009] Lipman, J. & Hashimoto, M. (2009). *Foundations of Grothendieck duality for diagrams of schemes*, volume 1960 des *Lecture Notes in Mathematics*. Springer-Verlag. 158

[Liu & Zheng, 2014] Liu, Y. & Zheng, W. (2014). Enhanced adic formalism and perverse t-structures for higher Artin stacks. arXiv:1404.1128v1. 186

[Matsumura, 1980a] Matsumura, H. (1980a). *Commutative algebra*, volume 56 des *Mathematics Lecture Note Series*. The Benjamin/Cummings Publishing Company, seconde édition. 72, 82, 89, 169, 173, 177, 189, 193, 196

[Matsumura, 1980b] Matsumura, H. (1980b). 可換環論. 共立出版. 34, 312, 430

[Matsumura, 1988] Matsumura, H. (1988). On the dimension of formal fibres of a local ring. In *Algebraic geometry and commutative algebra, vol. I* (pp. 261–266). Kinokuniya. 2

[Matsumura, 1989] Matsumura, H. (1989). *Commutative ring theory*, volume 8 des *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, seconde édition. Traduction anglaise de [**Matsumura, 1980b**]. 5, 29, 170, 176, 190, 196, 370, 372

[Mochizuki, 1999] Mochizuki, S. (1999). Extending families of curves over log regular schemes. *J. reine angew. Math.*, 511, 43–71. 51, 52

[Morel & Levine, 2001] Morel, F. & Levine, M. (2001). Cobordisme algébrique I. *C. R. Acad. sci. Paris*, 332(8), 723–728. 205

[Nagata, 1958] Nagata, M. (1958). An example of a normal local ring which is analytically reducible. *Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math.*, 31, 83–85. 312

[Nagata, 1962] Nagata, M. (1962). *Local rings*, volume 13 des *Interscience Tracts in Pure and Applied Mathematics*. John Wiley & Sons. 11, 12, 312, 328, 341, 363

[Nakayama, 2009] Nakayama, C. (2009). Quasi-sections in log geometry. *Osaka J. Math*, 46, 1163–1173. 121

[Neeman, 2001] Neeman, A. (2001). *Triangulated categories*, volume 148 des *Annals of Mathematics Studies*. Princeton University Press. 183, 297

[Nishimura & Nishimura, 1987] Nishimura, J.-i. & Nishimura, T. (1987). Ideal-adic completion of Noetherian rings II. In *Algebraic geometry and commutative algebra, vol. II* (pp. 453–467). Kinokuniya. 8

[Nisnevič, 1989] Nisnevič, Y. A. (1989). The completely decomposed topology on schemes and associated descent spectral sequences in algebraic K-theory. In *Algebraic K-theory: connections with geometry and topology* (*Lake Louise, 1987*), volume 279 des *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.* (pp. 241–342). Kluwer Acad. Publ. 15

[Nizioł, 2006] Nizioł, W. (2006). Toric singularities: log-blow-ups and global resolutions. *J. Algebraic Geom.*, 15(1), 1–29. vi, 51, 52, 66, 81, 85, 87, 88, 89, 107, 222, 231

[Ogoma, 1980] Ogoma, T. (1980). Noncatenary pseudogeometric normal rings. *Japan. J. Math.*, 6(1), 147–163. 10, 11

[Ogus, 2013] Ogus, A. (2013). Lectures on Logarithmic Algebraic Geometry. Projet de livre, non publié. 77, 79, 80

[Olivier, 1968] Olivier, J.-P. (1967-1968). Anneaux absolument plats universels et épimorphismes à buts réduits. In *Séminaire Samuel. Algèbre commutative*, volume 2, exp. n° 6. 354

[Olivier, 1971] Olivier, J.-P. (1971). Montée des propriétés par morphismes absolument plats. In *Comptes-rendus des journées d'algèbre pure et appliquée* (pp. 86–109). Univ. Sci. Tech. Languedoc. 8

[Orgogozo, 2003] Orgogozo, F. (2003). Altérations et groupe fondamental premier à p. *Bull. Soc. math. France*, 131, 123–147. 163, 181

[Orgogozo, 2006] Orgogozo, F. (2006). Modifications et cycles évanescents sur une base de dimension supérieure à un. *Int. Math. Res. Notices*, 2006, 1–38. 17, 148, 152, 153

[Orgogozo, 2011] Orgogozo, F. (2011). Sur les propriétés d'uniformité des images directes en cohomologie étale. Prépublication. 186

[Panin & Smyrnov, 2003] Panin, I. & Smyrnov, A. (2003). Push-forwards in oriented cohomology theories of algebraic varieties II. K-theory:0619. 207

[Popescu, 1986] Popescu, D. (1986). General Néron desingularization and approximation. *Nagoya Math. J.*, 104, 85–115. v, 24

[Ramero, 2005] Ramero, L. (2005). Local monodromy in non-Archimedean analytic geometry. *Publications mathématiques de l'IHÉS*, 102, 167–280. 373

[Rapoport & Zink, 1982] Rapoport, M. & Zink, T. (1982). Über die lokale Zetafunktion von Shimuravarietäten, Monodromiefiltration und verschwindende Zyklen in ungleicher Charakteristik. *Invent. math.*, 68(1), 21–101. 229

[Ratliff, 1971] Ratliff, J. L. J. (1971). Characterizations of catenary rings. *Amer. J. Math.*, 93, 1070–1108. 7

[Raynaud, 1967] Raynaud, M. (1967). Un critère d'effectivité de descente. In *Séminaire Samuel. Algèbre commutative*, volume 2, exp. nº 5 (pp. 1–22). 70

[Raynaud, 1970] Raynaud, M. (1970). *Anneaux locaux henséliens*, volume 169 des *Lecture Notes in Mathematics*. Springer-Verlag. 46, 146

[Raynaud & Gruson, 1971] Raynaud, M. & Gruson, L. (1971). Critères de platitude et de projectivité. Techniques de « platification » d'un module. *Invent. math.*, 13, 1–89. 18, 128, 131, 311

[Rotthaus, 1977] Rotthaus, C. (1977). Komplettierung semilokaler quasiausgezeichneter ringe. *Nagoya Math. J.*, 76(2), 173–180. 8

[Rotthaus, 1979] Rotthaus, C. (1979). Universell japanische Ringe mit nicht offenem regulärem Ort. *Nagoya Math. J.*, 74, 123–135. 10, 11

[Rotthaus, 1990] Rotthaus, C. (1990). Rings with approximation property. *Math. Ann.*, 287(3), 455–466. 9

[Rotthaus, 1991] Rotthaus, C. (1991). On rings with low-dimensional formal fibres. *J. Pure Appl. Algebra*, 71(2-3), 287–296. 2

[Rotthaus, 1997] Rotthaus, C. (1997). Excellent rings, henselian rings, and the approximation property. *Rocky Mountain J. Math.*, 27, 317–344. 12

[Šafarevič, 1966] Šafarevič, I. R. (1966). *Lectures on minimal models and birational transformations of two dimensional schemes*, volume 37 des *Lectures on Mathematics and Physics*. Tata Institute of Fundamental Research. Notes de C. P. Ramanujam. 254

[Serre, 1965] Serre, J.-P. (1965). *Algèbre locale. Multiplicités*, volume 11 des *Lecture Notes in Mathematics*. Springer-Verlag. 48, 252

[Serre, 1968] Serre, J.-P. (1968). *Corps locaux*, volume VIII des *Publications de l'Université de Nancago*. Hermann, seconde édition. 44, 228, 235

[Serre, 1978] Serre, J.-P. (1978). *Représentations linéaires des groupes finis*. Collection Méthodes. Hermann. 113

[Serre, 1994] Serre, J.-P. (1994). *Cohomologie galoisienne*, volume 5 des *Lecture Notes in Mathematics*. Springer-Verlag. 225, 241

[Seydi, 1970] Seydi, H. (1970). Anneaux henséliens et conditions de chaînes. *Bull. Soc. math. France*, 98, 9–31. 8, 9, 11

[Seydi, 1972] Seydi, H. (1972). Exemple d'un anneau local noethérien japonais normal qui n'est pas formellement réduit. *C. R. Acad. sci. Paris*, 274, 1334–1337. 11

[Spaltenstein, 1988] Spaltenstein, N. (1988). Resolutions of unbounded complexes. *Compositio Math.*, 65(2), 121–154. 292, 295

[Spivakovsky, 1999] Spivakovsky, M. (1999). A new proof of D. Popescu's theorem on smoothing of ring homomorphisms. *J. Amer. Math. Soc.*, 12(2), 381–444. 9

[Stix, 2002] Stix, J. (2002). *Projective Anabelian Curves in Positive Characteristic and Descent Theory for Log-Étale Covers*, volume 354 des *Bonner Mathematische Schriften*. Math. Inst. der Univ. Bonn. 53, 54

[Swan, 1998] Swan, R. G. (1998). Néron-Popescu desingularization. In *Algebra and geometry* (*Taipei, 1995*), volume 2 des *Lectures in Algebra and Geometry* (pp. 135–192). International Press of Boston. 9, 76, 345

[Temkin, 2008] Temkin, M. (2008). Desingularization of quasi-excellent schemes in characteristic zero. *Adv. Math.*, 219(2), 488–522. 3, 129, 138

[Temkin, 2010] Temkin, M. (2010). Stable modification of relative curves. *J. Algebraic Geom.*, 19, 603–677. 109, 129, 131

[Temkin, 2012] Temkin, M. (2012). Functorial desingularization of quasi-excellent schemes in characteristic zero: the nonembedded case. *Duke Math. J.*, 161(11), 2207–2254. 69, 73, 75, 129

[Tsuji, 1997] Tsuji, T. (1997). Saturated morphisms of logarithmic schemes. Manuscrit non publié. 51, 136

[Vidal, 2001a] Vidal, I. (2001a). Groupe fondamental logarithmique. Manuscrit non publié. 53

[Vidal, 2001b] Vidal, I. (2001b). Log schémas réguliers. Manuscrit non publié. 52

[Vidal, 2004] Vidal, I. (2004). Théorie de Brauer et conducteur de Swan. *J. Algebraic Geom.*, 13(2), 349–391. 109, 122, 125, 126, 129, 222, 231

[Włodarczyk, 2005] Włodarczyk, J. (2005). Simple Hironaka resolution in characteristic zero. *J. Amer. Math. Soc.*, 18(4), 779–822. 74

[Zariski & Samuel, 1975] Zariski, O. & Samuel, P. (1975). *Commutative algebra*, volumes 28 & 29. Graduate Texts in Mathematics. Springer-Verlag. 21, 170, 190

[Zheng, 2009] Zheng, W. (2009). Sur l'indépendance de ℓ en cohomologie ℓ-adique sur les corps locaux. *Ann. sci. École norm. sup.*, 42(2), 291–334. 122, 125

Sigles

[ÉGA] = [Grothendieck, 1967]
[ÉGA ɪ′] = [Grothendieck & Dieudonné, 1971]
[SGA 1] = [Grothendieck, 2003b]
[SGA 2] = [Grothendieck, 2003a]
[SGA 4] = [Artin et al., 1973]
[SGA 4½] = [Deligne, 1977]
[SGA 5] = [Grothendieck, 1977]
[SGA 6] = [Berthelot et al., 1971]
[SGA 7] = [Deligne et al., 1972]

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