SÉMINAIRE DE GÉOMÉTRIE ALGÉBRIQUE DU BOIS MARIE

1962

COHOMOLOGIE LOCALE DES FAISCEAUX COHÉRENTS ET THÉORÈMES DE LEFSCHETZ LOCAUX ET GLOBAUX

(SGA 2)

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(rédigé par un groupe d'auditeurs)

Augmenté d'un exposé de
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Édition recomposée et annotée du volume 2
des Advanced Studies in Pure Mathematics
publié en 1968 par North-Holland Publishing Company

PREFACE

The present text is a recomposed and annotated edition of the book "Cohomologie locale des faisceaux cohérents et théorèmes de Lefschetz locaux et globaux (SGA 2)", Advanced Studies in Pure Mathematics 2, North-Holland Publishing Company - Amsterdam, 1968, by A. Grothendieck *et al.* It is the second part of the project started by B. Edixhoven who re-edited SGA 1. This version reproduces the original text with some formal modifications correcting typographical errors on the one hand, and on the other hand, comments in footnotes, called "editor's notes" (N.D.E.), specifying the current state of knowledge on the questions raised in the original version. We have also given here and there some additional details on the proofs. To avoid risks of confusion, the original notes retain their numbering system with stars while the new ones are numbered with integers. The page numbers of the original version are indicated in the margin.

I thank the mathematicians who carried out most of the initial typesetting in LaTeX2e, namely L. Bayle, N. Borne, O. Brinon, J. Buresi, M. Chardin, F. Ducrot, P. Graftiaux, F. Han, P. Karwasz, L. Koelblen, D. Madore, S. Morel, D. Naie, B. Osserman, J. Riou and V. Sécherre as well as C. Sabbah for having formatted this text to the SMF layout. I also thank J.-B. Bost, P. Colmez, O. Gabber, W. Fulton, S. Kleiman, F. Orgogozo, M. Raynaud and J.-P. Serre for their comments and advice.

The editor, Yves Laszlo.

The present text is a new updated edition of the book "Cohomologie locale des faisceaux cohérents et théorèmes de Lefschetz locaux et globaux (SGA 2)", Advanced Studies in Pure Mathematics 2, North-Holland Publishing Company - Amsterdam, 1968, by A. Grothendieck *et al.* It is the second part of the SGA project initiated by B. Edixhoven who made a new edition of SGA 1. This version is meant to reproduce the original text with some modifications including minor typographical corrections and footnotes from the editor (N.D.E.) explaining the current status of questions raised in the first edition. One has also given more details about some proofs. To avoid possible confusion, the original footnotes are numbered using stars whereas the new ones are numbered using integers. The page numbers of the original version are written in the margin of the text.

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TABLE DES MATIÈRES

INTRODUCTION

We present here, in a revised and completed form, a photo-offset re-edition of the second Séminaire de Géométrie Algébrique of the Institut des Hautes Études Scientifiques held in 1962 (mimeographed).

The reader should refer to the Introduction of the first of these Seminars (cited as SGA 1 hereafter) for the goals of these seminars, and their relations with the Éléments de Géométrie Algébrique.

The text of Exposés I to XI was written as we went along, based on my oral presentations and handwritten notes, by a group of auditors, including I. Giorgiutti, J. Giraud, Miss M. Jaffe (who became Mrs. M. Hakim), and A. Laudal. These notes were originally considered to be provisional and for very limited circulation, pending their absorption by the EGA (an absorption that has now become problematic, to say the least, as for the other parts of the SGA). As was stated in the foreword to the original edition, this "confidential" character of the notes was to excuse certain "weaknesses of style," undoubtedly more manifest in the present Seminar SGA 2 than in the others. I have tried as much as possible to remedy this in the present re-edition, by a relatively tight revision of the initial text. In particular, I have harmonized the numbering systems of the statements used in the different exposés, by introducing everywhere the same decimal system, already used in most of the original exposés of the present SGA 2, as well as in all other parts of the SGA. This led me in particular to completely revise the numbering[(2)] of the statements of Exposés III to VIII, (and consequently, of the references to said exposés)[(*)]. I also tried to remove from the original text the main typographical or syntactical errors (which were numerous and

[(*)] It goes without saying that all references to SGA 2 that appear in the parts of SGA published in the Series in Pure Mathematics will refer to the present volume, and not to the original edition of SGA 2 !

[(2)] N.D.E. : the original numbering has been preserved as much as possible, adding the adverb bis when ambiguous duplicates occurred here and there.

annoying). Furthermore, Mrs. M. Hakim was kind enough to rewrite Exposé IV in a less telegraphic style than the initial exposé. As in the other re-editions of the SGA, I have also added a certain number of footnotes, either to give additional references, or to indicate the status of a question for which progress has been made since the writing of the original text. Finally, this Seminar has been augmented with a new exposé, namely Exposé XIV, written by Mrs. Michèle Raynaud in 1967, which takes up and completes suggestions contained in the "Comments on Exposé XIII" (XIII 6) (written in March 1963). This exposé revisits the Lefschetz-type theorems from the point of view of étale cohomology, using the results on étale cohomology presented in SGA 4 and SGA 5 (to appear in this same collection Series in Pure Mathematics)(3) ; it is therefore in this respect of a less "elementary" nature than the other exposés of the present volume, which hardly use more than the substance of chapters I to III of the EGA. Here is a sketch of the content of the present volume. Exposé I contains the sorites of "cohomology with support in Y" $\mathrm{H}^*_{\mathrm{Y}}(\mathrm{X}, \mathrm{F})$, where Y is a closed set of a space X, cohomology which can be interpreted as a cohomology of X *modulo* the open set $\mathrm{X} - \mathrm{Y}$, and which is the result of a very useful "*spectral sequence for passing from local to global*" I 2.6, involving *sheaves* of cohomology "with support in Y" $\underline{\mathrm{H}}^*_{\mathrm{Y}}(\mathrm{F})$(4). This formalism can in many questions play a "localization" role analogous to that played by the consideration of "tubular" neighborhoods of Y in differential geometry. Exposé II studies the preceding notions in the case of quasi-coherent sheaves on preschemes, Exposé III gives their relation with the classical notion of *depth* (III 3.3).

Exposés IV and V give notions of *local duality*, which can be compared to Serre's projective duality theorem (XII 1.1) ; let us point out that these two types of duality theorems are substantially generalized in Hartshorne's seminar (cited in a footnote at the end of Exp IV)(5)

Exposés VI and VII give easy technical notions, used in Exposé VIII to prove the *finiteness theorem* (VIII 2.3), giving necessary and sufficient conditions, for a

(3)N.D.E. : in fact, these seminars are published by Springer (numbers 269, 279, 305 and 589), but, alas, are out of print.

(4)N.D.E. : we have kept the underlined notations for the sheaved versions of functors, the calligraphic analogue of Γ not being clear.

(5)N.D.E. : Hartshorne's book has errors in signs and above all does not really prove the compatibility of the trace with base change. Conrad has completely reworked this, proving this crucial and highly non-trivial compatibility (*Grothendieck duality and base change*, Lect. Notes in Math., vol. 1750, Springer-Verlag, Berlin, 2000). Alas, errors remain (cf. two prepublications (Conrad B., "Clarifications and corrections to '*Grothendieck duality and base change*'" and "An addendum to Chapter 5 of '*Grothendieck duality and base change*'")). For a more concrete aspect, with particular attention to the notion of residue, see the works of Lipman, in particular (Lipman J., *Dualizing sheaves, differentials and residues on algebraic varieties*, Astérisque, vol. 117, Société mathématique de France, 1984). A categorical proof of the duality theorem, based on Brown's representability theorem, was obtained by Neeman (Neeman A., "The Grothendieck duality theorem via Bousfield's techniques and Brown representability", *J. Amer. Math. Soc.* **9** (1996), number 1, pp. 205–236).

coherent sheaf F on a noetherian scheme X, for the local cohomology sheaves $\underline{\mathrm{H}}^i_{\mathrm{Y}}(\mathrm{F})$ to be coherent for $i \leqslant n$ (or what amounts to the same, for the sheaves $\mathrm{R}^i f_*(\mathrm{F}|\mathrm{X}-\mathrm{Y})$ to be coherent for $i \leqslant n-1$, where $f\colon \mathrm{X}-\mathrm{Y} \to \mathrm{X}$ is the inclusion). This theorem is one of the central technical results of the Seminar, and we show in Exposé IX how a theorem of this nature can be used to establish a "comparison theorem" and an "existence theorem" in formal geometry, by modeling and generalizing the use made in (EGA III §§ 4 and 5) of the finiteness theorem for a proper morphism.

We apply these last results in X and XI, dedicated respectively to Lefschetz-type theorems for the fundamental group, and for the Picard group. These theorems consist in comparing, under certain conditions, the invariants (π_1 or Pic) attached respectively to a scheme X and to a subscheme Y (playing the role of a hyperplane section), and in particular in giving conditions under which they are isomorphic. Roughly speaking, the hypotheses made serve to pass from Y to the formal completion of X along Y, and then to be able to apply the results of IX to pass from there to an open *neighborhood* U of Y in X. To be able to pass from U to X, one still needs information (of "purity" or "parafactoriality" type) for the local rings of X at the points of $\mathrm{Z} = \mathrm{X} - \mathrm{U}$, (which is a finite discrete set in the cases considered). This explains the interaction in the proofs of Exposés X, XI, XII between local and global results, especially in certain recurrences. The main results obtained in X and XI are the theorems *of a local nature* X 3.4 (*purity theorem*) and XI 3.14 (*parafactoriality theorem*). We note that these theorems are demonstrated by cohomological techniques, of an essentially global nature. In XII we obtain, using the preceding local results, the global variants of these results for projective schemes over a field, or more generally over a more or less arbitrary base scheme ; among the typical statements, let us mention XII 3.5 and XII 3.7.

In XIII, we review some of the many problems and conjectures suggested by the results and methods of the Seminar. Perhaps the most interesting concern the cohomological and homotopic Lefschetz-type theorems for complex analytic spaces, cf. XIII pages 26 and following[(6)]. In the context of the étale cohomology of schemes, the corresponding conjectures are proven in XIV by a duality technique that should also apply in the complex analytic case (cf. comments XIII p. 25 and XIV 6.4). But the corresponding homotopic statements in the case of analytic spaces (and particularly the statements involving the fundamental group) seem to require entirely new techniques (cf. XIV 6.4).

I am happy to thank all those who, in various capacities, have helped in the publication of the present volume, including the collaborators already cited in this Introduction. In particular, I wish to thank Miss Chardon for the good grace with which she accomplished the thankless task of the material preparation of the final manuscript for photo-offset.

(6) N.D.E. : essentially all the conjectures stated in XIII and XIV are now proven ; see the footnotes in these sections for references and comments.

Bures-sur-Yvette, April 1968
A. Grothendieck.

EXPOSÉ I

GLOBAL AND LOCAL COHOMOLOGICAL INVARIANTS RELATIVE TO A CLOSED SUBSPACE

1. The functors Γ_Z, $\underline{\Gamma}_Z$

Let X be a topological space, $\underline{C}_X$ the category of abelian sheaves on X. Let Φ be a family of supports in the sense of Cartan ; one defines the functor Γ_Φ on $\underline{C}_X$ by :

(1) $\quad \Gamma_\Phi(F) =$ subgroup of $\Gamma(F)$ formed by sections f such that support $f \in \Phi$.

If Z is a closed part of X, we denote by abuse of language by $\underline{\Gamma}_Z$ the functor Γ_Φ, where Φ is the set of closed parts of X contained in Z. Thus we have :

(2) $\quad \Gamma_Z(F) =$ subgroup of $\Gamma(F)$ formed by sections f such that support $f \subset Z$.

We want to generalize this definition to the case where Z is a *locally closed* part of X, thus closed in a suitable open part V of X. We will set in this case :

$$\Gamma_Z(F) = \Gamma_Z(F_{|V}). \tag{3}$$

It is necessary to verify that $\Gamma_Z(F)$ "does not depend" on the chosen open set. It is sufficient to show that if V', $V \supset V' \supset Z$ is an open set, then the application $\rho^V_{V'} : F(V) \to F(V')$ maps $\Gamma_Z(F_{|V})$ isomorphically onto $\Gamma_Z(F_{|V'})$. But

$$\Gamma_Z(F_{|V}) = \ker \rho^V_{V-Z} \tag{4}$$

so if $f \in \Gamma_Z(F_{|V})$ and if $\rho^V_{V'}(f) = \rho^V_{V-Z}(f) = 0$ then $f = 0$ since $(V', V - Z)$ is a cover of V. Similarly, if $f' \in \Gamma_Z(F_{|V'})$, then $f' \in F(V')$ and $0 \in F(V - Z)$ define an $f \in F(V)$ such that $\rho^V_{V'}(f) = f', f \in \Gamma_Z(F_{|V})$, so $\rho^V_{V'}$ induces an isomorphism $\Gamma_Z(F_{|V}) \to \Gamma_Z(F_{|V'})$.

Note that any open set W of Z is induced by an open set U of X in which W is closed. It follows that $W \mapsto \Gamma_W(F)$ defines a presheaf on Z, and one verifies that it is a sheaf that we will denote by $i^!(F)$, where $i : Z \to X$ is the canonical immersion. We find :

$$\Gamma_Z(F) = \Gamma(i^!(F)). \tag{5}$$

The sheaf $i^!(\mathrm{F})$ is a subsheaf of $i^*(\mathrm{F})$; indeed the canonical homomorphism :

$$\Gamma(\mathrm{F}_{|\mathrm{U}}) = \Gamma(\mathrm{U},\mathrm{F}) \longrightarrow \Gamma(\mathrm{U}\cap\mathrm{Z}, i^*(\mathrm{F}))$$

is injective on $\Gamma_{\mathrm{U}\cap\mathrm{Z}}(\mathrm{F}_{|\mathrm{U}}) \subset \Gamma(\mathrm{F}_{|\mathrm{U}})$. In summary, we have the following result :

Proposition 1.1. — *There exists a unique subsheaf* $i^!(\mathrm{F})$ *of* $i^*(\mathrm{F})$ *such that for any open set* U *of* X *such that* $\mathrm{U}\cap\mathrm{Z}$ *is closed in* U,

$$\Gamma(\mathrm{F}_{|\mathrm{U}}) = \Gamma(\mathrm{U},\mathrm{F}) \longrightarrow \Gamma(\mathrm{U}\cap\mathrm{Z}, i^*(\mathrm{F}))$$

induces an isomorphism $\Gamma_{\mathrm{U}\cap\mathrm{Z}}(\mathrm{F}_{|\mathrm{U}}) \to \Gamma(\mathrm{U}\cap\mathrm{Z}, i^!(\mathrm{F}))$.

Note that if Z is an open set we will simply have

(6) $$i^!(\mathrm{F}) = i^*(\mathrm{F}) = \mathrm{F}_{|\mathrm{Z}},\ \Gamma_{\mathrm{Z}}(\mathrm{F}) = \Gamma(\mathrm{Z},\mathrm{F}).$$

Suppose again that Z is arbitrary. Then for a variable open set U of X, we see that

$$\mathrm{U} \longmapsto \Gamma_{\mathrm{U}\cap\mathrm{Z}}(\mathrm{F}_{|\mathrm{U}}) = \Gamma(\mathrm{U}\cap\mathrm{Z}, i^!(\mathrm{F}))$$

is a sheaf on X, which we will denote by $\underline{\Gamma}_{\mathrm{Z}}(\mathrm{F})$; precisely, according to the preceding formula (expressing that $i^!$ commutes with restriction to open sets) we have an isomorphism

(7) $$\underline{\Gamma}_{\mathrm{Z}}(\mathrm{F}) = i_*(i^!(\mathrm{F}))$$

by definition, we have, for any open set U of X,

(8) $$\Gamma(\mathrm{U}, \underline{\Gamma}_{\mathrm{Z}}(\mathrm{F})) = \Gamma_{\mathrm{U}\cap\mathrm{Z}}(\mathrm{F}_{|\mathrm{U}}).$$

Note here a characteristic difference between the case where Z is closed, and where Z is open. In the first case, formula (8) shows us that $\underline{\Gamma}_{\mathrm{Z}}(\mathrm{F})$ can be considered as a subsheaf of F, and we thus have a *canonical immersion*

(8′) $$\underline{\Gamma}_{\mathrm{Z}}(\mathrm{F}) \hookrightarrow \mathrm{F}.$$

In the case where Z is open, on the contrary, we see from (6) that the second member of (8) is $\Gamma(\mathrm{U}\cap\mathrm{Z},\mathrm{F})$, thus receives $\Gamma(\mathrm{U},\mathrm{F})$, so we have a *canonical homomorphism* in the opposite direction of the preceding :

(8”) $$^{(1)}\mathrm{F} \longrightarrow \underline{\Gamma}_{\mathrm{Z}}(\mathrm{F}),$$

which is none other than the canonical homomorphism

$$\mathrm{F} \longrightarrow i_*i^*(\mathrm{F}),$$

taking into account the isomorphism

(6 bis) $$\underline{\Gamma}_{\mathrm{Z}}(\mathrm{F}) \simeq i_*i^*(\mathrm{F})$$

deduced from (6) and (7).

Of course, for variable F, $\Gamma_{\mathrm{Z}}(\mathrm{F})$, $\underline{\Gamma}_{\mathrm{Z}}(\mathrm{F})$, $i^!(\mathrm{F})$ can be considered as functors in F, with values respectively in the category of abelian groups, abelian sheaves on X, abelian sheaves on Z. It is sometimes convenient to interpret the functor

$$i^! : \underline{C}_X \longrightarrow \underline{C}_Z$$

as the adjoint functor of a well-known functor

$$i_! : \underline{C}_Z \longrightarrow \underline{C}_X$$

defined by the following proposition :

Proposition 1.2. — *Let* G *be an abelian sheaf on* Z. *Then there exists a unique subsheaf of* $i_*(G)$*, denoted* $i_!(G)$*, such that for any open set* U *of* X *the* (*identical*) *isomorphism*

$$\Gamma(U \cap Z, G) = \Gamma(U, i_*(G))$$

defines an isomorphism

$$\Gamma_{\Phi_{U\cap Z,U}}(U \cap Z, G) = \Gamma(U, i_!(G)),$$

where $\Phi_{U\cap Z,U}$ *denotes the set of parts of* $U \cap Z$ *that are closed in* U.

The verification reduces to noting that the first member is a sheaf for variable U, i.e. that the property for a section of $i_*(G)$ on U, considered as a section of G on $U \cap Z$, of being with support closed *in* U is of *local nature on* U. The sheaf $i_!(G)$ that we have just defined is also known under the name : *sheaf deduced from* G *by extending by* 0 outside of Z, cf. [**Godement**]. In particular, if Z is closed, we have

$$i_!(G) = i_*(G); \tag{9}$$

but in the general case, the canonical injection $i_!(G) \to i_*(G)$ is not an isomorphism, as is well known already for Z open. Obviously, $i_!(G)$ depends functorially on G (and it is even an exact functor in G). That said, we have :

Proposition 1.3. — *There exists an isomorphism of bifunctors in* G, F (G *abelian sheaf on* Z, F *abelian sheaf on* X) *:*

$$\mathrm{Hom}(i_!(G), F) = \mathrm{Hom}(G, i^!(F)). \tag{10}$$

To define such an isomorphism, it is equivalent to define functorial homomorphisms

$$i_! i^!(F) \longrightarrow F, \;\; G \longrightarrow i^! i_!(G),$$

satisfying the well-known compatibility conditions (cf. for example Shih's exposé at the Cartan seminar on cohomological operations).

Recalling that $i_!$ is exact, thus transforms monomorphisms into monomorphisms, we conclude :

Corollary 1.4. — *If* F *is injective,* $i^!(F)$ *is injective, so* $\underline{\Gamma}_Z(F) = i_* i^!(F)$ *is also injective.*

Replacing X with a variable open set U of X, we also conclude from 1.3

Corollary 1.5. — *We have a functorial isomorphism in* F,G *:*

$$\underline{\mathrm{Hom}}\,(i_!(\mathrm{G}),\mathrm{F}) = i_*(\underline{\mathrm{Hom}}\,(\mathrm{G},i^!(\mathrm{F})). \tag{11}$$

Taking for G the constant sheaf on Z defined by $\mathbf{Z}$, i.e. $\mathbf{Z}_{\mathrm{Z}}$, 1.3 and 1.5 specialize to

Corollary 1.6. — *We have functorial isomorphisms in* F *:*

$$\begin{aligned}\Gamma_{\mathrm{Z}}(\mathrm{F}) &= \mathrm{Hom}(\mathbf{Z}_{\mathrm{Z},\mathrm{X}},\mathrm{F}),\\ \underline{\Gamma}_{\mathrm{Z}}(\mathrm{F}) &= \underline{\mathrm{Hom}}(\mathbf{Z}_{\mathrm{Z},\mathrm{X}},\mathrm{F}),\end{aligned} \tag{12}$$

where $\mathbf{Z}_{\mathrm{Z},\mathrm{X}} = i_!(\mathbf{Z}_{\mathrm{Z}})$ *is the abelian sheaf on* X *deduced from the constant sheaf on* Z *defined by* $\mathbf{Z}$*, by extending by* 0 *outside of* Z.

Remark 1.7. — Suppose that X is a ringed space, and let us equip Z with the sheaf of rings $\mathscr{O}_{\mathrm{Z}} = i^{-1}(\mathscr{O}_{\mathrm{X}})$, finally let us denote by $\underline{\mathrm{C}}_{\mathrm{X}}$ and $\underline{\mathrm{C}}_{\mathrm{Z}}$ the category of Modules on X and Z respectively. Then the preceding considerations extend word for word, by taking for F a Module on X and for G a module on Z, and interpreting the statements 1.3 to 1.6 accordingly.

To finish these generalities, let us examine what happens when we change the locally closed part Z. Let $\mathrm{Z}' \subset \mathrm{Z}$ be another locally closed part, and let

$$j : \mathrm{Z}' \longrightarrow \mathrm{Z}, i' : \mathrm{Z}' \longrightarrow \mathrm{X}, i' = ij$$

be the canonical inclusions. Then we have functorial isomorphisms :

$$(ij)^! = j^! i^!, \quad (ij)_! = i_! j_!. \tag{13}$$

The first isomorphism (13) defines a functorial isomorphism

$$\Gamma_{\mathrm{Z}'}(\mathrm{F}) = \Gamma(\mathrm{Z}',(ij)^!(\mathrm{F})) \simeq \Gamma(\mathrm{Z}',j^!(i^!(\mathrm{F}))) = \Gamma_{\mathrm{Z}'}(i^!(\mathrm{F})). \tag{14}$$

Suppose now that Z' is *closed* in Z, and

$$\mathrm{Z}'' = \mathrm{Z} - \mathrm{Z}'$$

its complement in Z, which is open in Z and thus locally closed in X. The canonical inclusion (8′) applied to $i^!(\mathrm{F})$ on Z equipped with Z' defines for us, thanks to (14), a canonical injective functorial homomorphism

$$\Gamma_{\mathrm{Z}'}(\mathrm{F}) \longrightarrow \Gamma_{\mathrm{Z}}(\mathrm{F}). \tag{15}$$

If we replace in (14) Z' with Z'' and use (8”), we find a canonical functorial homomorphism :

$$\Gamma_{\mathrm{Z}}(\mathrm{F}) \longrightarrow \Gamma_{\mathrm{Z}''}(\mathrm{F}). \tag{15′}$$

Proposition 1.8. — *Under the preceding conditions, the sequence of functorial homomorphisms :*

$$0 \longrightarrow \Gamma_{\mathrm{Z}'}(\mathrm{F}) \longrightarrow \Gamma_{\mathrm{Z}}(\mathrm{F}) \longrightarrow \Gamma_{\mathrm{Z}''}(\mathrm{F}) \tag{16}$$

is exact. If F *is flasque, the sequence remains exact by adding a zero on the right.*

Démonstration. — Replacing X with an open set V in which Z is closed, we are reduced to the case where Z is closed, so Z′ is closed. Then Z″ is closed in the open set X − Z′, and we have a canonical inclusion

$$\Gamma_{\mathrm{Z}''}(\mathrm{F}) \longrightarrow \Gamma(\mathrm{X}-\mathrm{Z}',\mathrm{F}),$$

and the exactness of (16) simply means that the sections of F with support in Z′ are those whose restriction to X − Z′ is zero.

When F is flasque, any element of $\Gamma_{\mathrm{Z}''}(\mathrm{F})$, considered as a section of F on X − Z′, can be extended to a section of F on X, and the latter will obviously have its support in Z, which proves that then the last homomorphism in (16) is surjective.

Corollary 1.9. — *We have a functorial exact sequence*

$$0 \longrightarrow \underline{\Gamma}_{\mathrm{Z}'}(\mathrm{F}) \longrightarrow \underline{\Gamma}_{\mathrm{Z}}(\mathrm{F}) \longrightarrow \underline{\Gamma}_{\mathrm{Z}''}(\mathrm{F}), \tag{16 bis}$$

and if F *is flasque, this sequence remains exact by adding a* 0 *on the right.*

We can interpret (1.8) in terms of results on the functors Hom and $\underline{\mathrm{Hom}}$ via 1.6, in the following way. Let us first note that if G is an abelian sheaf on Z, inducing the sheaves $j^*(\mathrm{G})$ and $k^*(\mathrm{G})$ on Z′ and Z″ respectively (where $j : \mathrm{Z}' \to \mathrm{Z}$ and $k : \mathrm{Z}'' \to \mathrm{Z}$ are the canonical injections), we have a canonical exact sequence of sheaves on X :

$$0 \longrightarrow k^*(\mathrm{G})_{\mathrm{X}} \longrightarrow \mathrm{G}_{\mathrm{X}} \longrightarrow j^*(\mathrm{G})_{\mathrm{X}} \longrightarrow 0 \tag{17}$$

where, to simplify notations, the index X denotes the sheaf on X obtained by extending by 0 in the complement of the definition space of the considered sheaf. The exact sequence (17) generalizes a well-known exact sequence when Z = X (cf. [**Godement**]), and is moreover deduced from it by writing the exact sequence in question on Z, and applying the functor $i_!$. Taking $\mathrm{G} = \mathbf{Z}_{\mathrm{Z}}$, we conclude in particular :

Proposition 1.10. — *Under the preceding conditions, we have an exact sequence of abelian sheaves on* X *:*

$$0 \longrightarrow \mathbf{Z}_{\mathrm{Z}'',\mathrm{X}} \longrightarrow \mathbf{Z}_{\mathrm{Z},\mathrm{X}} \longrightarrow \mathbf{Z}_{\mathrm{Z}',\mathrm{X}} \longrightarrow 0. \tag{18}$$

This being said, the two exact sequences 1.8 *and* 1.9 *are none other than the exact sequences deduced from* (18) *by applying the functor* Hom(−, F) *and* $\underline{\mathrm{Hom}}$(−, F) *respectively.*

This gives an obvious new proof of the fact that the sequences (16) and (16 bis) remain exact by adding a zero on the right, provided that F is *injective.*

2. The functors $\mathrm{H}^*_{\mathrm{Z}}(\mathrm{X},\mathrm{F})$ and $\underline{\mathrm{H}}^*_{\mathrm{Z}}(\mathrm{F})$

Definition 2.1. — We denote by $\mathrm{H}^*_{\mathrm{Z}}(\mathrm{X},\mathrm{F})$ and $\underline{\mathrm{H}}^*_{\mathrm{Z}}(\mathrm{F})$ the derived functors in F of the functors $\Gamma_{\mathrm{Z}}(\mathrm{F})$ and $\underline{\Gamma}_{\mathrm{Z}}(\mathrm{F})$ respectively.

These are cohomological functors, with values in the category of abelian groups and in the category of abelian sheaves on X respectively. When Z is closed, $\mathrm{H}^*_{\mathrm{Z}}(\mathrm{X},\mathrm{F})$ is by definition none other than $\mathrm{H}^*_{\Phi}(\mathrm{X},\mathrm{F})$ where Φ denotes the family of closed parts of X contained in Z. When Z is open, we will see that $\mathrm{H}^*_{\mathrm{Z}}(\mathrm{X},\mathrm{F})$ is none other than $\mathrm{H}^*(\mathrm{Z},\mathrm{F}) = \mathrm{H}^*(\mathrm{Z},\mathrm{F}_{|\mathrm{Z}})$, thanks to the following proposition.

***Proposition 2.2* (Excision Theorem)**. — *Let* V *be an open part of* X *containing* Z*. Then we have an isomorphism of cohomological functors in* F *:*

(19) $$\mathrm{H}^*_{\mathrm{Z}}(\mathrm{X},\mathrm{F}) \longrightarrow \mathrm{H}^*_{\mathrm{Z}}(\mathrm{V},\mathrm{F}_{|\mathrm{V}}).$$

Indeed, we have a functorial isomorphism $\Gamma^{\mathrm{X}}_{\mathrm{Z}} \simeq \Gamma^{\mathrm{V}}_{\mathrm{Z}} j^!$, where $j : \mathrm{V} \to \mathrm{X}$ is the inclusion and where $j^!$ is therefore the restriction functor (cf. (14)). The latter is exact, and transforms injectives into injectives by 1.4, whence immediately the isomorphism (19).

When Z is open, we can take $\mathrm{V} = \mathrm{Z}$ and we find :

Corollary 2.3. — *Suppose* Z *is open, then we have an isomorphism of cohomological functors :*

(20) $$\mathrm{H}^*_{\mathrm{Z}}(\mathrm{X},\mathrm{F}) = \mathrm{H}^*(\mathrm{Z},\mathrm{F}).$$

We conclude from the isomorphisms 1.6 and the definitions (cf. [**Tôhoku**]) :

Proposition 2.3 bis(2). — *We have isomorphisms of cohomological functors :*

(21) $$\mathrm{H}^*_{\mathrm{Z}}(\mathrm{X},\mathrm{F}) \simeq \mathrm{Ext}^*(\mathrm{X};\mathbf{Z}_{\mathrm{Z},\mathrm{X}},\mathrm{F}),$$

(21 bis) $$\underline{\mathrm{H}}^*_{\mathrm{Z}}(\mathrm{F}) \simeq \underline{\mathrm{Ext}}^*(\mathbf{Z}_{\mathrm{Z},\mathrm{X}},\mathrm{F}).$$

We can therefore apply the results of [**Tôhoku**] on the Ext of Modules. Let us first point out the following interpretation of the sheaves $\underline{\mathrm{H}}^*_{\mathrm{Z}}(\mathrm{F})$ in terms of global groups $\mathrm{H}^*_{\mathrm{Z}}(\mathrm{X},\mathrm{F})$:

Corollary 2.4. — $\underline{\mathrm{H}}^*_{\mathrm{Z}}(\mathrm{F})$ *is canonically isomorphic to the sheaf associated to the presheaf*

$$\mathrm{U} \longmapsto \mathrm{H}^*_{\mathrm{Z}\cap\mathrm{U}}(\mathrm{U},\mathrm{F}_{|\mathrm{U}}).$$

In particular, using corollary 2.3, we find :

Corollary 2.5. — *Suppose* Z *is open, then we have an isomorphism of cohomological functors :*

(22) $$\underline{\mathrm{H}}^*_{\mathrm{Z}}(\mathrm{F}) = \mathrm{R}^*\, i_* i^*(\mathrm{F})$$

(*where* $i : \mathrm{Z} \to \mathrm{X}$ *is the inclusion*).

The spectral sequence of Ext gives the important spectral sequence :

(2) N.D.E. : the proposition is numbered 2.3 in the original edition.

Theorem 2.6. — *We have a spectral sequence functorial in* F, *converging to* $\mathrm{H}^*_{\mathrm{Z}}(\mathrm{X},\mathrm{F})$ *and with initial term*

$$\mathrm{E}_2^{p,q}(\mathrm{F}) = \mathrm{H}^p(\mathrm{X}, \underline{\mathrm{H}}^q_{\mathrm{Z}}(\mathrm{F})). \tag{23}$$

Remarks 2.7. — It follows immediately from 2.4 that the sheaves $\underline{\mathrm{H}}^q_{\mathrm{Z}}(\mathrm{F})$ are zero in $\mathrm{X}-\overline{\mathrm{Z}}$, and also zero in the interior of Z for $q \neq 0$ (so for such a q, $\underline{\mathrm{H}}^q_{\mathrm{Z}}(\mathrm{F})$ is even supported on the boundary of Z).

Consequently, the second member of (23) can be interpreted as a cohomology group on $\overline{\mathrm{Z}}$. We will use 2.6 in the case where Z is closed in X, and where the second member of (23) can be interpreted as a cohomology group calculated on Z :

$$^{(3)}\mathrm{E}_2^{p,q}(\mathrm{F}) = \mathrm{H}^p(\mathrm{Z}, \underline{\mathrm{H}}^q_{\mathrm{Z}}(\mathrm{F})). \tag{23 bis}$$

Note also that when Z is open, the spectral sequence 2.6 is none other than the Leray spectral sequence for the continuous map $i : \mathrm{Z} \to \mathrm{X}$, taking into account the interpretation 2.5 in the calculation of the initial term of the Leray spectral sequence.

Let us return to the exact sequence (18)[(4)], it gives rise to an exact sequence of Ext (cf. [**Tôhoku**]) :

Theorem 2.8. — *Let* Z *be a locally closed part of* X, Z' *a closed part of* Z *and* $\mathrm{Z}'' = \mathrm{Z}-\mathrm{Z}'$. *Then we have an exact sequence functorial in* F *:*

$$\begin{gathered} 0 \longrightarrow \mathrm{H}^0_{\mathrm{Z}'}(\mathrm{X},\mathrm{F}) \longrightarrow \mathrm{H}^0_{\mathrm{Z}}(\mathrm{X},\mathrm{F}) \longrightarrow \mathrm{H}^0_{\mathrm{Z}''}(\mathrm{X},\mathrm{F}) \xrightarrow{\partial} \mathrm{H}^1_{\mathrm{Z}'}(\mathrm{X},\mathrm{F}) \longrightarrow \mathrm{H}^1_{\mathrm{Z}}(\mathrm{X},\mathrm{F}) \dots \\ \dots \mathrm{H}^i_{\mathrm{Z}'}(\mathrm{X},\mathrm{F}) \longrightarrow \mathrm{H}^i_{\mathrm{Z}}(\mathrm{X},\mathrm{F}) \longrightarrow \mathrm{H}^i_{\mathrm{Z}''}(\mathrm{X},\mathrm{F}) \xrightarrow{\partial} \mathrm{H}^{i+1}_{\mathrm{Z}'}(\mathrm{X},\mathrm{F}) \dots . \end{gathered} \tag{24}$$

Let us recall how one can obtain this exact sequence. Let C(F) be an injective resolution of F, then the exact sequence (18)[(5)] gives rise to the exact sequence

$$0 \longrightarrow \Gamma_{\mathrm{Z}'}(\mathrm{C}(\mathrm{F})) \longrightarrow \Gamma(\mathrm{C}(\mathrm{F})) \longrightarrow \Gamma_{\mathrm{Z}''}(\mathrm{C}(\mathrm{F})) \longrightarrow 0, \tag{25}$$

(which is none other than that defined in 1.8). We conclude a cohomology exact sequence, which is none other than (24).

The most important case for us is where Z is closed (and we can always reduce to this case by replacing X with an open set V in which Z is closed). Then Z' is closed, Z'' is closed in the open set $\mathrm{X}-\mathrm{Z}'$, and we can write

$$\mathrm{H}^i_{\mathrm{Z}''}(\mathrm{X},\mathrm{F}) = \mathrm{H}^i_{\mathrm{Z}''}(\mathrm{X}-\mathrm{Z}', \mathrm{F}_{|\mathrm{X}-\mathrm{Z}'}), \tag{26}$$

which allows us to write the exact sequence (24) in terms of cohomology with support in a given closed set. The most frequent case is where $\mathrm{Z} = \mathrm{X}$. Setting then for simplicity $\mathrm{Z}' = \mathrm{A}$, we find :

(4) N.D.E. : the original reference was (1.10).
(5) N.D.E. : see previous note.

Corollary 2.9. — *Let* A *be a closed part of* X*. Then we have an exact sequence functorial in* F *:*

$$(27)\qquad \begin{aligned} &0 \longrightarrow \mathrm{H}^0_{\mathrm{A}}(\mathrm{X},\mathrm{F}) \longrightarrow \mathrm{H}^0(\mathrm{X},\mathrm{F}) \longrightarrow \mathrm{H}^0(\mathrm{X}-\mathrm{A},\mathrm{F}) \xrightarrow{\partial} \mathrm{H}^1_{\mathrm{A}}(\mathrm{X},\mathrm{F})\ldots \\ &\ldots \mathrm{H}^i_{\mathrm{A}}(\mathrm{X},\mathrm{F}) \longrightarrow \mathrm{H}^i(\mathrm{X},\mathrm{F}) \longrightarrow \mathrm{H}^i(\mathrm{X}-\mathrm{A},\mathrm{F}) \xrightarrow{\partial} \mathrm{H}^{i+1}_{\mathrm{A}}(\mathrm{X},\mathrm{F})\ldots. \end{aligned}$$

This exact sequence shows that the cohomology group $\mathrm{H}^i_{\mathrm{A}}(\mathrm{X},\mathrm{F})$ plays the role of a relative cohomology group of X mod X − A, with coefficients in F. It is in this capacity that it was naturally introduced in applications. By "sheafifying" (24) and (27), or by proceeding directly, we find, taking into account that the sheaf associated to $\mathrm{U} \mapsto \mathrm{H}^i(\mathrm{U},\mathrm{F})$ is zero if $i > 0$:

Corollary 2.10. — *Under the conditions of* 2.8*, we have an exact sequence functorial in* F *:*

$$(24\text{ bis})\qquad \ldots \underline{\mathrm{H}}^i_{\mathrm{Z}'}(\mathrm{F}) \longrightarrow \underline{\mathrm{H}}^i_{\mathrm{Z}}(\mathrm{F}) \longrightarrow \underline{\mathrm{H}}^i_{\mathrm{Z}''}(\mathrm{F}) \xrightarrow{\partial} \underline{\mathrm{H}}^{i+1}_{\mathrm{Z}'}(\mathrm{F})\ldots$$

Corollary 2.11. — *Let* A *be a closed part of* X*, then we have an exact sequence functorial in* F *:*

$$(28)\qquad 0 \longrightarrow \underline{\mathrm{H}}^0_{\mathrm{A}}(\mathrm{F}) \longrightarrow \mathrm{F} \longrightarrow f_*(\mathrm{F}_{|\mathrm{X}-\mathrm{A}}) \xrightarrow{\partial} \underline{\mathrm{H}}^1_{\mathrm{A}} \longrightarrow 0,$$

and canonical isomorphisms, for $i \geqslant 2$ *:*

$$(29)\qquad \underline{\mathrm{H}}^i_{\mathrm{A}}(\mathrm{F}) = \underline{\mathrm{H}}^{i-1}_{\mathrm{X}-\mathrm{A}}(\mathrm{F}) = \mathrm{R}^{i-1} f_*(\mathrm{F}_{|\mathrm{X}-\mathrm{A}}),$$

where $f : (\mathrm{X}-\mathrm{A}) \to \mathrm{X}$ *is the inclusion.*

This therefore defines $\underline{\mathrm{H}}^0_{\mathrm{A}}(\mathrm{F})$ and $\underline{\mathrm{H}}^1_{\mathrm{A}}(\mathrm{F})$ respectively as ker and coker of the canonical homomorphism

$$\mathrm{F} \longrightarrow f_*f^*(\mathrm{F}) = f_*(\mathrm{F}_{|\mathrm{X}-\mathrm{A}}),$$

and the $\underline{\mathrm{H}}^i_{\mathrm{A}}(\mathrm{F})$ $(i \geqslant 2)$ in terms of the derived functors of f_*.

Corollary 2.12. — *Let* F *be an abelian sheaf on* X*. If* F *is flasque, then for any locally closed part* Z *of* X *and any integer* $i \neq 0$*, we have* $\mathrm{H}^i_{\mathrm{Z}}(\mathrm{X},\mathrm{F}) = 0$*,* $\underline{\mathrm{H}}^i_{\mathrm{Z}}(\mathrm{F}) = 0$*. Conversely, if for any closed part* Z *of* X *we have* $\mathrm{H}^1_{\mathrm{Z}}(\mathrm{X},\mathrm{F}) = 0$*, then* F *is flasque.*

Suppose that F is flasque, then F induces a flasque sheaf on any open set, so to prove $\mathrm{H}^i_{\mathrm{Z}}(\mathrm{X},\mathrm{F}) = 0$ for $i > 0$, we can suppose Z is closed, and then the assertion results from the exact sequence (27)[(6)]. We conclude from this for any locally closed Z, by "sheafifying" i.e. applying 2.4, that $\underline{\mathrm{H}}^i_{\mathrm{Z}}(\mathrm{F}) = 0$ for $i > 0$. Conversely, suppose $\mathrm{H}^1_{\mathrm{Z}}(\mathrm{X},\mathrm{F}) = 0$ for any closed set Z, then the exact sequence (27)[(7)] proves that for any such Z, $\mathrm{H}^0(\mathrm{X},\mathrm{F}) \to \mathrm{H}^0(\mathrm{X}-\mathrm{Z},\mathrm{F})$ is surjective, which means that F is flasque.

Combining 2.6 and 2.8, we will deduce :

(6) N.D.E. : the original reference was (2.9).
(7) N.D.E. : the original reference was (2.9).

Proposition 2.13. — *Let* F *be an abelian sheaf on* X, Z *a closed part of* X, $\mathrm{U}=\mathrm{X}-\mathrm{Z}$, N *an integer. The following conditions are equivalent :*

(i) $\underline{\mathrm{H}}_{\mathrm{Z}}^{i}(\mathrm{F})=0$ *for* $i \leqslant \mathrm{N}$.

(ii) *For any open set* V *of* X, *considering the canonical homomorphism*

$$\mathrm{H}^{i}(\mathrm{V},\mathrm{F}) \longrightarrow \mathrm{H}^{i}(\mathrm{V}\cap\mathrm{U},\mathrm{F}),$$

this homomorphism is :

a) *bijective for* $i<\mathrm{N}$,

b) *injective for* $i=\mathrm{N}$.

(*When* $\mathrm{N}>0$, *one can in* (ii) *restrict to requiring* a)).

To prove (i) IMPLIES (ii), we are reduced, thanks to the local nature of $\underline{\mathrm{H}}_{\mathrm{Z}}^{i}(\mathrm{F})$, to proving the

Corollary 2.14. — *If condition* 2.13 (*i*) *is verified, then*

$$\mathrm{H}^{i}(\mathrm{X},\mathrm{F}) \longrightarrow \mathrm{H}^{i}(\mathrm{U},\mathrm{F})$$

is bijective for $i<\mathrm{N}$, *injective for* $i=\mathrm{N}$.

Indeed, by virtue of the exact sequence (27), this also means $\mathrm{H}_{\mathrm{Z}}^{i}(\mathrm{X},\mathrm{F})=0$ for $i\leqslant \mathrm{N}$, and this relation is an immediate consequence of the spectral sequence (23 bis)[(8)].

Conversely, hypothesis 2.13 (ii) means that for any open set V of X, we have

$$\mathrm{H}_{\mathrm{Z}\cap\mathrm{V}}^{i}(\mathrm{V},\mathrm{F}_{|\mathrm{V}})=0 \text{ for } i\leqslant \mathrm{N},$$

which implies 2.13 (i) thanks to 2.4. If moreover $\mathrm{N}>0$, the hypothesis b) is superfluous. Indeed, if $\mathrm{N}=1$, hypothesis a) and (28) ensure the nullity of $\underline{\mathrm{H}}_{\mathrm{Z}}^{i}(\mathrm{F})=0$ for $i\leqslant\mathrm{N}$. If $\mathrm{N}>1$, hypothesis a) for $i=\mathrm{N}-1>0$ and (29) ensure the nullity of $\underline{\mathrm{H}}_{\mathrm{Z}}^{i}(\mathrm{F})$ for $i\leqslant\mathrm{N}$.

Taking into account 2.11 this also proves 2.13 (i)...

Remarque. — Let $\mathrm{Y}\to\mathrm{X}$ be a closed immersion, and suppose that locally it has the form $\{0\}\times\mathrm{Y}\subset\mathrm{R}^{n}\times\mathrm{Y}$. Suppose that F is a locally constant sheaf on x, then we find

$$\underline{\mathrm{H}}_{\mathrm{Y}}^{i}(\mathrm{F})\simeq\begin{cases}0 & \text{if } i\neq n\\ \mathrm{F}\otimes\underline{\mathrm{T}}_{\mathrm{Y},\mathrm{X}} & \text{if } i=n, \text{ where } \underline{\mathrm{T}}_{\mathrm{Y},\mathrm{X}}\simeq\underline{\mathrm{H}}_{\mathrm{Y}}^{n}(\mathbf{Z}_{\mathrm{X}})\end{cases}\tag{30}$$

is a sheaf extension to X of a sheaf on Y locally isomorphic to $\mathbf{Z}_{\mathrm{Y}}$, called "sheaf of normal orientation of Y in X".

Using the spectral sequence (23 bis)[(9)], we find in this case :

$$\mathrm{H}_{\mathrm{Y}}^{i}(\mathrm{X},\mathrm{F})\simeq\mathrm{H}^{i-n}(\mathrm{Y},\mathrm{F}\otimes\underline{\mathrm{T}}_{\mathrm{Y},\mathrm{X}}),\tag{31}$$

(8) N.D.E. : see previous note.
(9) N.D.E. : the original reference was 2.6.

and we find the "Gysin homomorphism" :

$$\mathrm{H}^j(\mathrm{Y}, \mathrm{F} \otimes \underline{\mathrm{T}}_{\mathrm{Y},\mathrm{X}}) \longrightarrow \mathrm{H}^{j+n}(\mathrm{X}, \mathrm{F}). \tag{32}$$

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EXPOSÉ II

APPLICATION TO QUASI-COHERENT SHEAVES ON PRESCHEMES

Proposition 1. — *Let* X *be a prescheme,* Z *a locally closed part of the form* $Z = U - V$, *where* U *and* V *are two open parts of* X *such that* $V \subset U$ *and the canonical immersions* $U \to X$, $V \to X$ *are quasi-compact. Then for any quasi-coherent Module* F *on* X, *the sheaves* $\underline{H}^i_Z(F)$ *are quasi-coherent.*

According to (I 24), there exists a relative cohomology exact sequence

$$\longrightarrow \underline{H}^{i-1}_U(F) \longrightarrow \underline{H}^i_V(F) \longrightarrow \underline{H}^i_Z(F) \longrightarrow \underline{H}^i_U(F) \longrightarrow \underline{H}^{i+1}_V(F) \longrightarrow .$$

According to (EGA III 1.4.17), for the $\underline{H}^i_Z(F)$ to be quasi-coherent it is sufficient that the $\underline{H}^i_U(F)$ and $\underline{H}^i_V(F)$ are. We can therefore suppose Z is open and the canonical immersion $j : Z \to X$ is quasi-compact.

Since Z is open we have (I 22) a canonical isomorphism :

$$\underline{H}^i_Z(F) \simeq R^i j_*(F_{|Z})$$

but j is separated (EGA I 5.5.1) and quasi-compact, so (EGA III 1.4.10) the $R^i j_*(F|Z) = \underline{H}^i_Z(F)$ are quasi-coherent, which completes the proof.

Corollary 2. — *Let* Z *be a closed part of* X *such that the canonical immersion* $X - Z \to X$ *is quasi-compact, then the Modules* $\underline{H}^i_Z(F)$ *are quasi-coherent.*

Corollary 3. — *If* X *is locally noetherian, then for any locally closed part* Z *of* X, *and any quasi-coherent Module* F *on* X, *the* $\underline{H}^i_Z(F)$ *are quasi-coherent.*

Results immediately from corollary 2 and from (EGA I 6.6.4).

Corollary 4. — *Suppose that* X *is the spectrum of a ring* A *and let* U *be a quasi-compact open set of* X, $Y = X - U$, F *a quasi-coherent Module on* X, *there exists an isomorphism of cohomological functors in* F *:*

$$\underline{H}^i_Y(F) = (\widetilde{H^i_Y(X, F)}). \tag{4.1}$$

We also have a functorial exact sequence in F *:*

$$0 \longrightarrow \mathrm{H}^0_{\mathrm{Y}}(\mathrm{X},\mathrm{F}) \longrightarrow \mathrm{H}^0(\mathrm{X},\mathrm{F}) \longrightarrow \mathrm{H}^0(\mathrm{U},\mathrm{F}) \longrightarrow \mathrm{H}^1_{\mathrm{Y}}(\mathrm{X},\mathrm{F}) \longrightarrow 0 \tag{4.2}$$

and functorial isomorphisms in F *:*

$$\mathrm{H}^i_{\mathrm{Y}}(\mathrm{X},\mathrm{F}) \simeq \mathrm{H}^{i-1}(\mathrm{U},\mathrm{F}), \quad i \geqslant 2. \tag{4.3}$$

According to corollary 2, the $\underline{\mathrm{H}}^i_{\mathrm{Y}}(\mathrm{F})$ are quasi-coherent, since X is affine we therefore have $\mathrm{H}^p(\mathrm{X},\underline{\mathrm{H}}^i_{\mathrm{Y}}(\mathrm{F})) = 0$ if $p > 0$. The spectral sequence (I 23) degenerates, so

$$\mathrm{H}^i_{\mathrm{Y}}(\mathrm{X},\mathrm{F}) = \Gamma(\underline{\mathrm{H}}^i_{\mathrm{Y}}(\mathrm{F})).$$

The equality (4.1) then results from (EGA I 1.1.3.7), (4.2) and (4.3) from the cohomology exact sequence (I 27) and from the fact that $\mathrm{H}^i(\mathrm{X},\mathrm{F}) = 0$ if $i > 0$, since X is affine.

With the hypotheses of 4[1], U is a finite union of affine open sets X_f, so we can find an ideal I generated by a finite number of elements f_α and defining Y, let $\boldsymbol{f} = (f_\alpha)$. With the notations of (EGA III 1)[2] we have :

Proposition 5. — *Suppose that* X *is the spectrum of a ring* A*, let* $\boldsymbol{f} = (f_\alpha)$ *be a finite family of elements of* A*,* Y *the closed part of* X *that they define,* M *an* A*-module,* F *the sheaf associated to* M*. We then have isomorphisms of* ∂*-functors in* M *:*

$$\mathrm{H}^i((\boldsymbol{f}),\mathrm{M}) \simeq \mathrm{H}^i_{\mathrm{Y}}(\mathrm{X},\mathrm{F}). \tag{5.1}$$

(We will also denote $\mathrm{H}^i_{\mathrm{J}}(\mathrm{M}) = \mathrm{H}^i_{\mathrm{Y}}(\mathrm{X},\mathrm{F})$, if Y is the closed part of X = Spec A defined by an ideal J of A).

For $i = 0$ and $i = 1$, we use the exact sequences (4.2) and (EGA III 1.4.3.2) ; if $i \geqslant 2$, we use (4.3) and (EGA III 1.4.3.1). This gives us functorial isomorphisms in M. We verify that up to a sign depending only on i, they are compatible with the boundary operator, whence the existence of the isomorphism of ∂-functors (5.1).

Let now X be a prescheme, Y a closed part of X and $f : \mathrm{Y} \to \mathrm{X}$ the inclusion, I a quasi-coherent ideal defining Y in X. Let F be a sheaf on X.

We have seen that there exist isomorphisms of ∂-functors in F

$$\mathrm{Ext}^i_{\mathscr{O}_{\mathrm{X}}}(\mathrm{X}; f_*f^{-1}(\mathscr{O}_{\mathrm{X}}),\mathrm{F}) \longrightarrow \mathrm{H}^i_{\mathrm{Y}}(\mathrm{X},\mathrm{F}) \tag{$*$}$$

$$\underline{\mathrm{Ext}}^i_{\mathscr{O}_{\mathrm{X}}}(f_*f^{-1}(\mathscr{O}_{\mathrm{X}}),\mathrm{F}) \longrightarrow \underline{\mathrm{H}}^i_{\mathrm{Y}}(\mathrm{F}). \tag{$**$}$$

Let n, m be integers such that $m \geqslant n \geqslant 0$, we denote by $i_{n,m}$ the canonical map : $\mathscr{O}_{\mathrm{Y}_m} = \mathscr{O}_{\mathrm{X}}/\mathrm{I}^{m+1} \to \mathscr{O}_{\mathrm{X}}/\mathrm{I}^{n+1} = \mathscr{O}_{\mathrm{Y}_n}$, and by j_n the map : $f_*f^{-1}(\mathscr{O}_{\mathrm{X}}) \to \mathscr{O}_{\mathrm{Y}_n}$. The $(\mathscr{O}_{\mathrm{Y}_n}, i_{n,m})$ form a projective system and the j_n are compatible with the $i_{n,m}$.

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[1] N.D.E. : for consistency and clarity, only equations have been numbered in parentheses.

[2] N.D.E. : recall that $\mathrm{H}^\bullet(\boldsymbol{f},\mathrm{M})$ is the Koszul cohomology $\mathrm{H}^\bullet(\mathrm{Hom}(\mathrm{K}_\bullet(\boldsymbol{f}),\mathrm{M}))$ of $\boldsymbol{f}$ (EGA III 1.1.2) with values in M and that $\mathrm{H}^\bullet((\boldsymbol{f}),\mathrm{M})$ is the limit (*loc. cit.*, 1.1.6.5) $\varinjlim_n \mathrm{H}^\bullet(\boldsymbol{f}^n,\mathrm{M})$, the transition morphisms being induced by the natural morphisms $\mathrm{K}_\bullet(\boldsymbol{f}^{n+1}) \to \mathrm{K}_\bullet(\boldsymbol{f}^n)$ (*loc. cit.*, 1.1.6).

By applying the functor $\mathrm{Ext}^i_{\mathscr{O}_\mathrm{X}}(\mathrm{X};\cdot,\mathrm{F})$, we deduce a morphism

$$\varphi' : \varinjlim_n \mathrm{Ext}^i_{\mathscr{O}_\mathrm{X}}(\mathrm{X};\mathscr{O}_{\mathrm{Y}_n},\mathrm{F}) \longrightarrow \mathrm{Ext}^i_{\mathscr{O}_\mathrm{X}}(\mathrm{X};f_*f^{-1}(\mathscr{O}_\mathrm{X}),\mathrm{F});$$

it is easily shown that this is a morphism of cohomological functors in F. The morphism

$$\varphi : \varinjlim_n \mathrm{Ext}^i_{\mathscr{O}_\mathrm{X}}(\mathrm{X};\mathscr{O}_{\mathrm{Y}_n},\mathrm{F}) \longrightarrow \mathrm{H}^i_\mathrm{Y}(\mathrm{X},\mathrm{F}),$$

composed of φ' and of $(*)$, is therefore also a morphism of cohomological functors in F.

One defines similarly

$$\underline{\varphi} : \varinjlim_n \underline{\mathrm{Ext}}^i_{\mathscr{O}_\mathrm{X}}(\mathscr{O}_{\mathrm{Y}_n},\mathrm{F}) \longrightarrow \underline{\mathrm{H}}^i_\mathrm{Y}(\mathrm{F}).$$

We have in view the following theorem :

Theorem 6. — a) *Let* X *be a locally noetherian prescheme,* Y *a closed part of* X *defined by a coherent Ideal* I*,* F *a quasi-coherent Module. Then* $\boldsymbol{\varphi}$ *is an isomorphism.*

b) *If* X *is noetherian* φ *is an isomorphism.*

Theorem 6 will result from 6.a) and the following

Lemme 7. — *If the underlying topological space of* X *is noetherian and if* $\boldsymbol{\varphi}$ *is an isomorphism, then so is* φ*.*

We will first prove lemma 7. We know that there exists a spectral sequence

$$\mathrm{H}^p(\mathrm{X},\underline{\mathrm{H}}^q_\mathrm{Y}(\mathrm{F})) \Longrightarrow \mathrm{H}^*_\mathrm{Y}(\mathrm{X},\mathrm{F}). \tag{7.1}$$

We have on the other hand an inductive system of spectral sequences

$$\mathrm{H}^p(\mathrm{X},\underline{\mathrm{Ext}}^q_{\mathscr{O}_\mathrm{X}}(\mathscr{O}_{\mathrm{Y}_n},\mathrm{F})) \Longrightarrow \mathrm{Ext}^*_{\mathscr{O}_\mathrm{X}}(\mathrm{X};\mathscr{O}_{\mathrm{Y}_n},\mathrm{F}). \tag{7.2$_n$}$$

It results from the definition of φ and of $\underline{\varphi}$ that these morphisms are associated to a homomorphism Φ of spectral sequences from the inductive limit of (7.2_n) to (7.1). If the underlying space of X is noetherian, by (God. 4.12.1)[(*)]

$$\varinjlim_n \mathrm{H}^p(\mathrm{X},\underline{\mathrm{Ext}}^q_{\mathscr{O}_\mathrm{X}}(\mathscr{O}_{\mathrm{Y}_n},\mathrm{F})) \xrightarrow{\sim} \mathrm{H}^p(\mathrm{X},\varinjlim_n \underline{\mathrm{Ext}}^q_{\mathscr{O}_\mathrm{X}}(\mathscr{O}_{\mathrm{Y}_n},\mathrm{F})),$$

then Φ_2 can be written as a morphism :

$$\mathrm{H}^p(\mathrm{X},\varinjlim_n \underline{\mathrm{Ext}}^q_{\mathscr{O}_\mathrm{X}}(\mathscr{O}_{\mathrm{Y}_n},\mathrm{F})) \longrightarrow \mathrm{H}^p(\mathrm{X},\underline{\mathrm{H}}^q_\mathrm{Y}(\mathrm{F}))$$

which is none other than the one deduced from $\underline{\varphi}$.

If $\underline{\varphi}$ is an isomorphism then so is Φ_2, hence φ according to (EGA 0_{III} 11.1.5), thus lemma 7 is proven.

We will now prove 6.a), it is a local question on X. According to corollary 4 and (EGA I 1.3.9 and 1.3.12) we can suppose that X is the spectrum of a ring A. It is

(*) Cf. first bibliographic reference, at the end of Exp I.

therefore sufficient to prove that under the hypotheses of theorem 6.a), the canonical homomorphism :

$$\varinjlim_n \mathrm{Ext}^i_{\mathrm{A}}(\mathrm{A}/\mathrm{I}^n, \mathrm{M}) \longrightarrow \mathrm{H}^i_{\mathrm{Y}}(\mathrm{X}, \mathrm{M}) \tag{7.3}$$

is an isomorphism.

Let f_α be a finite number of elements of A generating I, $\boldsymbol{f} = (f_\alpha)$; then the sequence of ideals $(\boldsymbol{f}^n)$ is decreasing and cofinal with the sequence of I^n, so that (7.3) is equivalent to a morphism of ∂-functors in M :

$$\varinjlim_n \mathrm{Ext}^i_{\mathrm{A}}(\mathrm{A}/(\boldsymbol{f}^n), \mathrm{M}) \longrightarrow \mathrm{H}^i_{\mathrm{Y}}(\mathrm{X}, \mathrm{M}). \tag{7.4}$$

We have on the other hand canonical isomorphisms :

$$\varinjlim_n \mathrm{Hom}_{\mathrm{A}}(\mathrm{A}/(\boldsymbol{f}^n), \mathrm{M}) \simeq \varinjlim_n (m \in \mathrm{M} \mid (\boldsymbol{f}^n)m = 0) \simeq \mathrm{H}^0((\boldsymbol{f}), \mathrm{M}). \tag{7.5}$$

Since $\varinjlim_n \mathrm{Ext}^i_{\mathrm{A}}(\mathrm{A}/(\boldsymbol{f}^n), \mathrm{M})$ is a universal ∂-functor in M, there exists a unique morphism of ∂-functors in M :

$$\varinjlim_n \mathrm{Ext}^i_{\mathrm{A}}(\mathrm{A}/(\boldsymbol{f}^n), \mathrm{M}) \longrightarrow \mathrm{H}^i((\boldsymbol{f}), \mathrm{M}), \tag{7.6}$$

which coincides in degree zero with (7.5).

Since the composite of (7.3) and of (5.1) is a morphism of ∂-functors in M which coincides with (7.6) in degree 0, it coincides with (7.6) in any degree. Theorem 6.a) is therefore an immediate consequence of the following

Lemme 8. — *Let* A *be a noetherian ring,* I *an ideal generated by a finite system* $\boldsymbol{f} = (f_\alpha)$ *of elements,* M *an* A*-module. Then the homomorphisms* (7.6) *are isomorphisms.*

Lemme 9. — *Let* A *be a ring,* $\boldsymbol{f} = (f_\alpha)$ *a finite system of elements of* A*,* I *the ideal generated by* $\boldsymbol{f}$*,* i *an integer* > 0*. The following conditions are equivalent :*

a) *The homomorphism* (7.6) *is an isomorphism for any* M.

b) $\mathrm{H}^i((\boldsymbol{f}), \mathrm{M}) = 0$ *for* M *injective.*

c) *The projective system* $(\mathrm{H}_i(\boldsymbol{f}^n, \mathrm{A})) = \mathrm{H}_{i,n}$ *is essentially null, that is : for any* n*, there exists* $n' > n$ *such that* $\mathrm{H}_{i,n'} \to \mathrm{H}_{i,n}$ *is null.*

a) implies b) trivially.

b) implies a), indeed b) implies that $\mathrm{M} \mapsto \mathrm{H}^i((\boldsymbol{f}), \mathrm{M})$ is a universal cohomological functor, (7.6) is then a morphism of universal cohomological functors. It is an isomorphism in degree zero, thus in any degree.

c) implies b), indeed if M is injective, we have for any n

$$\mathrm{H}^i(\boldsymbol{f}^n, \mathrm{M}) = \mathrm{Hom}(\mathrm{H}_i(\boldsymbol{f}^n, \mathrm{A}), \mathrm{M}) = \mathrm{Hom}(\mathrm{H}_{i,n}, \mathrm{M}),$$

c) thus implies that for any i the inductive system $(\mathrm{H}^i(\boldsymbol{f}^n, \mathrm{M}))_{n \in \mathbf{Z}}$ is essentially null, whence b).

b) implies c). Let indeed $n > 0$, and j a monomorphism from $\mathrm{H}_{i,n}$ into an injective module M. Let $n' \geqslant n$ and let $j_{n'} \in \mathrm{Hom}(\mathrm{H}_{i,n'}, \mathrm{M})$ be the composite of j and of the transition homomorphism $t_{n',n} : \mathrm{H}_{i,n'} \to \mathrm{H}_{i,n}$. The $j_{n'}$ define an element of $\mathrm{H}^i((\boldsymbol{f}), \mathrm{M})$ which is null by hypothesis. There thus exists n_0 such that $j_{n'} = 0$ if $n' > n_0$. But since j is a monomorphism, $j_{n'} = 0$ implies $t_{n',n} = 0$, whence the proposition.

Corollary 10. — *Suppose that the underlying space of* $\mathrm{X} = \mathrm{Spec}(\mathrm{A})$ *is noetherian. For the preceding conditions to be verified for any finite family of elements of* A *and any* $i > 0$ (*or : for* $i = 1$), *it is necessary and sufficient that for any injective* A*-module* M*, the sheaf* F *associated to* M *is flasque.*

It is necessary : let indeed $\boldsymbol{f} = (f_\alpha)$ be a finite system of elements of A, Y the closed set defined by $\boldsymbol{f}$ and $\mathrm{U} = \mathrm{X} - \mathrm{Y}$, we then have the exact sequence

$$\mathrm{H}^0(\mathrm{X}, \mathrm{F}) \longrightarrow \mathrm{H}^0(\mathrm{U}, \mathrm{F}) \longrightarrow \mathrm{H}^1((\boldsymbol{f}), \mathrm{M}) \longrightarrow 0,$$

and thanks to 9.b, $\mathrm{H}^0(\mathrm{X}, \mathrm{F}) \to \mathrm{H}^0(\mathrm{U}, \mathrm{F})$ is surjective.

It is sufficient by virtue of (5.1) and the fact that for any closed part Y of X and any flasque sheaf F on X, $\mathrm{H}^i_{\mathrm{Y}}(\mathrm{X}, \mathrm{F}) = 0$ for $i > 0$.

Lemme 11. — *Under the hypotheses of lemma* 9*, for any noetherian* A*-module* N *and for any* $i > 0$*, the projective system* $(\mathrm{H}_{i,n}(\mathrm{N}))_{n \in \mathbf{Z}}$*, where* $\mathrm{H}_{i,n}(\mathrm{N}) = \mathrm{H}_i(\boldsymbol{f}^n, \mathrm{N})$*, is essentially null.*

Proof by recurrence on the number m of elements of $\boldsymbol{f}$.

If $m = 1$, $\boldsymbol{f}$ is reduced to a single element, let it be f, $\mathrm{H}_{i,n}(\mathrm{N})$ is null if $i > 1$ and $\mathrm{H}_{1,n}(\mathrm{N})$ is canonically isomorphic to the annihilator $\mathrm{N}(n)$ of f^n in N, the transition homomorphism $\mathrm{N}(n') \to \mathrm{N}(n)$, $n' \geqslant n$, being multiplication by $f^{n'-n}$. The $\mathrm{N}(n)$ form an increasing sequence of submodules of N, and since N is noetherian there exists n_0 such that $\mathrm{N}(n) = \mathrm{N}(n_0)$ if $n \geqslant n_0$. Therefore all the $\mathrm{N}(n)$ are annihilated by f^{n_0} and the transition homomorphisms $\mathrm{N}(n') \to \mathrm{N}(n)$ are all null if $n' \geqslant n + n_0$. The lemma is thus proven for $m = 1$.

We now suppose that $m > 1$ and that the lemma is proven for integers $m' < m$; let then $\boldsymbol{g} = (f_1, \ldots, f_{m-1})$ and $\boldsymbol{h} = f_m$.

For any $n > 0$, we have (EGA III 1.1.4.1) an exact sequence :

$$0 \longrightarrow \mathrm{H}_0(\boldsymbol{h}^n, \mathrm{H}_i(\boldsymbol{g}^n, \mathrm{N})) \longrightarrow \mathrm{H}_i(\boldsymbol{f}^n, \mathrm{N}) \longrightarrow \mathrm{H}_1(\boldsymbol{h}^n, \mathrm{H}_{i-1}(\boldsymbol{g}^n, \mathrm{N})) \longrightarrow 0,$$

and for variable n a projective system of exact sequences. It results from the recurrence hypotheses that for $i > 0$ the $\mathrm{H}_i(\boldsymbol{g}^n, \mathrm{N})$ form an essentially null projective system, thus also the $\mathrm{H}_0(\boldsymbol{h}^n, \mathrm{H}_i(\boldsymbol{g}^n, \mathrm{N}))$ which are identified with quotients of $\mathrm{H}_i(\boldsymbol{g}^n, \mathrm{N})$. For the right-hand terms we will factor the transition morphisms from n' to n by :

$$\mathrm{H}_1(\boldsymbol{h}^{n'}, \mathrm{H}_{i-1}(\boldsymbol{g}^{n'}, \mathrm{N})) \longrightarrow \mathrm{H}_1(\boldsymbol{h}^{n'}, \mathrm{H}_{i-1}(\boldsymbol{g}^{n}, \mathrm{N})) \longrightarrow \mathrm{H}_1(\boldsymbol{h}^{n}, \mathrm{H}_{i-1}(\boldsymbol{g}^{n}, \mathrm{N})).$$

Since $\mathrm{H}_{i-1}(\boldsymbol{g}^n, \mathrm{N})$ is a noetherian module it results from the case $m = 1$ that there exists, for a given n, $n' > n$ such that the second arrow is null. We thus see that in this projective system of exact sequences, the extreme projective systems are essentially null, so the same is true for the middle projective system.

We have therefore proven lemma 11, thus lemma 8 and hence theorem 6.

Remarque. — One can also obtain theorem 6 by demonstrating the condition of corollary 10 with the help of the structure theorems of injective modules over a noetherian ring (Matlis, Gabriel).

EXPOSÉ III

COHOMOLOGICAL INVARIANTS AND DEPTH

1. Recalls

We will state some definitions and results that the reader can find, for example, in chapter I of the course taught by J.-P. Serre at the Collège de France in 1957-58.[(1)]

Definition 1.1. — Let A be a ring (commutative with unity as in all that follows) and let M be an A-module (unitary as in all that follows), we call :

– annihilator of M and denote by Ann M the set of $a \in \mathrm{A}$, such that for any $m \in \mathrm{M}$ we have $am = 0$.

– support of M and denote by Supp M the set of prime ideals $\mathfrak{p}$ of A such that the localization $\mathrm{M}_{\mathfrak{p}}$ is non-zero.

– "assassin of M" or "set of associated prime ideals of M" and denote by Ass M the set of prime ideals $\mathfrak{p}$ of A such that there exists a non-zero element of M whose annihilator is $\mathfrak{p}$.

If $\mathfrak{a}$ is an ideal of A, we will denote by $\mathfrak{r}(\mathfrak{a})$ the radical of $\mathfrak{a}$ in A, i.e. the set of elements of A of which a power is in $\mathfrak{a}$.

The following results are valid if one supposes that A *is noetherian and* M *is of finite type.*

Proposition 1.1

(i) Ass M *is a finite set.*

(ii) *For an element of* A *to annihilate a non-zero element of* M*, it is necessary and sufficient that it belongs to one of the associated ideals of* M.

(iii) *The radical of the annihilator of* M*,* $\mathfrak{r}(\mathrm{Ann\,M})$*, is the intersection of the associated ideals of* M *that are minimal* (*for the inclusion relation in* Ass M).

(1) N.D.E. : the re-edition of Serre's text (Serre J.-P., *Algèbre locale. Multiplicités*, Cours au Collège de France, 1957–1958, written by Pierre Gabriel, second edition, Lect. Notes in Math., vol. 11, Springer-Verlag, 1965) no longer contains the proofs of these statements. One can refer to (Bourbaki N., *Algèbre commutative*, Masson), as suggested by Serre himself.

Proposition 1.2. — *Let $\mathfrak{p}$ be a prime ideal of* A*, the following assertions are equivalent :*

(i) $\mathfrak{p} \in \operatorname{Supp} \mathrm{M}$.
(ii) *There exists* $\mathfrak{q} \in \operatorname{Ass} \mathrm{M}$ *such that* $\mathfrak{q} \subset \mathfrak{p}$.
(iii) $\mathfrak{p} \supset \operatorname{Ann} \mathrm{M}$.
(iii bis) $\mathfrak{p} \supset \mathfrak{r}(\operatorname{Ann} \mathrm{M})$.

Proposition 1.3. — *Let* N *be an* A*-module of finite type, we have the formula :*

$$\operatorname{Ass} \operatorname{Hom}_{\mathrm{A}}(\mathrm{N}, \mathrm{M}) = \operatorname{Supp} \mathrm{N} \cap \operatorname{Ass} \mathrm{M}.$$

2. Depth

In this whole paragraph, A denotes a commutative ring, I an ideal of A, M and N two A-modules. We will denote by X the prime spectrum of A (we will not use its structure sheaf in this paragraph) and by Y the variety of I, $\mathrm{Y} = \operatorname{Supp}(\mathrm{A}/\mathrm{I}) = \{\mathfrak{p} \in \mathrm{X},\ \mathfrak{p} \supset \mathrm{I}\}$.

Lemma 2.1. — *Suppose that* A *is noetherian and that the modules* M *and* N *are of finite type. Suppose moreover that* $\operatorname{Supp} \mathrm{N} = \mathrm{Y}$*. Then the following assertions are equivalent :*

(i) $\operatorname{Hom}_{\mathrm{A}}(\mathrm{N}, \mathrm{M}) = 0$.
(ii) $\operatorname{Supp} \mathrm{N} \cap \operatorname{Ass} \mathrm{M} = \emptyset$.
(iii) *The ideal* I *is not a zero-divisor in* M*, which means that for any* $m \in \mathrm{M}$, $\mathrm{I}m = 0$ *implies* $m = 0$.
(iv) *There exists in* I *an* M*-regular element.* (*An element* a *of* A *is called* M*-regular if the homothety of ratio* a *in* M *is injective.*)
(v) *For any* $\mathfrak{p} \in \mathrm{Y}$*, the maximal ideal* $\mathfrak{p}\mathrm{A}_{\mathfrak{p}}$ *of the local ring* $\mathrm{A}_{\mathfrak{p}}$ *is not associated to* $\mathrm{M}_{\mathfrak{p}}$*. In formula :* $\mathfrak{p}\mathrm{A}_{\mathfrak{p}} \notin \operatorname{Ass} \mathrm{M}_{\mathfrak{p}}$.

Démonstration

(i) IFF (ii) because $\operatorname{Ass} \operatorname{Hom}_{\mathrm{A}}(\mathrm{N}, \mathrm{M}) = \emptyset$ is equivalent to (ii) by proposition 1.3 and to (i) by an easy consequence of proposition 1.2.

(iii) IMPLIES (ii) by contradiction : "there exists $\mathfrak{p} \in \operatorname{Supp} \mathrm{N} \cap \operatorname{Ass} \mathrm{M}$" implies that $\mathfrak{p} \supset \mathrm{I}$ and that there exists $m \in \mathrm{M}$ whose annihilator is $\mathfrak{p}$, so $\mathrm{I}m \subset \mathfrak{p}m = 0$, which contradicts (iii).

(iv) IMPLIES (iii) trivially.

(ii) IFF (iv) because $\operatorname{Supp} \mathrm{N} = \mathrm{Y}$, so (ii) means that I is not contained in any ideal associated to M or, in other words, (because the ideals associated to M are prime and finite in number), that I is not contained in the union of the ideals associated to M. But, by proposition 1.1 (ii), this set is the set of elements of A that are not M-regular.

(i) IMPLIES (v) ; indeed, if $\mathrm{Hom}_{\mathrm{A}}(\mathrm{N},\mathrm{M}) = 0$ and if $\mathfrak{p} \in \mathrm{Y}$, we deduce from this, by virtue of the formula

$$\big(\mathrm{Hom}_{\mathrm{A}}(\mathrm{N},\mathrm{M})\big)_{\mathfrak{p}} = \mathrm{Hom}_{\mathrm{A}_{\mathfrak{p}}}\big(\mathrm{N}_{\mathfrak{p}},\mathrm{M}_{\mathfrak{p}}\big),$$

that $\mathrm{Hom}_{\mathrm{A}_{\mathfrak{p}}}\big(\mathrm{N}_{\mathfrak{p}},\mathrm{M}_{\mathfrak{p}}\big) = 0$, so, thanks to proposition 1.3,

$$\mathrm{Supp}\,\mathrm{N}_{\mathfrak{p}} \cap \mathrm{Ass}\,\mathrm{M}_{\mathfrak{p}} = \emptyset,$$

but $\mathfrak{p}\mathrm{A}_{\mathfrak{p}} \in \mathrm{Supp}\,\mathrm{N}_{\mathfrak{p}}$, so $\mathfrak{p}\mathrm{A}_{\mathfrak{p}} \notin \mathrm{Ass}\,\mathrm{M}_{\mathfrak{p}}$.

(v) IMPLIES (i) ; indeed, if $\mathfrak{p} \in \mathrm{Ass}\,\mathrm{M}$, there exists $m \in \mathrm{M}$ whose annihilator is $\mathfrak{p}$, so the canonical image of m in $\mathrm{M}_{\mathfrak{p}}$ is non-zero, so its annihilator is an ideal that contains $\mathfrak{p}$, thus $\mathfrak{p}\mathrm{A}_{\mathfrak{p}}$, thus is equal to it. The ideal $\mathfrak{p}\mathrm{A}_{\mathfrak{p}}$ is therefore associated to $\mathrm{M}_{\mathfrak{p}}$, so $\mathfrak{p} \notin \mathrm{Y}$ according to (v), whence (i). C.Q.F.D.

We will work on these conditions by replacing the functor Hom with its derivatives.

Theorem 2.2. — *Let* A *be a commutative ring,* I *an ideal of* A*,* M *an* A*-module. Let* n *be an integer.*

a) *If there exists a sequence* $f_1, \dots, f_{n+1}$*, of elements of* I *which forms an* M*-regular sequence* (*i.e. if* f_1 *is* M*-regular and if* f_{i+1} *is regular in* $\mathrm{M}/(f_1,\dots,f_i)\mathrm{M}$ *for* $i \leqslant n$)*, for any* A*-module* N *annihilated by a power of* I*, we have :*

$$\mathrm{Ext}^i_{\mathrm{A}}(\mathrm{N},\mathrm{M}) = 0 \textit{ for } i \leqslant n.$$

b) *If moreover* A *is noetherian, if* M *is of finite type, and if there exists an* A*-module* N *of finite type such that* $\mathrm{Supp}\,\mathrm{N} = \mathrm{V}(\mathrm{I})$ *and such that* $\mathrm{Ext}^i_{\mathrm{A}}(\mathrm{N},\mathrm{M}) = 0$ *for* $i \leqslant n$*, then there exists a sequence* $f_1, \dots, f_{n+1}$*, of elements of* I *which is* M*-regular.*

Let us first prove a), by recurrence. If $n < 0$ the statement is empty.

If $n \geqslant 0$, suppose that a) is proven for $n' < n$; by hypothesis there exists $f_1 \in \mathrm{I}$ which is M-regular. Let us denote by f^i_1 the multiplication by f_1 in $\mathrm{Ext}^i_{\mathrm{A}}(\mathrm{N},\mathrm{M})$ and by f^{M}_1 the multiplication by f_1 in M. The sequence

$$0 \longrightarrow \mathrm{M} \xrightarrow{f^{\mathrm{M}}_1} \mathrm{M} \longrightarrow \mathrm{M}/f_1\mathrm{M} \longrightarrow 0 \tag{2.1}$$

is exact, so also the sequence :

$$\mathrm{Ext}^{i-1}_{\mathrm{A}}(\mathrm{N},\mathrm{M}/f_1\mathrm{M}) \xrightarrow{\delta} \mathrm{Ext}^i_{\mathrm{A}}(\mathrm{N},\mathrm{M}) \xrightarrow{f^i_1} \mathrm{Ext}^i_{\mathrm{A}}(\mathrm{N},\mathrm{M}).$$

By hypothesis $\mathrm{I}^r\mathrm{N} = 0$, so f^0_1 is nilpotent ; Ext^i is a universal functor, so is f^i_1 for any i. Furthermore, there exists a regular sequence in $\mathrm{M}/f_1\mathrm{M}$ which has n elements, so, by recurrence hypothesis,

$$Ext^{i-1}_{\mathrm{A}}(\mathrm{N},\mathrm{M}) = 0 \text{ if } i \leqslant n-1.$$

We deduce from this that if $i \leqslant n$, f^i_1 is both nilpotent and injective so $\mathrm{Ext}^i_{\mathrm{A}}(\mathrm{N},\mathrm{M}) = 0$.

Let us prove b), also by recurrence. If $n < 0$, the statement is empty.

If $n = 0$, b) results from assertion (i) IMPLIES (iv) of lemma 2.1.

If $n > 0$, according to b) for $n = 0$, there exists an element $f_1 \in \mathrm{I}$ which is M-regular ; from the exact sequence (2.1), we deduce the exact sequence :

$$\mathrm{Ext}^{i-1}_{\mathrm{A}}(\mathrm{N}, \mathrm{M}) \longrightarrow \mathrm{Ext}^{i-1}_{\mathrm{A}}(\mathrm{N}, \mathrm{M}/f_1\mathrm{M}) \longrightarrow \mathrm{Ext}^{i}_{\mathrm{A}}(\mathrm{N}, \mathrm{M}). \tag{2.2}$$

We conclude that the hypotheses of b) are verified for the module $\mathrm{M}/f_1\mathrm{M}$ and for the integer $n-1$. By the recurrence hypothesis, there exists a sequence of n elements of I which is regular for $\mathrm{M}/f_1\mathrm{M}$, which implies that there exists a sequence of $n+1$ elements of I, starting with f_1, and which is M-regular.

This theorem invites us to generalize in the following way the classical definition of the depth of a module of finite type over a noetherian ring :

Definition 2.3. — Let A be a commutative ring with unity, let M be an A-module, let I be an ideal of A. We call the I-depth of M, and we denote by $\mathrm{prof}_{\mathrm{I}}\,\mathrm{M}$, the upper bound in $\mathbf{N} \cup \{+\infty\}$ of the set of natural numbers n, which are such that for any A-module of finite type N annihilated by a power of I, we have

$$\mathrm{Ext}^{i}_{\mathrm{A}}(\mathrm{N}, \mathrm{M}) = 0 \text{ for all } i < n.$$

We deduce from the preceding theorem that if n is the upper bound of the lengths of the M-regular sequences of elements of I, we have $n \leqslant \mathrm{prof}_{\mathrm{I}}\,\mathrm{M}$.
More precisely :

Proposition 2.4. — *Let* A *be a commutative ring,* I *an ideal of* A *and let* M *be an* A*-module, finally let* $n \in \mathbf{N}$. *Consider the assertions :*

(1) $n \leqslant \mathrm{prof}_{\mathrm{I}}\,\mathrm{M}$.

(2) *For any* A*-module of finite type* N *which is annihilated by a power of* I*, we have :*

$$\mathrm{Ext}^{i}_{\mathrm{A}}(\mathrm{N}, \mathrm{M}) = 0 \textit{ for } i < n.$$

(3) *There exists an* A*-module of finite type* N *such that* $\mathrm{Supp}\,\mathrm{N} = \mathrm{V(I)}$ *and such that* $\mathrm{Ext}^{i}_{\mathrm{A}}(\mathrm{N}, \mathrm{M}) = 0$ *if* $i < n$.

(4) *There exists an* M*-regular sequence of length* n *formed by elements of* I.

We have the following logical implications :

$$\begin{array}{ccccc} (1) & \Longleftrightarrow & (2) & \Longleftarrow & (4) \\ & & \Downarrow & & \\ & & (3) & & \end{array}$$

Moreover if A *is noetherian and* M *of finite type, these conditions are equivalent.*

Démonstration

(1) IFF (2) by definition and (2) IMPLIES (3) by taking $\mathrm{N} = \mathrm{A}/\mathrm{I}$; Moreover (4) IMPLIES (2) by theorem 2.2 a). Finally, if A is noetherian and M of finite type, (3) IMPLIES (4) by theorem 2.2 b).

We suppose A is noetherian and M is of finite type until the end of this paragraph.

Corollary 2.5. — *Let* $f \in \mathrm{I}$ *be an* M*-regular element, we have :*

$$\mathrm{prof}_{\mathrm{I}}\,\mathrm{M} = \mathrm{prof}_{\mathrm{I}}(\mathrm{M}/f\mathrm{M}) + 1.$$

Indeed, if $n \leqslant \mathrm{prof}_{\mathrm{I}}(\mathrm{M}/f\mathrm{M})$, there exists a sequence of elements of I, $f_1, \ldots, f_n$, which is $(\mathrm{M}/f\mathrm{M})$-regular ; therefore the sequence $f, f_1, \ldots, f_n$ is M-regular, so $n+1 \leqslant \mathrm{prof}_{\mathrm{I}}\,\mathrm{M}$, so $\mathrm{prof}_{\mathrm{I}}\,\mathrm{M} \geqslant \mathrm{prof}_{\mathrm{I}}(\mathrm{M}/f\mathrm{M}) + 1$. On the other hand, according to the exact sequence (2.2), if $i \leqslant \mathrm{prof}_{\mathrm{I}}\,\mathrm{M}$, we have $\mathrm{Ext}^{i-1}_{\mathrm{A}}(\mathrm{N}, \mathrm{M}/f\mathrm{M}) = 0$, so $\mathrm{prof}_{\mathrm{I}}\,\mathrm{M} - 1 \leqslant \mathrm{prof}_{\mathrm{I}}(\mathrm{M}/f\mathrm{M})$.

Corollary 2.6. — *Any finite* M*-regular sequence, formed by elements of* I*, can be extended to a maximal* M*-regular sequence, whose length is necessarily equal to the* I*-depth of* M*.*

Remark 2.7. — One can hardly resist saying that an A-module is more beautiful the greater its depth. A module whose support does not meet V(I) is most beautiful ; indeed, one can prove that for $\mathrm{prof}_{\mathrm{I}}\,\mathrm{M}$ to be finite, it is necessary and sufficient that $\mathrm{Supp}\,\mathrm{M} \cap \mathrm{V}(\mathrm{I}) \neq \emptyset$.

Remark 2.8. — If A is a semi-local ring, let $\mathfrak{r}(\mathrm{A})$ be its radical and $k = \mathrm{A}/\mathfrak{r}(\mathrm{A})$ its residue ring. The interesting notion of depth is obtained by taking for I the radical of A. We will therefore agree to simply denote by prof M the $\mathfrak{r}(\mathrm{A})$-depth of an A-module M. We find in this case the notion of "homological codimension", (cf. Serre, *op. cit.* note (1), page 25), which was denoted $\mathrm{codh}_{\mathrm{A}}\,\mathrm{M}$, and which is defined as the infimum of the integers i such that $\mathrm{Ext}^{i}_{\mathrm{A}}(k, \mathrm{M}) \neq 0$; indeed $\mathrm{Supp}\,k = \mathrm{V}(\mathfrak{r}(\mathrm{A}))$.

Proposition 2.9. — *If* A *is noetherian and* M *of finite type, we have :*

$$\mathrm{prof}_{\mathrm{I}}\,\mathrm{M} = \inf_{\mathfrak{p} \in \mathrm{V}(\mathrm{I})} \mathrm{prof}\,\mathrm{M}_{\mathfrak{p}}.$$

Corollary 2.10. — *If* A *is a noetherian semi-local ring, and if* M *is an* A*-module of finite type, we have :*

$$\mathrm{prof}\,\mathrm{M} = \inf_{\mathfrak{m}} \mathrm{prof}\,\mathrm{M}_{\mathfrak{m}},$$

where $\mathfrak{m}$ *ranges over the set of maximal ideals of* A*.*

The corollary results immediately from proposition 2.9 ; indeed the prime ideals that contain the radical are the maximal ideals.

On the other hand, let $f \in \mathrm{I}$; if f is M-regular, if $\mathfrak{p} \in \mathrm{X}$ and if $\mathfrak{p} \supset \mathrm{I}$, the image g of f in $\mathrm{A}_{\mathfrak{p}}$ belongs to $\mathfrak{p}\mathrm{A}_{\mathfrak{p}}$, maximal ideal of $\mathrm{A}_{\mathfrak{p}}$; moreover g is $\mathrm{M}_{\mathfrak{p}}$-regular, as it results from the exact sequence

$$0 \longrightarrow \mathrm{M}_{\mathfrak{p}} \xrightarrow{\ g'\ } \mathrm{M}_{\mathfrak{p}} \longrightarrow (\mathrm{M}/f\mathrm{M})_{\mathfrak{p}} \longrightarrow 0, \tag{2.3}$$

where g' denotes the homothety of ratio g in $\mathrm{M}_{\mathfrak{p}}$. This exact sequence also gives that $(\mathrm{M}/f\mathrm{M})_{\mathfrak{p}}$ is isomorphic to $\mathrm{M}_{\mathfrak{p}}/g\mathrm{M}_{\mathfrak{p}}$; by applying corollary 2.5 to M and to $\mathrm{M}_{\mathfrak{p}}$, we deduce, by recurrence, that $\operatorname{prof}_{\mathrm{I}} \mathrm{M} \leqslant \nu(\mathrm{M})$, where we have set for all M :

$$\nu(\mathrm{M}) = \inf_{\mathfrak{p}\in \mathrm{V(I)}} \operatorname{prof} \mathrm{M}_{\mathfrak{p}}.$$

More precisely, still by recurrence, we know, if f is M-regular, that $\nu(\mathrm{M}) = \nu(\mathrm{M}/f\mathrm{M})+1$; it thus remains to prove that if $\nu(\mathrm{M}) \geqslant 1$, there exists an M-regular element in I. But by applying lemma 2.1 to $\mathrm{M}_{\mathfrak{p}}$, $\mathrm{A}_{\mathfrak{p}}$ and $\mathfrak{p}\mathrm{A}_{\mathfrak{p}}$ for any $\mathfrak{p} \in \mathrm{V(I)}$, we see that $\mathfrak{p}\mathrm{A}_{\mathfrak{p}} \notin \operatorname{Ass} \mathrm{M}_{\mathfrak{p}}$, so, by applying lemma 2.1 to A, M and I, we have the conclusion.

Proposition 2.11. — *Let* $u : \mathrm{A} \to \mathrm{B}$ *be a homomorphism of noetherian rings. Let* I *be an ideal of* A*,* M *an* A*-module of finite type. Set* $\mathrm{I_B} = \mathrm{I} \otimes_{\mathrm{A}} \mathrm{B}$ *and* $\mathrm{M_B} = \mathrm{M} \otimes_{\mathrm{A}} \mathrm{B}$*. If* B *is* A*-flat, we have :*

$$\operatorname{prof}_{\mathrm{I_B}} \mathrm{M_B} \geqslant \operatorname{prof}_{\mathrm{I}} \mathrm{M};$$

moreover if B *is faithfully flat over* A*, we have equality.*

Indeed, let $\mathrm{N} = \mathrm{A}/\mathrm{I}$; by flatness we have : $\mathrm{N} \otimes_{\mathrm{A}} \mathrm{B} = \mathrm{B}/\mathrm{I_B}$; set $\mathrm{N_B} = \mathrm{N} \otimes_{\mathrm{A}} \mathrm{B}$. Still by flatness and noetherian hypotheses, we have :

$$\operatorname{Ext}^i_{\mathrm{B}}(\mathrm{N_B}, \mathrm{M_B}) = \operatorname{Ext}^i_{\mathrm{A}}(\mathrm{N}, \mathrm{M}) \otimes_{\mathrm{A}} \mathrm{B},$$

so $\operatorname{Ext}^i_{\mathrm{A}}(\mathrm{N}, \mathrm{M}) = 0$ implies $\operatorname{Ext}^i_{\mathrm{B}}(\mathrm{N_B}, \mathrm{M_B}) = 0$, and the converse is true if B is faithfully flat over A.

3. Depth and topological properties

Lemma 3.1. — *Let* X *be a topological space,* Y *a closed subspace, let* F *be a sheaf of abelian groups on* X*. Set* $\mathrm{U} = \mathrm{X} - \mathrm{Y}$*. If* n *is an integer, the following conditions are equivalent :*

(i) $\underline{\mathrm{H}}^i_{\mathrm{Y}}(\mathrm{X}, \mathrm{F}) = 0$ *if* $i < n$*.*

(ii) *For any open set* V *of* X*, the homomorphism of groups*

$$\mathrm{H}^i(\mathrm{V}, \mathrm{F}) \longrightarrow \mathrm{H}^i(\mathrm{V} \cap \mathrm{U}, \mathrm{F})$$

is bijective if $i < n-1$ *and injective if* $i = n-1$*.*

(iii) *For any open set* V *of* X*,*

$$\mathrm{H}^i_{\mathrm{Y}\cap\mathrm{V}}(\mathrm{V}, \mathrm{F}_{|\mathrm{V}}) = 0 \textit{ if } i < n.$$

Démonstration

(ii) IFF (iii) ; indeed, let V be an open set of X, set $\mathrm{X}' = \mathrm{V}$, $\mathrm{Y}' = \mathrm{Y} \cap \mathrm{V}$, $\mathrm{F}' = \mathrm{F}_{|\mathrm{V}}$, $\mathrm{U}' = \mathrm{X}' - \mathrm{Y}'$; Y' is closed in X', we thus have an exact sequence :

$$\mathrm{H}^i_{\mathrm{Y}'}(\mathrm{X}', \mathrm{F}') \longrightarrow \mathrm{H}^i(\mathrm{X}', \mathrm{F}') \xrightarrow{\rho_i} \mathrm{H}^i(\mathrm{U}', \mathrm{F}') \longrightarrow \mathrm{H}^{i+1}_{\mathrm{Y}'}(\mathrm{X}', \mathrm{F}').$$

If the extreme terms are null, the homomorphism ρ_i is bijective, and if the left term is null, ρ_i is injective. So (iii) IMPLIES (ii). Conversely, if $i < n$, $\mathrm{H}^i_{\mathrm{Y}'}(\mathrm{X}', \mathrm{F}')$ is null because ρ_i is injective and ρ_{i-1} surjective.

(i) IMPLIES (iii) ; indeed the spectral sequence "from local to global" gives :

$$\mathrm{H}^p\big(\mathrm{X}, \underline{\mathrm{H}}^q_{\mathrm{Y}}(\mathrm{X},\mathrm{F})\big) \Longrightarrow \mathrm{H}^*_{\mathrm{Y}}(\mathrm{X},\mathrm{F}).$$

But, by hypothesis $\underline{\mathrm{H}}^q_{\mathrm{Y}}(\mathrm{X},\mathrm{F}) = 0$ if $q < n$, so $\mathrm{H}^{p+q}_{\mathrm{Y}}(\mathrm{X},\mathrm{F}) = 0$ if $p+q < n$.

(iii) IMPLIES (i) ; indeed (iii) expresses that the presheaf

$$\mathrm{V} \longmapsto \mathrm{H}^i_{\mathrm{Y}\cap\mathrm{V}}(\mathrm{V}, \mathrm{F}_{|\mathrm{V}})$$

is null, thus also the associated sheaf which is $\underline{\mathrm{H}}^i_{\mathrm{Y}}(\mathrm{X},\mathrm{F})$, because Y is closed.

Remark 3.2. — The equivalence of (i) and (ii) was proven in proposition I 2.13. As was remarked then, if $n \geqslant 2$, one can omit the condition that ρ_{n-1} be injective.[2]

Proposition 3.3. — *Let* X *be a locally noetherian prescheme,* Y *a closed subprescheme of* X, F *a coherent* $\mathscr{O}_{\mathrm{X}}$*-module. The conditions of lemma* 3.1 *are equivalent to each of the following conditions :*

(iv) *For any* $x \in \mathrm{Y}$*, we have* $\operatorname{prof} \mathrm{F}_x \geqslant n$ *;*

(v) *For any coherent* $\mathscr{O}_{\mathrm{X}}$*-module* G *on* X*, with support contained in* Y *we have*

$$\underline{\mathrm{Ext}}^i_{\mathscr{O}_{\mathrm{X}}}(\mathrm{G},\mathrm{F}) = 0 \ \textit{if}\ i < n;$$

(vi) *There exists a coherent* $\mathscr{O}_{\mathrm{X}}$*-module* G *whose support is equal to* Y *and such that*

$$\underline{\mathrm{Ext}}^i_{\mathscr{O}_{\mathrm{X}}}(\mathrm{G},\mathrm{F}) = 0 \ \textit{if}\ i < n.$$

If X is affine, we have done all that is necessary (cf. proposition 2.4) to prove the equivalence of the three conditions of proposition 3.3 ; but they are local, apart from the implication (v) IMPLIES (vi), but one can then take $\mathrm{G} = \mathscr{O}_{\mathrm{Y}}$ and invoke proposition 2.4 again. It is therefore sufficient to prove (i) IMPLIES (vi) and (v) IMPLIES (i).

Let J be the ideal of Y, it is a coherent sheaf of ideals ; let $\mathscr{O}_{\mathrm{Y}_m} = \mathscr{O}_{\mathrm{X}}/\mathrm{J}^{m+1}$, it is a coherent $\mathscr{O}_{\mathrm{X}}$-module whose support is equal to Y, and we know (theorem II 6.b) that

$$\underline{\mathrm{H}}^i_{\mathrm{Y}}(\mathrm{X},\mathrm{F}) = \varinjlim_m \underline{\mathrm{Ext}}^i_{\mathscr{O}_{\mathrm{X}}}(\mathscr{O}_{\mathrm{Y}_m},\mathrm{F}),$$

so (v) IMPLIES (i). On the other hand, the transition morphisms are epimorphisms in the projective system of the $\mathscr{O}_{\mathrm{Y}_m}$.

If the functor $\underline{\mathrm{Ext}}^i$ is left exact in its first argument, at least when it is in the category of coherent $\mathscr{O}_{\mathrm{X}}$-modules with support contained in Y, the transition morphisms of the inductive system obtained by applying $\underline{\mathrm{Ext}}^i$ to the $\mathscr{O}_{\mathrm{Y}_m}$ will be injective, but (i) implies that the limit is null, so (i) will imply that the modules $\underline{\mathrm{Ext}}^i_{\mathscr{O}_{\mathrm{X}}}(\mathscr{O}_{\mathrm{Y}_m},\mathrm{F})$ are null for any m. Let us reason by recurrence. The statement is trivial for $n < 0$.

[2] N.D.E. : the original edition gave a new proof, not entirely correct.

Suppose that (i) IMPLIES (vi) for $n < q$, then (i) IMPLIES (v), so, by the exact sequence of $\underline{\mathrm{Ext}}$, $\underline{\mathrm{Ext}}^q$ is left exact in its first argument, so the modules $\underline{\mathrm{Ext}}^q_{\mathscr{O}_\mathrm{X}}(\mathscr{O}_{\mathrm{Y}_m}, \mathrm{F})$ are null for any m. So (i) IMPLIES (vi) for $n \leqslant q$. QED

Example 3.4. — Let A be a noetherian local ring, $\mathfrak{m}$ its maximal ideal, M an A-module of finite type, and finally n an integer. Set $\mathrm{X} = \mathrm{Spec}(\mathrm{A})$, $\mathrm{Y} = \{\mathfrak{m}\}$, $\mathrm{U} = \mathrm{X} - \mathrm{Y}$. Let F be the sheaf associated to M. The following conditions are equivalent :

1) $\mathrm{prof}\,\mathrm{M} \geqslant n$;
2) the natural homomorphism

$$\mathrm{H}^i(\mathrm{X}, \mathrm{F}) \longrightarrow \mathrm{H}^i(\mathrm{U}, \mathrm{F})$$

is injective if $i = n - 1$, bijective if $i < n - 1$;

3) $\mathrm{Ext}^i_\mathrm{A}(k, \mathrm{M}) = 0$ if $i < n$, where $k = \mathrm{A}/\mathfrak{m}$;
4) $\mathrm{H}^i_\mathrm{Y}(\mathrm{X}, \mathrm{F}) = 0$ if $i < n$.

Taking into account remark 3.2, we obtain :

Corollary 3.5. — *Let* X *be a locally noetherian prescheme,* Y *a closed subprescheme of* X*,* F *a coherent* $\mathscr{O}_\mathrm{X}$*-module ; the following conditions are equivalent :*

1) *for any* $x \in \mathrm{Y}$*,* $\mathrm{prof}\,\mathrm{F}_x \geqslant 2$ *;*
2) *for any open set* V *of* X*, the natural homomorphism*

$$\mathrm{H}^0(\mathrm{V}, \mathrm{F}) \longrightarrow \mathrm{H}^0(\mathrm{V} \cap (\mathrm{X} - \mathrm{Y}), \mathrm{F})$$

is bijective.

***Theorem 3.6* (Hartshorne)**. — *Let* X *be a locally noetherian prescheme,* Y *a closed subprescheme of* X*. Suppose that, for any* $x \in \mathrm{Y}$*,* $\mathrm{prof}\,\mathscr{O}_{\mathrm{X},x} \geqslant 2$ *; then the natural map*

$$\pi_0(\mathrm{X}) \longrightarrow \pi_0(\mathrm{X} - \mathrm{Y})$$

is bijective.

Démonstration. — Since X is locally noetherian, X is locally connected ; it is therefore sufficient to prove that for X to be connected it is necessary and sufficient that $\mathrm{X} - \mathrm{Y}$ be so. But, for a ringed space in local rings $(\mathrm{X}, \mathscr{O}_\mathrm{X})$ to be connected, it is necessary and sufficient that $\mathrm{H}^0(\mathrm{X}, \mathscr{O}_\mathrm{X})$ not be a direct composite of two non-zero rings. But the hypothesis implies, according to corollary 3.5 applied to $\mathrm{F} = \mathscr{O}_\mathrm{X}$, that the homomorphism

$$\mathrm{H}^0(\mathrm{X}, \mathscr{O}_\mathrm{X}) \longrightarrow \mathrm{H}^0(\mathrm{X} - \mathrm{Y}, \mathscr{O}_\mathrm{X})$$

is an isomorphism, whence the conclusion.

Corollary 3.7. — *Let* X *be a locally noetherian prescheme. Let* d *be an integer such that* $\dim \mathscr{O}_{\mathrm{X},x} \geqslant d$ *implies* $\mathrm{prof}\,\mathscr{O}_{\mathrm{X},x} \geqslant 2$*. Then, if* X *is connected,* X *is connected in codimension* $d-1$*, i.e. if* X' *and* X'' *are two irreducible components of* X*, there exists* *a sequence of irreducible components of* X *:*

$$\mathrm{X}' = \mathrm{X}_0, \mathrm{X}_1, \ldots, \mathrm{X}_n = \mathrm{X}''$$

such that for any i*,* $0 \leqslant i < n$*, the codimension of* $\mathrm{X}_i \cap \mathrm{X}_{i+1}$ *in* X *is less than or equal to* $d - 1$*.*

Let us first note that if X is Cohen-Macaulay, $d = 2$ will enjoy the property mentioned above. In this regard, recall that we define the codimension of Y in X as the infimum of the dimensions of the local rings in X of the points of Y.

Démonstration. — Let $\mathscr{F}$ be the set of closed parts of X whose codimension in X is greater than or equal to d. We note that $\mathscr{F}$ is an antifilter of closed parts of X. On the other hand, for a closed set Y of X to be an element of $\mathscr{F}$, it is necessary and sufficient that, for any $y \in \mathrm{Y}$ there exists an open neighborhood V of X such that $\mathrm{Y} \cap \mathrm{V}$ is of codimension $\geqslant d$ in V. Finally, if X is connected and if $\mathrm{Y} \in \mathscr{F}$, $\mathrm{X} - \mathrm{Y}$ is connected according to Hartshorne's theorem. The corollary thus results from the following lemma, which is of a purely topological nature.

Lemma 3.8. — *Let* X *be a connected and locally noetherian topological space, and let* $\mathscr{F}$ *be an antifilter of closed parts of* X*. We suppose that any closed set* $\mathrm{Y} \subset \mathrm{X}$ *which belongs locally to* $\mathscr{F}$*, (i.e. for any point* $x \in \mathrm{X}$ *there exists an open neighborhood* V *of* x *in* X *and a* $\mathrm{Y}' \in \mathscr{F}$ *such that* $\mathrm{V} \cap \mathrm{Y} = \mathrm{V} \cap \mathrm{Y}'$*), belongs to* $\mathscr{F}$*. The following conditions are equivalent :*

(i) *for any* $\mathrm{Y} \in \mathscr{F}$*,* $\mathrm{X} - \mathrm{Y}$ *is connected ;*

(ii) *if* X' *and* X'' *are two distinct irreducible components of* X*, there exists a sequence of irreducible components of* X*,* $\mathrm{X}_0, \mathrm{X}_1, \ldots, \mathrm{X}_n$*, such that* $\mathrm{X}' = \mathrm{X}_0$*,* $\mathrm{X}'' = \mathrm{X}_n$ *and, for any* i*,* $1 \leqslant i < n$*,* $\mathrm{X}_i \cap \mathrm{X}_{i+1} \notin \mathscr{F}$*.*

(ii) IMPLIES (i). Let $\mathrm{Y} \in \mathscr{F}$, we must prove that the open set $\mathrm{U} = \mathrm{X} - \mathrm{Y}$ is connected. But, if U' and U'' are two irreducible components of U, there exist two irreducible components X' and X'' of X such that $\mathrm{X}'' \cap \mathrm{U} = \mathrm{U}''$ and $\mathrm{X}' \cap \mathrm{U} = \mathrm{U}'$; let $\mathrm{X}_0, \ldots \mathrm{X}_n$ be a sequence of irreducible components of X possessing the property mentioned above; if we set $\mathrm{U}_i = \mathrm{X}_i \cap \mathrm{U}$, $0 \leqslant i \leqslant n$, the U_i will be irreducible components of U, moreover $\mathrm{U}_i \cap \mathrm{U}_{i+1}$ is non-empty if $0 \leqslant i < n$, because otherwise, $\mathrm{X}_i \cap \mathrm{X}_{i+1} \subset \mathrm{Y}$ would be an element of $\mathscr{F}$, which is contrary to the choice of the sequence of the X_i. This implies that U is connected.

(i) IMPLIES (ii). Let $\mathrm{Y} = \bigcup_{\mathrm{X}',\mathrm{X}''} \mathrm{X}' \cap \mathrm{X}''$ where we impose that X' and X'' be two *distinct* irreducible components of X such that $\mathrm{X}' \cap \mathrm{X}'' \in \mathscr{F}$. The family of the $\mathrm{X}' \cap \mathrm{X}''$ is locally finite because X is locally noetherian, moreover the $\mathrm{X}' \cap \mathrm{X}''$ are closed, so Y is closed. On the other hand, Y belongs locally to $\mathscr{F}$, so $\mathrm{Y} \in \mathscr{F}$. So $\mathrm{U} = \mathrm{X} - \mathrm{Y}$ is connected. Let X' and X'' be two distinct irreducible components of X, let U' and U'' be their traces on U, which are non-empty by construction of Y. These are irreducible components of U, but U is connected, so U being locally noetherian, there exists a sequence of irreducible components $\mathrm{U}_0, \ldots, \mathrm{U}_n$ of U, such that $\mathrm{U}_0 = \mathrm{U}'$, $\mathrm{U}_n = \mathrm{U}''$ and $\mathrm{U}_i \cap \mathrm{U}_{i+1} \neq \emptyset$ and $\neq \mathrm{U}_i, 0 \leqslant i < n$. Let $\mathrm{X}_0, \ldots, \mathrm{X}_n$ be the sequence of irreducible components of X such that $\mathrm{X}_i \cap \mathrm{U} = \mathrm{U}_i$; if $\mathrm{X}_i \cap \mathrm{X}_{i+1} \in \mathscr{F}$, by construction of $\mathscr{F}$, $\mathrm{U}_i \cap \mathrm{U}_{i+1} = \emptyset$ or $\mathrm{U}_i = \mathrm{U}_{i+1}$ which is not possible according to the choice of the U_i. C.Q.F.D.

Corollary 3.9. — *Let* A *be a noetherian local ring. We suppose that for any prime ideal* $\mathfrak{p}$ *of* A*, we have :*

$$(\dim \mathrm{A}_\mathfrak{p} \geqslant 2) \Longrightarrow (\operatorname{prof} \mathrm{A}_\mathfrak{p} \geqslant 2)$$

We suppose moreover that A *satisfies the chain condition*(*)*. Then, for any* $\mathfrak{p}$*, minimal prime ideal of* A*,* $\dim \mathrm{A}/\mathfrak{p} = \dim \mathrm{A}$*, or in other words, all irreducible components of* Spec A *have the same dimension : that of* A.

If X' and X'' are two irreducible components of X, we join them by a chain having the properties enumerated in corollary 3.7; it is then sufficient to prove that two successive components have the same dimension, which results from the second hypothesis.

Example 3.10. — Let X be the union of two supplementary vector subspaces of respective dimensions 2 and 3 in a vector space of dimension 5; more precisely, let $\mathrm{X} = \operatorname{Spec} \mathrm{A}$, with $\mathrm{A} = \mathrm{B}/\mathfrak{p} \cap \mathfrak{q}$, where $\mathrm{B} = k[\mathrm{X}_1, \ldots, \mathrm{X}_5]$, $\mathfrak{p}$ is the ideal generated by $\mathrm{X}_1, \mathrm{X}_2, \mathrm{X}_3$ and $\mathfrak{q}$ the ideal generated by X_4 and X_5 ; X can be disconnected by the intersection point x of the two vector subspaces, so the depth of $\mathscr{O}_{\mathrm{X},x}$ is equal to 1, because it cannot be $\geqslant 2$ by virtue of theorem 3.6. Another reason : the conclusion of equidimensionality of the preceding corollary is in default.

More generally, taking a union X of two vector subspaces of dimension $p, q \geqslant 2$ in a vector space of dimension $p+q$, for no embedding of X in a regular scheme, is X even set-theoretically a complete intersection at the origin, because (up to modifying it without changing the underlying topological space in the neighborhood of the origin), X would be Cohen-Macaulay thus of depth $\geqslant 2$ at the origin, which is not the case.

Remark 3.11. — Let X be a locally noetherian prescheme, Y a closed subprescheme of X, F an $\mathscr{O}_\mathrm{X}$-module. The depth is a purely topological notion which is expressed in terms of the nullity of the $\underline{\mathrm{H}}^i_\mathrm{Y}(\mathrm{X}, \mathrm{F})$ for $i < n$. We also wish to study these sheaves for a given i, or for $i > n$. We prove in this regard the following result :

Lemma 3.12. — *Let* m *be an integer, for* $\underline{\mathrm{H}}^i_\mathrm{Y}(\mathrm{X}, \mathrm{F}) = 0$*,* $i > m$ *for any coherent* F*, it is necessary and sufficient that it be true for* $\mathrm{F} = \mathscr{O}_\mathrm{X}$.

By inductive limit it is then true for any quasi-coherent sheaf. For example, if Y can be described locally by m equations, or, as one says, if Y is locally set-theoretically

(*) Cf. EGA 0_{IV} 14.3.2

a complete intersection (which happens for example if X and Y are *non-singular*) it results from the calculation of the $\underline{\mathrm{H}}^i_{\mathrm{Y}}(\mathrm{X}, \mathrm{F})$ by the Koszul complex that these sheaves are null for $i > m$. We have moreover used this fact implicitly in example 3.10. This cohomological condition is however not sufficient, as the example below proves :

Example 3.13. — Let $\mathrm{X} = \mathrm{Spec}(\mathrm{A})$, where A is a normal noetherian local ring of dimension 2. Let Y be a curve in X. One can prove that the complement of the curve is an affine open set thus[(3)] $\underline{\mathrm{H}}^i_{\mathrm{Y}}(\mathscr{O}_{\mathrm{X}}) \approx \underline{\mathrm{H}}^{i-1}_{\mathrm{X}-\mathrm{Y}}(\mathscr{O}_{\mathrm{X}}) = 0$ for $i > 1$ because $\mathrm{H}^{i-1}(\mathrm{X} - \mathrm{Y}, \mathscr{O}_{\mathrm{X}}) = 0$. However one can construct a curve that is not described by an equation.

We will seek[(*)] conditions for the $\underline{\mathrm{H}}^i_{\mathrm{Y}}(\mathrm{X}, \mathrm{F})$ to be coherent for a given i, which is not the case in general, as shown by obvious examples, for example $\mathrm{H}^n_{\mathfrak{m}}(\mathrm{A})$ for A a noetherian local ring of dimension $n > 0$; when for example A is a discrete valuation ring with fraction field K, one finds $\mathrm{H}^1_{\mathfrak{m}}(\mathrm{A}) \simeq \mathrm{K}/\mathrm{A}$, which is not a module of finite type over A. To enlighten the reader, let us say that the problem posed is equivalent to the following : let $f : \mathrm{U} \to \mathrm{X}$ be an open immersion, let G be a coherent sheaf on U, find criteria for the higher direct images $\mathrm{R}^i f_* \mathrm{G}$ to be coherent sheaves on X for a given i. These conditions are necessary for the use of formal geometry that we have seen in exposé IX and the following ones.

(*) Cf. Exp VIII.

(3) N.D.E. : There was a typo in the original edition.

EXPOSÉ IV

DUALIZING MODULES AND FUNCTORS

1. Generalities on module functors

Let A be a noetherian commutative ring,
$\underline{\mathrm{C}}$ the category of finitely generated A-modules,
$\underline{\mathrm{C}}'$ the category of arbitrary A-modules,
$\underline{\mathrm{Ab}}$ the category of abelian groups.

The purpose of this paragraph is the study of certain properties of functors $\mathrm{T} : \underline{\mathrm{C}}^{\circ} \to \underline{\mathrm{Ab}}$ (assumed to be additive).

Note that if $\mathrm{M} \in \mathrm{Ob}\,\underline{\mathrm{C}}$, T(M) can be canonically equipped with an A-module structure as follows : if f_{M} is the homothety of M associated with $f \in \mathrm{A}$, A operates on T(M) by $f_{\mathrm{T(M)}}$. In other words, T factors as :

$$\begin{tikzcd} \underline{\mathrm{C}}^{\circ} \arrow[rr, "\mathrm{T}"] \arrow[dr, "\mathrm{T}_{\circ}"'] & & \underline{\mathrm{Ab}} \\ & \underline{\mathrm{C}}' \arrow[ur] & \end{tikzcd}$$

where $\underline{\mathrm{C}}' \to \underline{\mathrm{Ab}}$ is the canonical functor.

In what follows, T(M) will be considered as equipped with its A-module structure.

By composing with the isomorphism $\mathrm{M} \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{A}}(\mathrm{A}, \mathrm{M})$ the morphism $\mathrm{Hom}_{\mathrm{A}}(\mathrm{A}, \mathrm{M}) \to \mathrm{Hom}_{\mathrm{A}}(\mathrm{T(M)}, \mathrm{T(A)})$, one obtains the following morphisms which are deduced from each other in an obvious way :

$$\begin{aligned} \mathrm{M} &\longrightarrow \mathrm{Hom}_{\mathrm{A}}(\mathrm{T(M)}, \mathrm{T(A)}), \\ \mathrm{M} \times \mathrm{T(M)} &\longrightarrow \mathrm{T(A)}, \\ \mathrm{T(M)} &\longrightarrow \mathrm{Hom}_{\mathrm{A}}(\mathrm{M}, \mathrm{T(A)}), \end{aligned}$$

which defines for us a morphism φ_{T} of contravariant functors :

$$\varphi_{\mathrm{T}} : \mathrm{T} \longrightarrow \mathrm{Hom}_{\mathrm{A}}(\mathrm{M}, \mathrm{T(A)}).$$

Proposition 1.1. — *The following two properties are equivalent :*

(i) φ_{T} *is an isomorphism of functors.*
(ii) T *is left exact.*

The implication (i) IMPLIES (ii) is trivial.

The implication (ii) IMPLIES (i) results from the fact that for a morphism $u : \mathrm{F} \to \mathrm{F}'$ of two additive left exact functors F and F′ from $\underline{\mathrm{C}}^{\circ}$ to $\underline{\mathrm{Ab}}$, if $u(\mathrm{A})$ is an isomorphism, u is an isomorphism (we use the fact that A is noetherian and thus that every finitely generated A-module is of finite presentation).

Remark 1.2. — This shows in particular that the functors $\mathrm{T} : \underline{\mathrm{C}'}^{\circ} \to \underline{\mathrm{Ab}}$ which are representable are the functors which commute with arbitrary projective limits (over a preordered set not necessarily filtered).

If $\underline{\mathrm{Hom}}(\underline{\mathrm{C}}^{\circ}, \underline{\mathrm{Ab}})_g$ denotes the full subcategory of $\underline{\mathrm{Hom}}(\underline{\mathrm{C}}^{\circ}, \underline{\mathrm{Ab}})$ whose objects are the left exact functors, we have demonstrated the equivalence of the categories

$$\underline{\mathrm{C}}' \xrightarrow{\sim} \underline{\mathrm{Hom}}(\underline{\mathrm{C}}^{\circ}, \underline{\mathrm{Ab}})_g$$

by the quasi-inverse functors of each other

$$\mathrm{H} \longmapsto \mathrm{Hom}_{\mathrm{A}}(\ ,\mathrm{H})$$

and

$$\mathrm{T}(\mathrm{A}) \longleftarrow\!\shortmid \mathrm{T}.$$

Now let J be an ideal of A, $\mathrm{Y} = \mathrm{V}(\mathrm{J}) \subset \operatorname{Spec} \mathrm{A}$, and let $\underline{\mathrm{C}}_{\mathrm{Y}}$ be the full subcategory of $\underline{\mathrm{C}}$ whose objects are the finitely generated A-modules M such that $\operatorname{Supp} \mathrm{M} \subset \mathrm{Y}$. We have :

$$\underline{\mathrm{C}}_{\mathrm{Y}} = \bigcup_n \underline{\mathrm{C}}^{(n)},$$

where $\underline{\mathrm{C}}^{(n)}$ is the full subcategory of $\underline{\mathrm{C}}_{\mathrm{Y}}$ of modules M such that $\mathrm{J}^n\mathrm{M} = 0$.

Proposition 1.3. — *With the same notations as previously, let* $\mathrm{T} : \underline{\mathrm{C}}_{\mathrm{Y}} \to \underline{\mathrm{Ab}}$ *be a functor. To* $\mathrm{H} = \varinjlim \mathrm{T}(\mathrm{A}/\mathrm{J}^n)$[(1)] *is associated a natural morphism*

$$\varphi_{\mathrm{T}} : \mathrm{T} \longrightarrow \mathrm{Hom}_{\mathrm{A}}(\ ,\mathrm{H}),$$

and the following conditions are equivalent :

(i) φ_{T} *is an isomorphism.*
(ii) *The functor* T *is left exact.*

Démonstration. — a) Definition of φ_{T}.

Let $\mathrm{M} \in \operatorname{Ob} \underline{\mathrm{C}}_{\mathrm{Y}}$. There exists an integer n such that $\mathrm{J}^n\mathrm{M} = 0$. Then M is an A/J^n-module, and if T_n denotes the restriction of T to $\underline{\mathrm{C}}^{(n)}$, we know how to define

(1) N.D.E. : the definition of H is implicit in the original text.

the morphism

$$\mathrm{T}_n \longrightarrow \mathrm{Hom}_{\mathrm{A}}(\ ,\mathrm{H}_n),\ \text{where } \mathrm{H}_n = \mathrm{T}(\mathrm{A}/\mathrm{J}^n)\ ;$$

$$\text{whence}\quad \mathrm{T}(\mathrm{M}) = \mathrm{T}_n(\mathrm{M}) \longrightarrow \mathrm{Hom}_{\mathrm{A}}(\mathrm{M}, \varinjlim \mathrm{H}_n) = \mathrm{Hom}_{\mathrm{A}}(\mathrm{M},\mathrm{H})$$

$$\text{and}\quad \varphi_{\mathrm{T}} : \mathrm{T} \longrightarrow \mathrm{Hom}_{\mathrm{A}}(\ ,\mathrm{H}).$$

b) Equivalence of (i) and (ii).

It is clear that (i) implies (ii). Suppose (ii) is verified and let $\mathrm{M} \in \mathrm{Ob}\,\underline{\mathrm{C}}^{(n)}$. We have seen that $\mathrm{T}_n(\mathrm{M}) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{A}}(\mathrm{M},\mathrm{H}_n)$, so for any integer $n' > n$ we have

$$\mathrm{T}(\mathrm{M}) = \mathrm{T}_n(\mathrm{M}) = \mathrm{T}_{n'}(\mathrm{M}) = \varinjlim \mathrm{T}_n(\mathrm{M})$$

$$\text{and}\quad \mathrm{T}(\mathrm{M}) = \varinjlim \mathrm{Hom}_{\mathrm{A}}(\mathrm{M},\mathrm{H}_n).$$

Since these are filtered inductive limits, we also have the isomorphism

$$\varinjlim \mathrm{Hom}_{\mathrm{A}}(\mathrm{M},\mathrm{H}_n) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{A}}(\mathrm{M}, \varinjlim \mathrm{H}_n) = \mathrm{Hom}_{\mathrm{A}}(\mathrm{M},\mathrm{H}).$$

If $\underline{\mathrm{C}}'_{\mathrm{Y}}$ denotes the category of A-modules, with support contained in Y, but not necessarily of finite type, we still have the natural equivalence of categories : $\underline{\mathrm{C}}'_{\mathrm{Y}} \xrightarrow{\sim} \mathrm{Hom}(\underline{\mathrm{C}}^{\circ}_{\mathrm{Y}}, \underline{\mathrm{Ab}})_g$.

Application. — The notations being the same as previously, let

$$\mathrm{T}^* : \underline{\mathrm{C}}^{\circ}_{\mathrm{Y}} \longrightarrow \underline{\mathrm{Ab}}$$

be an exact ∂-functor. For all $i \in \mathbf{Z}$, we set $\mathrm{H}^i_n = \mathrm{T}^i(\mathrm{A}/\mathrm{J}^n)$ and $\mathrm{H}^i = \varinjlim \mathrm{H}^i_n$.

Theorem 1.4. — *Let $n \in \mathbf{Z}$. If there exists $i_0 \in \mathbf{Z}$ such that $\mathrm{T}^i = 0$ for all $i < i_0$, then the following three conditions are equivalent :*

(i) $\mathrm{T}^i = 0$ *for all* $i < n$.

(ii) $\mathrm{H}^i = 0$ *for all* $i < n$.

(iii) *There exists a module* M_0 *of* $\underline{\mathrm{C}}_{\mathrm{Y}}$ *such that* $\mathrm{Supp}\,\mathrm{M}_0 = \mathrm{Y}$ *and* $\mathrm{T}^i(\mathrm{M}_0) = 0$ *for all* $i < n$.

Démonstration. — It is evident that (i) implies (ii) and (iii) (one takes $\mathrm{M}_0 = \mathrm{A}/\mathrm{J}$). Let us show by recurrence on n that (ii) implies (i); this is true for $n = i_0$, and we suppose it demonstrated up to rank n. Suppose then that $\mathrm{H}^i = 0$ for all $i < n+1$; by the recurrence hypothesis we have $\mathrm{T}^i = 0$ for $i < n$, but $\mathrm{T}^{n-1} = 0$ implies that T^n is a left exact functor and

$$\mathrm{T}^n \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{A}}(\ ,\mathrm{H}^n) = 0.$$

Let us then show that (iii) implies (ii). This is again true for $n = i_0$; suppose it demonstrated up to rank n : let M_0 be an A-module of $\underline{\mathrm{C}}_{\mathrm{Y}}$ such that $\mathrm{Supp}\,\mathrm{M}_0 = \mathrm{Y}$ and $\mathrm{T}^i(\mathrm{M}_0) = 0$ for all $i < n+1$; by the recurrence hypothesis we then have $\mathrm{H}^i = 0$ for all $i < n$; it remains to show that $\mathrm{H}^n = 0$. But "$\mathrm{H}^i = 0$ for all $i < n$" implies that $\mathrm{T}^{n-1} = 0$ and thus that $\mathrm{T}^n \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{A}}(\ ,\mathrm{H}^n)$. We then have

$$\mathrm{Ass}\,\mathrm{H}^n = \mathrm{Ass}\,\mathrm{Hom}(\mathrm{M}_0,\mathrm{H}^n) = \mathrm{Supp}\,\mathrm{M}_0 \cap \mathrm{Ass}\,\mathrm{H}^n = \mathrm{Ass}\,\mathrm{H}^n$$

because
$$\operatorname{Ass} \mathrm{H}^n \subset \operatorname{Supp} \mathrm{H}^n \subset \mathrm{Y} = \operatorname{Supp} \mathrm{M}_0.$$
Whence $\mathrm{T}^n(\mathrm{M}_0) = 0 \Leftrightarrow \operatorname{Ass} \mathrm{H}^n = \emptyset \Leftrightarrow \mathrm{H}^n = 0$; which completes the proof.

2. Characterization of exact functors

The ring A is still supposed to be noetherian and commutative. The notations are those of proposition 1.3 :
$$\mathrm{Y} = \mathrm{V}(\mathrm{J}), \quad \mathrm{T} : \underline{\mathrm{C}}_{\mathrm{Y}}^{\circ} \longrightarrow \underline{\mathrm{Ab}}, \quad \mathrm{H} = \varinjlim \mathrm{T}(\mathrm{A}/\mathrm{J}^n),$$
where we suppose that T is a left exact functor, whence :
$$\mathrm{T}(\mathrm{M}) \xrightarrow{\sim} \operatorname{Hom}_{\mathrm{A}}(\mathrm{M}, \mathrm{H}).$$

Proposition 2.1. — *The following properties are equivalent :*

(i) *The functor* T *is exact,*

(ii) H *is injective in* $\underline{\mathrm{C}}'$.

Démonstration. — It is obviously sufficient to show that (i) implies (ii), that is to say, to prove that if the restriction to $\underline{\mathrm{C}}_{\mathrm{Y}}$ of the functor $\operatorname{Hom}_{\mathrm{A}}(\ ,\mathrm{H})$ is an exact functor, then H is injective. But since A is noetherian, to show that H is injective, we can limit ourselves to proving that any homomorphism $f : \mathrm{N} \to \mathrm{H}$ from a source A-module N *of finite type*, a submodule of an A-module M *of finite type*, extends to a homomorphism $\overline{f} : \mathrm{M} \to \mathrm{H}$.

The definition of H and the fact that N is of finite type imply that there exists an integer n such that $\mathrm{J}^n.f(\mathrm{N}) = 0$. Let us then equip M and N with the J-adic topology. The J-adic topology of N is equivalent to the topology induced by the J-adic topology of M (Krull's theorem). There thus exists $\mathrm{V} = \mathrm{J}^k \cdot \mathrm{M}$ such that
$$\mathrm{U} = \mathrm{V} \cap \mathrm{N} \subset \mathrm{J}^n \mathrm{N}.$$
We then have the factorization
$$\begin{tikzcd} \mathrm{N} \arrow[r] \arrow[d, "f"'] & \mathrm{N}/\mathrm{U} \arrow[dl, "u"] \\ \mathrm{H} & \end{tikzcd},$$
N/U and M/V are in $\underline{\mathrm{C}}_{\mathrm{Y}}$. The hypothesis thus allows extending u to $\overline{u}$
$$\begin{tikzcd} \mathrm{N}/\mathrm{U} \arrow[r, hook] \arrow[d, "u"'] & \mathrm{M}/\mathrm{V} \arrow[dl, "\overline{u}"] \\ \mathrm{H} & \end{tikzcd},$$
and $\mathrm{M} \to \mathrm{M}/\mathrm{V} \xrightarrow{\overline{u}} \mathrm{H}$ gives the sought-after extension $\overline{f}$.

Corollary 2.2. — *Let* K *be an injective* A*-module, then the submodule* $\mathrm{H}^0_\mathrm{J}(\mathrm{K})$ *of* K *formed by the elements annihilated by a suitable power of* J *is injective.*

Démonstration. — It suffices to verify that the restriction to $\underline{\mathrm{C}}_\mathrm{Y}$ of the functor $\mathrm{Hom}_\mathrm{A}(\ ,\mathrm{H}^0_\mathrm{J}(\mathrm{K}))$ is an exact functor. Now let $\mathrm{M} \in \mathrm{Ob}\,\underline{\mathrm{C}}_\mathrm{Y}$, there exists k such that $\mathrm{J}^k \cdot \mathrm{M} = 0$, and the inclusion

$$\mathrm{Hom}_\mathrm{A}(\ ,\mathrm{H}^0_\mathrm{J}(\mathrm{K})) \longrightarrow \mathrm{Hom}_\mathrm{A}(\mathrm{M},\mathrm{K})$$

is then an isomorphism. Whence the result since $\mathrm{Hom}_\mathrm{A}(\ ,\mathrm{K})$ is exact.

3. Study of the case where T is left exact and T(M) is of finite type for all M

Let as above

$$\mathrm{T} : \underline{\mathrm{C}}^\circ_\mathrm{Y} \longrightarrow \underline{\mathrm{Ab}},$$

we now suppose that T is left exact and that we have the factorization

$$\begin{tikzcd} \underline{\mathrm{C}}^\circ_\mathrm{Y} \arrow[rr, "\mathrm{T}"] \arrow[dr] & & \underline{\mathrm{Ab}}\,. \\ & \underline{\mathrm{C}}_\mathrm{Y} \arrow[ur] & \end{tikzcd}$$

where, as above, $\underline{\mathrm{C}}_\mathrm{Y} \to \underline{\mathrm{Ab}}$ is the forgetful functor. We thus know how to define $\mathrm{T}(\mathrm{T}(\mathrm{M})) = \mathrm{T} \circ \mathrm{T}(\mathrm{M})$, and the canonical morphism

$$\mathrm{M} \longrightarrow \mathrm{Hom}_\mathrm{A}(\mathrm{Hom}_\mathrm{A}(\mathrm{M},\mathrm{H}),\mathrm{H})$$

defines a morphism

$$\mathrm{M} \longrightarrow \mathrm{T} \circ \mathrm{T}(\mathrm{M}).$$

Proposition 3.1. — *The ring* A *still being supposed noetherian, if we make the additional hypothesis that* A/J *is artinian, the following conditions are equivalent :*

(i) T *is left exact and, for any* $\mathrm{M} \in \mathrm{Ob}\,\underline{\mathrm{C}}_\mathrm{Y}$, T(M) *is of finite type and* $\mathrm{M} \to \mathrm{T} \circ \mathrm{T}(\mathrm{M})$ *is an isomorphism.*

(ii) T *is exact and, for any residue field* k *associated to a maximal ideal containing* J, *we have* $\mathrm{T}(k) \xrightarrow{\sim} k$.

(iii) *We have* $\mathrm{T} \xrightarrow{\sim} \mathrm{Hom}_\mathrm{A}(\ ,\mathrm{H})$ *with* H *injective and, for any* k *as in* (ii), *we have* $\mathrm{Hom}_\mathrm{A}(k,\mathrm{H}) \xrightarrow{\sim} k$.

(iv) T *is exact and, for any* $\mathrm{M} \in \mathrm{Ob}\,\underline{\mathrm{C}}_\mathrm{Y}$, *we have* $\mathrm{long}\,\mathrm{T}(\mathrm{M}) = \mathrm{long}\,\mathrm{M}$.

Démonstration. — We have already shown the equivalence of (ii) and (iii) (prop. 2.1). Let us show that (ii) implies (iv) : first if $\mathrm{M} \in \mathrm{Ob}\,\underline{\mathrm{C}}_\mathrm{Y}$, since M is an A/J^n-module with A/J^n artinian, $\mathrm{long}\,\mathrm{M}$ is finite. Let us reason by recurrence on the length of M. Condition (iv) is true if $\mathrm{long}\,\mathrm{M} = 1$, because then M is a residue field covered by (ii). If $\mathrm{long}\,\mathrm{M} > 1$, there exists a submodule M′ of M such that $\mathrm{M}' \neq 0$ and $\mathrm{long}\,\mathrm{M}' < \mathrm{long}\,\mathrm{M}$.

Let us then form the exact sequence

$$0 \longrightarrow \mathrm{M}' \longrightarrow \mathrm{M} \longrightarrow \mathrm{M}'' \longrightarrow 0.$$

Since T is exact, we have the sequence

$$0 \longrightarrow \mathrm{T}(\mathrm{M}') \longrightarrow \mathrm{T}(\mathrm{M}) \longrightarrow \mathrm{T}(\mathrm{M}'') \longrightarrow 0,$$

and $\operatorname{long} \mathrm{T}(\mathrm{M}) = \operatorname{long} \mathrm{T}(\mathrm{M}') + \operatorname{long} \mathrm{T}(\mathrm{M}'') = \operatorname{long} \mathrm{M}' + \operatorname{long} \mathrm{M}'' = \operatorname{long} \mathrm{M}$.

(ii) implies (i) : Since (ii) implies (iv), let M be an A-module of $\underline{\mathrm{C}}_{\mathrm{Y}}$, we have $\operatorname{long} \mathrm{T}(\mathrm{M}) = \operatorname{long} \mathrm{M}$; so T(M) is of finite length and thus of finite type.

It remains to show that $\mathrm{M} \to \mathrm{T} \circ \mathrm{T}(\mathrm{M})$ is an isomorphism ; we reason again by recurrence on $\operatorname{long} \mathrm{M}$. For $\mathrm{M} = k$ this is true. In the general case we write the commutative diagram whose two rows are exact :

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{M}' \arrow[r] \arrow[d] & \mathrm{M} \arrow[r] \arrow[d] & \mathrm{M}'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{T} \circ \mathrm{T}(\mathrm{M}') \arrow[r] & \mathrm{T} \circ \mathrm{T}(\mathrm{M}) \arrow[r] & \mathrm{T} \circ \mathrm{T}(\mathrm{M}'') \arrow[r] & 0,
\end{tikzcd}$$

where M' is a submodule of M such that $\mathrm{M}' \neq 0$ and $\operatorname{long} \mathrm{M}' < \operatorname{long} \mathrm{M}$. By the recurrence hypothesis, the extreme arrows are isomorphisms, so

$$\mathrm{M} \longrightarrow \mathrm{T} \circ \mathrm{T}(\mathrm{M})$$

is an isomorphism.

(i) implies (ii) : let

$$0 \longrightarrow \mathrm{M}' \longrightarrow \mathrm{M} \longrightarrow \mathrm{M}'' \longrightarrow 0$$

be an exact sequence of A-modules of $\underline{\mathrm{C}}_{\mathrm{Y}}$, and let Q be the cokernel of $\mathrm{T}(\mathrm{M}) \to \mathrm{T}(\mathrm{M}')$. Applying T to the exact sequence

$$0 \longrightarrow \mathrm{T}(\mathrm{M}') \longrightarrow \mathrm{T}(\mathrm{M}) \longrightarrow \mathrm{T}(\mathrm{M}'') \longrightarrow \mathrm{Q} \longrightarrow 0,$$

we obtain :

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{T}(\mathrm{Q}) \arrow[r] & \mathrm{T} \circ \mathrm{T}(\mathrm{M}') \arrow[r] & \mathrm{T} \circ \mathrm{T}(\mathrm{M}) \\
& & \mathrm{M}' \arrow[r] \arrow[u, "s"'] & \mathrm{M} \arrow[u, "s"']
\end{tikzcd}$$

so $\mathrm{T}(\mathrm{Q}) = 0$ and $\mathrm{Q} \xrightarrow{\sim} \mathrm{T}(\mathrm{T}(\mathrm{Q})) = 0$.

On the other hand let k be a residue field, $k = \mathrm{A}/\mathfrak{m}$, $\mathrm{J} \subset \mathfrak{m}$. We must show that $\mathrm{T}(k) \xrightarrow{\sim} k$. For this it suffices to remark that $\mathrm{T}(k)$ is a k-vector space. We deduce :

$$\mathrm{T}(k) \simeq k \oplus \mathrm{V},$$
$$\mathrm{T}(\mathrm{T}(k)) \simeq \mathrm{T}(k) \oplus \mathrm{T}(\mathrm{V}) \simeq k \oplus \mathrm{V} \oplus \mathrm{T}(\mathrm{V}) \simeq k,$$

whence $\mathrm{V} = 0$.

Finally let us show that (iv) implies (iii) : it suffices to show that $\mathrm{T}(k) \xrightarrow{\sim} k$; but $\operatorname{long} \mathrm{T}(k) = \operatorname{long} k = 1$, so $\mathrm{T}(k) = k'$ is a residue field and $\operatorname{Supp} k' = \operatorname{Supp} \operatorname{Hom}_{\mathrm{A}}(k, \mathrm{H}) \subset \operatorname{Supp} k$. So $k \simeq k'$.

Remark 3.2. — One can show that condition (iv) is equivalent to condition

(iv)′ For any $\mathrm{M} \in \operatorname{Ob} \underline{\mathrm{C}}_{\mathrm{Y}}$, we have $\operatorname{long} \mathrm{T}(\mathrm{M}) = \operatorname{long} \mathrm{M}$.

4. Dualizing module. Dualizing functor

Definition 4.1. — Let A be a noetherian local ring and $\mathfrak{m}$ its maximal ideal. We call a dualizing functor for A any functor

$$\mathrm{T} : \underline{\mathrm{C}}_{\mathfrak{m}}^{\circ} \longrightarrow \underline{\mathrm{Ab}},$$

where we write $\underline{\mathrm{C}}_{\mathfrak{m}}$ instead of $\underline{\mathrm{C}}_{\mathrm{Y}}$ for $\mathrm{Y} = \mathrm{V}(\mathfrak{m})$, which satisfies the equivalent conditions of proposition 3.1. We say that an A-module I is dualizing for A if the functor $\mathrm{M} \to \operatorname{Hom}_{\mathrm{A}}(\mathrm{M}, \mathrm{I})$ is dualizing.

We can generalize definition 4.1 to the case where we no longer suppose that A is a local ring.

Definition 4.2. — Let A be a noetherian ring and let $\overline{\underline{\mathrm{C}}}$ be the full subcategory of $\underline{\mathrm{C}}$ formed by the A-modules of finite length; we call a dualizing functor any functor T, A-linear, from $\overline{\underline{\mathrm{C}}}^{\circ}$ to $\overline{\underline{\mathrm{C}}}$, which is exact and such that the functor morphism

$$\mathrm{id} \longrightarrow \mathrm{T} \circ \mathrm{T}$$

is an isomorphism.

We will prove an existence theorem and also that the module I which represents such a functor is locally artinian. We will also show that, for any maximal ideal $\mathfrak{m}$ of A, the $\mathfrak{m}$-primary component of the socle of I is of length 1.

Proposition 4.3. — *Let* A *and* B *be two noetherian local rings with maximal ideals* $\mathfrak{m}_{\mathrm{A}}$ *and* $\mathfrak{m}_{\mathrm{B}}$*, such that* B *is a finite* A*-algebra. Then, if* I *is a dualizing module for* A*,* $\operatorname{Hom}_{\mathrm{A}}(\mathrm{B}, \mathrm{I})$ *is a dualizing module for* B.

Démonstration. — Let

$$\mathrm{R} : \underline{\mathrm{C}}_{\mathfrak{m}_{\mathrm{B}}} \longrightarrow \underline{\mathrm{C}}_{\mathfrak{m}_{\mathrm{A}}}$$

be the restriction of scalars functor; it is exact. Let T be a dualizing functor for A,

$$\mathrm{T} : \underline{\mathrm{C}}_{\mathfrak{m}_{\mathrm{A}}} \longrightarrow \underline{\mathrm{Ab}};$$

it is exact and, for any $\mathrm{M} \in \operatorname{Ob} \underline{\mathrm{C}}_{\mathfrak{m}_{\mathrm{A}}}$, the natural morphism $\mathrm{M} \to \mathrm{T} \circ \mathrm{T}(\mathrm{M})$ is an isomorphism; so $\mathrm{T} \circ \mathrm{R}$ is a dualizing functor for B. If I represents T, according to the classical formula $\operatorname{Hom}_{\mathrm{A}}(\mathrm{M}, \mathrm{I}) = \operatorname{Hom}_{\mathrm{B}}(\mathrm{M}, \operatorname{Hom}_{\mathrm{A}}(\mathrm{B}, \mathrm{I}))$, valid for any B-module M, we deduce that $\operatorname{Hom}_{\mathrm{A}}(\mathrm{B}, \mathrm{I})$ is a dualizing module for B.

Corollary 4.4. — *Let* A *be a noetherian local ring and* $\mathfrak{a}$ *an ideal of* A *; let* $B = A/\mathfrak{a}$. *If* I *is a dualizing module for* A*, the annihilator of* $\mathfrak{a}$ *in* I *is a dualizing module for* B.

Lemma 4.5. — *Let* A *be a noetherian local ring and* I *a locally artinian* A*-module. There exists a canonical isomorphism*

$$I \longrightarrow \widehat{I} = I \otimes_A \widehat{A}.$$

Démonstration. — Let I_n be the annihilator of $\mathfrak{m}^n$ in I, where $\mathfrak{m}$ is the maximal ideal of A. To say that I is locally artinian is to say that I is the inductive limit of the I_n and that these are of finite length. But the tensor product commutes with inductive limits, so we are reduced to the case where I is artinian. In this case I is annihilated by a power of the maximal ideal, say $\mathfrak{m}^k$; so for $p \geqslant k$, $I \xrightarrow{\sim} I \otimes_A A/\mathfrak{m}^p$, so $I \xrightarrow{\sim} I \otimes_A \widehat{A}$ because A is noetherian and I is of finite type.

We conclude from this that the restriction of scalars functor from $\widehat{A}$ to A and the extension of scalars functor induce quasi-inverse *equivalences* of each other between the category of locally artinian $\widehat{A}$*-modules* and the category of locally artinian A-*modules.*

Proposition 4.6. — *Let* A *be a noetherian local ring,* $\widehat{A}$ *its completion,* I *a dualizing module for* A (*resp. for* $\widehat{A}$) *and* J *the completion of* I (*resp. the* A*-module obtained by restriction of scalars*).[2] *Then* J *is a dualizing module for* $\widehat{A}$ (*resp. for* A). *Moreover the underlying abelian groups of* I *and* J *are isomorphic.*

Démonstration. — We simply remark that the equivalence between the category of locally artinian A-modules and the category of locally artinian $\widehat{A}$-modules induces an isomorphism between the bifunctors $\mathrm{Hom}_A(\ ,\ )$ and $\mathrm{Hom}_{\widehat{A}}(\ ,\ )$, and that the characterization of a dualizing functor or module only involves these bifunctors.

Theorem 4.7. — *Let* A *be a noetherian local ring.*

a) *There always exists a dualizing module* I.

b) *Two dualizing modules are isomorphic* (*by a non-canonical isomorphism*).

c) *For a module* I *to be dualizing, it is necessary and sufficient that it be an injective hull of the residue field* k *of* A.

Remark 4.8. — Proposition 4.6 allows reducing to the case of a complete noetherian local ring. According to a structure theorem of Cohen[3], such a ring is a quotient of a regular ring. Proposition 4.3 then allows supposing A is regular. As we will see later,

[2] N.D.E. : one must understand here the tensor product $\widehat{I} = I \otimes_A \widehat{A}$ (cf. lemma 4.5), namely I equipped with its canonical $\widehat{A}$-module structure, and not the $\mathfrak{m}$-adic completion. For example, if p is a prime number and $A = \widehat{A} = \mathbf{Z}_p$ is the ring of p-adic integers. Then, the injective hull of the residue field $k = \mathbf{Z}/p\mathbf{Z}$ is the discrete $\mathbf{Z}_p$-module $\mathbf{Q}_p/\mathbf{Z}_p$, whose completion for the p-adic topology is null.

[3] N.D.E. : see Cohen I.S., "On the structure and ideal theory of complete local rings", *Trans. Amer. Math. Soc.* **59** (1946), pp. 54–106.

this remark allows an explicit calculation of the dualizing module[(*)] ; we will however prove theorem 4.7 by other means.

Recalls. — Before proving the theorem, we will recall a few things about the notion of *injective hull*. Cf. Gabriel, Thesis, Paris 1961, *Des Catégories Abéliennes*, ch. II § 5.

Let $\mathscr{C}$ be an abelian category in which $\varinjlim$ exist and are exact[(4)] (ex. $\mathscr{C}$ =category of modules). Any object M embeds in an injective object and we call an injective hull of M any minimal injective object containing M. We have the following properties :

(i) Any object M has an injective hull I.

(ii) If I and J are two injective hulls of M, there exists between I and J an isomorphism (in general not unique) which induces the identity on M.

(iii) I is an essential extension of M, that is to say, $\mathrm{P} \subset \mathrm{I}$ and $\mathrm{P} \cap \mathrm{M} = \{0\}$ implies $(\mathrm{P} = \{0\})$. Moreover if I is injective and an essential extension of M, I is an injective hull of M.

These results being admitted, to prove theorem 4.7, it is obviously sufficient to prove c).

Démonstration. — Let I be a dualizing module for A. Then I is injective and $\mathrm{Hom}_{\mathrm{A}}(k, \mathrm{I})$ is isomorphic to k. By composing the isomorphism $k \simeq \mathrm{Hom}_{\mathrm{A}}(k, \mathrm{I})$ with the inclusion

$$\mathrm{Hom}_{\mathrm{A}}(k, \mathrm{I}) \hookrightarrow \mathrm{Hom}_{\mathrm{A}}(\mathrm{A}, \mathrm{I}) \simeq \mathrm{I},$$

we obtain the inclusion

$$k \hookrightarrow \mathrm{I}.$$

Let us show that I is an injective hull of k. Let J be an injective module such that

$$k \subset \mathrm{J} \subset \mathrm{I}.$$

Since J is injective, there exists an injective A-submodule J' of I such that $\mathrm{I} = \mathrm{J} \oplus \mathrm{J}'$. Let us show that $\mathrm{Hom}_{\mathrm{A}}(k, \mathrm{J}') = 0$. We have

$$\mathrm{Hom}_{\mathrm{A}}(k, \mathrm{I}) \simeq \mathrm{Hom}_{\mathrm{A}}(k, \mathrm{J}) \oplus \mathrm{Hom}_{\mathrm{A}}(k, \mathrm{J}');$$

$\mathrm{Hom}_{\mathrm{A}}(k, \mathrm{J})$ is a vector subspace of $\mathrm{Hom}_{\mathrm{A}}(k, \mathrm{I}) \simeq k$ not reduced to zero (since it contains the inclusion $k \subset \mathrm{J}$), so $\mathrm{Hom}_{\mathrm{A}}(k, \mathrm{J}) \simeq k$ and consequently $\mathrm{Hom}_{\mathrm{A}}(k, \mathrm{J}') = 0$.

[(*)] This was the method followed by Grothendieck (in 1957). The method by injective hulls which follows is due, it seems, to K. Morita, "Duality for modules and its applications to the theory of rings with minimum conditions", *Sc. Rep. Tokyo Kyoiku Daigaku* **6** (1958/59), pp. 83-142. Morita's work is moreover independent of Grothendieck's and well before the present seminar, and is not limited to the case of commutative base rings.

[(4)] N.D.E. : of course, it is the small filtered inductive limits that are supposed to be exact ; one should also suppose the existence of a generator. Cf. [**Tôhoku**]. As for the category of modules, sufficient for our purposes, one can also refer to chapter 10 of Bourbaki's *Algebra*.

By reasoning by recurrence on the length, we deduce that $\mathrm{Hom}_{\mathrm{A}}(\mathrm{M}, \mathrm{J}') = 0$ for any A-module M of finite length ; as I is the inductive limit of the modules $\mathrm{Hom}(\mathrm{A}/\mathfrak{m}^n, \mathrm{I})$ (cf. proposition 1.3) which are of finite length by hypothesis, the projection $\mathrm{I} \to \mathrm{J}'$ is null, and consequently $\mathrm{J}' = 0$.

Conversely, let I be an injective hull of k. To see that I is a dualizing module, it is sufficient to show according to 2.1 and 3.1 (ii) that $\mathrm{V} = \mathrm{Hom}_{\mathrm{A}}(k, \mathrm{I})$ is isomorphic to k. But we have the double inclusion

$$k \subset \mathrm{V} \subset \mathrm{I};$$

V is a vector space over k which decomposes into the direct sum of k and a vector subspace V' of I such that $\mathrm{V}' \cap k = 0$. But I is an essential extension of k, whence $\mathrm{V}' = 0$ and $\mathrm{V} = k$.

Corollary 4.9. — *Let* A *be a noetherian local ring ; any dualizing module for* A *is locally artinian.*

Démonstration. — Let I be a dualizing module ; it is an injective hull of k. Let us use the notations and the result of corollary 2.2. We have

$$k \subset \mathrm{H}^0_{\mathfrak{m}}(\mathrm{I}) \subset \mathrm{I},$$

and $\mathrm{H}^0_{\mathfrak{m}}(\mathrm{I})$ is injective. We deduce that $\mathrm{I} = \mathrm{H}^0_{\mathfrak{m}}(\mathrm{I})$ and thus that I is locally artinian.(5)

5. Consequences of the theory of dualizing modules

The functor

$$\mathrm{T} = \mathrm{Hom}_{\mathrm{A}}(\ ,\mathrm{I}) \colon \underline{\mathrm{C}}_{\mathfrak{m}} \longrightarrow \underline{\mathrm{C}}_{\mathfrak{m}}$$

is an anti-equivalence. Indeed $\mathrm{T} \circ \mathrm{T}$ is isomorphic to the identity functor and the argument is formal from there.

We deduce from this the usual properties of the notion of orthogonality :

Let $\mathrm{M}^* = \mathrm{Hom}_{\mathrm{A}}(\mathrm{M}, \mathrm{I}) = \mathrm{T}(\mathrm{M})$ and let $\mathrm{N} \subset \mathrm{M}$ be a submodule. We call *orthogonal of* N the submodule N' of M^* formed by the elements of M^* null on N. We thus obtain a bijection between the set of submodules of M and the set of submodules of M^*, which reverses the notion of order.

We have in particular :

– $\mathrm{long}_{\mathrm{M}} \mathrm{N} = \mathrm{colong}_{\mathrm{M}^*} \mathrm{N}'$.

– To monogenic modules, i.e. such that $\mathrm{M}/\mathfrak{m}\mathrm{M}$ is of dimension 0 or 1, correspond in the duality modules whose socle is of length 0 or 1.

– If A is artinian, the ideals of A correspond to the submodules of I.

etc.

(5) N.D.E. : as already seen, one can also simply observe that I is the inductive limit of the modules $\mathrm{Hom}_{\mathrm{A}}(\mathrm{A}/\mathfrak{m}^n, \mathrm{I})$.

Let A be a noetherian local ring, DA the category of A-modules M such that, for any $n \in \mathbf{N}$, $\mathrm{M}^{(n)} = \mathrm{M}/\mathfrak{m}^{n+1}\mathrm{M}$ is of finite length and such that $\mathrm{M} = \varprojlim_n \mathrm{M}^{(n)}$, and let $\widehat{\mathrm{A}}$ be the completion of A. The restriction of scalars functor and the completion functor are quasi-inverse *equivalences* between DA and $\mathrm{D}\widehat{\mathrm{A}}$, which commute up to isomorphism with the formation of the underlying abelian groups of the considered modules. Let us denote by CA the category of locally artinian A-modules with finite dimensional socle.

Proposition 5.1. — *Let* A *be a noetherian local ring and let* I *be a dualizing module for* A. *The functors*

$$\mathrm{Hom}_{\mathrm{A}}(\ ,\mathrm{I})\colon (\mathrm{CA})^{\circ} \longrightarrow \mathrm{DA}$$

and

$$\mathrm{Hom}_{\widehat{\mathrm{A}}}(\ ,\mathrm{I})\colon \mathrm{DA} \longrightarrow (\mathrm{CA})^{\circ}$$

are equivalences of categories, quasi-inverse to each other.

Moreover, if one transports these functors by the equivalences of categories between DA *and* $\mathrm{D}\widehat{\mathrm{A}}$ *on the one hand, and* CA *and* $\mathrm{C}\widehat{\mathrm{A}}$ *on the other hand, one finds the functor* $\mathrm{Hom}_{\widehat{\mathrm{A}}}(\ ,\mathrm{I})$.

Démonstration. — Let $\mathrm{X} \in \mathrm{Ob}\,\mathrm{CA}$. By definition, we have :

$$\mathrm{X} = \varinjlim_{k\in\mathbf{N}} \mathrm{X}_k, \quad \mathrm{X}_k = \mathrm{Hom}_{\mathrm{A}}(\mathrm{A}/\mathfrak{m}^{k+1}, \mathrm{X}),$$

so

$$\mathrm{Hom}_{\mathrm{A}}(\mathrm{X},\mathrm{I}) = \varprojlim \mathrm{Hom}_{\mathrm{A}}(\mathrm{X}_k,\mathrm{I}).$$

So $\mathrm{Y} = \varprojlim \mathrm{X}_k$ is a finitely generated $\widehat{\mathrm{A}}$-module as results from EGA 0_{I} 7.2.9. We remark in this regard that $\mathrm{D}\widehat{\mathrm{A}}$ is also the category of finitely generated $\widehat{\mathrm{A}}$-modules or, if you will, that DA is the category of complete and finitely generated A-modules over $\widehat{\mathrm{A}}$. Let then Y be such a module, let $f : \mathrm{Y} \to \mathrm{I}$ be an $\widehat{\mathrm{A}}$-homomorphism. The image of f is a finitely generated submodule, so is annihilated by $\mathfrak{m}^k$ for a certain k ; indeed any $x \in \mathrm{I}$ is annihilated by a power of $\mathfrak{m}$. So f factors through $\mathrm{Y}/\mathfrak{m}^k\mathrm{Y}$, from which it results that

$$\begin{aligned}\mathrm{Hom}_{\widehat{\mathrm{A}}}(\mathrm{Y},\mathrm{I}) &= \varinjlim_k \mathrm{Hom}_{\widehat{\mathrm{A}}}(\mathrm{Y}^{(k)},\mathrm{I}) \quad \text{with } \mathrm{Y}^{(k)} = \mathrm{Y}/\mathfrak{m}^{k+1}\mathrm{Y}\\ &= \varinjlim_k (\mathrm{Y}^{(k)})^*\end{aligned}$$

belongs to $\mathrm{Ob}\,\mathrm{CA}$. From which it results immediately that the two functors of the statement are quasi-inverse to each other.

It results from the preceding considerations that we do not change anything to the categories or to the considered functors, nor to the underlying abelian groups of the considered modules, by replacing A with $\widehat{\mathrm{A}}$; proposition 5.1 then states as follows :

The restriction of the functor $\operatorname{Hom}_{\widehat{A}}(\ ,I)$ to the category of finitely generated $\widehat{A}$-modules takes its values in the category of locally artinian $\widehat{A}$-modules with finite dimensional socle, and admits a quasi-inverse functor, which is the restriction of the functor $\operatorname{Hom}_{\widehat{A}}(\ ,I)$. On the intersection of these two categories, these two functors coincide (obviously!) and establish a self-duality of the category of $\widehat{A}$-modules of finite length.

***Example 5.2* (Macaulay)**. — Let A be a local ring with residue field k. Let k_0 be a subfield of A such that k is finite over k_0, $[k : k_0] = d$. Any A-module of finite length can be considered as a k_0-vector space of finite dimension equal to $d \cdot \operatorname{long}(M)$. The functor T :

$$M \longrightarrow \operatorname{Hom}_{k_0}(M, k_0)$$

is then exact and preserves length, thus is dualizing for A. The associated dualizing module is therefore :

$$A' = \varinjlim_{n} \operatorname{Hom}_{k_0}(A/\mathfrak{m}^n, k_0),$$

it is the topological dual of A equipped with the $\mathfrak{m}$-adic topology.

Example 5.3. — Let A be a noetherian local *regular ring of dimension* n. Let $\mathfrak{m}$ be its maximal ideal, let k be its residue field. There exists a regular system of parameters, $(x_1, x_2, \ldots, x_n)$, which generates $\mathfrak{m}$, and which is an A-*regular sequence*. We can therefore calculate the $\operatorname{Ext}^i_A(k, A)$ by the Koszul complex; we find :

$$\operatorname{Ext}^i_A(k, A) = 0 \quad \text{if } i \neq n,$$
$$\operatorname{Ext}^n_A(k, A) \simeq k.$$

The depth of A being n, for any M annihilated by a power of $\mathfrak{m}$, $\operatorname{Ext}^i_A(M, A) = 0$ if $i < n$; moreover $\operatorname{Ext}^i_A(M, A) = 0$ if $i > n$ because the global cohomological dimension of A is equal to n. So $\operatorname{Ext}^n_A(\ ,A)$ is exact and moreover $\operatorname{Ext}^n_A(k, A) \simeq k$; it results that :

Proposition 5.4. — *If* A *is a noetherian local regular ring of dimension n, the functor*

$$M \longrightarrow \operatorname{Ext}^n_A(M, A)$$

is dualizing. The associated dualizing module is

$$I = \varinjlim_{r} \operatorname{Ext}^n_A(A/\mathfrak{m}^r, A),$$

it is isomorphic to $H^n_{\mathfrak{m}}(A)$ (*Exposé* II, *thm.* 6)(*).

Remark 5.5. — If A verifies at the same time the hypotheses of the two preceding examples, the two dualizing modules found are isomorphic. Suppose for example that

(*)Let A be a ring, J an ideal of A, M an A-module, $i \in \mathbf{Z}$; we will then set $H^i_J(M) = H^i_Y(X, F)$, where $X = \operatorname{Spec}(A)$, $Y = V(J)$ and $F = \widetilde{M}$.

A is regular of dimension n, complete and of equal characteristics. There then exists a field of representatives, say K. If one chooses a system of parameters $(x_1, \dots, x_n)$ of A, one can construct an isomorphism between A and the ring of formal series : $\mathrm{K}[[\mathrm{T}_1, \dots, \mathrm{T}_n]]$; whence, as we will see, an *explicit* isomorphism between the two dualizing modules

$$v\colon \mathrm{H}^n_{\mathfrak{m}}(\mathrm{A}) \longrightarrow \mathrm{A}'.$$

One can find an *intrinsic* interpretation of this isomorphism with the help of the module $\Omega^n = \Omega^n(\mathrm{A}/\mathrm{K})$ of completed relative differentials of maximum degree. Indeed, we know that Ω^n admits a basis formed by the element $dx_1 \wedge dx_2 \cdots \wedge dx_n$.

Whence an isomorphism

$$u\colon \mathrm{H}^n_{\mathfrak{m}}(\Omega^n) \longrightarrow \mathrm{H}^n_{\mathfrak{m}}(\mathrm{A}).$$

A remarkable fact is then that the composite

$$vu = w\colon \mathrm{H}^n_{\mathfrak{m}}(\Omega^n) \longrightarrow \mathrm{A}'$$

no longer depends on the choice of the system of parameters and commutes with the change of base field.

To construct v we calculate $\mathrm{H}^n_{\mathfrak{m}}(\mathrm{A})$ thanks to the Koszul complex associated to the x_i, we find :

$$\mathrm{H}^n_{\mathfrak{m}}(\mathrm{A}) = \varinjlim_{r} \mathrm{A}/(x_1^r, \dots, x_n^r);$$

where the transition morphisms are defined as follows : set $\mathrm{I}_r = \mathrm{A}/(x_1^r, \dots, x_n^r)$; let $e^r_{a_1,\dots,a_n}$ be the image of $x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$ in I_r. The $e^r_{a_1,\dots,a_n}$, for $0 \leqslant a_i < r$ form a basis of I_r.

That said, if s is an integer, the transition morphism

$$t_{r,r+s}\colon \mathrm{I}_r \longrightarrow \mathrm{I}_{r+s}$$

is multiplication by $x_1^s x_2^s \cdots x_n^s$, so :

$$u_{r,r+s}(e^r_{a_1,\dots,a_n}) = e^{r+s}_{a_1+s,\dots,a_n+s}.$$

Note that giving an A-homomorphism w from an A-module M to A' is equivalent to giving a K-linear form $w' : \mathrm{M} \to \mathrm{K}$ which is *continuous* on the finitely generated submodules. In the case $\mathrm{M} = \mathrm{H}^n_{\mathfrak{m}}(\Omega^n)$, the definition of w is thus equivalent to that of a linear form

$$\rho\colon \mathrm{H}^n_{\mathfrak{m}}(\Omega^n) \longrightarrow \mathrm{K},$$

called *residue form*(*). To construct ρ, it suffices to define forms $\rho_r : \mathrm{I}_r \to \mathrm{K}$ which glue, and we will take

$$\rho_r(e^r_{a_1,\dots,a_n}) = \begin{cases} 1 & \text{if } a_i = r-1 \text{ for } 1 \leqslant i \leqslant n \\ 0 & \text{otherwise.} \end{cases}$$

(*) For a more detailed study of the notion of residue, Cf. R. Hartshorne, *Residues and Duality*, Lect. Notes in Math., vol. 20, Springer, 1966.

EXPOSÉ V

LOCAL DUALITY AND STRUCTURE OF THE $\mathrm{H}^i(\mathrm{M})$

1. Homomorphism complexes

1.1. Let $\mathrm{F}^\bullet$ and $\mathrm{G}^\bullet$ be two graded modules, then we denote by : 1

(1) $$\mathrm{Hom}^\bullet(\mathrm{F}^\bullet, \mathrm{G}^\bullet)$$

the graded module of graded module homomorphisms from $\mathrm{F}^\bullet$ to $\mathrm{G}^\bullet$. Thus we have :

(2) $$\mathrm{Hom}^s(\mathrm{F}^\bullet, \mathrm{G}^\bullet) = \prod_k \mathrm{Hom}(\mathrm{F}^k, \mathrm{G}^{k+s}).$$

Let $\mathrm{F}^\bullet$ (resp. $\mathrm{G}^\bullet$) be a complex, and let d_1 (resp. d_2) be its differential, then for $h \in \mathrm{Hom}^s(\mathrm{F}^\bullet, \mathrm{G}^\bullet)$ we will set[(1)]

(3) $$d(h) = h \circ d_1 + (-1)^{s+1} d_2 \circ h.$$

One trivially verifies that $d \circ d = 0$, so that $\mathrm{Hom}^\bullet(\mathrm{F}^\bullet, \mathrm{G}^\bullet)$ equipped with d is a complex. The cohomology group of this complex is denoted

(4) $$\underset{\sim}{\mathrm{H}}^\bullet(\mathrm{F}^\bullet, \mathrm{G}^\bullet).$$

If $\mathrm{G}^\bullet$ is injective in each degree, then

$$\mathrm{F}^\bullet \longmapsto \underset{\sim}{\mathrm{H}}^\bullet(\mathrm{F}^\bullet, \mathrm{G}^\bullet)$$

is an exact ∂-functor. Similarly, for any $\mathrm{F}^\bullet$,

$$\mathrm{G}^\bullet \longmapsto \underset{\sim}{\mathrm{H}}^\bullet(\mathrm{F}^\bullet, \mathrm{G}^\bullet)$$

is an exact δ-functor on the category of complexes $\mathrm{G}^\bullet$ that are injective in each degree.

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(1) N.D.E. : the original sign convention was different ; but, it is not compatible with the convention of exposé VIII, which seems more reasonable, since in this case the cohomology in degree 0 is then the set of homotopy classes of morphisms from $\mathrm{F}^\bullet$ to $\mathrm{G}^\bullet$. The calculations have been modified accordingly in what follows.

Remark 1.2. — The cycles of $\mathrm{Hom}^\bullet(F^\bullet, G^\bullet)$ are the homomorphisms from $F^\bullet$ to $G^\bullet$ that commute or anticommute, depending on the degree, with the differentials. The boundaries of $\mathrm{Hom}^\bullet(F^\bullet, G^\bullet)$ are the homomorphisms from $F^\bullet$ to $G^\bullet$ that are homotopic to zero.

2 Let A be a ring and let M (resp. N) be an A-module, R(M) (resp. R(N)) an injective resolution of M (resp. N), then there exists a canonical isomorphism[(2)]

$$\underset{\sim}{\mathrm{H}}^s(\mathrm{R(M)}, \mathrm{R(N)}) \simeq \mathrm{Ext}^s(\mathrm{M}, \mathrm{N}). \tag{1.3}$$

Indeed, let $i\colon \mathrm{M} \to \mathrm{R(M)}$ be the canonical augmentation, and let $h \in \mathrm{Hom}^s(\mathrm{R(M)}, \mathrm{R(N)})$, then we will denote by t_s the map

$$h \longmapsto h_0 \circ i$$

from $\mathrm{Hom}^s(\mathrm{R(M)}, \mathrm{R(N)})$ to $\mathrm{Hom}(\mathrm{M}, \mathrm{R(N)}^s)$. The family $(t_s)_{s\geqslant 0}$ defines a homomorphism of (ordinary) complexes[(3)]

$$t\colon \mathrm{Hom}^\bullet(\mathrm{R(M)}, \mathrm{R(N)}) \longrightarrow \mathrm{Hom}^\bullet(\mathrm{M}, \mathrm{R(N)}),$$

i.e., we have $(dh)_0 \circ i = d_2 \circ h_0 \circ i$.

One easily verifies that, passing to cohomology, t gives an isomorphism. In particular, it follows that

$$\underset{\sim}{\mathrm{H}}^\bullet(\mathrm{R(M)}, \mathrm{R(N)})$$

"does not depend" on the chosen injective resolution R(M) (resp. R(N)) of M (resp. N).

To any exact sequence of A-modules

$$0 \longrightarrow \mathrm{M}' \longrightarrow \mathrm{M} \longrightarrow \mathrm{M}'' \longrightarrow 0 \tag{5}$$

we associate an exact sequence of injective resolutions

$$0 \longrightarrow \mathrm{R(M')} \longrightarrow \mathrm{R(M)} \longrightarrow \mathrm{R(M'')} \longrightarrow 0. \tag{6}$$

One verifies that the isomorphism (1.3) commutes with the homomorphisms :

$$\underset{\sim}{\mathrm{H}}^s(\mathrm{R(M')}, \mathrm{R(N)}) \longrightarrow \underset{\sim}{\mathrm{H}}^{s+1}(\mathrm{R(M'')}, \mathrm{R(N)}), \tag{8}$$

$$\mathrm{Ext}^s(\mathrm{R(M')}, \mathrm{R(N)}) \longrightarrow \mathrm{Ext}^{s+1}(\mathrm{R(M'')}, \mathrm{R(N)}), \tag{9}$$

deduced from (6) and (5).

3 Let P be a third A-module, R(P) an injective resolution of P, then the composition of graded morphisms gives a pairing

$$\mathrm{Hom}^i(\mathrm{R(N)}, \mathrm{R(M)}) \times \mathrm{Hom}^j(\mathrm{R(M)}, \mathrm{R(P)}) \longrightarrow \mathrm{Hom}^{i+j}(\mathrm{R(N)}, \mathrm{R(P)}), \tag{10}$$

(2) N.D.E. : we have kept the strange original numbering.

(3) N.D.E. : we still denote by M the complex M[0] consisting of M placed in degree 0.

which defines a pairing :

$$\underset{\sim}{\mathrm{H}}^i(\mathrm{R(N)},\mathrm{R(M)}) \times \underset{\sim}{\mathrm{H}}^j(\mathrm{R(M)},\mathrm{R(P)}) \longrightarrow \underset{\sim}{\mathrm{H}}^{i+j}(\mathrm{R(N)},\mathrm{R(P)}), \tag{11}$$

hence a homomorphism of functors in M :

$$\underset{\sim}{\mathrm{H}}^i(\mathrm{R(N)},\mathrm{R(M)}) \longrightarrow \mathrm{Hom}(\underset{\sim}{\mathrm{H}}^j(\mathrm{R(M)},\mathrm{R(P)}),\underset{\sim}{\mathrm{H}}^{i+j}(\mathrm{R(N)},\mathrm{R(P)})). \tag{1.4}$$

We will see that (1.4) is a homomorphism of δ-functors in M. The exact sequences (5) and (6) give a commutative diagram :

$$\begin{tikzcd}
\mathrm{Hom}^i(\mathrm{R(N)},\mathrm{R(M')}) \arrow[r] \arrow[d] & \mathrm{Hom}(\mathrm{Hom}^j(\mathrm{R(M')},\mathrm{R(P)}),\mathrm{Hom}^{i+j}(\mathrm{R(N)},\mathrm{R(P)})) \arrow[d, "\mathrm{Hom}(q,\mathrm{id})"] \\
\mathrm{Hom}^i(\mathrm{R(N)},\mathrm{R(M)}) \arrow[r] \arrow[d, "p"] & \mathrm{Hom}(\mathrm{Hom}^j(\mathrm{R(M)},\mathrm{R(P)}),\mathrm{Hom}^{i+j}(\mathrm{R(N)},\mathrm{R(P)})) \arrow[d] \\
\mathrm{Hom}^i(\mathrm{R(N)},\mathrm{R(M'')}) \arrow[r] & \mathrm{Hom}(\mathrm{Hom}^j(\mathrm{R(M'')},\mathrm{R(P)}),\mathrm{Hom}^{i+j}(\mathrm{R(N)},\mathrm{R(P)})).
\end{tikzcd}$$

Let $h \in \mathrm{Hom}^i(\mathrm{R(N)},\mathrm{R(M'')})$ (resp. $g \in \mathrm{Hom}^j(\mathrm{R(M')},\mathrm{R(P)})$) be a cycle, and let $h' \in \mathrm{Hom}^i(\mathrm{R(N)},\mathrm{R(M)})$ (resp. $g' \in \mathrm{Hom}^j(\mathrm{R(M)},\mathrm{R(P)})$) be such that $p(h') = h$ (resp. $q(g') = g$), then to say that (1.4) is a homomorphism of δ-functors in M, is to say that

$$g \circ dh' - dg' \circ h \tag{12}$$

is a coboundary in $\mathrm{Hom}^\bullet(\mathrm{R(N)},\mathrm{R(P)})$.

Now we have :

$$\begin{aligned} dh' &= h' \circ d_1 + (-1)^{i+1} d_2 \circ h' \\ dg' &= g' \circ d_2 + (-1)^{j+1} d_3 \circ g' \end{aligned}$$

with the obvious notations. So (12) is written as :

$$g \circ h' \circ d_1 + (-1)^{i+1} g \circ d_2 \circ h' - g' \circ d_2 \circ h - (-1)^{j+1} d_3 \circ g' \circ h.$$

On the other hand, since h and g are cycles we have : 4

$$\begin{aligned} g \circ d_2 &= (-1)^j d_3 \circ g, \\ d_2 \circ h &= (-1)^i h \circ d_1, \end{aligned}$$

so, finally, (12) is written as :

$$d(g \circ h' + (-1)^{i+1} g' \circ h),$$

which completes the proof.

(1.3) and (1.4) thus give a homomorphism of δ-functors in M :

$$\mathrm{Ext}^i(\mathrm{N},\mathrm{M}) \longrightarrow \mathrm{Hom}(\mathrm{Ext}^j(\mathrm{M},\mathrm{P}),\mathrm{Ext}^{i+j}(\mathrm{N},\mathrm{P})). \tag{1.5}$$

2. The local duality theorem for a regular local ring

Let A be a regular local ring of dimension r, $\mathfrak{m}$ the maximal ideal of A and M a finitely generated A-module. We set $\mathrm{H}^i(\mathrm{M}) = \mathrm{H}^i_{\mathfrak{m}}(\mathrm{M})$ (so $\mathrm{H}^i(\mathrm{M}) = \varinjlim \mathrm{Ext}^i(\mathrm{A}/\mathfrak{m}^k, \mathrm{M})$). We have seen (IV 5.4) that $\mathrm{I} = \mathrm{H}^r(\mathrm{A})$ is a dualizing module for A, let us denote by D the associated dualizing functor. In (1.5) let us set $\mathrm{N} = \mathrm{A}/\mathfrak{m}^k$, $\mathrm{P} = \mathrm{A}$, then we obtain a homomorphism of δ-functors in M

$$\varphi_k\colon\ \mathrm{Ext}^i(\mathrm{A}/\mathfrak{m}^k, \mathrm{M}) \longrightarrow \mathrm{Hom}(\mathrm{Ext}^{r-i}(\mathrm{M}, \mathrm{A}), \mathrm{Ext}^r(\mathrm{A}/\mathfrak{m}^k, \mathrm{A})). \tag{13}$$

Passing to the inductive limit over k, we find a homomorphism of δ-functors

$$\varphi\colon\ \mathrm{H}^i(\mathrm{M}) \longrightarrow \mathrm{D}(\mathrm{Ext}^{r-i}(\mathrm{M}, \mathrm{A})). \tag{14}$$

***Theorem 2.1* (Theorem of local duality).** — *The preceding functorial homomorphism φ is an isomorphism.*

5 *Démonstration.* — If $i > r$, the second member of (14) is trivially zero, and the first member is zero because $\mathrm{H}^i(\mathrm{M}) = \varinjlim_k \mathrm{Ext}^i(\mathrm{A}/\mathfrak{m}^k, \mathrm{M})$, and this is true for each $\mathrm{Ext}^i(\mathrm{A}/\mathfrak{m}^k, \mathrm{M})$ (Syzygy Theorem).

If $i = r$, according to the preceding, the two functors in M, $\mathrm{H}^r(\mathrm{M})$ and $\mathrm{D}(\mathrm{Hom}(\mathrm{M}, \mathrm{A}))$ are right exact; since A is noetherian, and M is finitely generated, it suffices to verify the isomorphism for $\mathrm{M} = \mathrm{A}$, which is immediate.

To show that φ is a functorial isomorphism, it is now sufficient, by proceeding by descending induction on i, to note that every finitely generated module admits a finite presentation, and that for $i < r$ both members of (14) are zero if M is a free module of finite type. This is evident for the second member, and since H^i commutes with finite sums it is sufficient, for the first, to show that $\mathrm{H}^i(\mathrm{A}) = 0$ for $i < r$. Now this results, since $\mathrm{prof}(\mathrm{A}) = r$, from (III 3.4).

3. Application to the structure of the $\mathrm{H}^i(\mathrm{M})$

***Theorem 3.1*.** — *Let* A *be a noetherian local ring,* D *a dualizing functor for* A*,* M *a finitely generated* A*-module* $\neq 0$*, and of dimension* n*, then we have :*

(i) $\mathrm{H}^i(\mathrm{M}) = 0$ *if* $i < 0$ *or if* $i > n$.

(ii) $\mathrm{D}(\widehat{\mathrm{H}^i(\mathrm{M})})$ *is a finitely generated module over* $\widehat{\mathrm{A}}$*, of dimension* $\leqslant i$.

(iii) $\mathrm{H}^n(\mathrm{M}) \neq 0$*, and if* A *is complete,* $\mathrm{D}(\mathrm{H}^n(\mathrm{M}))$ *is of dimension* n *and* $\mathrm{Ass}(\mathrm{D}(\mathrm{H}^n(\mathrm{M}))) = \{\mathfrak{p} \in \mathrm{Ass}(\mathrm{M}) \mid \dim \mathrm{A}/\mathfrak{p} = n\}$.

Démonstration. — Let I be the dualizing module associated with D. We know that $\widehat{\mathrm{I}}$ is a dualizing module for $\widehat{\mathrm{A}}$. On the other hand, we have :

$\widehat{\mathrm{H}^i(\mathrm{M})} = \mathrm{H}^i(\widehat{\mathrm{M}})$,
$\mathrm{D}(\widehat{\mathrm{H}^i(\mathrm{M})}) = \mathrm{Hom}(\mathrm{H}^i(\widehat{\mathrm{M}}), \widehat{\mathrm{I}})$ and
$\dim \widehat{\mathrm{M}} = \dim \mathrm{M}$,

so we can assume A is complete. Now, according to a theorem of Cohen, any complete 6
local ring is a quotient of a regular local ring. To reduce to this case, we need the following lemma :

Lemma 3.2. — *Let* X (*resp.* Y) *be a ringed space,* X′ (*resp.* Y′) *a closed subspace of* X (*resp.* Y), *and* $f\colon X \to Y$ *a morphism of ringed spaces such that* $f^{-1}(Y') = X'$. *Let* F *be an* $\mathscr{O}_X$*-Module and let us denote by* A (*resp.* B) *the ring* $\Gamma(\mathscr{O}_X)$ (*resp.* $\Gamma(\mathscr{O}_Y)$) *and by* $\overline{f}\colon B \to A$ *the ring homomorphism corresponding to* f. *There exists a spectral sequence of* B*-modules, with initial term*

$$E_2^{p,q} = H^p_{Y'}(Y, R^q f_*(F)), \tag{15}$$

converging to the B*-module* $H^\bullet_{X'}(X, F)_{[\overline{f}]}$.

Démonstration. — Let $\mathscr{O}_{Y,Y'}$ be the sheaf $\mathscr{O}_{Y|Y'}$ extended by 0 outside of Y′ (see Exp I). We have an isomorphism of B-modules :

$$\operatorname{Hom}(\mathscr{O}_{Y,Y'}, f_*(F)) \simeq \operatorname{Hom}(f^*(\mathscr{O}_{Y,Y'}), F)_{[\overline{f}]}. \tag{16}$$

Now, we have :

$$f^*(\mathscr{O}_{Y,Y'}) = \mathscr{O}_{X,X'}, \tag{17}$$

and furthermore if G is an injective $\mathscr{O}_X$-Module, then $f_*(G)$ is an injective $\mathscr{O}_Y$-Module, at least if f is flat, a case to which we can easily reduce by replacing $\mathscr{O}_X$ etc. with the constant ring sheaves **Z**. So the spectral sequence of the composite functor

$$F \longrightarrow \operatorname{Hom}(\mathscr{O}_{Y,Y'}, f_*(F)),$$

with initial term

$$E_2^{p,q} = \operatorname{Ext}^p(Y; \mathscr{O}_{Y,Y'}, R^q f_*(F)),$$

converges, taking into account (16) and (17), to :

$$\operatorname{Ext}^\bullet(X; \mathscr{O}_{X,X'}, F)_{[\overline{f}]}.$$

The lemma then results from (I 2 bis). C.Q.F.D. 7

Now let $f\colon B \to A$ be a surjective homomorphism of local rings. Let

$$f\colon \operatorname{Spec}(A) \longrightarrow \operatorname{Spec}(B)$$

be the corresponding morphism of affine schemes. Let $X = \operatorname{Spec}(A)$ (resp. $X' = \{\mathfrak{m}_A\}$), $Y = \operatorname{Spec}(B)$ (resp. $Y' = \{\mathfrak{m}_B\}$), and let M be an A-module and $\widetilde{M}$ the corresponding $\mathscr{O}_X$-Module. Since $R^q f_*(\widetilde{M}) = 0$ for $q > 0$, the spectral sequence (15) degenerates, and we obtain from (3.2) an isomorphism of B-modules :

$$H^n_{\{\mathfrak{m}_B\}}(Y, f_*(\widetilde{M})) \simeq H^n_{\{\mathfrak{m}_A\}}(X, \widetilde{M})_{[f]}, \tag{18}$$

hence an isomorphism of B-modules :

$$H^n_{\mathfrak{m}_B}(M_{[f]}) \simeq H^n_{\mathfrak{m}_A}(M)_{[f]}. \tag{19}$$

On the other hand if $\mathrm{D_A}$ (resp. $\mathrm{D_B}$) is the dualizing functor for A (resp. B), we have :

$$\mathrm{D_A}(\mathrm{M})_{[f]} \simeq \mathrm{D_B}(\mathrm{M}_{[f]}). \tag{20}$$

Finally, sincewe have a ring isomorphism

$$\mathrm{B}/\operatorname{Ann}\mathrm{M}_{[f]} \simeq \mathrm{A}/\operatorname{Ann}\mathrm{M}, \tag{21}$$

we see that the contemplated change of base rings changes nothing. Let us therefore suppose that A is regular of dimension r.

According to (2.1) we have :

$$\mathrm{D}(\mathrm{H}^i(\mathrm{M})) = \operatorname{Ext}^{r-i}(\mathrm{M},\mathrm{A}). \tag{22}$$

8 We are going to prove the equivalence between the following properties :

(a) $\dim \operatorname{Ext}^j(\mathrm{M},\mathrm{A}) \leqslant r-j$;

(b) for all $\mathfrak{p} \in \mathrm{X} = \operatorname{Spec}(\mathrm{A})$ such that $\dim \mathrm{A}_\mathfrak{p} < j$, we have $\operatorname{Ext}^j(\mathrm{M},\mathrm{A})_\mathfrak{p} = 0$;

(c) $\operatorname{codim}(\operatorname{Supp}(\operatorname{Ext}^j(\mathrm{M},\mathrm{A})),\mathrm{X}) \geqslant j$.

To prove (a) $\Rightarrow$ (b), let $\mathfrak{p} \in \mathrm{X}$, $\dim \mathrm{A}_\mathfrak{p} < j$, then $\dim \mathrm{A}/\mathfrak{p} > r-j$, so by (a) $\operatorname{Ann}(\operatorname{Ext}^j(\mathrm{M},\mathrm{A})) \not\subset \mathfrak{p}$, which implies $\operatorname{Ext}^j(\mathrm{M},\mathrm{A})_\mathfrak{p} = 0$. Let $\mathfrak{p} \in \operatorname{Supp}(\operatorname{Ext}^j(\mathrm{M},\mathrm{A}))$, then $\operatorname{Ext}^j(\mathrm{M},\mathrm{A})_\mathfrak{p} \neq 0$, so by (b) $\dim \mathrm{A}_\mathfrak{p} \geqslant j$. So $\operatorname{codim}(\operatorname{Supp}(\operatorname{Ext}^j(\mathrm{M},\mathrm{A})),\mathrm{X}) = \inf\{\dim \mathrm{A}_\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Supp}(\operatorname{Ext}^j(\mathrm{M},\mathrm{A}))\} \geqslant j$, that is (b) $\Rightarrow$ (c). Finally (c) trivially implies (a).

Let us now prove the theorem.

(i) Let $x = (x_1,\ldots,x_r)$ be a system of parameters for A such that $x_i \in \operatorname{Ann}\mathrm{M}$ for $i = 1,\ldots,r-n$. Let $\mathrm{K}^\bullet((x^k),\mathrm{M})$ be the Koszul complex. We easily see that the map $\mathrm{K}^i((x^k),\mathrm{M}) \to \mathrm{K}^i((x^{k'}),\mathrm{M})$ for $k < k'$ is zero if $i > n$. It follows that $\mathrm{H}^i(\mathrm{M}) = \varinjlim \mathrm{H}^i((x^k),\mathrm{M}) = 0$ if $i > n$. On the other hand, it is trivial that $\mathrm{H}^i(\mathrm{M}) = 0$ if $i < 0$, so (i) is proven.

(ii) Since A is regular, $\dim \mathrm{A}_\mathfrak{p} < j$ implies that the global homological dimension of $\mathrm{A}_\mathfrak{p}$ is strictly smaller than j and thus $\operatorname{Ext}^j(\mathrm{M},\mathrm{A})_\mathfrak{p} = \operatorname{Ext}^j_{\mathrm{A}_\mathfrak{p}}(\mathrm{M}_\mathfrak{p},\mathrm{A}_\mathfrak{p}) = 0$, so we have proven (b) and consequently (a). (ii) then results from (22) and from (a).

(iii) There exists a $\mathfrak{p} \in \operatorname{Supp}(\mathrm{M})$ such that $\dim \mathrm{A}_\mathfrak{p} = r-n$ and such that $\operatorname{Supp}(\mathrm{M}_\mathfrak{p}) = \{\mathfrak{m}\mathrm{A}_\mathfrak{p}\}$. Since $\mathrm{A}_\mathfrak{p}$ is regular if A is, we find $\operatorname{prof}\mathrm{A}_\mathfrak{p} = r-n$, hence

$$\operatorname{Ext}^{r-n}_{\mathrm{A}}(\mathrm{M},\mathrm{A})_\mathfrak{p} = \operatorname{Ext}^{r-n}_{\mathrm{A}_\mathfrak{p}}(\mathrm{M}_\mathfrak{p},\mathrm{A}_\mathfrak{p}) \neq 0. \tag{23}$$

9 This implies, taking into account aof (22), that on the one hand :

$$\mathrm{H}^n(\mathrm{M}) \neq 0,$$

on the other hand

$$\dim \mathrm{D}(\mathrm{H}^n(\mathrm{M})) \geqslant n,$$

so according to (ii)

$$\dim \mathrm{D}(\mathrm{H}^n(\mathrm{M})) = n.$$

Now let $\mathrm{Y} = \operatorname{Supp}(\mathrm{M})$. According to (i) we know that $\mathrm{D}(\mathrm{H}^n(\mathrm{M}')) = \operatorname{Ext}^{r-n}(\mathrm{M}',\mathrm{A})$ is a functor in M', left exact, on the category $(\mathbf{C}_\mathrm{Y})^\circ$. So there exists an A-module H and an isomorphism of functors in M' :

$$\mathrm{Ext}^{r-n}(M', A) = \mathrm{Hom}(M', H).$$

Let Y_i, $i = 1, \dots, k$ be the irreducible components of Y of maximum dimension. We will see that the assertion $\mathrm{Ext}^{r-n}(M', A) \neq 0$ is equivalent to the assertion : there exists an i such that $\mathrm{Supp}\, M' \supset Y_i$. Indeed if $\mathrm{Supp}\, M' \supset Y_i$ then $\dim(M') = n$ so $\mathrm{Ext}^{r-n}(M', A) \neq 0$.

If $\mathrm{Supp}\, M' \not\supset Y_i$ for all $i = 1, \dots, k$, then $\dim M' < n$

$$D(H^n(M')) = \mathrm{Ext}^{r-n}(M', A) = 0.$$

Since $\mathrm{Ass}(\mathrm{Ext}^{r-n}(M, A)) = \mathrm{Supp}\, M \cap \mathrm{Ass}(H)$ we see that the last assertion of (iii) results from the following lemma :

Lemma 3.3. — *Let* $X = \mathrm{Spec}(A)$, *and let* Y *be a closed subset of* X, $T\colon (\mathbf{C}_Y)^\circ \to \underline{\mathrm{Ab}}$ *a left exact functor, and* Y_i, $i = 1, \dots, k$ *a family of irreducible components of* Y *such that the assertion :* $T(M) = 0$ *is equivalent to the assertion :* $\forall i$ $\mathrm{Supp}\, M \not\supset Y_i$. *Then* T *is representable by a module* H *such that* $\mathrm{Ass}(H) = \bigcup_{i=1}^{k} \{y_i\}$, *where* y_i *is the generic point of* Y_i, $i = 1, \dots, k$.

Démonstration. — Let $y \in Y$, we construct an A-module $M(y)$ such that 10
$\mathrm{Supp}(M(y)) = \overline{\{y\}}$. Suppose that $y \neq y_i$ for all $i = 1, \dots, k$, then $Y_i \not\subset \mathrm{Supp}(M(y))$ for all $i = 1, \dots, k$, so $T(M(y)) = 0$. It follows that :

$$\mathrm{Ass}(T(M(y))) = \mathrm{Supp}(M(y)) \cap \mathrm{Ass}(H) = \emptyset,$$

so $y \notin \mathrm{Ass}(H)$. If $y = y_i$, then $Y_i \subset \mathrm{Supp}(M(y))$, so $T(M(y)) \neq 0$, thus :

$$\mathrm{Ass}(T(M(y))) = \mathrm{Supp}(M(y)) \cap \mathrm{Ass}(H) \neq \emptyset.$$

According to the first part of the proof, this implies $y \in \mathrm{Ass}(H)$, whence the lemma, Q.E.D.

Example 3.4. — Let A be a noetherian ring, let $X = \mathrm{Spec}(A)$, and Y a closed subset of X, such that $X - Y$ is affine, then for any irreducible component Y_α of Y we have $\mathrm{codim}(Y_\alpha, X) \leqslant 1$.

Indeed, let us consider X as a prescheme over X. Let $y_\alpha \in Y_\alpha$ be a generic point, and consider the morphism $\mathrm{Spec}(\mathscr{O}_{X,y_\alpha}) \to X$. The affine scheme obtained by extension of the base scheme of X to $\mathrm{Spec}(\mathscr{O}_{X,y_\alpha})$ is canonically isomorphic to $\mathrm{Spec}(\mathscr{O}_{X,y_\alpha})$.

According to (EGA I 3.2.7) we see that if y_0 is the unique closed point of $Y_0 = \mathrm{Spec}(\mathscr{O}_{X,y_\alpha})$ then $Y_0 - y_0$ is affine. By (EGA III 1.3.1) we find :

$$H^i(Y_0 - y_0, \mathscr{O}_{Y_0}) = 0 \quad \text{if } i > 0,$$

so by (I 2.9)

$$H^{i-1}(\mathscr{O}_{X,y_\alpha}) = H^i_{\{y_0\}}(Y_0, \mathscr{O}_{Y_0}) = 0 \quad \text{if } i \geqslant 2.$$

Taking into account 3.1 (iii), it follows :

$$\dim \mathscr{O}_{\mathrm{X},y_\alpha} \leqslant 1,$$

so $\operatorname{codim}(\mathrm{Y}_\alpha, \mathrm{X}) = \inf_{y\in \mathrm{Y}_\alpha} \dim \mathscr{O}_{\mathrm{X},y} \leqslant 1,$ Q.E.D.

11 Let A be a noetherian local ring, $\mathfrak{m}$ the maximal ideal and M a finitely generated A-module. Suppose that A is a quotient of a regular local ring. Let $\mathrm{X} = \operatorname{Spec}(\mathrm{A})$, and for any $x \in \mathrm{X}$, $\mathfrak{m}_x = \mathfrak{m}\mathrm{A}_x$.

Proposition 3.5. — *The following two conditions are equivalent :*

a) $\mathrm{H}^i(\mathrm{M})$ *is of finite length,*

b) $\forall\, x \in \mathrm{X} - \{\mathfrak{m}\}$, $\mathrm{H}^{i-\dim \overline{\{x\}}}_{\mathfrak{m}_x}(\mathrm{M}_x) = 0$.

Démonstration. — Taking into account (3.2) we can assume A is regular. According to (2.1) we have :

$$\mathrm{H}^i(\mathrm{M}) = \mathrm{D}(\operatorname{Ext}^{r-i}(\mathrm{M}, \mathrm{A})),$$

where $r = \dim \mathrm{A}$. According to (IV 4.7), a) is equivalent[(4)] to :

$$\operatorname{Ext}^{r-i}(\mathrm{M}, \mathrm{A}) \text{ is of finite length.} \tag{24}$$

Now (24) is equivalent to :

$$\forall\, x \in \mathrm{X} - \{\mathfrak{m}\}, \text{ we have } \operatorname{Ext}^{r-i}(\mathrm{M}, \mathrm{A})_x = 0. \tag{25}$$

On the other hand A_x is regular of dimension $r - \dim \overline{\{x\}}$, so according to (2.1)

$$\mathrm{H}^{i-\dim \overline{\{x\}}}_{\mathfrak{m}_x}(\mathrm{M}_x) = \mathrm{D}(\operatorname{Ext}^{(r-\dim \overline{\{x\}})-(i-\dim \overline{\{x\}})}_{\mathrm{A}_x}(\mathrm{M}_x, \mathrm{A}_x)) = \mathrm{D}(\operatorname{Ext}^{r-i}_{\mathrm{A}_x}(\mathrm{M}_x, \mathrm{A}_x)). \tag{26}$$

Since M is finitely generated we have :

$$\operatorname{Ext}^{r-i}_{\mathrm{A}}(\mathrm{M}, \mathrm{A})_x = \operatorname{Ext}^{r-i}_{\mathrm{A}_x}(\mathrm{M}_x, \mathrm{A}_x)$$

whence the proposition.

Corollary 3.6. — *For* $\mathrm{H}^i(\mathrm{M})$ *to be of finite length for* $i \leqslant n$, *it is necessary and sufficient that*

$$\operatorname{prof}(\mathrm{M}_x) > n - \dim \overline{\{x\}}$$

for all $x \in \mathrm{X} - \{\mathfrak{m}\}$.

(4) N.D.E. : indeed, it is a matter of showing that, E being a finitely generated A-module, if D(E) is of finite length then E is of finite length. Let K (resp. Q) be the kernel (resp. cokernel) of the canonical morphism $\epsilon : \mathrm{E} \to \mathrm{DD}(\mathrm{E})$. The composition of $\mathrm{D}(\epsilon)$ and the canonical morphism $\gamma :\ \mathrm{D}(\mathrm{E}) \xrightarrow{\gamma} \mathrm{DDD}(\mathrm{E})$ is the identity on D(E). As D(E) is of finite length, γ is an isomorphism and so is $\mathrm{D}(\epsilon)$. As D is exact, we deduce that D(K) and D(Q) are zero. It suffices to prove that if M is an A-module with zero dual, then M is zero, because we will then have $\mathrm{E} = \mathrm{DD}(\mathrm{E})$ of finite length like D(E). Indeed, let M_0 be a finitely generated submodule of M. As D is exact, $\mathrm{D}(\mathrm{M}_0)$ is a quotient of D(M), which is therefore zero. Still by exactness of D, we have $\mathrm{D}(\mathrm{M}_0/\mathfrak{m}_{\mathrm{A}}\mathrm{M}_0) = 0$, and thus, by biduality, the module of finite length $\mathrm{M}_0/\mathfrak{m}_{\mathrm{A}}\mathrm{M}_0$ is zero. Nakayama's lemma then ensures the nullity of M_0 and finally we obtain that of M.

Démonstration. — Results from (3.5) and from (III 3.1).

EXPOSÉ VI

THE FUNCTORS $\mathrm{Ext}^\bullet_Z(X; F, G)$ AND $\underline{\mathrm{Ext}}^\bullet_Z(F, G)$

1. Generalities

1.1. Let $(X, \mathscr{O}_X)$ be a ringed space and Z a locally closed subset of X. Let F and G be $\mathscr{O}_X$-Modules, we will denote by $\mathrm{Ext}^i_Z(X; F, G)$ (resp. $\underline{\mathrm{Ext}}^i_Z(F, G)$) the i-th derived functor of the functor $G \mapsto \Gamma_Z(\underline{\mathrm{Hom}}_{\mathscr{O}_X}(F, G))$ (resp. $\underline{\Gamma}_Z(\underline{\mathrm{Hom}}_{\mathscr{O}_X}(F, G))$). 12

Lemma 1.2. — *The sheaf* $\underline{\mathrm{Ext}}^i_Z(F, G)$ *is canonically isomorphic to the sheaf associated with the presheaf* $U \mapsto \mathrm{Ext}^i_{Z\cap U}(U; F_{|U}, G_{|U})$.

This results from ([**Tôhoku**], 3.7.2) and from the fact that $\Gamma(U; \underline{\Gamma}_Z(\underline{\mathrm{Hom}}_{\mathscr{O}_X}(F, G)))$ is canonically isomorphic to $\Gamma_{Z\cap U}(\underline{\mathrm{Hom}}_{\mathscr{O}_{X|U}}(F_{|U}, G_{|U}))$

Theorem 1.3 **(Excision Theorem)**. — *Let* V *be an open set of* X *containing* Z*, we then have an isomorphism of cohomological functors*

$$\mathrm{Ext}^\bullet_X(X; F, G) \simeq \mathrm{Ext}^\bullet_V(V; F_{|V}, G_{|V}) \tag{1.3.1}$$

Indeed, if $G^\bullet$ is an injective resolution of G, then $G^\bullet_{|V}$ is an injective resolution of $G_{|V}$. The theorem follows immediately.

1.4. Let $\mathscr{O}_{X,Z}$ be the $\mathscr{O}_X$-Module defined by the following conditions ([**Godement**], 2.9.2) : $\mathscr{O}_{X,Z|X-Z} = 0$ and $\mathscr{O}_{X,Z|Z} = \mathscr{O}_{X|Z}$. We have seen that for any $\mathscr{O}_X$-Module H, there exists a functorial isomorphism : $\Gamma_Z(H) \simeq \mathrm{Hom}_{\mathscr{O}_X}(\mathscr{O}_{X,Z}, H)$. We therefore deduce functorial isomorphisms in F and G :

$$\Gamma_Z(\underline{\mathrm{Hom}}_{\mathscr{O}_X}(F, G)) \simeq \mathrm{Hom}_{\mathscr{O}_X}(\mathscr{O}_{X,Z}, \underline{\mathrm{Hom}}_{\mathscr{O}_X}(F, G)), \tag{1.4.1}$$

$$\Gamma_Z(\underline{\mathrm{Hom}}_{\mathscr{O}_X}(F, G)) \simeq \mathrm{Hom}_{\mathscr{O}_X}(\mathscr{O}_{X,Z} \otimes_{\mathscr{O}_X} F, G), \tag{1.4.2}$$

$$\Gamma_Z(\underline{\mathrm{Hom}}_{\mathscr{O}_X}(F, G)) \simeq \mathrm{Hom}_{\mathscr{O}_X}(F, \underline{\mathrm{Hom}}_{\mathscr{O}_X}(\mathscr{O}_{X,Z}, G)) = \mathrm{Hom}_{\mathscr{O}_X}(F, \underline{\Gamma}_Z(G)). \tag{1.4.3}$$

It follows in particular from (1.4.2) that there exists a ∂-functorial isomorphism in F and G 13

$$\theta\colon \mathrm{Ext}^i_{\mathscr{O}_X}(\mathscr{O}_{X,Z} \otimes_{\mathscr{O}_X} F, G) \xrightarrow{\sim} \mathrm{Ext}^i_Z(X; F, G).$$

1.5. By definition the functor $G \mapsto \Gamma_Z(\underline{\mathrm{Hom}}_{\mathscr{O}_X}(F, G))$ is the composition of the functor $G \mapsto \underline{\mathrm{Hom}}_{\mathscr{O}_X}(F, G)$ and the functor Γ_Z. As Γ_Z is left exact (I 1.9) and as $\underline{\mathrm{Hom}}_{\mathscr{O}_X}(F, G)$ is flasque if G is injective, and Γ_Z is exact on flasque sheaves (I 2.12), it results from ([**Tôhoku**], 2.4.1) that there exists a spectral functor converging to $\mathrm{Ext}^{\bullet}_{Z}(X; F, G)$ and whose initial term is $H^p_Z(X, \underline{\mathrm{Ext}}^q_{\mathscr{O}_X}(F, G))$.

On the other hand, it results from (1.4.3) that $\Gamma_Z(\underline{\mathrm{Hom}}_{\mathscr{O}_X}(F, G))$ is the composition of $\underline{\Gamma}_Z$ and the functor $H \mapsto \mathrm{Hom}_{\mathscr{O}_X}(F, H)$.

Since the functor $\underline{\Gamma}_Z$ transforms injectives into injectives (I 1.4), it results from ([**Tôhoku**], 2.4.1) that there exists a spectral functor converging to $\mathrm{Ext}^{\bullet}_{Z}(X; F, G)$ and whose initial term is $\mathrm{Ext}^p_{\mathscr{O}_X}(X; F, \underline{H}^q_Z(G))$.

Finally, it results from (1.4.2) and the spectral sequence of Ext, that there exists a spectral functor converging to $\mathrm{Ext}^{\bullet}_{Z}(X; F, G)$ and whose initial term is $H^p(X; \underline{\mathrm{Ext}}^{\bullet}_{Z}(F, G))$. Whence the

Theorem 1.6. — *There exist three spectral functors converging to* $\mathrm{Ext}^{\bullet}_{Z}(X; F, G)$ *and whose initial terms are respectively*

$$H^p_Z(X, \underline{\mathrm{Ext}}^q_{\mathscr{O}_X}(F, G)) \tag{1.6.1}$$

$$H^p(X, \underline{\mathrm{Ext}}^q_Z(F, G)) \tag{1.6.2}$$

$$\mathrm{Ext}^p_{\mathscr{O}_X}(X; F, \underline{H}^q_Z(G)). \tag{1.6.3}$$

14 **1.7.** Now let Z' be a closed subset of Z and let $Z'' = Z - Z'$. We have an exact sequence

$$0 \longrightarrow \mathscr{O}_{X,Z''} \longrightarrow \mathscr{O}_{X,Z} \longrightarrow \mathscr{O}_{X,Z'} \longrightarrow 0 \tag{1.7.1}$$

which generalizes the exact sequence of ([**Godement**], 2.9.3). This exact sequence splits locally, we thus have, for any $\mathscr{O}_X$-Module F, another exact sequence :

$$0 \longrightarrow F \otimes_{\mathscr{O}_X} \mathscr{O}_{X,Z''} \longrightarrow F \otimes_{\mathscr{O}_X} \mathscr{O}_{X,Z} \longrightarrow F \otimes_{\mathscr{O}_X} \mathscr{O}_{X,Z'} \longrightarrow 0 \tag{1.7.2}$$

Now let G be an $\mathscr{O}_X$-Module; if we apply the functor $\mathrm{Hom}_{\mathscr{O}_X}(\bullet, G)$ to the exact sequence (1.7.2), we deduce from (1.4.2) and from the exact sequence of Ext the following theorem :

Theorem 1.8. — *Let* Z *be a locally closed subset of* X*,* Z' *a closed subset of* Z *and* $Z'' = Z - Z'$. *We then have a functorial exact sequence in* F *and* G *:*

$$\begin{gathered} 0 \longrightarrow \mathrm{Hom}_{Z'}(F, G) \longrightarrow \mathrm{Hom}_Z(F, G) \longrightarrow \mathrm{Hom}_{Z''}(F, G) \longrightarrow \mathrm{Ext}^1_{Z'}(F, G) \longrightarrow \cdots \\ \cdots \mathrm{Ext}^i_Z(F, G) \longrightarrow \mathrm{Ext}^i_{Z''}(F, G) \longrightarrow \mathrm{Ext}^{i+1}_{Z'}(F, G) \longrightarrow \cdots \end{gathered}$$

Corollary 1.9. — *Let* Y *be a closed subset of* X *and let* $U = X - Y$. *We then have a functorial exact sequence in* F *and* G *:*

$$\begin{gathered} 0 \longrightarrow \mathrm{Hom}_Y(F, G) \longrightarrow \mathrm{Hom}_{\mathscr{O}_X}(F, G) \longrightarrow \mathrm{Hom}_{\mathscr{O}_{X|U}}(F_{|U}, G_{|U}) \longrightarrow \mathrm{Ext}^1_Y(F, G) \longrightarrow \cdots \\ \cdots \mathrm{Ext}^i_{\mathscr{O}_X}(F, G) \longrightarrow \mathrm{Ext}^i_{\mathscr{O}_{X|U}}(F_{|U}, G_{|U}) \longrightarrow \mathrm{Ext}^{i+1}_Y(F, G) \longrightarrow \cdots \end{gathered}$$

This corollary is an immediate consequence of theorem (1.3) and theorem (1.8).

2. Applications to quasi-coherent sheaves on preschemes

15

Proposition 2.1. — *Let* X *be a locally noetherian prescheme; for any locally closed subset* Z *of* X*, for any coherent Module* F *and any quasi-coherent Module* G *on* X*, the* $\underline{\mathrm{Ext}}^i_{\mathrm{Z}}(\mathrm{F},\mathrm{G})$ *are quasi-coherent.*

It is shown, as for (1.6.3), that the Modules $\underline{\mathrm{Ext}}^i_{\mathrm{Z}}(\mathrm{F},\mathrm{G})$ are the abutment of a spectral sequence with initial term $\underline{\mathrm{Ext}}^p_{\mathscr{O}_{\mathrm{X}}}(\mathrm{F},\underline{\mathrm{H}}^q_{\mathrm{Z}}(\mathrm{G}))$. According to (II, corol. 3), the $\underline{\mathrm{H}}^q_{\mathrm{Z}}(\mathrm{G})$ are quasi-coherent, and thus so are the $\underline{\mathrm{Ext}}^p_{\mathscr{O}_{\mathrm{X}}}(\mathrm{F},\underline{\mathrm{H}}^q_{\mathrm{Z}}(\mathrm{G}))$, since F is coherent. The proposition then follows immediately.

2.2. Now let Y be a closed subprescheme of X and $\mathscr{I}$ an ideal of definition of Y. Let m and n be integers such that $m \geqslant n \geqslant 0$; we denote by $i_{n,m}$ the canonical map $\mathscr{O}_{\mathrm{Y}_m} = \mathscr{O}_{\mathrm{X}}/\mathscr{I}^{m+1} \to \mathscr{O}_{\mathrm{X}}/\mathscr{I}^{n+1} = \mathscr{O}_{\mathrm{Y}_n}$ and by j_n the map $\mathscr{O}_{\mathrm{X},\mathrm{Y}} \to \mathscr{O}_{\mathrm{Y}_n}$. The $(\mathscr{O}_{\mathrm{Y}_n}, i_{n,m})$ form a projective system and the j_n are compatible with the $i_{n,m}$.

By applying the functor $\mathrm{Ext}^i_{\mathscr{O}_{\mathrm{X}}}(\mathrm{F}\otimes\bullet,\mathrm{G})$, we deduce a morphism

$$\varphi'\colon \varinjlim_n \mathrm{Ext}^i_{\mathscr{O}_{\mathrm{X}}}(\mathrm{X};\mathrm{F}\otimes\mathscr{O}_{\mathrm{Y}_n},\mathrm{G}) \longrightarrow \mathrm{Ext}^i_{\mathscr{O}_{\mathrm{X}}}(\mathrm{X};\mathrm{F}\otimes\mathscr{O}_{\mathrm{X},\mathrm{Y}},\mathrm{G});$$

it is a morphism of cohomological functors in G. The morphism

$$\varphi\colon \varinjlim_n \mathrm{Ext}^i_{\mathscr{O}_{\mathrm{X}}}(\mathrm{X};\mathrm{F}\otimes\mathscr{O}_{\mathrm{Y}_n},\mathrm{G}) \longrightarrow \mathrm{Ext}^i_{\mathrm{Y}}(\mathrm{X};\mathrm{F},\mathrm{G})$$

composed of φ' and of θ (cf. 1.4) is therefore also a functor of cohomological morphisms in G.

We similarly define

$$\underline{\varphi}\colon \varinjlim_n \underline{\mathrm{Ext}}^i_{\mathscr{O}_{\mathrm{X}}}(\mathrm{F}\otimes\mathscr{O}_{\mathrm{Y}_n},\mathrm{G}) \longrightarrow \underline{\mathrm{Ext}}^i_{\mathrm{Y}}(\mathrm{X};\mathrm{F},\mathrm{G})$$

Theorem 2.3. — *Let* X *be a locally noetherian prescheme,* Y *a closed subset of* X 16
defined by a coherent ideal $\mathscr{I}$*,* F *a coherent Module,* G *a quasi-coherent Module. Then,*

a) $\underline{\varphi}$ *is an isomorphism.*

b) *If* X *is noetherian,* φ *is an isomorphism.*

The proof of b) being almost word for word that of (II 6b)), thanks to the spectral sequence 1.6.2, we will not reproduce it.

For the proof of a), we can, according to (2.1), assume X is affine with ring A, F (resp. G) defined by an A-module M (resp. N) and $\mathscr{I}$ by an ideal I. It suffices to prove that the homomorphism

$$\varinjlim_n \mathrm{Ext}^i_{\mathrm{A}}(\mathrm{M}/\mathrm{I}^n\mathrm{M},\mathrm{N}) \longrightarrow \mathrm{Ext}^i_{\mathrm{Y}}(\mathrm{X},\mathrm{F},\mathrm{G}) \tag{2.3.1}$$

deduced from $\underline{\varphi}$ is an isomorphism

Indeed, for $i = 0$, we can canonically identify the two members of (2.3.1) with the submodule of $\mathrm{Hom}_{\mathrm{A}}(\mathrm{M},\mathrm{N})$ defined by the elements of $\mathrm{Hom}_{\mathrm{A}}(\mathrm{M},\mathrm{N})$ annihilated by a power of I. We then see that the homomorphism (2.3.1) is none other than the identity map.

The functor $\mathrm{N} \mapsto \varinjlim_n \mathrm{Ext}^{\bullet}_{\mathrm{A}}(\mathrm{M}/\mathrm{I}^n\mathrm{M},\mathrm{N})$ is a universal ∂-functor. We will show that the same is true for the functor $\mathrm{N} \mapsto \mathrm{Ext}^{\bullet}_{\mathrm{Y}}(\mathrm{M},\mathrm{N})$. Indeed if N is an injective module, according to (9 and 11), $\underline{\mathrm{H}}^q_{\mathrm{Y}}(\mathrm{N}) = 0$ if $q \neq 0$; and according to (IV.2.2), $\mathrm{H}^0_{\mathrm{Y}}(\mathrm{N})$ is injective.

It then follows that $\mathrm{Ext}^p_{\mathscr{O}_{\mathrm{X}}}(\mathrm{X};\mathrm{M},\underline{\mathrm{H}}^q_{\mathrm{Y}}(\mathrm{N})) = 0$ if $p+q \neq 0$; we thus have, according to (1.6.3), $\mathrm{Ext}^i_{\mathrm{Y}}(\mathrm{M},\mathrm{N}) = 0$ for $i \neq 0$ and N injective. This completes the proof.

Bibliography

Same references as those listed at the end of Exp I, cited respectively as [**Tôhoku**] and [**Godement**].

EXPOSÉ VII

VANISHING CRITERIA, COHERENCE CONDITIONS FOR THE SHEAVES $\underline{\mathrm{Ext}}^i_{\mathrm{Y}}(\mathrm{F},\mathrm{G})$

1. Study for $i < n$

Let us prove a lemma : 17

Lemma 1.1. — *Let* X *be a locally Noetherian prescheme,* Y *a closed subset of* X*, and* G *a quasi-coherent* $\mathscr{O}_{\mathrm{X}}$*-Module. Suppose that for any coherent* $\mathscr{O}_{\mathrm{X}}$*-Module* F *with support contained in* Y*, we have :*

$$\underline{\mathrm{Ext}}^{n-1}(\mathrm{F},\mathrm{G}) = 0.$$

Then for any coherent $\mathscr{O}_{\mathrm{X}}$*-Module* F *and any closed subset* Z *of* X *such that* $\mathrm{Y} \supset \operatorname{Supp}\mathrm{F}\cap\mathrm{Z}$*, we have*

$$\underline{\mathrm{Ext}}^n_{\mathrm{Z}}(\mathrm{F},\mathrm{G}) \approx \underline{\mathrm{Hom}}(\mathrm{F},\underline{\mathrm{H}}^n_{\mathrm{Y}}(\mathrm{G})).$$

We first note that

$$\underline{\mathrm{Ext}}^i_{\mathrm{Z}}(\mathrm{F},\mathrm{G}) = \underline{\mathrm{Ext}}^i_{\mathrm{Z}\cap\operatorname{Supp}\mathrm{F}}(\mathrm{F},\mathrm{G}).$$

(trivial, cf. Exposé VI). We first give the proof for $\mathrm{Z}=\mathrm{X}$, hence $\operatorname{Supp}\mathrm{F}\subset\mathrm{Y}$. The functor

$$\mathrm{F}\longmapsto \underline{\mathrm{Ext}}^n(\mathrm{F},\mathrm{G}),$$

defined on the category of coherent $\mathscr{O}_{\mathrm{X}}$-Modules with support contained in Y, is left exact. By virtue of (IV 1.3), it is representable by

$$\mathrm{I} = \varinjlim_k \underline{\mathrm{Ext}}^n(\mathscr{O}_{\mathrm{X}}/\mathscr{I}^{k+1},\mathrm{G}),$$

where $\mathscr{I}$ is the ideal of definition of Y. Now, according to (II 6), we know that :

$$\underline{\mathrm{H}}^n_{\mathrm{Y}}(\mathrm{G}) \approx \varinjlim_k \underline{\mathrm{Ext}}^n(\mathscr{O}_{\mathrm{X}}/\mathscr{I}^{k+1},\mathrm{G}).$$

Hence the conclusion if $\mathrm{Z}=\mathrm{X}$. Again according to (VI 2.3), we know that : 18

$$\underline{\mathrm{Ext}}^n_{\mathrm{Z}}(\mathrm{F},\mathrm{G}) \approx \varinjlim_k \underline{\mathrm{Ext}}^n(\mathrm{F}/\mathscr{J}^{k+1}\mathrm{F},\mathrm{G}),$$

where $\mathscr{J}$ is the ideal of definition of Z. The support of $\mathrm{F}/\mathscr{J}^{k+1}\mathrm{F}$ is contained in Y if $\mathrm{Z}\cap\operatorname{Supp}\mathrm{F}\subset\mathrm{Y}$; according to what we have just shown, we thus have :

$$\underline{\mathrm{Ext}}^n_{\mathrm{Z}}(\mathrm{F},\mathrm{G})\approx\varinjlim_k\underline{\mathrm{Hom}}(\mathrm{F}/\mathscr{J}^{k+1}\mathrm{F},\underline{\mathrm{H}}^n_{\mathrm{Y}}(\mathrm{G})).$$

It remains to be shown that the natural homomorphism :

$$\varinjlim_k\underline{\mathrm{Hom}}(\mathrm{F}/\mathscr{J}^{k+1}\mathrm{F},\underline{\mathrm{H}}^n_{\mathrm{Y}}(\mathrm{G}))\longrightarrow\underline{\mathrm{Hom}}(\mathrm{F},\underline{\mathrm{H}}^n_{\mathrm{Y}}(\mathrm{G})),$$

is an isomorphism when $\mathrm{Z}\cap\operatorname{Supp}\mathrm{F}\subset\mathrm{Y}$. Now, X can be covered by Noetherian affine open sets ; we are thus reduced to the case where X is affine Noetherian ; then F(X) is a finitely generated $\mathscr{O}_{\mathrm{X}}(\mathrm{X})$-Module and $\operatorname{Supp}\underline{\mathrm{H}}^n_{\mathrm{Y}}(\mathrm{G})\subset\mathrm{Y}$. Thus any homomorphism $u:\mathrm{F}(\mathrm{X})\to\underline{\mathrm{H}}^n_{\mathrm{Y}}(\mathrm{G})(\mathrm{X})$ is annihilated by a power of $\mathscr{I}$ and therefore by a power of $\mathscr{J}$, Q.E.D.

Proposition 1.2. — *Let* X *be a locally Noetherian prescheme,* Y *a closed subset of* X*,* G *a quasi-coherent* $\mathscr{O}_{\mathrm{X}}$*-Module, and* n *an integer. For any closed subsets* Z *and* S *of* X *such that* $\mathrm{Z}\cap\mathrm{S}=\mathrm{Y}$*, the following conditions are equivalent :*

(i) $\underline{\mathrm{H}}^i_{\mathrm{Y}}(\mathrm{G})=0$ *if* $i<n$ *;*

(ii) *there exists a coherent* $\mathscr{O}_{\mathrm{X}}$*-Module* F*, with support* S*, such that :*

$$\underline{\mathrm{Ext}}^i_{\mathrm{Z}}(\mathrm{F},\mathrm{G})=0 \textit{ if } i<n;$$

(iii) *for any coherent* $\mathscr{O}_{\mathrm{X}}$*-Module* F *with support contained in* S (*i.e.,* $\operatorname{Supp}\mathrm{F}\cap\mathrm{Z}=\operatorname{Supp}\mathrm{F}\cap\mathrm{Y}$)*, we have :*

$$\underline{\mathrm{Ext}}^i_{\mathrm{Z}}(\mathrm{F},\mathrm{G})=0 \textit{ if } i<n;$$

19 (iv) *for any coherent* $\mathscr{O}_{\mathrm{X}}$*-Module* F*, we have :*

$$\underline{\mathrm{Ext}}^i_{\mathrm{Y}}(\mathrm{F},\mathrm{G})=0 \textit{ if } i<n.$$

Moreover, if they are satisfied, then for any coherent $\mathscr{O}_{\mathrm{X}}$*-Module F and any closed subset* Z' *of* X *such that* $\mathrm{Z}'\cap\operatorname{Supp}\mathrm{F}=\mathrm{Y}\cap\operatorname{Supp}\mathrm{F}$*, we have isomorphisms :*

$$\underline{\mathrm{Ext}}^n_{\mathrm{Z}}(\mathrm{F},\mathrm{G})\approx\underline{\mathrm{Ext}}^n_{\mathrm{Y}}(\mathrm{F},\mathrm{G})\approx\underline{\mathrm{Hom}}(\mathrm{F},\underline{\mathrm{H}}^n_{\mathrm{Y}}(\mathrm{G})).$$

Démonstration. — Let us argue by induction. The proposition is trivial for $n<0$. Suppose it is proven for $n<q$. If one of the conditions is satisfied for $n=q$, and for two subsets Z and S as stated, by the induction hypothesis we have, for any closed subset Z' of X and any coherent $\mathscr{O}_{\mathrm{X}}$-Module F such that $\mathrm{Z}'\cap\operatorname{Supp}\mathrm{F}=\mathrm{Y}\cap\operatorname{Supp}\mathrm{F}$, isomorphisms :

$$\underline{\mathrm{Ext}}^{q-1}_{\mathrm{Z}'}(\mathrm{F},\mathrm{G})\approx\underline{\mathrm{Hom}}(\mathrm{F},\underline{\mathrm{H}}^{q-1}_{\mathrm{Y}}(\mathrm{G}))\approx\underline{\mathrm{Ext}}^{q-1}_{\mathrm{Y}}(\mathrm{F},\mathrm{G}). \tag{1.1}$$

Therefore :

(i) $\Rightarrow$ (iv), because we take $\mathrm{Z}'=\mathrm{Y}$ in (1.1) ;

(iv) $\Rightarrow$ (iii), because we take $\mathrm{Z}'=\mathrm{Z}$ in (1.1) ;

(iii) $\Rightarrow$ (ii), because we take $\mathrm{F}=\mathscr{O}_{\mathrm{S}}$;

(ii) $\Rightarrow$ (i), because we take $Z' = Z$ in (1.1) ; hence $\underline{\mathrm{Hom}}(F, \underline{H}^{q-1}_Y(G)) = 0$; we then note that :

$$\operatorname{Supp} \underline{H}^{q-1}_Y(G) \subset Y = Z \cap S \subset S = \operatorname{Supp} F,$$

and we apply the following lemma :

Lemma 1.3. — *Let* X *be a prescheme, let* P *be a coherent* $\mathscr{O}_X$*-Module, and let* H *be a quasi-coherent* $\mathscr{O}_X$*-Module such that :*

$$\underline{\mathrm{Hom}}(P, H) = 0 \text{ and } \operatorname{Supp} P \supset \operatorname{Supp} H.$$

Then $H = 0$.

It suffices to prove the lemma when X is affine, because the affine open sets form a basis for the topology of X and the hypotheses are preserved by restriction to an open set. Now, in this case, we are reduced to a problem on A-modules, where $X = \operatorname{Spec}(A)$. We apply the formula (valid under the sole hypothesis that M is of finite type) : 20

$$\operatorname{Ass} \operatorname{Hom}_A(P, H) = \operatorname{Supp} P \cap \operatorname{Ass} H;$$

We know that $\operatorname{Ass} H \subset \operatorname{Supp} H \subset \operatorname{Supp} P$ and that $\operatorname{Ass} \operatorname{Hom}_A(P, H) = \emptyset$; therefore $\operatorname{Ass} H = \emptyset$, hence $H = 0$.

To complete the proof of the proposition, it remains to note that (iv) allows us to apply 1.1.

Corollary 1.4. — *Let* G *be a coherent Cohen-Macaulay* $\mathscr{O}_X$*-Module, let* $n \in \mathbf{Z}$. *The conditions of* 1.2 *are equivalent to :*

(v) $$\operatorname{codim}(Y \cap \operatorname{Supp} G, \operatorname{Supp} G) \geqslant n.$$

Let us first recall that an $\mathscr{O}_X$-module is said to be Cohen-Macaulay if, for any $x \in X$, the fiber G_x is a Cohen-Macaulay $\mathscr{O}_{X,x}$-module, i.e., we have for any $x \in S = \operatorname{Supp} G$:

(1.2) $$\operatorname{prof} G_x = \dim G_x = \dim \mathscr{O}_{S,x}.$$

According to proposition III 3.3, condition (i) of 1.2 is equivalent to :

(1.3) $$\operatorname{prof}_Y G = \inf_{x \in Y} \operatorname{prof} G_x \geqslant n,$$

and therefore also to :

$$\operatorname{prof}_Y G = \inf_{x \in Y \cap S} \operatorname{prof} G_x \geqslant n;$$

because the depth of a zero module is infinite.

Now, by definition :

$$\operatorname{codim}(Y \cap S, S) = \inf_{x \in S \cap Y} \dim \mathscr{O}_{S,x},$$

hence the conclusion, by applying formula (1.2)).

We will now prove a result that allows us to deduce the coherence conditions we are aiming for from certain vanishing criteria. 21

Lemma 1.5. — *Let* X *be a locally Noetherian prescheme. Let* T^* *be a contravariant exact* ∂*-functor, defined on the category of coherent* $\mathscr{O}_{\mathrm{X}}$*-Modules, with values in the category of* $\mathscr{O}_{\mathrm{X}}$*-Modules. Let* Y *be a closed subset of* X*. Let* $i \in \mathbf{Z}$*. Suppose that, for any coherent* $\mathscr{O}_{\mathrm{X}}$*-Module with support contained in* Y, $\mathrm{T}^i\mathrm{F}$ *and* $\mathrm{T}^{i-1}\mathrm{F}$ *are coherent. Let* F *be a coherent* $\mathscr{O}_{\mathrm{X}}$*-Module. For* $\mathrm{T}^i\mathrm{F}$ *to be coherent, it is necessary and sufficient that* $\mathrm{T}^i\mathrm{F}''$ *be coherent, where we have set :*

$$\mathrm{F}'' = \mathrm{F}/\Gamma_{\mathrm{Y}}(\mathrm{F}).$$

Indeed, $\mathrm{F}' = \Gamma_{\mathrm{Y}}(\mathrm{F})$ is coherent because X is locally Noetherian ; the exact cohomology sequence of T^* then gives :

$$\mathrm{T}^{i-1}\mathrm{F}' \longrightarrow \mathrm{T}^i\mathrm{F}'' \longrightarrow \mathrm{T}^i\mathrm{F} \longrightarrow \mathrm{T}^i\mathrm{F}'$$

where the outer terms are coherent, hence the conclusion.

Lemma 1.6. — *If* F *and* G *are coherent, and if* $\operatorname{Supp}\mathrm{F} \subset \mathrm{Y}$, $\underline{\mathrm{Ext}}^i_{\mathrm{Y}}(\mathrm{F},\mathrm{G})$ *is coherent.*

Indeed, $\underline{\mathrm{Ext}}^i_{\mathrm{Y}}(\mathrm{F},\mathrm{G})$ is isomorphic to $\underline{\mathrm{Ext}}^i(\mathrm{F},\mathrm{G})$; this holds on any ringed space X, moreover : if Z is a closed set containing $\mathrm{Y} \cap \operatorname{Supp}\mathrm{F}$, $\underline{\mathrm{Ext}}^i_{\mathrm{Z}}(\mathrm{F},\mathrm{G})$ is isomorphic to $\underline{\mathrm{Ext}}^i_{\mathrm{Y}}(\mathrm{F},\mathrm{G})$ (cf. Exposé VI).

Proposition 1.7. — *Suppose* F *and* G *are coherent and let* $\operatorname{Supp}\mathrm{F} = \mathrm{S}$, $\mathrm{S}' = \overline{\mathrm{S} \cap (\mathrm{X} - \mathrm{Y})}$*. Suppose that, for any* $x \in \mathrm{Y} \cap \mathrm{S}'$*, we have* $\operatorname{prof} \mathrm{G}_x \geqslant n$*, then* $\underline{\mathrm{Ext}}^i_{\mathrm{Y}}(\mathrm{F},\mathrm{G})$ *is coherent for* $i < n$*.*

Indeed, 1.6 allows us to apply 1.5 to $\mathrm{T}^*(\mathrm{F}) = \underline{\mathrm{Ext}}^*_{\mathrm{Y}}(\mathrm{F},\mathrm{G})$. Setting $\mathrm{F}'' = \mathrm{F}/\Gamma_{\mathrm{Y}}(\mathrm{F})$, we see that $\operatorname{Supp}\mathrm{F}'' = \mathrm{S}'$. Now, according to III 3.3, the hypothesis on the depth of G ensures the vanishing of $\mathrm{H}_{\mathrm{Y}\cap\mathrm{S}'}(\mathrm{G})$ for $i < n$; according to 1.2, we deduce the vanishing of $\mathrm{T}^i\mathrm{F}''$ for $i < n$, hence the conclusion thanks to 1.5.

2. Study for $i > n$

22 Let X be a *regular* locally Noetherian prescheme, that is, one whose local rings are all regular. Let Y be a closed subset of X. Let F and G be two coherent $\mathscr{O}_{\mathrm{X}}$-Modules. Let $\mathrm{S} = \operatorname{Supp}\mathrm{F}$, $\mathrm{S}' = \overline{\mathrm{S} \cap (\mathrm{X} - \mathrm{Y})}$. Let :

$$m = \sup_{x \in \mathrm{Y}\cap\mathrm{S}} \dim \mathscr{O}_{\mathrm{X},x},$$
$$n = \sup_{x \in \mathrm{Y}\cap\mathrm{S}'} \dim \mathscr{O}_{\mathrm{X},x};$$

we have $n \leqslant m$.

Proposition 2.1. — *In the situation described above, we have :*

(1) $\underline{\mathrm{Ext}}^i_{\mathrm{Y}}(\mathrm{F},\mathrm{G}) = 0$ *if* $i > m$*,*
(2) $\underline{\mathrm{Ext}}^i_{\mathrm{Y}}(\mathrm{F},\mathrm{G})$ *is coherent if* $i > n$*.*

Let us first note that $\underline{\mathrm{Ext}}^i_{\mathrm{Y}}(\mathrm{F},\mathrm{G})$ is coherent for all i when $\mathrm{Supp}\,\mathrm{F} \subset \mathrm{Y}$. Furthermore, by setting as above $\mathrm{F}'' = \mathrm{F}/\Gamma_{\mathrm{Y}}(\mathrm{F})$, we see that $\mathrm{Supp}\,\mathrm{F}'' = \mathrm{S}'$, so (2) results from (1) and 1.3.

To prove (1), we first note that

$$\underline{\mathrm{Ext}}^i_{\mathrm{Y}}(\mathrm{F},\mathrm{G}) \approx \varinjlim_k \underline{\mathrm{Ext}}^i(\mathrm{F}/\mathscr{J}^k\mathrm{F},\mathrm{G}),$$

where $\mathscr{J}$ is the ideal of definition of Y. Furthermore, it results from Theorem 4.2.2. of (A. Grothendieck, "On some points of homological algebra", *Tôhoku Mathematical Journal* **9** (1957), p. 119–221.) that the $\underline{\mathrm{Ext}}$ commute with the formation of fibers, at least when X is a locally Noetherian prescheme and when the first argument is coherent; as the same is true for inductive limits, we find isomorphisms :

$$(\underline{\mathrm{Ext}}^i_{\mathrm{Y}}(\mathrm{F},\mathrm{G}))_x \approx \varinjlim_k \mathrm{Ext}_{\mathscr{O}_{\mathrm{X},x}}((\mathrm{F}/\mathscr{J}^k\mathrm{F})_x,\mathrm{G}_x)$$

for all $x \in \mathrm{X}$. Since $\mathrm{Supp}\,\mathrm{Ext}^i_{\mathrm{Y}}(\mathrm{F},\mathrm{G}) \subset \mathrm{S}\cap\mathrm{Y}$, it suffices, to conclude, to note that 23
$x \in \mathrm{Y}\cap\mathrm{S}$ implies $\dim\mathscr{O}_{\mathrm{X},x} \leqslant m$, therefore :

$$\mathrm{Ext}^i_{\mathscr{O}_{\mathrm{X},x}}((\mathrm{F}/\mathscr{J}^k\mathrm{F})_x,\mathrm{G}_x) = 0 \text{ if } i > m,$$

because the global cohomological dimension of a regular local ring is equal to its dimension(*).

Let X be a locally Noetherian prescheme; for any subset P of X, let :

$$\mathrm{D}(\mathrm{P}) = \{\dim\mathscr{O}_{\mathrm{X},p} \mid p \in \mathrm{P}\}\,.$$

Lemma 2.2. — *If* P *is the underlying space of a connected sub-prescheme of* A*,* D(P) *is an interval.*

Indeed, let a and B belong to D(P), corresponding to points p and q of P. Let us show that there exists a sequence of points in P : $(p = p_1,\ldots,p_n = q)$ such that, for $1 \leqslant i < n$, we have $\left|\dim\mathscr{O}_{\mathrm{X},p_i} - \dim\mathscr{O}_{\mathrm{X},p_{i+1}}\right| = 1$; it will follow that D(P) contains the interval $[p,q]$. To do this, we note that p and q can be joined by a sequence of irreducible components of P, such that two successive components intersect. We are reduced to the case where p is the generic point of an irreducible component Q of P, and where $q \in \mathrm{Q}$ and thus $q \supset p$, as ideals of $\mathscr{O}_q$, where it is trivial by the definition of dimension.

Proposition 2.3. — *Let* X *be a regular locally Noetherian prescheme,* Y *a closed subset of* X*, and* F *a coherent* $\mathscr{O}_{\mathrm{X}}$*-Module. Let* $\mathrm{P} = \mathrm{Y}\cap\mathrm{Supp}\,\mathrm{F}\cap(\mathrm{X}-\mathrm{Y})$*. Let* $n \in \mathbf{Z}$*, and suppose that* $n \notin \mathrm{D}(\mathrm{P})$*. Then* $\underline{\mathrm{Ext}}^n_{\mathrm{Y}}(\mathrm{F},\mathscr{O}_{\mathrm{X}})$ *is coherent.*

The conclusion is local and the hypotheses are preserved by restriction to an open set. Now P is closed, hence locally Noetherian, hence locally connected; we can therefore assume X is affine and Noetherian, and P is connected. Let $\mathrm{D}(\mathrm{P}) = [a,b[$, which is

(*) Cf. EGA 0_{IV} 17.3.1

permissible according to the preceding lemma ; if $n > b$, we conclude by 2.1 ; if $n < a$, we have $n < \dim \mathscr{O}_{\mathrm{X},x} = \operatorname{prof} \mathscr{O}_{\mathrm{X},x}$ for all $x \in \mathrm{P}$, and we conclude by 1.7.

EXPOSÉ VIII

THE FINITENESS THEOREM

1. A biduality spectral sequence[(*)]

Let us state the result we wish to arrive at : **24**

Proposition 1.1. — *Let* A *be a Noetherian ring and let* I *be an ideal of* A. *Let* $\mathrm{X} = \mathrm{Spec}(\mathrm{A})$ *and* $\mathrm{Y} = \mathrm{V}(\mathrm{I})$. *Let* M *be an* A*-module* of finite type and finite projective dimension. *Let* $\mathrm{F} = \widetilde{\mathrm{M}}$ *be the* $\mathscr{O}_{\mathrm{X}}$*-Module associated with* M.

1) *There exists a spectral sequence :*

$$\mathrm{H}_{\mathrm{Y}}(\mathrm{X}, \mathrm{F}) \Longleftarrow \mathrm{Ext}^{p}_{\mathrm{Y}}(\mathrm{Ext}^{-q}(\mathrm{M}, \mathrm{A}), \mathrm{A}).$$

2) *There exists a spectral sequence :*

$$\underline{\mathrm{H}}_{\mathrm{Y}}(\mathrm{X}, \mathrm{F}) \Longleftarrow \underline{\mathrm{Ext}}^{p}_{\mathrm{Y}}(\underline{\mathrm{Ext}}^{-q}(\mathrm{F}, \mathscr{O}_{\mathrm{X}}), \mathscr{O}_{\mathrm{X}}).$$

Of course, 2) is deduced from 1) by noting that, if M and N are two finitely generated A-modules and if we set $\mathrm{F} = \widetilde{\mathrm{M}}$ and $\mathrm{G} = \widetilde{\mathrm{N}}$, we have isomorphisms :

$$\underline{\mathrm{H}}_{\mathrm{Y}}(\mathrm{F}) \approx \mathrm{H}_{\mathrm{Y}}\widetilde{(\mathrm{X}, \mathrm{F})},$$

$$\underline{\mathrm{Ext}}_{\mathrm{Y}}(\mathrm{F}, \mathrm{G}) \approx \mathrm{Ext}_{\mathrm{Y}}\widetilde{(\mathrm{F}, \mathrm{G})},$$

$$\underline{\mathrm{Ext}}_{\mathscr{O}_{\mathrm{X}}}(\mathrm{F}, \mathrm{G}) \approx \mathrm{Ext}_{\mathrm{A}}\widetilde{(\mathrm{M}, \mathrm{N})}.$$

Let $\underline{\mathrm{C}}$ be the category of A-modules, and $\underline{\mathrm{Ab}}$ that of abelian groups. Let $\underline{\mathrm{F}}$ be the functor :

$$\underline{\mathrm{F}} : \underline{\mathrm{C}} \longrightarrow \underline{\mathrm{Ab}} \quad \text{defined by}$$

$$\mathrm{M} \longmapsto \Gamma_{\mathrm{Y}}(\widetilde{\mathrm{M}}).$$

We know since Exposé II that there exists an isomorphism of ∂-functors : **25**

$$\mathrm{H}^{*}_{\mathrm{Y}}(\mathrm{X}, \widetilde{\mathrm{M}}) \approx \mathrm{R}^{*}\,\underline{\mathrm{F}}(\mathrm{M}).$$

[(*)] The reader familiar with the language of Verdier's derived categories will recognize the spectral sequence associated with a *biduality isomorphism*. Cf. SGA 6 I.

Moreover, let $\mathrm{Ext}^*_{\mathrm{Y}}$ be the right derived functors in the second argument of

$$\underline{\mathrm{F}} \circ \mathrm{Hom} \colon \underline{\mathrm{C}}^\circ \times \underline{\mathrm{C}} \longrightarrow \underline{\mathrm{Ab}}.$$

We know since Exposé VI that we have an isomorphism of ∂-functors :

$$\mathrm{Ext}^*_{\mathrm{Y}}(\mathrm{M},\mathrm{N}) \simeq \mathrm{Ext}^*_{\mathrm{Y}}(\widetilde{\mathrm{M}},\widetilde{\mathrm{N}}).$$

Finally, let us retain the following result from Exposé VI : if C is an injective A-module and if N is a finitely generated A-module, the sheaf $\underline{\mathrm{Hom}}(\widetilde{\mathrm{N}},\widetilde{\mathrm{C}}) \approx \widetilde{\mathrm{Hom}(\mathrm{N},\mathrm{C})}$ is *flasque*, hence $\mathrm{R}^1\,\underline{\mathrm{F}}(\mathrm{Hom}(\mathrm{N},\mathrm{C})) = 0$.

It remains for us to prove the following result :

Lemma 1.2. — *Let* A *be a Noetherian ring and let* $\underline{\mathrm{C}}$ *be the category of* A*-modules. Let* $\underline{\mathrm{F}} \colon \underline{\mathrm{C}} \to \underline{\mathrm{Ab}}$ *be a left exact additive functor such that, for any finitely generated* A*-module* N *and any* injective A*-module* C*, we have* $\mathrm{R}^1\,\underline{\mathrm{F}}(\mathrm{Hom}(\mathrm{N},\mathrm{C})) = 0$*. Let* M *be a finitely generated* A*-module of finite projective dimension. There exists a spectral sequence :*

$$\mathrm{R}^*\,\underline{\mathrm{F}}(\mathrm{M}) \Longleftarrow \mathrm{Ext}^p_{\underline{\mathrm{F}}}(\mathrm{Ext}^{-q}(\mathrm{M},\mathrm{A}),\mathrm{A}),$$

where $\mathrm{Ext}^p_{\underline{\mathrm{F}}}$ *denotes the* p*-th right derived functor of* $\underline{\mathrm{F}} \circ \mathrm{Hom}$.

We will only consider complexes whose differential is of degree $+1$. According to the hypothesis made on M, there exists a finite length *projective* resolution of M :

$$u \colon \mathrm{L}^\bullet \longrightarrow \mathrm{M},$$

where, moreover, the L^p are *finitely generated* modules and $\mathrm{L}^p = 0$ if $p \notin [-n, 0]$.
26 Furthermore, let $v \colon \mathrm{M} \to \mathrm{I}^\bullet$ be an injective resolution of M. I claim that

$$v \circ u \colon \mathrm{L}^\bullet \longrightarrow \mathrm{I}^\bullet \tag{1.1}$$

is an injective resolution of $\mathrm{L}^\bullet$. It is necessary to clarify what this means.

Definition 1.3. — Let $\mathrm{X}^\bullet$ be a complex of A-modules ; an injective resolution of $\mathrm{X}^\bullet$ is a homomorphism of complexes :

$$x \colon \mathrm{X}^\bullet \longrightarrow \mathrm{CX}^\bullet,$$

such that CX^p is injective for all $p \in \mathbf{Z}$, and such that x induces an isomorphism in homology.

Proposition 1.4. — *Any complex bounded on the left, i.e., such that there exists* $q \in \mathbf{Z}$ *with* $\mathrm{X}^p = 0$ *for* $p < q$*, admits an injective resolution. Moreover, if* $u \colon \mathrm{X}^\bullet \to \mathrm{Y}^\bullet$ *is a homomorphism of (left-bounded) complexes and if* $x \colon \mathrm{X}^\bullet \to \mathrm{CX}^\bullet$ *and* $y \colon \mathrm{Y}^\bullet \to \mathrm{CY}^\bullet$ *are injective resolutions of* $\mathrm{X}^\bullet$ *and* $\mathrm{Y}^\bullet$*, there exists a homomorphism of complexes :*

$$\mathrm{C}u \colon \mathrm{CX}^\bullet \longrightarrow \mathrm{CY}^\bullet,$$

unique up to homotopy, such that the diagram :

$$\begin{tikzcd} X^\bullet \arrow[r, "x"] \arrow[d, "u"'] & CX^\bullet \arrow[d, "Cu"] \\ Y^\bullet \arrow[r, "y"] & CY^\bullet \end{tikzcd}$$

is commutative up to homotopy.

The proof is left to the reader[(*)].

Let us recall a notation introduced in Exposé V.

Notation. — Let $X^\bullet$ and $Y^\bullet$ be two complexes. We denote by $\mathrm{Hom}^\bullet(X^\bullet, Y^\bullet)$ the 27
simple complex whose component of degree n is

$$(\mathrm{Hom}^\bullet(X^\bullet, Y^\bullet))^n = \prod_{-p+q=n} \mathrm{Hom}(X^p, Y^q)$$

also denoted $\mathrm{Hom}^n(X^\bullet, Y^\bullet)$, and whose differential is given by :

$$\partial_n \colon \mathrm{Hom}^n(X^\bullet, Y^\bullet) \longrightarrow \mathrm{Hom}^{n+1}(X^\bullet, Y^\bullet)$$
$$\partial_n = d' + (-1)^{n+1} d'',$$

where d' and d'' are the (degree +1) differentials induced by those of $X^\bullet$ and $Y^\bullet$.

Let then $A^\bullet$ be the complex defined by $A^p = 0$ if $p \neq 0$ and $A^0 = A$. Let

$$a \colon A^\bullet \longrightarrow CA^\bullet$$

be an injective resolution of $A^\bullet$. Consider the double complex :

$$Q^{p,q} = \mathrm{Hom}(\mathrm{Hom}(L^{-q}, A), CA^p). \tag{1.2}$$

The first spectral sequence of the bicomplex $\underline{F}Q^{\bullet\bullet}$ will give the conclusion of lemma 1.2.

Let

$$L'^\bullet = \mathrm{Hom}^\bullet(L^\bullet, A^\bullet), \tag{1.3}$$

and

$$P^\bullet = \mathrm{Hom}^\bullet(L'^\bullet, CA^\bullet). \tag{1.4}$$

One easily sees that $P^\bullet$ is the simple complex associated with $Q^{\bullet\bullet}$. Let us calculate the abutment of the spectral sequence, i.e., the homology of $\underline{F}P^\bullet$. For this, using the fact that $L^\bullet$ is projective of finite type in every dimension, one proves that $L^\bullet$ is isomorphic to $\mathrm{Hom}^\bullet(L'^\bullet, A^\bullet)$. From the homomorphism $a \colon A^\bullet \to CA^\bullet$, we deduce a 28
homomorphism :

$$b \colon \mathrm{Hom}^\bullet(L'^\bullet, A^\bullet) \longrightarrow \mathrm{Hom}^\bullet(L'^\bullet, CA^\bullet),$$

[(*)] Cf. also H. Cartan & S. Eilenberg, *Homological Algebra*, Princeton Math. Series, vol. 19, Princeton University Press, 1956.

or, a homomorphism

$$c\colon \mathrm{L}^{\bullet} \longrightarrow \mathrm{P}^{\bullet}. \tag{1.5}$$

That said, it is easy to see, using the fact that $\mathrm{L}'^{\bullet}$ is projective of finite type in every dimension and bounded on the left, that (1.5) is an injective resolution of $\mathrm{L}^{\bullet}$. Using proposition 1.4., we conclude that $\mathrm{P}^{\bullet}$ is homotopically equivalent to $\mathrm{I}^{\bullet}$, where $\mathrm{I}^{\bullet}$ is the injective resolution of M introduced above (1.1). We deduce that the *abutment* of the first spectral sequence of the double complex $\underline{\mathrm{F}}\mathrm{Q}^{\bullet\bullet}$, which is $\mathrm{H}^*(\underline{\mathrm{F}}\mathrm{P}^{\bullet})$, *is isomorphic* to $\mathrm{R}^*\,\underline{\mathrm{F}}(\mathrm{M})$.

The initial term of the first spectral sequence of the bicomplex $\underline{\mathrm{F}}\mathrm{Q}^{\bullet\bullet}$ is :

$$\mathrm{E}_2^{p,q} = {}'\mathrm{H}^p({}''\mathrm{H}^q(\underline{\mathrm{F}}\mathrm{Q}^{\bullet\bullet})).$$

For any $p \in \mathbf{Z}$, CA^p is injective. According to the hypothesis made on $\underline{\mathrm{F}}$, the functor (restricted to the category of finitely generated modules) :

$$\mathrm{N} \longmapsto \underline{\mathrm{F}}\,\mathrm{Hom}(\mathrm{N}, \mathrm{CA}^p)$$

is exact. From which we deduce isomorphisms :

$${}''\mathrm{H}^q(\underline{\mathrm{F}}\,\mathrm{Hom}(\mathrm{L}'^{\bullet}, \mathrm{CA}^p)) \approx \underline{\mathrm{F}}\,\mathrm{Hom}(\mathrm{H}^{-q}(\mathrm{L}'^{\bullet}), \mathrm{CA}^p).$$

According to the definition of $\mathrm{Ext}^*_{\underline{\mathrm{F}}}$ as a derived functor of $\underline{\mathrm{F}}\circ\mathrm{Hom}$, we deduce isomorphisms :

$$\mathrm{E}_2^{p,q} \approx \mathrm{Ext}^p_{\underline{\mathrm{F}}}(\mathrm{H}^q(\mathrm{L}'^{\bullet}), \mathrm{A}).$$

Now $\mathrm{L}'^{\bullet} = \mathrm{Hom}^{\bullet}(\mathrm{L}^{\bullet}, \mathrm{A}^{\bullet})$, where $\mathrm{L}^{\bullet}$ is a projective resolution of M, hence isomorphisms :

$$\mathrm{Ext}^q(\mathrm{M}, \mathrm{A}) \approx \mathrm{H}^q(\mathrm{L}'^{\bullet}),$$

which gives the conclusion. C.Q.F.D.

29

2. The Finiteness Theorem

Theorem 2.1[(1)]. — *Let* X *be a locally Noetherian prescheme,* Y *a closed subset of* X*, and* F *a coherent* $\mathscr{O}_{\mathrm{X}}$*-Module. Suppose that* X *is locally immersible in a regular prescheme*[(*)]. *Let* $i \in \mathbf{Z}$. *Suppose that :*

a) *for any* $x \in \mathrm{U} = \mathrm{X} - \mathrm{Y}$*, we have*

$$\mathrm{H}^{i-c(x)}(\mathrm{F}_x) = 0,$$

(*)This condition can be generalized to the hypothesis of the local existence on X of a "dualizing complex", in the sense defined in R. Hartshorne, *Residues and duality* (cited in note (∗) of Exp. IV p. 49).

(1)N.D.E. : for an analogous statement, but in a slightly more general situation, see Mrs. Raynaud (Raynaud M., « Lefschetz theorems in cohomology of coherent sheaves and in étale cohomology. Application to the fundamental group », *Ann. Sci. Éc. Norm. Sup.* (*4*) **7** (1974), p. 29–52, proposition II.2.1).

where we have set[2] *:*

$$c(x) = \operatorname{codim}(\overline{\{x\}} \cap \mathrm{Y}, \overline{\{x\}}). \tag{2.1}$$

Then :

b) $\underline{\mathrm{H}}^i_{\mathrm{Y}}(\mathrm{F})$ *is coherent.*

Corollary 2.2[3]. — *Under the hypotheses of the preceding theorem, condition* a) *is equivalent to :*

c) *for any* $x \in \mathrm{U}$ *such that* $c(x) = 1$*, we have* $\mathrm{H}^{i-1}(\mathrm{F}_x) = 0$.

Corollary 2.3. — *Let* X *be a locally Noetherian prescheme locally immersible in a regular prescheme,* Y *a closed subset of* X*,* F *a coherent* $\mathscr{O}_{\mathrm{X}}$*-Module,* n *an integer. The following conditions are equivalent :*

(i) *for any* $x \in \mathrm{U}$*, we have* $\operatorname{prof} \mathrm{F}_x > n - c(x)$ *;*
(ii) *for any* $x \in \mathrm{U}$ *such that* $c(x) = 1$*, we have* $\operatorname{prof} \mathrm{F}_x \geqslant n$ *;*
(iii) *for any* $i \in \mathbf{Z}$*,* $\underline{\mathrm{H}}^i_{\mathrm{Y}}(\mathrm{F})$ *is coherent if* $i \leqslant n$.
(iv) $\mathrm{R}^i\, i_*(\mathrm{F}_{|\mathrm{U}})$ *is coherent for* $i < n$ [4].

Suppose these results are established when X is the spectrum of a regular Noetherian ring A and when F is the sheaf associated with an A-module of finite projective dimension.

Let us first note that, if $(\mathrm{X}_j)_{j\in \mathrm{J}}$ is an open cover of X by open sets immersible in a regular scheme, each of the conditions above is equivalent to the conjunction of the analogous conditions obtained by replacing X by X_j, Y by $\mathrm{Y}_j = \mathrm{Y} \cap \mathrm{X}_j$, and F by $\mathrm{F}_{|}\mathrm{X}_j$. Indeed, only the conditions involving $c(x)$ might pose a difficulty. Let $x \in \mathrm{U}$. If $x \in \mathrm{X}_j$, let us set **30**

$$c_j(x) = \operatorname{codim}(\mathrm{X}_j \cap \overline{\{x\}} \cap \mathrm{Y}, \mathrm{X}_j \cap \overline{\{x\}}),$$

we necessarily have $c_j(x) \geqslant c(x)$. Let $y \in \overline{\{x\}} \cap \mathrm{Y}$, which "gives the codimension", i.e., such that $c(x) = \dim \mathscr{O}_{\overline{\{x\}},y}$, let X_j be an open set of the cover such that $y \in \mathrm{X}_j$, then $x \in \mathrm{X}_j$, so $c_j(x) = c(x)$, which allows us to conclude that a) for the X_j implies

[2]N.D.E. : as in Exposé V, $\mathrm{H}^*(\mathrm{F}_x)$ denotes the local cohomology $\mathrm{H}_{\mathfrak{m}_x}(\mathrm{F}_x)$.

[3]N.D.E. : *strictly speaking*, this is a corollary of the following proof and not of the statement. The implication c) ⇒ a) is tautological. The other direction is not, but results from the proof. Let us be more precise. As below, we cover X with open sets immersible in regular schemes, which allows, as explained below, to reduce to the case where $\mathrm{X} = \operatorname{Spec}(\mathrm{A})$ is regular affine and $\mathrm{F} = \widetilde{\mathrm{M}}$ where M is an A-module of finite projective dimension. It is shown in this case that conditions a) and c) are equivalent to the dual conditions a′) and c′). We then show that c′) implies question d) (see *infra*) which itself implies a′). See the considerations following 2.4.

[4]N.D.E. : this condition was only in the body of the proof but not in the statement of the corollary ; since it is used in 3, we have added it.

a) for X. At this stage, we only have a partial converse, namely that a) for X implies a) for the X_j such that $c(x) = c_j(x)$, which is sufficient for our purpose.[(5)]

We choose a cover of X by open sets immersible in a regular prescheme. Applying the preceding, we see that we can assume X is closed in a regular X′. The reduction to X′ is then immediate.

We can therefore assume X is regular, and even affine by covering X with affine open sets. That we can assume $F = \widetilde{M}$, where M is of finite projective dimension will result from the following lemma :

Lemma 2.4. — *Let* X *be a regular Noetherian prescheme. Let* F *be a coherent* $\mathscr{O}_X$*-Module. The function that to each* $x \in X$ *associates the projective dimension of* F_x *is bounded above.*

Indeed, let $x \in X$ and let U be an affine open neighborhood of x. Let $L^\bullet$ be a projective resolution of the module F(U), where the L^i are of finite type. By hypothesis, the ring $\mathscr{O}_{X,x}$ is regular, so the projective dimension of F_x is finite ; let d be this integer. Let

$$K = \ker(L^{-d} \longrightarrow L^{-d+1}).$$

The module K_x is free, because d is the projective dimension of F_x ([**M**], Ch. VI. Prop. 2.1). According to (EGA 0_I 5.4.1 Errata), we deduce that the $\mathscr{O}_U$-Module $\widetilde{K}$ is
31 free on a neighborhood U′ of x, $U' \subset U$. Choosing $f \in \mathscr{O}_X(U)$ such that $x \in D(f) \subset U'$, we thus have a projective resolution of $M_{f'}$ $(M = F(U))$:

$$0 \longrightarrow K_f \longrightarrow (L^{d-1})_f \longrightarrow \cdots \longrightarrow M_f \longrightarrow 0,$$

which proves that the studied function is upper semi-continuous. Now X is quasi-compact, hence the conclusion.

We now assume X is affine, Noetherian, and regular, and we assume that $F = \widetilde{M}$, where M is a finitely generated A-module, necessarily of finite projective dimension. We will proceed in several steps. First, we find a condition d), equivalent to a), and prove that it is also equivalent to c). Then, with the help of the spectral sequence of the preceding section, we prove d) ⇒ b). It then remains to prove that (iii) ⇒ (ii) ; indeed, (i) ⇔ (ii) ⇒ (iii) results immediately from a) ⇔ c) ⇒ b).

Let $x \in U$, by hypothesis $\mathscr{O}_{X,x}$ is a regular local ring ; denoting by D the relative dualizing functor for the local ring $\mathscr{O}_{X,x}$, it follows from (V 2.1) that :

$$D\,H^{i-c(x)}(F_x) \approx \operatorname{Ext}^{d(x)-i}_{\mathscr{O}_{X,x}}(F_x, \mathscr{O}_{X,x}),$$

(5) N.D.E. : in fact, a) for X implies a) for all the X_j as stated in the original text, but to see this one must read the following proof in detail. This implication does not seem formal at this stage. Let us denote by an index J the conjunctions of a property a), b) or c) for the X_j. It is shown in the proof *infra* c_J) ⇒ a_J) (this is the sequence of implications c′) ⇒ d) ⇒ a′)). Now, we have tautologically a) ⇒ c), and c) ⇔ c_J), hence a) ⇒ a_J).

where we have set

$$d(x) = \dim \mathscr{O}_{\mathrm{X},x} + c(x) = \dim \mathscr{O}_{\mathrm{X},x} + \operatorname{codim}(\overline{\{x\}} \cap \mathrm{Y}, \overline{\{x\}}). \tag{2.2}$$

Now X is Noetherian and F is coherent, so :

$$\mathrm{D}\,\mathrm{H}^{i-c(x)}(\mathrm{F}_x) \approx (\underline{\mathrm{Ext}}^{d(x)-i}_{\mathscr{O}_\mathrm{X}}(\mathrm{F}, \mathscr{O}_\mathrm{X}))_x. \tag{2.3}$$

Moreover, for a module to be zero, it is necessary and sufficient that its dual be zero (cf. the editor's note (4) on page 58). For any $q \in \mathbf{Z}$, let us set : 32

$$\begin{cases} \mathrm{S}_q = \operatorname{Supp} \underline{\mathrm{Ext}}^q_{\mathscr{O}_\mathrm{X}}(\mathrm{F}, \mathscr{O}_\mathrm{X}), \\ \mathrm{S}'_q = \mathrm{S}_q \cap \mathrm{U},\ (\mathrm{U} = \mathrm{X} - \mathrm{Y}), \\ \mathrm{Z}_q = \overline{\mathrm{S}'_q} \cap \mathrm{Y}. \end{cases} \tag{2.4}$$

From formula (2.3), it follows that a) and c) are respectively equivalent to

a′) for any $q \in \mathbf{Z}$ and any $x \in \mathrm{S}'_q$, we have $q + i \neq d(x)$.
c′) for any $q \in \mathbf{Z}$ and any $x \in \mathrm{S}'_q$, if $c(x) = 1$, we have $q + i \neq d(x)$.

Here is the promised condition d) :

d) for any $q \in \mathbf{Z}$ and any $y \in \mathrm{Z}_q$, we have $q + i \neq \dim \mathscr{O}_{\mathrm{X},y}$.

These conditions are equivalent :

a′) ⇒ c′) for the record.

d) ⇒ a′). Indeed, let $q \in \mathbf{Z}$ and let $x \in \mathrm{S}'_q$; let $y \in \overline{\{x\}} \cap \mathrm{Y}$ which[(6)] "gives the codimension", i.e., such that :

$$\dim \mathscr{O}_{\overline{\{x\}},y} = \operatorname{codim}(\overline{\{x\}} \cap \mathrm{Y}, \overline{\{x\}}) = c(x). \tag{2.5}$$

From the fact that X is regular at y, we deduce :

$$\dim \mathscr{O}_{\mathrm{X},y} = d(x) \quad (\text{cf. } (2.2)). \tag{2.6}$$

But $y \in \overline{\{x\}}$, so $y \in \mathrm{Z}_q$, hence the conclusion.

c′) ⇒ d). Let $q \in \mathbf{Z}$ and let $y \in \mathrm{Z}_q$. Let us provisionally admit that there exists $x \in \mathrm{S}'_q$ such that :

$$y \in \overline{\{x\}} \text{ and } \dim \mathscr{O}_{\overline{\{x\}},y} = 1\ ;$$

(we also say that x *follows* y). It follows that $c(x) = 1$, because y "gives the codimension of $\overline{\{x\}} \cap \mathrm{Y}$ in $\overline{\{x\}}$", since $x \notin \mathrm{Y}$. According to c′) we deduce

$$q + i \neq d(x).$$

Hence the conclusion, if we note that $d(x) = \dim \mathscr{O}_{\mathrm{X},y}$ (2.6). The admitted result is expressed in the following lemma : 33

Lemma 2.5. — *Let* X *be a locally Noetherian prescheme and let* Y *be a closed subset of* X. *Let* $\mathrm{U} = \mathrm{X} - \mathrm{Y}$ *and suppose that* U *is dense in* X. *For any* $y \in \mathrm{Y}$, *there exists*

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(6) N.D.E. : in all that follows, the closures of points are given the reduced structure.

$x \in \mathrm{U}$ *"which follows it", i.e., such that :*

$$y \in \overline{\{x\}} \text{ and } \dim \mathscr{O}_{\overline{\{x\}},y} = 1.$$

We have applied the lemma by taking for X the prescheme $\overline{\mathrm{S}'_q}$ and for Y the subset $\mathrm{Y} \cap \overline{\mathrm{S}'_q}$.

Proof of 2.5. — There exists $x \in \mathrm{U}$ such that $y \in \overline{\{x\}}$; let us therefore choose an $x \in \mathrm{U}$ such that $y \in \overline{\{x\}}$ and such that $\dim \mathscr{O}_{\overline{\{x\}},y} = r$ is minimal. We must prove that $r = 1$. Since we have chosen x such that any $z \in \mathrm{Spec}(\mathscr{O}_{\overline{\{x\}},y})$, $z \neq x$, is in Y, $\{x\}$ is open in $\mathrm{Spec}(\mathscr{O}_{\overline{\{x\}},y})$. Hence the conclusion.

The second step consists in deducing b) *from* d).
Let $\mathrm{D}(\mathrm{Z}_q) = \{\dim \mathscr{O}_{\mathrm{X},y} \mid y \in \mathrm{Z}_q\}$. According to d), we know that, for any $q \in \mathbf{Z}$, we have $q + i \notin \mathrm{D}(\mathrm{Z}_q)$. We then apply VII.2.3, and we see that

$$\underline{\mathrm{Ext}}^{q+i}_{\mathrm{Y}}(\underline{\mathrm{Ext}}^q(\mathrm{F}, \mathscr{O}_{\mathrm{X}}), \mathscr{O}_{\mathrm{X}}) \text{ is coherent.}$$

The initial term of the spectral sequence of the preceding section is given by :

$$\mathrm{E}_2^{p,q} = \underline{\mathrm{Ext}}^p_{\mathrm{Y}}(\underline{\mathrm{Ext}}^{-q}(\mathrm{F}, \mathscr{O}_{\mathrm{X}}), \mathscr{O}_{\mathrm{X}}).$$

We deduce that $\mathrm{E}_2^{p,q}$ is coherent for all $p \in \mathbf{Z}$ and all $q \in \mathbf{Z}$ such that $p + q = i$. Now there is only a finite number of pairs (p, q) such that $p+q = i$, and this spectral sequence converges to $\underline{\mathrm{H}}^*_{\mathrm{Y}}(\mathrm{F})$, hence the conclusion.

It remains for us to prove that (iii) $\Rightarrow$ (ii). Let :

$$i \colon \mathrm{U} \longrightarrow \mathrm{X}$$

be the canonical immersion of U into X. Taking into account the exact homology sequence of the closed set Y (I 2.11) we see that (iii) *is equivalent to* :

(iv) $\mathrm{R}^i\, i_*(\mathrm{F}_{|\mathrm{U}})$ is coherent for $i < n$.

34 Indeed, we have an exact sequence :

$$0 \longrightarrow \underline{\mathrm{H}}^0_{\mathrm{Y}}(\mathrm{F}) \longrightarrow \mathrm{F} \longrightarrow i_*(\mathrm{F}_{|\mathrm{U}}) \longrightarrow \underline{\mathrm{H}}^1_{\mathrm{Y}}(\mathrm{F}) \longrightarrow 0.$$

Now $\underline{\mathrm{H}}^0_{\mathrm{Y}}(\mathrm{F})$ is a quasi-coherent subsheaf of F which is coherent, so it is coherent. Therefore $\underline{\mathrm{H}}^1_{\mathrm{Y}}(\mathrm{F})$ is coherent if and only if $i_*(\mathrm{F}_{|\mathrm{U}})$ is. Furthermore, if $p > 0$, the exact cohomology sequence of the closed set Y reduces to isomorphisms :

$$\mathrm{R}^p\, i_*(\mathrm{F}_{|\mathrm{U}}) \xrightarrow{\approx} \underline{\mathrm{H}}^{p+1}_{\mathrm{Y}}(\mathrm{F}).$$

We will show that (iv) $\Rightarrow$ (ii). For this, let us recall (ii) :

(ii) for any $x \in \mathrm{U}$ such that $c(x) = 1$, we have $\operatorname{prof} \mathrm{F}_x \geqslant n$.

Let us argue by induction on n.

If $n = 0$, both conditions are empty.

If $n = 1$, we suppose that $i_*(\mathrm{F}_{|\mathrm{U}})$ is coherent. Let us argue by contradiction and suppose that there exists $x \in \mathrm{U}$ such that $c(x) = 1$ and $\operatorname{prof} \mathrm{F}_x = 0$, i.e., $x \in \operatorname{Ass} \mathrm{F}_x$.

Let $y \in \overline{\{x\}} \cap \mathrm{Y}$ such that $\dim \mathscr{O}_{\overline{\{x\}},y} = 1$. Let :

$$\mathrm{A} = \mathscr{O}_{\mathrm{X},y} \text{ and } \mathrm{X}' = \mathrm{Spec}(\mathrm{A}).$$

Let us perform the base change $v \colon \mathrm{X}' \to \mathrm{X}$, which is flat.

(2.7)
$$\begin{tikzcd} \mathrm{U}' = \mathrm{X}' \times_{\mathrm{X}} \mathrm{U} \arrow[r, "v'"] \arrow[d, "i'"'] & \mathrm{U} \arrow[d, "i"] \\ \mathrm{X}' \arrow[r, "v"] & \mathrm{X}\,. \end{tikzcd}$$

The morphism i is separated (because it is an immersion), and of finite type (because it is an open immersion and X is locally Noetherian), the base change is flat so (EGA III 1.4.15) we have an isomorphism :

$$v^*(i_*(\mathrm{F}_{|\mathrm{U}})) \approx i'_*(v'^*(\mathrm{F}_{|\mathrm{U}})). \tag{2.8}$$

Let us denote by $\underline{x}$ (resp. $\underline{y}$) the ideal of A corresponding to x (resp. y). **35**

Let $\mathrm{G} = v'^*(\mathrm{F}_{|\mathrm{U}})$; G is coherent and $\underline{x} \in \mathrm{Ass}\,\mathrm{G}$, so there exists a monomorphism $\mathscr{O}_{\overline{\{x\}}} \to \mathrm{G}$, and consequently $i'_*(\mathscr{O}_{\overline{\{x\}}|\mathrm{U}'})$ is coherent. By the choice of y, $\dim \mathrm{A}/\underline{x} = 1$, and consequently the support of $\mathscr{O}_{\overline{\{x\}}}$ is reduced to $\overline{\{\underline{x}\}} = \{\underline{x}\} \cup \{\underline{y}\}$, because $\overline{\{\underline{x}\}} = \mathrm{Spec}(\mathrm{A}/\underline{x})$ as a scheme. It follows that

$$(\mathscr{O}_{\overline{\{x\}}|\mathrm{U}'})(\mathrm{U}') = \mathrm{Frac}(\mathrm{A}/\underline{x}),$$

the field of fractions of $\mathrm{A}/\underline{x}$, and

$$i'_*(\mathscr{O}_{\overline{\{x\}}|\mathrm{U}'})(\mathrm{X}') = \mathrm{Frac}(\mathrm{A}/\underline{x}).$$

But $\mathrm{Frac}(\mathrm{A}/\underline{x})$ is not a finitely generated A-module because $\underline{x}$ is different from the maximal ideal of A. Hence a contradiction.

Suppose $n > 1$ and the result is established for $n' < n$. By the induction hypothesis, for any $x \in \mathrm{U}$ such that $c(x) = 1$, we have $x \notin \mathrm{Ass}\,\mathrm{F}_x$. Let such an x be given, and let $y \in \overline{\{x\}} \cap \mathrm{Y}$ such that x follows y, i.e. $\dim \mathscr{O}_{\overline{\{x\}},y} = 1$. We perform the base change $v \colon \mathrm{Spec}(\mathscr{O}_{\mathrm{X},y}) \to \mathrm{X}$ while keeping the notations of diagram (2.7). We find, by applying (EGA III 1.4.15), isomorphisms :

$$v^*(\mathrm{R}^p\, i_*(\mathrm{F}_{|\mathrm{U}})) \simeq \mathrm{R}^p\, i'_*(v'^*(\mathrm{F}_{|\mathrm{U}})),\ p \in \mathbf{Z}.$$

We are thus reduced to the case where X *is the spectrum of a local ring* A *in which* $\underline{x}$ *is a prime ideal of dimension* 1, i.e., $\dim \mathrm{A}/x = 1$. Let us then set $\mathrm{F}' = \underline{\Gamma}_{\mathrm{Y}}(\mathrm{F})$ and $\mathrm{F}'' = \mathrm{F}/\mathrm{F}'$.
We see that $\mathrm{F}_x \simeq \mathrm{F}''_x$ and that $y \notin \mathrm{Ass}\,\mathrm{F}''$. Furthermore $\mathrm{F}'_{|\mathrm{U}} = 0$, hence, by the exact sequence of $\mathrm{R}^p\, i_*$, isomorphisms :

$$\mathrm{R}^p\, i_*(\mathrm{F}_{|\mathrm{U}}) \simeq \mathrm{R}^p\, i_*(\mathrm{F}''_{|\mathrm{U}}),\ p \in \mathbf{Z}.$$

Since $n > 1$, we deduce that neither x nor y belong to $\mathrm{Ass}\,\mathrm{F}''$. Now, $\underline{x}, \underline{y}$ are the only prime ideals of A which contain $\underline{x}$; it follows (III 2.1) that there exists an element **36**

$g \in \underline{x}$ which is M-regular, where we have set $\mathrm{F} = \widetilde{\mathrm{M}}$, $\mathrm{M} = \mathrm{F}(\mathrm{X})$. Hence an exact sequence :

$$0 \longrightarrow \mathrm{M} \xrightarrow{g'} \mathrm{M} \longrightarrow \mathrm{N} \longrightarrow 0,$$

in which g' denotes multiplication by g in M. By the exact homology sequence, we see that :

$\mathrm{R}^p\, i_*(\widetilde{\mathrm{N}}_{|\mathrm{U}})$ is coherent for $p < n-1$,

therefore, by the induction hypothesis, $\operatorname{prof}(\widetilde{\mathrm{N}})_x \geqslant n-1$, so $\operatorname{prof} \mathrm{F}_x \geqslant n$, C.Q.F.D.

3. Applications

From these results we deduce a coherence condition for the higher direct images of a coherent sheaf by a *morphism which is not proper.*

Theorem 3.1. — *Let* $f\colon \mathrm{X} \to \mathrm{Y}$ *be a morphism of preschemes. Suppose that* Y *is* locally Noetherian *and that* f *is* proper. *Suppose that* X *is* locally immersible in a regular prescheme. *Let* $n \in \mathbf{Z}$. *Let* U *be an open subset of* X *and let* F *be a* coherent $\mathscr{O}_{\mathrm{U}}$*-Module. Suppose that, for any* $x \in \mathrm{U}$ *such that* $\operatorname{codim}(\overline{\{x\}} \cap (\mathrm{X}-\mathrm{U}), \overline{\{x\}}) = 1$, *we have* $\operatorname{prof} \mathrm{F}_x \geqslant n$. Then *the* $\mathscr{O}_{\mathrm{Y}}$*-Modules* $\mathrm{R}^p(f \circ g)_*(\mathrm{F})$ *are coherent for* $p < n$, *where* g *is the canonical immersion of* U *into* X.

Indeed, there exists a Leray spectral sequence whose abutment is $\mathrm{R}^*(f \circ g)_*(\mathrm{F})$ and whose initial term is given by :

$$\mathrm{E}_2^{p,q} = \mathrm{R}^p f_*(\mathrm{R}^q\, g_*(\mathrm{F})).$$

Furthermore, there exists a *coherent* $\mathscr{O}_{\mathrm{X}}$-Module G such that $\mathrm{G}_{|\mathrm{U}} \simeq \mathrm{F}$, (EGA I 9.4.3).
37 It then follows from the preceding paragraph that condition (iv) on page 78 is satisfied, i.e., that $\mathrm{R}^q\, g_*(\mathrm{G}_{|\mathrm{U}})$ is coherent for $q < n$. We then apply the finiteness theorem of EGA III 3.2.1 to f and to the sheaves $\mathrm{R}^q\, g_*(\mathrm{F})$, and we find that $\mathrm{E}_2^{p,q}$ is coherent for $q < n$, hence the conclusion.

Proposition 3.2. — *Let* X *be a locally Noetherian prescheme locally immersible in a regular prescheme. Let* U *be an open subset of* X *and let* $i\colon \mathrm{U} \to \mathrm{X}$ *be the canonical immersion. Let* $n \in \mathbf{Z}$. *Finally, let* F *be a coherent and* Cohen-Macaulay $\mathscr{O}_{\mathrm{U}}$*-Module* (*on* U *!*). *The following conditions are equivalent :*

(a) $\mathrm{R}^p\, i_*(\mathrm{F})$ *is coherent for* $p < n$ *;*

(b) *for any irreducible component* S' *of the closure* $\overline{\mathrm{S}}$ *of the support* S *of* F, *we have :*

$$\operatorname{codim}(\mathrm{S}' \cap (\mathrm{X}-\mathrm{U}), \mathrm{S}') > n.$$

Let G be a coherent $\mathscr{O}_{\mathrm{X}}$-Module such that $\mathrm{G}_{|\mathrm{U}} \simeq \mathrm{F}$ (EGA I 9.4.3). Applying corollary 2.3 to G, we find that condition (a) is equivalent to (c) :

(c) for any $x \in \mathrm{S}$, we have $\operatorname{prof} \mathrm{F}_x > n - c(x)$, with

$$c(x) = \operatorname{codim}(\overline{\{x\}} \cap \mathrm{Y}, \overline{\{x\}}).$$

(a) $\Rightarrow$ (b). Indeed, let S' be an irreducible component of $\overline{\mathrm{S}}$ and let s be its generic point. Since F is Cohen-Macaulay, we have $\operatorname{prof} \mathrm{F}_s = \dim \mathscr{O}_{\mathrm{S},s} = 0$. Moreover, $\overline{\{s\}} = \mathrm{S}'$, hence the conclusion.

(b) $\Rightarrow$ (a). Let $x \in \mathrm{S}$ and let S' be an irreducible component of $\overline{\mathrm{S}}$ such that $x \in \mathrm{S}'$. Let s be the generic point of S'. Since F is Cohen-Macaulay, we know that :

$$\operatorname{prof} \mathrm{F}_x = \dim \mathscr{O}_{\overline{\{s\}},x}.$$

If $c(x) = +\infty$, there is nothing to prove. Otherwise, there exists $y \in \mathrm{Y} \cap \overline{\{x\}}$, such 38
that :

$$c(x) = \dim \mathscr{O}_{\overline{\{x\}},y}.$$

Now $\mathscr{O}_{\mathrm{X},y}$ is a quotient of a regular local ring by hypothesis, so :

$$\dim \mathscr{O}_{\overline{\{s\}},y} = \dim \mathscr{O}_{\overline{\{s\}},x} + \dim \mathscr{O}_{\overline{\{x\}},y} > n.$$

Q.E.D.

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EXPOSÉ IX

ALGEBRAIC GEOMETRY AND FORMAL GEOMETRY

The goal of this exposé is to generalize theorems 3.4.2 and 4.1.5 of EGA III to the case of a morphism that is not proper. 39

1. The comparison theorem

Let $f\colon \mathrm{X} \to \mathrm{X}'$ be a *separated* morphism of *finite type* between preschemes. Suppose that X' is locally Noetherian. Let Y' be a closed subset of X' and let $\mathrm{Y} = f^{-1}(\mathrm{Y}')$. Let $\widehat{\mathrm{X}}$ and $\widehat{\mathrm{X}'}$ be the formal completions of X and X' along Y and Y'. Let $\widehat{f}$ be the morphism deduced from f by passing to completions.

(1.1)
$$
\begin{array}{ccc}
\mathrm{X} & \longleftarrow & \mathrm{Y} \\
{\scriptstyle f}\downarrow & & \downarrow \\
\mathrm{X}' & \longleftarrow & \mathrm{Y}'
\end{array}
\quad , \quad
\begin{array}{ccc}
\widehat{\mathrm{X}} & \xrightarrow{j} & \mathrm{X} \\
{\scriptstyle \widehat{f}}\downarrow & & \downarrow{\scriptstyle f} \\
\widehat{\mathrm{X}'} & \xrightarrow{i} & \mathrm{X}'.
\end{array}
$$

We denote by j (resp. i) the homomorphism from $\widehat{\mathrm{X}}$ to X (resp. from $\widehat{\mathrm{X}'}$ to X'). We know that i and j are *flat*.

Let $\mathscr{I}'$ be an ideal of definition of Y' and let $\mathscr{J} = f^*(\mathscr{I}').\mathscr{O}_{\mathrm{X}}$, it is an ideal of definition of Y. We thus have :

(1.2)
$$\widehat{\mathrm{X}'} = (\mathrm{Y}', \varprojlim_{k\in\mathbf{N}} \mathscr{O}_{\mathrm{X}'}/\mathscr{I}'^{k+1}), \quad \widehat{\mathrm{X}} = (\mathrm{Y}, \varprojlim_{k\in\mathbf{N}} \mathscr{O}_{\mathrm{X}}/\mathscr{J}^{k+1}).$$

For any $k \in \mathbf{N}$, let :

(1.3)
$$\mathrm{Y}'_k = (\mathrm{Y}', \mathscr{O}_{\mathrm{X}'}/\mathscr{I}'^{k+1}), \quad \mathrm{Y}_k = (\mathrm{Y}, \mathscr{O}_{\mathrm{X}}/\mathscr{J}^{k+1}).$$

Let F be a *coherent* $\mathscr{O}_{\mathrm{X}}$-Module. For any $k \in \mathbf{N}$, we set :

(1.4)
$$\mathrm{F}_k = \mathrm{F}/\mathscr{J}^{k+1}\mathrm{F}, \qquad \widehat{\mathrm{F}} = j^*(\mathrm{F}) \simeq \varprojlim \mathrm{F}_k.$$

If we set : 40

(1.5)
$$\mathrm{R}^i f_*(\mathrm{F})^\wedge = \varprojlim_{k\in\mathbf{N}} (\mathrm{R}^i f_*(\mathrm{F}) \otimes_{\mathscr{O}_{\mathrm{X}'}} \mathscr{O}_{\mathrm{Y}'_k}), \quad i \in \mathbf{Z},$$

we have a natural homomorphism :

$$r_i : i^*(\mathrm{R}^i f_*(\mathrm{F})) \longrightarrow \mathrm{R}^i f_*(\mathrm{F})^\wedge, \tag{1.6}$$

which is an *isomorphism* when $\mathrm{R}^i f_*(\mathrm{F})$ is *coherent.*

As explained in EGA III 4.1.1, we have a commutative diagram :

$$\begin{tikzcd} i^*(\mathrm{R}^i f_*(\mathrm{F})) \arrow[r, "\rho_i"] \arrow[d, "r_i"] & \mathrm{R}^i \widehat{f}_*(\widehat{\mathrm{F}}) \arrow[d, "\psi_i"] \\ \mathrm{R}^i f_*(\mathrm{F})^\wedge \arrow[r, "\varphi_i"] & \varprojlim_{k\in\mathbf{N}} \mathrm{R}^i f_*(\mathrm{F}_k). \end{tikzcd} \tag{1.7}$$

In *loc. cit.* we find a commutative diagram, because we know that $\mathrm{R}^i f_*(\mathrm{F})$ is coherent, and we identify the source and target of (1.6). In our case, $\mathrm{R}^i f_*(\mathrm{F})$ will only be coherent for certain values of i, for which we will study (1.7).

Consider the graded ring

$$\mathscr{S} = \bigoplus_{k\in\mathbf{N}} \mathscr{I}'^k, \tag{1.8}$$

and the graded $\mathscr{S}$-Module :

$$\mathscr{H}^i = \bigoplus_{k\in\mathbf{N}} \mathrm{R}^i f_*(\mathscr{J}^k\mathrm{F}), \quad i \in \mathbf{Z}, \tag{1.9}$$

whose $\mathscr{S}$-Module structure is defined as follows.

41 The sheaf $\mathrm{R}^i f_*(\mathscr{J}^k\mathrm{F})$ is associated with the presheaf which, to any *affine* open set U' of X', associates :

$$\mathrm{H}^i(f^{-1}(\mathrm{U}'), \mathscr{J}^k\mathrm{F}_{|f^{-1}(\mathrm{U}')}). \tag{1.10}$$

Let U' be an affine open set of X', let

$$\mathrm{U} = f^{-1}(\mathrm{U}'),$$

and let $x' \in \mathscr{I}'^m(\mathrm{U}')$. Let x be the image of x' in $\mathscr{J}^m(\mathrm{U})$. The homothety with ratio x in $\mathrm{F}_{|\mathrm{U}}$ maps $\mathscr{J}^k\mathrm{F}_{|\mathrm{U}}$ into $\mathscr{J}^{k+m}\mathrm{F}_{|\mathrm{U}}$, hence, by functoriality, a morphism :

$$\mu^i_{x',k}(\mathrm{U}') : \mathrm{H}^i(\mathrm{U}, \mathscr{J}^k\mathrm{F}_{|\mathrm{U}}) \longrightarrow \mathrm{H}^i(\mathrm{U}, \mathscr{J}^{k+m}\mathrm{F}_{|\mathrm{U}}), \tag{1.11}$$

defined for all $i \in \mathbf{Z}$ and all $k \in \mathbf{N}$, which gives, by passing to the associated sheaf, the graded $\mathscr{S}$-Module structure of $\mathscr{H}^i$.

Theorem 1.1. — *Let n be an integer. Suppose that the graded $\mathscr{S}$-Module $\mathscr{H}^i$ is of finite type for $i = n-1$ and $i = n$, then :*

(0) *r_n and r_{n-1} are isomorphisms and $\mathrm{R}^i\widehat{f}_*(\widehat{\mathrm{F}})$ is coherent for $i = n-1$;*

(1) *for $i = n-1$, ρ_i, φ_i and ψ_i are topological isomorphisms (in particular the filtration defined on $\mathrm{R}^{n-1}f_*(\mathrm{F})$ by the kernels of the homomorphisms*

$$\mathrm{R}^{n-1}f_*(\mathrm{F}) \longrightarrow \mathrm{R}^{n-1}f_*(\mathrm{F}_k) \tag{1.12}$$

is $\mathscr{I}'$-good) ;

(2) *for* $i = n$, ρ_i, φ_i *and* ψ_i *are monomorphisms, moreover the filtration on* $\mathrm{R}^n f_*(\mathrm{F})$ *is* $\mathscr{I}'$*-good and* ψ_n *is an isomorphism;*

(3) *The projective system of* $\mathrm{R}^i f_*(\mathrm{F}_k)$ *satisfies, for* $i = n-2, n-1$, *the uniform* 42
Mittag-Leffler condition, i.e., there exists a fixed *integer* $k \geqslant 0$ *such that, for any* $p \geqslant 0$ *and any* $p' \geqslant p + k$, *we have :*

$$\mathrm{Im}[\mathrm{R}^i f_*(\mathrm{F}_{p'}) \longrightarrow \mathrm{R}^i f_*(\mathrm{F}_p)] = \mathrm{Im}[\mathrm{R}^i f_*(\mathrm{F}_{p+k}) \longrightarrow \mathrm{R}^i f_*(\mathrm{F}_p)].$$

Proceeding as in EGA III 4.1.8, it is easy to *reduce to the case where* X' *is the spectrum of a Noetherian ring* A. In this case, we know that

$$\mathrm{R}^i f_*(\mathrm{F}) = \widetilde{\mathrm{H}^i(\mathrm{X}, \mathrm{F})} \quad \text{(cf. 1.10)}. \tag{1.13}$$

Let I be the ideal of A such that $\widetilde{\mathrm{I}} = \mathscr{I}'$ and let

$$\mathrm{S} = \bigoplus_{k \in \mathbf{N}} \mathrm{I}^k, \tag{1.14}$$

$$\mathrm{H}^i = \bigoplus_{k \in \mathbf{N}} \mathrm{H}^i(\mathrm{X}, \mathscr{J}^k \mathrm{F}), \quad i \in \mathbf{Z}, \tag{1.15}$$

where H^i is equipped with the graded S-module structure defined by 1.11, where we have taken $\mathrm{U} = \mathrm{X}'$.

The proof is modeled on that of EGA III 4.1.5, let us give a summary.

We work on φ_i and ψ_i, to which correspond module homomorphisms :

$$\begin{array}{ccc} & & \mathrm{H}^i(\widehat{\mathrm{X}}, \widehat{\mathrm{F}}) \\ & & \Big\downarrow{\scriptstyle \psi_i} \\ \mathrm{H}^i(\mathrm{X}, \mathrm{F})^\wedge & \xrightarrow{\varphi_i} & \varprojlim_k \mathrm{H}^i(\mathrm{X}, \mathrm{F}_k). \end{array} \tag{1.16}$$

(a) We only assume that H^i is a finitely generated graded S-module. We deduce 43
that *the filtration defined on* $\mathrm{H}^i(\mathrm{X}, \mathrm{F})$ *by the modules* :

$$\mathrm{R}^i_k = \ker(\mathrm{H}^i(\mathrm{X}, \mathrm{F}) \longrightarrow \mathrm{H}^i(\mathrm{X}, \mathrm{F}_k)) \tag{1.17}$$

is I-*good*. For this we use the exact cohomology sequence :

$$\mathrm{H}^i(\mathrm{X}, \mathscr{J}^{k+1}\mathrm{F}) \longrightarrow \mathrm{H}^i(\mathrm{X}, \mathrm{F}) \longrightarrow \mathrm{H}^i(\mathrm{X}, \mathrm{F}_k), \tag{1.18}$$

which proves that *the graded* S-*module* $\bigoplus_{k \in \mathbf{N}} \mathrm{R}^i_k$ is a *quotient* of the graded sub-S-module

$$\bigoplus_{k \in \mathbf{N}} \mathrm{H}^i(\mathrm{X}, \mathscr{J}^{k+1}\mathrm{F})$$

of H^i, so *is of finite type*, because S is Noetherian. Hence this first point.

Let :

$$\mathrm{M}^i = \mathrm{H}^i(\mathrm{X}, \mathrm{F}), \quad \mathrm{H}^i_k = \mathrm{H}^i(\mathrm{X}, \mathrm{F}_k). \tag{1.19}$$

We have a commutative diagram :

$$\begin{tikzcd} \mathrm{H}^i(\mathrm{X},\mathrm{F})^\wedge \arrow[r, "s_i"] \arrow[dr, "\varphi_i"'] & \varprojlim_k(\mathrm{M}^i/\mathrm{R}^i_k) \arrow[d, "t_i"] \\ & \varprojlim_k \mathrm{H}^i_k , \end{tikzcd} \tag{1.20}$$

in which s_i is an *isomorphism* ; indeed the filtration of $\mathrm{H}^i(\mathrm{X},\mathrm{F})$ is I-good. Moreover t_i is a *monomorphism* ; indeed the functor $\varprojlim$ is left exact, and, for any $k \geqslant 0$, the natural morphism $\mathrm{M}^i/\mathrm{R}^i_k \to \mathrm{H}^i_k$ is a monomorphism, by definition of R^i_k.

To study the surjectivity of t_i we introduce :

$$\mathrm{Q}^i_k = \mathrm{coker}(\mathrm{H}^i(\mathrm{X},\mathrm{F}) \longrightarrow \mathrm{H}^i(\mathrm{X},\mathrm{F}_k)), \tag{1.21}$$

44 hence a projective system of exact sequences :

$$0 \longrightarrow \mathrm{R}^i_k \longrightarrow \mathrm{M}^i \longrightarrow \mathrm{H}^i_k \longrightarrow \mathrm{Q}^i_k \longrightarrow 0. \tag{1.22}$$

Using the exact cohomology sequence :

$$\mathrm{H}^i(\mathrm{X},\mathrm{F}) \longrightarrow \mathrm{H}^i(\mathrm{X},\mathrm{F}_k) \longrightarrow \mathrm{H}^{i+1}(\mathrm{X},\mathscr{I}^{k+1}\mathrm{F}), \tag{1.23}$$

we see that *the graded* S*-module*

$$\mathrm{Q}^i = \bigoplus_{k\in\mathbf{N}} \mathrm{Q}^i_k \tag{1.24}$$

is a graded sub-S*-module of* H^{i+1}. Moreover, for any $k \geqslant 0$, we have :

$$\mathrm{I}^{k+1}\mathrm{Q}^i_k = 0 \tag{1.25}$$

because Q^i_k is the image of H^i_k.

(b) We only assume that H^{i+1} is of finite type and we are interested in t_i (forgetting s_i). Since S is Noetherian, Q^i is of finite type ; as $\mathrm{I}^{k+1}\mathrm{Q}^i_k$ is zero, we find that there exists an integer $r \geqslant 0$ and an integer $k_0 \geqslant 0$ such that

$$\mathrm{I}^r\mathrm{Q}^i_k = 0 \quad \text{for } k \geqslant k_0. \tag{1.26}$$

It follows that the *projective system* $(\mathrm{Q}^i_k)_{k\in\mathbf{N}}$ *is essentially zero*, so that the *projective system* $(\mathrm{H}^i_k)_{k\in\mathbf{N}}$ *satisfies the uniform Mittag-Leffler condition.* From the exact sequence (1.22) we deduce the exact sequence

$$0 \longrightarrow \mathrm{M}^i/\mathrm{R}^i_k \longrightarrow \mathrm{H}^i_k \longrightarrow \mathrm{Q}^i_k \longrightarrow 0, \tag{1.27}$$

hence the exact sequence :

$$0 \longrightarrow \varprojlim_k \mathrm{M}^i/\mathrm{R}^i_k \xrightarrow{t_i} \varprojlim_k \mathrm{H}^i_k \longrightarrow \varprojlim_k \mathrm{Q}^i_k. \tag{1.28}$$

45 Now the projective system $(\mathrm{Q}^i_k)_{k\in\mathbf{N}}$ is essentially zero, so t_i is an *isomorphism.*

(c) Let us prove that, if H^i is of finite type, ψ_i is an isomorphism. It suffices to apply EGA 0_{III} 13.3.1 taking for the basis of open sets of X the affine open sets. This is licit ;

indeed, according to (b), the projective system $(\mathrm{H}_k^{i-1})_{k\in\mathbf{N}}$ satisfies the Mittag-Leffler condition.

The theorem follows formally from (a), (b) and (c). We note that in fact the proof only uses, at each step, the finiteness of H^i for a single value of i.

Let us give some examples where the hypothesis of theorem 1.1 is satisfied.

Corollary 1.2. — *Suppose that $\mathscr{I}'$ is generated by a section t' of $\mathscr{O}_{\mathrm{X}'}$, and let t be the corresponding section of $\mathscr{O}_{\mathrm{X}}$. Let* F *be a coherent $\mathscr{O}_{\mathrm{X}}$-module and let n be an integer.*

Suppose that :

(i) *t is* F*-regular* (*i.e., the homothety with ratio t in* F *is a monomorphism*).

(ii) $\mathrm{R}^i f_*(\mathrm{F})$ *is coherent for $i = n-1$ and $i = n$.*

Then the hypothesis of theorem 1.1 *is satisfied.*

Indeed, we note that multiplication by t^k defines an isomorphism $\mathrm{F} \xrightarrow{\sim} \mathscr{J}^k\mathrm{F}$ and we deduce that

$$\mathscr{H}^i \simeq \mathrm{R}^i f_*(\mathrm{F}) \otimes_{\mathscr{O}_{\mathrm{X}'}} \mathscr{O}_{\mathrm{X}'}[\mathrm{T}], \tag{1.29}$$

where T is an indeterminate. Hence the conclusion.

Corollary 1.3[(1)]. — *Suppose that $\mathrm{X}' = \mathrm{Spec}(\mathrm{A})$, where* A *is a Noetherian ring, sepa-* 46
rated and complete for the I*-adic topology. Suppose that the* S*-module H^i is of finite type for $i = n-1$ and $i = n$.* (*cf.* 1.14 *and* 1.15). *Then the hypotheses of theorem* 1.1 *are satisfied and we find a commutative diagram of isomorphisms :*

$$\begin{tikzcd}
\mathrm{H}^i(\mathrm{X},\mathrm{F}) \arrow[rr, "\rho'_i"] \arrow[dr, "\varphi'_i"'] & & \mathrm{H}^i(\widehat{\mathrm{X}},\widehat{\mathrm{F}}) \arrow[dl, "\psi_i"] \\
& \varprojlim_k \mathrm{H}^i(\mathrm{X},\mathrm{F}_k) &
\end{tikzcd} \tag{1.1}$$

for $i = n-1$.

We simply note that $\mathrm{H}^i(\mathrm{X},\mathrm{F})$ is of finite type, hence isomorphic to its completion. We obtain (1.1) by transcribing into the category of A-modules the diagram of Modules (1.7), and replacing the left vertical arrow by $\mathrm{H}^i(\mathrm{X},\mathrm{F})$.

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(1) N.D.E. : in the same vein, see Chow's article, (Chow W.-L., « Formal functions on homogeneous spaces », *Invent. Math.* **86** (1986), N° 1, p. 115–130). The author proves the following result. Let X be an algebraic variety over a field, homogeneous under an algebraic group G and let Z be a complete subvariety of X of dimension > 0. We suppose that Z generates X in the following sense : given $p \in \mathrm{Z}$, let Γ_p be the set of elements of G sending p into Z. We then say that Z generates if the group generated by the connected component of 1 of Γ_p is all of G. In this case, any formal rational function of X along Z is algebraic ; to be compared with the results of Hironaka and Matsumura cited in the editor's note (3) page 142. In line with the techniques introduced by these authors, let us point out the very nice algebrization result due to Gieseker (Gieseker D., « On two theorems of Griffiths about embeddings with ample normal bundle », *Amer. J. Math.* **99** (1977), N° 6, p. 1137–1150, theorems 4.1 and 4.2). Let X be a connected projective variety of dimension > 0, locally a complete intersection

Proposition 1.4. — *Let* A *be a Noetherian ring. Let* $t \in \mathrm{A}$ *and suppose that* A *is separated and complete for the* $(t\mathrm{A})$*-adic topology. Let :*

$$\mathrm{X}' = \mathrm{Spec}(\mathrm{A}), \quad \mathrm{Y}' = \mathrm{V}(t), \quad \mathrm{I} = (t\mathrm{A}). \tag{1.31}$$

Let T *be a closed subset of* X'*, let*

$$\mathrm{X} = \mathrm{X}' - \mathrm{T}, \quad \mathrm{Y} = \mathrm{Y}' \cap \mathrm{X} = \mathrm{Y}' - (\mathrm{Y}' \cap \mathrm{T}). \tag{1.32}$$

Let F *be a coherent* $\mathscr{O}_{\mathrm{X}}$*-Module. Finally, let*

$$\mathrm{T}' = \{x \in \mathrm{X}' \mid \mathrm{codim}(\{\overline{x}\} \cap \mathrm{T}, \{\overline{x}\}) = 1\}. \tag{1.33}$$

Suppose that :

a) t *is* F*-regular,*
b) $\mathrm{prof}_{\mathrm{T}'}(\mathrm{F}) \geqslant n+1$,
c) A *is a quotient of a regular Noetherian ring.*

Then, in diagram (1.1)*, the morphisms* ρ'_i, φ'_i *and* ψ_i *are isomorphisms for* $i < n$ *and monomorphisms for* $i = n$*. Moreover* ψ_n *is an isomorphism.*

47 By virtue of 1.3 and 1.2, it suffices to prove that $\mathrm{R}^i f_*(\mathrm{F})$ is coherent for $i \leqslant n$, which results from the finiteness theorem 2.1.

In particular :

Example 1.5. — We will apply 1.4 when A is a *local* ring and t belongs to the radical $\mathfrak{r}(\mathrm{A})$ of A. We will then take $\mathrm{T} = \{\mathfrak{r}(\mathrm{A})\}$. *In this case*, for $n = 1$ we find the following statement :

If A *Noetherian is separated and complete for the* t*-adic topology and a quotient of a regular ring* (*for example if* A *is complete*)*, if moreover* t *is* F*-regular and if* $\mathrm{prof}\,\mathrm{F}_x \geqslant 2$ *for all* $x \in \mathrm{Spec}(\mathrm{A})$ *such that* $\dim \mathrm{A}/x = 1$*, then the natural homomorphism*

$$\Gamma(\mathrm{X}, \mathrm{F}) \longrightarrow \Gamma(\widehat{\mathrm{X}}, \widehat{\mathrm{F}})$$

is an isomorphism.

Indeed, keeping the notations of 1.4, we have $\mathrm{T} = \{\mathfrak{r}(\mathrm{A})\}$ and formula (1.33) says that

$$\mathrm{T}' = \{x \in \mathrm{Spec}(\mathrm{A}) \mid \dim \mathrm{A}/x = 1\}.$$

2. Existence theorem

Let us first state EGA III 3.4.2 in a slightly more general form.

(over an algebraically closed field). Suppose we have two embeddings of X in smooth projective varieties Y, W. Then, if the formal completions of X in Y and W are equivalent, there exists a scheme U containing X (as a closed subscheme) which embeds in Y and W as an étale neighborhood of X in Y and W. In other words, formally equivalent implies étale-equivalent. See also the article by Faltings (Faltings G., « Formale Geometrie and homogene Raüme », *Invent. Math.* **64** (1981), p. 123-165).

Let $f : \mathfrak{X} \to \mathfrak{X}'$ be an adic morphism[(*)] of formal preschemes, with $\mathfrak{X}'$ Noetherian. Let $\mathscr{I}'$ be an ideal of definition of $\mathfrak{X}'$; since f is adic, $f^*\mathscr{I}' = \mathscr{J}$ is[(2)] an ideal of definition of $\mathfrak{X}$.

For any $n \in \mathbf{N}$, let

$$\mathfrak{X}_n = (\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}/\mathscr{J}^{n+1}), \tag{2.1}$$

it is an ordinary prescheme which has the same underlying topological space as $\mathfrak{X}$.

Let $\mathfrak{F}$ be a *coherent* $\mathscr{O}_{\mathfrak{X}}$-Module. For any $k \in \mathbf{N}$, the $\mathscr{O}_{\mathfrak{X}_k}$-Modules 48

$$\mathrm{F}_k = \mathfrak{F}/\mathscr{J}^{k+1}\mathfrak{F} \tag{2.2}$$

are *coherent*. For any i, we have a homomorphism

$$\psi_i : \mathrm{R}^i f_*(\mathfrak{F}) \longrightarrow \varprojlim_k \mathrm{R}^i f_*(\mathrm{F}_k), \tag{2.3}$$

deduced by functoriality from the natural homomorphism :

$$\mathfrak{F} \longrightarrow \mathrm{F}_k. \tag{2.4}$$

Let

$$\mathbf{S} = \mathrm{gr}_{\mathscr{I}'} \mathscr{O}_{\mathfrak{X}'} = \bigoplus_{k\in\mathbf{N}} \mathscr{I}'^k/\mathscr{I}'^{k+1}, \tag{2.5}$$

$$\mathrm{gr}_{\mathscr{J}}(\mathfrak{F}) = \bigoplus_{k\in\mathbf{N}} \mathscr{J}^k\mathfrak{F}/\mathscr{J}^{k+1}\mathfrak{F}, \tag{2.6}$$

$$\mathbf{K}^i = \mathrm{R}^i f_*(\mathrm{gr}_{\mathscr{J}}(\mathfrak{F})) = \bigoplus_{k\in\mathbf{N}} \mathrm{R}^i f_*(\mathscr{J}^k\mathfrak{F}/\mathscr{J}^{k+1}\mathfrak{F}\,). \tag{2.7}$$

It is clear that $\mathbf{K}^i$ is equipped with a graded $\mathbf{S}$-Module structure.

Theorem 2.1. — *Suppose that* $\mathbf{K}^i$ *is a* finitely generated *graded* $\mathbf{S}$*-Module for* $i = n-1$, $i = n$, $i = n+1$, *then :*

(i) $\mathrm{R}^n f_*(\mathfrak{F})$ *is coherent.*

(ii) *The homomorphism* ψ_n (2.3) *is a topological isomorphism. The natural filtration of the second member of* (2.3) *is* $\mathscr{I}'$*-good.*

(iii) *The projective system of* $\mathrm{R}^n f_*(\mathrm{F}_k)$ *satisfies the uniform Mittag-Leffler condition.*

The proof is very easy from EGA 0_{III}13.7.7 (cf. EGA III 3.4.2), provided that the text on page 78 is corrected as indicated in (EGA III 2, $\mathrm{Err}_{\mathrm{III}}$24). 49

Theorem 2.2[(3)]. — *Let* A *be a Noetherian adic ring and let* I *be an ideal of definition of* A. *Let* T *be a closed subset of* $\mathrm{X}' = \mathrm{Spec}(\mathrm{A})$. *Suppose that* I *is generated by a* $t \in \mathrm{A}$.

(*) This hypothesis is not essential, cf. XII, p. 122.

(2) N.D.E. : by definition, cf. EGA I 10.12.1.

Let us reuse the notations 1.31, 1.32 *and* 1.33*. Let* $\mathfrak{F}$ *be a coherent* $\mathscr{O}_{\widehat{\mathrm{X}}}$*-Module. Let*

$$\mathrm{F}_0 = \mathfrak{F}/\mathscr{J}\mathfrak{F}, \tag{2.1}$$

where $\mathscr{J} = t\mathscr{O}_{\widehat{\mathrm{X}}}$ *is an ideal of definition of* $\widehat{\mathrm{X}}$. Suppose that A *is a quotient of a regular Noetherian ring and that :*

(1) t *is* $\mathfrak{F}$*-regular,*

(2) $\mathrm{prof}_{\mathrm{T}'} \mathrm{F}_0 \geqslant 2$.

Then there exists a coherent $\mathscr{O}_{\mathrm{X}}$*-Module* F *such that* $\widehat{\mathrm{F}} \simeq \mathfrak{F}$.

It suffices to prove that $\widehat{f}_*(\mathfrak{F})$ is a coherent $\mathscr{O}_{\widehat{\mathrm{X}}}$-Module, where $\widehat{f} : \widehat{\mathrm{X}} \to \widehat{\mathrm{X}'}$ is the morphism of formal preschemes deduced from the injection of X into X′ by completion with respect to t. Indeed, A is separated and complete for the t-adic topology, so there will exist an A-module F′ whose completion will be isomorphic to $\widehat{f}_*(\mathfrak{F})$. Since X is an open subset of X′, we can take $\mathrm{F} = \widetilde{\mathrm{F}'}_{|\mathrm{X}}$.

It remains to show that 2.1 is applicable to the morphism of formal preschemes $\widehat{f}$ and to $\mathfrak{F}$. Now, according to hypothesis (1), for any $k \in \mathbf{N}$ we have an isomorphism :

$$\mathscr{J}^k\mathfrak{F}/\mathscr{J}^{k+1}\mathfrak{F} \longrightarrow \mathfrak{F}/\mathscr{J}\mathfrak{F},$$

50 from which it follows that the hypothesis of 2.1 will be satisfied if we know that

$$\mathrm{R}^i f_*(\mathrm{F}_0) \text{ is coherent for } i \leqslant 1.$$

Now this results from (2) and from the finiteness theorem 2.1. Hence the conclusion.

It remains to specialize this statement by assuming that A is a local ring.

Corollary 2.3. — *Let* A *be a Noetherian local ring and let* $t \in \mathfrak{r}(\mathrm{A})$*, an element of the radical of* A*. Suppose that* A *is separated and complete for the* t*-adic topology, and, moreover, a quotient of a regular ring* (*for example, suppose that* A *is a complete Noetherian local ring*)*. Let*

$$\mathrm{X}' = \operatorname{Spec} \mathrm{A}, \quad \mathrm{T} = \{\mathfrak{r}(\mathrm{A})\}, \tag{2.9}$$

and let us reuse the notations (1.31)*,* (1.32) *and* (1.33)*. Let* $\mathfrak{F}$ *be an* $\mathscr{O}_{\widehat{\mathrm{X}}}$*-Module. Suppose that :*

[(3)]N.D.E. : many algebrization statements have been obtained since, not to mention those cited below, cf. the articles by Faltings or Mrs. Raynaud cited in the editor's note (22) p. 158 and (7) p. 207 respectively. One thinks in particular of the results of Artin (see in particular Artin M., « Algebraization of formal moduli. I », in *Global Analysis* (*Papers in Honor of K. Kodaira*), Univ. Tokyo Press, Tokyo, 1969, p. 21-71), but also of recent results on the algebraicity of leaves of foliations ; see in particular Bost J.-B., « Algebraic leaves of algebraic foliations over number fields », *Publ. Math. Inst. Hautes Études Sci.* **93** (2001), p. 161-221 and Chambert-Loir A. « Théorèmes d'algébricité en géométrie diophantienne (d'après J.-B. Bost, Y. André, D. & G. Chudnovsky) », in *Séminaire Bourbaki, Vol. 2000/2001*, Astérisque , vol. 282, Société mathématique de France, Paris, 2002, Exp. 886, p. 175–209 and the references cited therein. In particular, one will find in these two articles discussions on the link between algebrization questions and the theory of Diophantine approximation.

(1) *t is $\mathfrak{F}$-regular,*
(2) $\operatorname{prof}_{\mathrm{T}'} \mathrm{F}_0 \geqslant 2$, *with* $\mathrm{F}_0 = \mathfrak{F}/\mathscr{J}\mathfrak{F}$ *and* $\mathscr{J} = t\mathscr{O}_{\widehat{\mathrm{X}}}$.
Then there exists a coherent $\mathscr{O}_{\mathrm{X}}$-Module F *such that* $\widehat{\mathrm{F}} \simeq \mathfrak{F}$.

Note that here T' is the set of prime ideals $\mathfrak{p}$ of A such that $\dim \mathrm{A}/\mathfrak{p} = 1$.

EXPOSÉ X

APPLICATION TO THE FUNDAMENTAL GROUP

Throughout this exposé, we will denote by X a locally Noetherian prescheme, by Y a closed subset of X, by U a variable open neighborhood of Y in X, and by $\widehat{X}$ the formal completion of X along Y (EGA I 10.8). For any prescheme Z, we will denote by $\mathbf{Et}(Z)$ the category of étale coverings of Z, and by $\mathbf{L}(Z)$ the category of locally free coherent Modules on Z. 51

1. Comparison of $\mathbf{Et}(\widehat{X})$ and $\mathbf{Et}(Y)$

Let $\mathscr{I}$ be an ideal of definition of Y in X. Let, for any $n \in \mathbf{N}$, $Y_n = (Y, (\mathscr{O}_X/\mathscr{I}^{n+1})_{|Y})$. The Y_n form an inductive system of usual preschemes, or also of formal preschemes, by equipping the structure sheaves with the discrete topology. We know (EGA I 10.6.2) that $\widehat{X}$ is an inductive limit, in the category of formal preschemes, of the inductive system of the Y_n. We also know (EGA I 10.13) that to give an $\widehat{X}$-formal prescheme of *finite type* R, is to give an inductive system of usual Y_n-preschemes R_n of *finite type*, such that $R_n \simeq (R_{n+1}) \times_{(Y_{n+1})} (Y_n)$. Moreover, for R to be an étale covering of $\widehat{X}$, it is necessary and sufficient that for any n, R_n be an étale covering of Y_n. That said, it is easy to see that nilpotent elements do not count for étale coverings (SGA 1 8.3), that is to say that the base change functor :

$$\mathbf{Et}(Y_{n+1}) \longrightarrow \mathbf{Et}(Y_n)$$

is an equivalence of categories for all $n \in \mathbf{N}$. So :

Proposition 1.1. — *With the notations introduced above, the natural functor* $\mathbf{Et}(\widehat{X}) \to \mathbf{Et}(Y)$ *is an equivalence of categories* (*cf.* SGA 1 8.4).

2. Comparison of $\mathbf{Et}(Y)$ and $\mathbf{Et}(U)$, for variable U

We will introduce two conditions from which the announced comparison theorem will easily follow. Let X be a locally Noetherian prescheme and let Y be a closed subset of X. We say that the pair (X, Y) satisfies the *Lefschetz condition*, which we write as Lef(X, Y), if, for any open set U of X containing Y and any locally free coherent sheaf 52

E on U, the natural homomorphism

$$\Gamma(\mathrm{U},\mathrm{E}) \longrightarrow \Gamma(\widehat{\mathrm{X}},\widehat{\mathrm{E}})$$

is an isomorphism.

We say that the pair (X,Y) satisfies the *effective Lefschetz condition*, which we write as $\mathrm{Leff}(\mathrm{X},\mathrm{Y})$, if we have $\mathrm{Lef}(\mathrm{X},\mathrm{Y})$ and if moreover, for any locally free coherent sheaf $\mathscr{E}$ on $\widehat{\mathrm{X}}$, there exists an open neighborhood U of Y and a locally free coherent sheaf E on U and an isomorphism $\widehat{\mathrm{E}} \simeq \mathscr{E}$.

These conditions are satisfied in two important examples :

Example 2.1[(1)]. — Let A be a Noetherian ring and let $t \in \mathfrak{r}(\mathrm{A})$ be an A-regular element belonging to the radical $\mathfrak{r}(\mathrm{A})$ of A. Suppose that A is a quotient of a regular local ring and that A is complete for the t-adic topology (for example A complete for the $\mathfrak{r}(\mathrm{A})$-adic topology). Let $\mathrm{X}' = \mathrm{Spec}(\mathrm{A})$ and $\mathrm{Y}' = \mathrm{V}(t)$, moreover let $x = \mathfrak{r}(\mathrm{A})$ and $\mathrm{X} = \mathrm{X}' - \{x\}$, $\mathrm{Y} = \mathrm{Y}' - \{x\}$. So X is open in X′ and $\mathrm{Y} = \mathrm{X} \cap \mathrm{Y}'$. Then :

(i) If, for any prime ideal $\mathfrak{p}$ of A such that $\dim \mathrm{A}/\mathfrak{p} = 1$ (i.e. for any closed point of X) we have $\mathrm{prof}\,\mathrm{A}_{\mathfrak{p}} \geqslant 2$, then we have $\mathrm{Lef}(\mathrm{X},\mathrm{Y})$;

(ii) if moreover, for any prime ideal $\mathfrak{p}$ of A such that $t \in \mathfrak{p}$ and $\dim \mathrm{A}/\mathfrak{p} = 1$ (i.e. for any closed point of Y), we have $\mathrm{prof}\,\mathrm{A}_{\mathfrak{p}} \geqslant 3$, then we have $\mathrm{Leff}(\mathrm{X},\mathrm{Y})$.

Let us first show that for any open neighborhood U of Y in X, the complement of U in X is the union of a finite number of closed points (in X). Note that U is open in
53 X, hence in X′, so $\mathrm{Z}' = \mathrm{X}' - \mathrm{U}$ is closed. Let I be an ideal of definition of Z′ ; it suffices to prove that A/I has dimension 1. Now $\mathrm{Z}' \cap \mathrm{Y}' = \{x\}$, so $\mathrm{A}/(\mathrm{I} + (t))$ is Artinian, hence the conclusion thanks to the "Hauptidealsatz".

The *first hypothesis is equivalent to* : "for any prime ideal $\mathfrak{p}$ of A, $\mathfrak{p} \neq \mathfrak{r}(\mathrm{A})$, we have $\mathrm{prof}\,\mathrm{A}_{\mathfrak{p}} \geqslant 3 - \dim \mathrm{A}/\mathfrak{p}$". Indeed A is a quotient of a regular ring, so we can apply VIII 2.3 to the prescheme X′, to the closed subset $\{x\}$ and to the coherent sheaf $\mathscr{O}_{\mathrm{X}'}$, observing that, $c(\mathfrak{p}) = \dim(\mathrm{A}/\mathfrak{p})$ for $\mathfrak{p} \in \mathrm{U} = \mathrm{X}' - \{x\}$ (because x is the closed point of X′).

Let U be an open neighborhood of Y in X and E a locally free $\mathscr{O}_{\mathrm{U}}$-module. Let $\mathrm{Z} = \mathrm{X} - \mathrm{U}$ and let $u\colon \mathrm{U} \to \mathrm{X}$ be the canonical immersion. We will first prove that $u_*(\mathrm{E})$ is a *coherent* $\mathscr{O}_{\mathrm{X}}$-Module, or, which amounts to the same thing, that $\underline{\mathrm{H}}^i_{\mathrm{Z}}(\mathrm{E}')$ is coherent for $i = 0, 1$, where E′ is a coherent extension of E to X. For this we apply theorem VIII 2.1 to the prescheme X, to the closed subset Z and to the coherent sheaf E′. It suffices to verify that for any point $p \in \mathrm{U}$ such that $c(p) = 1$, we have

[(1)]N.D.E. : one can slightly improve (i) : see Mrs. Raynaud (Raynaud M., « Lefschetz theorems in cohomology of coherent sheaves and in étale cohomology. Application to the fundamental group », *Ann. Sci. Éc. Norm. Sup.* (*4*) **7** (1974), p. 29–52, corollaries I.1.4 and I.5) ; condition (ii) can be improved to get rid of the depth conditions along Y (see theorem 3.3 of *loc. cit.* for a precise statement). The proof of this last point is very technical, the above article giving moreover only hints of proof, referring to a detailed, earlier version, published in the Bulletin de la Société mathématique de France prof $\mathrm{E}'_p \geqslant 1$, where we have set

$$c(p) = \operatorname{codim}(\overline{\{p\}} \cap \mathrm{Z}, \overline{\{p\}}).$$

Now if $p \in \mathrm{U}$ and if $c(p) = 1$, denoting by $\mathfrak{p}$ the ideal of A corresponding to p, we see that $\dim \mathrm{A}/\mathfrak{p} = 2$, because the complement of U is the union of a finite number of closed points and A is a quotient of a regular ring. Moreover, E is locally free, so, for any $p \in \operatorname{Supp} \mathrm{E}$, we have $\operatorname{prof} \mathrm{E}_p = \operatorname{prof} \mathscr{O}_{\mathrm{U},p}$. Finally, if $p \in \mathrm{U}$ and if $c(p) = 1$, we have

$$\operatorname{prof} \mathrm{E}'_p = \operatorname{prof} \mathrm{E}_p = \operatorname{prof} \mathscr{O}_{\mathrm{U},p} = \operatorname{prof} \mathrm{A}_{\mathfrak{p}} \geqslant 3 - 2 = 1.$$

We must now prove that the natural homomorphism

$$\Gamma(\mathrm{U}, \mathrm{E}) \longrightarrow \Gamma(\widehat{\mathrm{X}}, \widehat{\mathrm{E}}) \tag{1}$$

is an isomorphism. Setting then $\overline{\mathrm{E}} = u_*(\mathrm{E})$, we note that $\overline{\mathrm{E}}$ is coherent and has depth $\geqslant 2$ at all closed points of X. It follows that $\mathrm{R}^i f_*(\overline{\mathrm{E}})$ is coherent for $i = 0, 1$, where $f \colon \mathrm{X} \to \mathrm{X}'$ denotes the canonical immersion of X into $\mathrm{X}' = \operatorname{Spec}(\mathrm{A})$ (Exp. VIII). We then apply (IX 1.5), and we conclude that 54

$$\Gamma(\mathrm{U}, \overline{\mathrm{E}}) \longrightarrow \Gamma(\widehat{\mathrm{X}}, \widehat{\overline{\mathrm{E}}}) \tag{2}$$

is an isomorphism, because A is complete for the t-adic topology.

We have a commutative diagram :

$$\begin{tikzcd}
\Gamma(\mathrm{X}, \overline{\mathrm{E}}) \arrow[rr, "\simeq"] \arrow[dr, "\simeq"'] & & \Gamma(\mathrm{U}, \mathrm{E}) \arrow[dl] \\
& \Gamma(\widehat{\mathrm{X}}, \widehat{\mathrm{E}}) &
\end{tikzcd}$$

hence the conclusion.

Now let $\mathscr{E}$ be a locally free coherent sheaf on $\widehat{\mathrm{X}}$. If we have proved that $\mathscr{E}$ is algebrizable, i.e., is isomorphic to the formal completion of a coherent $\mathscr{O}_{\mathrm{X}}$-Module E, it is easy to see that E is locally free in a neighborhood of Y, and thus to prove $\operatorname{Leff}(\mathrm{X}, \mathrm{Y})$. Let $\widehat{\mathrm{X}'}$ be the formal spectrum of A for the t-adic topology, which identifies with the formal completion of X' along Y'. Let us denote by f the canonical immersion of X into X', by f' the canonical immersion of Y into Y', and by $\widehat{f}$ the morphism deduced by passing to completions. For $\mathscr{E}$ to be algebrizable it suffices that $\widehat{f}_*(\mathscr{E})$ be a coherent $\mathscr{O}_{\widehat{\mathrm{X}}}$-Module, because A is complete for the t-adic topology. Let $\mathscr{I} = t\mathscr{O}_{\widehat{\mathrm{X}}}$, it is an ideal of definition of $\widehat{\mathrm{X}}$.

For any $n \geqslant 0$, let $\mathrm{E}_n = \mathscr{E}/\mathscr{I}^{n+1}\mathscr{E}$. At any closed point $y \in \mathrm{Y}$, the depth of E_0 is $\geqslant 2$; indeed t is an A-regular element, so $\operatorname{prof} \mathscr{O}_{\mathrm{Y}_0,y} = \operatorname{prof} \mathscr{O}_{\mathrm{X},y} - 1 \geqslant 2$. We conclude that $\widehat{f}_*(\mathscr{E})$ is coherent (IX 2.3). C.Q.F.D.

Example 2.2 **(**Will allow comparing the fundamental group of a projective variety and a hyperplane section**)**

Let K be a field and let X be a proper K-prescheme. Let $\mathscr{L}$ be an ample invertible
55 $\mathscr{O}_{\mathrm{X}}$-Module. Let $t \in \Gamma(\mathrm{X}, \mathscr{L})$ be an $\mathscr{O}_{\mathrm{X}}$-regular element, which means that, for any open set U and any isomorphism $u \colon \mathscr{L}_{|\mathrm{U}} \to \mathscr{O}_{\mathrm{U}}$, $u(t)$ is a non-zero-divisor in $\mathscr{O}_{\mathrm{U}}$ (a condition that does not depend on u). Let $\mathrm{Y} = \mathrm{V}(t)$ be the subscheme of X with equation $t = 0$.[(2)] Then :

(i) If, for any point x closed in X, we have $\operatorname{prof} \mathscr{O}_{\mathrm{X},x} \geqslant 2$, we have $\mathrm{Lef}(\mathrm{X}, \mathrm{Y})$;

(ii) if moreover, for any closed point $y \in \mathrm{Y}$, we have $\operatorname{prof} \mathscr{O}_{\mathrm{X},y} \geqslant 3$, we have $\mathrm{Leff}(\mathrm{X}, \mathrm{Y})$.

This example will be treated in detail in Exp. XII.

Let S be a prescheme, we know (EGA II 6.1.2) that the functor which, to any finite and flat covering $r \colon \mathrm{R} \to \mathrm{S}$, associates the $\mathscr{O}_{\mathrm{X}}$-Algebra $r_*(\mathscr{O}_{\mathrm{R}})$, induces an equivalence between the category of finite and flat coverings of S and the category of coherent and locally free $\mathscr{O}_{\mathrm{X}}$-Algebras. Let U be an open neighborhood of Y, and let $r \colon \mathrm{R} \to \mathrm{U}$ be a finite and flat covering of U. Let $\widehat{\mathrm{R}}$ be the finite and flat covering of $\widehat{\mathrm{X}}$ which is deduced from it by base change. We have $\widehat{r}_*(\mathscr{O}_{\widehat{\mathrm{R}}}) \simeq \widehat{r_*(\mathscr{O}_{\mathrm{R}})}$.

Suppose then $\mathrm{Lef}(\mathrm{X}, \mathrm{Y})$. This implies that, for any U, the inverse image functor :

$$\mathbf{L}(\mathrm{U}) \longrightarrow \mathbf{L}(\widehat{\mathrm{X}})$$

is *fully faithful*. Indeed, let E and F be two locally free coherent $\mathscr{O}_{\mathrm{U}}$-Modules ; $\underline{\mathrm{Hom}}(\mathrm{E}, \mathrm{F})$ is also coherent and locally free ; $\underline{\mathrm{Hom}}(\mathrm{E}, \mathrm{F})$ is also coherent and locally free. By hypothesis, the natural application

$$\Gamma_{\mathrm{U}}(\underline{\mathrm{Hom}}(\mathrm{E}, \mathrm{F})) \longrightarrow \Gamma_{\widehat{\mathrm{X}}}(\widehat{\underline{\mathrm{Hom}}(\mathrm{E}, \mathrm{F})})$$

is an isomorphism, hence the conclusion, because $\underline{\mathrm{Hom}}$ commutes with $\widehat{\ }$ since everything is locally free. Now the $\widehat{\ }$ commutes with the tensor product, we deduce that the functor which to any coherent and locally free $\mathscr{O}_{\mathrm{U}}$-Algebra $\mathscr{A}$ associates the $\mathscr{O}_{\widehat{\mathrm{X}}}$-
56 Algebra $\widehat{\mathscr{A}}$, is fully faithful. Better, if E is a locally free coherent $\mathscr{O}_{\mathrm{U}}$-Module, there is a one-to-one correspondence between the commutative $\mathscr{O}_{\widehat{\mathrm{X}}}$-Algebra structures on $\widehat{\mathrm{E}}$.

Proposition 2.3. — *Let* X *be a locally Noetherian prescheme and let* Y *be a closed subset of* X*. Let* $\widehat{\mathrm{X}}$ *be the formal completion of* X *along* Y*. For any open subset* U *of* X*,* $\mathrm{U} \supset \mathrm{Y}$*, let us denote by* L_{U} *(resp.* P_{U}*, resp.* E_{U}*) the functor which to any locally free coherent* $\mathscr{O}_{\mathrm{U}}$*-Module (resp. to any finite and flat covering of* U*, resp. to any étale covering of* U*) associates its inverse image by* $\widehat{\mathrm{X}} \to \mathrm{X}$*.*

(i) *If we have* $\mathrm{Lef}(\mathrm{X}, \mathrm{Y})$*, then for any open neighborhood* U *of* Y*, the functors* L_{U}*,* P_{U} *and* E_{U} *are fully faithful.*

[(2)] N.D.E. : condition (ii) is superfluous, see note page 94.

(ii) *If we have* $\mathrm{Leff}(X, Y)$*, then for any locally free coherent* $\mathscr{O}_{\widehat{X}}$*-Module* $\mathscr{E}$ *(resp. ...), there exists an open set* U *and a locally free coherent* $\mathscr{O}_U$*-Module* E *(resp. ...), such that* $L_U(E) \simeq \mathscr{E}$ *(resp. ...).*

(i) Has been seen.

(ii) Results from (i) and the hypothesis, at least for L_U and P_U. Moreover if R is an *étale* covering of $\widehat{X}$, there exists an open neighborhood U of Y in X and a finite and flat covering R' of U such that $\widehat{R'} \simeq R$. We deduce a covering R'' of Y which is étale according to 1.1, so R' is étale in a neighborhood U' of Y. C.Q.F.D.

Corollary 2.4. — *If we have* $\mathrm{Lef}(X, Y)$*, for a finite and flat covering* R *of an open neighborhood* U *of* Y *to be connected, it is necessary and sufficient that* $R \times_U \widehat{X}$ *be so. In particular, for* Y *to be connected, it is necessary and sufficient that the open neighborhood* U *of* Y *be so, or that* X *be so.*

Indeed, for a ringed space with local rings $(X, \mathscr{O}_X)$ to be connected, it is necessary and sufficient that $\Gamma(X, \mathscr{O}_X)$ is not a direct composition of two non-zero rings. Now we have 57

$$\Gamma(U, r_*(\mathscr{O}_R)) \simeq \Gamma(\widehat{X}, \widehat{r}_*(\mathscr{O}_{\widehat{R}}))$$

by $\mathrm{Lef}(X, Y)$.

Corollary 2.5. — *If we have* $\mathrm{Lef}(X, Y)$*, then for any* U*, the functor*

$$\mathbf{Et}(U) \longrightarrow \mathbf{Et}(Y)$$

is fully faithful. If we have $\mathrm{Leff}(X, Y)$*, then for any étale covering* R *of* Y*, there exists an open neighborhood* U *of* Y *and a covering* R' *of* U *such that* $R' \times_U Y \simeq R$.

Corollary 2.6[(3)]. — *If we have* $\mathrm{Lef}(X, Y)$ *and if* Y *is connected, any open neighborhood* U *of* Y *is connected and the natural homomorphism* $\pi_1(Y) \to \pi_1(U)$ *is surjective. If moreover we have* $\mathrm{Leff}(X, Y)$*, the natural homomorphism*

$$\pi_1(Y) \longrightarrow \varprojlim_{U} \pi_1(U)$$

is an isomorphism. (N.B. We assume a "base-point" has been chosen in Y*, which we also take as the base-point in* X*, for the definition of the fundamental groups.)*

All this follows trivially from prop. 1.1 and prop. 2.3.

(3) N.D.E. : joined with 3.3 and criteria 2.4 and 3.4, we obtain the following relative Lefschetz theorem. Let $f : X \to S$ be a projective and flat morphism of connected Noetherian schemes and let D be an effective relative Cartier divisor in X and relatively ample. If, for any $s \in S$, the depth of X_s at each closed point is $\geqslant 2$, then D is connected and, for any open set U of X containing D, the map $i_U : \pi_1(D) \to \pi_1(U)$ is surjective. If moreover, the depth of X_s along each closed point of D_s is $\geqslant 3$ and if the local rings of X at its closed points are pure, for example of complete intersection (cf. X 3.4), then i_X is an isomorphism. Cf. Bost J.-B., « Lefschetz theorem for Arithmetic Surfaces », *Ann. Sci. Éc. Norm. Sup.* (*4*) **32** (1999), p. 241-312, theorems 1.1 and 2.1. In the case where X is simply a

3. Comparison of $\pi_1(\mathrm{X})$ and $\pi_1(\mathrm{U})$

Definition 3.1. — Let X be a prescheme and Z a closed subset of X. Let $\mathrm{U} = \mathrm{X} - \mathrm{Z}$. We say that the pair (X, Z) is pure if, for any open set V of X, the functor

$$\begin{aligned} \mathbf{Et}(\mathrm{V}) &\longrightarrow \mathbf{Et}(\mathrm{V} \cap \mathrm{U}) \\ \mathrm{V}' &\longmapsto \mathrm{V}' \times_{\mathrm{V}} (\mathrm{V} \cap \mathrm{U}) \end{aligned}$$

is an equivalence of categories[(*)].

58 ***Definition 3.2***. — Let A be a Noetherian local ring. Let $\mathrm{X} = \operatorname{Spec} \mathrm{A}$. Let $\mathfrak{r}(\mathrm{A})$ be the radical of A and let $x = \mathfrak{r}(\mathrm{A})$ be the closed point of X. We say that A is pure if the pair $(\mathrm{X}, \{x\})$ is.

We leave it to the reader not to prove the following proposition :

Proposition 3.3. — *Let* X *be a locally Noetherian prescheme and let* Z *be a closed subset of* X*. For the pair* (X, Z) *to be pure it is necessary and sufficient that, for any* $z \in \mathrm{Z}$*, the ring* $\mathscr{O}_{\mathrm{X},z}$ *be pure*[(†)]*.*

That said, the following theorem is the essential result of this section :

***Theorem 3.4* (Purity Theorem[(4)])**. — (i) *A regular Noetherian local ring of dimension* $\geqslant 2$ *is pure (Zariski-Nagata purity theorem).*

(ii) *A Noetherian local ring of dimension* $\geqslant 3$ *which is a complete intersection is pure.*

Let us recall that a local ring is said to be a *complete intersection* if there exists a *regular* Noetherian local ring B and a B-regular sequence $(t_1, \ldots, t_k)$ of elements of

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(*) For a more satisfactory notion in some respects, cf. the comment XIV 1.6 d).

(†) Compare the non-commutative case of XIV 1.8, whose proof is essentially the same as that of 3.3.

smooth and geometrically connected projective surface over a field, we always have connectedness of D and surjectivity of $\pi_1(\mathrm{D}) \to \pi_1(\mathrm{U})$ (where U is open containing D) for D only nef with square > 0 (cf. *loc. cit.*, theorem 2.3 and also theorem 2.4 for surfaces that are only normal and complete). In the case of an arithmetic surface (normal and quasi-projective) X over a ring of integers $\mathscr{O}_{\mathrm{K}}$, Bost, improving results of Ihara (Ihara Y., « Horizontal divisors on arithmetic surfaces associated with Belyĭ uniformizations », in *The Grothendieck theory of dessins d'enfants* (*Luminy, 1993*), London Math. Soc. Lect. Note Series, vol. 200, Cambridge Univ. Press, Cambridge, 1994, 245–254 or *loc. cit.*, corollary 7.2), has shown that if a point $\mathrm{P} \in \mathrm{X}(\mathscr{O}_{\mathrm{K}})$, which plays the role of the divisor D in the geometric situation, satisfies certain positivity conditions, then the map $\pi_1(\mathrm{X}) \to \pi_1(\operatorname{Spec} \mathscr{O}_{\mathrm{K}})$ deduced from the projection was invertible with inverse the map $\pi_1(\operatorname{Spec} \mathscr{O}_{\mathrm{K}}) \to \pi_1(\mathrm{X})$ deduced from P (*loc. cit.*, theorem 1.2).

(4) N.D.E. : for the history of the methods used, see the letter of October 1, 1961 from Grothendieck to Serre, *Correspondance Grothendieck-Serre*, edited by Pierre Colmez and Jean-Pierre Serre, Documents Mathématiques, vol. 2, Société Mathématique de France, Paris, 2001.

the radical $\mathfrak{r}(\mathrm{B})$ of B such that

$$\mathrm{A} \simeq \mathrm{B}/(t_1, \dots, t_k).$$

In this regard, let us note that it would be less ambiguous to say that A is an *absolute* complete intersection, as opposed to the situation, which we have already encountered, where X is a locally Noetherian prescheme (which need not be regular) and where Y is a closed subset of X, which is said to be "locally set-theoretically a complete intersection *in* X" .

Let us first prove some lemmas.

Lemma 3.5. — *Let* X *be a locally Noetherian prescheme and let* U *be an open subset of* X*. Let* $\mathrm{Z} = \mathrm{X} - \mathrm{U}$*. Let* $i\colon \mathrm{U} \to \mathrm{X}$ *be the canonical immersion of* U *into* X*. The following conditions are equivalent :* 59

(i) *For any open set* V *of* X*, if we set* $\mathrm{V}' = \mathrm{V} \cap \mathrm{U}$*, the functor* $\mathrm{F} \mapsto \mathrm{F}_{|\mathrm{V}'}$ *from the category of locally free coherent* $\mathscr{O}_\mathrm{V}$*-Modules to the category of locally free coherent* $\mathscr{O}_{\mathrm{V}'}$*-Modules is fully faithful ;*

(ii) *the natural homomorphism* $\mathscr{O}_\mathrm{X} \to i_*(\mathscr{O}_\mathrm{U})$ *is an isomorphism ;*

(iii) *for any* $z \in \mathrm{Z}$*, we have* $\operatorname{prof} \mathscr{O}_{\mathrm{X},z} \geqslant 2$.

We have already seen (III 3.3) the equivalence of (ii) and (iii). Let us show that (ii) implies (i). Let F and G be two locally free coherent $\mathscr{O}_\mathrm{V}$-Modules, $\underline{\mathrm{Hom}}(\mathrm{F}, \mathrm{G})$ is also one, so $\underline{\mathrm{Hom}}(\mathrm{F}, \mathrm{G}) \to i_*(\underline{\mathrm{Hom}}(\mathrm{F}_{|\mathrm{V}'}, \mathrm{G}_{|\mathrm{V}'}))$ is an isomorphism, so $\mathrm{Hom}(\mathrm{F}, \mathrm{G}) \simeq \mathrm{Hom}(\mathrm{F}_{|\mathrm{V}'}, \mathrm{G}_{|\mathrm{V}'})$. Conversely, we take $\mathrm{F} = \mathrm{G} = \mathscr{O}_\mathrm{X}$ and apply (i) to any open set V of X.

Here is a useful "descent lemma" :

Lemma 3.6. — *Let* X *be a locally Noetherian prescheme and let* Z *be a closed subset of* X*. Let* $\mathrm{U} = \mathrm{X} - \mathrm{Z}$*. Suppose that the homomorphism* $\mathscr{O}_\mathrm{X} \to i_*(\mathscr{O}_\mathrm{U})$ *is an isomorphism. Let* $f\colon \mathrm{X}_1 \to \mathrm{X}$ *be a faithfully flat and quasi-compact morphism. Let* $\mathrm{Z}_1 = f^{-1}(\mathrm{Z})$*. If the pair* $(\mathrm{X}_1, \mathrm{Z}_1)$ *is pure, the same is true of* (X, Z)*.*

Note that the hypothesis $\mathscr{O}_\mathrm{X} \simeq i_*(\mathscr{O}_\mathrm{U})$ is preserved by flat base extension, because i is a quasi-compact morphism and, in this case, the direct image commutes with the inverse image. Now this hypothesis implies that the functor

$$\mathbf{Et}(\mathrm{V}) \longrightarrow \mathbf{Et}(\mathrm{U} \cap \mathrm{V})$$

defined by

$$\mathrm{V}' \longmapsto \mathrm{V}' \times_\mathrm{V} (\mathrm{V} \cap \mathrm{U})$$

is fully faithful, as shown by the interpretation of an étale covering in terms of a locally free coherent Algebra. It remains to prove effectivity. One can for example introduce the square X_2 and the cube X_3 of X_1 over X and note that a faithfully flat and quasi-compact morphism is a morphism of *universal effective descent* for the 60 fibered category of étale coverings, over the category of preschemes. The conclusion is formal from there[*].

Remark 3.7. — We have proved along the way that if $\mathscr{O}_{\mathrm{X}} \to i_*(\mathscr{O}_{\mathrm{U}})$ is an isomorphism, X is connected if and only if U is, and then $\pi_1(\mathrm{U}) \to \pi_1(\mathrm{X})$ is surjective.

Corollary 3.8. — *Let* A *be a Noetherian local ring. Suppose that* $\operatorname{prof} \mathrm{A} \geqslant 2$*. Then if* $\widehat{\mathrm{A}}$ *is pure,* A *is pure.*

Results from lemma 3.5 and lemma 3.6.

The following lemma is the essential point of the proof of the purity theorem :

Lemma 3.9. — *Let* A *be a Noetherian local ring and let* $t \in \mathfrak{r}(\mathrm{A})$ *be an* A*-regular element. Suppose that* A *is complete for the* t*-adic topology and moreover a quotient of a regular local ring* (*for example* A *complete*)*. Let* $\mathrm{B} = \mathrm{A}/t\mathrm{A}$.

(i) *If for any prime ideal* $\mathfrak{p}$ *of* A *such that* $\dim \mathrm{A}/\mathfrak{p} = 1$*, we have* $\operatorname{prof} \mathrm{A}_{\mathfrak{p}} \geqslant 2$*, then* B *pure implies* A *pure.*

(ii) *If for any prime ideal* $\mathfrak{p}$ *of* A *such that* $\dim \mathrm{A}/\mathfrak{p} = 1$*, we have* $\operatorname{prof} \mathrm{A}_{\mathfrak{p}} \geqslant 2$*, if* $\mathrm{A}_{\mathfrak{p}}$ *is pure when* $t \notin \mathfrak{p}$*, and if*[5] $\operatorname{prof} \mathrm{A}_{\mathfrak{p}} \geqslant 3$ *when* $t \in \mathfrak{p}$*, then* A *pure implies* B *pure.*

Let $\mathrm{X}' = \operatorname{Spec}(\mathrm{A})$ and let $\mathrm{Y}' = \mathrm{V}(t)$, which we identify with the spectrum of B. Let $x = \mathfrak{r}(\mathrm{A})$, and let $\mathrm{X} = \mathrm{X}' - \{x\}$ and $\mathrm{Y} = \mathrm{Y}' - \{x\} = \mathrm{X} \cap \mathrm{Y}'$. Let $\widehat{\mathrm{X}'}$ be the formal spectrum of A for the t-adic topology, which identifies with the formal completion of X' along Y'.

61 Since A is complete for the t-adic topology, we note that $\mathbf{Et}(\mathrm{X}') \to \mathbf{Et}(\widehat{\mathrm{X}'})$ is an equivalence of categories. Similarly $\mathbf{Et}(\widehat{\mathrm{X}'}) \to \mathbf{Et}(\mathrm{Y}')$ by prop. 1.1, so $\mathbf{Et}(\mathrm{X}') \to \mathbf{Et}(\mathrm{Y}')$ is an equivalence of categories.

Let us show (i). Consider the diagrams

$$\begin{tikzcd} \mathrm{X}' & \mathrm{X} \arrow[l] \\ \mathrm{Y}' \arrow[u] & \mathrm{Y} \arrow[l] \arrow[u] \end{tikzcd} \qquad \begin{tikzcd} \mathbf{Et}(\mathrm{X}') \arrow[r, "a"] \arrow[d, "c"'] & \mathbf{Et}(\mathrm{X}) \arrow[d, "b"] \\ \mathbf{Et}(\mathrm{Y}') \arrow[r, "d"] & \mathbf{Et}(\mathrm{Y}) \end{tikzcd}$$

We have just seen that c is an equivalence, d is also one by the hypothesis that B is pure, and finally b is fully faithful as we saw in example 2.1, cf. 2.3 (i).

Let us show (ii). This time we suppose that A is pure so a is an equivalence; similarly for c. Let us see that b is an equivalence. According to example 2.1 we know that we have $\operatorname{Leff}(\mathrm{X}, \mathrm{Y})$, so b is already fully faithful, let us prove that it is essentially surjective. We use 2.3 (ii) noting that, if U is an open neighborhood of Y in X, the

[*] Cf. J. Giraud, *Méthode de la descente*, Mémoire N° 2 du Bulletin de la Société Mathématique de France (1964).

[5] N.D.E. : this last condition can be improved, cf. the editor's note (1) on page 94.

complement of U in X is a union of a finite number of closed points; the pair $(\mathrm{X}, \mathrm{X}-\mathrm{U})$ is therefore pure according to prop. 3.3, because at such a point $\mathfrak{p}$, $\mathscr{O}_{\mathrm{X},\mathfrak{p}} = \mathrm{A}_{\mathfrak{p}}$ is pure by hypothesis. Hence the conclusion.

Proof of the purity theorem. — Let us first show (i) by induction on the dimension. Let A be a Noetherian local ring of *dimension* 2. Let $\mathrm{X}' = \mathrm{Spec}(\mathrm{A})$, $x = \mathfrak{r}(\mathrm{A})$, $\mathrm{X} = \mathrm{X}' - \{x\}$. We have $\operatorname{prof} \mathrm{A} = 2$. We can therefore apply lemma 3.5 to the pair $(\mathrm{X}', \{x\})$ and so $\mathbf{Et}(\mathrm{X}') \to \mathbf{Et}(\mathrm{X})$ is fully faithful. Now let $r \colon \mathrm{R} \to \mathrm{X}$ be an étale covering defined by a locally free and étale coherent $\mathscr{O}_{\mathrm{X}}$-Algebra $\mathscr{A} = r_*(\mathscr{O}_{\mathrm{R}})$. Let $i \colon \mathrm{X} \to \mathrm{X}'$ be the canonical immersion of X into X′. I claim that $i_*(\mathscr{A}) = \mathscr{B}$ is a *coherent* $\mathscr{O}_{\mathrm{X}'}$-Algebra. Indeed, it suffices to apply the "finiteness theorem" VIII 2.3. I claim that this algebra has depth $\geqslant 2$ at x. Indeed, it is the direct image of an 62 $\mathscr{O}_{\mathrm{X}}$-Module, with $\mathrm{X} = \mathrm{X}' - \{x\}$. Since A is a *regular* ring of dimension 2 we have : $\operatorname{dp} \mathscr{B} + \operatorname{prof} \mathscr{B} = \dim \mathrm{A} = 2$, where $\operatorname{dp} \mathscr{B}$ denotes the projective dimension of $\mathscr{B}$. So $\operatorname{dp} \mathscr{B} = 0$, so $\mathscr{B}$ is projective, hence free. It follows that $\mathscr{B}$ defines a finite and flat covering of $\mathrm{X}' = \mathrm{Spec}(\mathrm{A})$. The set of points of X′ where this covering is not étale is a closed subset of X′ whose equation is a principal ideal : the discriminant ideal of $\mathscr{B}/\mathrm{A}$. Now, by construction, this closed set is contained in $x = \mathfrak{r}(\mathrm{A})$, so it is empty because $\dim \mathrm{A} = 2$.

Let A be a regular Noetherian local ring, $\dim \mathrm{A} = n \geqslant 3$. Suppose (i) is proved for rings of dimension $< n$. To prove that A is pure, we can assume A is complete by 3.8. Let $t \in \mathfrak{r}(\mathrm{A})$ whose image in $\mathfrak{r}(\mathrm{A})/\mathfrak{r}(\mathrm{A})^2$ is non-zero. Then $\mathrm{B} = \mathrm{A}/t\mathrm{A}$ is a *regular* Noetherian local ring of dimension $n-1$, so it is pure, because $n-1 \geqslant 2$. We conclude by lemma 3.9 (i), which is applicable because A is complete.

Let us show (ii). Let A be a Noetherian local ring of dimension $\geqslant 3$. Suppose there exists a *regular* Noetherian local ring B and a B-sequence $(t_1, \ldots, t_k)$ such that $\mathrm{A} \simeq \mathrm{B}/(t_1, \ldots, t_k)$. We can show that A is pure, by induction on k. If $k = 0$, we know this by (i). *Suppose $k \geqslant 1$ and the result is established for $k' < k$.* According to corollary 3.9 we can suppose that A (and therefore also B) is complete. Let $\mathrm{C} = \mathrm{B}/(t_1, \ldots, t_{k-1})$, so $\mathrm{A} \simeq \mathrm{C}/t_k\mathrm{C}$ and t_k is C-regular. By the induction hypothesis we know that C is pure, it suffices to prove that lemma 3.9 (ii) is applicable. Notation : A and B of the lemma become C and A. We have $\dim \mathrm{C} \geqslant 4$, so for any prime ideal $\mathfrak{p}$ of C such that $\dim \mathrm{C}/\mathfrak{p} = 1$, we have $\operatorname{prof} \mathrm{C}_{\mathfrak{p}} \geqslant 3$. Moreover, $\mathrm{C}_{\mathfrak{p}}$ is a complete intersection with $k' \leqslant k-1$, so is pure by the induction hypothesis. C.Q.F.D.

Theorem 3.10. — *Let* X *be a locally Noetherian prescheme and let* Y *be a closed subset* 63 *of* X. *Suppose we have* Leff(X, Y) (*cf. Examples* 2.1 *and* 2.2). *Suppose moreover that, for any open neighborhood* U *of* Y *and any* $x \in \mathrm{X} - \mathrm{U}$, *the local ring* $\mathscr{O}_{\mathrm{X},x}$ *is regular of dimension* $\geqslant 2$ *or a complete intersection of dimension* $\geqslant 3$. *Then*

$$\pi_0(\mathrm{Y}) \longrightarrow \pi_0(\mathrm{X})$$

is a bijection, and if X *is connected*

$$\pi_1(\mathrm{Y}) \longrightarrow \pi_1(\mathrm{X})$$

is an isomorphism.

There is nothing left to prove. We note that, in the two cited examples 2.1 and 2.2, the complement of U is a union of a finite number of closed points, from which it follows that the hypothesis on the dimension of $\mathscr{O}_{\mathrm{X},x}$ is not facetious.

EXPOSÉ XI

APPLICATION TO THE PICARD GROUP

This exposition is modeled on the preceding one, but this time the result of number 1 is less strong.

In this entire exposition, X will denote a locally noetherian prescheme, $\mathscr{I}$ a quasi-coherent ideal of $\mathscr{O}_{\mathrm{X}}$, $(\mathrm{Y} = \mathrm{V}(\mathscr{I})$ is thus a closed subset of X), U a variable open neighborhood of Y in X, and $\widehat{\mathrm{X}}$ the formal completion of X along Y. For any ringed space $(\mathrm{Z}, \mathscr{O}_{\mathrm{Z}})$, we denote by $\mathbf{P}(\mathrm{Z})$ the category of invertible $\mathscr{O}_{\mathrm{Z}}$-Modules, in other words locally free of rank 1, and by Pic(Z) the group of isomorphism classes of invertible Modules on Z. 64

1. Comparison of $\mathrm{Pic}(\widehat{\mathrm{X}})$ and of $\mathrm{Pic}(\mathrm{Y})$

For any $n \in \mathbf{N}$, let us set $\mathrm{X}_n = (\mathrm{Y}, \mathscr{O}_{\mathrm{X}}/\mathscr{I}^{n+1})$ and $\mathrm{P}_n = \mathscr{I}^{n+1}/\mathscr{I}^{n+2}$. The sequence of sheaves of abelian groups on Y

$$(1.1) \qquad 0 \longrightarrow \mathrm{P}_n \xrightarrow{\ u\ } \mathscr{O}^*_{\mathrm{X}_{n+1}} \xrightarrow{\ v\ } \mathscr{O}^*_{\mathrm{X}_n} \longrightarrow 1$$

is *exact*. Let us specify that the group structure of P_n is the additive structure, that $u(x) = 1 + x$ for any $x \in \mathrm{P}_n$, and that v is the homomorphism deduced from the injection $\mathscr{I}^{n+2} \to \mathscr{I}^{n+1}$. We see that v is surjective by noting that, for any $y \in \mathrm{Y}$, $\mathscr{O}_{\mathrm{X}_n,y}$ is a local ring, quotient of $\mathscr{O}_{\mathrm{X}_{n+1},y}$ by a nilpotent ideal; the rest is just as trivial. We deduce from (1.1) an exact cohomology sequence :

$$(*) \qquad \mathrm{H}^1(\mathrm{Y}, \mathrm{P}_n) \xrightarrow{\ u^1\ } \mathrm{H}^1(\mathrm{Y}, \mathscr{O}^*_{\mathrm{X}_{n+1}}) \xrightarrow{\ v^1\ } \mathrm{H}^1(\mathrm{Y}, \mathscr{O}^*_{\mathrm{X}_n}) \xrightarrow{\ d\ } \mathrm{H}^2(\mathrm{Y}, \mathrm{P}_n).$$

Furthermore, for any $n \in \mathbf{N}$, we can identify $\mathrm{Pic}(\mathrm{X}_n)$ and $\mathrm{H}^1(\mathrm{Y}, \mathscr{O}^*_{\mathrm{X}_n})$; moreover, if E is an invertible $\mathscr{O}_{\mathrm{X}_{n+1}}$-Module, corresponding to a cohomology class $c(\mathrm{E})$, the cohomology class corresponding to the inverse image of E on X_n is equal to $v^1(c(\mathrm{E}))$. Whence the following proposition : 65

Proposition 1.1. — *Let us keep the notations introduced above. Let* $p \in \mathbf{N}$. *The map* $\mathrm{Pic}(\widehat{\mathrm{X}}) \to \mathrm{Pic}(\mathrm{Y}_n)$ *:*

(i) *is injective for* $n \geqslant p$*, if* $\mathrm{H}^1(\mathrm{Y}, \mathrm{P}_n) = 0$ *for* $n \geqslant p$ *;*

(ii) *is an isomorphism for* $n \geqslant p$*, if* $\mathrm{H}^i(\mathrm{Y}, \mathrm{P}_n) = 0$ *for* $n \geqslant p$ *and* $i = 1, 2$.

Of course, the exact sequence $(*)$ contains more information than the above proposition. The reader will have noticed that we have said nothing about the functor $\mathbf{P}(\widehat{\mathrm{X}}) \to \mathbf{P}(\mathrm{Y})$. Given two invertible $\mathscr{O}_{\widehat{\mathrm{X}}}$-Modules E, F, $\mathrm{H} = \underline{\mathrm{Hom}}(\mathrm{E}, \mathrm{F})$ is also invertible. If we indicate by an index n the reduction modulo $\mathscr{I}^{n+1}$, we find an exact sequence :

$$0 \longrightarrow \mathrm{H}_0 \otimes \mathrm{P}_n \longrightarrow \underline{\mathrm{Hom}}(\mathrm{E}_{n+1}, \mathrm{F}_{n+1}) \longrightarrow \underline{\mathrm{Hom}}(\mathrm{E}_n, \mathrm{F}_n) \longrightarrow 0.$$

Whence an exact cohomology sequence that we will not write out and whose interpretation is obvious; we can use this remark to study the functor $\mathbf{P}$.

2. Comparison of $\mathrm{Pic}(\mathrm{X})$ and of $\mathrm{Pic}(\widehat{\mathrm{X}})$

The reader will find in exposition X, number 2, the proof of what follows :

Proposition 2.1. — *Suppose that we have* $\mathrm{Lef}(\mathrm{X}, \mathrm{Y})$ *; then for any open neighborhood* U *of* Y *in* X*, the functor*

$$\mathbf{P}(\mathrm{U}) \longrightarrow \mathbf{P}(\widehat{\mathrm{X}}) \tag{2.1}$$

is fully faithful, the map

$$\mathrm{Pic}(\mathrm{U}) \longrightarrow \mathrm{Pic}(\widehat{\mathrm{X}}) \tag{2.2}$$

66 *is therefore injective. If we have* $\mathrm{Leff}(\mathrm{X}, \mathrm{Y})$*, the map* (2.3) *is an isomorphism :*

$$\varinjlim_{\mathrm{U}} \mathrm{Pic}(\mathrm{U}) \longrightarrow \mathrm{Pic}(\widehat{\mathrm{X}}). \tag{2.3}$$

Corollary 2.2. — *Suppose that we have* $\mathrm{Lef}(\mathrm{X}, \mathrm{Y})$ *and that for any integer* $n \geqslant p$*, we have* $\mathrm{H}^1(\mathrm{Y}, \mathrm{P}_n) = 0$ *; then for any open* $\mathrm{U} \supset \mathrm{Y}$*, the maps*

$$\mathrm{Pic}(\mathrm{X}) \longrightarrow \mathrm{Pic}(\mathrm{U}) \longrightarrow \mathrm{Pic}(\mathrm{Y}_n)$$

are injective if $n \geqslant p$. *If we have* $\mathrm{Leff}(\mathrm{X}, \mathrm{Y})$ *and if moreover, for any integer* $n \geqslant p$*, we have* $\mathrm{H}^i(\mathrm{Y}, \mathrm{P}_n) = 0$ *for* $i = 1$ *and* $i = 2$*, then the map*

$$\varinjlim_{\mathrm{U}} \mathrm{Pic}(\mathrm{U}) \longrightarrow \mathrm{Pic}(\mathrm{Y}_n)$$

is an isomorphism for $n \geqslant p$.

3. Comparison of $\mathbf{P}(\mathrm{X})$ and of $\mathbf{P}(\mathrm{U})$

A definition :

Definition 3.1[(*)]. — Let X be a prescheme and let Z be a closed subset of X. Let $\mathrm{U} = \mathrm{X} - \mathrm{Z}$. We say that X is *parafactorial* at the points of Z if, for any open set V of X, the functor $\mathbf{P}(\mathrm{V}) \to \mathbf{P}(\mathrm{V} \cap \mathrm{U})$ is an equivalence of categories. We also say that the pair (X, Z) is parafactorial.

Recall that $\mathbf{P}(\mathrm{Z})$ denotes the category of locally free Modules of rank 1 on Z.

Definition 3.2. — A noetherian local ring is said to be *parafactorial* if the pair $(\mathrm{Spec}(\mathrm{A}), \{\mathfrak{r}(\mathrm{A})\})$ is parafactorial.

The following proposition is proven, which shows that the notion is "pointwise" :

Proposition 3.3. — *Suppose* X *is locally noetherian. For the pair* (X, Z) *to be parafactorial, it is necessary and sufficient that, for any* $z \in \mathrm{Z}$*, the local ring* $\mathscr{O}_{\mathrm{X},z}$ *is so.* 67

Note that in parafactorial there is "fully faithful". The following is proven as in lemma 3.5 of exposition X :

Lemma 3.4. — *If* X *is a locally noetherian prescheme and if* $\mathrm{Z} = \mathrm{X} - \mathrm{U}$ *is a closed subset of* X*, the following conditions are equivalent :*

(i) *for any open set* V *of* X*, the functor* $\mathbf{P}(\mathrm{V}) \to \mathbf{P}(\mathrm{V} \cap \mathrm{U})$ *is fully faithful ;*
(ii) *the homomorphism* $\mathscr{O}_{\mathrm{X}} \to i_*(\mathscr{O}_{\mathrm{U}})$ *is an isomorphism ;*
(iii) *for any* $z \in \mathrm{Z}$*, we have* $\mathrm{prof}(\mathscr{O}_{\mathrm{X},z}) \geqslant 2$.

So "parafactorial" means that the conditions of 3.4 are satisfied and that, for any open set V of X, the homomorphism $\mathrm{Pic}(\mathrm{V}) \to \mathrm{Pic}(\mathrm{V} \cap \mathrm{U})$, is surjective. In particular, if X is the spectrum of a noetherian local ring, we find :

Proposition 3.5. — *Let* A *be a noetherian local ring ; for it to be parafactorial it is necessary and sufficient that* $\mathrm{prof}\,\mathrm{A} \geqslant 2$ *and* $\mathrm{Pic}(\mathrm{X}' - \{x\}) = 0$*, where we have set* $\mathrm{X}' = \mathrm{Spec}(\mathrm{A})$ *and where* x *is the unique closed point of* X'.

Note that a local ring of dimension $\leqslant 1$ is never parafactorial because its depth is $\leqslant 1$. So *factorial does not imply parafactorial* ; it is however true for noetherian local rings of dimension $\geqslant 2$, as we will see below.

Lemma 3.6. — *Let* X *be a locally noetherian prescheme and let* Z *be a closed subset of* X*. Let* $f : \mathrm{X}_1 \to \mathrm{X}$ *be a faithfully flat and quasi-compact morphism. Let* $\mathrm{Z}_1 = f^{-1}(\mathrm{Z})$*. If* $(\mathrm{X}_1, \mathrm{Z}_1)$ *is parafactorial, then* (X, Z) *is so.*

We first note that, if $i : (\mathrm{X} - \mathrm{Z}) \to \mathrm{X}$ denotes the canonical immersion of $\mathrm{U} =$ 68
$\mathrm{X} - \mathrm{Z}$ in X, the formation of the direct image by i of a quasi-coherent $\mathscr{O}_{\mathrm{U}}$-Module commutes with the base change f, because the latter is flat. It is therefore equivalent

[(*)] For a more detailed study of the notion of parafactoriality, and the proof of 3.3, cf. EGA IV 21.13, 21.14.

to assume the equivalent conditions of lemma 3.5 for (X, Z) or for (X_1, Z_1), because f is a morphism of descent for the category of quasi-coherent sheaves. It remains to prove that, for any open set V of X, $\mathrm{Pic}(V) \to \mathrm{Pic}(V \cap U)$ is surjective. We make the base change $V \to X$, which changes nothing (*sic*), and we are reduced to $V = X$. We then note that, if L is an invertible $\mathscr{O}_U$-Module and if L admits a locally free extension, this extension is isomorphic to $i_*(L)$, because of what we have just seen. It remains to prove that $i_*(L)$ is invertible. Using again the fact that the direct image by i commutes with flat base change, and that "locally free of rank 1" is a property that descends by a faithfully flat and quasi-compact morphism, we have won.

Corollary 3.7. — *Let* A *be a noetherian local ring ; if* $\widehat{A}$ *is parafactorial, so is* A.

Do not believe that, if A is parafactorial, $\widehat{A}$ is so[1].

Before tackling the statement of the main theorem of this n°, let us make the connection with the theory of divisors and the notion of a factorial ring[*].

Let X be a *noetherian* and *normal* prescheme. Let $Z^1(X)$ be the free abelian group generated by the $x \in X$ such that $\dim \mathscr{O}_{X,x} = 1$. The local ring of such a point is a discrete valuation ring. We will denote by v_x the corresponding normalized valuation. Let $K(X)$ be the ring of rational functions of X and let

$$p\colon K(X)^* \longrightarrow Z^1(X)$$

be the map that to any $f \in K(X)^*$ associates the 1-codimensional cycle :

$$(f) = \sum_{x \in X,\ \dim \mathscr{O}_{X,x} = 1} v_x(f) \cdot x.$$

69 The image of p is denoted $P(X)$ and its elements are called *principal divisors*[†]. We set

$$\mathrm{Cl}(X) = Z^1(X)/P(X).$$

Let $Z'^1(X)$ be the subgroup of $Z^1(X)$ whose elements are the *locally principal divisors*. We know that

$$\mathrm{Pic}(X) \simeq Z'^1(X)/P(X),$$

and consequently $\mathrm{Pic}(X)$ is identified with a subgroup of $\mathrm{Cl}(X)$.

Note that if U is a dense open set of X, $K(X) \to K(U)$ is an isomorphism, and that if $\mathrm{codim}(X - U, X) \geqslant 2$, i.e. if every $x \in X$ such that $\dim \mathscr{O}_{X,x} \leqslant 1$ belongs to U, the homomorphism $Z^1(X) \to Z^1(U)$ and consequently $\mathrm{Cl}(X) \to \mathrm{Cl}(U)$ is also an

[*] For the generalities that follow, cf. also EGA IV 21.

[†] In accordance with the terminology of EGA IV 21, we now prefer to reserve the name "divisors" for "locally principal divisors" or "Cartier divisors".

[1] N.D.E. : for a precise study of the link between the factoriality of A and of its completion, see (Heitmann R., « Characterization of completions of unique factorization domains », *Trans. Amer. Math. Soc.* **337** (1993), N° 1, p. 379–387).

isomorphism. Finally, if every $x \in U$ is factorial, i.e. $\mathscr{O}_{X,x}$ is so, then $Z^1(U) = Z'^1(U)$ so $\mathrm{Pic}(U) \simeq \mathrm{Cl}(U)$.

Proposition 3.7.1. — *Let* X *be a noetherian and normal prescheme. Let* $(U_i)_{i\in I}$ *be a family of open sets of* X *such that :*

a) *the* U_i *form a filter base*[2] *;*

b) *if we set* $Y_i = X - U_i$*, we have* $\mathrm{codim}(Y_i, X) \geqslant 2$ *for any* i*,*

c) *if* $x \in U_i$ *for any* $i \in I$*, then* $\mathscr{O}_{X,x}$ *is factorial.*

Then we have an isomorphism :

$$\varinjlim_{i\in I} \mathrm{Pic}(U_i) \xrightarrow{\approx} \mathrm{Cl}(X).$$

Note that b) implies that any $x \in X$ such that $\dim \mathscr{O}_{X,x} \leqslant 1$ belongs to U_i for any i. So the U_i are dense and moreover the homomorphism $Z^1(U_i) \to Z^1(X)$ is an isomorphism, as is $K(U_i) \to K(X)$. So $\mathrm{Pic}(U) \subset \mathrm{Cl}(U_i) \simeq \mathrm{Cl}(X)$. To prove what we want, it is therefore sufficient to show that any $D \in Z^1(X)$ belongs to $Z'^1(U_i)$ for a suitable i. It is sufficient to do it for irreducible and positive "divisors". So let $x \in X$ such that $\dim \mathscr{O}_{X,x} = 1$. It is sufficient to prove that there exists an $i \in I$ such that 70
$\overline{\{x\}}$ is locally principal at the points of U_i. Let $\mathscr{I}$ be the largest ideal of definition of the closed set $\overline{\{x\}}$. The set of points in the neighborhood of which $\mathscr{I}$ is free is an open set U. But $U \supset \bigcap_{i\in I} U_i$ according to c). If we set $Y = X - U$, we have $Y \subset \bigcup_{i\in I} Y_i$ with $Y_i = X - U_i$, but Y is closed, so admits a *finite* number of generic points, so is contained in the union of a finite number of Y_i, so in Y_j for some $j \in I$, because the U_i form a filter base. So $U \supset U_j$. C.Q.F.D.

Corollary 3.8. — *Let* X *be a noetherian and normal prescheme and let* Y *be a closed subset of codimension* $\geqslant 2$*. Suppose that, for any* $p \in X - Y$*,* $\mathscr{O}_{X,p}$ *is factorial, then*

$$\mathrm{Pic}(X - Y) \longrightarrow \mathrm{Cl}(X - Y) \longrightarrow \mathrm{Cl}(X)$$

are isomorphisms.

Corollary 3.9. — *Let* A *be a noetherian and normal local ring. Let* $X' = \mathrm{Spec}(A)$ *and* $x = \mathfrak{r}(A)$*. For* A *to be* factorial *it is necessary and sufficient that* $\mathrm{Pic}(X' - \{x\}) = 0$ *and that* $\mathfrak{p} \in X' - \{x\}$ *implies that* $A_{\mathfrak{p}}$ *is factorial.*

Indeed, for A to be factorial it is necessary and sufficient that $\mathrm{Cl}(X') = 0$ [3].

Corollary 3.10. — *Let* A *be a noetherian local ring of dimension* $\geqslant 2$*. Let* $X' = \mathrm{Spec}(A)$ *and let* $x = \mathfrak{r}(A)$*. Let* $X = X' - \{x\}$*. The following conditions are equivalent :*

(i) A *is factorial;*

[2] N.D.E. : i.e. a filtered decreasing family.

[3] N.D.E. : see Bourbaki, *Algèbre commutative* VII.1.4, cor. of th. 2 and VII.3.2, th. 1

(ii) a) *for any* $y \in \mathrm{X}$, $\mathscr{O}_{\mathrm{X},p}$ *is factorial, and*

b) A *is parafactorial, i.e.* $\operatorname{prof} \mathrm{A} \geqslant 2$ *and* $\operatorname{Pic}(\mathrm{X}) = 0$.

71 Before proving this corollary, let us state

Serre's criterion for normality 3.11[(*)]. — *Let* A *be a noetherian local ring. For* A *to be normal, it is necessary and sufficient that*

(i) *for any prime ideal* $\mathfrak{p}$ *of* A *such that* $\dim \mathrm{A}_{\mathfrak{p}} \leqslant 1$, $\mathrm{A}_{\mathfrak{p}}$ *is normal,*

(ii) *for any prime ideal* $\mathfrak{p}$ *of* A *such that* $\dim \mathrm{A}_{\mathfrak{p}} \geqslant 2$, *we have* $\operatorname{prof} \mathrm{A}_{\mathfrak{p}} \geqslant 2$.

Let us prove 3.10.

(i) $\Rightarrow$ (ii). Knowing that a localization of a factorial ring is also factorial, we have (ii) a). Moreover A is normal thus $\operatorname{prof} \mathrm{A} \geqslant 2$ because $\dim \mathrm{A} \geqslant 2$ (3.11 (ii)). Finally A is parafactorial; indeed $\operatorname{Pic}(\mathrm{X}) \simeq \operatorname{Cl}(\mathrm{X}') = 0$ (cf. 3.9).

(ii) $\Rightarrow$ (i). Let us first prove that A is *normal* by applying Serre's criterion. Since $\dim \mathrm{A} \geqslant 2$, condition (i) of the criterion is in the hypotheses. Moreover, for any $\mathfrak{p} \in \mathrm{X}$, $\mathrm{A}_{\mathfrak{p}}$ is factorial, thus normal, thus of depth $\geqslant 2$, at least if $\dim \mathrm{A}_{\mathfrak{p}} \geqslant 2$. Finally $\operatorname{prof} \mathrm{A} \geqslant 2$ according to (ii) b). It remains to apply 3.9.

Let us summarize what precedes :

Proposition 3.12. — *Let* X *be a locally noetherian prescheme and let* $\mathscr{I}$ *be a quasi-coherent ideal of* X. *Let* $\mathrm{Y} = \mathrm{V}(\mathscr{I})$. *Let* $p \in \mathbf{N}$. *Suppose that :*

1) *We have* $\operatorname{Leff}(\mathrm{X}, \mathrm{Y})$ (Exposition X) *;*

2) $\mathrm{H}^i(\mathrm{X}, \mathscr{I}^{n+1}/\mathscr{I}^{n+2}) = 0$ *if* $i = 1$ *or* 2 *and if* $n \geqslant p$ *;*

3) *for any open neighborhood* U *of* Y *in* X *and any* $x \in \mathrm{X} - \mathrm{U}$, *the ring* $\mathscr{O}_{\mathrm{X},x}$ *is parafactorial.*

Then, for any $n \geqslant p$, *and any open neighborhood* U *of* Y, *the homomorphisms*

$$\begin{tikzcd} \operatorname{Pic}(\mathrm{X}) \arrow[rr] \arrow[dr] & & \operatorname{Pic}(\mathrm{X}_n) \\ & \operatorname{Pic}(\mathrm{U}) \arrow[ur] & \end{tikzcd}$$

are isomorphisms.

72 We know of parafactorial rings :

Theorem 3.13

(i) (*Auslander-Buchsbaum*)[(4)] *A regular noetherian local ring is factorial* (*thus parafactorial if its dimension is* $\geqslant 2$).

[(*)] Cf. EGA IV 5.8.6.

[(4)] N.D.E. : to be compared with the following purity result, due to Gabber. Let X be the spectrum of a regular local ring A of dimension 3, a with non-zero differential, i.e. $a \in \mathfrak{m} - \mathfrak{m}^2$ and U the

(ii) *A noetherian local ring of dimension* $\geqslant 4$ *and which is a* complete intersection *is parafactorial.*

***Corollary 3.14* (Conjecture of Samuel[5]).** — *A noetherian local ring* A *which is a complete intersection and which is factorial in codimension* $\geqslant 3$ (*i.e.* $\dim \mathrm{A}_{\mathfrak{p}} \leqslant 3$ *implies that* $\mathrm{A}_{\mathfrak{p}}$ *is factorial*) *is factorial.*

Let us prove the corollary

We argue by induction on the dimension of A.

If $\dim \mathrm{A} \leqslant 3$, A is factorial by hypothesis.

If $\dim \mathrm{A} > 3$, by the induction hypothesis, noting that a localization of a complete intersection is also one, all localizations of A other than A are factorial. By theorem 3.13 (ii), A is parafactorial, thus factorial by 3.10.

Let us prove 3.13 (i) (following Kaplansky)[*].

Let A be a regular noetherian local ring, let $\dim \mathrm{A} = n$. If $n = 0$ or 1, the result is known. Suppose $n \geqslant 2$, let us argue by induction on n and suppose $n \geqslant 2$ and the theorem is proven for rings of dimension $< n$. Let $\mathrm{X}' = \mathrm{Spec}(\mathrm{A})$ and $\mathrm{X} = \mathrm{X}' - \{x\}$, where $x = \mathfrak{r}(\mathrm{A})$. The localizations of A other than A are regular and of dimension $< n$, thus factorial. Moreover $\operatorname{prof} \mathrm{A} = \dim \mathrm{A} \geqslant 2$. It is therefore sufficient to prove that $\mathrm{Pic}(\mathrm{X}) = 0$ (cor. 3.10). So let L be an invertible $\mathscr{O}_{\mathrm{X}}$-Module, we know that we can extend it to a coherent $\mathscr{O}_{\mathrm{X}'}$-Module L'. There exists a resolution of L' by free $\mathscr{O}_{\mathrm{X}}$-Modules :

$$0 \longleftarrow \mathrm{L}' \longleftarrow \mathrm{L}'_1 \longleftarrow \cdots \longleftarrow \mathrm{L}'_n \longleftarrow 0,$$

because the cohomological dimension of A is finite. By restriction to X' we obtain a **73**
finite free resolution. It is therefore sufficient to prove the following lemma :

***Lemma 3.15*.** — *Let* $(\mathrm{X}, \mathscr{O}_{\mathrm{X}})$ *be a ringed space and let* L *be a locally free* $\mathscr{O}_{\mathrm{X}}$*-Module that admits a finite resolution by* free modules of finite type. *Then* $\det(\mathrm{L}) \simeq \mathscr{O}_{\mathrm{X}}$.

Recall that we define $\det(\mathrm{L})$ as the highest exterior power of L. In the case considered, $\det(\mathrm{L}) \simeq \mathrm{L}$ because L is invertible, so the lemma allows us to conclude. Let us

[*] This is the proof reproduced in EGA IV 21.11.1.

complement of $\mathrm{V}(a)$. Then, a vector bundle on U is free (for a simple proof, see (Swan R.G., « A simple proof of Gabber's theorem on projective modules over a localized local ring » *Proc. Amer. Math. Soc.* **103** (1988), N° 4, p. 1025-1030). The rank 1 case is a particular case of theorem 3.13. For purity results concerning vector bundles of arbitrary rank, whether in the analytic or algebraic setting, see (Gabber O., « On purity theorems for vector bundles », *Internat. Math. Res. Notices* (2002), N° 15, p. 783–788).

[5] N.D.E. : for a proof in the same vein, but more elementary, see Call F. & Lyubeznik G., « A simple proof of Grothendieck's theorem on the parafactoriality of local rings », in *Commutative algebra : syzygies, multiplicities, and birational algebra* (*South Hadley, MA, 1992*), Contemp. Math., vol. 159, American Mathematical Society, Providence, RI, 1994, p. 15–18.

prove this lemma. Let

$$0 \longleftarrow \mathrm{L}_0 \longleftarrow \mathrm{L}_1 \longleftarrow \mathrm{L}_2 \longleftarrow \cdots \longleftarrow \mathrm{L}_n \longleftarrow 0,$$

be the announced exact sequence, where $\mathrm{L}_0 = \mathrm{L}$. Since everything is locally free, we have :

$$\bigotimes_{0\leqslant i\leqslant n} (\det(\mathrm{L}_i))^{(-1)^i} \simeq \mathscr{O}_\mathrm{X}$$

but all the L_i, $i > 0$, are free, so also their determinants, so also that of $\mathrm{L}_0 = \mathrm{L}$. C.Q.F.D.

We still need to prove (ii) of the theorem. Before that, let us prove a lemma that will allow us to proceed by induction :

Lemma 3.16. — *Let* A *be a noetherian local ring, quotient of a regular one. Let* $t \in \mathfrak{r}(\mathrm{A})$ *be an* A*-regular element. Suppose that* A *is complete for the* t*-adic topology. Let* $\mathrm{X}' = \mathrm{Spec}(\mathrm{A})$, $\mathrm{Y}' = \mathrm{V}(t) \simeq \mathrm{Spec}(\mathrm{B})$, $\mathrm{B} = \mathrm{A}/t\mathrm{A}$, $\mathrm{X} = \mathrm{X}' - \{x\}$, $\mathrm{Y} = \mathrm{Y}' - \{x\}$, $x = \mathfrak{r}(\mathrm{A})$*. Suppose that :*

a) *for any* $y \in \mathrm{X}$ *closed in* X *we have* prof $\mathscr{O}_{\mathrm{X},y} \geqslant 2$*,*

b) prof $\mathrm{A}/t\mathrm{A} \geqslant 3$*,*

74 *then the map* $\mathrm{Pic}(\mathrm{X}) \to \mathrm{Pic}(\mathrm{Y})$ *is injective. In particular, if* B *is parafactorial, so is* A*.*

We know that a) implies Lef(X, Y) thanks to X 2.1. If we prove that $\mathrm{H}^1(\mathrm{Y}, \mathrm{P}_n) = 0$ for any $n \geqslant 0$, we will know thanks to (2.2) that $\mathrm{Pic}(\mathrm{X}) \to \mathrm{Pic}(\mathrm{Y})$ is injective. If, moreover, B is parafactorial, we will know that $\mathrm{Pic}(\mathrm{Y}) = 0$ (3.5) so $\mathrm{Pic}(\mathrm{X}) = 0$, but $\mathrm{prof}(\mathrm{A}) = 3 + 1 \geqslant 2$ because t is A-regular, so A will be parafactorial by 3.5.

Let $\mathscr{I} = \widetilde{(t\mathrm{A})}$ be the $\mathscr{O}_{\mathrm{X}'}$-Module associated with the ideal $t\mathrm{A}$. In number 1, we set $\mathrm{P}_n = (\mathscr{I}^{n+1}/\mathscr{I}^{n+2})_{|\mathrm{Y}}$, for any $n \geqslant 0$. But t is A-regular so $\mathrm{P}_n \simeq \mathscr{O}_\mathrm{Y}$. *It therefore remains for us to prove that* $\mathrm{H}^1(\mathrm{Y}, \mathscr{O}_\mathrm{Y}) = 0$. But $\mathrm{Y} = \mathrm{Y}' - \{x\}$ is an open set of Y', so we have an exact sequence (I (27)) :

$$\mathrm{H}^1(\mathrm{Y}', \mathscr{O}_{\mathrm{Y}'}) \longrightarrow \mathrm{H}^1(\mathrm{Y}, \mathscr{O}_\mathrm{Y}) \longrightarrow \mathrm{H}^2_x(\mathrm{Y}', \mathscr{O}_{\mathrm{Y}'}),$$

whose right term is zero by virtue of hypothesis b), and the left one because Y' is affine. C.Q.F.D.

Lemma 3.17. — *Keeping the hypotheses of* 3.16*, suppose moreover that :*

c) *for any* y *closed in* Y*, we have* prof $\mathscr{O}_{\mathrm{X},y} \geqslant 3$*,*

d) prof $\mathrm{A}/t\mathrm{A} \geqslant 4$ (*stronger than b*)*,*

e) *for any* y *closed in* X*,* $y \in \mathrm{Y}$*, the ring* $\mathscr{O}_{\mathrm{X},y}$ *is parafactorial.*

Then the map $\mathrm{Pic}(\mathrm{X}) \to \mathrm{Pic}(\mathrm{Y})$ *is an isomorphism, in particular for* A *to be parafactorial, it is necessary and sufficient that* B *is so.*

4. Comparison of $\mathbf{Et}(\widehat{\mathrm{X}})$ and $\mathbf{Et}(\mathrm{Y})$

We know (X 2.1) that a) and c) imply Leff(X, Y). Moreover, by the argument just made, d) implies that $\mathrm{H}^1(\mathrm{Y}, \mathrm{P}_n) = 0$ for any $n \geqslant 0$ and $i = 1$ or $i = 2$. Moreover, for any open neighborhood U of Y in X, the complement of U in X is formed of a finite number of closed points. Thanks to e) and theorem 3.12, we deduce that $\mathrm{Pic(X)} \to \mathrm{Pic(Y)}$ is an isomorphism. On the other hand $\operatorname{prof} \mathrm{A} \geqslant \operatorname{prof} \mathrm{B} \geqslant 2$; by the criterion 3.5 we deduce that A is parafactorial if and only if B is so. 75

Let us now prove 3.13 (ii). Let R be a regular noetherian local ring. Let $(t_1, \ldots, t_k)$ be an R-sequence. Let $\mathrm{B} = \mathrm{R}/(t_1, \ldots, t_k)$ and suppose that $\dim \mathrm{B} \geqslant 4$. We must prove that B is parafactorial. Let us argue by induction on k. If $k = 0$, B is regular, thus factorial by 3.13 (i), thus parafactorial by 3.10. Suppose $k \geqslant 1$ and the theorem is proven for $k' < k$. Let $\mathrm{A} = \mathrm{R}/(t_1, \ldots, t_{k-1})$ so $\mathrm{B} = \mathrm{A}/t_k\mathrm{A}$. We can assume B is complete according to 3.7. By the induction hypothesis, A is parafactorial. Let us prove that we can apply lemma 3.17. We have assumed B is complete, so also A, and thus A is complete for the t-adic topology. If $x \in \mathrm{X}$, and if x is closed in X, A_x is a complete intersection of dimension $\geqslant 4$, with $k' < k$. According to the induction hypothesis A_x is parafactorial, and moreover of depth $\geqslant 4$. This gives a), c) and e). Moreover, $\dim \mathrm{A} \geqslant 5$, whence d). C.Q.F.D.

Theorem 4.1. — *Let* X *be a locally noetherian prescheme and let* $\mathscr{I}$ *be a coherent sheaf of ideals of* X. *Let* $\mathrm{Y} = \mathrm{V}(\mathscr{I})$. *Let* n *be an integer. Suppose that :*

(i) *we have* Leff(X, Y) (*cf. examples* X 2.1 *and* X 2.2),

(ii) *for any* $p \geqslant n$, *we have* $\mathrm{H}^i(\mathrm{Y}, \mathscr{I}^{p+1}/\mathscr{I}^{p+2}) = 0$ *for* $i = 1$ *and* $i = 2$,

(iii) *for any open* $\mathrm{U} \supset \mathrm{Y}$ *and any* $x \in \mathrm{X} - \mathrm{U}$, *the ring* $\mathscr{O}_{\mathrm{X},x}$ *is regular of dimension* $\geqslant 2$ *or a complete intersection of dimension* $\geqslant 4$.

Then, for any open $\mathrm{U} \supset \mathrm{Y}$ *and any integer* $p \geqslant n$, *the homomorphisms*

$$\mathrm{Pic(X)} \longrightarrow \mathrm{Pic(U)} \longrightarrow \mathrm{Pic}(\mathrm{Y}_p)$$

are isomorphisms, where Y_p *denotes the prescheme* $(\mathrm{Y}, \mathscr{O}_{\mathrm{X}}/\mathscr{I}^{p+1})$.

It suffices to combine 3.12 and 3.13.

EXPOSÉ XII

APPLICATIONS TO PROJECTIVE ALGEBRAIC SCHEMES

1. Projective duality theorem and finiteness theorem[*]

The following theorem, essentially contained in FAC[†] (except that at the time, Serre did not have the language of Ext of sheaves of modules[(1)]), is the global analogue of the local duality theorem (Exposition IV), which was modeled on it. 76

Theorem 1.1. — *Let k be a field, $\mathrm{X} = \mathbf{P}^r_k$ the projective space of dimension r over k, F a variable coherent module on X; then we have an isomorphism of ∂-functors in F :*

$$\mathrm{H}^i(\mathrm{X}, \mathrm{F})' \xleftarrow{\sim} \mathrm{Ext}^{r-i}(\mathrm{X}; \mathrm{F}, \Omega^r_{\mathrm{X}/k}), \tag{1}$$

where we set

$$\Omega^r_{\mathrm{X}/k} = \mathscr{O}_{\mathbf{P}^r_k}(-r-1). \tag{2}$$

Remarque. — Of course, this Module is also the Module of relative differentials of degree r of X over k. In this form, the theorem remains true if X is a proper and smooth scheme over k (for the projective case, see A. Grothendieck, "*Théorèmes de dualité pour les faisceaux algébriques cohérents* » ", Séminaire Bourbaki May 1957)[‡]. When F is locally free, we recover Serre's duality theorem $\mathrm{H}^i(\mathrm{X}, \mathrm{F})' \xrightarrow{\sim} \mathrm{H}^{r-i}(\underline{\mathrm{Hom}}_{\mathscr{O}_\mathrm{X}}(\mathrm{F}, \Omega^r_{\mathrm{X}/k}))$. Theorem 1.1 (which moreover gives back the case of an X projective over k, as in *loc. cit.*) will suffice for our purpose.

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[*] The present exposition, written in January 1963, is clearly more detailed than the oral presentation was, in June 1962.

[†] J.-P. Serre, « Faisceaux algébriques cohérents », *Ann. of Math.* **61** (1955), p. 197-278.

[‡] For a more general duality theorem, cf. the Hartshorne Seminar cited at the end of Exp. IV.

[(1)] N.D.E. : the reader fond of the History of Mathematics will consult with interest the letter that Grothendieck wrote to Serre on December 15, 1955 and the latter's reply of December 22 of the same year ; see *Correspondance Grothendieck-Serre*, edited by Pierre Colmez and Jean-Pierre Serre, Documents Mathématiques, vol. 2, Société Mathématique de France, Paris, 2001.

77 The homomorphism (1) is deduced from the Yoneda pairing

$$\mathrm{H}^i(\mathrm{X},\mathrm{F}) \times \mathrm{Ext}^{r-i}(\mathrm{X};\mathrm{F},\Omega^r_{\mathrm{X}/k}) \longrightarrow \mathrm{H}^r(\mathrm{X},\Omega^r_{\mathrm{X}/k}), \tag{3}$$

and from a well-known isomorphism (cf. FAC, or EGA III 2.1.12) :

$$\mathrm{H}^r(\mathrm{X},\Omega^r_{\mathrm{X}/k}) = \mathrm{H}^r(\mathbf{P}^r_k, \mathscr{O}_{\mathbf{P}^r_k}(-r-1)) \xrightarrow{\sim} k. \tag{4}$$

To show that (1) is an isomorphism, we proceed as in the case of the local duality theorem, noting that $\mathrm{H}^r(\mathrm{X},\mathrm{F})$ as a functor in F is right exact (since $\mathrm{H}^n(\mathrm{X},\mathrm{F}) = 0$ for $n > r$), and that any coherent Module is isomorphic to a quotient of a direct sum of Modules of the form $\mathscr{O}(-m)$, with m large. This reduces us by descending induction on i to doing the verification for a sheaf of the form $\mathscr{O}(-m)$, where this is contained in the well-known explicit computations (FAC, or EGA III 2.1.12). We can moreover assume $-m \leqslant -r-1$, in which case $\mathrm{H}^i(\mathrm{X},\mathscr{O}(-m)) = 0$ for $i \neq r$.

Corollary 1.2. — *For a given coherent* F*, and* m *large enough, we have a canonical isomorphism*

$$\mathrm{H}^i(\mathrm{X},\mathrm{F}(-m))' \xrightarrow{\sim} \mathrm{H}^0(\mathrm{X},\underline{\mathrm{Ext}}^{r-i}_{\mathscr{O}_\mathrm{X}}(\mathrm{F},\Omega^r_{\mathrm{X}/k})(m)) \tag{5}$$

(*where the* $'$ *denotes the dual vector space*).

Indeed, on the space projective X, we have for any pair of coherent sheaves F,G and for n large enough a canonical isomorphism :

$$\mathrm{Ext}^n(\mathrm{X};\mathrm{F}(-m),\mathrm{G}) \simeq \mathrm{Ext}^n(\mathrm{X};\mathrm{F},\mathrm{G}(m)) \xrightarrow{\sim} \mathrm{H}^0(\mathrm{X},\underline{\mathrm{Ext}}^n_{\mathscr{O}_\mathrm{X}}(\mathrm{F},\mathrm{G})(m)), \tag{6}$$

(the isomorphism of the first two members being trivially true for any m), as results from the spectral sequence of global Ext

$$\mathrm{H}^p(\mathrm{X},\underline{\mathrm{Ext}}^q_{\mathscr{O}_\mathrm{X}}(\mathrm{F},\mathrm{G}(m))) \Longrightarrow \mathrm{Ext}^\bullet(\mathrm{X};\mathrm{F},\mathrm{G}(m)),$$

78 which degenerates for m large enough thanks to the fact that

$$\underline{\mathrm{Ext}}^q_{\mathscr{O}_\mathrm{X}}(\mathrm{F},\mathrm{G}(m)) \simeq \underline{\mathrm{Ext}}^q_{\mathscr{O}_\mathrm{X}}(\mathrm{F},\mathrm{G})(m),$$

and that the $\underline{\mathrm{Ext}}^q_{\mathscr{O}_\mathrm{X}}(\mathrm{F},\mathrm{G})$ are coherent sheaves. So (5) results from (6) and (1).

Corollary 1.3. — *For given* i,F*, the following conditions are equivalent :*

(i) $\mathrm{H}^i(\mathrm{X},\mathrm{F}(-m)) = 0$ *for large* m.

(i bis) $\mathrm{H}^i(\mathrm{X},\mathrm{F}(\cdot)) = \bigoplus_{m\in\mathbf{Z}} \mathrm{H}^i(\mathrm{X},\mathrm{F}(-m))$ *is a finitely generated* S*-module, where* $\mathrm{S} = k[t_o,\ldots,t_r]$.

(ii) $\underline{\mathrm{Ext}}^{r-i}_{\mathscr{O}_\mathrm{X}}(\mathrm{F},\Omega^r_{\mathrm{X}/k}) = 0$.

(ii bis) $\underline{\mathrm{Ext}}^{r-i}_{\mathscr{O}_\mathrm{X}}(\mathrm{F},\mathscr{O}_\mathrm{X}) = 0$.

(iii) $\mathrm{H}^i_x(\mathrm{F}_x) = 0$ *for any closed point* x *of* X.

(iv) $\mathrm{H}^{i+1}_x(\widetilde{\mathrm{F}}_x) = 0$ *for any closed point* x *of the punctured projecting cone* $\widetilde{\mathrm{X}} = \mathrm{Spec}(\mathrm{S}) - \mathrm{Spec}(k)$ *of* X*, where* $\widetilde{\mathrm{F}}$ *denotes the inverse image of* F *by the canonical morphism* $\widetilde{\mathrm{X}} \to \mathrm{X}$.

Démonstration. — (i) $\Leftrightarrow$ (i bis) since the submodule of $\mathrm{H}^i(\mathrm{X}, \mathrm{F}(\cdot))$ which is the sum of the homogeneous components of degree $\geqslant \nu$ is of finite type over S (in fact, for $i \neq 0$, it is even of finite type over k), (cf. FAC or EGA III 2.2.1 and 2.3.2).

(i) $\Leftrightarrow$ (ii) by virtue of corollary 1.2.

(ii) $\Leftrightarrow$ (ii bis) because $\Omega^r_{\mathrm{X}/k}$ is locally isomorphic to $\mathscr{O}_{\mathrm{X}}$.

(ii bis) $\Leftrightarrow$ (iii) by virtue of the local duality theorem for $\mathscr{O}_{\mathrm{X},x}$ (which is a regular 79
local ring of dimension r), according to which the "dual" of $\underline{\mathrm{Ext}}^{r-i}_{\mathscr{O}_{\mathrm{X}}}(\mathrm{F}, \mathscr{O}_{\mathrm{X}})_x$ is identified with $\mathrm{H}^i_x(\mathrm{F}_x)$ (V 2.1).

(ii bis) is equivalent to the analogous relation

$$\underline{\mathrm{Ext}}^{r-i}_{\mathscr{O}_{\widetilde{\mathrm{X}}}}(\widetilde{\mathrm{F}}, \mathscr{O}_{\widetilde{\mathrm{X}}}) = 0$$

(thanks to the fact that $\widetilde{\mathrm{X}} \to \mathrm{X}$ is faithfully flat, hence the inverse image of $\underline{\mathrm{Ext}}^{r-i}_{\mathscr{O}_{\mathrm{X}}}(\mathrm{F}, \mathscr{O}_{\mathrm{X}})$ is isomorphic to $\underline{\mathrm{Ext}}^{r-i}_{\mathscr{O}_{\widetilde{\mathrm{X}}}}(\widetilde{\mathrm{F}}, \mathscr{O}_{\widetilde{\mathrm{X}}})$) and this last relation is equivalent to (iv) by the local duality theorem for the local ring $\mathscr{O}_{\mathrm{X},x}$, which is regular of dimension $r+1$.

In particular, applying this to all $i \leqslant n$, we find :

Corollary 1.4. — *Equivalent conditions for given* n, F *:*

(i) $\mathrm{H}^i(\mathrm{X}, \mathrm{F}(-m)) = 0$ *for* $i \leqslant n$ *and large* m.

(i bis) $\mathrm{H}^i(\mathrm{X}, \mathrm{F}(\cdot))$ *is a finitely generated* S*-module for* $i \leqslant n$.

(ii) $\mathrm{prof}(\mathrm{F}_x) > n$ *for any closed point* x *of* X.

(iii) $\mathrm{prof}(\widetilde{\mathrm{F}}_x) > n+1$ *for any closed point* x *of* $\widetilde{\mathrm{X}}$.

The interest of corollaries 1.3 and 1.4 is to express a global condition (i) or (i bis) in terms of local conditions, namely the vanishing of local invariants like $\mathrm{H}^i_x(\mathrm{X}, \mathrm{F}_x)$ or $\mathrm{H}^i_x(\mathrm{X}, \widetilde{\mathrm{F}}_x)$, or an inequality on the depth. In this form, these results remain trivially valid for any projective scheme X, and a very ample invertible sheaf $\mathscr{O}_{\mathrm{X}}(1)$ on X, as can be seen by inducing the latter with a suitable projective immersion $\mathrm{X} \to \mathbf{P}^r_k$. (Of course, conditions 1.3 (ii) and 1.3 (ii bis) are no longer equivalent to the others in this general case, unless we assume that X is regular for example). We can moreover generalize to the case of a projective morphism $\mathrm{X} \to \mathrm{S}$ in the following way :

Proposition 1.5. — *Let* $f : \mathrm{X} \to \mathrm{S}$ *be a projective morphism, with* S *noetherian,* $\mathscr{O}_{\mathrm{X}}(1)$ 80
an invertible Module on X *very ample relative to* S, F *a coherent Module on* X, *flat over* S, s *an element of* S, X_s *the fiber of* X *at* s (*considered as a projective scheme over* $k(s)$), F_s *the sheaf induced on* X_s *by* F, *finally* i *an integer. Suppose that for any closed point* x *of* X_s, *we have* $\mathrm{H}^i_x(\mathrm{F}_{s,x}) = 0$ (*for example* $\mathrm{prof}(\mathrm{F}_{s,x}) > i$). *Then there exists an open neighborhood* U *of* s, *such that the same condition is satisfied for* $s' \in \mathrm{U}$. *Moreover, for such a* U, *we have*

$$\mathrm{R}^i f_*(\mathrm{F}(-m)) = 0 \text{ for } m \text{ large,}$$

and if $\mathscr{S}$ *is a quasi-coherent graded algebra over* S*, generated by* $\mathscr{S}^1$*, and which defines* X *with* $\mathscr{O}_X(1)$ *as* $X = \mathrm{Proj}(\mathscr{S})$*,* $\mathscr{O}_X(1) = \mathrm{Proj}(\mathscr{S}(1))$*, then the* S*-module*

$$R^i f_*(F(\cdot)) = \bigoplus_{m\in\mathbf{Z}} R^i f_*(F(m))$$

is of finite type over U*.*

Let us embed X in an $X' = \mathbf{P}^r_S$ so that $\mathscr{O}_X(1)$ is induced by $\mathscr{O}_{X'}(1)$ (this is possible provided we replace S by an affine neighborhood of s). Let us set[(*)] for any integer j, and any $t \in S$:

$$\underline{E}^j(t) = \underline{\mathrm{Ext}}^j_{\mathscr{O}_{X'_t}}(F_{t'}, \mathscr{O}_{X'_t}(-r-1)) \tag{7}$$

So $\underline{E}^j(t)$ is a coherent Module on X_t, I say that for variable t, the family of these Modules is "constructible" in the following sense : there exists for any $t \in S$ a non-empty open set V of the closure $\bar{t}$, which we equip with the reduced induced structure, and a coherent Module $\underline{E}^j(V)$ on $X_V = X \times_S V$, *flat* relative to V, such that for any $t' \in V$, $\underline{E}^j(t)$ is isomorphic to the Module induced by $\underline{E}^j(V)$ on X_t. To verify this assertion, setting $Z = \bar{t}$ equipped with the induced structure, we consider the coherent
81 Modules

$$\underline{E}^j(Z) = \underline{\mathrm{Ext}}^j_{\mathscr{O}_{X'_Z}}(F_Z, \mathscr{O}_{X'_Z}(-r-1))$$

(where the index Z again means that we induce above Z), and we take for V a non-empty open set of Z such that the Modules $\underline{E}^j(Z)$ are flat above V : this is possible, because we immediately check that $\underline{E}^j(Z) = 0$ for j not in the interval $[0, r]$, and we can apply SGA 1 IV 6.11. We then take for $\underline{E}^j(V) = \underline{E}^j(Z)|_{X_V}$, and we easily check that it answers the question.

From the preceding remark it follows that there exists a finite partition of S formed by sets V_α of the form $V = V(t)$ as above, (noetherian induction), and applying Serre's theorem EGA III 2.2.1 to the $\underline{E}^j(V_i)$, we see that there exists an integer m_0 such that

$$R^i f_{V_\alpha *}(\underline{E}^j(V_\alpha)) = 0 \text{ for } i \neq 0, m \geqslant m_0, \text{ for } j,$$

whence it follows, using the flatness of $\underline{E}^j(V_\alpha)$ over V_α and easy Künneth-like relations (cf. EGA III par. 7), that

$$H^i(X_t, \underline{E}^j(t)(m)) = 0 \text{ for } i \neq 0, m \geqslant m_0, \text{ for } j,$$

for any $t \in V_\alpha$, thus for any t since the V_α cover S. From this and the spectral sequence of global Ext follows, thanks to 1.1, as in the proof of 1.2, an isomorphism

$$H^i(X_t, F_t(-m))' \xleftarrow{\sim} H^0(X_t, \underline{E}^{r-i}(t)(m)) \text{ for } m \geqslant m_0, \tag{8}$$

any integer i, and any $t \in S$.

[(*)] The first part of 1.5 can be obtained immediately by applying the purely local statement EGA IV 12.3.4 to the preceding $\underline{E}^j$, which short-circuits most of the proof that follows.

Let us now use the hypothesis made on F_s, which is written

$$\underline{\mathrm{E}}^{r-i}(s) = 0, \tag{9}$$

and thanks to (8) is equivalent to

$$\mathrm{H}^i(\mathrm{X}_s, \mathrm{F}_s(-m)) = 0 \text{ for } m \geqslant m_0. \tag{10}$$

Since F hence $\mathrm{F}(-m)$ is flat over S, it follows by the already invoked Künneth-like relations, that (for a given m) the same relation (10) holds when replacing s by a t 82
neighboring s, in particular for any generalization t of s. By virtue of (8), we will thus have, for such a generalization

$$\underline{\mathrm{E}}^{r-i}(t) = 0, \tag{11}$$

but the set of $t \in s$ for which this relation is true is obviously a constructible set (because it induces an open set on each V_α) ; since it contains the generalizations of s, it contains an open neighborhood U of s. This proves the first assertion of 1.5. Moreover, for $t \in \mathrm{U}$, we conclude from (11) and (8) that

$$\mathrm{H}^i(\mathrm{X}_t, \mathrm{F}_t(-m) = 0 \text{ for } m \geqslant m_0,\ t \in \mathrm{U}, \tag{12}$$

which by virtue of the Künneth-like relations, implies (in fact, is much stronger than)

$$\mathrm{R}^i f_*(\mathrm{F}(-m)) = 0 \text{ on } \mathrm{U}, \text{ for } m \geqslant m_0. \tag{13}$$

This proves the second assertion of 1.5. Finally the last one results from it immediately, by proceeding as at the beginning of the proof of 1.3.

Remark 1.6(*). — The proof simplifies notably (by eliminating all consideration of constructibility) when we already assume that the hypothesis made for $s \in \mathrm{S}$ is satisfied for all $s' \in \mathrm{S}$. In fact, when we make the hypothesis that F_s has depth $> i$ at the closed points of X, we have a general statement, *of a local nature on* X, which says that the same condition is satisfied for all X_t, provided we replace X by a suitable open neighborhood of the fiber X_s (in other words, a certain part of X, defined by conditions on the Modules induced by F on the fibers, is open, cf. EGA IV). Since f is here proper, we can therefore take this neighborhood to be of the form $f^{-1}(\mathrm{U})$, which gives back the first assertion of 1.5 without tedious unwinding. In this general case, we can still prove by the method of *loc. cit.* that the first assertion of 1.5 (proven here by global means, using that X is projective over S) still results from a purely local statement on X (which the reader will make explicit if they deem it useful).

2. Lefschetz theory for a projective morphism : Grauert's comparison theorem

This is the following theorem : 83

(*)This remark is made more precise by footnote 116.

Theorem 2.1. — *Let* $f : \mathrm{X} \to \mathrm{S}$ *be a projective morphism, with* S *noetherian,* $\mathscr{O}_{\mathrm{X}}(1)$ *an invertible Module on* X, *ample relative to* S, Y *the prescheme of zeros of a section* t *of* $\mathscr{O}_{\mathrm{X}}(1)$, J *the ideal defining* Y, X_n *the sub-prescheme of* X *defined by* J^{n+1}, $\widehat{\mathrm{X}}$ *the formal completion of* X *along* Y, $\widehat{f} : \widehat{\mathrm{X}} \to \mathrm{S}$ *the composite morphism* $\widehat{\mathrm{X}} \to \mathrm{X} \to \mathrm{S}$, F *a coherent Module on* X, *flat relative to* S. *We suppose moreover that for any* $s \in \mathrm{S}$, *the Module* F_s *induced on the fiber* X_s *has depth* $> n$ *at the points of said fiber, and that* t *is* F*-regular. Under these conditions :*

(i) *The canonical homomorphism*

$$\mathrm{R}^i f_*(\mathrm{F}) \longrightarrow \mathrm{R}^i \widehat{f}_*(\widehat{\mathrm{F}})$$

is an isomorphism for $i < n$, *a monomorphism for* $i = n$.

(ii) *The canonical homomorphism*

$$\mathrm{R}^i \widehat{f}_*(\widehat{\mathrm{F}}) \longrightarrow \varprojlim_m \mathrm{R}^i f_*(\mathrm{F}_m)$$

is an isomorphism for $i \leqslant n$.

Démonstration. — We immediately reduce to the case where S is affine, and then to proving the

Corollary 2.2. — *Under the conditions of* 2.1 *suppose moreover* S *is affine. then :*

(i) *The canonical homomorphism*

$$\mathrm{H}^i(\mathrm{X}, \mathrm{F}) \longrightarrow \mathrm{H}^i(\widehat{\mathrm{X}}, \widehat{\mathrm{F}})$$

84 *is an isomorphism for* $i < n$, *a monomorphism for* $i = n$.

(ii) *The canonical homomorphism*

$$\mathrm{H}^i(\widehat{\mathrm{X}}, \widehat{\mathrm{F}}) \longrightarrow \varprojlim_m \mathrm{H}^i(\mathrm{X}_m, \mathrm{F}_m)$$

is an isomorphism for $i \leqslant n$.

Up to replacing $\mathscr{O}_{\mathrm{X}}(1)$ by a tensor power, and t by a power of t, we can suppose $\mathscr{O}_{\mathrm{X}}(1)$ is very ample relative to S. On the other hand, t hence t^m being F-regular, the multiplication by t^m, considered as a homomorphism from $\mathrm{F}(-m)$ to F, is injective, so we have for any $m \geqslant 0$ an exact sequence :

$$0 \longrightarrow \mathrm{F}(-m) \xrightarrow{t^m} \mathrm{F} \longrightarrow \mathrm{F}_m \longrightarrow 0, \tag{14}$$

whence an exact cohomology sequence

$$\mathrm{H}^i(\mathrm{X}, \mathrm{F}(-m)) \longrightarrow \mathrm{H}^i(\mathrm{X}, \mathrm{F}) \longrightarrow \mathrm{H}^i(\mathrm{X}, \mathrm{F}_m) \longrightarrow \mathrm{H}^{i+1}(\mathrm{X}, \mathrm{F}(-m)).$$

But by virtue of 1.5 we have $\mathrm{H}^i(\mathrm{X}, \mathrm{F}(-m)) = 0$ for $i \leqslant n$, and m large enough, which proves the

Lemma 2.3. — *For large* m, *the canonical homomorphism*

$$\mathrm{H}^i(\mathrm{X}, \mathrm{F}) \longrightarrow \mathrm{H}^i(\mathrm{X}, \mathrm{F}_m)$$

is bijective if $i < n$*, injective if* $i = n$.

This shows that for $i < n$, the projective system $(\mathrm{H}^i(\mathrm{X}_m, \mathrm{F}_m))_{m\geqslant 0}$ is essentially constant, a fortiori satisfies the Mittag-Leffler condition, so (taking into account that $\widehat{\mathrm{F}} = \varprojlim \mathrm{F}_m$) we conclude (ii) by EGA 0_{III} 13.3. On the other hand (i) results from it trivially, taking into account 2.3.

Corollary 2.4[2]. — *Let* $f : \mathrm{X} \to \mathrm{S}$ *be a projective and flat morphism, with* S *locally noetherian,* $\mathscr{O}_{\mathrm{X}}(1)$ *an invertible Module on* X*, ample relative to* S*,* t *a section of this Module which is* $\mathscr{O}_{\mathrm{X}}$*-regular,* Y *the sub-prescheme of zeros of* t*,* $\widehat{\mathrm{X}}$ *the formal completion of* X *along* Y*. We suppose that for any* $s \in \mathrm{S}$*,* X_s *has depth* $\geqslant 1$ (*resp. depth* $\geqslant 2$) *at its closed points. Then for any open neighborhood* U *of* Y*, the functor* 85

$$\mathrm{F} \longmapsto \widehat{\mathrm{F}}$$

from the category of coherent locally free Modules on U*, to the category of coherent locally free Modules on* $\widehat{\mathrm{X}}$*, is faithful* (*resp. fully faithful, i.e. the Lefschetz condition* (Lef) *of* X 2 *is satisfied*).

Introducing for two locally free Modules F and G on U the Module

$$\mathrm{H} = \underline{\mathrm{Hom}}_{\mathscr{O}_{\mathrm{U}}}(\mathrm{F}, \mathrm{G})$$

we are reduced to proving that the canonical homomorphism

$$\mathrm{H}^0(\mathrm{U}, \mathrm{H}) \longrightarrow \mathrm{H}^0(\widehat{\mathrm{U}}, \widehat{\mathrm{H}}) \tag{15}$$

is injective (resp. bijective). But the Modules H_t have depth $\geqslant 1$ (resp. $\geqslant 2$) at the closed points of X_t, so we can apply 2.1, which implies the conclusion 2.4 in the case where $\mathrm{U} = \mathrm{X}$. In the case of an arbitrary U, we note that the question is local on S, so we can assume S is affine. Then any coherent Module on X is a quotient of a coherent locally free Module, (since $\mathscr{O}_{\mathrm{X}}(1)$ is an relatively ample invertible Module on X). Since the dual Module $\mathrm{H}' = \underline{\mathrm{Hom}}(\mathrm{H}, \mathscr{O}_{\mathrm{U}})$ extends to a coherent Module on X, which is thus isomorphic to a cokernel of a homomorphism of locally free Modules on X, it follows by transposition that we can find a homomorphism

$$u' : \mathrm{L}'^0 \longrightarrow \mathrm{L}'^1$$

of locally free Modules on X, inducing a homomorphism 86

$$u : \mathrm{L}^0 \longrightarrow \mathrm{L}^1$$

of locally free Modules on U, such that we have an exact sequence

$$0 \longrightarrow \mathrm{H} \longrightarrow \mathrm{L}^0 \xrightarrow{u} \mathrm{L}^1.$$

[2] N.D.E. : see corollary I.1.4 of the article by Mme Raynaud (Raynaud M., « Théorèmes de Lefschetz en cohomologie des faisceaux cohérents et en cohomologie étale. Application au groupe fondamental », *Ann. Sci. Éc. Norm. Sup.* (*4*) **7** (1974), p. 29–52).

Using the five lemma (which becomes the three lemma), and the left exactness of the functor H^0, we are reduced to proving that (15) is injective (resp. bijective) when we replace H by L^0, L^1, which reduces us to the case where H is induced by a locally free Module H' on X. Moreover in the non-resp. case this reduction is even useless, because the kernel of (15) is in any case formed of sections of H on U which vanish in a suitable open neighborhood V of Y, but the restriction homomorphism $\mathrm{H}^0(\mathrm{U},\mathrm{H}) \to \mathrm{H}^0(\mathrm{V},\mathrm{H})$ is injective, because H has depth $\geqslant 1$ at the points of any closed subset Z of X not meeting Y (cf. lemma below). In the resp. case, we are reduced to proving that

$$\mathrm{H}^0(\mathrm{X},\mathrm{H}') \longrightarrow \mathrm{H}^0(\mathrm{U},\mathrm{H}')$$

is bijective, which results from the fact that H' has depth $\geqslant 2$ at all points of a closed set $\mathrm{Z} = \mathrm{X} - \mathrm{U}$ of X not meeting Y. So we simply need to prove the

Lemma 2.5. — *Let* F *be a coherent Module on* X*, flat over* S*, such that for any* $s \in \mathrm{S}$*,* F_s *has depth* $\geqslant n$ *at any closed point of* X_s*. Then for any closed subset* Z *of* X *not meeting* Y*,* F *has depth* $\geqslant n$ *at all points of* Z*.*

Indeed, for any $x \in \mathrm{X}$, letting $s = f(x)$, we have

$$\operatorname{prof}(\mathrm{F}_x) \geqslant \operatorname{prof}(\mathrm{F}_{s,x}), \tag{16}$$

87 as can be seen by lifting in any way a maximal regular $\mathrm{F}_{s,x}$-sequence of elements of $\mathfrak{r}(\mathscr{O}_{\mathrm{X}_s,x})$, which gives an F_x-regular sequence by virtue of SGA 1 IV 5.7. But if x belongs to a Z as in lemma 2.5, then x is necessarily closed in X_s, in other words, Z is *finite* over S. Indeed Z (equipped with a structure induced by X) is projective over S as a closed sub-prescheme of X which is so, and Z is affine over S as a closed sub-prescheme of $\mathrm{X} - \mathrm{Y}$, which is so.

Remark 2.6. — Suppose that for any $s \in \mathrm{S}$, the section t_s of $\mathscr{O}_{\mathrm{X}_s}(1)$ induced by t is $\mathscr{O}_{\mathrm{X}_s}$-regular (which implies by SGA 1 IV 5.7 that t is $\mathscr{O}_{\mathrm{X}}$-regular). Then the hypotheses made are stable under base extension $\mathrm{S}' \to \mathrm{S}$ (S' locally noetherian). So the conclusion remains valid after any base change.

3. Lefschetz theory for a projective morphism : existence theorem

Theorem 3.1[(3)]. — *Let* $f : \mathrm{X} \to \mathrm{S}$ *be a projective morphism, with* S *noetherian,* $\mathscr{O}_{\mathrm{X}}(1)$ *an invertible Module on* X *ample relative to* S*,* X_0 *the sub-prescheme of zeros of a section* t *of* $\mathscr{O}_{\mathrm{X}}(1)$*,* $\widehat{\mathrm{X}}$ *the formal completion of* X *along* X_0*,* $\mathfrak{F}$ *a coherent Module on* $\widehat{\mathrm{X}}$*,* F_0 *the Module it induces on* X_0*. We suppose moreover :*

a) $\mathfrak{F}$ *is flat over* S.

(3) N.D.E. : for a version without flatness hypothesis, see (Raynaud M., « Théorèmes de Lefschetz en cohomologie des faisceaux cohérents et en cohomologie étale. Application au groupe fondamental », *Ann. Sci. Éc. Norm. Sup.* (*4*) **7** (1974), p. 29–52, theorem II.3.3).

b) *For any* $s \in \mathrm{S}$, *the section* t_s *induced by* t *on* X_s *is* $\mathfrak{F}_s$*-regular* (*which implies that* F_0 *is also flat over* S, *cf.* SGA 1 IV 5.7).

c) *For any* $s \in \mathrm{S}$, F_{0s} *has depth* $\geqslant 2$ *at the closed points of* X_{0s}.

We suppose moreover that S *admits an ample invertible sheaf. Under these conditions, there exists a coherent Module* F *on* X, *and an isomorphism of its formal completion* $\widehat{\mathrm{F}}$ *with* $\mathfrak{F}$.

This statement will result from the following :

Corollary 3.2. — *Under the conditions* a), b), c) *above we have the following :*

(i) *The Module* $\widehat{f}_*(\mathfrak{F})$ *on* S *is coherent, so for any* n, *the Module* $\widehat{f}_*(\mathfrak{F}(n))$ *on* S *is* 88
coherent.

(ii) *For large* n, *the canonical homomorphism* $\widehat{f}^*\widehat{f}_*(\mathfrak{F}(n)) \to \mathfrak{F}(n)$ *is surjective.*

Let us admit the corollary for now, and prove 3.1. Thanks to the last hypothesis made in 3.1, we can reduce to the case where $\mathrm{X} = \mathbf{P}^r_{\mathrm{S}}$, up to replacing $\mathscr{O}_{\mathrm{X}}(1)$, t by a suitable power. I say that we can moreover suppose that for any s, we have $t_s \neq 0$. Otherwise, we indeed have $\mathfrak{F}_s = 0$ by b), or what amounts to the same by Nakayama, $\mathrm{F}_{0s} = 0$ i.e. s does not belong to the image of $\operatorname{Supp}\mathrm{F}_0$ by the morphism $f_0 : \mathrm{X}_0 \to \mathrm{S}$ induced by f. But this image S' is open by virtue of a), b) since F_0 is flat over S, but it is obvious that it suffices to prove the conclusion of 3.1 in the situation obtained by restricting above S', because the coherent Module F' on $\mathrm{X}|_{\mathrm{S}'}$ obtained will be a restriction of a coherent Module F on X, which will answer the question. We can therefore suppose that in addition to hypotheses a), b), c), the following hypotheses are also satisfied :

a$'$) $\mathscr{O}_{\mathrm{X}}$ is flat over S.

b$'$) For any $s \in \mathrm{S}$, the section t_s is $\mathscr{O}_{\mathrm{X}_s}$-regular.

c$'$) For any $s \in \mathrm{S}$, $\mathscr{O}_{\mathrm{X}_{0s}}$ has depth $\geqslant 2$ at the closed points of X_{0s}. (It suffices to choose $\mathrm{X} = \mathbf{P}^r_{\mathrm{S}}$ with $r \geqslant 3$, which is permissible).

But 3.2 implies that we can find an epimorphism

$$\widehat{\mathrm{L}} \longrightarrow \mathfrak{F} \longrightarrow 0, \tag{17}$$

where L is a Module on X of the form $f_*(\mathrm{G})(-n)$, G being a coherent locally free Module on S : it suffices indeed; for large n, to represent the coherent Module $\widehat{f}_*(\mathfrak{F})$ on S as a quotient of such a G. On the other hand, the hypotheses a), b), c) on f, 89
t imply that $\widehat{\mathrm{L}}$ satisfies the same conditions a), b), c) as $\mathfrak{F}$. We easily conclude from this that the same is true of the kernel of (17), to which we can therefore apply the same argument, so that $\mathfrak{F}$ is represented as the cokernel of a homomorphism

$$\widehat{\mathrm{L}'} \longrightarrow \widehat{\mathrm{L}}, \tag{18}$$

where L, L' are locally free Modules on X. But by virtue of a$'$) and the second part of c$'$), and of 2.1 or 2.4 at choice, the homomorphism (18) comes from a homomorphism

$L' \to L$ of Modules on X. It now suffices to take for F the cokernel of $L' \to L$, and we win.

It remains to prove 3.2. This had been done in the seminar by a somewhat tedious expedient, consisting in interpreting everything in terms of cohomology on the punctured projecting cone of X relative to S, in order to be able to reduce to theorem 2.1. A more direct and more satisfactory way (although substantially identical), now seems to me the following. It consists in noting that in IX, number 2 (and with the notations of this exposition) the hypothesis that the morphism $f : \mathscr{X} \to \mathscr{X}'$ is adic does not intervene anywhere in the proof of 2.1, via EGA 0_{III} 13.7.7; it suffices to suppose instead that $\mathscr{X}$ is also adic, and to choose two ideals of definition $\mathscr{J}$ for $\mathscr{X}'$, $\mathscr{I}$ for $\mathscr{X}$, such that $\mathscr{J}\mathscr{O}_{\mathscr{X}} \subset \mathscr{I}$, and to define $\mathscr{S} = \operatorname{gr}_{\mathscr{J}}(\mathscr{O}_{\mathscr{X}'})$, and consider $\operatorname{gr}_{\mathscr{I}}(\mathfrak{F})$. In any case, 2.1 can be applied directly to the morphism $\widehat{f} : \widehat{X} \to S$ considered in the present number, where we simply take $\mathscr{J} = 0$. Thus to verify that $\widehat{f}_*(\mathfrak{F})$ is coherent, it suffices by virtue of *loc. cit.* to verify that $R^i f_{0*}(\operatorname{gr}_{\mathscr{I}}(\mathfrak{F}))$ is coherent over S for $i = 0, 1$; for this we note that by virtue of a) and b), the Module considered is none other than $\bigoplus_{m\geqslant 0} R^i f_{0*}(F_0(-m))$, which is indeed coherent by virtue of hypothesis c) and of 1.5.

This proves 3.2 (i). For 3.2 (ii), we will need the

90 ***Lemma 3.3***. — *Under the conditions of* a), b), c) *of* 3.1, *let*

$$G_m = \widehat{f}_*(\mathfrak{F}(\cdot)_m) = \bigoplus_n \widehat{f}_*(\mathfrak{F}_m(n))$$

Then the projective system (G_m) *satisfies the Mittag-Leffler condition.*

We can suppose S is affine, with ring A. Let then $\mathscr{S}$ be a finitely generated graded A-algebra with positive degrees, and $t' \in \mathscr{S}_1$, such that X embeds in $\operatorname{Proj}(\mathscr{S})$, $\mathscr{O}_X(1)$ being induced by $\mathscr{O}(1)$ and the section t being the image of t'. Let us equip $\mathscr{S}$ with the $\mathscr{J}$-adic filtration, where $\mathscr{J} = t'\mathscr{S}$, and consider the projective system of the $\mathfrak{F}(\cdot)_m$ in the category of abelian sheaves on X_0. We are still under the preliminary conditions of EGA 0_{III} 13.7.7[(*)] and moreover $H^i(X_0, \operatorname{gr}(\mathfrak{F}(\cdot))$ is a Module of finite type over $\operatorname{gr}_{\mathscr{J}}(\mathscr{S})$ for $i = 0, 1$. Indeed, since t' is $\mathfrak{F}$-regular, we immediately see that as a Module over $(\mathscr{S}/t'\mathscr{S})[T]$ (of which $\operatorname{gr}_{\mathscr{J}}(\mathscr{S})$ is a quotient), the Module considered is identified with $H^i(X_0, F_0(\cdot)) \otimes_{\mathscr{S}/t'\mathscr{S}} (\mathscr{S}/t'\mathscr{S})[T]$, but by virtue of 1.5, $H^i(X_0, F_0(\cdot))$ is of finite type over $\mathscr{S}$, thus over $\mathscr{S}/t'\mathscr{S}$, for $i = 0, 1$, which proves our assertion. Consequently we are under the conditions for applying 0_{III} 13.7.7 with $n = 1$, which proves 3.3.

This point acquired (and still assuming S is affine, which is permissible to prove 3.2 (ii)) let m_0 be such that $m \geqslant m_0$ implies $\operatorname{Im}(G_m \to G_0) = \operatorname{Im}(G_{m_0} \to G_0)$, so that the two members are also equal to $\operatorname{Im}(\varprojlim G_m \to G_0) = \operatorname{Im}(\widehat{f}_*(\mathfrak{F}(\cdot)) \to f_*\mathfrak{F}_0(\cdot))$. Let us now note that for large n, $\mathfrak{F}_{m_0}(n)$ is generated by its sections, so $\mathfrak{F}_0(n)$ is generated

(*) Rectified as indicated in IX p. 89.

by sections that lift to $\mathfrak{F}_{m_0}$, thus (thanks to the choice of m_0) that lift to $\mathfrak{F}$. So the sections of $\mathfrak{F}$ generate $\mathfrak{F}_0$, thus also $\mathfrak{F}$ thanks to Nakayama. This proves 3.2 (ii), hence 3.1.

Corollary 3.4. — *Let $f : \mathrm{X} \to \mathrm{S}$ be a projective and flat morphism, with* S *locally noetherian, $\mathscr{O}_{\mathrm{X}}(1)$ an invertible Module on* X, *ample relative to* S, *t a section of this Module such that for any $s \in \mathrm{S}$, the section t_s induced on the fiber X_s is $\mathscr{O}_{\mathrm{X}_s}$-regular,* **91**
X_0 the sub-prescheme of zeros of t, $\widehat{\mathrm{X}}$ the formal completion of X *along X_0. We suppose that for any $s \in \mathrm{S}$, X_{0s} has depth $\geqslant 2$ at its closed points,* (*i.e.* X_s *has depth* $\geqslant 3$ *at the closed points of* X_{0s}) *and X_s has depth $\geqslant 2$ at its closed points. Under these conditions, the pair* $(\mathrm{X}, \mathrm{X}_0)$ *satisfies the effective Lefschetz condition* (Leff) *of paragraph* 2 *of exposition* X, *i.e. :*

a) *For any open neighborhood* U *of* X_0 *in* X, *the functor*

$$\mathrm{F} \longmapsto \widehat{\mathrm{F}}$$

from the category of coherent locally free Modules on U *to the category of coherent locally free Modules on* $\widehat{\mathrm{X}}$ *is fully faithful.*

b) *For any coherent locally free Module* $\mathfrak{F}$ *on* $\widehat{\mathrm{X}}$, *there exists an open neighborhood* U *of* X_0, *and a coherent locally free Module* F *on* U *such that* $\mathfrak{F}$ *is isomorphic to* $\widehat{\mathrm{F}}$.

Indeed, a) has already been noted in 2.4 under weaker conditions. For b), we apply 3.1 which gives the conclusion, at least if S is noetherian and admits a relatively ample invertible Module, in particular if S is affine. Indeed, if F is a coherent module on X such that $\widehat{\mathrm{F}}$ is isomorphic to $\mathfrak{F}$ hence locally free, it follows that F is locally free on a neighborhood U of X_0, and $\mathrm{F}|_{\mathrm{U}}$ will satisfy the desired condition. But let us now note that by virtue of 2.5, for such an F, its image by the immersion $\mathrm{U} \to \mathrm{X}$ is coherent, and moreover independent of the chosen solution (U, F) (taking into account that two solutions coincide in a neighborhood of X_0, by virtue of a)). Precisely, we can find a coherent Module F on X and an isomorphism $\widehat{\mathrm{F}} \xrightarrow{\sim} \mathfrak{F}$, such that F has depth $\geqslant 2$ at all points of X which are closed in their fiber, and this determines F up to a unique isomorphism. Thanks to this uniqueness property, the solutions to the problem that we find by inducing above the affine open sets of S glue together, whence a coherent F on the whole of X and an isomorphism $\widehat{\mathrm{F}} \simeq \mathfrak{F}$. Restricting F to the open set U of points where it is free, we find what we were looking for.

Thanks to 2.4 and 3.4, we can exploit, in the situation of a projective algebraic **92**
scheme and a "hyperplane section" thereof, the general facts established in expositions X and XI concerning the conditions (Lef) and (Leff). Thus :

Corollary 3.5. — *Let* X *be a projective algebraic scheme equipped with an ample invertible Module $\mathscr{O}_{\mathrm{X}}(1)$, let t be a section of this Module which is $\mathscr{O}_{\mathrm{X}}$-regular, and let X_0 be the subscheme of zeros of t. Suppose that* X *has depth* $\geqslant 2$ *at its closed points* (*resp. and depth* $\geqslant 3$ *at the closed points of* X_0). *Then* $\pi_0(\mathrm{X}_0) \to \pi_0(\mathrm{X})$ *is* *bijective, in particular* X *is connected if and only if* X_0 *is, and choosing a geometric base-point in* X_0*,* $\pi_1(X_0) \to \pi_1(X)$ *is surjective, and more generally for any open* $U \supset X_0$*, the homomorphism* $\pi_1(X_0) \to \pi_1(U)$ *is surjective* (*resp. the homomorphism* $\pi_1(X_0) \to \varprojlim_U \pi_1(U)$ *is bijective*)*. In the resp. case, if we suppose moreover that the local ring of any closed point of* X *not in* X_0 *is pure* (3.2) (*for example is regular, or only a complete intersection*) *then* $\pi_1(X_0) \to \pi_1(X)$ *is an isomorphism.*

We apply 2.4 and 3.4. Note that in the resp. case, the hypothesis that X has depth $\geqslant 3$ at the closed points of X_0, implies that all irreducible components of dimension $\neq 0$ of X have dimension $\geqslant 3$ (as can be seen by noting that such a component necessarily meets X_0, and by looking at a closed point of the intersection).

Remarque. — When X is normal, of dimension $\geqslant 2$ at all its points, it has depth $\geqslant 2$ at its closed points and we are under the non-resp. conditions of 3.5. In this case, we have a more elementary proof of the surjectivity of $\pi_1(X_0) \to \pi_1(X)$ using Bertini's theorem (cf. SGA 1 X.2.10). When we suppose moreover that X_0 is normal, and X has dimension $\geqslant 3$ at all its points, then we are under the resp. conditions of 3.5. In this case, 3.5 has been established by Grauert (it turns out that thanks to the hypothesis of normality, one can then manage to do without the existence theorem 3.1
93 by expedients). It is this proof by Grauert that was the starting point of the "Lefschetz theory" which is the subject of the present seminar.

Corollary 3.6. — *Let* X*,* $\mathscr{O}_X(1)$*,* t*,* X_0 *be as in* 3.5*. Suppose that* X *has depth* $\geqslant 2$ *at its closed points, and that*

$$H^i(X_0, \mathscr{O}_{X_0}(-n)) = 0$$

for $n > 0$ *and for* $i = 1$ (*resp. for* $i = 1$ *and for* $i = 2$)*, which implies by virtue of* 1.4 *that* X_0 *has depth* $\geqslant 2$ (*resp.* $\geqslant 3$) *at its closed points, i.e. that* X *has depth* $\geqslant 3$ (*resp.* $\geqslant 4$) *at the closed points of* X_0*. Under these conditions, for any open neighborhood* U *of* X_0*,* $\mathrm{Pic}(U) \to \mathrm{Pic}(X_0)$ *is injective, in particular* $\mathrm{Pic}(X) \to \mathrm{Pic}(X_0)$ *is injective,* (*resp.* $\varprojlim_U \mathrm{Pic}(U) \to \mathrm{Pic}(X_0)$ *is bijective*)*. In the resp. case, if we suppose moreover that the local ring of* X *at any closed point not in* X_0 *is parafactorial* (3.1) (*for example is regular, or more generally a complete intersection*)*, then* $\mathrm{Pic}(X) \to \mathrm{Pic}(X_0)$ *is bijective.*

We apply XI 3.12 and 3.13, noting that the resp. hypothesis implies that the irreducible components of dimension $\neq 0$ of X have dimension $\geqslant 4$. We find in particular, by applying this to the case where X is a global complete intersection of dimension $\geqslant 4$ in projective space :

Corollary 3.7. — *Let* X *be an algebraic scheme of dimension* $\geqslant 3$*, which is a complete intersection in a scheme* $\mathbf{P}^r_k$*. Then* $\mathrm{Pic}(X)$ *is the free group generated by the class of the sheaf* $\mathscr{O}_X(1)$*.*

We argue by induction on the number of hypersurfaces of which X is the intersection, by applying 3.6 and noting that for a complete intersection X of dimension $\geqslant 3$, we have $\mathrm{H}^i(\mathrm{X}, \mathscr{O}_{\mathrm{X}}(n)) = 0$ for $i = 1, 2$, and any n.

Remark 3.8. — In the case where X is a non-singular hypersurface, 3.7 is due to Andreotti. The result 3.7 is also expressed (when X is non-singular) by saying that the homogeneous coordinate ring of X is factorial, and in this form is contained in XI 3.13 (ii). Let us also point out that Serre had given a proof of 3.7 in the non-singular case, by transcendental means, using a specialization argument to reduce to the case of characteristic 0, where we have Lefschetz's theorem in its classical form. Of course, the fact that the purely algebraic proof given here allows us to get rid of hypotheses of non-singularity in the statement of Lefschetz's theorem, invites us to reconsider it also in the classical case. Cf. the following exposition which proposes conjectures in this direction. 94

In corollaries 3.5 and 3.7 we have placed ourselves over a base field, whereas the key theorems 2.4 and 3.4 are valid over any base. To generalize to a general S these corollaries 3.5 and 3.6, we must give serviceable criteria for a point of X (flat over S) to have a "pure" resp. parafactorial local ring. This will be the subject of the next number.

4. Formal completion and normal flatness

Theorem 4.1. — *Let* X *be a locally noetherian prescheme, locally immersible in a regular scheme,* Y *a closed subset of* X, $\mathrm{U} = \mathrm{X} - \mathrm{Y}$, X_0 *a closed sub-prescheme of* X *defined by an ideal* $\mathscr{J}$, $\widehat{\mathrm{X}}$ *the formal completion of* X *along* X_0, U_0 *the trace of* X_0 *on* U, $\widehat{\mathrm{U}}$ *the formal completion of* U *along* U_0, $i : \mathrm{U} \to \mathrm{X}$ *and* $\widehat{i} : \widehat{\mathrm{U}} \to \widehat{\mathrm{X}}$ *the canonical immersions,* n *an integer. We suppose*

a) X *is normally flat along* X_0 *at the points of* $\mathrm{Y} \cap \mathrm{X}_0$, *i.e. at these points the Modules* $\mathscr{J}^n/\mathscr{J}^{n+1}$ *on* X_0 *are flat i.e. locally free.*

b) *For any* $x \in \mathrm{Y} \cap \mathrm{X}_0$, *we have* prof $\mathscr{O}_{\mathrm{X}_0,x} \geqslant n+2$.

Under these conditions, we have the following : 95

(1) *Let* F *be a coherent Module on* U, *suppose that we have :*

c) *For any* $x \in \mathrm{Y} - \mathrm{Y} \cap \mathrm{X}_0$, *we have* prof $\mathscr{O}_{\mathrm{X},x} \geqslant n+2$.

d) F *is free at the points of* U_0, *and of depth* $\geqslant n+1$ *at all points of* U *where it is not free.*

Then the graded Module

$$\bigoplus_{m\geqslant 0} \mathrm{R}^p\, i_*(\mathscr{J}^m \mathrm{F})$$

over $\bigoplus_{m\geqslant 0} \mathscr{J}^m$ *is of finite type for* $p \leqslant n$.

(2) *Let* $\mathfrak{F}$ *be a coherent Module on* $\widehat{\mathrm{U}}$*, then the graded Module*

$$\bigoplus_{m\geqslant 0} \mathrm{R}^p\, i_*(\mathscr{J}^m\mathfrak{F}/\mathscr{J}^{m+1}\mathfrak{F})$$

over $\bigoplus_{m\geqslant 0} \mathscr{J}^m/\mathscr{J}^{m+1} = \mathrm{gr}_{\mathscr{J}}(\mathscr{O}_{\mathrm{X}})$ *is of finite type for* $p \leqslant n$.

Démonstration. — 1) Let $\mathrm{X}' = \mathrm{Spec}(\bigoplus_{m\geqslant 0} \mathscr{J}^m)$; the base change $f : \mathrm{X}' \to \mathrm{X}$ then defines $\mathrm{U}' = \mathrm{X}' - \mathrm{Y}'$, X'_0, $\mathrm{U}'_0 = \mathrm{X}'_0\mathrm{U}'$, and immersions $i' : \mathrm{U}' \to \mathrm{X}'$, $i'_0 : \mathrm{U}'_0 \to \mathrm{X}'_0$. We thus have a cartesian square

$$\begin{array}{ccc} \mathrm{X}' & \xleftarrow{\;i'\;} & \mathrm{U}' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ \mathrm{X} & \xleftarrow{\;i\;} & \mathrm{U} \end{array}$$

and we have

$$\bigoplus_{m\geqslant 0} \mathrm{R}^p\, i_*(\mathscr{J}^m\mathrm{F}) = \mathrm{R}^p\, i_*\Big(\bigoplus_{m\geqslant 0} \mathscr{J}^m\mathrm{F}\Big) = \mathrm{R}^p\, i_*(g_*(\mathrm{F}')), \tag{19}$$

96 where $\mathrm{F}' = g^*\mathrm{F}$, so that we indeed have a canonical isomorphism

$$g_*(\mathrm{F}') \xrightarrow{\sim} \bigoplus_{m\geqslant 0} \mathscr{J}^m\mathrm{F},$$

because this is true at the points of U_0, due to the fact that F is free there by virtue of d), and also at the points of U_0, due to the fact that there we have $\mathscr{J}^m = \mathscr{O}_{\mathrm{U}}$, (so that in both cases, $\mathscr{J}^m \otimes_{\mathscr{O}_{\mathrm{U}}} \mathrm{F} \to \mathscr{J}^m\mathrm{F}$ is an isomorphism).

On the other hand, since f and consequently g are affine, we have

$$\mathrm{R}^p\, i_*(g_*(\mathrm{F}')) = \mathrm{R}^p(ig)_*(\mathrm{F}') = \mathrm{R}^p(fi')_*(\mathrm{F}') = f_*(\mathrm{R}^p i'_*(\mathrm{F}')) \tag{20}$$

so comparing (19) and (20), we see that assertion 1) is equivalent to the following : $\mathrm{R}^p i'_*(\mathrm{F}')$ is a Module of finite type i.e. coherent over X', for any $p \leqslant n$. But since X is locally immersible in a regular scheme, the same is true of X' which is of finite type over X, and we can apply the coherence criterion VIII 2.3 to a coherent extension F'' of F' : we want to express that $\underline{\mathrm{H}}^p_{\mathrm{Y}}(\mathrm{F}'')$ is coherent for $p \leqslant n+1$, and this is also equivalent to saying that for any $x' \in \mathrm{U}'$ such that

$$\mathrm{codim}(\overline{x'} \cap \mathrm{Y}', \overline{x'}) = 1, \tag{20 bis}$$

we have

$$\mathrm{prof}\, \mathrm{F}'_{x'} \geqslant n+1. \tag{21}$$

But this condition is satisfied at the points x' where F' is not free, because for such an x' we have $x' \notin \mathrm{U}'_0$ by virtue of d), so g is an isomorphism there, and by virtue of d) again, F has depth $\geqslant n+1$ at $g(x')$, so F' has depth $\geqslant n+1$ at x'. It therefore suffices to check condition (21) at the $x' \in \mathrm{U}'$ satisfying (21), and at which F' is free.

97 But for this it is sufficient to prove that we have

$$\mathrm{prof}\, \mathscr{O}_{\mathrm{X}',x'} \geqslant n+1 \tag{21 bis}$$

at these points, a fortiori it is sufficient to establish that we have this relation at all points x of U' satisfying (20 bis). But, again by virtue of criterion 2.3 of exposition VIII, this is equivalent to the assertion that the Modules

$$\underline{\mathrm{H}}^p_{\mathrm{Y}'}(\mathscr{O}_{\mathrm{X}'}) \qquad \text{for } p \leqslant n+1$$

are coherent. In fact, we are going to prove that they are even zero, or what amounts to the same by virtue of exposition III, that we have

$$\text{(22)} \qquad \operatorname{prof} \mathscr{O}_{\mathrm{X}',x'} \geqslant n+2 \qquad \text{for any } \ x' \in \mathrm{Y}'.$$

For this, we distinguish two cases. If $x' \notin \mathrm{X}'_0$, then f is an immersion at x', and we must check that F has depth $\geqslant n+2$ at the image $x = f(x')$, which is none other than condition c). If on the other hand $x' \in \mathrm{X}'_0$, i.e. $x = f(x') \in \mathrm{X}_0$ so $x \in \mathrm{Y} \cap \mathrm{X}_0$, we apply conditions a) and b) thanks to the

Lemma 4.2. — *Let* X *be a locally noetherian prescheme,* X_0 *a closed sub-prescheme of* X *defined by an ideal* $\mathscr{J}$, $\mathrm{X}' = \operatorname{Spec}(\bigoplus_{m \geqslant 0} \mathscr{J}^m)$, $\mathrm{X}'_0 = \operatorname{Spec}(\bigoplus_{m \geqslant 0} \mathscr{J}^m/\mathscr{J}^{m+1}) = \mathrm{X}' \times_{\mathrm{X}} \mathrm{X}_0$, x *a point of* X_0 *at which* X *is normally flat along* X_0 *i.e. such that* $\operatorname{gr}_{\mathscr{J}}(\mathscr{O}_{\mathrm{X}}) = \bigoplus_{m \geqslant 0} \mathscr{J}^m/\mathscr{J}^{m+1}$ *is flat there as a Module over* X_0. *Then for any sequence of elements* f_i $(1 \leqslant i \leqslant m)$ *of* $\mathscr{O}_{\mathrm{X},x}$ *whose images in* $\mathscr{O}_{\mathrm{X}_0,x}$ *form an* $\mathscr{O}_{\mathrm{X}_0,x}$*-regular sequence, and any* $x' \in \mathrm{X}'$ *above* x, *the images of the* f_i *in* $\mathscr{O}_{\mathrm{X}',x'}$ (*resp. in* $\mathscr{O}_{\mathrm{X}'_0,x'}$) *also form an* $\mathscr{O}_{\mathrm{X}',x'}$*-regular sequence* (*resp.* $\mathscr{O}_{\mathrm{X}'_0,x'}$*-regular sequence*), *in particular we have*

$$\text{(23)} \qquad \operatorname{prof} \mathscr{O}_{\mathrm{X}',x'} \geqslant \operatorname{prof} \mathscr{O}_{\mathrm{X}_0,x}, \quad \operatorname{prof} \mathscr{O}_{\mathrm{X}'_0,x'} \geqslant \operatorname{prof} \mathscr{O}_{\mathrm{X}_0,x}.$$

To prove it, we can suppose that X is local with closed point x, thus affine with ring 98
$\mathrm{A} = \mathscr{O}_{\mathrm{X},x}$, $\mathscr{J}$ being defined by the ideal J, and it is sufficient to prove that for any sequence f_i $(1 \leqslant i \leqslant m)$ of elements of A whose images in A/J form an A/J-regular sequence, the f_i also form a $(\bigoplus_{m \geqslant 0} \mathrm{J}^m)$-regular sequence and a $(\bigoplus_{m \geqslant 0} \mathrm{J}^m/\mathrm{J}^{m+1})$-regular sequence, i.e. for any m, it forms a J^m-regular sequence and a $\mathrm{J}^m/\mathrm{J}^{m+1}$-regular sequence. The second assertion is trivial, since $\mathrm{J}^m/\mathrm{J}^{m+1}$ is a free module over A/J. The first follows, by looking at the J-adic filtration of J^m, and noting that for the graded module associated with J^m for said filtration, the sequence of f_i is regular.

This proves 4.2 and consequently 4.1 1).

Let us prove 4.1 2). For this, we use the cartesian square

$$\begin{tikzcd} \mathrm{X}'_0 \arrow[d, "f_0"'] & \mathrm{U}'_0 \arrow[l, "i'_0"'] \arrow[d, "g_0"] \\ \mathrm{X}_0 & \mathrm{U}_0 \arrow[l, "i_0"'] \end{tikzcd}$$

and proceeding as at the beginning of the proof of 1), we find that we have

$$\text{(24)} \qquad \bigoplus_{m \geqslant 0} \mathrm{R}^p\, i_*(\mathscr{J}^m \mathfrak{F}/\mathscr{J}^{m+1}\mathfrak{F}) \simeq f_{0*}(\mathrm{R}^p\, i'_{0*}(\mathfrak{F}'_0)),$$

where $\mathfrak{F}_0 = \mathfrak{F}/\mathscr{J}\mathfrak{F}$ and where $\mathfrak{F}'_0 = g_0^*(\mathfrak{F}_0)$, (using the fact that $\mathfrak{F}$ is locally free). So the conclusion of 2) is equivalent to saying that for $p \leqslant n$, $\mathrm{R}^p\, i'_{0*}(\mathfrak{F}'_0)$ is a coherent Module.

Here again, taking into account that $\mathfrak{F}'_0$ is locally free, criterion VIII 2.3 allows us to reduce to proving that it is so if we replace $\mathfrak{F}'_0$ by $\mathscr{O}_{\mathrm{U}'_0}$, i.e. to proving that the
99 Modules

$$\underline{\mathrm{H}}^p_{\mathrm{Y}'_0}(\mathscr{O}_{\mathrm{X}'_0}) \quad \text{for } p \leqslant n+1 \quad (\text{where } \mathrm{Y}'_0 = \mathrm{Y}' \cap \mathrm{X}'_0 = \mathrm{X}'_0 - \mathrm{U}'_0)$$

are coherent. We again prove that they are in fact zero, i.e. that we have

$$\text{(25)} \qquad \operatorname{prof} \mathscr{O}_{\mathrm{X}'_0,x'} \geqslant n+2 \quad \text{for any } x' \in \mathrm{Y}'_0.$$

But this results indeed from conditions a) and b), taking into account 4.2. This completes the proof of 4.1.

Remark 4.3. — We see immediately, by descent, that the hypothesis : X locally immersible in a regular scheme, can be replaced by the following weaker one : there exists a morphism $\overline{\mathrm{X}} \to \mathrm{X}$, faithfully flat and quasi-compact, such that $\overline{\mathrm{X}}$ is locally immersible in a regular scheme.

Theorem 4.1 puts us in a position to apply the results of exposition IX (comparison and existence theorems). We will be particularly interested in the

Corollary 4.4. — *Suppose conditions a), b), c) of theorem* 4.1 *are satisfied, with* $n = 1$, *and* $\mathrm{X} = \operatorname{Spec}(\mathrm{A})$, *with* A *being separated and complete for the* J*-adic topology. Then*

(1) *The functor* $\mathrm{F} \mapsto \widehat{\mathrm{F}}$ *from the category of coherent locally free Modules on* U, *to the category of coherent locally free Modules on* $\widehat{\mathrm{U}}$, *is fully faithful.*

(2) *For any coherent locally free Module* $\mathfrak{F}$ *on* $\widehat{\mathrm{U}}$, *there exists a coherent Module* F *on* U *and an isomorphism* $\widehat{\mathrm{F}} \xrightarrow{\sim} \mathfrak{F}$.

In particular, if for any $x \in \mathrm{U}$ *whose closure in* U *does not meet* U_0 *i.e. such that* $\overline{x} \cap \mathrm{X}_0 \subset \mathrm{Y}$, *we have* $\operatorname{prof} \mathscr{O}_{\mathrm{U},x} \geqslant 2$, *then the pair* $(\mathrm{U}, \mathrm{U}_0)$ *satisfies the effective Lefschetz condition* (Leff) *of exposition* X.

(For the last assertion, we proceed as in X 2.1)

Particular case of 4.4 :

100 ***Corollary 4.5***. — *Let* A *be a noetherian ring,* J *an ideal of* A *contained in the radical,* $\mathrm{A}_0 = \mathrm{A}/\mathrm{J}$. *We suppose*

(i) $\operatorname{prof} \mathrm{A}_0 \geqslant 3$.

(ii) $\operatorname{gr}_{\mathrm{J}}(\mathrm{A})$ *is a free* A_0*-module.*

(iii) A *is complete for the* J*-adic topology.*

Let $\mathrm{X} = \operatorname{Spec}(\mathrm{A})$, $\mathrm{X}_0 = \operatorname{Spec}(\mathrm{A}_0) = \mathrm{V}(\mathrm{J})$, *a the closed point of* X, $\mathrm{U} = \mathrm{X} - \{a\}$, $\mathrm{U}_0 = \mathrm{X}_0 - \{a\}$, $\widehat{\mathrm{U}}$ *the formal completion of* U *along* U_0. *Then the functor* $\mathrm{F} \mapsto \widehat{\mathrm{F}}$ *from the category of coherent locally free Modules on* U *to the category of coherent locally* *free Modules on* $\widehat{\mathrm{U}}$*, is fully faithful. Moreover, for any coherent locally free Module* $\mathfrak{F}$ *on* $\widehat{\mathrm{U}}$*, there exists a coherent Module* (*not necessarily locally free!*) F *on* U*, and an isomorphism* $\widehat{\mathrm{F}} \simeq \mathfrak{F}$.

Note that thanks to 4.3, we did not have to assume that A is a quotient of a regular ring, because the completion of A for the $\mathfrak{r}$(A)-adic topology satisfies this condition in any case.

Proceeding as in expositions X and XI, we conclude from 4.5

Corollary 4.6. — *Under the conditions of* 4.5*, we have this :*

a) U *and* U_0 *are connected* (III 3.1).

Choosing a geometric base-point in U_0*, the homomorphism*

$$\pi_1(\mathrm{U}_0) \longrightarrow \pi_1(\mathrm{U})$$

is surjective.

b) *The homomorphism*

$$\mathrm{Pic}(\mathrm{U}) \longrightarrow \mathrm{Pic}(\mathrm{U}_0)$$

is injective.

To prove b), taking into account 4.5, this amounts to checking that any isomorphism $\mathrm{L}'_0 \xrightarrow{\sim} \mathrm{L}_0$ lifts to an isomorphism $\widehat{\mathrm{L}'} \xrightarrow{\sim} \widehat{\mathrm{L}}$. But for this we lift step by step to isomorphisms $\mathrm{L}'_n \xrightarrow{\sim} \mathrm{L}_n$, the obstructions are in $\mathrm{H}^1(\mathrm{U}_0, \mathscr{J}^n/\mathscr{J}^{n+1})$, but these modules are zero due to the fact that $\mathrm{J}^n/\mathrm{J}^{n+1}$ is free and $\mathrm{prof}\,\mathrm{A}_0 \geqslant 3$. 101

We are now in a position to prove the

Theorem 4.7. — *Let* A *be a noetherian local ring,* J *an ideal of* A *contained in its radical,* $\mathrm{A}_0 = \mathrm{A}/\mathrm{J}$. *We suppose*

(i) $\mathrm{prof}\,\mathrm{A}_0 \geqslant 3$.

(ii) $\mathrm{gr}_{\mathrm{J}}(\mathrm{A})$ *is a free module over* A_0.

Then, if A_0 *is "pure"* (X 3.1) (*resp. parafactorial* (XI 3.1))*, the same is true of* A.

Démonstration. — By descent, we can suppose that we also have

(iii) A is complete for the J-adic topology.

Indeed, by virtue of (i) and (ii), we have $\mathrm{prof}(\mathrm{A}) \geqslant 3$ so $\mathrm{prof}(\widehat{\mathrm{A}}) \geqslant 3$, where $\widehat{\mathrm{A}}$ is the completion of A for the J-adic topology, and we apply X 3.6 and XI 3.6. We are thus under the conditions of 4.5. Since $\mathrm{prof}(\mathrm{A}) \geqslant 3 \geqslant 2$, saying that A is parafactorial simply means that $\mathrm{Pic}(\mathrm{U}) = 0$, and by virtue of 4.6 b) it is sufficient for this that $\mathrm{Pic}(\mathrm{U}_0) = 0$, i.e. that A_0 is parafactorial. To prove that A is "pure" if A_0 is so, we must prove that if V is an étale covering of U, defined by an algebra $\mathscr{B}$ over U, then $\mathrm{H}^0(\mathrm{U}, \mathscr{B})$ is a finite étale algebra over A. But since A_0 is pure, the same is true of the A_n (which differ from it only by nilpotent elements), so for any n, $\mathrm{B}_n = \mathrm{H}^0(\mathrm{U}, \mathscr{B}_n)$ is an étale algebra over $\mathrm{A}/\mathrm{J}^{n+1}$, and of course these algebras glue together, so that

$\varprojlim \mathrm{B}_n$ is an étale algebra over A. But by virtue of 4.5, this algebra is none other than $\mathrm{H}^0(\mathrm{U}, \mathscr{B})$, which establishes our assertion.

102 ***Corollary 4.8***. — *Let $f : \mathrm{X} \to \mathrm{Y}$ be a flat morphism of locally noetherian preschemes, $x \in \mathrm{X}$, $y = f(x)$, we suppose that $\mathscr{O}_{\mathrm{X}_y,x}$ is a "pure" (resp. parafactorial) local ring of depth $\geqslant 3$, then the same is true of $\mathscr{O}_{\mathrm{X},x}$.*

This is the result of the type promised at the end of the preceding number, to generalize corollaries 3.5 and following. We thus find, using 3.4, the

Corollary 4.9. — *Let $f : \mathrm{X} \to \mathrm{S}$ be a projective and flat morphism, with* S *locally noetherian, $\mathscr{O}_{\mathrm{X}}(1)$ an invertible Module on* X *ample over* S, *t a section of $\mathscr{O}_{\mathrm{X}}(1)$ such that for any $s \in \mathrm{S}$, the section t_s induced on X_s is $\mathscr{O}_{\mathrm{X}_s}$-regular, X_0 the subscheme of zeros of t, X_m the sub-scheme of zeros of t^{m+1}. We suppose that for any $s \in \mathrm{S}$, X_s has depth $\geqslant 3$ at all its closed points. Then :*

a) *If the local rings of the closed points of $\mathrm{X}_s - \mathrm{X}_{0,s}$ ($s \in \mathrm{S}$) are "pure", for example are complete intersections, then the functor $\mathrm{X}' \mapsto \mathrm{X}'_0 = \mathrm{X}' \times_{\mathrm{X}} \mathrm{X}_0$ from the category of étale coverings of* X *to the category of étale coverings of X_0 is an equivalence of categories ; in particular, choosing a geometric base point in X_0, the homomorphism*

$$\pi_1(\mathrm{X}_0) \longrightarrow \pi_1(\mathrm{X})$$

is an isomorphism[4]

b) *If the local rings of the closed points of the $\mathrm{X}_s - \mathrm{X}_{0,s}$ ($s \in \mathrm{S}$) are "parafactorial", for example regular, or complete intersections of dimension $\geqslant 4$, then for any integer m such that $\mathrm{R}^i f_{0*}(\mathscr{O}_{\mathrm{X}_0}(-n)) = 0$ for $n > m$ and $i = 1, 2$, the map $\mathrm{Pic}(\mathrm{X}) \to \mathrm{Pic}(\mathrm{X}_m)$ is bijective.*

Moreover if S *is noetherian, and the $\mathrm{X}_{0,s}$ have depth $\geqslant 3$ at their closed points, such m exist* (*cf.* 1.5).

[4] N.D.E. : let us mention the spectacular connectedness result obtained since by Fulton and Hansen, in the case where $\mathrm{S} = \mathrm{Spec}(k)$ (k algebraically closed field). Let $g : \mathrm{X} \to \mathbf{P}^m_k \times \mathbf{P}^m_k$ be such that $\dim g(\mathrm{X}) > m$; then, the inverse image of the diagonal is connected. This allows among other things to generalize corollary 4.9 when f is the structural morphism of the projective $\mathbf{P}^m_k$ over $\mathrm{S} = \mathrm{Spec}(k)$: precisely, an irreducible subvariety X of $\mathbf{P}^m_k$ of dimension $> m/2$ has a trivial fundamental group ! (cf. Fulton W. & Hansen J., « A connectedness theorem for projective varieties, with applications to intersections and singularities of mappings », *Ann. of Math.* (*2*) **110** (1979), N° 1, p. 159-166). For generalizations to the case of grassmannians or abelian varieties, see Debarre O., « Théorèmes de connexité pour les produits d'espaces projectifs et les grassmanniennes », *Amer. J. Math.* **118** (1996), N° 6, p. 1347-1367 and « Théorèmes de connexité et variétés abéliennes », *Amer. J. Math.* **117** (1995), N° 3, p. 787–805. The result of triviality of the fundamental group of X as above has been obtained independently by Faltings, who proves moreover that Pic(X) has no torsion prime to the characteristic of k, by methods of algebrization of formal bundles, more in the line of Grothendieck's techniques, cf. (Faltings G., « Algebraization of some formal vector bundles » *Ann. of Math.* (*2*) **110** (1979), N° 3, p. 501–514).

1.5 that there exists an m such that $n > m$ implies even $\mathrm{H}^i(\mathrm{X}_0, \mathscr{O}_{\mathrm{X}_0}(-n)) = 0$ for $i = 1, 2$, and even for $i \leqslant 2$). This condition is stronger than $\mathrm{R}^i f_*(\mathscr{O}_{\mathrm{X}_0}(-n)) = 0$ for $i = 1, 2$, and it has moreover the advantage of being stable under base change. The same is true of the depth hypotheses we made in 4.9, and also of a hypothesis of the type "the X_s are locally complete intersections". It follows then, under these conditions that 4.9 b) also implies that the morphism of functors

$$\mathbf{Pic}_{\mathrm{X}/\mathrm{S}} \longrightarrow \mathbf{Pic}_{\mathrm{X}_n/\mathrm{S}}$$

in $\mathbf{Sch}_{/\mathrm{S}}$ is an isomorphism, so also the morphism for the relative Picard schemes, when these exist :

$$\mathrm{Pic}_{\mathrm{X}/\mathrm{S}} \longrightarrow \mathrm{Pic}_{\mathrm{X}_n/\mathrm{S}}\,.$$

Even in the case where S is the spectrum of an algebraically closed field, this statement is clearly more precise than the statement consisting in saying only that $\mathrm{Pic}(\mathrm{X}) \to \mathrm{Pic}(\mathrm{X}_n)$ is bijective.

One can wonder if we can always take $n = 0$ in the preceding conclusions (supposing therefore the $\mathrm{X}_{0,s}$ have depth $\geqslant 3$ at their closed points). When X_0 is smooth over S and the residual characteristics of S are zero, it is indeed so, by virtue of Kodaira's "vanishing theorem", (proven by transcendental means, using a kähler metric) which implies that for any connected smooth projective scheme of dimension n over a field k of characteristic zero, and any ample invertible Module L on X, we have $\mathrm{H}^i(\mathrm{X}, \mathrm{L}^{-1}) = 0$ for $i \neq n$. It is not known[(5)] at present if this theorem can be replaced by a generalization in characteristic $p > 0$, and if the smoothness hypothesis can be replaced by a hypothesis of a more general nature (concerning depth, or of the "complete intersection" type...).

(5) N.D.E. : as Raynaud has remarked, the decomposition result of the de Rham complex of Deligne and Illusie easily implies the vanishing of the group $\mathrm{H}^i(\mathrm{X}, \mathrm{L}^{-1})$ (with L ample on X smooth projective over k of characteristic $p > 0$) for $i < \inf(p, \dim(\mathrm{X}))$ as soon as we suppose that X is liftable to a flat scheme over $\mathrm{W}_2(k)$ (cf. Deligne P. & Illusie L., « Relèvements modulo p^2 et décomposition du complexe de de Rham », *Invent. Math.* **89** (1987), N° 2, p. 247–270) ; this gives a purely algebraic proof of Kodaira's result for projective varieties in characteristic zero. If X is not liftable, it is well known that the "vanishing theorem" is false ; cf. the example in (Raynaud M., « Contre-exemple au "vanishing theorem" en caractéristique $p > 0$ », in *C.P. Ramanujam—a tribute*, Tata Inst. Fund. Res. Studies in Math., vol. 8, Springer, Berlin-New York, 1978, p. 273–278) ; see also the very nice examples in (Haboush W. & Lauritzen N., « Varieties of unseparated flags », in *Linear algebraic groups and their representations* (*Los Angeles, CA, 1992*), Contemp. Math., vol. 153, American Mathematical Society, Providence, RI, 1993, p. 35–57), simplified in (Lauritzen N. & Rao A.P., « Elementary counterexamples to Kodaira vanishing in prime characteristic », *Proc. Indian Acad. Sci. Math. Sci.* **107** (1997), N° 1, p. 21–25). On the other hand, I do not know of an example where the arrow $\mathrm{Pic}(\mathrm{X}_{n+1}) \to \mathrm{Pic}(\mathrm{X}_n)$ is not surjective for $n > 1$ in positive characteristic, where X_n denotes a thickened hyperplane section of X smooth projective as above.

5. Universal finiteness conditions for a non-proper morphism

Let us recall for the record the 104

Proposition 5.1. — *Let* $f : \mathrm{X} \to \mathrm{S}$ *be a proper morphism of preschemes, with* S *locally noetherian,* U *an open subset of* X*,* $g : \mathrm{U} \to \mathrm{X}$ *the canonical immersion,* $h = fg : \mathrm{U} \to \mathrm{S}$*,* F *a Module on* U*. Suppose that the Modules* $\mathrm{R}^i\, g_*(\mathrm{F})$ *are coherent for* $i \leqslant n$ *(hypothesis of a local nature on* X*, which is practically checked using criterion* VIII 2.3*). Then* $\mathrm{R}^i\, h_*(\mathrm{F})$ *is coherent for* $i \leqslant n$*.*

This results immediately from the Leray spectral sequence

$$\mathrm{E}_2^{p,q} = \mathrm{R}^p\, f_*(\mathrm{R}^q\, g_*(\mathrm{F})) \Longrightarrow \mathrm{R}^*\, h_*(\mathrm{F}), \tag{26}$$

and from the fact that the higher direct images by f of a coherent Module on X are coherent (EGA III 3.2.1).

Proposition 5.2. — *Let* S *be a locally noetherian prescheme,* $\mathscr{S}$ *a quasi-coherent graded Algebra, of finite type over* S*, generated by* $\mathscr{S}_1$*,* X *a sub-prescheme of* $\mathrm{Proj}(\mathscr{S})$*,* $\mathscr{O}_{\mathrm{X}}(1)$ *the invertible Module on* X *very ample relative to* S *induced by* $\mathrm{Proj}(\mathscr{S}(1))$*,* U *an open subset of* X*,* $g : \mathrm{U} \to \mathrm{X}$ *the canonical immersion,* $h = fg : \mathrm{U} \to \mathrm{S}$*,* F *a quasi-coherent Module on* U*, whence twisted Modules* $\mathrm{F}(m) = \mathrm{F} \otimes \mathscr{O}_{\mathrm{X}}(m))$ $(m \in \mathbf{Z})$*,* n *an integer,* m_0 *an integer. The following conditions are equivalent :*

(i) $\mathrm{R}^i\, g_*(\mathrm{F})$ *is coherent for* $i \leqslant n$*.*

(ii) $\bigoplus_{m>m_0} \mathrm{R}^i\, h_*(\mathrm{F}(m))$ *is a finitely generated* $\mathscr{S}$*-Module for* $i \leqslant n$*.*

Démonstration. — Replacing in the spectral sequence above F by $\mathrm{F}(m)$ we find a
105 spectral sequence of graded $\mathscr{S}$-Modules

$$\mathrm{E}_2^{p,q} = \bigoplus_{m \geqslant m_0} \mathrm{R}^p\, f_*(\mathrm{R}^q\, g_*(\mathrm{F}(m)) \Longrightarrow \bigoplus_{m \geqslant m_0} \mathrm{R}^*\, h_*(\mathrm{F}(m)).$$

Since we have

$$\mathrm{R}^q\, g_*(\mathrm{F}(m)) \simeq \mathrm{R}^q\, g_*(\mathrm{F})(m),$$

we see that if the $\mathrm{R}^i\, g_*(\mathrm{F})$ are coherent, $\mathrm{E}_2^{p,q}$ is of finite type over $\mathscr{S}$ for $q \leqslant n$, thanks to part a) of lemma 5.3 below, which implies that the abutment is of finite type over $\mathscr{S}$ in degree $i \leqslant n$. This proves (i) ⇒ (ii). Moreover, arguing in the abelian category of graded $\mathscr{S}$-Modules modulo the thick subcategory $\mathscr{C}$ of those which are quasi-coherent of finite type, we find by the preceding spectral sequence

$$\bigoplus_{m \geqslant m_0} \mathrm{R}^{n+1}\, h_*(\mathrm{F}(m)) \simeq \bigoplus_{m \geqslant m_0} f_*(\mathrm{R}^{n+1}\, g_*(\mathrm{F})(m)) \mod \mathscr{C},$$

which proves that if the left hand side is a finitely generated $\mathscr{S}$-Module, then $\mathrm{R}^{n+1}\, g_*(\mathrm{F})$ is coherent, by virtue of part b) of lemma 5.3. This proves the implication (ii) ⇒ (i) by induction on n. It remains to prove :

Lemma 5.3. — *Let* S*,* $\mathscr{S}$*,* X*,* f *be as in* 5.2*, and* G *a quasi-coherent Module on* X*,* m_0 *an integer. Then*

a) *If* G *is coherent, then for any integer* i*, the graded Module*

$$\bigoplus_{m\geqslant m_0} \mathrm{R}^i f_*(\mathrm{G}(m))$$

over $\mathscr{S}$ *is of finite type.*

b) *Conversely, suppose that the Module* $\bigoplus_{m\geqslant m_0} \mathrm{R}^i f_*(\mathrm{G}(m))$ *over* $\mathscr{S}$ *is of finite type, then* G *is coherent.*

Proof of 5.3. — For a), the case $i = 0$ is given in EGA III 2.3.2, the case $i > 0$ in EGA III 2.2.1 (i)(ii) which says that the $\mathrm{R}^i f_*\mathrm{G}(m)$ are coherent, and zero for large 106
m (if we suppose S is noetherian, which is permissible). For b), we note that G is isomorphic to $\mathbf{Proj}(\bigoplus_{m\geqslant m_0} f_*(\mathrm{G}(m))$ (EGA II 3.4.4 and 3.4.2), which proves that G is coherent if $\bigoplus_{m\geqslant m_0} f_*(\mathrm{G}(m))$ is of finite type over $\mathscr{S}$, by virtue of *loc. cit.* 3.4.4.

Corollary 5.4 (S noetherian). — *Suppose that* $\mathrm{R}^i g_*(\mathrm{F})$ *is coherent for* $i \leqslant n$*, then for* $i \leqslant n+1$*, and large* m*, we have a canonical isomorphism :*

$$\mathrm{R}^i h_*(\mathrm{F}(m)) \simeq f_*(\mathrm{R}^i g_*(\mathrm{G})(m)).$$

Indeed the spectral sequence (26) for $\mathrm{F}(m)$ then degenerates in degree $\leqslant n$, by EGA III 2.2.1 (ii), whence the result immediately (which moreover gives back the implication (ii) $\Rightarrow$ (i) of 5.2).

Corollary 5.5. — *Under the preliminary conditions of* 5.2*, with* S *noetherian, the following conditions are equivalent :*

(i) $\bigoplus_{m\geqslant m_0} h_*(\mathrm{F}(m)$ *is of finite type over* S*, and* $\mathrm{R}^i h_*(\mathrm{F}(m)) = 0$ *for* $0 < i \leqslant n$ *and large* m*.*

(ii) $g_*(\mathrm{F})$ *is coherent, and* $\mathrm{R}^i g_*(\mathrm{F}) = 0$ *for* $0 < i \leqslant n$*.*

(ii bis) $g_*(\mathrm{F})$ *is coherent, and* $\mathrm{prof}_{\mathrm{Y}}\, g_*(\mathrm{F}) > n+1$*.*

The equivalence of (ii) and (ii bis) is contained in III 3.3. Moreover, by virtue of 5.2 conditions (i) and (ii) both imply that the $\mathrm{R}^i g_*(\mathrm{F})$ $(i \leqslant n)$ are coherent. The equivalence of (i) and (ii) then results from 5.4, taking into account the fact that for any coherent Module G on X, we have $\mathrm{G} = 0$ if and only if $f_*(\mathrm{G}(m)) = 0$ for large m, for example by virtue of EGA III 2.2.1 (iii).

Remark 5.6. — We can interpret criteria 5.2 and 5.5 by saying that the "simultaneous 107
finiteness condition" 5.2 (ii) is expressed by local regularity properties (in terms of depth thanks to VIII 2.1) of F at the points of U neighboring $\mathrm{Y} = \mathrm{X} - \mathrm{U}$, whereas the "asymptotic vanishing condition" 5.5 (i) is of a clearly stronger nature, and is expressed by local regularity conditions of $g_*(\mathrm{F})$ at the points of Y itself. It would be interesting, to generalize the Lefschetz-like theorems for projective morphisms to quasi-projective morphisms, to find local criteria on X that are necessary and sufficient for the $\mathscr{S}$-Modules $\bigoplus_{m\geqslant 0} \mathrm{R}^i h_*(\mathrm{F}(m))$ for $i \leqslant n$ to be of finite type. When S is the spectrum of a field (and doubtless more generally, if it is the spectrum of an artinian

ring) and $\mathrm{Y} = \mathrm{X} - \mathrm{U}$ is finite, one can show that it is necessary and sufficient that the following conditions are satisfied :

1°) $\operatorname{prof} \mathrm{F}_x > n$ for any closed point x of U (compare 1.4).

2°) $\mathrm{R}^i\, g_*(\mathrm{F})$ is coherent for $i \leqslant n$, or what amounts to the same, there exists an open neighborhood V of Y such that for any closed point x of $\mathrm{U} \cap \mathrm{V}$, we have $\operatorname{prof} \mathrm{F}_x > n+1$.

Proposition 5.7. — *Let* S *be a locally noetherian prescheme,* $g : \mathrm{U} \to \mathrm{X}$ *a morphism of preschemes of finite type over* $\mathrm{S}^{(*)}$*, with structural morphisms* h *and* f*,* F *a quasi-coherent Module on* U*,* n *an integer. The following conditions are equivalent :*

(i) *For any base change* $\mathrm{S}' \to \mathrm{S}$*, with* S' *noetherian, the module* $\mathrm{R}^n\, g'_*(\mathrm{F}')$ *on* X' *is coherent.*

(ii) *For any base change as above, and any coherent Ideal* $\mathscr{J}$ *on* S'*, denoting by* $\mathscr{I}$ *the Ideal* $\mathscr{J}\mathscr{O}_{\mathrm{X}'}$ *on* X'*, the graded Module*

$$\bigoplus_{m \geqslant 0} \mathrm{R}^n\, g'_*(\mathscr{I}^m \mathrm{F}')$$

over $\bigoplus_{m\geqslant 0} \mathscr{I}^m$ *is of finite type.*

108 (iii) *For any base change* $\mathrm{S}' \to \mathrm{S}$*, and* $\mathscr{J}$ *as above, the graded Module*

$$\bigoplus_{m \geqslant 0} \mathrm{R}^n\, g'_*(\mathscr{I}^m \mathrm{F}'/\mathscr{I}^{m+1}\mathrm{F}')$$

over $\operatorname{gr}_{\mathrm{I}}(\mathscr{O}_{\mathrm{X}'}) = \bigoplus_{m\geqslant 0} \mathscr{I}^m/\mathscr{I}^{m+1}$ *is of finite type.*

Obviously (ii) $\Rightarrow$ (i) and (iii) $\Rightarrow$ (i), as can be seen by taking $\mathscr{I} = 0$ in conditions (ii) and (iii). The inverse implications are obtained by applying (i) to the composite base change $\mathrm{S}'' \to \mathrm{S}' \to \mathrm{S}$, where S'' is equal to $\operatorname{Spec} \bigoplus_{m\geqslant 0} \mathscr{J}^m$ resp. $\operatorname{Spec} \bigoplus_{m\geqslant 0} \mathscr{J}^m/\mathscr{J}^{m+1}$.

The interest of this proposition is that conditions of the form (ii) are those that intervene in the "algebraic-formal comparison theorems", whereas conditions of the form (iii) intervene in the "existence theorems" that complete them, cf. exposition IX. A first interesting case is that where $f : \mathrm{X} \to \mathrm{S}$ is the identity, and where it is therefore a question of conditions on a morphism $h : \mathrm{U} \to \mathrm{S}$ locally of finite type and a Module F quasi-coherent on U flat over S. To obtain sufficient conditions, we will suppose that U embeds via $g : \mathrm{U} \to \mathrm{X}$, as an open sub-prescheme of an X which is *proper* over S. Applying 5.1, we thus see :

Corollary 5.8. — *Let* $f : \mathrm{X} \to \mathrm{S}$ *be a proper morphism, with* S *locally noetherian,* U *an open set of* X*,* $g : \mathrm{U} \to \mathrm{X}$ *the canonical immersion,* $h = fg : \mathrm{U} \to \mathrm{S}$*,* F *a quasi-coherent Module on* X*, flat over* S*. Suppose that for any base change* $\mathrm{S}' \to \mathrm{S}$*, with*

(*) It is in fact sufficient that g be quasi-compact and quasi-separated (EGA IV1.2.1), without condition on U, X.

S' *locally noetherian, we have* $\mathrm{R}^i g'_*(\mathrm{F}')$ *coherent on* X' *for* $i \leqslant n$. *Then we have the following :*

(i) *For any base change* $\mathrm{S}' \to \mathrm{S}$, *with* S' *locally noetherian,* $\mathrm{R}^i h'_*(\mathrm{F}')$ *is coherent on* S' *for* $i \leqslant n$.

(ii) *For any* $\mathrm{S}' \to \mathrm{S}$ *as above, and any coherent Ideal* $\mathscr{J}$ *on* S', *the graded Modules* 109

$$\bigoplus_{m\geqslant 0} \mathrm{R}^i h'_*(\mathscr{J}^m \mathrm{F}')$$

over $\bigoplus_{m\geqslant 0} \mathscr{J}^m$ *are of finite type for* $i \leqslant n$.

(iii) *For any* $\mathrm{S}' \to \mathrm{S}$ *and* $\mathscr{J}$ *as above, the graded Modules*

$$\bigoplus_{m\geqslant 0} \mathrm{R}^i h'_*(\mathscr{J}^m \mathrm{F}'/\mathscr{J}^{m+1}\mathrm{F}')$$

over $\mathrm{gr}_{\mathscr{J}}(\mathscr{O}_{\mathrm{S}'}) = \bigoplus_{m\geqslant 0} \mathscr{J}^m/\mathscr{J}^{m+1}$ *are of finite type for* $i \leqslant n$.

Moreover, under the conditions of (ii), and by virtue of the comparison theorem 1.1, *denoting by* $\widehat{\mathrm{S}'}$ *the formal completion of* S' *with respect to* $\mathscr{J}$, *and by* $\widehat{\mathrm{U}'}$ *that of* U' *with respect to* $\mathscr{J}\mathscr{O}_{\mathrm{U}'}$, *the canonical homomorphisms*

$$\widehat{\mathrm{R}^i h'_*(\mathrm{F}')} \longrightarrow \mathrm{R}^i \widehat{h}'_*(\widehat{\mathrm{F}}') \longrightarrow \varprojlim_k \mathrm{R}^i h'_*(\mathrm{F}'_k)$$

are isomorphisms for $i \leqslant n-1$.

Remark 5.9. — Suppose that we are moreover under the conditions of 5.8 with F *coherent*, and consider a base change $\mathrm{S}' \to \mathrm{S}$ as in 5.9 (i). Suppose moreover that S' is locally immersible in a regular scheme, or more generally, that there exists a morphism $\mathrm{S}'' \to \mathrm{S}$ faithfully flat and quasi-compact, such that S'' is locally immersible in a regular scheme; this condition is satisfied in particular if S' is local. Then the conclusion of 5.8 (i) and (ii) remains valid when we replace F' by a Module G' on U', such that any point of U' has an open neighborhood on which G' is isomorphic to a Module of the form F'^n. Indeed, we are reduced to the case where S' itself is locally immersible in a regular scheme, so that the same is true of $\mathrm{S}'' = \mathrm{Spec}\left(\bigoplus_{m\geqslant 0} \mathscr{J}^m\right)$ and of $\mathrm{X}'' = \mathrm{X}' \times_{\mathrm{S}'} \mathrm{S}'' = \mathrm{X} \times_{\mathrm{S}} \mathrm{S}''$, which are of finite type over it. We then apply the finiteness criterion VIII 2.3 for the direct images for $i \leqslant n$ of G'' under the immersion $\mathrm{U}'' \to \mathrm{X}''$, noting that they are satisfied by hypotheses for F'', hence also for G'' 110
since they are expressed in terms of depth and G'' is locally isomorphic to a F''^n. The same argument shows that if $\mathscr{G}'$ is a coherent Module on $\widehat{\mathrm{U}'}$ (completion of U' with respect to the Ideal $\mathscr{J}\mathscr{O}_{\mathrm{U}'}$), such that $\mathscr{G}'_0 = \mathscr{G}'/\mathscr{J}\mathscr{G}'$ is locally of the form F'^n_0, then the conclusion of (iii) remains valid when replacing F' by $\mathscr{G}'$. We thus obtain the following result, using the results of exposition IX :

Corollary 5.10. — *Let* $f : \mathrm{X} \to \mathrm{S}$ *be a proper morphism, with* S *locally noetherian,* U *an open subset of* X*; we suppose* U *is flat over* S, *and that for any base change* $\mathrm{S}' \to \mathrm{S}$, *with* S' *locally noetherian, we have* $\mathrm{R}^i g'_*(\mathscr{O}_{\mathrm{U}'})$ *coherent on* X' *for* $i = 0, 1$. *Suppose then that* S' *is of the form* $\mathrm{Spec}(\mathrm{A})$, *where* A *is a noetherian ring equipped* *with an ideal* J *such that* A *is separated and complete for the* J*-adic topology. Under these conditions :*

(i) *The functor* $\mathrm{F} \mapsto \widehat{\mathrm{F}}$ *from the category of locally free Modules on* U' *to the category of locally free Modules on* $\widehat{\mathrm{U}'}$ *is fully faithful.*

(ii) *For any locally free Module* $\mathfrak{F}$ *on* $\widehat{\mathrm{U}'}$*, there exists a coherent Module* F *on* U' (*not necessarily locally free, alas*)*, and an isomorphism* $\widehat{\mathrm{F}} \simeq \mathfrak{F}$*.*

It only remains to prove (ii), thanks to 5.9. But according to this remark and 2.1 it follows that $\mathfrak{F}$ is induced by a coherent Module $\mathfrak{G}$ on $\widehat{\mathrm{X}'}$. According to the existence theorem EGA III 5.1.4, $\mathfrak{G}$ is of the form $\widehat{\mathrm{F}}$, where F is coherent on X, whence the conclusion.

Remarks 5.11. — 1°) Using 5.10, 4.7 and a suitable hypothesis, saying that certain local rings of the geometric fibers of $\mathrm{X}' \to \mathrm{S}'$ are "pure" resp. parafactorial, one should be able to obtain statements saying that the functor $\mathrm{Z}' \mapsto \widehat{\mathrm{Z}'}$ from the category of
111 étale coverings of X' to the category of étale coverings of $\widehat{\mathrm{X}'}$ (or what amounts to the same, of X'_0) is an equivalence of categories, resp. that the functor $\mathrm{L} \mapsto \widehat{\mathrm{L}}$ from the category of invertible Modules on X' to the category of invertible Modules on $\widehat{\mathrm{X}'}$ is an equivalence. Using recent results of Murre, it is probable that one should be able to deduce from the existence theorems the existence of Picard schemes for certain *non-proper* algebraic schemes[6]. In general, the elimination of purity hypotheses in various existence theorems, notably of representability of functors like the Hilbert

[6] N.D.E. : of course, one will refer in the projective case to Grothendieck's existence theorems from FGA ; cf. Grothendieck A., « Technique de descente et théorèmes d'existence en géométrie algébrique. VI. Les schémas de Picard : propriétés générales », in *Séminaire Bourbaki*, vol. 7, Société mathématique de France, Paris, 1995, Exp. 236, p. 221–243 and « Technique de descente et théorèmes d'existence en géométrie algébrique. V. Les schémas de Picard : théorèmes d'existence », in *Séminaire Bourbaki*, vol. 7, Société mathématique de France, Paris, 1995, Exp. 232, 143–161. The nine finiteness conjectures found therein are proven in expositions XII and XIII of Mme Raynaud and of Kleiman from SGA 6. For an excellent elementary text on the subject, see the exposition article by Kleiman (Kleiman S., « The Picard scheme », to appear in Contemp. math.). For an application of these techniques to global generalized jacobians of a relative smooth curve, see (Contou-Carrère C., « La jacobienne généralisée d'une courbe relative ; construction et propriété universelle de factorisation », *C. R. Acad. Sci. Paris Sér. A-B* **289**, (1979), N° 3, A203–A206 and « Jacobiennes généralisées globales relatives », in *The Grothendieck Festschrift, Vol. II*, Progr. Math., vol. 87, Birkhäuser, Boston, 1990, p. 69–109). See also by the same author, in the purely local setting, the construction and study of the "local generalized jacobian" functor (« Jacobienne locale, groupe de bivecteurs de Witt universel, et symbole modéré », *C. R. Acad. Sci. Paris Sér. I Math.* **318** (1994), N° 8, p. 743–746). Furthermore, if in the case of a projective and smooth morphism, the connected components of the Picard scheme are proper, this is no longer the case in the singular case. The problem of the compactification of Picard schemes naturally arises : this problem has been studied in detail, notably in (Altman A.B. & Kleiman S., « Compactifying the Picard scheme », *Adv. in Math.* **35** (1980), N° 1, p. 50–112. and « Compactifying the Picard scheme. II », *Amer. J. Math.* **101** (1979), N° 1, p. 10–41). The case of curves had been studied previously (D'Souza C., « Compactification of generalised

or Picard functors etc., with the help of the techniques developed in this seminar, deserves a systematic study.

2°) One can propose to give manageable necessary and sufficient conditions, in terms of depth, for the universal finiteness condition considered in 5.10 to be satisfied. When S is the spectrum of a field, it results easily from EGA III 1.4.15 that it is necessary and sufficient that the $\mathrm{R}^i\, g_*(\mathrm{F})$ $(i \leqslant n)$ be coherent, which is well expressed in terms of depth thanks to VIII 2.3. In the general case, we note however that it is not sufficient to require that the preceding condition be satisfied for all fibers $\mathrm{U}_s \subset \mathrm{X}_s$ $(s \in \mathrm{S})$, even in the case where $n = 0$. Take for example $\mathrm{X} = \mathrm{S}$, with S the spectrum of a discrete valuation ring, U the open set reduced to the generic point, $\mathrm{F} = \mathscr{O}_{\mathrm{U}}$.

3°) Here is however a *sufficient* condition ensuring that we are under the conditions of the hypothesis of 5.10 : It is sufficient that f be flat, and that for any $s \in \mathrm{S}$ and any $x \in \mathrm{Y}_s = \mathrm{X}_s - \mathrm{U}_s$, we have

$$\operatorname{prof} \mathscr{O}_{\mathrm{X}_s,x} \geqslant n + 2.$$

Jacobians », *Proc. Indian Acad. Sci. Sect. A Math. Sci.* **88** (1979), N° 5, p. 419–457). We even know exactly when the compactified jacobian of a curve is irreducible (Rego C.J., « The compactified Jacobian » *Ann. Sci. Éc. Norm. Sup. (4)* **13** (1980), N° 2, p. 211–223, this being the closure of the (ordinary) jacobian when the curve is geometrically integral and locally planar ; for a construction in family of compactified jacobians, see (Esteves E., « Compactifying the relative Jacobian over families of reduced curves », *Trans. Amer. Math. Soc.* **353** (2001), N° 8, p. 3045–3095). Since then, the existence results for the Picard scheme in the proper case have progressed since the original edition of SGA 2 ; cf. (Murre J.P., « On contravariant functors from the category of pre-schemes over a field into the category of abelian groups (with an application to the Picard functor) », *Publ. Math. Inst. Hautes Études Sci.* **23** (1964), p. 5-43) and especially (Artin M., « Algebraization of formal moduli. I », in *Global Analysis* (*Papers in Honor of K. Kodaira*), Univ. Tokyo Press, Tokyo, 1969, p. 21–71). See also (Raynaud M., « Spécialisation du foncteur de Picard », *Publ. Math. Inst. Hautes Études Sci.* **38** (1970), p. 27–76) in the case of a proper scheme over a discrete valuation ring, but not necessarily flat. For a discussion of more recent results, in particular those of Artin, for the Picard functor of proper and flat schemes, in particular in the cohomologically flat case in dimension 0, see chapter VIII of (Bosch S., Lütkebohmert W. & Raynaud M., *Néron models*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), vol. 21, Springer-Verlag, Berlin, 1990) and the references cited. Much more recently, very fine results have been obtained in the case of relative curves $f : \mathrm{X} \to \mathrm{S}$ above the spectrum S of a discrete valuation ring with perfect residue field. More precisely, one supposes that f is proper and flat, X regular and $f_*\mathscr{O}_{\mathrm{X}} = \mathscr{O}_{\mathrm{S}}$. On the other hand, one does not suppose f is cohomologically flat in dimension 0, i.e. one does not suppose $\mathrm{H}^1(\mathrm{X}, \mathscr{O})$ is torsion-free. The Picard scheme is then not representable, whether by a scheme or an algebraic space, as soon as the torsion in question is non-zero. Let J be the Néron model of the generic fiber of f : it is a quotient of the Picard functor P. Then, Raynaud has shown that the kernel of the tangent map $\mathrm{H}^1(\mathrm{X}, \mathscr{O}) = \mathrm{Lie}(\mathrm{P}) \to \mathrm{Lie}(\mathrm{J})$ coincides with the torsion subgroup of the H^1 and that the cokernel has the same length (see theorem 3.1 of (Liu Q., Lorenzini D. & Raynaud M., « Néron models, Lie algebras, and reduction of curves of genus one », *Invent. Math.* **157** (2004), p. 455–518). This result allows the aforementioned authors to study the link between the Birch-Swinnerton-Dyer and Artin-Tate conjectures (see th. 6.6 of *loc. cit.*). Concerning the local Picard scheme, see the thesis of Boutot, cited in the editor's note (13) page 153.

Indeed, taking into account lemma 2.5 (cf. relation (16) after 2.5), it follows that we then have $g_*(\mathscr{O}_{\mathrm{U}}) \simeq \mathscr{O}_{\mathrm{X}}$ and $\mathrm{R}^i\, g_*(\mathscr{O}_{\mathrm{U}}) = 0$ for $0 < i \leqslant n$, and the same relations will obviously be satisfied after any base change $\mathrm{S}' \to \mathrm{S}$.

EXPOSÉ XIII

PROBLEMS AND CONJECTURES

1. Relations between global and local results. Affine problems related to duality

It is well known that many statements concerning a projective scheme X can be formulated in terms of statements concerning a certain graded ring, or better a complete local ring, namely the homogeneous coordinate ring of X (i.e. the affine ring of the projecting cone $\widetilde{X}$ of X), or its completion (i.e. the completion of the local ring of the vertex of $\widetilde{X}$). The interest of this reformulation is that it often allows, starting from known global results, to conjecture, and even to prove, analogous results for more general complete noetherian local rings than those which really intervene in the global statement, for example for local rings which are not necessarily of equal characteristic. Thus, Serre's duality theorem for projective space XII 1.1 suggested the useful local duality theorem V 2.1. The fundamental theorem of Serre on the cohomology of coherent Modules on projective space (finiteness, asymptotic behavior for large n of $\mathrm{H}^i(\mathrm{X}, \mathrm{F}(n))$, cf. EGA III 2.2.1) generalizes to a structure theorem for the local invariants $\mathrm{H}^i_{\mathfrak{m}}(\mathrm{M})$, see V 3. Similarly, the Lefschetz theorems for the fundamental group, and the Picard group (" equivalence criteria"), very familiar in the classical case and later extended to a base field of any characteristic, have suggested the "local" Lefschetz theorems of expositions X and XI. Of course, the local theorems in their turn are precious tools for obtaining global statements. For example, local duality allows to formulate a global asymptotic property XII 1.3 (i) by the vanishing of certain local invariants $\mathrm{H}^i(\mathrm{F}_x)$. More substantially, the local Lefschetz theorems, implying for example the "purity" or the parafactoriality of certain local rings which are complete intersections (X 3.4 and XI 3.13) allow in the global Lefschetz theorems to get rid of certain hypotheses of non-singularity, as in X 3.5, 3.6, 3.7. 112

An other useful generalization of the theorems concerning projective schemes over a field k consists in replacing k by a general base scheme. Thus, the sequel to EGA III 113 will give[1] a generalization in this sense of Serre's duality[*] ; the finiteness and asymptotic behavior theorems for the $\mathrm{H}^i(\mathrm{X}, \mathrm{F}(n))$ have been stated in EGA III 2.2.1 over a general base scheme, finally the Lefschetz theorems can also be developed for a projective morphism, as we have seen in XII 4.9, thanks to the local theorem XII 4.7. Of course, working over a general base scheme also leads to essentially new statements, such as the "comparison theorem" EGA III 4.15 and the existence theorem for sheaves EGA III 5.1.4 (which, as we have seen elsewhere in IX, fall under the same key theorems of a cohomological nature as the Lefschetz theorems for π_1 and Pic).

It is then necessary to extract from theorems that simultaneously encompass the two generalizations we have just pointed out of statements concerning projective schemes over a field. The natural objects for such a common generalization are the *noetherian rings separated and complete for an* I-*adic topology*. Their study, from this point of view, has not yet been seriously undertaken, and seems to me at present the most interesting subject in the local theory of coherent sheaves. Here is a typical problem in this direction :

Conjecture 1.1 **("Second affine finiteness theorem"[**][2])**. — *Let* M *be a finitely generated module over a noetherian ring* A (*which we will assume if necessary to be a quotient of a regular one*), J *an ideal of* A, *prove that the modules* $\mathrm{H}^i_\mathrm{J}(\mathrm{M})$ *are "*J*-cofinite", i.e. that the modules*

$$\mathrm{Hom}_\mathrm{A}(\mathrm{A}/\mathrm{J}, \mathrm{H}^i_\mathrm{J}(\mathrm{M}))$$

[*]cf. Hartshorne Seminar, cited at the end of Exp. IV.

[*]This conjecture, and conjecture 1.2, below, are false, as shown by R. Hartshorne, « Affine duality and cofinite modules », *Invent. Math.* **9** (1969/70), p. 145-164, section 3.

[1]N.D.E. : in fact, this generalization is not found there ; see note below, and the editor's note (5) on page 6

[2]N.D.E. : however, if A is local complete (resp. regular of positive characteristic) and J is the maximal ideal, the statement is true for M of finite type (resp. M = A), cf. (Hartshorne R., « Affine duality and cofinite modules », *Invent. Math.* **9** (1969/70), p. 145-164, corollary 1.4) (resp. (Huneke C. & Sharp R., « Bass numbers of local cohomology modules », *Trans. Amer. Math. Soc.* **339** (1993), N° 2, p. 765–779), which moreover contains much stronger results). For completely different methods (D-modules) allowing to approach characteristic zero, see (Lyubeznik G., « Finiteness properties of local cohomology modules (an application of D-modules to commutative algebra) », *Invent. Math.* **113** (1993), N° 1, p. 41–55) ; see also by the same author « Finiteness properties of local cohomology modules for regular local rings of mixed characteristic : the unramified case », *Comm. Algebra* **28** (2000), N° 12, p. 5867–5882, Special issue in honor of Robin Hartshorne, and « Finiteness properties of local cohomology modules : a characteristic-free approach », *J. Pure Appl. Algebra* **151** (2000), N° 1, p. 43–50. The notion of cofinite module has since evolved under the guidance of Hartshorne. We say that M is J-cofinite if its support is contained in V(J) and if all the $\mathrm{Ext}^i_\mathrm{A}(\mathrm{A}/\mathrm{J}, \mathrm{H}^i_\mathrm{J}(\mathrm{M}))$ are of finite type. On this subject, see for example (Delfino D. & Marley Th., « Cofinite modules and local cohomology », *J. Pure Appl. Algebra* **121** (1997), N° 1, p. 45–52).

are of finite type.

Recall that we denote by $\mathrm{H}^i_{\mathrm{J}}(\mathrm{M})$ the module $\mathrm{H}^i_{\mathrm{Y}}(\mathrm{X}, \widetilde{\mathrm{M}})$ (where $\mathrm{X} = \mathrm{Spec}(\mathrm{A})$, $\mathrm{Y} = \mathrm{V}(\mathrm{J})$) of Exp. I, interpreted in II in terms of inductive limit of cohomologies of Koszul **114** complexes, or else for $i \geqslant 2$ the module $\mathrm{H}^{i-1}(\mathrm{X}-\mathrm{Y}, \widetilde{\mathrm{M}})$. À To be honest, 1.1 should be a consequence of a more precise statement, implying that the $\mathrm{H}^i_{\mathrm{J}}(\mathrm{M})$ are in a suitable abelian subcategory $\underline{\mathrm{D}}_{\mathrm{J}}$ of the category $\underline{\mathrm{C}}_{\mathrm{J}}$ of A-modules with support $\subset \mathrm{Y} = \mathrm{V}(\mathrm{J})$, such that $\mathrm{H} \in \mathrm{Ob}\, \underline{\mathrm{D}}_{\mathrm{J}}$ implies that H is J-cofinite. (N.B. The category of modules H with support contained in V(J) and which are J-cofinite is unfortunately not stable under taking quotients!). The essential problem would then consist in defining $\underline{\mathrm{D}}_{\mathrm{J}}$. More precisely, the solution to problem 1.1 should come out (at least if A is a quotient of a regular one) from a duality theory, generalizing both local duality, and the duality theory of projective morphisms to which we alluded earlier, and which would be of the following kind :

***Conjecture 1.2* ("Affine duality"(*)).** — *Suppose* A *is regular, separated and complete for the* J*-adic topology. Let* $\mathrm{C}^\bullet(\mathrm{A})$ *be an injective resolution of* A.

(i) *Prove that the functor*

$$\mathrm{D}_{\mathrm{J}} : \mathrm{L}_\bullet \longmapsto \mathrm{Hom}_{\mathrm{J}}(\mathrm{L}_\bullet, \mathrm{C}^\bullet(\mathrm{A}))$$

from the category of complexes of A*-modules, free of finite type in every dimension and with degrees bounded above,* (*oùwhere the morphisms are homomorphisms of complexes up to homotopy*) *to the category of complexes of* A*-modules* $\mathrm{K}^\bullet$*, injective in every dimension and with degrees bounded above* (*oùwhere the homomorphisms are defined in the same way*)*, is fully faithful.*

(ii) *Prove that for any* $\mathrm{K}^\bullet$ *of the form* $\mathrm{D}_{\mathrm{J}}(\mathrm{L}_\bullet)$*, the* $\mathrm{H}^i(\mathrm{K}^\bullet)(= \mathrm{Ext}^i_{\mathrm{Y}}(\mathrm{X}; \mathrm{L}_\bullet, \mathscr{O}_{\mathrm{X}}))$ *are* J*-cofinite.*

(iii) *More precisely, prove that the* $\mathrm{K}^\bullet$ *which are homotopic to a complex of the form* $\mathrm{D}_{\mathrm{J}}(\mathrm{L}_\bullet)$ *can be characterized by finiteness properties of the* $\mathrm{H}^i(\mathrm{K}^\bullet)$*, stronger than that considered in* (*ii*)*, for example by the property* $\mathrm{H}^i(\mathrm{K}^\bullet) \in \mathrm{Ob}\, \underline{\mathrm{D}}_{\mathrm{J}}$*, where* $\underline{\mathrm{D}}_{\mathrm{J}}$ *is a* **115** *suitable abelian category, as considered above.*

Note that the problem is solved in the affirmative when A is local and J is an ideal of definition (cf. Exp. IV), and also when J is the zero ideal. In these two cases, exceptionally, we can limit ourselves to taking for $\underline{\mathrm{D}}_{\mathrm{J}}$ the category of Modules with support V(J) which are J-cofinite, (which in the second case simply means that we take the category of Modules of finite type over A). An affirmative solution to conjecture 1.2 in general would give one for 1.1, by taking for $\mathrm{L}_\bullet$ the dual of a free resolution of finite type of M. On the other hand, an affirmative solution to 1.1 would give an

(*)This conjecture, false as stated, has however been established in a fairly close form by R. Hartshorne, « Affine duality and cofinite modules », *Invent. Math.* **9** (1969/70), p. 145-164.

affirmative answer to the first part of the following conjecture which we formulate in "global" form :

Conjecture 1.3. — *Let* $\mathrm{X} \subset \mathbf{P}_k^r$ *be a closed subscheme of the projective space which is locally a complete intersection and whose every irreducible component has codimension* $\geqslant s$. *Let* $\mathrm{U} = \mathbf{P}_k^r - \mathrm{X}$.

(i) *Prove that for any coherent Module* F *on* U, *we have*

$$\dim \mathrm{H}^i(\mathrm{U}, \mathrm{F}) < +\infty \ \textit{for}\ i \geqslant s.^{(*)}$$

(ii) *Give an example, with* X *connected and regular, where we have*

$$\mathrm{H}^s(\mathrm{U}, \mathrm{F}) \neq 0.$$

To see that (i) is a particular case of 1.1 we consider

$$\mathrm{H}^i(\mathrm{U}, \mathrm{F}(\cdot)) = \bigoplus_n \mathrm{H}^i(\mathrm{U}, \mathrm{F}(n)) = \mathrm{H}^i(\mathrm{E}^{r+1} - \widetilde{\mathrm{X}}, \widetilde{\mathrm{F}})$$

as a module over the affine ring $k[t_0, \ldots, t_r]$ of the projecting cone E^{r+1} of $\mathbf{P}^r$. This module is none other than $\mathrm{H}_{\mathrm{J}}^{i+1}(\mathrm{M})$, where J is the ideal of the projecting cone $\widetilde{\mathrm{X}}$ of X in E^{r+1}. On the other hand, from the hypothesis made on X, which implies that
116 $\widetilde{\mathrm{X}}$ is also a complete intersection of codimension $\geqslant s$ at any point of E^{r+1} distinct from the origin, it results that $\mathrm{H}_{\mathrm{J}}^{i+1}(\mathrm{M})$ is zero outside the origin for $i \geqslant s$. If it is therefore J-cofinite as 1.1 wants, it is a fortiori $\mathfrak{m}$-cofinite, which easily implies that it is of finite dimension in every degree(3) Note moreover that conjecture 1.3 already arises for an irreducible non-singular curve X in $\mathbf{P}^3$, it is not known if in this case the $\mathrm{H}^2(\mathbf{P}^3 - \mathrm{X}, \mathscr{O}_{\mathrm{X}}(n))$ are of finite dimension, or if they are necessarily zero(†). It is not even known if there exists an irreducible curve in $\mathbf{P}^3$ which is not set-theoretically the intersection of two hypersurfaces(4).

(†)The question has just been solved in the affirmative by R. Hartshorne (Hartshorne R., « Ample vector bundles », *Publ. Math. Inst. Hautes Études Sci.* **29** (1966), p. 63–94, theorem 8.1) and H. Hironaka.

(3)N.D.E. : Hartshorne has proven (Hartshorne R., « Cohomological dimension of algebraic varieties », *Ann. of Math.* (*2*) **88** (1968), p. 403–450) that the cohomology $\mathrm{H}^{n-1}(\mathbf{P}_k^n - \mathrm{X}, \mathrm{F})$ is zero for F coherent and X of positive dimension (k algebraically closed). In fact, thanks essentially to Serre duality and Lichtenbaum's theorem — vanishing of the cohomology of coherent sheaves in maximal dimension of irreducible non-complete quasi-projective varieties —, we are reduced to proving that the formal completion $\widehat{\mathbf{P}}_k^n$ and X have the same field of rational functions. This is the difficult point (theorem 7.2 of *loc. cit.*), in other words $\mathbf{P}_k^n$ is G3 in the terminology of (Hironaka H. & Matsumura H., « Formal functions and formal embeddings », *J. Math. Soc. Japan* **20** (1968), p. 52–82). These authors have independently proven the preceding results, and in fact much more. They have proven that X is universally G3 and have calculated the field of rational functions of the formal completion of an abelian variety along a subvariety of positive dimension. It is in this article that the now classic conditions G1, G2, G3 appear for the first time.

(4)N.D.E. : see conjecture 3.5 and the corresponding note.

Problem 1.4. — *Give an affine variant of the "comparison theorem"* EGA III 4.1.5 *as a theorem of commutation of the functors* $\mathrm{H}^i_{\mathrm{J}}$ *with certain projective limits.*

Finally, in the present order of ideas, I had posed the following problem : let A be a complete regular noetherian local ring, K its field of fractions, prove that $\mathrm{Ext}^i_{\mathrm{A}}(\mathrm{K}, \mathrm{A}) = 0$ for any i. An affirmative answer was immediately given by M. Auslander, the hypothesis of regularity can be replaced by that A is an integral domain and it is in fact true that $\mathrm{Ext}^i_{\mathrm{A}}(\mathrm{K}, \mathrm{M}) = 0$ for any i, as soon as M is of finite type over A. This results immediately from the following statement, due to Auslander : if A is a complete noetherian local ring, then for any finitely generated module M over A, the functors $\mathrm{Ext}^i_{\mathrm{A}}(., \mathrm{M})$ transform inductive limits into projective limits[5].

2. Problems related to π_0 : local Bertini theorems

Let A be a complete noetherian local ring, f an element of its maximal ideal, $\mathrm{X} = \mathrm{Spec}(\mathrm{A})$, $\mathrm{Y} = \mathrm{Spec}(\mathrm{A}/f\mathrm{A})$. The use of the local "Lefschetz" technique allows to give criteria for $\mathrm{Y}' = \mathrm{X}' \cap \mathrm{Y}$ (where $\mathrm{X}' = \mathrm{X} - \{\mathfrak{m}\}$) to be connected, in terms of hypotheses on X′. Thus, it is sufficient that we have : a) X′ connected, b) prof $\mathscr{O}_{\mathrm{X}',x} \geqslant 2$ 117
for any closed point x of X′, c) f is A-regular. Note however that hypotheses b) and c) are not of a purely topological nature, for example are not invariantes when replacing X by $\mathrm{X}_{\mathrm{red}}$. In the analogous situation for a projective scheme X′ over a field and a hyperplane section Y′ of X′, the use of "Bertini's theorem" and of Zariski's "connection theorem" allows to obtain in fact results of a clearly more satisfactory allure, which had led me in the oral seminar to state a conjecture, which I have since solved in the affirmative. Let us therefore state here :

Theorem 2.1. — *Let* A *be a complete noetherian local ring,* X *its spectrum,* $\mathfrak{a}$ *the closed point of* X*,* $\mathrm{X}' = \mathrm{X} - \{\mathfrak{a}\}$*. Suppose that* X *satisfies the conditions* (*where k denotes an integer* $\geqslant 1$) *:*

a_k) *The irreducible components of* X′ *have dimension* $\geqslant k+1$.

b_k) X′ *is connected in dimension* $\geqslant k$*, i.e. one cannot disconnect* X′ *by a closed subset of dimension* $< k$ (*cf.* III 3.8).

Let m be an integer, $0 \leqslant m \leqslant k$*, and let* $f_1, \ldots, f_m \in \mathfrak{r}(\mathrm{A})$*, let* $\mathrm{B} = \mathrm{A}/\sum_i f_i\mathrm{A}$, $\mathrm{Y} = \mathrm{Spec}(\mathrm{B}) = \mathrm{V}(f_1) \cap \cdots \cap \mathrm{V}(f_m)$*,* $\mathrm{Y}' = \mathrm{X}' \cap \mathrm{Y} = \mathrm{Y} - \{\mathfrak{a}\}$*. Then* Y *satisfies conditions* a_{k-m}), b_{k-m})*. In particular, for any sequence of* $m \leqslant k$ *elements* $f_1, \ldots, f_m$ *of* $\mathfrak{r}(\mathrm{A})$*,* $\mathrm{Y}' = \mathrm{X}' \cap \mathrm{V}(f_1) \cap \cdots \cap \mathrm{V}(f_m)$ *is connected.*

It is moreover easy to see that if the last conclusion is valid (it is obviously sufficient to take $m = k$), and excluding the case where X would be irreducible of dimension 0

[5] N.D.E. : write $\mathrm{K} = \varinjlim_{a \neq 0} \mathrm{A}[1/a]$ and observe that $a \neq 0$ is A-regular.

or 1, it follows that the irreducible components of X' have dimension $\geqslant k+1$, and X' is connected in dimension $\geqslant k$, so that in a sense, 2.1 is a "best possible" result.

Let us give the principle of the proof of 2.1. Only condition b_{k-m}) for Y presents a problem. We are easily reduced for a given k to the case where X is an integral domain,
118 and even (by passing to the normalization, which is finite over X) to the case where X is *normal.* If $k=1$, so $\dim X' \geqslant 2$, then X' has depth $\geqslant 2$ at its closed points and we can apply the result recalled at the beginning of the n°, which shows that $Y' = X' \cap V(f)$ is connected. In the case $k \geqslant 1$, we suppose the theorem is proven for $k' < k$. By induction on m, we are reduced to the case where $m=1$, i.e. to verifying that for $f_1 \in \mathfrak{r}(A)$, $X' \cap V(f_1)$ is connected in dimension $\geqslant k-1$. If it were not, i.e. if it were disconnected by a Z' of dimension $< k-1$, there would exist a sequence $f_2, \dots, f_k$ such that $X' \cap V(f_1) \cap \cdots \cap V(f_k)$ is disconnected, and in this sequence we can arbitrarily choose $f_2 \in \mathfrak{r}(A)$ subject to the sole condition of not vanishing on any point of a certain finite subset F of X' (namely the set of maximal points of Z'). Moreover, we easily check, using the fact that X' is normal, thus satisfies Serre's condition (S_2)(*) that there exists a finite subset F' of X' such that $f \in \mathfrak{r}(A)$, $V(f) \cap F' = \emptyset$ implies that $V(f) \cap X'$ also satisfies the condition (S_2). We can then choose f_2 such that f_2 vanishes neither on F nor on F', so that $X' \cap V(f_2)$ satisfies (S_2). But then, by virtue of Hartshorne's theorem III 3.6, $X' \cap V(f_2)$ is connected in codimension 1 thus (since every component of $X' \cap V(f_2)$ has dimension $\geqslant k$) it is connected in dimension $\geqslant k-1$. Applying the induction hypothesis to $V(f_2) = \operatorname{Spec}(A/f_2A)$, it follows that $X' \cap V(f_2) \cap V(f_1) \cap V(f_3) \cap \cdots \cap V(f_k)$ is connected, whereas we had constructed it disconnected, which is absurd.

Let us mention some interesting corollaries :

Corollary 2.2. — *Let $f\colon X \to Y$ be a proper morphism of locally noetherian preschemes, with Y an integral domain, $y_0 \in Y$, y_1 the generic point of Y. We suppose*

a) Y *is unibranch at y_0, every irreducible component of* X *dominates* Y.

b) *The irreducible components of X_1 have dimension $\geqslant k+1$, and X_1 is connected in dimension $\geqslant k$.*

119 *Then the irreducible components of X_0 have dimension $\geqslant k+1$, and X_0 is connected in dimension $\geqslant k$.*

Indeed, Zariski's connection theorem (cf. EGA III 4.3.1) implies that X_0 is connected; to show that it is not disconnected by a closed subset of dimension $< k$, we are reduced to showing that its local rings at points $x \in X_0$ such that $\dim \overline{x} < k$ have a spectrum not disconnected by x. But this is true without assuming either f proper, or Y unibranch at y_0. We are reduced to see this to the case where X is an integral domain dominating Y, and if one wishes, Y affine of finite type over **Z**, so that we are

(*) Cf. EGA IV 5.7.2.

under the conditions of the dimension formula for $\mathscr{O}_{X,x}$ over $\mathscr{O}_{Y,y_0}$. Using in this case the finiteness of the normal closure, we can even suppose X is *normal*, thus by virtue of a theorem of Nagata[(†)], the completion of a local ring $\mathscr{O}_{X,x}$ of X′ is still normal, thus (if $\mathscr{O}_{X,x}$ has dimension N) $\mathrm{Spec}(\widehat{\mathscr{O}}_{X,x})$ is connected in dimension $\geqslant N-1$. Let $n = \dim \mathscr{O}_{Y,y_0}$, then $\deg\mathrm{tr}\ k(x)/k(y) < k$ implies $\dim \mathscr{O}_{X,x} > n+(k+1)-k = n+1$, taking into account that $\dim X_1 \geqslant k+1$, and taking a system $f_1, \ldots, f_n$ of parameters of $\mathscr{O}_{Y,y_0}$ which we lift to elements of $\mathscr{O}_{X,x}$, we see by 2.1 that $\mathrm{Spec}(\widehat{\mathscr{O}}_{X,x}/\sum f_i \widehat{\mathscr{O}}_{X,x})$ is connected in dimension $\geqslant 1$, i.e. is not disconnected by its closed point, or what amounts to the same, $\mathrm{Spec}(\widehat{\mathscr{O}}_{X_0,x})$ is not disconnected by its closed point ; a fortiori the same is true of $\mathrm{Spec}(\mathscr{O}_{X_0,x})$.

As in the case of the ordinary connection theorem, one can vary 2.2 by taking the geometric fibers (over the algebraic closures of the residue fields), provided one supposes Y is *geometrically* unibranch at y_0, or (without other hypothesis than Y noetherian) that f is universally open. Applying this to the case where Y is the dual scheme of a projective scheme $\mathbf{P}^r_k$ over a field, we recover a strengthened form of the global result that had inspired 2.1, namely :

Corollary 2.3[(6)]. — *Let* X *be a closed subscheme of* $\mathbf{P}^r_k$ *(*k *a field), we suppose the irreducible components of* X *have dimension* $\geqslant l+1$*, and* X *is geometrically connected* 120
in dimension $\geqslant l$*. Then for any sequence* $H_1, \ldots, H_m$ *of* m *hyperplanes of* $\mathbf{P}^r_k$ $(0 \leqslant m \leqslant l-1)$, $X \cap H_1 \cap \cdots \cap H_m$ *satisfies the same condition with* $l-m$*, in particular is geometrically connected in dimension* $\geqslant l-1$.

One can moreover modify this statement in an obvious way, for the case where we are given a proper morphism $X \to \mathbf{P}^r_k$, which is not necessarily an immersion, and a similar extension is possible for 2.1 (by considering a proper scheme over X′). These statements are moreover formally deduced from the statements given here, taking into account the ordinary connection theorem which reduces us to the case of a finite morphism.

Corollary 2.4. — *Let* A *be a complete normal noetherian local ring of dimension* $\geqslant k+2$*. Let* $X = \mathrm{Spec}(A)$, $X' = X - \{\mathfrak{a}\}$, $f_1, \ldots, f_k$ *elements of* $\mathfrak{r}(A)$*, then* $Y' = X' \cap V(f_1) \cap \cdots \cap V(f_k)$ *is connected, and* $\pi_1(Y') \to \pi_1(X')$ *is surjective.*

We proceed as in SGA 1 X 2.11.

In all this, it was only a question of *connectedness* questions. But in the global case, well-known theorems state that for an irreducible projective variety $X \subset \mathbf{P}^r_k$, with k

(†) Cf. EGA IV 7.8.3 (i) (ii) (v).

(6) N.D.E. : for a very nice direct proof, see (Fulton W. & Lazarsfeld. R., « Connectivity and its applications in algebraic geometry », in *Algebraic geometry* (*Chicago, Ill., 1980*), Lect. Notes in Math., vol. 862, Springer, Berlin-New York, 1981, p. 26–92, theorem 2.1). Cf. also [**HL**], cited in the editor's note (22) page 158.

algebraically closed, its intersection with a sufficiently "general" hyperplane H is irreducible, (and not only connected) : this is *Bertini's theorem*, proven by Zariski, which in turn implies by Zariski's connection theorem, that for *any* H, $\mathrm{H} \cap \mathrm{X}$ is connected (although not necessarily irreducible). One can moreover proceed in the reverse direction, by proving this last result by a Lefschetz-like technique, and deducing Bertini's theorem, by reducing to the case where X is normal, and using the following result : for a "sufficiently general" H, $\mathrm{X} \cap \mathrm{H}$ is also normal. This suggests the

Conjecture 2.5[7]. — *Let* A *be a complete normal noetherian local ring. Show that*
121 *there exists a non-zero* $f \in \mathfrak{r}(\mathrm{A})$ *such that* $\mathrm{Y}' = \mathrm{X}' \cap \mathrm{V}(f) = \mathrm{Y} - \{\mathfrak{a}\}$ (*where* $\mathrm{Y} = \mathrm{Spec}(\mathrm{A}/f\mathrm{A})$) *is normal* (*thus irreducible by* 2.1 *if* $\dim \mathrm{A} \geqslant 3$).

To do this well, one would have to show that in a suitable sense, there are even "many" elements f having the property in question, for example that one can choose f in an arbitrary power of the maximal ideal. Using Serre's criterion for normality, and the remark made above for Serre's property (S_2), we see that we would have an affirmative answer to 2.5 if we had one to the

Conjecture 2.6[8]. — *Let* A *be a complete noetherian local ring,* U *an open subset of its spectrum* X, F *a finite subset of* $\mathrm{X}' = \mathrm{X} - \{\mathfrak{a}\}$. *We suppose that* U *is regular. Prove that there exists an* $f \in \mathfrak{r}(\mathrm{A})$ *such that* $\mathrm{V}(f) \cap \mathrm{U}$ *is regular, and* $\mathrm{V}(f) \cap \mathrm{F} = \emptyset$.

See, for a result of "local Bertini" type, Chow [**2**]

3. Problems related to π_1

Here again, we have many questions, suggested by global results or transcendental results.

Conjecture 3.1[9]. — *Let* A *be a complete noetherian local ring with algebraically closed residue field,* $\mathrm{X} = \mathrm{Spec}(\mathrm{A})$, $\mathrm{X}' = \mathrm{X} - \{\mathfrak{a}\}$, $\mathfrak{a}$ *the closed point. Suppose the irreducible components of* X *have dimension* $\geqslant 2$, *and* X′ *is connected.*

(i) *Prove that* $\pi_1(\mathrm{X}')$ *is topologically finitely generated.*

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[7] N.D.E. : see the following editor's note.

[8] N.D.E. : one now finds a proof of this conjecture in the literature, and so the preceding one should also be considered proven as indicated above. One can also find two attempts at proofs, published earlier but unfortunately unsuccessful, by Flenner and Trivedi. See Trivedi V., « Erratum : "A local Bertini theorem in mixed characteristic" », *Comm. Algebra* **25** (1997), N° 5, p. 1685-1686. However, the editor has not checked that the proof is now complete.

[9] N.D.E. : the analogous statement is true for (connected) schemes X of finite type over a separably closed field k under the hypothesis of strong desingularization for all $\overline{k}$-schemes (of finite type), in particular if k is of characteristic zero or X has dimension $\leqslant 2$. We reduce to the case of quasi-projective surfaces by Lefschetz-type techniques developed by Mme Raynaud, cf. notes *supra* ; see SGA 7.1, theorem II.2.3.1.

(ii) *If p is the characteristic exponent of the residue field k of* A*, prove that the largest topological quotient group of $\pi_1(\mathrm{X}')$ which is "of order prime to p" is finitely presented.*

For part (i), using descent theory SGA 1 IX 5.2 and theorem 2.4, we are reduced to the case where A is normal of dimension 2. In this case, a systematic method to study the fundamental group of X′, inaugurated by Mumford [**5**] in the transcendental setting, consists in *desingularizing* X i.e. in considering a projective birational morphism $\mathrm{Z} \to \mathrm{X}$, with Z a regular integral domain, inducing an isomorphism $\mathrm{Z}' = \mathrm{Z}|\mathrm{X}' \to \mathrm{X}'$; it is plausible that such a Z always exists, this is in any case what the method of Abhyankar [**1**] proves in the case of "equal characteristics"(*). Let C be the fiber of the closed point of X by $\mathrm{Z} \to \mathrm{X}$, it is an algebraic curve over the residue field k, connected by virtue of the connection theorem. The solution of 3.1 then seems linked to the 122

Problem 3.2. — *With the preceding notations, relate $\pi_1(\mathrm{X}')$ with the topological invariants of* C*, in particular its fundamental group,* (*to bring out the finite topological generation of $\pi_1(\mathrm{X}')$, using for example* SGA I*, theorem* X 2.6).

Another method would be to consider A as a finite algebra over a complete *regular* local ring B of dimension 2, ramified along a curve C contained in $\mathrm{Spec(B)} = \mathrm{Y}$. We are thus led to the

Problem 3.3. — *Let* A *be a complete regular local ring of dimension* 2*, with algebraically closed residue field k,* X *its spectrum,* C *a closed subset of* X *of dimension* 1*. Define local invariants of the immersed curve* C*, having a meaning independent of the residual characteristic, and whose knowledge allows to calculate the fundamental group of* $\mathrm{X} - \mathrm{C}$ *by generators and relations when k is of characteristic zero. Prove that when k is of characteristic $p > 0$, the "tame" fundamental group of* $\mathrm{X} - \mathrm{C}$ *is a quotient of the preceding one, and the two fundamental groups* (*in characteristic* 0*, and in characteristic $p > 0$*) *have the same maximal quotient of order prime to p.*

Of course, 3.3 shows us that in 3.1, one should also replace X′ by a scheme of the form $\mathrm{X} - \mathrm{Y}$, where Y is a closed subset of X which has codimension $\geqslant 2$ in any 123
component of X containing it. When we abandon this restriction on Y, there should still exist an analogous finiteness result, provided we put "tame"-like restrictions on the ramification at the maximal points of the irreducible components of Y which are of codimension 1.

Problem 3.4. — *Let* A *be a complete noetherian local ring of dimension* 2*, with algebraically closed residue field. Let again* $\mathrm{X} = \mathrm{Spec(A)}$, $\mathrm{X}' = \mathrm{X} - \{\mathfrak{a}\}$. *Find particular structural properties of $\pi_1(\mathrm{X}')$ for the case where* A *is a complete intersection.*

(*) The possibility of "resolving" A is now proven in full generality by Abhyankar [**8**].

A satisfactory solution to this problem would perhaps allow to solve the following old problem :

Conjecture 3.5[10]. — *Find an irreducible curve in* $\mathbf{P}^3_k$ *(k an algebraically closed field), preferably non-singular, which is not set-theoretically the intersection of two hypersurfaces.*

(Kneser [**4**] shows that one can always obtain it as the intersection of three hypersurfaces).

4. Problems related to the higher π_i : local and global Lefschetz theorems for complex analytic spaces[11]

Let X be a scheme locally of finite type over the field of complex numbers $\mathbf{C}$, one can associate to it an analytic space X^h over $\mathbf{C}$, whence homotopy and homology invariants $\pi_i(\mathrm{X}^h)$, $\mathrm{H}_i(\mathrm{X}^h)$, $\mathrm{H}^i(\mathrm{X}^h)$ etc. We know moreover that X is connected if and only if X^h is, so that we have a bijection

$$\pi_0(\mathrm{X}^h) \longrightarrow \pi_0(\mathrm{X})$$

124 Similarly, since any étale covering X' of X defines an étale covering X'^h of X^h, we have a canonical homomorphism

$$\pi_1(\mathrm{X}^h) \longrightarrow \pi_1(\mathrm{X}),$$

of which we know, using a theorem of Grauert-Remmert, that it identifies the second group with the compactification of the first for the topology of finite index subgroups (which simply expresses the fact that $\mathrm{X}' \mapsto \mathrm{X}'^h$ is an equivalence of the category of étale coverings of X with the category of *finite* étale coverings of X^h). It follows that the results of this seminar (by purely algebraic means) on $\pi_0(\mathrm{X})$ and $\pi_1(\mathrm{X})$ imply results for $\pi_0(\mathrm{X}^h)$ and $\pi_1(\mathrm{X}^h)$ (which are of a transcendental nature). If moreover X is proper, the well-known exact sequence $0 \to \mathbf{Z} \to \mathbf{C} \to \mathbf{C}^* \to 0$ allows to show that the Néron-Severi group of X (quotient of its Picard group by the connected component of the neutral element) is isomorphic to a subgroup of $\mathrm{H}^2(\mathrm{X}^h, \mathbf{Z})$; in the non-singular kählerian case, it is the subgroup denoted $\mathrm{H}^{(1,1)}(\mathrm{X}^h, \mathbf{Z})$ (classes of type $(1,1)$) :

$$\mathrm{Pic}(\mathrm{X})/\mathrm{Pic}^0(\mathrm{X}) \subset \mathrm{H}^2(\mathrm{X}, \mathbf{Z}).$$

Consequently, the information we have obtained on the Picard groups implies information, very partial it is true, on the groups $\mathrm{H}^2(\mathrm{X}^h, \mathbf{Z})$. It is tempting to complete all these fragmentary results with conjectures.

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(10) N.D.E. : this problem is, in the fall of 2004, still open.

(11) N.D.E. : the statements are made precise in the Comments (section 6). The conjectures that appear there have become theorems, cf. the footnotes of section 6.

Very precise indications, going in the same direction as those we have just pointed out, are provided by a classic theorem of Lefschetz [**7**]. It states that if X is a non-singular irreducible projective analytic space of dimension n, and if Y is a non-singular hyperplane section, then the injection

$$\mathrm{Y}^{n-1} \longrightarrow \mathrm{X}^n$$

induces a homomorphism 125

$$\pi_i(\mathrm{Y}^{n-1}) \longrightarrow \pi_i(\mathrm{X}^n)$$

which is an *isomorphism for* $i \leqslant n-2$, an *epimorphism for* $i = n-1$. The analogous statement for the homomorphisms

$$\mathrm{H}_i(\mathrm{Y}^{n-1}) \longrightarrow \mathrm{H}_i(\mathrm{X}^n)$$

on homology (integer for fixing ideas) results from this, while in cohomology,

$$\mathrm{H}^i(\mathrm{X}^n) \longrightarrow \mathrm{H}^i(\mathrm{Y}^{n-1})$$

is an isomorphism in dimension $i \leqslant n-2$, a monomorphism in dimension $i = n-1$. We have obtained variants of these results in the context of schemes, for π_0, π_1, Pic, valid moreover without hypotheses of non-singularity to a large extent, cf. Exp. XII. Moreover, in the elimination of non-singularity hypotheses, we have used in an essential way "local" variants of these global Lefschetz theorems. All this suggests the following problems, which will doubtless have to be attacked simultaneously(*).

Problem 4.1. — *Let* X *be an analytic space,* Y *a closed analytic subset of* X (*or simply a closed subset ?*)(12) *such that for any* $x \in \mathrm{Y}$*, the local ring* $\mathscr{O}_{\mathrm{X},x}$ *is a complete intersection. Let* n *be the complex codimension of* Y *in* X*. Is the canonical homomorphism*

$$\pi_i(\mathrm{X} - \mathrm{Y}) \longrightarrow \pi_i(\mathrm{X})$$

an isomorphism for $i \leqslant n-2$*, and an epimorphism for* $i = n-1$ *?*

In this problem and the following ones, it is obviously implicitly assumed that a base-point has been chosen to define the homotopy groups. To state the following problem, one must define, for an analytic space X (more generally, for a locally arcwise connected space) and an $x \in \mathrm{X}$, local invariants $\pi_i^x(\mathrm{X})$(†). For this, we choose a non-constant map f from the interval $[0,1]$ to X, such that $f(0) = x$ and $f(t) \neq x$ for $t \neq 0$ (such a map exists if x is not an isolated point). Then for any neighborhood U of x, there exists an $\varepsilon > 0$ such that $0 < t < \varepsilon$ implies $f(t) \in \mathrm{U}$, and the homotopy 126

(*)The formulations 4.1 to 4.3 that follow are provisional. See conjectures A to D below, in "comments on Exp. XIII" for more satisfactory formulations, as well as Exp. XIV.
(†)If $i \geqslant 2$. For the case $i \leqslant 1$, cf. *Comments* in number 6 below, page 157.

(12)N.D.E. : the meaning of this question is not clear; indeed, the very statement of the problem does not seem to make sense in this case since the codimension of Y in X is not defined when Y is no longer assumed to be analytic.

groups $\pi_i(\mathrm{U}-x, f(t))$ are essentially independent of t (they are, for variable t, related by a transitive system of isomorphisms), one can denote them $\pi_i(\mathrm{U}-x, f)$. We then set

$$\pi_i^x(\mathrm{X}) = \varprojlim_{\mathrm{U}} \pi_{i-1}(\mathrm{U}-x, f)$$

the projective limit being taken over the system of open neighborhoods U of x. Strictly speaking, this limit depends on f, and should be denoted $\pi_i^x(\mathrm{X}, f)$, but we check that for variable f, these groups are isomorphic to each other[(*)], precisely they form a local system on the space of paths of the type considered starting from x. These invariants are the homotopic version of the local cohomology invariants $\mathrm{H}_x^i(\mathrm{F})$ for a sheaf F on X, introduced in I, and should play the role of *relative local homotopy groups* of X modulo $\mathrm{X}-x$. Their vanishing for $i \leqslant n$ and for any $x \in \mathrm{Y}$, where Y is a closed subset of X of topological dimension $\leqslant d$, should imply that the homomorphisms

$$\pi_i(\mathrm{X}-\mathrm{Y}) \longrightarrow \pi_i(\mathrm{X})$$

are bijective for $i < n-d$, and surjective for $i = n-d$[(†)]. From this point of view, 4.1 would imply (for Y reduced to a point) a conjecture of a purely local nature, expressing itself by

$$\pi_i^x(\mathrm{X}) = 0 \text{ for } i \leqslant n-1$$

when X is a complete intersection of dimension n at x.

À As an example of local invariants $\pi_i^x(\mathrm{X})$, note that if x is a non-singular point
127 of X of complex dimension n, then

$$\pi_i^x(\mathrm{X}) = \pi_{i-1}(\mathrm{S}^{2n-1}),$$

where S^{2n-1} denotes the sphere of dimension $2n-1$. In particular in this case $\pi_i^x(\mathrm{X}) = 0$ for $i \leqslant 2n-1$, which corresponds to the fact that if from a topological *manifold* X one removes a closed subset Y of codimension $\geqslant m$, then $\pi_i(\mathrm{X}-\mathrm{Y}) \to \pi_i(\mathrm{X})$ is an isomorphism for $i \leqslant m-2$ and an epimorphism for $i = m-1$.

This being said :

Problem 4.2. — *Let* X *be an analytic space,* $x \in \mathrm{X}$*,* t *a section of* $\mathscr{O}_\mathrm{X}$ *vanishing at* x*,* Y *the set of zeros of* t*. Suppose the following conditions are satisfied :*

a) *t is regular at x (i.e. not a zero divisor at x, a possibly superfluous hypothesis, moreover).*

b) *At the points x' of* $\mathrm{X}-\mathrm{Y}$ *neighboring x,* $\mathscr{O}_{\mathrm{X},x'}$ *is a complete intersection (a hypothesis that should be replaceable by the following more general one if* 4.1 *is true : for x' as above,* $\pi_i^{x'}(\mathrm{X}) = 0$ *for* $i \leqslant n-1$*).*

c) *At the points y of* $\mathrm{Y}-\{x\}$ *neighboring x, we have*

$$\operatorname{prof} \mathscr{O}_{\mathrm{X},y} \geqslant n$$

(*) At least if x does not disconnect X in a neighborhood of x, cf. *Comments* below, page 157.

(†) For a corrected formulation, cf. *Comments* below, page 157.

(*it is sufficient for example that we have* prof $\mathscr{O}_{X,x} \geqslant n$).

Under these conditions, is the canonical homomorphism

$$\pi_i^x(Y) \longrightarrow \pi_i^x(X)$$

an isomorphism for $i \leqslant n-2$, *an epimorphism for* $i = n-1$ *?*

Here is finally a global variant of 4.2, which should be deduced from it by consideration of the projecting cone at its origin, and which would generalize the classical Lefschetz theorems :

Problem 4.3. — *Let* X *be a projective analytic space, equipped with an ample invertible* **128**
Module L, t *a section of* L, Y *the set of zeros of* t. *Suppose :*

a) t *is a regular section* (*hypothesis possibly superfluous*).

b) *For any* $x \in X - Y$, $\mathscr{O}_{X,x}$ *is a complete intersection* (*should be replaceable by* $\pi_i^x(X) = 0$ *for* $i \leqslant n-1$).

c) *For any* $x \in Y$, prof $\mathscr{O}_{X,x} \geqslant n$.

Under these conditions, is the homomorphism

$$\pi_i(Y) \longrightarrow \pi_i(X)$$

an isomorphism for $i \leqslant n-2$, *an epimorphism for* $i = n-1$ *?*

We will leave to the reader the task of stating analogous conjectures of a *cohomological* nature(*), the hypotheses and conclusions then bearing on the local cohomological invariants (with coefficients in a given group). In any case, the key result seems to have to be 4.2, when hypothesis *b*) is taken there in the resp. form, — whether one places oneself from the point of view of homology, or homotopy.

We have stated these conjectures in the transcendental setting, in the hope of interesting topologists in them and convincing them that questions of the "Lefschetz" type are far from closed. Of course, now that we are on the point of having a good theory of the cohomology of schemes (with finite coefficients), thanks to the recent work of M. Artin, the same questions arise in the context of schemes, and it is difficult to doubt that they will not receive a positive answer, in the near future(†).

5. Problems related to local Picard groups

A first fundamental problem, pointed out for the first time by Mumford [**5**] in a **129**
particular case, is the following. Let A be a complete local ring with residue field k, $X = \mathrm{Spec}(A)$, $U = \mathrm{Spec}(A) - \{a\}$, where a is the maximal ideal of A i.e. the closed point of Spec(A). We propose to construct a strict projective system G of locally

(*) Cf. Exp. XIV the corresponding results in the theory of schemes.

(†) see preceding note.

algebraic groups G_i over k, and a natural isomorphism

$$\mathrm{Pic}(U) \simeq G(k) \tag{+}$$

where we obviously set $G(k) = \varprojlim G_i(k)$. Heuristically, we propose to "put an algebraic group structure" (or, at least, pro-algebraic, in a suitable sense) on the group $\mathrm{Pic}(U)$.

It is obvious that as stated, the problem is not precise enough, because the given of an isomorphism (+) is far from characterizing the pro-object G. If A contains a subfield also denoted k, which is a field of representatives, one can make the problem more precise, by requiring that for a variable extension k' of k, we have an isomorphism, functorial in k' :

$$\mathrm{Pic}(U') \simeq G(k') \tag{+$'$}$$

where U' is the open set analogous to U in $\mathrm{Spec}(A')$, $A' = A\widehat{\otimes}_k k'$. One can proceed in a similar way even if A does not have a field of representatives, provided that k is perfect, which then allows to construct functorially an A' "by residual extension k'/k". Moreover, when A admits a field of representatives, the algebraic structure that one will find on $\mathrm{Pic}(U)$ will essentially depend on the choice of this field of representatives (as one already sees on the projecting cone of an elliptic curve), it seems therefore that one must start from a "pro-algebraic ring over k" in the sense of Greenberg [**3**], to succeed in defining the pro-object G. It is moreover conceivable that in the case where
130 there is no given field of representatives, one only finds a projective system of *quasi*-algebraic groups in the sense of Serre, or rather quasi-locally algebraic (the groups G_i obtained will not in general be of finite type over k, but only locally of finite type over k). It is even possible that one will in general only find an even weaker structure on $\mathrm{Pic}(U)$, of the kind encountered by Néron [**6**] in his theory of degeneration of abelian varieties defined over local fields.

A method to attack the problem, also introduced by Mumford, consists in desingularizing X, i.e. in considering a projective birational morphism $Y \to X$ with Y regular. When U is regular (i.e. a is an isolated singular point), one can often find Y in such a way that $Y_{|}U = V \to U$ is an isomorphism. In this case, we will thus have

$$\mathrm{Pic}(U) \simeq \mathrm{Pic}(V) \simeq \mathrm{Pic}(Y)/\operatorname{Im}\mathbf{Z}^{I},$$

where I is the set of irreducible components of the fiber Y_a (each of these defining an element of $\mathrm{Pic}(Y)$, being a locally principal divisor, thanks to Y being regular). On the other hand, using the technique of Formal Geometry EGA III 4 and 5, notably the existence theorem, one finds

$$\mathrm{Pic}(Y) \simeq \varprojlim \mathrm{Pic}(Y_n),$$

where $Y_n = Y \otimes_A A_n$, $A_n = A/\mathfrak{m}^{n+1}$. When A admits a field of representatives k, we have the theory of Picard schemes of the projective schemes Y_n over k, so we have

$$\mathrm{Pic}(Y_n) \simeq \mathbf{Pic}_{Y_n/k}(k).$$

This thus provides a construction of a projective system of locally algebraic groups $\mathbf{Pic}_{\mathrm{Y}_n/k}/\operatorname{Im}\mathbf{Z}^{\mathrm{I}}$, which is the sought system.[(13)] In the case considered here, one can moreover see (using that a is an isolated singular point) that the connected components of the universal image subgroups in this projective system form an *essentially* **131**
constant projective system, so in this case one finds a locally algebraic group G as a solution to the problem. If we even suppose A is normal of dimension 2, then a remark by Mumford (saying that the intersection matrix of the components of Y_a in X is negative definite[(14)]) implies that G is even an algebraic group, i.e. of finite type over k (the number of its connected components being moreover equal to the determinant of the intersection matrix considered just now).

If on the other hand a is not an isolated singularity, one is convinced on examples (with A of dimension 2) that one finds a projective system of algebraic groups, not reducing to a single algebraic group.

Once we have a good notion of "*local Picard scheme*", one would need to strengthen the notion of parafactoriality, by saying that A is "*geometrically parafactorial*", when not only A and even $\widehat{\mathrm{A}}$ are parafactorial, but that the local Picard scheme $\mathrm{G}(\widehat{\mathrm{A}})$ is the trivial group (which is stronger, when the residue field is not algebraically closed, than saying that G has no other rational point over k than the identity). We realize the necessity of a strengthened notion of parafactoriality, by recalling that there exist normal complete local rings of dimension 2 which are factorial, but which admit finite étale algebras which are not[(15)]. A "*geometrically factorial*" local ring would then be a normal ring A such that all lits localizations of dimension $\geqslant 2$ are geometrically

[(13)] N.D.E. : the question has been greatly clarified by the results of Boutot (Boutot J.-F., *Schéma de Picard local*, Lect. Notes in Math., vol. 632, Springer, Berlin, 1978). In particular, if A is a complete local (noetherian) k-algebra of depth $\geqslant 2$ such that $\mathrm{H}^2_{\mathfrak{m}}(\mathrm{A})$ has finite dimension over k, the local Picard group is a group scheme locally of finite type over k, with tangent space at the origin $\mathrm{H}^2_{\mathfrak{m}}(\mathrm{A})$. If A is moreover normal of dimension $\geqslant 3$, Serre's criterion for normality XI 3.11 together with corollary V 3.6 ensure the required finiteness and, consequently, the existence of the local Picard scheme. See also (Lipman J., « The Picard group of a scheme over an Artin ring », *Publ. Math. Inst. Hautes Études Sci.* **46** (1976), p. 15–86) for an approach closer to that of Grothendieck sketched above.

[(14)] N.D.E. : Mumford D., « The topology of normal singularities of an algebraic surface and a criterion for simplicity », *Publ. Math. Inst. Hautes Études Sci.* **9** (1961), p. 5–22.

[(15)] N.D.E. : factorial rings with non-factorial strict henselization arise naturally when studying moduli spaces of vector bundles. See for example (Drézet J.-M., « Groupe de Picard des variétés de modules de faisceaux semi-stables sur $\mathbf{P}_2$ », in *Singularities, representation of algebras, and vector bundles* (*Lambrecht, 1985*), Lect. Notes in Math., vol. 1273, Springer, Berlin, 1987, p. 337–362). *Stricto sensu*, Drézet shows that the completion is not factorial, but in fact the proof gives the result for the henselization : the point is that Luna's étale slice theorem (Luna D., « Slices étales », in *Sur les groupes algébriques*, Mém. Soc. math. France, vol. 33, Société mathématique de France, Paris, 1973, p. 81–105) describes the local ring of a quotient in the sense of invariant geometry near a semi-stable point locally for the étale topology.

parafactorial, or better, such that the localizations of $\widehat{\mathrm{A}}$ are parafactorial[(*)]. Of course, it would be interesting to find a "good" definition of these notions, independent of the theory, still to be done, of local Picard schemes.[(16)]

It is in any case plausible that we will need these notions if we wish to obtain statements of the following type : Let A be a "good ring" (for example a finitely generated algebra over $\mathbf{Z}$, or over a complete local ring, for example over a field).
132 Let U be the set of $x \in \mathrm{X} = \mathrm{Spec}(\mathrm{A})$ such that $\mathscr{O}_{\mathrm{X},x}$ is "geometrically factorial", then U is open ? Or again : Let $f\colon \mathrm{X} \to \mathrm{Y}$ be a flat morphism of finite type with Y locally noetherian, let U be the set of $x \in \mathrm{X}$ such that $\mathscr{O}_{\mathrm{X}_{f(x)},x}$ is "geometrically factorial", then U is open, at least under nice supplementary conditions on f ? I doubt that with the usual notion of factorial ring, there exist true statements of this type.

We have raised here, in a particular case, the question of the study of the geometric properties of "variable" local rings, for example the $\mathscr{O}_{\mathrm{X},x}$ for x varying over a prescheme X. When X is a scheme of finite type over a field, for example, we know[(17)] that there exists on X a projective system of finite algebras $\mathrm{P}^n_{\mathrm{X}/k}$ (obtained by completing $\mathrm{X}\times_k\mathrm{X}$ along the diagonal), whose fiber at any point $x \in \mathrm{X}$ rational over k is isomorphic to the projective system of the $\mathscr{O}_{\mathrm{X},x}/\mathfrak{m}_{\mathrm{X}}^{n+1}$. It is then natural to link the study of the completions of the local rings $\mathscr{O}_{\mathrm{X},x}$, for variable x, to that of the "algebraic family of complete local rings" given by the P^n, by noting that for any $x \in \mathrm{X}$ (rational over k or not), one obtains a complete local ring

$$\mathrm{P}^{\infty}(x) = \varprojlim \mathrm{P}^n(x)$$

(where $\mathrm{P}^n(x) =$ reduced fiber $\mathrm{P}^n \otimes_{\mathscr{O}_{\mathrm{X},x}} k(x)$). A particular interest will be attached for example to the complete ring associated in this way to the generic point, and one would expect that its algebraic-geometric properties (expressing themselves for example by its Picard, or homotopy, or homology, local groups), will essentially be those of the completions $\widehat{\mathscr{O}}_{\mathrm{X},x}$ for x in a suitable dense open set U.

One can, in a general way, propose to make a simultaneous study of the complete local rings obtained thus from an adic projective system (P_n) of finite algebras over a given scheme X. It is plausible that one will find, subject to certain regularity
133 conditions (such as the flatness of the P_n) that the local homotopy groups come from a projective system of finite group schemes over X, and that one will have analogous results for the local Picard groups. Concerning these latter, a first interesting case that deserves to be investigated is that where one starts from an algebraic surface X having singular curves, and one proposes to study the local Picard schemes at the variable points on these, in terms of a suitable pro-group scheme defined on the singular locus.

(*) For a more flexible notion of "geometrically factorial" local ring, cf. *Comments*, page 155.

(16) N.D.E. : see page 155 : a local ring is geometrically factorial (resp. parafactorial) if its strict henselization is factorial (resp. parafactorial).

(17) N.D.E. : see EGA IV.16.

6. Comments[(*)]

The point of view of the "Étale Cohomology" of schemes and recent progress in this theory, leadsus to specify and at the same time to broaden some of the problems posed. For the notion of "topology" and of "étale topology of a scheme", I refer to M. Artin, *Grothendieck Topologies*, Harvard University 1962 (mimeographed notes)[(†)]. 134

This theory, by a finer notion of localization than that provided by the traditional "Zariski topology", leads to attaching a particular interest to *strictly local* rings, i.e. henselian local rings with separably closed residue field. For any local ring A with residue field k, and any separable closure k' of k, one can find a local homomorphism from A to a strictly local ring A', the *strict henselization* of A, with residue field k', having an obvious universal property. A' is henselian, flat over A, and $\mathrm{A}' \otimes_{\mathrm{A}} k \simeq k'$; it is noetherian if and only if A is. (Cf. *loc. cit.* Chap. III, section 4)[(*)]. If X is a prescheme, and x a point of X, x' a point above x, spectrum of a separable closure k' of $k = k(x)$, one is led to define the *strictly local ring* of X at x', $\mathscr{O}'_{\mathrm{X},x'}$, as the strict henselization of the usual local ring $\mathscr{O}_{\mathrm{X},x}$, relative to the residual extension k'/k. It is the *strictly* local rings of the "geometric" points of X which, from the point of view of the étale topology, are supposed to reflect the local properties of the prescheme X. They also play, in many respects, the role that was given to the *completions* of the local rings of X (say, at points with algebraically closed residue field), while remaining "closer" to X and allowing an easier passage to "neighboring points".

One should then reconsider a good number of questions, that are generally posed for complete local rings (possibly restricted to having an algebraically closed residue 135
field), for henselian noetherian local rings (resp. strictly local noetherian rings). Thus the topological problems raised in numbers 2 and 3, arise more generally for strictly local rings. One can moreover state as a conjecture, for "good" strictly local rings, certain properties of simple connectedness and acyclicity for the geometric fibers of the canonical morphism $\mathrm{Spec}(\widehat{\mathrm{A}}) \to \mathrm{Spec}(\mathrm{A})$, which would show that for many properties of a "topological" nature, it comes to the same thing to prove them for the ring A, or for its completion $\widehat{\mathrm{A}}$. Certain results already obtained in this direction[(†)] give hope that we will soon have complete results in this direction.

The notion of étale localization provides a definition that seems reasonable for the notion of "*geometrically parafactorial*" or "*geometrically factorial*" local ring (the need for which was pointed out in number5, p. 153) : we will call so a local ring whose strict henselization is parafactorial, resp. factorial. Hypotheses of this nature are indeed

[(*)] Written in March 1963.
[(†)] Or preferably, to SGA 4.
[(*)] Or EGA IV 18.8.
[(†)] Cf. M. Artin in SGA 4 XIX

introduced in a natural way in the study of the étale cohomology of preschemes[18]. Thus, if X is a locally noetherian prescheme whose strictly local rings are factorial (i.e. whose ordinary local rings are "geometrically factorial"), one shows that the $\mathrm{H}^i(\mathrm{X}_{\text{ét}}, \mathbf{G}_m)$ are torsion groups for $i \geqslant 2$ (which sometimes allows to express these groups in terms of the cohomology groups with coefficients in the groups $\boldsymbol{\mu}_n$ of the n-th roots of unity), and if X is an integral domain with field of fractions K, the natural homomorphism $\mathrm{H}^2(\mathrm{X}_{\text{ét}}, \mathbf{G}_m) \to \mathrm{H}^2(\mathrm{K}, \mathbf{G}_m) = \mathrm{Br}(\mathrm{K})$ is injective[19], examples show that these conclusions can be at fault, even for local X, if one only supposes X is factorial instead of geometrically factorial[‡].

Concerning the problems of local and global Lefschetz type raised in 3.4, and their analogues in the theory of schemes, the homological version of these questions has been considerably clarified, all resulting formally from three general theorems,
136 one concerning the cohomological dimension of certain affine schemes (resp. of Stein spaces), such as affine schemes X of finite type over an algebraically closed field : their cohomological dimension is $\leqslant \dim \mathrm{X}$ ("affine Lefschetz theorem")[*] : the other being a duality theorem for the cohomology (with discrete coefficients) of a projective morphism[†], finally the last a theorem of *local duality* of a similar nature[‡]. In Algebraic Geometry, only this last one is not proven at the time of writing these lines (it is however in characteristic 0, using resolution of singularities by Hironaka). Moreover,

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[‡]Cf. A. Grothendieck, le groupe de Brauer II (Séminaire Bourbaki N° 297, Nov. 1965), notably 1.8 and 1.11 b.

[*]Cf. SGA 4 XIV.

[†]Cf. SGA 4 XVIII.

[‡]Cf. SGA 5 I.

[18]N.D.E. : see for example (Strano R., « The Brauer group of a scheme », *Ann. Mat. Pura Appl.* (*4*) **121** (1979), p. 157–169) where the hypothesis of geometric parafactoriality of the local rings of a scheme X sometimes allows to show the coincidence of the Brauer groups of X (calculated in terms of Azumaya algebras) and the cohomological Brauer group of X.

[19]N.D.E. : the link between the Brauer group and the Picard group is intimate. Let us cite in this respect the following results of Saito (Saito S., « Arithmetic on two-dimensional local rings », *Invent. Math.* **85** (1986), N° 2, p. 379–414) in the case of surfaces, the first being local the other global. Let A be an excellent local ring of dimension 2, normal and henselian with *finite* residue field and X the complement of the closed point in Spec(A). Then, we have a perfect duality of *torsion* groups $\mathrm{Pic}(\mathrm{X}) \times \mathrm{Br}(\mathrm{X}) \to \mathbf{Q}/\mathbf{Z}$ — by Brauer group of X, we mean *cohomological* Brauer group $\mathrm{Br}(\mathrm{X}) = \mathrm{H}^2_{\text{ét}}(\mathrm{X}, \mathrm{G}_m)$. In the global case, we have the following generalization of a result of Lichtenbaum (Lichtenbaum S., « Duality theorems for curves over p-adic fields », *Invent. Math.* **7** (1969), p. 120–136) : let k be the field of fractions of a complete discrete valuation ring $\mathscr{O}$ with finite residue field and X a projective curve, smooth and geometrically complete over k. The group $\mathrm{Pic}^0(\mathrm{X})$ is equipped with the topology induced from the adic topology of k and Pic(X) is the topological group that makes $\mathrm{Pic}^0(\mathrm{X})$ an open subgroup. Then, we have a perfect duality of topological groups $\mathrm{Pic}(\mathrm{X}) \times \mathrm{Br}(\mathrm{X}) \to \mathbf{Q}/\mathbf{Z}$. Note that this statement, which concerns curves, is of course proven by considering a regular model (proper and flat) of X over $\mathscr{O}$: it is a result on surfaces.

in the transcendental setting, we already have global and local duality, recently proven by Verdier[(20)]. Let us limit ourselves to indicating that in the statement of the homological versions of problems 4.2 and 4.3 (which henceforth deserve the name of conjectures), the conditions "at infinity" a) and c) are certainly superfluous, only the *local cohomological structure* of $\mathrm{X}-\mathrm{Y}$ being important, which we will suppose for example to be locally a complete intersection of dimension $\geqslant n$. Moreover, in 4.3 let's say, the fact that Y is a hyperplane section should not play a role, and should be able to be replaced by the sole hypothesis that X is compact and $\mathrm{X}-\mathrm{Y}$ is Stein (i.e. in the case of Algebraic Geometry, X is proper over k and $\mathrm{X}-\mathrm{Y}$ affine ; as we were saying, the homological version of this conjecture is proven for algebraic spaces over the field $\mathbf{C}$)[(§)].

In the definition (p. 150) of the $\pi_i^x(\mathrm{X})$, one must suppose $i \geqslant 2$. For $i = 0, 1$, there is no reasonable definition of the $\pi_i^x(\mathrm{X})$; they should be replaced by $\mathrm{H}_0^x(\mathrm{X})$ and $\mathrm{H}_1^x(\mathrm{X})$, defined respectively as the cokernel and the kernel in the natural homomorphism

$$\varprojlim \mathrm{H}_0(\mathrm{U}-(x), \mathbf{Z}) \longrightarrow \varprojlim \mathrm{H}_0(\mathrm{U}, \mathbf{Z}).$$

À For the sake of rigor and for the convenience of formulations, we can set $\pi_i^x(\mathrm{X}) = \mathrm{H}_i^x(\mathrm{X})$ for $i \leqslant 1$, otherwise we must complete the subsequent assertions concerning the π_i^x with the corresponding assertions for $\mathrm{H}_0^x, \mathrm{H}_1^x$. If x is an isolated point of X, it is appropriate to set $\pi_i^x(\mathrm{X}) = 0$ for $i \neq 0$, $\pi_0^x(\mathrm{X}) = \mathrm{H}_0^x(\mathrm{X}) = \mathbf{Z}$.

The assertion that the $\pi_i^x(\mathrm{U}, f)$ are isomorphic to each other is only true when X **137**
is not disconnected by x in a neighborhood of x, i.e. if $\pi_i^x(\mathrm{X}) = 0$ for $i = 0, 1$. In the general case $\pi_i^x(\mathrm{X})$ can only denote a *family* of groups, not necessarily isomorphic to each other ; however the writing $\pi_i^x(\mathrm{X}) = 0$ keeps an obvious meaning.

Page 150, where I predict that the vanishing of the local homotopic invariants $\pi_i^x(\mathrm{X})$ for $x \in \mathrm{Y}$, $i \leqslant n$ must imply the bijectivity of $\pi_i(\mathrm{X}-\mathrm{Y}) \to \pi_i(\mathrm{X})$ for $i < n-d$, the surjectivity for $i = n-d$, it is appropriate to be cautious, for lack of being able to dispose in the present context (as in Algebraic Geometry) of "general" points at which the local conditions will also have to apply. It will doubtless be necessary, for this reason, to appeal to relative local homotopic invariants

$$\pi_i^{\mathrm{Y}}(\mathrm{X}, f) = \pi_i^{\mathrm{Y}}(\mathrm{X}, x) = \varprojlim_{\mathrm{U}} \pi_{i-1}(\mathrm{U}-\mathrm{U}\cap\mathrm{Y}, f(t)) \qquad \text{for } i \geqslant 2,$$

(and *ad hoc* definition as above for $i = 0, 1$), where Y is a closed subset of X ; or to supplement the absence of general points by expressing the hypotheses on X in terms of properties of a topological nature (for the étale topology) of the spectra of the local rings of X, which allows to recover general points. The same reservation applies to

[(§)] Cf. Exp. XIV

[(20)] N.D.E. : see Verdier J.-L., « Dualité dans la cohomologie des espaces localement compacts », in *Séminaire Bourbaki*, vol. 9, Société mathématique de France, Paris, 1995, Exp. 300, p. 337–349.

the generalization of conjectures 4.2 and 4.3 to the case where $X - Y$ is not supposed to be locally a complete intersection, a generalization suggested in the statement of conditions b) of these conjectures.

To formulate the expurgated versions of conjectures 4.2 and 4.3 suggested by the results to which we have alluded above, it is appropriate to state the

Definition 1. — Let X be a topological space, Y a locally closed subset of X, and n an integer. We say that X has *homotopical depth* $\geqslant n$ along Y, and we write $\operatorname{prof\,hpt}_{Y}(X) \geqslant n$, if for any $x \in Y$, we have $\pi_i^{Y}(X, x) = 0$ for $i < n$.

138 It should be equivalent to say that for any open set X' of X, and any $x \in X' \cap U = U'$ (where $U = X - Y$), the canonical homomorphism

$$\pi_i(U', x) \longrightarrow \pi_i(X', x)$$

is an isomorphism for $i < n - 1$, a monomorphism for $i = n - 1$[(21)].

Definition 2. — Let X be a complex analytic space, n an integer, we say that the *rectified homotopical depth* of X is n[(*)], if for any locally closed analytic subset Y of X, we have

$$(x) \qquad \operatorname{prof\,htp}_{Y}(X) \geqslant n - \dim Y$$

(where, of course, $\dim Y$ denotes the *complex* dimension of Y).

It should be equivalent to say that for any irreducible locally closed analytic subset Y in X, there exists a closed analytic subset Z of Y, of dimension $< \dim Y$, such that the relation (x) is valid for $Y - Z$ instead of Y. This would allow for example in definition 2 to limit oneself to the case where Y is non-singular.[(22)]

The following conjecture, of a purely topological nature, is in the nature of a "*local Hurewicz theorem*".

(*) In the first edition of these notes, we had used the term : "true homotopical depth". In the present version, we follow EGA IV 10.8.1.

(21) N.D.E. : when the pair (X, Y) is moreover polyhedral, this equivalence is true ; cf. (Eyral C., « Profondeur homotopique et conjecture de Grothendieck », *Ann. Sci. Éc. Norm. Sup.* (*4*) **33** (2000), N° 6, p. 823–836).

(22) N.D.E. : all the conjectures that follow, suitably rectified if I dare say so, have become theorems thanks to the work of Hamm and Lê Dũng Tráng (Hamm H.A. & Lê Dũng Tráng, « Rectified homotopical depth and Grothendieck conjectures », in *The Grothendieck Festschrift, Vol. II*, Progr. Math., vol. 87, Birkhäuser, Boston, 1990, p. 311–351) cited as [**HL**] in what follows. Concerning the two conjecturally equivalent definitions of rectified depth, they are even equivalent to a third one, which is expressed in terms of Withney stratification (cf. *loc. cit.*, theorem 1.4).

Conjecture A **("local Hurewicz theorem"[23]).** — *Let* X *be a topological space,* Y *a locally closed subset, subject if necessary to "smoothness" conditions like local triangulability of the pair* (X, Y)*,* n *an integer* $\geqslant 3$*. For us to have* $\operatorname{prof}\operatorname{htp}_{\mathrm{Y}}(\mathrm{X}) \geqslant n$*, it is necessary and sufficient that we have* 139

$$\mathrm{H}^i_{\mathrm{Y}}(\mathbf{Z}_{\mathrm{X}}) = 0 \quad \text{for } i < n$$

(*we then say that* X *has* cohomological depth $\geqslant n$ *along* Y)*, and that the local fundamental groups*

$$\pi_2^{\mathrm{Y}}(\mathrm{X}, x) = \varprojlim_{\mathrm{U} \ni x} \pi_1(\mathrm{U} - \mathrm{U} \cap \mathrm{Y})$$

are zero (*we then say that* X *is "pure" along* Y)*.*

Note that if X is an analytic space, Y an analytic subspace, and if X is pure along Y, then for any $x \in \mathrm{Y}$, the local ring $\mathscr{O}_{\mathrm{X},x}$, as well as its localizations with respect to prime ideals containing the ideal defining the germ Y at x (i.e. in the inverse image Y_x of Y by $\operatorname{Spec}(\mathscr{O}_{\mathrm{X},x}) = \mathrm{X}_x \to \mathrm{X}$) are pure in the sense of Exp. X ; it seems plausible that the converse is also true. Similar remarks hold for cohomological depth, it being understood that we work with the étale topology on the $\operatorname{Spec}(\mathscr{O}_{\mathrm{X},x})$.

Conjecture 4.1 is then generalized to

Conjecture B **("Purity"[24]).** — *Let* E *be an analytic space,* X *an analytic subset of* E*. We suppose that* E *is non-singular of dimension* N *at* $x \in \mathrm{X}$*, and that* X *can be described by* p *analytic equations in a neighborhood of any point. Then the rectified homotopical depth of* X *is* $\geqslant \mathrm{N} - p$*.*

In particular, a local complete intersection of dimension n at every point would have rectified homotopical depth $\geqslant n$, which is none other than conjecture 4.1.

Conjectures 4.2 and 4.3 are generalized respectively to :

Conjecture C **("local Lefschetz"[25]).** — *Let* X *be an analytic space,* Y *a closed analytic subset,* x *a point of* Y*, we suppose that* $\mathrm{X} - \mathrm{Y}$ *is Stein in a neighborhood of* x (*for example* Y *defined by an equation at* x)*, and that* $\mathrm{X} - \mathrm{Y}$ *has rectified homotopical depth* $\geqslant n$ *in a neighborhood of* x (*for example, is at every point of* $\mathrm{X} - \mathrm{Y}$ *neighboring* x*, a complete intersection of dimension* $\geqslant n$*, cf. conjecture* B)*. Then the canonical homomorphism*

$$\pi_i^x(\mathrm{Y}) \longrightarrow \pi_i^x(\mathrm{X})$$

is an isomorphism for $i < n - 1$*, an epimorphism for* $i = n - 1$*.*

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[23] N.D.E. : as observed in [**HL**], example 3.1.3, this conjecture is already false for $\mathrm{X} = \{\boldsymbol{z} \in \mathbf{C}^n \mid z_1^2 + z_2^3 + \cdots + z_n^3 = 0\}, n \geqslant 4$ and Y reduced to the origin. But, suitably modified, it is true (theorem 3.1.4 of *loc. cit.*).

[24] N.D.E. : this conjecture is proven, even in the case where E is singular, in [**HL**] : it is theorem 3.2.1.

[25] N.D.E. : this conjecture is proven in [**HL**], even in its strong form from the following remark, cf. theorem 3.3.1 of *loc. cit.*

140 ***Conjecture D* ("global Lefschetz"[26]).** — *Let* X *be a* compact *analytic space,* Y *an analytic subspace of* X *such that* $\mathrm{U} = \mathrm{X} - \mathrm{Y}$ *is Stein, and has rectified homotopical depth* $\geqslant n$ (*for example a complete intersection of dimension* $\geqslant n$ *at every point*). *Then the canonical homomorphism*

$$\pi_i(\mathrm{Y}) \longrightarrow \pi_i(\mathrm{X})$$

is an isomorphism for $i < n-1$*, an epimorphism for* $i = n-1$.

***Remarque*.** — When, in statements C and D, we replace the hypothesis that $\mathrm{X} - \mathrm{Y}$ is Stein by the hypothesis that $\mathrm{X} - \mathrm{Y}$ is a union of $c+1$ Stein open sets (which will play the role of a topological "concavity" hypothesis), the conclusions must be modified simply by replacing n by $n - c$.[27]

Let us make explicit finally, in the "global case" D, the conjecture concerning the fundamental group (obtained by taking $n = 3$) :

***Conjecture D′* (Global Lefschetz for the fundamental group[29])**

Let X *be a compact analytic space over the field of complex numbers,* Y *a closed analytic subset such that* $\mathrm{U} = \mathrm{X} - \mathrm{Y}$ *is Stein. Suppose moreover the following conditions are satisfied :*

(i) *For any* $x \in \mathrm{U}$*, the local fundamental group* $\pi_2^x(\mathrm{X}, x)$ *is zero* (*i.e.* X is "pure at x »)*, or only the local ring* $\mathscr{O}_{\mathrm{X},x}$ *is pure.*

(ii) *The local rings of the points of* U *are "connected in dimension* $\geqslant 2$*".*

(iii) *The local rings of the points of* U *have dimension* $\geqslant 3$.

Under these conditions, for any $x \in \mathrm{Y}$*, the homomorphism*

$$\pi_1(\mathrm{Y}, x) \longrightarrow \pi_1(\mathrm{X}, x)$$

141 *is an isomorphism* (*and* $\pi_2(\mathrm{Y}, x) \to \pi_2(\mathrm{X}, x)$ *an epimorphism*).

Note that the local conditions (i) (ii) (iii) on U are satisfied if U is locally a complete intersection of dimension $\geqslant 3$. From the point of view of Algebraic Geometry, (when U comes from a scheme, also denoted U), conditions (i) to (iii) correspond to hypotheses on the local invariants $\pi_i^x(\mathrm{U})$, namely $\pi_i^x(\mathrm{U}) = 0$ for $i < 3 - \deg\operatorname{tr} k(x)/k$, for the points

[26] N.D.E. : this conjecture is again proven in [**HL**], even in its strong form from the following remark, cf. theorem 3.4.1 of *loc. cit.*

[27] N.D.E. : let us finally mention the following result of Fulton, to be compared with the result of Fulton-Hansen cited in the editor's note (4) page 130 : let X and H be closed subschemes of $\mathbf{P}^m_{\mathbf{C}}$, n the dimension of X and d the codimension of H. Then, the map

$$\pi_i(\mathrm{X}, \mathrm{X} \cap \mathrm{H}) \longrightarrow \pi_i(\mathbf{P}^n_{\mathbf{C}}, \mathrm{H})$$

is an isomorphism if $i \leqslant n - d$ and is surjective if $i = n - d - 1$; see (Fulton W., « Connectivity and its applications in algebraic geometry », in *Algebraic geometry* (*Chicago, Ill., 1980*), Lect. Notes in Math., vol. 862, Springer, Berlin-New York, 1981, p. 26–92).

[29] N.D.E. : this conjecture is proven in [**HL**], cf. theorem 3.5.1 of *loc. cit.*

x such that we have respectively $\deg \operatorname{tr} k(x)/k = 0, 1, 2$. The global condition on U (U Stein) will be satisfied if X is projective and Y a hyperplane section.

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EXPOSÉ XIV

DEPTH AND LEFSCHETZ THEOREMS IN ÉTALE COHOMOLOGY

by Mrs. M. Raynaud[(*)]

In section 1, we define a notion of "*étale depth*" which is the analogue in étale cohomology of the notion of depth studied in III, in the cohomology of coherent sheaves. After a technical part, we prove in section 4 some "Lefschetz theorems", the central theorem being 4.2. Let X be a scheme, Y a closed subset of X, U the open complement $X - Y$ and F an abelian sheaf on X for the étale topology ; generally speaking, the purpose of the Lefschetz theorems is to show that, if F satisfies certain local conditions on U, expressible in terms of étale depth at the points of U, then, under certain additional global conditions on U (for example U *affine*), the natural map of étale cohomology groups **143**

$$\mathrm{H}^i(\mathrm{X},\mathrm{F}) \longrightarrow \mathrm{H}^i(\mathrm{Y},\mathrm{F}_{|\mathrm{Y}})$$

is an isomorphism for values $i < n$, where n is a certain explicit integer. By taking F to be a constant sheaf, we thus obtain conditions for $\pi_0(\mathrm{X})$ to be equal to $\pi_0(\mathrm{Y})$ and conditions for the abelianized fundamental groups of X and Y to be the same. In section 5, the introduction of a notion of "geometric depth" allows us to give useful special cases of the Lefschetz theorems (5.7). Finally, in section 6, we mention some conjectures, concerning in particular "non-commutative" variants of the theorems obtained.

1. Cohomological and homotopical depth

1.0. Let us fix the following notation. Let X be a scheme[(†)], Y a closed subset of X, U the open complement and $i : \mathrm{Y} = \mathrm{X} - \mathrm{U} \to \mathrm{X}$ the canonical immersion. Let $\underline{\Gamma}_{\mathrm{Y}}$ be the functor which, to an abelian sheaf on X, associates the "sheaf of sections with support in Y", that is $\underline{\Gamma}_{\mathrm{Y}} = i_* i^!$ (cf. SGA 4 IV 3.8 and VIII 6.6) and Γ_{Y} the functor **144**

(*) According to unpublished notes by A. Grothendieck.

(†) In accordance with the new terminology (cf. re-edition of EGA I), we will here call "scheme" what was previously called "prescheme" and "separated scheme" what was called "scheme".

$\Gamma \cdot \underline{\Gamma}_{\mathrm{Y}}$ (where Γ is the "global sections" functor). Let us consider the derived category $\mathrm{D}^+(\mathrm{X})$ and the derived functor $\mathrm{R}\,\underline{\Gamma}_{\mathrm{Y}}$ (resp. $\mathrm{R}\,\Gamma_{\mathrm{Y}}$) of $\underline{\Gamma}_{\mathrm{Y}}$ (resp. of Γ_{Y}) (cf. [**3**]). Given a complex of abelian sheaves F on X, with degrees bounded below, we can consider it as an element of $\mathrm{D}^+(\mathrm{X})$; we will then denote by $\underline{\mathrm{H}}^p_{\mathrm{Y}}(\mathrm{F})$ the p-th cohomology sheaf of $\mathrm{R}\,\underline{\Gamma}_{\mathrm{Y}}(\mathrm{F})$ and by $\mathrm{H}^p_{\mathrm{Y}}(\mathrm{X},\mathrm{F})$ the p-th cohomology group of $\mathrm{R}\,\Gamma_{\mathrm{Y}}(\mathrm{F})$. The results of (SGA 4 V 4.3 and 4.4) extend trivially to $\underline{\mathrm{H}}^p_{\mathrm{Y}}(\mathrm{F})$ and $\mathrm{H}^p_{\mathrm{Y}}(\mathrm{X},\mathrm{F})$.

Proposition 1.1. — *Let* X *be a scheme,* Y *a closed subset of* X*,* U *the open complement and* $i : \mathrm{U} \to \mathrm{X}$ *the canonical immersion. Let* F *denote either a sheaf of sets on* X*, or a sheaf of groups on* X*, or a complex of abelian sheaves on* X*, with degrees bounded below. Let us fix the following notation : if* $\mathrm{X}' \to \mathrm{X}$ *is a morphism,* U' *and* F' *denote the inverse images of* U *and* F *on* X' *; moreover, if* $\overline{y}$ *is a geometric point of* X*,* $\overline{\mathrm{X}}$ *denotes the strict localization of* X *at* $\overline{y}$ *and* $\overline{\mathrm{U}}$ *and* $\overline{\mathrm{F}}$ *the inverse images of* U *and* F *in* $\overline{\mathrm{X}}$*.*

145 1°) *Let* F *be a sheaf of sets on* X *and* n *an integer* $\leqslant 2$ *; then the following conditions are equivalent :*

(i) *The canonical morphism*

$$\mathrm{F} \longrightarrow i_* i^* \mathrm{F}$$

is injective if $n \geqslant 1$*, bijective if* $n \geqslant 2$*.*

(ii) *For any scheme* X' *étale over* X*, the canonical morphism*

$$\mathrm{H}^0(\mathrm{X}',\mathrm{F}') \longrightarrow \mathrm{H}^0(\mathrm{U}',\mathrm{F}')$$

is injective if $n \geqslant 1$*, bijective if* $n \geqslant 2$*.*

Suppose furthermore that the open set U *is retrocompact in* X *; then the preceding conditions are equivalent to the following :*

(iii) *For any geometric point* $\overline{y}$ *of* Y*, the canonical morphism*

$$\mathrm{H}^0(\overline{\mathrm{X}},\overline{\mathrm{F}}) \longrightarrow \mathrm{H}^0(\overline{\mathrm{U}},\overline{\mathrm{F}})$$

is injective if $n \geqslant 1$*, and bijective if* $n \geqslant 2$*.*

2°) *Let* F *be a sheaf of groups on* X *and* n *an integer* $\leqslant 3$ *; then the following conditions are equivalent :*

(i) *The canonical morphism*

$$\mathrm{F} \longrightarrow i_* i^* \mathrm{F}$$

is injective if $n \geqslant 1$*, bijective if* $n \geqslant 2$*, and if* $n \geqslant 3$*, in addition to the preceding conditions, the sheaf of pointed sets* $\mathrm{R}^1\, i_*(i^*\mathrm{F})$ *is zero.*

(ii) *For any scheme* X' *étale over* X*, the canonical morphism*

$$\mathrm{H}^0(\mathrm{X}',\mathrm{F}') \longrightarrow \mathrm{H}^0(\mathrm{U}',\mathrm{F}')$$

146 *is injective if* $n \geqslant 1$*, bijective if* $n \geqslant 2$*, and furthermore the canonical morphism*

$$\mathrm{H}^1(\mathrm{X}',\mathrm{F}') \longrightarrow \mathrm{H}^1(\mathrm{U}',\mathrm{F}')$$

is injective if $n \geqslant 2$*, bijective if* $n \geqslant 3$*.*

(ii bis) *Identical to* (ii), *except in the case* $n = 2$ *where we only assume* $\mathrm{H}^0(\mathrm{X}', \mathrm{F}') \to \mathrm{H}^0(\mathrm{U}', \mathrm{F}')$ *to be bijective.*

Suppose furthermore that U *is retrocompact in* X *; then the preceding conditions are also equivalent to the following :*

(iii) *For any geometric point* $\overline{y}$ *of* Y*, the canonical morphism*

$$\mathrm{H}^0(\overline{\mathrm{X}}, \overline{\mathrm{F}}) \longrightarrow \mathrm{H}^0(\overline{\mathrm{U}}, \overline{\mathrm{F}})$$

is injective if $n \geqslant 1$*, bijective if* $n \geqslant 2$ *; finally if* $n \geqslant 3$*, in addition to the preceding conditions,* $\mathrm{H}^1(\overline{\mathrm{U}}, \overline{\mathrm{F}})$ *is zero.*

3°) *Let* F *be a complex of abelian sheaves, with degrees bounded below, and* n *be an integer ; then the following conditions are equivalent :*

(i) *We have* $\underline{\mathrm{H}}^p_{\mathrm{Y}}(\mathrm{F}) = 0$ *for* $p < n$ *(cf. 1.0).*

(ii) *For any scheme* X′ *étale over* X*, the canonical morphism*

$$\mathrm{H}^p(\mathrm{X}', \mathrm{F}') \longrightarrow \mathrm{H}^p(\mathrm{U}', \mathrm{F}')$$

is bijective for $p < n - 1$*, injective for* $p = n - 1$*.*

Suppose U *is retrocompact in* X*, then the preceding conditions are also equivalent to the following :*

(iii) *For any geometric point* $\overline{y}$ *of* Y*, the canonical morphism*

$$\mathrm{H}^p(\overline{\mathrm{X}}, \overline{\mathrm{F}}) \longrightarrow \mathrm{H}^p(\overline{\mathrm{U}}, \overline{\mathrm{F}})$$

is bijective for $p < n - 1$*, injective for* $p = n - 1$*.*

In the case where F *is an abelian sheaf and* $n \geqslant 2$*, the conditions* (i) *and* (ii) *are* **147**
also equivalent to the following :

(ii bis) *For any scheme* X′ *étale over* X*, the canonical morphism*

$$\mathrm{H}^p(\mathrm{X}', \mathrm{F}') \longrightarrow \mathrm{H}^p(\mathrm{U}', \mathrm{F}')$$

is bijective for $p < n - 1$*.*

Démonstration

1°) It is clear that (i) ⇔ (ii). Let us show that, if U is retrocompact in X, (i) ⇔ (iii). Indeed, (i) is equivalent to saying that, for any geometric point $\overline{y}$ of X, the morphism $\mathrm{F}_{\overline{y}} \to (i_* i^* \mathrm{F})_{\overline{y}}$ is injective if $n \leqslant 1$ and bijective if $n \leqslant 2$ (SGA 4 VIII 3.6). Since this morphism is bijective anyway when $\overline{y}$ is a geometric point of U, we can restrict to the geometric points $\overline{y}$ of Y. Now it follows from the fact that i is quasi-compact and from (SGA 4 VIII 5.3) that the morphism

$$\mathrm{F}_{\overline{y}} \longrightarrow (i_* i^* \mathrm{F})_{\overline{y}}$$

is canonically identified with the morphism

$$\mathrm{H}^0(\overline{\mathrm{X}}, \overline{\mathrm{F}}) \longrightarrow \mathrm{H}^0(\overline{\mathrm{U}}, \overline{\mathrm{F}}),$$

hence the equivalence of (i) and (iii).

2°) (i) $\Rightarrow$ (ii). The assertions on H^0 follow from 1°). Let then i' be the canonical immersion of U' into X' ; the assertions on H^1 follow from the exact sequence (SGA 4 XII 3.2)

$$0 \longrightarrow \mathrm{H}^1(\mathrm{X}', i'_* i'^* \mathrm{F}') \longrightarrow \mathrm{H}^1(\mathrm{U}', \mathrm{F}') \longrightarrow \mathrm{H}^0(\mathrm{X}', \mathrm{R}^1\, i'_*(i'^*\mathrm{F}')).$$

148 (ii bis) $\Rightarrow$ (i). According to 1°), it suffices to show that, for $n \geqslant 3$, we have $\mathrm{R}^1\, i_*(i^*\mathrm{F}) = 0$. Now $\mathrm{R}^1\, i_*(i^*\mathrm{F})$ is the sheaf associated to the presheaf $\mathrm{X}' \mapsto \mathrm{H}^1(\mathrm{U}', \mathrm{F}')$, which is, by hypothesis, the sheaf associated to the presheaf $\mathrm{X}' \mapsto \mathrm{H}^1(\mathrm{X}', \mathrm{F}')$, which is zero.

(i) $\Leftrightarrow$ (iii). Taking 1°) into account, the only thing left to see is that the relation $\mathrm{R}^1\, i_*(i^*\mathrm{F}) = 0$ is equivalent to the fact that $\mathrm{H}^1(\overline{\mathrm{U}}, \overline{\mathrm{F}}) = 0$ for any geometric point $\overline{y}$ of Y. Since $\mathrm{R}^1\, i_*(i^*\mathrm{F}))$ is zero outside of Y, it amounts to the same thing to say that $\mathrm{R}^1\, i_*(i^*\mathrm{F}) = 0$ or that $(\mathrm{R}^1\, i_*(i^*\mathrm{F}))_{\overline{y}} = 0$ for any geometric point $\overline{y}$ of Y. It then suffices to note that, since i is quasi-compact, we have $(\mathrm{R}^1\, i_*(i^*\mathrm{F}))_{\overline{y}} = \mathrm{H}^1(\overline{\mathrm{U}}, \overline{\mathrm{F}})$ (SGA 4 VIII 5.3).

3°) (i) $\Rightarrow$ (ii). Let X' be a scheme étale over X ; we have the exact sequence (SGA 4 V 4.5)

$$(*) \qquad \longrightarrow \mathrm{H}^p_{\mathrm{Y}'}(\mathrm{X}', \mathrm{F}') \longrightarrow \mathrm{H}^p(\mathrm{X}', \mathrm{F}') \longrightarrow \mathrm{H}^p(\mathrm{U}', \mathrm{F}') \longrightarrow;$$

thus (ii) is equivalent to $\mathrm{H}^p_{\mathrm{Y}'}(\mathrm{X}', \mathrm{F}') = 0$ for $p < n$ and for any scheme X' étale over X. Let us then consider the spectral sequence

$$\mathrm{E}_2^{pq} = \mathrm{H}^p(\mathrm{X}', \underline{\mathrm{H}}^q_{\mathrm{Y}}(\mathrm{F})) \Longrightarrow \mathrm{H}^*_{\mathrm{Y}'}(\mathrm{X}', \mathrm{F}');$$

by hypothesis, $\underline{\mathrm{H}}^q_{\mathrm{Y}}(\mathrm{F}) = 0$ for $q < n$, hence $\mathrm{E}_2^{pq} = 0$ for $p + q < n$ and consequently $\mathrm{H}^p_{\mathrm{Y}'}(\mathrm{X}', \mathrm{F}') = 0$ for $p < n$.

(ii) $\Rightarrow$ (i). The sheaf $\underline{\mathrm{H}}^p_{\mathrm{Y}}(\mathrm{F})$ is associated to the presheaf $\mathrm{X}' \mapsto \mathrm{H}^p_{\mathrm{Y}'}(\mathrm{X}', \mathrm{F}')$; as we have already remarked that (ii) is equivalent to the relation $\mathrm{H}^p_{\mathrm{Y}'}(\mathrm{X}', \mathrm{F}') = 0$ for $p < n$ and for any scheme X' étale over X, we do have $\underline{\mathrm{H}}^p_{\mathrm{Y}}(\mathrm{F}) = 0$ for $p < n$.

(i) $\Leftrightarrow$ (iii). The sheaves $\underline{\mathrm{H}}^p_{\mathrm{Y}}(\mathrm{F})$ are concentrated on Y ; consequently it amounts to
149 the same thing to say that $\underline{\mathrm{H}}^p_{\mathrm{Y}}(\mathrm{F}) = 0$ or to say that, for any geometric point $\overline{y}$ of Y, the fiber $(\underline{\mathrm{H}}^p_{\mathrm{Y}}(\mathrm{F}))_{\overline{y}}$ is zero. Now, since i is quasi-compact, we deduce from (SGA 4 VIII 5.2) that we have $(\underline{\mathrm{H}}^p_{\mathrm{Y}}(\mathrm{F}))_{\overline{y}} = \underline{\mathrm{H}}^p_{\overline{\mathrm{Y}}}(\overline{\mathrm{X}}, \overline{\mathrm{F}})$. The equivalence of (i) and (iii) follows from this, taking into account the analogue on $\overline{\mathrm{X}}$ of the exact sequence $(*)$.

(ii bis) $\Rightarrow$ (ii) in the case where F is an abelian sheaf. The only thing left to show is that $\underline{\mathrm{H}}^{n-1}_{\mathrm{Y}}(\mathrm{F}) = 0$. Now, for $n > 2$, the sheaf $\underline{\mathrm{H}}^{n-1}_{\mathrm{Y}}(\mathrm{F})$ is associated to the presheaf $\mathrm{X}' \mapsto \mathrm{H}^{n-2}(\mathrm{U}', \mathrm{F}') = \mathrm{H}^{n-2}(\mathrm{X}', \mathrm{F}')$ so it is indeed zero. The case $n = 2$ follows from the fact that $\underline{\mathrm{H}}^1_{\mathrm{Y}}(\mathrm{F})$ is the cokernel of the morphism $\mathrm{F} \to i_* i^* \mathrm{F}$.

Definition 1.2. — The notation is that of 1.1. We say that F has Y-étale depth $\geqslant n$ and we write

$$\mathrm{prof}_{\mathrm{Y}}(\mathrm{F}) \geqslant n$$

if F satisfies the equivalent conditions (i) and (ii) of 1.1. If F is a complex of abelian sheaves, we call the Y-étale depth of F the upper bound of the n for which $\mathrm{prof}_{\mathrm{Y}}(\mathrm{F}) \geqslant n$; we will use the same notation if F is a sheaf of sets, resp. of not necessarily commutative groups (so that we then have $0 \leqslant \mathrm{prof}_{\mathrm{Y}}(\mathrm{F}) \leqslant 2$, resp. $0 \leqslant \mathrm{prof}_{\mathrm{Y}}(\mathrm{F}) \leqslant 3$, when the context does not allow for confusion on which of the three variants considered here is being used).

If $\mathbf{L}$ is a set of prime numbers, we say that the Y-étale depth for $\mathbf{L}$ of X is $\geqslant n$ and we write

$$\mathrm{prof}^{\mathbf{L}}_{\mathrm{Y}}(\mathrm{X}) \geqslant n$$

if, for any constant sheaf of the form $\mathbf{Z}/\ell\mathbf{Z}$ with $\ell \in \mathbf{L}$, we have $\mathrm{prof}_{\mathrm{Y}}(\mathbf{Z}/\ell\mathbf{Z}) \geqslant n$. **150**
The Y-étale depth for $\mathbf{L}$ of X is defined in an obvious way. If $\mathbf{L} = \mathbf{P}$, the set of all prime numbers, and if there is no risk of confusion with the notation of (EGA IV 5.7.1) (relative to the case where $\mathrm{F} = \mathscr{O}_{\mathrm{X}}$), we omit $\mathbf{L}$ in the notation; otherwise we write prof ét(X).

Finally we say that X has homotopical Y-depth $\geqslant 3$ for $\mathbf{L}$ and we write

$$\mathrm{prof\,hop}^{\mathbf{L}}_{\mathrm{Y}}(\mathrm{X}) \geqslant 3$$

if, for any finite constant sheaf of $\mathbf{L}$-groups F on X, we have $\mathrm{prof}_{\mathrm{Y}}(\mathrm{F}) \geqslant 3$. If $\mathbf{L} = \mathbf{P}$, we omit $\mathbf{L}$ in the notation.

Corollary 1.3. — *Under the conditions of* 1.1, *if* $\mathrm{prof}_{\mathrm{Y}}(\mathrm{F}) \geqslant n$, *then, for any closed subset* Z *of* Y, *we have*

$$\mathrm{prof}_{\mathrm{Z}}(\mathrm{F}) \geqslant n.$$

Let's do the reasoning for instance in the case where F is a complex of abelian sheaves with degrees bounded below. We use 1.1 3°) (ii). Let $\mathrm{V} = \mathrm{X} - \mathrm{Z}$ and consider, for any integer p, the sequence of morphisms

$$\mathrm{H}^p(\mathrm{X},\mathrm{F}) \xrightarrow{f} \mathrm{H}^p(\mathrm{V},\mathrm{F}) \xrightarrow{g} \mathrm{H}^p(\mathrm{U},\mathrm{F}).$$

By hypothesis g and $f \circ g$ are bijective for $p < n-1$ and injective for $p = n-1$; so is f. As the reasoning is valid when we replace X by a scheme X′ étale over X, this proves 1.3.

Corollary 1.4. — *The notation is that of* 1.1 2°). *If* X′ *is a scheme over* X, *let* Φ' *be* **151**
the functor which associates to an étale covering of X′ *its restriction to* U′, *and* $\Phi'_{\mathrm{F}'}$, *the functor which associates to a torsor*(*) *under* F′ *its restriction to* U′. *Then the following conditions are equivalent :*

(i) *We have* $\mathrm{prof}_{\mathrm{Y}}(\mathrm{F}) \geqslant 1$ (*resp.* $\mathrm{prof}_{\mathrm{Y}}(\mathrm{F}) \geqslant 2$, *resp.* $\mathrm{prof}_{\mathrm{Y}}(\mathrm{F}) \geqslant 3$).

(ii) *For any scheme* X′ *étale over* X, *the functor* $\Phi'_{\mathrm{F}'}$ *is faithful* (*resp. fully faithful, resp. an equivalence of categories*).

(*) i.e. a "principal homogeneous bundle" in an older terminology.

In particular, for us to have $\mathrm{prof}_{\mathrm{Y}}(\mathrm{X}) \geqslant 1$ (*resp.* $\mathrm{prof}_{\mathrm{Y}}(\mathrm{X}) \geqslant 2$, *resp.* $\mathrm{prof\,hop}_{\mathrm{Y}}(\mathrm{X}) \geqslant 3$), *it is necessary and sufficient that the functor* Φ' *be faithful* (*resp. fully faithful, resp. an equivalence of categories*).

This follows indeed from 1.1 2°) (ii), taking into account the interpretation of $\mathrm{H}^1(\mathrm{X}', \mathrm{F}')$ as the set of classes (mod. isomorphisms) of torsors under F' (SGA 4 VII 2), and of étale coverings Z of degree n of a scheme as being associated to principal Galois coverings with group the symmetric group $\mathfrak{S}_n$, with Z being associated to the covering $\underline{\mathrm{Isom}}_{\mathrm{X}}(\mathrm{Z}_0, \mathrm{Z})$, where Z_0 is the trivial covering of X of degree n.

Corollary 1.5. — *Under the conditions of* 1.1 3°), *suppose we have* $\mathrm{prof}_{\mathrm{Y}}(\mathrm{F}) \geqslant n$; *then we have*

$$\mathrm{H}^n_{\mathrm{Y}}(\mathrm{X}, \mathrm{F}) \simeq \mathrm{H}^0(\mathrm{X}, \underline{\mathrm{H}}^n_{\mathrm{Y}}(\mathrm{F})).$$

The corollary follows from the spectral sequence

$$\mathrm{E}_2^{pq} = \mathrm{H}^p(\mathrm{X}, \underline{\mathrm{H}}^q_{\mathrm{Y}}(\mathrm{F})) \Longrightarrow \mathrm{H}^*_{\mathrm{Y}}(\mathrm{X}, \mathrm{F}).$$

Indeed we have by hypothesis $\mathrm{E}_2^{pq} = 0$ for $q < n$, from which it follows that

$$\mathrm{H}^n_{\mathrm{Y}}(\mathrm{X}, \mathrm{F}) = \mathrm{E}_2^{0n} = \mathrm{H}^0(\mathrm{X}, \underline{\mathrm{H}}^n_{\mathrm{Y}}(\mathrm{F})).$$

Remarks 1.6

152 a) The notion of Y-depth, in the form of the equivalent conditions (i) and (ii) of 1.1, makes sense for any site. In the particular case where X is a locally noetherian scheme, endowed with the Zariski topology, and F is a sheaf of coherent $\mathscr{O}_{\mathrm{X}}$-modules, we find the usual notion of Y-depth as the lower bound of the depths at the points of Y (III).

b) For $n \leqslant 2$, the notion of Y-étale depth of X is independent of $\mathbf{L}$. For $n = 1$, it simply means that U is dense in X. Indeed this condition is necessary for us to have $\mathrm{prof}_{\mathrm{Y}}(\mathrm{F}) \geqslant 1$, and it is also sufficient because we can assume X is reduced, a case in which the condition U dense in X is preserved by étale base change (EGA IV 11.10.5) (ii) b)). If U is retrocompact in X, the relation $\mathrm{prof}_{\mathrm{Y}}(\mathrm{X}) \geqslant 1$ is also equivalent to saying that Y contains no maximal point of X (EGA I 6.6.5). For $n = 2$ and U retrocompact in X, the condition $\mathrm{prof}_{\mathrm{Y}}(\mathrm{X}) \geqslant 2$ is equivalent to the fact that, for any geometric point $\overline{y}$ of Y, $\overline{\mathrm{U}}$ is connected and non-empty, that is "Y does not disconnect X, locally for the étale topology".

c) If X has Y-depth $\geqslant n$ for $\mathbf{L}$ and U is retrocompact in X, then for any locally constant abelian sheaf of $\mathbf{L}$-torsion F on X, we have $\mathrm{prof}_{\mathrm{Y}}(\mathrm{X}) \geqslant n$. Indeed, since the property $\mathrm{prof}_{\mathrm{Y}}(\mathrm{F}) \geqslant n$ is local for the étale topology, we can assume F is constant; then F is a filtered inductive limit of sheaves which are finite sums of sheaves of the form $\mathbf{Z}/p^m\mathbf{Z}$, where m is an integer > 0 and $p \in \mathbf{L}$. Using 1.1 (iii) and (SGA 4 VII
153 3.3), we see that we can reduce to the case where $\mathrm{F} = \mathbf{Z}/p^m\mathbf{Z}$, then, by induction on m, to the case where $\mathrm{F} = \mathbf{Z}/p\mathbf{Z}$ for which the assertion follows from the definition.

d) According to 1.4, if we have $\mathrm{prof}_{\mathrm{Y}}(\mathrm{X}) \geqslant 3$, the pair (X, Y) is *pure* in the sense of X 3.1. In fact the pure pairs encountered in practice (cf. X 3.4) satisfy the stronger condition of homotopical depth $\geqslant 3$, and this notion can therefore be advantageously substituted for that of a pure pair.

e) Let F be a complex of abelian sheaves and T(F) the complex obtained by applying the translation functor to F ([**3**]) ; then we obviously have :

$$\mathrm{prof}_{\mathrm{Y}}(\mathrm{T}(\mathrm{F})) = \mathrm{prof}_{\mathrm{Y}}(\mathrm{F}) - 1.$$

f) Let us mention that the recent works of Artin-Mazur ([**1**]) allow to define the notion of homotopical depth $\geqslant n$, for any integer n (not only if $n \geqslant 3$).

g) Under the conditions of 1.1 3), for us to have $\mathrm{prof}_{\mathrm{Y}}(\mathrm{F}) = \infty$, it is necessary and sufficient that we have $\mathrm{F} \xrightarrow{\sim} \mathrm{R}i_*(i^*\mathrm{F})$ in $\mathrm{D}^+(\mathrm{X})$. Indeed, the $\underline{\mathrm{H}}^p_{\mathrm{Y}}(\mathrm{F})$ are the cohomology sheaves of the cone (= mapping cylinder) of the canonical morphism $\mathrm{F} \to \mathrm{R}i_*i^*(\mathrm{F})$.

Definition 1.7. — Let X be a scheme, x a point of X, $\overline{x}$ a geometric point over x and $\overline{\mathrm{X}}$ the strict localization of X at $\overline{x}$. As before F denotes, either a sheaf of sets on X, or a sheaf of groups on X, or a complex of abelian sheaves on X with degrees bounded below, $\overline{\mathrm{F}}$ its inverse image on $\overline{\mathrm{X}}$ and $\mathbf{L}$ a set of prime numbers. We say that F has *étale depth* $\geqslant n$ *at the point* x (resp. that the *étale depth for* $\mathbf{L}$ *of* X *at* x *is* $\geqslant n$, resp. 154
that the *homotopical depth for* $\mathbf{L}$ *of* X *at* x *is* $\geqslant 3$) and we write $\mathrm{prof}_x(\mathrm{F}) \geqslant n$ (resp. $\mathrm{prof}^{\mathbf{L}}_x(\mathrm{X}) \geqslant n$, resp. $\mathrm{prof\,hop}^{\mathbf{L}}_x(\mathrm{X}) \geqslant 3$) if we have $\mathrm{prof}_{\overline{x}}(\overline{\mathrm{F}}) > n$ (resp. $\mathrm{prof}^{\mathbf{L}}_{\overline{x}}(\overline{\mathrm{X}}) > n$, resp. $\mathrm{prof\,hop}^{\mathbf{L}}_{\overline{x}}(\overline{\mathrm{X}}) \geqslant 3$). The integer $\mathrm{prof}^{\mathbf{L}}_x(\mathrm{X})$ is defined in an obvious way and, if F is a complex of abelian sheaves, the integer $\mathrm{prof}_x(\mathrm{F})$. If $\mathbf{L}$ is the set of all prime numbers, we omit $\mathbf{L}$ in the notation $\mathrm{prof}^{\mathbf{L}}_x(\mathrm{X})$, unless there is a risk of confusion with the notation of (EGA IV 5.7.1), in which case we write $\mathrm{prof\,\acute{e}t}_x(\mathrm{X})$.

We then have the following *pointwise characterization* of depth :

Theorem 1.8. — *Let* X *be a scheme,* Y *a closed subset of* X *such that the open set* $\mathrm{U} = \mathrm{X} - \mathrm{Y}$ *is retrocompact in* X*. If* F *is, either a sheaf of sets on* X*, or a sheaf of groups on* X*, or a complex of abelian sheaves on* X *with degrees bounded below, then we have*

$$\mathrm{prof}_{\mathrm{Y}}(\mathrm{F}) = \inf_{y \in \mathrm{Y}} \mathrm{prof}_y(\mathrm{F}).$$

1.8.1. Let us first show that, for any point y of Y, we have the inequality $\mathrm{prof}_y(\mathrm{F}) \geqslant \mathrm{prof}_{\mathrm{Y}}(\mathrm{F})$. Let indeed $\overline{y}$ be a geometric point over y, $\overline{\mathrm{X}}$ the strict localization of X at $\overline{y}$, $\overline{\mathrm{F}}$ and $\overline{\mathrm{Y}}$ the inverse images of F and Y on $\overline{\mathrm{X}}$. We have according to 1.7 and 1.3

$$\mathrm{prof}_y(\mathrm{F}) = \mathrm{prof}_{\overline{y}}(\overline{\mathrm{F}}) \geqslant \mathrm{prof}_{\overline{\mathrm{Y}}}(\overline{\mathrm{F}}) \geqslant \mathrm{prof}_{\mathrm{Y}}(\mathrm{F}),$$

the last inequality using the hypothesis "U retrocompact", via the conditions (iii) in 1.1 and transitivity in the formation of strict localizations.

1.8.2. Conversely suppose that, for any point y of Y, we have $\mathrm{prof}_y(\mathrm{F}) \geqslant n$ (n integer) and let us show that we then have $\mathrm{prof}_{\mathrm{Y}}(\mathrm{F}) \geqslant n$.

155 Let us first recall the following well-known results (SGA 4 VIII) :

Lemma 1.8.2.1. — *Let* X *be a scheme,* F *a sheaf of sets on* X (*resp.* $\mathrm{G} \to \mathrm{F}$ *a monomorphism of sheaves of sets on* X). *Then, for two sections s and s' of* F *to coincide* (*resp. for a section s of* F *to come from a section of* G), *it is necessary and sufficient that this be the case locally. In particular, if s and s' are two sections of* F, *there exists a largest open set* V *of* X *on which they coincide* (*resp. if s is a section of* F *on* X, *there exists a largest open set* V *of* X *such that $s|\mathrm{V}$ comes from a section of* G *on* V). *This open set is also the set of points x of* X *such that, letting $\overline{x}$ be a geometric point over x, the sections s and s' have the same image in the fiber $\mathrm{F}_{\overline{x}}$ (resp. that the image of s in $\mathrm{F}_{\overline{x}}$ comes from an element of $\mathrm{G}_{\overline{x}}$).*

Let us return to the proof of 1.8.

1°) Case where F is a sheaf of sets. If $n = 1$, it suffices to show that the canonical morphism

$$\mathrm{H}^0(\mathrm{X}, \mathrm{F}) \longrightarrow \mathrm{H}^0(\mathrm{U}, \mathrm{F})$$

is injective, the result still applying when we replace X by a scheme étale over X. Let s and s' be two sections of F over X which have the same image in $\mathrm{H}^0(\mathrm{U}, \mathrm{F})$ and let V be the largest open set over which they are equal ; we obviously have $\mathrm{V} \supset \mathrm{U}$. Suppose

156 $\mathrm{V} \neq \mathrm{X}$ and let y be a maximal point of $\mathrm{X} - \mathrm{V}$, $\overline{y}$ a geometric point over y, $\overline{\mathrm{X}}$ the strict localization of X at $\overline{y}$ and $\overline{\mathrm{V}}$ and $\overline{\mathrm{F}}$ the inverse images of V and F on $\overline{\mathrm{X}}$. According to the choice of y, we have $\overline{\mathrm{X}} - \overline{y} = \overline{\mathrm{V}}$ and by hypothesis, the morphism

$$\mathrm{H}^0(\overline{\mathrm{X}}, \overline{\mathrm{F}}) \longrightarrow \mathrm{H}^0(\overline{\mathrm{X}} - \overline{y}, \overline{\mathrm{F}}) = \mathrm{H}^0(\overline{\mathrm{V}}, \overline{\mathrm{F}})$$

is injective. It follows that s and s' coincide at the point y, which is absurd. If $n = 2$, it suffices to show, taking into account the preceding, that the morphism

$$\mathrm{H}^0(\mathrm{X}, \mathrm{F}) \longrightarrow \mathrm{H}^0(\mathrm{U}, \mathrm{F}) = \mathrm{H}^0(\mathrm{X}, i_* i^* \mathrm{F})$$

is surjective (where i is the canonical immersion of U into X). Let s be a section of $i_* i^* \mathrm{F}$ over X and V the largest open set over which it comes from a section of F. Suppose $\mathrm{V} \neq \mathrm{X}$ and let y be a maximal point of $\mathrm{X} - \mathrm{V}$; with the preceding notation, it follows from the hypothesis that the canonical morphism

$$\mathrm{H}^0(\overline{\mathrm{X}}, \overline{\mathrm{F}}) \longrightarrow \mathrm{H}^0(\overline{\mathrm{X}} - \overline{y}, \overline{\mathrm{F}}) = \mathrm{H}^0(\overline{\mathrm{V}}, \overline{\mathrm{F}})$$

is bijective ; consequently $s|\overline{\mathrm{V}}$ extends to the point $\overline{y}$, which is absurd and completes the proof in the case 1°).

2°) Case where F is a sheaf of groups. Taking 1°) into account, the only thing left to show is that, in the case $n = 3$, the morphism

$$\mathrm{H}^1(\mathrm{X}, \mathrm{F}) = \mathrm{H}^1(\mathrm{X}, i_* i^* \mathrm{F}) \longrightarrow \mathrm{H}^1(\mathrm{U}, \mathrm{F})$$

is bijective. We already know that it is injective thanks to 1°) and to 1.1 2°) (ii bis). For surjectivity, we use the exact sequence (SGA 4 XII 3.2)

$$0 \longrightarrow \mathrm{H}^1(\mathrm{X}, i_* i^*\mathrm{F}) \longrightarrow \mathrm{H}^1(\mathrm{U}, \mathrm{F}) \xrightarrow{d} \mathrm{H}^0(\mathrm{X}, \mathrm{R}^1\, i_*(i^*\mathrm{F})).$$

Let $s \in \mathrm{H}^1(\mathrm{U}, \mathrm{F})$ and $\mathrm{V} \supset \mathrm{U}$ be the largest open set over which $d(s) = 0$; this is also the largest open set such that s comes from an element of $\mathrm{H}^1(\mathrm{V}, \mathrm{F})$. Suppose $\mathrm{V} \neq \mathrm{X}$ and let y be a maximal point $\mathrm{X} - \mathrm{V}$; if $\overline{\mathrm{X}}$ is the strict localization of X at a geometric point $\overline{y}$ over y, we have, with obvious notation, the exact sequence 157

$$0 \longrightarrow \mathrm{H}^1(\overline{\mathrm{X}}, \overline{i}_*(\overline{i}^*\overline{\mathrm{F}})) \longrightarrow \mathrm{H}^1(\overline{\mathrm{U}}, \overline{\mathrm{F}}) \xrightarrow{\overline{d}} \mathrm{H}^0(\overline{\mathrm{X}}, \mathrm{R}^1\, \overline{i}_*(\overline{i}^*\overline{\mathrm{F}})).$$

Since $i : \mathrm{U} \to \mathrm{X}$ is quasi-compact, $\mathrm{R}^1\, \overline{i}_*(\overline{i}^*\overline{\mathrm{F}})$ is the inverse image of $\mathrm{R}^1\, i_*(i^*\mathrm{F})$ by the morphism $\overline{\mathrm{X}} \to \mathrm{X}$, hence $\mathrm{H}^0(\overline{\mathrm{X}}, \mathrm{R}^1\, \overline{i}_*(\overline{i}^*\overline{\mathrm{F}})) = (\mathrm{R}^1\, i_*(i^*\mathrm{F}))_{\overline{y}}$. By hypothesis and since $y \in \mathrm{Y}$, the morphism

$$\mathrm{H}^1(\overline{\mathrm{X}}, \overline{\mathrm{F}}) \longrightarrow \mathrm{H}^1(\overline{\mathrm{V}}, \overline{\mathrm{F}})$$

is bijective. The image $\overline{s}$ of s in $\mathrm{H}^1(\overline{\mathrm{U}}, \overline{\mathrm{F}})$, which extends to $\overline{\mathrm{V}}$ by definition of V, therefore also extends to $\overline{\mathrm{X}}$; it follows that $\overline{d}(\overline{s}) = 0$ so the image of $d(s)$ in the geometric fiber $(\mathrm{R}^1\, i_*(i^*\mathrm{F}))_{\overline{y}}$ is zero ; but this contradicts the definition of V, hence the case 2°).

3°) Case where F is a complex of abelian sheaves, with degrees bounded below. We argue by induction on n. The conclusion is satisfied for n small enough, since F has degrees bounded below. Suppose then that $\mathrm{prof}_{\mathrm{Y}}(\mathrm{F}) \geqslant n-1$ and let us show that $\mathrm{prof}_{\mathrm{Y}}(\mathrm{F}) \geqslant n$, knowing that, for any point y of Y, we have $\mathrm{prof}_y(\mathrm{F}) \geqslant n$. It suffices to see that the canonical morphism

$$(*) \qquad \mathrm{H}^{n-2}(\mathrm{X}, \mathrm{F}) \longrightarrow \mathrm{H}^{n-2}(\mathrm{U}, \mathrm{F})$$

is surjective and that

$$(**) \qquad \mathrm{H}^{n-1}(\mathrm{X}, \mathrm{F}) \longrightarrow \mathrm{H}^{n-1}(\mathrm{U}, \mathrm{F})$$

is injective (the result applying when we replace X by a scheme étale over of X). 158

a) Surjectivity of $(*)$. The proof is analogous to that of 2°). Taking into account 1.5 and (SGA 4 V 4.5), we have the exact sequence

$$\mathrm{H}^{n-2}(\mathrm{X}, \mathrm{F}) \longrightarrow \mathrm{H}^{n-2}(\mathrm{U}, \mathrm{F}) \xrightarrow{d} \mathrm{H}^{n-1}_{\mathrm{Y}}(\mathrm{X}, \mathrm{F}) = \mathrm{H}^0(\mathrm{X}, \underline{\mathrm{H}}^{n-1}_{\mathrm{Y}}(\mathrm{F})).$$

Let $s \in \mathrm{H}^{n-2}(\mathrm{U}, \mathrm{F})$ and $\mathrm{V} \supset \mathrm{U}$ be the largest open set over which $d(s) = 0$, which is also the largest open set such that s extends to $\mathrm{H}^{n-2}(\mathrm{V}, \mathrm{F})$. Suppose $\mathrm{V} \neq \mathrm{X}$ and let y be a maximal point of $\mathrm{X} - \mathrm{V}$ and $\overline{\mathrm{X}}$ the strict localization of X at a geometric point $\overline{y}$ over y. Since $i : \mathrm{U} \to \mathrm{X}$ is quasi-compact, the formation of $\underline{\mathrm{H}}^{n-1}_{\mathrm{Y}}(\mathrm{F})$ commutes with the base change $\overline{\mathrm{X}} \to \mathrm{X}$ and we thus have (with obvious notation) the exact sequence

$$\mathrm{H}^{n-2}(\overline{\mathrm{X}}, \overline{\mathrm{F}}) \longrightarrow \mathrm{H}^{n-2}(\overline{\mathrm{U}}, \overline{\mathrm{F}}) \xrightarrow{\overline{d}} \mathrm{H}^{n-1}_{\overline{\mathrm{Y}}}(\overline{\mathrm{X}}, \overline{\mathrm{F}}) = (\underline{\mathrm{H}}^{n-1}_{\mathrm{Y}}(\mathrm{F}))_{\overline{y}},$$

the last equality resulting from the retrocompactness hypothesis on U.

Now we have by hypothesis the isomorphism

$$\mathrm{H}^{n-2}(\overline{\mathrm{X}}, \overline{\mathrm{F}}) \xrightarrow{\sim} \mathrm{H}^{n-2}(\overline{\mathrm{X}} - \overline{y}, \overline{\mathrm{F}}) = \mathrm{H}^{n-2}(\overline{\mathrm{V}}, \overline{\mathrm{F}});$$

consequently the image $\overline{s}$ of s in $\mathrm{H}^{n-2}(\overline{\mathrm{U}}, \overline{\mathrm{F}})$, which extends (by definition of V) to $\mathrm{H}^{n-2}(\overline{\mathrm{V}}, \overline{\mathrm{F}})$, also extends to $\mathrm{H}^{n-2}(\overline{\mathrm{X}}, \overline{\mathrm{F}})$; but this shows that $\overline{d}(\overline{s}) = 0$, that is $d(s)$ is zero at $\overline{y}$, which is absurd.

b) Injectivity of $(**)$. Using the surjectivity of $(*)$, we obtain the exact sequence

$$0 \longrightarrow \mathrm{H}^0(\mathrm{X}, \underline{\mathrm{H}}_{\mathrm{Y}}^{n-1}(\mathrm{F})) \longrightarrow \mathrm{H}^{n-1}(\mathrm{X}, \mathrm{F}) \longrightarrow \mathrm{H}^{n-1}(\mathrm{U}, \mathrm{F})$$

159 and we must show that any element $s \in \mathrm{H}^0(\mathrm{X}, \underline{\mathrm{H}}_{\mathrm{Y}}^{n-1}(\mathrm{F}))$ is zero. Let V be the largest open set over which $s = 0$. Suppose that we have $\mathrm{V} \neq \mathrm{X}$ and let y be a maximal point of $\mathrm{X} - \mathrm{V}$, $\overline{\mathrm{X}}$ a strict localization of X at a geometric point $\overline{y}$ over y. We have, by induction hypothesis and by 1.8.1, the relation $\mathrm{prof}_{\overline{\mathrm{Y}}}(\overline{\mathrm{F}}) \geqslant \mathrm{prof}_{\mathrm{Y}}(\mathrm{F}) \geqslant n-1$, whence the fact that the map e of the following diagram is injective :

$$\mathrm{H}^0(\overline{\mathrm{X}}, \underline{\mathrm{H}}_{\mathrm{Y}}^{n-1}(\overline{\mathrm{F}})) = (\underline{\mathrm{H}}_{\mathrm{Y}}^{n-1}(\mathrm{F}))_{\overline{y}} \xrightarrow{e} \mathrm{H}^{n-1}(\overline{\mathrm{X}}, \overline{\mathrm{F}}) \xrightarrow{f} \mathrm{H}^{n-1}(\overline{\mathrm{V}}, \overline{\mathrm{F}}).$$

The same is true for f by virtue of the hypothesis ; the left-hand equality follows from the retrocompactness hypothesis on U. Let $\overline{s}$ be the image of s in $(\underline{\mathrm{H}}_{\mathrm{Y}}^{n-1}(\mathrm{F}))_{\overline{y}}$; since s vanishes over V, we have $f \cdot e(\overline{s}) = 0$, hence $\overline{s} = 0$, which contradicts the choice of y and completes the proof.

Remark 1.9. — A result analogous to 1.8 is probably valid in the case where we replace the étale topos of a scheme X by a "topos of locally finite type", that is, definable by a site of locally finite type (SGA 4 VI 1.1). To see this, one must use a result of P. Deligne (SGA 4 VI.9), asserting that there are "enough fiber functors" in such a topos.

We will deduce from 1.8 important cases where the étale depth can be determined.

Theorem 1.10 **(Cohomological semi-purity theorem(1))**. — *Let* X *be either a smooth scheme over a field k, or an excellent regular scheme* (EGA IV 7.8.2) *of characteristic zero* (*N.B. if we admit the resolution of singularities in the sense of* (SGA 4 XIX),

(1)N.D.E. : Gabber has since proven — in 1994 — Grothendieck's absolute cohomological purity conjecture : if Y is a closed subscheme of absolute noetherian schemes of pure codimension c and n is an integer invertible on X, then $\underline{\mathrm{H}}_{\mathrm{Y}}^{q}(\Lambda)$ is zero if $q \neq 2c$ and is $\Lambda_{\mathrm{Y}}(-c)$ (Tate twist) otherwise, where we have set $\Lambda = \mathbf{Z}/n\mathbf{Z}$. See (Fujiwara K., "A Proof of the Absolute Purity Conjecture (after Gabber)", in *Algebraic geometry 2000, Azumino* (*Hotaka*), Adv. Stud. in Pure Math., vol. 36, 2002, p. 153-183). For applications to the existence of the dualizing complex, see (SGA 5, Lect. Notes in Math., vol. 589, Springer-Verlag, 1977, p. 1672), exposé 1 and *loc. cit.*, § 8. This conjecture had been proven in the case $n = \ell^\nu$ with ℓ a prime invertible on X large enough by crucially using K-theory (Thomason R.W., "Absolute cohomological purity", *Bull. Soc. Math. France* **112** (1984), N° 3, p. 397–406). We find K-theory again in Gabber's proof through the Atiyah-Hirzebruch-Thomason spectral sequence relating étale cohomology and K-theory, a method already used in Thomason's approach. Besides this result, the other fundamental argument is the generalization of the Lefschetz theorem cited in note (5), page 185.

it suffices to suppose, more generally, that X *is an excellent regular scheme of equal characteristic*)*. Let* Y *be a closed subset of* X *and* $\mathbf{L}$ *the set of prime numbers distinct* **160**
from the characteristic of X*. Then we have*

$$\mathrm{prof}^{\mathbf{L}}_{\mathrm{Y}}(\mathrm{X}) = 2\,\mathrm{codim}(\mathrm{Y},\mathrm{X}).$$

Démonstration. — It follows from 1.8 that we have

$$\mathrm{prof}^{\mathbf{L}}_{\mathrm{Y}}(\mathrm{X}) = \inf_{y\in \mathrm{Y}} \mathrm{prof}_y(\mathrm{X}).$$

As on the other hand $\mathrm{codim}(\mathrm{Y},\mathrm{X}) = \inf_{y\in\mathrm{Y}} \dim \mathscr{O}_{\mathrm{X},y}$, we are reduced to showing that

$$\mathrm{prof}^{\mathbf{L}}_{y}(\mathrm{X}) = 2\dim \mathscr{O}_{\mathrm{X},y},$$

which follows from (SGA 4 XVI 3.7 and XIX 3.2).

***Theorem 1.11* (Homotopical purity theorem[(2)]).** — *If* X *is a locally noetherian scheme which is regular* (*resp. whose local rings are complete intersections*)*,* Y *a closed subset of* X *such that* $\mathrm{codim}(\mathrm{Y},\mathrm{X}) \geqslant 2$ (*resp.* $\mathrm{codim}(\mathrm{Y},\mathrm{X}) \geqslant 3$)*, then we have*

$$\mathrm{prof\,hop}_{\mathrm{Y}}(\mathrm{X}) \geqslant 3.$$

It follows indeed from 1.8 that we have $\mathrm{prof\,hop}_{\mathrm{Y}}(\mathrm{X}) = \inf_{y\in\mathrm{Y}} \mathrm{prof\,hop}_y(\mathrm{X})$. Now the strict local rings of X at the different points of Y are regular rings of dimension $\geqslant 2$ (resp. of complete intersection and dimension $\geqslant 3$). It then follows from the purity theorem X 3.4 that $\mathrm{prof\,hop}_y(\mathrm{X}) \geqslant 3$, which proves the theorem.

***Example 1.12*.** — Let X be a locally noetherian scheme, Y a closed subset of X and $n = 1$ or 2. Then, if we have $\mathrm{prof}_{\mathrm{Y}}(\mathscr{O}_{\mathrm{X}}) \geqslant n$ ($\mathrm{prof}_{\mathrm{Y}}(\mathscr{O}_{\mathrm{X}})$ denoting the Y-depth in the sense of coherent sheaves (cf. 1.6 a)), we also have $\mathrm{prof}_{\mathrm{Y}}(\mathrm{X}) \geqslant n$; this is obvious for $n = 1$ and, for $n = 2$, this is nothing other than Hartshorne's theorem (III 1). On **161**
the other hand, the analogous assertion is false for $n \geqslant 3$. Take for example an affine space of dimension $\geqslant 3$ over a field of characteristic $\neq 2$ and let the group $\mathbf{Z}/2\mathbf{Z}$ act by symmetry with respect to the origin. Let X be the quotient and $\mathrm{Y} = \{x\}$ the image of the origin in X. Then $\mathscr{O}_{\mathrm{X},x}$ is a Cohen-Macaulay ring, so we have $\mathrm{prof}_x(\mathscr{O}_{\mathrm{X}}) \geqslant 3$; but the affine space deprived of the origin is an étale covering of $\mathrm{X} - \{x\}$ which does not extend to an étale covering of X ; so we have according to 1.4 $\mathrm{prof}_{\mathrm{Y}}(\mathrm{X}) = 2$.

The following theorem is the analogue of (EGA IV 6.3.1) :

[(2)] N.D.E. : recently, de Jong and Oort have obtained the following purity statement : let $\widetilde{\mathrm{S}} \to \mathrm{S}$ be a resolution of singularities of the spectrum S of a 2-dimensional normal noetherian local ring and let U be the complement of the closed point s in S. Suppose furthermore that $k(s)$ is algebraically closed. Then, for any prime number p, in particular if S is of characteristic p, the restriction morphism $\mathrm{H}^1_{\text{ét}}(\widetilde{\mathrm{S}}, \mathbf{Q}_p) \to \mathrm{H}^1_{\text{ét}}(\mathrm{U}, \mathbf{Q}_p)$ is bijective (de Jong A.J. & Oort F, "Purity of the stratification by Newton polygons", *J. Amer. Math. Soc.* **13** (2000), N° 1, p. 209–241, theorem 3.2). If $k = \mathbf{C}$ and A is the completion of a surface singularity, this result is due to Mumford (see page 161, [**5**]).

Theorem 1.13. — *Let $f : \mathrm{X} \to \mathrm{S}$ be a morphism of schemes,* Y *a closed subset of* X, Z *a closed subset of* S, *such that $f(\mathrm{Y}) \subset \mathrm{Z}$. We suppose that the local rings of* X *at the different points of* Y *are noetherian and that the open sets* $\mathrm{X} - \mathrm{Y}$ *and* $\mathrm{S} - \mathrm{Z}$ *are retrocompact in* X *and* S *respectively. Let p, q, r be integers such that $p \geqslant -r$, $q \geqslant 0$,* **L** *a set of prime numbers and* F *a complex of abelian sheaves of* **L***-torsion on* S, *such that the cohomology sheaves $\mathrm{H}^i(\mathrm{F})$ are zero for $i < -r$. We suppose that*

a) *The morphism f is locally $(p+q+r-2)$-acyclic for* **L** (SGA 4 XV 1.11).

b) *We have*

$$\mathrm{prof}_{\mathrm{Z}}(\mathrm{F}) \geqslant p.$$

c) *For any point s of* Z, *we have*

$$\mathrm{prof}^{\mathbf{L}}_{\mathrm{Y}_s}(\mathrm{X}_s) \geqslant q.$$

162 *Then we have*

$$\mathrm{prof}_{\mathrm{Y}}(f^*\mathrm{F}) \geqslant p+q.$$

We will need the following lemma :

Lemma 1.13.1. — *Let* **L** *be a set of prime numbers, n and r integers, $f : \mathrm{X} \to \mathrm{S}$ a locally n-acyclic morphism for* **L**. *Let* F *be a complex of abelian sheaves, with cohomology sheaves of* **L***-torsion, such that $\mathrm{H}^i(\mathrm{F}) = 0$ for $i < -r$,* Z *a closed subset of* S *such that* $\mathrm{S} - \mathrm{Z}$ *is retrocompact in* S *and* $\mathrm{T} = f^{-1}(\mathrm{Z})$. *Then the canonical morphism*

$$f^*(\underline{\mathrm{H}}^i_{\mathrm{Z}}(\mathrm{F})) \longrightarrow \underline{\mathrm{H}}^i_{\mathrm{T}}(f^*\mathrm{F})$$

is bijective for $i < n-r+2$ and injective for $i = n-r+2$.

Let $\mathrm{U} = \mathrm{S} - \mathrm{Z}$ and $\mathrm{V} = \mathrm{X} - \mathrm{T}$, so that we have the cartesian square

$$\begin{tikzcd} \mathrm{V} \arrow[r, "g"] \arrow[d, "k"'] & \mathrm{U} \arrow[d, "j"] \\ \mathrm{X} \arrow[r, "f"] & \mathrm{S} \end{tikzcd}$$

Let us consider the following commutative diagram whose rows are exact

$$\begin{tikzcd} \arrow[r] & f^*(\underline{\mathrm{H}}^i_{\mathrm{Z}}(\mathrm{F})) \arrow[r] \arrow[d] & f^*(\underline{\mathrm{H}}^i(\mathrm{F})) \arrow[r] \arrow[d, "\wr"] & f^*(\underline{\mathrm{H}}^i(\mathrm{R}\, j_*(j^*\mathrm{F}))) \arrow[r] \arrow[d] & {} \\ \arrow[r] & \underline{\mathrm{H}}^i_{\mathrm{T}}(f^*\mathrm{F}) \arrow[r] & \underline{\mathrm{H}}^i(f^*\mathrm{F}) \arrow[r] & \underline{\mathrm{H}}^i(\mathrm{R}\, k_*(k^*f^*\mathrm{F})) \arrow[r] & {} \end{tikzcd};$$

it follows that we are reduced to showing that the morphism

$$f^*(\underline{\mathrm{H}}^i(\mathrm{R}\, j_*(j^*\mathrm{F}))) \longrightarrow \underline{\mathrm{H}}^i(\mathrm{R}\, k_*(k^*f^*\mathrm{F}))$$

163 is bijective for $i < n-r+1$ and injective for $i = n-r+1$. Now such a morphism

comes from the following morphism between hypercohomology spectral sequences

$$\begin{array}{ccc} f^*\mathrm{E}_2^{p,q} = f^*(\mathrm{R}^p\, j_*(\underline{\mathrm{H}}^q(j^*\mathrm{F}))) & \Longrightarrow & f^*(\underline{\mathrm{H}}^*(\mathrm{R}\, j_*(j^*\mathrm{F}))) \\ \downarrow & & \downarrow \\ \mathrm{E}_2'^{p,q} = \mathrm{R}^p\, k_*(\underline{\mathrm{H}}^q(k^*f^*\mathrm{F})) & \Longrightarrow & \underline{\mathrm{H}}^*(\mathrm{R}\, k_*(k^*f^*\mathrm{F})). \end{array}$$

Since j is quasi-compact, it follows from (SGA 4 XV 1.10) that the morphism $f^*(\mathrm{E}_2^{p,q}) \to \mathrm{E}_2'^{p,q}$ is bijective for $p \leqslant n$ and injective for $p = n+1$; in particular it is bijective for $p+q \leqslant n-r$ and injective for $p+q = n-r+1$. The conclusion follows immediately.

Let us return to the proof of 1.13. Let $\mathrm{T} = f^{-1}(\mathrm{Z})$. According to 1.13.1 and condition a), the canonical morphism $f^*(\underline{\mathrm{H}}^i_{\mathrm{Z}}(\mathrm{F})) \to \underline{\mathrm{H}}^i_{\mathrm{T}}(f^*\mathrm{F})$ is an isomorphism for $i \leqslant p+q$. It follows therefore from b) that $\underline{\mathrm{H}}^i_{\mathrm{T}}(f^*\mathrm{F}) = 0$ for $i < p$ and, for $i < p+q$, $\underline{\mathrm{H}}^i_{\mathrm{T}}(f^*\mathrm{F})$ restricted to T is the inverse image of a sheaf G^i on Z. Let

$$f_{\mathrm{T}} : \mathrm{T} \longrightarrow \mathrm{Z}$$

be the restriction of f to T. It then follows from c) and the following corollary that $\underline{\mathrm{H}}^j_{\mathrm{Y}}(f^*_{\mathrm{T}}(\mathrm{G}^i)) = 0$ for $j < q$. We conclude that

$$\underline{\mathrm{H}}^j_{\mathrm{Y}}(\underline{\mathrm{H}}^i_{\mathrm{T}}(f^*\mathrm{F})) = 0 \text{ for } i+j < p+q,$$

because the inequality $i+j < p+q$ implies either $i < p$ and then $\underline{\mathrm{H}}^i_{\mathrm{T}}(f^*\mathrm{F}) = 0$, or $j < q$ and then $\underline{\mathrm{H}}^j_{\mathrm{Y}}(\underline{\mathrm{H}}^i_{\mathrm{T}}(f^*\mathrm{F})) = 0$. Given that we have, with the notation of 1.0, $\underline{\Gamma}_{\mathrm{Y}} = \underline{\Gamma}_{\mathrm{Y}}.\underline{\Gamma}_{\mathrm{T}}$, we have the spectral sequence

$$\mathrm{E}_2^{i,j} = \underline{\mathrm{H}}^j_{\mathrm{Y}}(\underline{\mathrm{H}}^i_{\mathrm{T}}(f^*\mathrm{F})) \Longrightarrow \underline{\mathrm{H}}^*_{\mathrm{Y}}(f^*\mathrm{F}); \tag{1.13.2}$$

as $\mathrm{E}_2^{i,j} = 0$ for $i+j < p+q$, we see that $\underline{\mathrm{H}}^k_{\mathrm{Y}}(f^*\mathrm{F}) = 0$ for $k < p+q$. The theorem will **164**
therefore be proven if we prove the following corollary (which is the particular case of 1.13 obtained by setting $\mathrm{Z} = \mathrm{S}$, $r = p = 0$ and F reduced to degree 0).

Corollary 1.14. — *Let $f : \mathrm{X} \to \mathrm{S}$ be a morphism, Y a closed subset such that the open complement $\mathrm{X} - \mathrm{Y}$ is retrocompact in X and the local rings of X at the different points of Y are noetherian. Let $\mathbf{L}$ be a set of prime numbers, q an integer and F an abelian sheaf of $\mathbf{L}$-torsion on S. Suppose that f is locally $(q-2)$-acyclic for $\mathbf{L}$ and that, for any point s of S, we have $\mathrm{prof}^{\mathbf{L}}_{\mathrm{Y}_s}(\mathrm{X}_s) \geqslant q$. Then, we have $\mathrm{prof}_{\mathrm{Y}}(f^*\mathrm{F}) \geqslant q$.*

1°) Reduction to the case where X and S are strictly local schemes, f a local morphism and Y reduced to a closed point of X.

According to 1.8, to establish 1.14, we must show that we have for any point y of Y :

$$\mathrm{prof}_y(f^*\mathrm{F}) \geqslant q.$$

Let $s = f(y)$, $\overline{s}$ a geometric point over s, $\overline{y}$ a geometric point over y and $\overline{s}$, $\overline{\mathrm{X}}$ and $\overline{\mathrm{S}}$ the strict localizations of X and S at $\overline{y}$ and $\overline{s}$ respectively, $\overline{f} : \overline{\mathrm{X}} \to \overline{\mathrm{S}}$ the canonical morphism and $\overline{\mathrm{F}}$ the inverse image of F on $\overline{\mathrm{S}}$. Since we have the relation

$\mathrm{prof}_y(f^*\mathrm{F}) = \mathrm{prof}_{\overline{y}}(\overline{f}^*\overline{\mathrm{F}})$, it suffices to show that the hypotheses of 1.14 are preserved when we replace f (resp. Y, resp. F) by $\overline{f}$ (resp. $\{\overline{y}\}$, resp. $\overline{\mathrm{F}}$). The retrocompactness
165 condition follows from the noetherian hypothesis on $\mathscr{O}_{\mathrm{X},x}$, implying that $\overline{\mathrm{X}}$ is noetherian. According to (SGA 4 XV 1.10 (i)), $\overline{f}$ is still locally $(q-2)$-acyclic for $\mathbf{L}$. On the other hand, the fiber $(\overline{\mathrm{X}})_{\overline{s}}$ of $\overline{\mathrm{X}}$ over $\overline{s}$ is identified with the strict localization of X_s at $\overline{y}$, so it satisfies the relation $\mathrm{prof}_{\overline{y}}((\overline{\mathrm{X}})_{\overline{s}}) \geqslant q$. As a similar relation is trivially verified for the fibers of the $\overline{\mathrm{S}}$-scheme $\overline{\mathrm{X}}$, other than the closed fiber, this completes the reduction.

2°) Case where X and S are strictly local, f a local homomorphism and Y reduced to the closed point of X. Let

$$g : \mathrm{Y} = \mathrm{X} - \{y\} \longrightarrow \mathrm{S}$$

be the structural morphism of U. We must show that the canonical morphism

$$u_i : \mathrm{H}^i(\mathrm{X}, f^*\mathrm{F}) \longrightarrow \mathrm{H}^i(\mathrm{U}, f^*\mathrm{F})$$

is bijective for $i \leqslant q-2$ and injective for $i = q-1$. Consider the commutative diagram

$$\begin{tikzcd}
\mathrm{H}^i(\mathrm{X}, f^*\mathrm{F}) \arrow[rr, "u_i"] & & \mathrm{H}^i(\mathrm{U}, f^*\mathrm{F}) \\
& \mathrm{H}^i(\mathrm{S}, \mathrm{F}) \arrow[ul, "v_i"] \arrow[ur, "w_i"'] &
\end{tikzcd}$$

The morphism v_i is obviously bijective for all i. On the other hand g is locally $(q-2)$-acyclic for $\mathbf{L}$; moreover its fibers are $(q-2)$-acyclic for $\mathbf{L}$, as follows from the fact that $\mathrm{prof}_y(\mathrm{X}_s) \geqslant q$ and that the fibers of f are $(q-2)$-acyclic for $\mathbf{L}$; as g is quasi-compact since X is noetherian, it follows from (SGA 4 XV 1.16) that g is $(q-2)$-acyclic for $\mathbf{L}$. Consequently w_i, and therefore also u_i, is bijective for $i \leqslant q-2$ and injective for
166 $i = q-1$, which completes the proof of 1.14.

Corollary 1.15. — *Let $f : \mathrm{X} \to \mathrm{S}$ be a morphism of schemes, $\mathbf{L}$ a set of prime numbers, m and r integers and F a complex of abelian sheaves of $\mathbf{L}$-torsion on S such that $\underline{\mathrm{H}}^i(\mathrm{F}) = 0$ for $i < -r$. Let x be a point of X, $s = f(x)$ and suppose that the local ring $\mathscr{O}_{\mathrm{X},x}$ is noetherian. Then, if f is locally m-acyclic for $\mathbf{L}$, we have the relation*

$$(*) \qquad \mathrm{prof}_x(f^*\mathrm{F}) \geqslant \inf(\mathrm{prof}_s(\mathrm{F}) + \mathrm{prof}_x^{\mathbf{L}}\,\mathrm{X}_s), n) \ \textit{where}\ n = m - r + 2.$$

In particular, if $n \geqslant \mathrm{prof}_s(\mathrm{F}) + \mathrm{prof}_x^{\mathbf{L}}(\mathrm{X}_s)$, for example if f is locally acyclic for $\mathbf{L}$, we have

$$(**) \qquad \mathrm{prof}_x(f^*\mathrm{F}) \geqslant \mathrm{prof}_s(\mathrm{F}) + \mathrm{prof}_x^{\mathbf{L}}(\mathrm{X}_s).$$

If $\mathbf{L}$ is reduced to an element ℓ and if we have $n \geqslant \mathrm{prof}_s(\mathrm{F}) + \mathrm{prof}_x^{\mathbf{L}}(\mathrm{X}_s)$, the preceding inequality is an equality.

We reduce to the case where s and x are closed points, by taking the strict localizations of S and x at geometric points $\overline{s}$ over s and $\overline{x}$ over x and $\overline{s}$. If we have

the inequality $n \geqslant \operatorname{prof}_s(\mathrm{F}) + \operatorname{prof}_x^{\mathbf{L}}(\mathrm{X}_s)$, then $(*)$ is obtained from 1.13 by setting $p = \operatorname{prof}_s(\mathrm{F})$ and $q = \operatorname{prof}_x^{\mathbf{L}}(\mathrm{X}_s)$ (the hypothesis that $\mathrm{S} - \{s\}$ is retrocompact in S follows from the fact that $\mathrm{X} - \mathrm{X}_s$ is retrocompact in X and that f is surjective since it is (-1)-acyclic (except perhaps if the conclusion of 1.15 is empty)). If we have $n < \operatorname{prof}_s(\mathrm{F}) + \operatorname{prof}_x^{\mathbf{L}}(\mathrm{X}_s)$, the inequality $(*)$ is still obtained from 1.13 by setting for example $p = \operatorname{prof}_s(\mathrm{F})$ and $q = n - p$. It remains to prove the last assertion. Let 167
$p = \operatorname{prof}_s(\mathrm{F})$ and $q = \operatorname{prof}_x(\mathrm{X}_s)$; it follows from (1.13.2) that we have

$$\underline{\mathrm{H}}_x^{p+q}(f^*(\mathrm{F})) \simeq \underline{\mathrm{H}}_x^q(f^*(\underline{\mathrm{H}}_s^p(\mathrm{F}))).$$

Since $\operatorname{prof}_s(\mathrm{F}) = p$, the sheaf $\underline{\mathrm{H}}_s^p(\mathrm{F})$ is a sheaf of ℓ-torsion, constant on s, different from zero. Consequently the sheaf $\mathrm{G} = f^*(\underline{\mathrm{H}}_s^p(\mathrm{F}))$ is a sheaf of ℓ-torsion, constant on X_s, non-zero, so it contains a subsheaf isomorphic to $\mathbf{Z}/\ell\mathbf{Z}$; since $\underline{\mathrm{H}}_x^q(\mathbf{Z}/\ell\mathbf{Z})$ is different from zero, we do have $\underline{\mathrm{H}}_x^q(\mathrm{G}) \neq 0$.

Corollary 1.16. — *Let $f : \mathrm{X} \to \mathrm{S}$ be a regular morphism of excellent schemes* (EGA IV 7.8.2) *of characteristic zero, ℓ a prime number and* F *a complex of sheaves of ℓ-torsion on* S. *Let $x \in \mathrm{X}$, $s = f(x)$; then we have*

$$\operatorname{prof}_x(f^*\mathrm{F}) = \operatorname{prof}_s(\mathrm{F}) + 2\dim(\mathscr{O}_{\mathrm{X},x}).$$

Indeed f is locally acyclic (SGA 4 XIX 4.1). It then follows from 1.15 that we have

$$\operatorname{prof}_x(f^*\mathrm{F}) = \operatorname{prof}_s(\mathrm{F}) + \operatorname{prof}_x(\mathrm{X}_s).$$

Now we have according to 1.10

$$\operatorname{prof}_x(\mathrm{X}_s) = 2\dim \mathscr{O}_{\mathrm{X}_s,x},$$

hence the result.

Remark 1.17. — It follows from 1.15 that 1.13 remains true when we replace b) and c) by the conditions : 168

b′) For any point $s \in f(\mathrm{Y})$, we have $\operatorname{prof}_s(\mathrm{F}) \geqslant p$.

c′) For any point $x \in \mathrm{Y}$, if $s = f(x)$, we have $\operatorname{prof}_x^{\mathbf{L}}(\mathrm{X}_s) \geqslant q$.

In the case of a sheaf of sets or groups, we have the following theorem analogous to 1.13.

Theorem 1.18. — *Let $f : \mathrm{X} \to \mathrm{S}$ be a morphism of schemes,* Y *a closed subset of* X *such that* $\mathrm{X} - \mathrm{Y}$ *is retrocompact in* X *and that, for any point x of* Y, *the local ring $\mathscr{O}_{\mathrm{X},x}$ is noetherian.*

1°) *Let* F *be a sheaf of sets on* S *and n an integer equal to* 1 *or* 2. *Suppose that f is locally $(n-2)$-acyclic and that, for any point s of $f(\mathrm{Y})$, we have :*

$$\operatorname{prof}_{\mathrm{Y}_s}(\mathrm{X}_s) + \operatorname{prof}_s(\mathrm{F}) \geqslant n.$$

Then we have :

$$\operatorname{prof}_{\mathrm{Y}}(f^*\mathrm{F}) \geqslant n.$$

2°) *Let* $\mathbf{L}$ *be a set of prime numbers and* F *a sheaf of ind-*$\mathbf{L}$*-groups. Suppose that* f *is locally* 1*-aspherical for* $\mathbf{L}$ (SGA 4 XV 1.11) *and that, for any point* s *of* $f(\mathrm{Y})$*, we have :*

$$\operatorname{prof}\operatorname{hop}^{\mathbf{L}}_{\mathrm{Y}_s}(\mathrm{X}_s) + \operatorname{prof}_s(\mathrm{F}) \geqslant 3.$$

Then, we have :

$$\operatorname{prof}_{\mathrm{Y}}(f^*\mathrm{F}) \geqslant 3.$$

169 *Démonstration.* — We reduce, as in 1.14 and 1.15, to the case where X and S are strictly local schemes, f a local homomorphism and Y the closed point x of X. Let $s = f(x)$ be the closed point of S ; we have the commutative diagram :

$$\begin{tikzcd}
\mathrm{X}-\mathrm{X}_s \arrow[r, "i"] \arrow[d, "h"'] & \mathrm{X}-\{x\} \arrow[rr, "j"] \arrow[dr, "g"] & & \mathrm{X} \arrow[dl, "f"] \\
\mathrm{S}-\{s\} \arrow[rr, "k"'] & & \mathrm{S}. &
\end{tikzcd}$$

1°) a) *Case* $n = 1$.

If we have $\operatorname{prof}_s(\mathrm{F}) \geqslant 1$, then the morphism $\mathrm{F} \to k_*k^*\mathrm{F}$ is injective, so the morphism $f^*\mathrm{F} \to f^*(k_*k^*\mathrm{F})$ is also injective. On the other hand, it follows from the fact that f is locally (-1)-acyclic, that the morphism $f^*(k_*k^*\mathrm{F}) \to (j.i)*(f^*\mathrm{F}_{|\mathrm{X}-\mathrm{X}_s})$ is injective. Finally, the composite morphism $f^*\mathrm{F} \to (j.i)*(f^*\mathrm{F}_{|\mathrm{X}-\mathrm{X}_s})$ is injective, which shows that we have $\operatorname{prof}_{\mathrm{X}_s}(f^*\mathrm{F}) \geqslant 1$, hence also $\operatorname{prof}_x(f^*\mathrm{F}) \geqslant 1$.

If we have $\operatorname{prof}_x(\mathrm{X}_s) \geqslant 1$, we consider the commutative diagram

$$(*) \qquad \begin{tikzcd}
\mathrm{H}^0(\mathrm{X}, f^*\mathrm{F}) \arrow[r, "v"] \arrow[d, "\wr"'] & \mathrm{H}^0(\mathrm{X}-\{x\}, f^*\mathrm{F}) \arrow[d] \\
\mathrm{H}^0(\mathrm{X}_s, f^*\mathrm{F}) \arrow[r, "v'"] & \mathrm{H}^0(\mathrm{X}-\{x\}, f^*\mathrm{F});
\end{tikzcd}$$

By hypothesis, v' is injective so is v.

b) *Case* $n = 2$. We consider the commutative diagram

$$(**) \qquad \begin{tikzcd}
\mathrm{H}^0(\mathrm{S}, \mathrm{F}) \arrow[rr, "u"] \arrow[d, "m"', "\wr"] & & \mathrm{H}^0(\mathrm{S}-\{s\}, \mathrm{F}) \arrow[d, "n"', "\wr"] \\
\mathrm{H}^0(\mathrm{X}, f^*\mathrm{F}) \arrow[r, "v"] & \mathrm{H}^0(\mathrm{X}-\{x\}, f^*\mathrm{F}) \arrow[r, "w"] & \mathrm{H}^0(\mathrm{X}-\mathrm{X}_s, f^*\mathrm{F});
\end{tikzcd}$$

170 we must show that v is bijective. The morphism m is obviously bijective, and, since f is 0-acyclic, n is also bijective.

If we have $\operatorname{prof}_s(\mathrm{F}) \geqslant 2$, u is bijective. As we have seen in a), the sole hypothesis $\operatorname{prof}_s(\mathrm{F}) \geqslant 1$ implies the relation $\operatorname{prof}_{\mathrm{X}_s}(f^*\mathrm{F}) \geqslant 1$; consequently v and w are injective ; it then follows from $(**)$ that v is bijective.

If we have $\mathrm{prof}_x(\mathrm{X}_s) \geqslant 2$, then g is 0-acyclic (because it is locally 0-acyclic and its fibers are 0-acyclic). It follows that $v \cdot m$ is bijective, hence v is bijective.

If we have $\mathrm{prof}_s(\mathrm{F}) \geqslant 1$ and $\mathrm{prof}_x(\mathrm{X}_s) \geqslant 1$, then we already know that v and w are injective. Let z be a maximal point of $\mathrm{X}_s - \{x\}$ (such a point exists by the hypothesis $\mathrm{prof}_x(\mathrm{X}_s) \geqslant 1$), $\overline{\mathrm{Z}}$ the strict localization of X at a geometric point over z and $\overline{f^*\mathrm{F}}$ the inverse image of $f^*\mathrm{F}$ on $\overline{\mathrm{Z}}$. Consider the commutative diagram

$$\begin{tikzcd}
 & \mathrm{H}^0(\mathrm{S},\mathrm{F}) \arrow[rr] \arrow[dl, "m"', "\sim"] & & \mathrm{H}^0(\mathrm{S}-\{s\},\mathrm{F}) \arrow[dr, "n", "\sim"'] & \\
\mathrm{H}^0(\mathrm{X}, f^*\mathrm{F}) \arrow[rr, "v"] \arrow[dr, "m'"', "\sim"] & & \mathrm{H}^0(\mathrm{X}-\{x\}, f^*\mathrm{F}) \arrow[rr, "w"] \arrow[dl, "r"] & & \mathrm{H}^0(\mathrm{X}-\mathrm{X}_s, f^*\mathrm{F}) \arrow[dl, "n'", "\sim"'] \\
 & \mathrm{H}^0(\overline{\mathrm{Z}}, f^*\mathrm{F}) \arrow[rr] & & \mathrm{H}^0(\overline{\mathrm{Z}}-\{\overline{z}\}, \overline{f^*\mathrm{F}}) &
\end{tikzcd}$$

the morphism $m' \cdot m$ is obviously bijective and it follows from the fact that f is locally 0-acyclic that $n' \cdot n$ is bijective; consequently m' and n' are also bijective. Since w is injective, r is also injective and consequently v is bijective.

2°) Taking b) into account, we already know that $\mathrm{prof}_x(f^*\mathrm{F}) \geqslant 2$.

If we have $\mathrm{prof}_s(\mathrm{F}) \geqslant 3$, then $\mathrm{R}^1 k_*(k^*\mathrm{F}) = 1$[(3)]. Since f is locally 1-aspherical, we have $\mathrm{R}^1(j \cdot i)_*(f^*\mathrm{F}_{|\mathrm{X}-\mathrm{X}_s}) = f^*(\mathrm{R}^1 k_*(k^*\mathrm{F})) = 1$. We thus have $\mathrm{prof}_{\mathrm{X}_s}(f^*\mathrm{F}) \geqslant 3$ and consequently we have $\mathrm{prof}_x(f^*\mathrm{F}) \geqslant 3$. 171

If we have $\mathrm{prof}_x(\mathrm{X}_s) \geqslant 3$, then g is 1-aspherical (because g is locally 1-aspherical and its fibers are 1-aspherical). We thus have $\mathrm{H}^1(\mathrm{X}-\{x\}, f^*\mathrm{F}) = \mathrm{H}^1(\mathrm{S},\mathrm{F}) = 1$ and consequently $\mathrm{prof}_x(f^*\mathrm{F}) \geqslant 3$.

If we have $\mathrm{prof}_s(\mathrm{F}) \geqslant 2$ and $\mathrm{prof}_x(\mathrm{X}_s) \geqslant 1$, we use the exact sequence (SGA 4 XII 3.2) :

$$1 \longrightarrow \mathrm{R}^1 j_*(i_*(f^*\mathrm{F}_{|\mathrm{X}-\mathrm{X}_s})) \longrightarrow \mathrm{R}^1(j.i)_*(f^*\mathrm{F}_{|\mathrm{X}-\mathrm{X}_s}) \longrightarrow j_*(\mathrm{R}^1 i_*(f * \mathrm{F}_{|\mathrm{X}-\mathrm{X}_s})).$$

Since f and g are locally 1-aspherical, we have

$$\mathrm{R}^1(j \cdot i)_*(f^*\mathrm{F}_{|\mathrm{X}-\mathrm{X}_s}) \simeq f^*(\mathrm{R}^1 k_*(k^*\mathrm{F}))$$
$$\mathrm{R}^1 i_*(f^*\mathrm{F}_{|\mathrm{X}-\mathrm{X}_s}) \simeq g^*(\mathrm{R}^1 k_*(k^*\mathrm{F}));$$

the preceding exact sequence is then written in the form

$$(***) \quad 1 \longrightarrow \mathrm{R}^1 j_*(i_*(f^*\mathrm{F}_{|\mathrm{X}-\mathrm{X}_s})) \longrightarrow f^*(\mathrm{R}^1 k_*(k^*\mathrm{F})) \xrightarrow{a} j_*(j^*(f^*(\mathrm{R}^1 k_*(k^*\mathrm{F})))).$$

The hypothesis $\mathrm{prof}_s(\mathrm{F}) \geqslant 2$ shows that the morphism $\mathrm{F} \to k_*k^*\mathrm{F}$ is bijective; by applying g^*, we find, taking into account the fact that g is locally 0-acyclic, $f^*\mathrm{F}_{|\mathrm{X}-\{x\}} = i_*(f^*\mathrm{F}_{|\mathrm{X}-\mathrm{X}_s})$. The hypothesis $\mathrm{prof}_x(\mathrm{X}_s) \geqslant 1$ shows that the morphism a is injective (note that $f^*(\mathrm{R}^1 k_*(k^*\mathrm{F}))$ is a sheaf equal to 1 outside of X_s

(3)N.D.E. : the trivial torsor is successively denoted by 0 or 1 in what follows; we have left this double notation, which, on reflection, brings no ambiguity.

and constant on X_s). It then follows from (∗∗∗) that we have $R^1 j_*(f^*F_{|X-\{x\}}) = 1$,
172 hence $\mathrm{prof}_x(f^*F) \geqslant 3$.

If we have $\mathrm{prof}_s(F) \geqslant 1$ and $\mathrm{prof}_x(f^*F) \geqslant 2$, we consider the sheaf of homogeneous spaces G defined by the exact sequence

$$1 \longrightarrow F \longrightarrow k_*k * F \longrightarrow G \longrightarrow 1.$$

By applying the exact functor g^* to this exact sequence and using (SGA 4 XII 3.1), we obtain the following commutative diagram whose rows are exact :

$$\begin{tikzcd}
f^*(k_*k^*F) \arrow[r] \arrow[d] & f^*G \arrow[r] \arrow[d, "b"] & 1 & \\
j_*(g^*(k_*k^*F)) \arrow[r, "u"] & j_*(g^*G) \arrow[r] & R^1 j_*(g^*F) \arrow[r] & R^1 j_*(g^*(k_*k^*F)).
\end{tikzcd}$$

Since $\mathrm{prof}_x(X_s) \geqslant 2$, the morphism b is bijective so u is surjective and we thus have a map with kernel reduced to the neutral element :

$$1 \longrightarrow R^1 j_*(g^*F) \longrightarrow R^1 j_*(g^*(k_*k^*F)) = R.$$

Since $g^*(k_*k^*F) \simeq i_*(f^*F_{|X-X_s})$ (because g is locally 0-acyclic), R is identified with the first term of the exact sequence (∗∗∗); now we have seen in the preceding case that $R = 1$ as soon as we have $\mathrm{prof}_x(X_s) \geqslant 1$, which proves that $\mathrm{prof}_x(f^*F) \geqslant 3$ and completes the proof of 1.18.

The following corollaries are generalizations of (SGA 4 XVI 3.2 and 3.3).

Corollary 1.19. — *Let $f : X \to S$ be a flat morphism, with separable fibers, of locally*
173 *noetherian schemes and Y a closed subset of X. Suppose that for any point $s \in f(Y)$, the fiber Y_s is rare*[(4)] *in X_s and that one of the two following conditions is verified :*

a) *the closure of $f(Y)$ is rare in S.*

b) *X_s is geometrically unibranch at the points of Y_s.*

Then we have

$$\mathrm{prof}_Y(X) \geqslant 2.$$

It follows indeed from the hypothesis made on f that f is locally 0-acyclic (SGA 4 XV 4.1). We then apply 1.13. The hypothesis Y_s rare in X_s (resp. $\overline{f(Y)}$ rare in S) is equivalent according to 1.6 b) to the relation $\mathrm{prof}_{Y_s}(X_s) \geqslant 1$ (resp. $\mathrm{prof}_{\overline{f(Y)}}(S) \geqslant 1$). The hypothesis X_s geometrically unibranch at each point of Y_s is equivalent to saying that the strict localization of X_s at a geometric point of Y_s is irreducible; knowing that Y_s is rare in X_s, this obviously implies $\mathrm{prof}_{Y_s}(X_s) \geqslant 2$, thanks to 1.8. In both cases 1.13 does give $\mathrm{prof}_Y(X) \geqslant 2$.

(4) N.D.E. : "rare" = "of empty interior", cf. Bourbaki TG IX.52.

Corollary 1.20. — *Let* $f : \mathrm{X} \to \mathrm{S}$ *be a regular morphism* (EGA IV 6.8.1) *of locally noetherian schemes,* Y *a closed subset of* X. *Suppose that, for any point* $s \in f(\mathrm{Y})$, *one of the following conditions is realized :*

a) *We have* $\operatorname{codim}(\mathrm{Y}_s, \mathrm{X}_s) \geqslant 2$.

b) *We have* $\operatorname{codim}(\mathrm{Y}_s, \mathrm{X}_s) \geqslant 1$ *and* $\operatorname{prof}_s(\mathrm{S}) \geqslant 1$.

c) *We have* $\operatorname{prof\,hop}_s(\mathrm{S}) \geqslant 3$.

Then we have 174

$$\operatorname{prof\,hop}_{\mathrm{Y}}(\mathrm{X}) \geqslant 3.$$

This follows indeed from 1.18, given that hypothesis a) implies $\operatorname{prof\,hop}_{\mathrm{Y}_s}(\mathrm{X}_s) \geqslant 3$ (cf. 1.11), and that the condition $\operatorname{codim}(\mathrm{Y}_s, \mathrm{X}_s) \geqslant 1$ obviously implies $\operatorname{prof}_{\mathrm{Y}}(\mathrm{X}) \geqslant 2$.

2. Technical lemmas

2.1. Let S be a locally noetherian scheme, $f : \mathrm{X} \to \mathrm{S}$ a locally of finite type morphism, t a point of S. If $x \in \mathrm{X}$ is such that $s = f(x) \in \operatorname{Spec} \mathscr{O}_{\mathrm{S},t}$, we set

$$\delta_t(x) = \operatorname{deg\,tr} k(x)/k(s) + \dim(\overline{\{s\}}),$$

where $\overline{\{s\}}$ denotes the closure of s in $\operatorname{Spec} \mathscr{O}_{\mathrm{S},t}$, $k(x)$ and $k(s)$ the residue fields of x and s respectively. If S is a local ring with closed point t, we also write $\delta(x)$ instead of $\delta_t(x)$ (cf. SGA 4 XIV 2.2).

Lemma 2.1.1. — *Let a cartesian square be given*

$$\begin{tikzcd} \mathrm{X}' \arrow[r, "h"] \arrow[d, "f'"'] & \mathrm{X} \arrow[d, "f"] \\ \mathrm{S}' \arrow[r, "g"] & \mathrm{S} \end{tikzcd},$$

where S *and* S′ *are noetherian local rings, with closed points* t *and* t' *respectively,* g *a faithfully flat morphism such that* $g^{-1}(t) = t'$, f *a locally of finite type morphism. Let* 175
$x' \in \mathrm{X}'$, $x = h(x')$, $s = f(x)$, $s' = f'(x')$ *; then we have*

$$\delta(x') \leqslant \delta(x).$$

Furthermore, the preceding inequality is an equality if and only if we have :

$$\operatorname{deg\,tr} k(x)/k(s) = \operatorname{deg\,tr} k(x')/k(s') \ \textit{and} \ \dim(\overline{\{s\}}) = \dim(\overline{\{s'\}}).$$

In particular, given $x \in \mathrm{X}$, *we can find* x' *such that we have* $\delta(x) = \delta(x')$.

We have indeed (EGA IV 6.11)

$$\dim(\overline{\{s\}}) = \dim g^{-1}(\overline{\{s\}}).$$

It follows that, for any point s' of $g^{-1}(s)$, we have the relation $\dim(\overline{\{s'\}}) \leqslant \dim(\overline{\{s\}})$, and that, given s, we can find $s' \in g^{-1}(s)$, such that we have equality. Let us then denote by Z the schematic closure of x in the fiber X_s of X at s, and let $\mathrm{Z}' =$

$\mathrm{Z} \times_{\mathrm{Spec}\, k(s)} \mathrm{Spec}\, k(s')$. Then, Z' is equidimensional of dimension $\deg \mathrm{tr}\, k(x)/k(s)$; we thus have, for any point $x' \in \mathrm{Z}'_x$,

$$\deg \mathrm{tr}\, k(x')/k(s') \leqslant \deg \mathrm{tr}\, k(x)/k(s), \text{ and we have equality}$$

when x' is a maximal point of Z'_x. Hence immediately the announced conclusion.

2.2. Let $f : \mathrm{X} \to \mathrm{S}$ be a locally of finite type morphism and T a closed subset of S. Let $x \in \mathrm{X}, s = f(x)$; we will set

$$\delta_{\mathrm{T}}(x) = \deg \mathrm{tr}\, k(x)/k(s) + \mathrm{codim}(\overline{\{s\}} \cap \mathrm{T}, \overline{\{s\}}) = \inf_{t \in \mathrm{T} \cap \overline{\{s\}}} \delta_t(x).$$

176 ***Lemma 2.2.1***. — *Let a cartesian square be given*

$$\begin{array}{ccc} \mathrm{X}' & \xrightarrow{h} & \mathrm{X} \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ \mathrm{S}' & \xrightarrow{g} & \mathrm{S} \end{array} \quad ,$$

where the schemes S *and* S' *are locally noetherian, catenary, the morphism f locally of finite type and g faithfully flat. Let* T *be a closed subset of* S, T' *a closed subset of* S', *such that* $g(\mathrm{T}') \subset \mathrm{T}$, x' *an element of* X', $x = h(x')$ *and*

$$h_{x'} : \mathrm{Spec}\, \mathscr{O}_{\mathrm{X}',x'} \longrightarrow \mathrm{Spec}\, \mathscr{O}_{\mathrm{X},x}$$

the morphism induced by h. Then we have :

$$\delta_{\mathrm{T}}(x) - \delta_{\mathrm{T}'}(x') \leqslant \dim h_{x'}^{-1}(x).$$

Let $s' = f'(x'), s = f(x)$. We have, by definition :

$$\begin{aligned} \delta_{\mathrm{T}}(x) - \delta_{\mathrm{T}'}(x') = {} & \deg \mathrm{tr}\, k(x)/k(s) - \deg \mathrm{tr}\, k(x')/k(s') \\ & + \mathrm{codim}(\overline{\{s\}} \cap \mathrm{T}, \overline{\{s\}}) - \mathrm{codim}(\overline{\{s'\}} \cap \mathrm{T}', \overline{\{s'\}}). \end{aligned}$$

Since g is faithfully flat, it follows from (EGA IV 6.1.4) that we have

$$(*) \qquad \begin{aligned} \mathrm{codim}(\overline{\{s\}} \cap \mathrm{T}, \overline{\{s\}}) &= \mathrm{codim}(g^{-1}(\overline{\{s\}}) \cap g^{-1}(\mathrm{T}), g^{-1}(\overline{\{s\}})) \\ &\leqslant \mathrm{codim}(g^{-1}(\overline{\{s\}}) \cap \mathrm{T}', g^{-1}(\overline{\{s\}})); \end{aligned}$$

as S' is catenary, we have, according to (EGA 0_{IV} 14.3.2 b)) :

$$\begin{aligned} \mathrm{codim}(\overline{\{s'\}} \cap \mathrm{T}', g^{-1}(\overline{\{s\}})) &= \mathrm{codim}(\overline{\{s'\}} \cap \mathrm{T}', \overline{\{s'\}}) + \mathrm{codim}(\overline{\{s'\}}, g^{-1}(\overline{\{s\}})) \\ &= \mathrm{codim}(\overline{\{s'\}} \cap \mathrm{T}', g^{-1}(\overline{\{s\}}) \cap \mathrm{T}') + \mathrm{codim}(g^{-1}(\overline{\{s\}}) \cap \mathrm{T}', g^{-1}(\overline{\{s\}})). \end{aligned}$$

177 We deduce from this relation and from $(*)$

$$\delta_{\mathrm{T}}(x) - \delta_{\mathrm{T}'}(x') \leqslant \deg \mathrm{tr}\, k(x)/k(s) - \deg \mathrm{tr}\, k(x')/k(s') + \mathrm{codim}(\overline{\{s'\}}, g^{-1}(\overline{\{s\}})).$$

Let us calculate $\mathrm{codim}(\overline{\{s'\}}, g^{-1}(\overline{\{s\}})) = \dim \mathscr{O}_{\mathrm{S}'_s, s'}$ (where S'_s is the fiber of S' at s). Let Z be the closed image of x in X_s and $\mathrm{Z}' \subset \mathrm{X}'_s$ the scheme defined by the cartesian square

$$\begin{tikzcd} Z' \arrow[r] \arrow[d] & Z \arrow[d] \\ S'_s \arrow[r] & \operatorname{Spec} k(s) \end{tikzcd} \quad .$$

The morphism $Z \to \operatorname{Spec} k(s)$ is flat, locally of finite type and we have $\dim Z = \deg\operatorname{tr} k(x)/k(s)$. It then follows from (EGA IV 6.1.2) that

$$\dim(\mathscr{O}_{Z',x'}) = \dim(\mathscr{O}_{S'_s,s'}) + \deg\operatorname{tr} k(x)/k(s) - \deg\operatorname{tr} k(x')/k(s');$$

taking into account the fact that $Z'_{s'} \simeq Z \otimes_{k(s)} k(s')$, we then obtain :

$$\delta_T(x) - \delta_{T'}(x') \leqslant \dim(\mathscr{O}_{Z',x'}).$$

Now $\operatorname{Spec}(\mathscr{O}_{Z',x'})$ is identified with the fiber at x of the morphism

$$(\operatorname{Spec}(\mathscr{O}_{X',x'}))_s \longrightarrow (\operatorname{Spec}(\mathscr{O}_{X,x}))_s,$$

so also with the fiber at x of $h_{x'}$, which proves the theorem.

2.3. The proofs of the theorems of section 4 are based on duality theory ; they use the following lemmas. Let m be an integer which is a power of a prime number ℓ ; 178
if X is a scheme, all the sheaves considered on X are sheaves of $\mathbf{Z}/m\mathbf{Z}$-modules ; we then have the notion of a dualizing complex on X (SGA 5 I 1.7). Suppose there exists such a complex K on X ; then, for each geometric point $\overline{x}$ over a point x of X, we deduce from K (cf. SGA 5 I 4.5) a dualizing complex $K_{\overline{x}}$ on $\operatorname{Spec} k(\overline{x})$, so that we have $K_{\overline{x}} \simeq \mathbf{Z}/m\mathbf{Z}[n]$ (the bracket denoting the translation functor) for a certain integer n depending only on x. We will set

$$\delta_K(x) = n.$$

If K is normalized at the point x (SGA 5 I 4.5), we therefore have $n = 0$.

Lemma 2.3.1. — *Let* X *be a locally noetherian scheme, endowed with a dualizing complex* K. *If x and x' are two points of* X*, such that x is a specialization of x' and we have* $\operatorname{codim}(\overline{\{x\}}, \overline{\{x'\}}) = 1$*, then we have*

$$\delta^K(x) = \delta^K(x') - 2.$$

We can first reduce to the case where X is a strictly local scheme. Let indeed $\overline{X}$ be the strict localization of X at a geometric point $\overline{x}$ over x, $i : \overline{X} \to X$ the canonical morphism, $\overline{x}'$ a geometric point of $\overline{X}$ over x'. Then i^*K is a dualizing complex on $\overline{X}$ and we have (SGA 5 I 4.5)

$$(i^*K)_{\overline{x}} \simeq K_{\overline{x}} \text{ and } (i^*K)_{\overline{x}'} \simeq K_{\overline{x}'},$$

which completes the reduction to the strictly local case. 179

If $j : \overline{\{x'\}} \to X$ denotes the immersion of the reduced closed subscheme of X, with underlying space $\overline{\{x'\}}$, then $R^! j(K)$ is a dualizing complex on $\overline{\{x'\}}$ and we see right away, using (SGA 5 I 4.5) that it suffices to prove the lemma for $\overline{\{x'\}}$. We are thus reduced to the case where X is an integral strictly local scheme of dimension 1.

Let then X′ be the normalization of X and $f : \mathrm{X}' \to \mathrm{X}$ the canonical morphism; f is an integral, surjective, radicial morphism, and it follows that $f^*\mathrm{K}$ is a dualizing complex on X′ and that it suffices to prove the lemma for X′ and for the points over x and x'. We are thus reduced to the case where X is a local, integral, regular scheme of dimension 1, but we know (cf. SGA 5 I 4.6.2 and 5.1) that then $\mu_m[2]$ and $\mathbf{Z}/m\mathbf{Z}$ are dualizing complexes, normalized respectively at the points x and x'; the lemma follows immediately.

Lemma 2.3.2. — *Let* S *be a noetherian local scheme,* $f : \mathrm{X} \to \mathrm{S}$ *a morphism of finite type. If* K *is a dualizing complex on* S*, normalized at the closed point* t *of* S *and if* $\mathrm{R}^! f(\mathrm{K}) = \mathrm{K}'$ *is a dualizing complex on* X (*cf.* SGA *5 I 3.4.3*)*, we have, for any point* x *of* X *:*

$$\delta^{\mathrm{K}'}(x) = 2\delta(x).$$

Indeed let $s = f(x)$ and x' a closed point of the fiber X_s; then we have $\delta^{\mathrm{K}'}(x') = \delta^{\mathrm{K}}(s)$ and according to 2.3.1

$$\delta^{\mathrm{K}}(s) = 2\,\mathrm{codim}(\overline{\{t\}}, \overline{\{s\}}) = 2\dim(\overline{\{s\}}).$$

180 As we can choose for x' a specialization of x, we have according to 2.3.1

$$\delta^{\mathrm{K}'}(x) = \delta^{\mathrm{K}'}(x') + 2\,\mathrm{codim}(\overline{\{x\}}, \overline{\{x'\}}) = \delta^{\mathrm{K}'}(x') + 2\deg\mathrm{tr}\, k(x)/k(s);$$

the lemma follows immediately.

The following lemma will only be used for the converse of the Lefschetz theorem, in section4 :

Lemma 2.3.3. — *Let a cartesian square be given*

$$\begin{array}{ccc} \mathrm{X}' & \xrightarrow{h} & \mathrm{X} \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ \mathrm{S}' & \xrightarrow{g} & \mathrm{S} \end{array},$$

where S *is an excellent strictly local scheme of characteristic zero,* S′ *the completion of* S *and* f *a morphism of finite type. Let* ℓ *be a prime number,* $x \in \mathrm{X}$, Z *the schematic closure of* X'_x *in* X′*, and* $i : \mathrm{X}'_x \to \mathrm{Z}$, $j : \mathrm{Z} \to \mathrm{X}$ *the canonical morphisms. Then, if* $k : \mathrm{X}' \to \mathrm{R}$ *is a closed immersion of* X′ *into an excellent regular scheme* R *of characteristic zero, the complex*

$$\mathrm{K}' = i^*(\mathrm{R}^!(k.j)(\mathbf{Z}/\ell\mathbf{Z}))$$

is a constant dualizing complex on X'_x (*that is, having only one non-zero cohomology sheaf, isomorphic to* $\mathbf{Z}/\ell\mathbf{Z}$).

Taking into account (SGA 5 I 3.4.3), the only thing to prove is that K′ is constant. Now, since Z is excellent, the set of points of Z whose local rings are regular is an 181
open set U (EGA IV 7.8.3 (iv)), and U obviously contains X'_x which is regular. Let then

$$u : \mathrm{U} \longrightarrow \mathrm{R}$$

be the canonical immersion of U into R ; it follows from the purity theorem (SGA 4 XIX 3.2 and 3.4) and the isomorphism

$$(\mu_l)_{\mathrm{S}} \simeq (\mathbf{Z}/\ell\mathbf{Z})_{\mathrm{S}}$$

(S strictly local) that we have

$$\mathrm{R}^! \, u(\mathbf{Z}/\ell\mathbf{Z}) \simeq \mathbf{Z}/\ell\mathbf{Z}[2c],$$

where c is a locally constant function on U, necessarily constant in a neighborhood of X'_x, because the fibers of g are geometrically integral according to (EGA IV 18.9.1) so X'_x is integral. The lemma follows immediately.

3. Converse of the affine Lefschetz theorem

The present section will be used in section 4 to prove a converse to the "Lefschetz theorem"; a reader who is only interested in the direct part of said theorem can therefore omit reading the present section.

3.1. Let us recall the statement of the affine Lefschetz theorem(5) (SGA 4 XIX 6.1 bis) :

Let S be an excellent strictly local scheme of characteristic zero, $f : \mathrm{X} \to \mathrm{S}$ an affine morphism of finite type and F a torsion sheaf on X. Then, if we set 182

$$\delta(\mathrm{F}) = \sup\{\delta(x) | x \in \mathrm{X} \text{ and } \mathrm{F}_{\overline{x}} \neq 0\},$$

we have

$$\mathrm{H}^q(\mathrm{X}, \mathrm{F}) = 0 \text{ for } q > \delta(\mathrm{F}).$$

Before stating the converse, let us prove some lemmas.

(5) N.D.E. : Gabber has proven the following generalization. Let Y be a strictly local scheme of arithmetic type over a regular noetherian scheme S of dimension $\leqslant 1$. Let $f : \mathrm{X} \to \mathrm{Y}$ be an affine morphism of finite type, $\Lambda = \mathbf{Z}/n\mathbf{Z}$ with n invertible on X and F a Λ-sheaf. Then, $\mathrm{H}^q(\mathrm{X}, \mathrm{F}) = 0$ if $q > \delta(\mathrm{F})$. We deduce the following local Lefschetz theorem. Let $\mathscr{O}$ be strictly local of arithmetic type over S. For any $f \in \mathscr{O}$ not a zero divisor and any Λ-sheaf F on $\mathrm{Spec}(\mathscr{O}[f^{-1}])$, we have $\mathrm{H}^q(\mathrm{Spec}(\mathscr{O}[f^{-1}]), \mathrm{F}) = 0$ for $q > \dim(\mathscr{O})$. Cf. (Fujiwara K., "A Proof of the Absolute Purity Conjecture (after Gabber)", in *Algebraic geometry 2000, Azumino* (*Hotaka*), Adv. Stud. in Pure Math., vol. 36, 2002, p. 153-183, § 5) and especially the article by Illusie (Illusie L., "Perversity and variation", *Manuscripta Math.* **112** (2003), p. 271-295). This result is one of the crucial points used by Gabber in his proof of Grothendieck's purity theorem (cf. note (1), page 172).

Lemma 3.2. — *Let* K *be a field,* ℓ *a prime number distinct from the characteristic of* K *and* F *a constructible, non-zero sheaf of* ℓ*-torsion on* K*. Suppose that the cohomological* ℓ*-dimension of* K (SGA 4 X 1) *is equal to* n (*this is realized for example if* K *is the field of fractions of an excellent, integral, strictly local ring, of characteristic zero, of dimension* n (SGA 4 XIX 6.3)*, or if* K *is a finitely generated extension of transcendence degree* n *of a separably closed field* (SGA 4 X 2.1))*. Then we can find a finite separable extension* L *of* K*, such that we have :*

$$\mathrm{H}^n(\mathrm{L}, \mathrm{F}_{|\mathrm{L}}) \neq 0.$$

We can find a finite extension K′ of K, such that the restrictions of F and of μ_ℓ to Spec K′ are constant sheaves. We then have $\mathrm{cd}_\ell(\mathrm{K}') = \mathrm{cd}_\ell(\mathrm{K}) = n$ (SGA 4 X 2.1), and it follows from ([**2**] II § 3 Prop. 4 (iii)) that we can find a finite extension L of K′
183 such that we have

$$\mathrm{H}^n(\mathrm{L}, \mu_\ell) \neq 0, \quad \text{i.e. } \mathrm{H}^n(\mathrm{L}, \mathbf{Z}/\ell\mathbf{Z}) \neq 0.$$

Now the functor $\mathrm{H}^n(\mathrm{L}, \cdot)$ is right exact on the category of sheaves of ℓ-torsion, since $\mathrm{cd}_\ell(\mathrm{L}) = n$; as F admits a quotient isomorphic to $\mathbf{Z}/\ell\mathbf{Z}$, we also have $\mathrm{H}^n(\mathrm{L}, \mathrm{F}_{|\mathrm{L}}) \neq 0$.

Corollary 3.3. — *Let* k *be a field,* K *a finitely generated extension of transcendence degree* n *of* k*,* F *a constructible, non-zero sheaf of* ℓ*-torsion on* K*, with* ℓ *prime to the characteristic of* k*. Then we can find a finite separable extension* L *of* K *such that, if* $u : \operatorname{Spec} \mathrm{L} \to \operatorname{Spec} k$ *denotes the canonical morphism, we have*

$$\mathrm{R}^n u_*(\mathrm{F}_{|\operatorname{Spec}\mathrm{L}}) \neq 0.$$

When the field k is separably closed, the corollary is a particular case of 3.2. In the general case, we can find a finite separable extension k_1 of k such that the irreducible components of $\mathrm{K} \otimes_k k_1$ are geometrically irreducible (EGA IV 4.5.11) ; let K_1 be one of them. If k' is a separable closure of k_1, then $\mathrm{K}' = \mathrm{K}_1 \otimes_{k_1} k'$ is a field, and we have according to (EGA IV 4.2)

$$\deg\operatorname{tr} \mathrm{K}'/k' = \deg\operatorname{tr} \mathrm{K}/k = n.$$

It then follows from 3.2 that we can find a finite separable extension L′ of K′ such that we have $\mathrm{H}^n(\mathrm{L}', \mathrm{F}_{|\mathrm{L}'}) \neq 0$. But we have $k' = \varinjlim_i k_i$, where k_i runs through the finite
184 extensions of k_1 contained in k', and consequently $\mathrm{K}' = \varinjlim_i (k_i \otimes_{k_1} \mathrm{K}_1)$. It follows that we can find an index i and a finite separable extension L of $k_i \otimes_{k_1} \mathrm{K}_1 = \mathrm{K}_i$, such that we have $\mathrm{L}' \simeq \mathrm{L} \otimes_{\mathrm{K}_i} \mathrm{K}'$. The extension L of K answers the question ; indeed it follows from the commutative diagram

$$\begin{tikzcd} & \operatorname{Spec}\mathrm{L} \arrow[dl, "v"'] \arrow[dr, "u"] & \\ \operatorname{Spec} k_i \arrow[rr, "w"] & & \operatorname{Spec} k\,, \end{tikzcd}$$

with w finite so $\mathrm{R}^q w_* = 0$ if $q > 0$, that we have

$$\mathrm{R}^n u_*(\mathrm{F}_{|\,\mathrm{Spec\,L}}) \simeq w_*(\mathrm{R}^n v_*(\mathrm{F}_{|\,\mathrm{Spec\,L}})).$$

Now $\mathrm{R}^n v_*(\mathrm{F}_{|\,\mathrm{Spec\,L}}) \neq 0$, since $\mathrm{H}^n(\mathrm{L}', \mathrm{F}_{|\mathrm{L}'}) \neq 0$; we thus also have $\mathrm{R}^n u_*(\mathrm{F}_{|\,\mathrm{Spec\,L}}) \neq 0$.

Let us recall the following known lemma (cf. EGA 0_{III} 10.3.1.2 and EGA IV 18.2.3) :

Lemma 3.4. — *Let* X *be a scheme,* x *a point of* X, K *a finite separable extension of* $k(x)$*. Then there exists an affine scheme* X_1 *étale over* X*, and a point* $x_1 \in \mathrm{X}_1$ *over* x*, such that* $k(x_1)$ *is* $k(x)$*-isomorphic to* K.

We will use in section 4 the following technical form of the converse of 3.1.

Proposition 3.5. — *Let a cartesian square be given* 185

$$\begin{tikzcd} \mathrm{X}' \arrow[r, "h"] \arrow[d, "f'"'] & \mathrm{X} \arrow[d, "f"] \\ \mathrm{S}' \arrow[r, "g"] & \mathrm{S}, \end{tikzcd}$$

where the schemes S *and* S′ *are excellent strictly local schemes of characteristic zero, the morphism* f *locally of finite type,* g *regular* (EGA IV 6.8.1) *surjective, the closed fiber of* g *reduced to the closed point of* S′*. Given an* S*-scheme* X_1 (*resp. an* S*-morphism* f_1*, etc.*)*, we will denote by* X'_1 (*resp.* f'_1*, etc.*) *the scheme* $\mathrm{X}_1 \times_{\mathrm{S}} \mathrm{S}'$ (*resp. the morphism* $(f_1)_{(\mathrm{S}')}$*, etc.*)*. Let* F *be a sheaf of* $\mathbf{Z}/m\mathbf{Z}$*-modules on* X′ (m *a power of a prime number* ℓ)*, constructible, satisfying the following conditions :*

(i) *For any point* $x \in \mathrm{X}$*, we can find a finite separable extension* K *of* $k(x)$ *such that the restriction of* F *to the fiber* $(\mathrm{X}')_{(\mathrm{Spec\,K})}$ *comes by inverse image from a constructible sheaf on* Spec K.

(ii) *For any morphism* $f_1 : \mathrm{X}_1 \to \mathrm{S}$*, with* X_1 *affine and étale over* X*, for any point* $s \in \mathrm{S}$ *and for any integer* $q > 0$*, we can find a finite separable extension* K *of* $k(s)$ *such that the restriction of* $\mathrm{R}^q f'_{1*}(\mathrm{F}|\mathrm{X}'_1)$ *to the fiber* $\mathrm{S}'_{(\mathrm{Spec\,K})}$ *comes by inverse image from a constructible sheaf on* Spec K.

Let n *be an integer, and suppose that for any affine scheme* X_1 *étale over* X*, we have* 186

$$\mathrm{H}^i(\mathrm{X}'_1, \mathrm{F}) = 0 \text{ for } i > n.$$

Then, if $\overline{x}'$ *is a geometric point over the point* $x' \in \mathrm{X}'$*, such that* $\mathrm{F}_{\overline{x}'} \neq 0$*, we have*

$$\delta(x') \leqslant n.$$

Let Z′ be the set of points x' of X′ such that we have $\mathrm{F}_{\overline{x}'} = 0$. Then, if $\mathrm{Z} = h(\mathrm{Z}')$, we have according to (i) $\mathrm{Z}' = h^{-1}(\mathrm{Z})$; let $x' \in \mathrm{X}'$, $x = h(x')$, $s' = f'(x')$, $s = f(x)$. It follows from 2.1.1 and from the fact that the function δ decreases by specialization, that it suffices to prove the inequality $\delta(x') \leqslant n$ when x is a maximal point of Z and

x' is such that we have

$$r = \deg \operatorname{tr} k(x)/k(s) = \deg \operatorname{tr} k(x')/k(s') \text{ and } d = \dim \overline{\{s\}} = \dim \overline{\{s'\}}$$

Let x' be such a point ; it suffices to show that we can find an affine scheme X_1 étale over X, such that we have

$$\mathrm{H}^{d+r}(\mathrm{X}'_1, \mathrm{F}) \neq 0.$$

The set Z' is constructible (SGA 4 IX 2.4), so is Z (EGA IV 1.9.12) ; we can then suppose, up to restricting X to a neighborhood of x, that Z is an irreducible closed subset with generic point x. Let $\mathrm{T} = f(\mathrm{Z})$; T is a constructible set contained in $\overline{\{s\}}$; we can therefore find an affine open set U of S, such that $s \in \mathrm{U}$ and that $\mathrm{T} \cap \mathrm{U} = \mathrm{T}_{\mathrm{U}}$ is an irreducible closed subset of U with generic point s.

187 Let then V be an affine scheme étale over X, whose image in X contains x and whose image in S is contained in U ; let Z_{V} be the inverse image of Z in V and $u : \mathrm{Z}_{\mathrm{V}} \to \mathrm{T}_{\mathrm{U}}$ the canonical morphism. Let W be an affine scheme étale over U, we then denote by T_{W} the inverse image of T_{U} in W and let $\mathrm{X}_1 = \mathrm{W} \times_{\mathrm{U}} \mathrm{V}$. Since F is zero outside of Z', we have the spectral sequence

$$\mathrm{E}_2^{pq} = \mathrm{H}^p((\mathrm{T}_{\mathrm{W}})', \mathrm{R}^q u'_*(\mathrm{F}_{|(\mathrm{Z}_{\mathrm{V}})'})) \Longrightarrow \mathrm{H}^*(\mathrm{X}'_1, \mathrm{F}).$$

We will show that we can choose V and W such that we have

a) $\mathrm{E}_2^{pq} = 0$ for $p > d$ and for $q > r$.

b) $\mathrm{E}_2^{dr} \neq 0$.

It will then follow from the spectral sequence that we have $\mathrm{H}^{d+r}(\mathrm{X}'_1, \mathrm{F}) \neq 0$.

1°) Let $\mathrm{G}_q = \mathrm{R}^q u'_*(\mathrm{F}_{|(\mathrm{Z}_{\mathrm{V}})'})$; then we have :

$$(\mathrm{G}_q)_{\overline{s}'} = \mathrm{H}^q((\mathrm{Z}_{\mathrm{V}})'_{\overline{s}'}, \mathrm{F}_{|(\mathrm{Z}_{\mathrm{V}})'_{\overline{s}'}}),$$

because s' is a maximal point of $(\mathrm{T}_{\mathrm{U}})'$. As the fiber $(\mathrm{Z}_{\mathrm{V}})'_{\overline{s}'}$ is an affine scheme of finite type of dimension r over a separably closed field, it follows from 3.1 that we have

$$(\mathrm{G}_q)_{\overline{s}'} = 0 \text{ for } q > r.$$

188 For $q > r$, let Y'_q be the set of points of $(\mathrm{T}_{\mathrm{U}})'$ where the geometric fiber of G_q is $\neq 0$ and $\mathrm{Y}_q = g(\mathrm{Y}'_q)$; then we have $\mathrm{Y}'_q = g^{-1}(\mathrm{Y}_q)$ according to (ii), so Y_q is a constructible subset of T_{U} (SGA 4 XIX 5.1 and EGA 1.9.12) which does not contain s ; up to restricting U to an open neighborhood of s, we can suppose that we have $\mathrm{G}_q = 0$ for $q > r$, hence $\mathrm{E}_2^{pq} = 0$ for $q > r$.

On the other hand, since $(\mathrm{T}_{\mathrm{W}})'$ is an affine scheme of finite type over $g^{-1}(\overline{\{s\}})$, we have for any q (cf. 3.1) :

$$\mathrm{H}^p((\mathrm{T}_{\mathrm{W}})', \mathrm{G}_q) = 0 \text{ for } p > \dim g^{-1}(\overline{\{s\}}) = d,$$

hence condition a).

2°) Let us show that we can choose V such that we have $(\mathrm{G}_r)_{\overline{s}'} \neq 0$. According to (i), there exists a constructible sheaf I, defined on a finite separable extension K of $k(x)$, whose inverse image on $(\mathrm{X}')_{(\operatorname{Spec} \mathrm{K})}$ is isomorphic to $\mathrm{F}_{|(\mathrm{X}')_{(\operatorname{Spec} \mathrm{K})}}$. According to

3.3, we find a finite separable extension L of K such that, if $v : \operatorname{Spec} \mathrm{L} \to \operatorname{Spec} k(s)$ denotes the canonical morphism, we have $\mathrm{R}^r v_*(\mathrm{I}) \neq 0$. As the morphism $\mathrm{S}'_s \to \operatorname{Spec} k(s)$ is regular, we have according to (SGA 4 XIX 4.2) :

$$\mathrm{R}^r v'_*(\mathrm{F}_{|(\operatorname{Spec}\mathrm{L})'}) \simeq (\mathrm{R}^r v_*(\mathrm{I}))' \neq 0.$$

According to lemma 3.4, we can find an affine scheme X_2 étale over X, and a point x_2 of X_2 over x, such that L is $k(x)$-isomorphic to $k(x_2)$ and we can suppose X_2 to be over U. Since x is a maximal point of Z, we have 189

$$\operatorname{Spec} \mathrm{L} \simeq \varprojlim_{\mathrm{V}} \mathrm{Z}_{\mathrm{V}},$$

where V runs through the affine open neighborhoods of x_2. We deduce by passing to the limit (SGA 4 VII 5.8), after restriction to the geometric fiber at $\overline{s}'$:

$$(\mathrm{R}^r v'_*(\mathrm{F}_{|\operatorname{Spec}\mathrm{I}})')_{\overline{s}'} = \varinjlim_{\mathrm{V}} (\mathrm{R}^r u'_*(\mathrm{F}_{|(\mathrm{Z}_{\mathrm{V}})'})_{\overline{s}'},$$

which shows that we can indeed find V such that we have $(\mathrm{G}_r)_{\overline{s}'} \neq 0$.

3°) The scheme V having been chosen in 2°), let us show that we can choose the scheme W such that we have

$$\mathrm{E}_2^{dr} = \mathrm{H}^d((\mathrm{T}_{\mathrm{W}})', \mathrm{G}_r) \neq 0.$$

According to (ii), there exists a constructible sheaf J, defined on a finite separable extension K of $k(s)$, whose inverse image on $(\mathrm{S}')_{(\operatorname{Spec}\mathrm{K})}$ is isomorphic to $\mathrm{G}_{r|(\mathrm{S}')_{(\operatorname{Spec}\mathrm{K})}}$. According to lemma 3.2, we can find a finite separable extension L of K, such that we have $\mathrm{H}^d(\operatorname{Spec}\mathrm{L}, \mathrm{J}) \neq 0$. As the morphism $(\mathrm{S}')_{(\operatorname{Spec}\mathrm{L})} \to \operatorname{Spec}\mathrm{L}$ is acyclic (SGA 4 XIX 4.1 and XV 1.10 and 1.16), we have

$$\mathrm{H}^d((\operatorname{Spec}\mathrm{L})', \mathrm{G}_{r|(\mathrm{S}')_{(\operatorname{Spec}\mathrm{L})'}}) = \mathrm{H}^d(\operatorname{Spec}\mathrm{L}, \mathrm{J}) \neq 0.$$

According to 3.4, we can find an affine scheme U_1 étale over U, and a point s_1 over s, such that $k(s_1)$ is $k(s)$-isomorphic to L. Now, since s is a maximal point of T_{U}, we have

$$\operatorname{Spec}\mathrm{L} \simeq \varprojlim_{\mathrm{W}} \mathrm{T}_{\mathrm{W}},$$

where W runs through the affine open neighborhoods of s_1. We deduce that $(\operatorname{Spec}\mathrm{L})' \simeq \varprojlim_{\mathrm{W}} (\mathrm{T}_{\mathrm{W}})'$, and by passing to the limit (SGA 4 VII 5.8) : 190

$$\mathrm{H}^d((\operatorname{Spec}\mathrm{L})', \mathrm{G}_{r|(\operatorname{Spec}\mathrm{L})'}) \simeq \varinjlim_{\mathrm{W}} \mathrm{H}^d((\mathrm{T}_{\mathrm{W}})', \mathrm{G}_{r|(\mathrm{T}_{\mathrm{W}})'});$$

consequently we can find W such that we have

$$\mathrm{H}^d((\mathrm{T}_{\mathrm{W}})', \mathrm{G}_{r|(\mathrm{T}_{\mathrm{W}})'}) \neq 0,$$

which completes the proof of the theorem.

Corollary 3.6. — *The hypotheses concerning* $\mathrm{S}, \mathrm{S}', f, f', m$ *are those of* 3.5. *Let us now denote by* F *a complex of sheaves of* $\mathbf{Z}/m\mathbf{Z}$*-modules on* X'*, with degrees bounded* *below and constructible cohomology, and whose cohomology sheaves satisfy conditions* (i) *and* (ii) *of* 3.5. *Let n be an integer, and suppose that, for any affine scheme* X_1 *étale over* X, *we have*

$$\mathrm{H}^i(\mathrm{X}'_1, \mathrm{F}) = 0 \text{ for } i > n.$$

Then, if $\overline{x}'$ is a geometric point over a point x' of X′, *such that we have, for an integer j, $(\underline{\mathrm{H}}^j(\mathrm{F}))_{\overline{x}'} \neq 0$, we have*

$$\delta(x') \leqslant n - j.$$

Let T′ be the set of points of X′ where the conclusion of 3.7 is at fault and suppose $\mathrm{T}' \neq \emptyset$; let $\mathrm{T} = f(\mathrm{T}')$, x a maximal point of T and x' a point of X′ over x. Let j be
191 the largest integer such that we have $(\underline{\mathrm{H}}^j(\mathrm{F}))_{\overline{x}'} \neq 0$; we thus have $r = \delta(x) > n - j$. Let Z'_q be the set of points where the geometric fiber of $\underline{\mathrm{H}}^q(\mathrm{F})$ is $= 0$ and $\mathrm{Z}_q = h(\mathrm{Z}'_q)$; we see as in the proof of 3.5 that Z_q is constructible. We obviously have $\mathrm{Z}'_q = \emptyset$ for $q > n$ and for q small enough. The other values of q are divided into three subsets. Let

$$\mathrm{Q}_1 = \{q \mid x \in \mathrm{Z}_q \text{ and a generalization of } x, \text{ distinct from } x, \notin \mathrm{Z}_q\}.$$

We have $j \in \mathrm{Q}_1$ and we can find an affine open neighborhood U_1 of x, such that, for any $q \in \mathrm{Q}_1, \mathrm{U}_1 \cap \mathrm{Z}_q$ is an irreducible closed subset with generic point x. If $q \in \mathrm{Q}_1$, we have

$$(*) \qquad \delta(\underline{\mathrm{H}}^q(\mathrm{F})_{|\mathrm{U}'_1}) = \delta(x) \quad \text{(for the definition of } \delta(\underline{\mathrm{H}}^q(\mathrm{F})) \text{ cf. 3.1).}$$

Let

$$\mathrm{Q}_2 = \{q \mid \text{no generalization of } x \text{ belongs to } \mathrm{Z}_q\}.$$

Then, if $j < q \leqslant n$, we have $q \in \mathrm{Q}_2$, and we can find an affine open neighborhood U_2 of x, such that, for any $q \in \mathrm{Q}_2$, we have $\mathrm{Z}_q \cap \mathrm{U}_2 = \emptyset$; we thus have

$$(**) \qquad \underline{\mathrm{H}}^q(\mathrm{F})_{|\mathrm{U}'_2} = 0 \text{ for } q \in \mathrm{Q}_2.$$

Finally, let

$$\mathrm{Q}_3 = \{q \mid \mathrm{Z}_q \text{ contains strict generalizations of } x\}.$$

Then we can find an affine open neighborhood of x, U_3, such that, for any $q \in \mathrm{Q}_3$, all the maximal points of $\mathrm{Z}_q \cap \mathrm{U}_3$ are generalizations of x. If $q \in \mathrm{Q}_3$, we have

$$(***) \qquad \delta(\underline{\mathrm{H}}^q(\mathrm{F})_{|\mathrm{U}'_3}) \leqslant n - q.$$

192 For any affine scheme X_1 étale over $\mathrm{U}_1 \cap \mathrm{U}_2 \cap \mathrm{U}_3$, let us consider the hypercohomology spectral sequence

$$\mathrm{E}_2^{pq} = \mathrm{H}^p(\mathrm{X}'_1, \underline{\mathrm{H}}^q(\mathrm{F})) \Longrightarrow \mathrm{H}^*(\mathrm{X}'_1, \mathrm{F}).$$

We have $\mathrm{E}_2^{pq} = 0$ for $q \in \mathrm{Q}_2$ according to $(**)$. We have $\mathrm{E}_2^{pq} = 0$ for $p + q \geqslant r + j$ except perhaps for $p = r, q = j$. Indeed this is clear if $q \in \mathrm{Q}_2$; if $q \in \mathrm{Q}_1$, we then have $p > r$ unless $p = r, q = j$ and this follows from 3.1 taking into account $(*)$; finally if $q \in \mathrm{Q}_3$, since $r > n - j$, we have $p > n - q$ and the assertion follows from 3.1 taking into account $(***)$. Since $\mathrm{H}^{r+j}(\mathrm{X}'_1, \mathrm{F}) = 0$, it follows from the spectral sequence that we have

$$\mathrm{H}^r(\mathrm{X}'_1, \underline{\mathrm{H}}^j(\mathrm{F})) = 0;$$

now this implies, according to 3.5, $\delta(x) < r$, which is absurd.

Corollary 3.7. — *Let* S *be an excellent strictly local scheme of characteristic zero,* f : $\mathrm{X} \to \mathrm{S}$ *a locally of finite type morphism,* m *a power of a prime number,* F *a complex of sheaves of* $\mathbf{Z}/m\mathbf{Z}$*-modules on* X*, bounded below, with constructible cohomology, and* n *an integer. Then the following conditions are equivalent :*

(i) *For any affine scheme* X_1 *étale over* X*, we have*

$$\mathrm{H}^i(\mathrm{X}_1, \mathrm{F}) = 0 \text{ for } i > n.$$

(ii) *For any geometric point* $\overline{x}$ *over the point* x *of* X*, and for any integer* j *such* 193
that we have $(\underline{\mathrm{H}}^j(\mathrm{F}))_{\overline{x}} \neq 0$*, we have*

$$\delta(x) \leqslant n - j.$$

(i) $\Rightarrow$ (ii) is the particular case of 3.6 obtained by setting $\mathrm{S} = \mathrm{S}'$.

(ii) $\Rightarrow$ (i) follows immediately from 3.1, using the hypercohomology spectral sequence

$$\mathrm{H}^p(\mathrm{X}_1, \underline{\mathrm{H}}^q(\mathrm{F})) \Longrightarrow \mathrm{H}^*(\mathrm{X}_1, \mathrm{F}).$$

4. Main theorem and variants

4.0. Let $g \colon \mathrm{X} \to \mathrm{S}$ be a separated morphism of finite type, T a closed subset of S, $\mathrm{Z} = g^{-1}(\mathrm{T})$ and F a complex of abelian sheaves on X, with degrees bounded below. We call i*-th cohomology group of* F*, with proper support, with support in* Z the group

$$\mathrm{H}^i_{\mathrm{Z}!}(\mathrm{X}/\mathrm{S}, \mathrm{F}) = \mathrm{H}^i_{\mathrm{T}}(\mathrm{S}, \mathrm{R}_! \, g(\mathrm{F})),$$

where $\mathrm{R}_! \, g$ denotes "the direct image with proper support" (SGA 4 XVII). In the particular case where g is proper, we simply have

$$\mathrm{H}^i_{\mathrm{Z}!}(\mathrm{X}/\mathrm{S}, \mathrm{F}) = \mathrm{H}^i_{\mathrm{Z}}(\mathrm{X}, \mathrm{F}).$$

Proposition 4.1. — *Let* $f \colon \mathrm{U} \to \mathrm{S}$ *be a morphism of finite type,* F *a complex of abelian sheaves on* U*, with degrees bounded below. Suppose that we have a factorization of* f : 194

$$\begin{tikzcd} \mathrm{U} \arrow[rr, "i"] \arrow[dr, "f"'] & & \mathrm{X} \arrow[dl, "g"] \\ & \mathrm{S} & \end{tikzcd}$$

where i *is an open immersion and* g *a separated morphism of finite type, and let* G *be a complex of abelian sheaves on* X*, with degrees bounded below, which extends* F*. Let* Y *be a closed subscheme of* X *with underlying space* $\mathrm{X} - \mathrm{U}$*, so that we have a* *commutative diagram :*

$$\begin{tikzcd} \mathrm{Y} \arrow[rr, "j"] \arrow[dr, "h"'] & & \mathrm{X} \arrow[dl, "g"] \\ & \mathrm{S} & \end{tikzcd}$$

Finally, let n be an integer and T a closed subset of S. Then the following conditions are equivalent :

(i) *We have* $\mathrm{prof}_{\mathrm{T}}(\mathrm{R}_!\, f(\mathrm{F})) \geqslant n$.

(ii) *The canonical morphism*

$$\underline{\mathrm{H}}^i_{\mathrm{T}}(\mathrm{R}_!\, g(\mathrm{G})) \longrightarrow \underline{\mathrm{H}}^i_{\mathrm{T}}(\mathrm{R}_!\, h(j^*\mathrm{G}))$$

is bijective for $i < n-1$, injective for $i = n-1$.

195 (iii) *For any scheme S' étale over S, if we denote by X' (resp. f', resp. etc.) the scheme $\mathrm{X} \times_{\mathrm{S}} \mathrm{S}'$ (resp. the morphism $f_{(\mathrm{S}')}$, resp. etc.), the canonical morphism*

$$\mathrm{H}^i_{g'^{-1}(\mathrm{T}')!}(\mathrm{X}'/\mathrm{S}', \mathrm{G}') \longrightarrow \mathrm{H}^i_{h'^{-1}(\mathrm{T}')!}(\mathrm{Y}'/\mathrm{S}', j'^*\mathrm{G}')$$

is bijective for $i < n-1$, injective for $i = n-1$.

Let us consider in the derived category $\mathrm{D}^+(\mathrm{X})$ (cf. [**3**]) the distinguished triangle

$$\begin{tikzcd} & j_*j^*\mathrm{G} \arrow[dl] & \\ i_!\mathrm{F} \arrow[rr] & & \mathrm{G}. \arrow[ul] \end{tikzcd}$$

By applying the functor $\mathrm{R}_!\, g$ to this triangle, we obtain the triangle

$$(*) \qquad \begin{tikzcd} & \mathrm{R}_!\, h(j^*\mathrm{G}) \arrow[dl] & \\ \mathrm{R}_!\, f(\mathrm{F}) \arrow[rr] & & \mathrm{R}_!\, g(\mathrm{G}). \arrow[ul] \end{tikzcd}$$

Let us show that (i) ⇔ (ii). Indeed, according to definition 1.2, (i) is equivalent to the relation

$$\underline{\mathrm{H}}^i_{\mathrm{T}}(\mathrm{R}_!\, f(\mathrm{F})) = 0 \quad \text{for } i < n\,;$$

now we deduce from $(*)$ the exact sequence of sheaves

$$\longrightarrow \underline{\mathrm{H}}^i_{\mathrm{T}}(\mathrm{R}_!\, f(\mathrm{F})) \longrightarrow \underline{\mathrm{H}}^i_{\mathrm{T}}(\mathrm{R}_!\, g(\mathrm{G})) \longrightarrow \underline{\mathrm{H}}^i_{\mathrm{T}}(\mathrm{R}_!\, h(j^*\mathrm{G})) \longrightarrow,$$

196 whence the equivalence of (i) and (ii).

(i) ⇔ (iii). Indeed (i) is equivalent to saying that, for any scheme S' étale over S, we have the relation

$$(**) \qquad \mathrm{H}^i_{\mathrm{T}'}(\mathrm{S}', \mathrm{R}_!\, f'(\mathrm{F}')) = 0 \quad \text{for } i < n.$$

Now we deduce from $(*)$ the exact sequence of abelian groups

$$\longrightarrow \mathrm{H}^i_{\mathrm{T}'}(\mathrm{S}', \mathrm{R}_! f'(\mathrm{F}')) \longrightarrow \mathrm{H}^i_{\mathrm{T}'}(\mathrm{S}', \mathrm{R}_! g'(\mathrm{G}')) \longrightarrow \mathrm{H}^i_{\mathrm{T}'}(\mathrm{S}', \mathrm{R}_! h'(j'^*\mathrm{G}')) \longrightarrow;$$

taking into account 4.0, this exact sequence is written in the form

$$\longrightarrow \mathrm{H}^i_{\mathrm{T}'}(\mathrm{S}', \mathrm{R}_! f'(\mathrm{F}')) \longrightarrow \mathrm{H}^i_{g'^{-1}(\mathrm{T}')!}(\mathrm{X}'/\mathrm{S}', \mathrm{G}') \longrightarrow \mathrm{H}^i_{h'^{-1}(\mathrm{T}')!}(\mathrm{Y}'/\mathrm{S}', j'^*\mathrm{G}') \longrightarrow .$$

The equivalence of (i) and (iii) follows from this, taking into account the form $(**)$ of (i).

4.2.0. When $f\colon \mathrm{U} \to \mathrm{S}$ is affine, we will give *local* conditions on F for the conditions (i) to (iii) of 4.1 to be verified. In what follows, the considered schemes are excellent schemes of characteristic zero, the sheaves are sheaves of $\mathbf{Z}/m\mathbf{Z}$-modules, where m is a power of a prime number. If we had at our disposal the resolution of singularities in the sense of (SGA 4 XIX), the stated results, as well as their proofs, would still be valid for excellent schemes of equal characteristic, with m prime to the characteristic.

Theorem 4.2. — *Let* S *be an excellent scheme of characteristic zero and* $f\colon \mathrm{U} \to \mathrm{S}$ *a separated morphism of finite type. Let* F *be a complex of sheaves of* $\mathbf{Z}/m\mathbf{Z}$*-modules on* U*, with degrees bounded below and constructible cohomology,* n *an integer and* T *a closed subset of* S*. Then the following conditions are equivalent :* **197**

(i) *For any scheme* U_1 *étale over* U*, affine over* S*, we have, denoting by* f_1 *the structural morphism of* U_1 *and by* F_1 *the restriction of* F *to* U_1 *:*

$$\operatorname{prof}_{\mathrm{T}}(\mathrm{R}_! f_1(\mathrm{F}_1)) \geqslant n$$

(*cf. prop.* 4.1 *on the meaning of this relation*).

(ii) *For any point* u *of* U*, we have :*

$$\operatorname{prof}_u(\mathrm{F}) \geqslant n - \delta_{\mathrm{T}}(u),$$

where we set (*cf.* 2.2) *:* $\delta_{\mathrm{T}}(u) = \deg\operatorname{tr}(k(x)/k(s)) + \operatorname{codim}\left(\overline{\{s\}} \cap \mathrm{T}, \overline{\{s\}}\right)$.

Démonstration. — 1°) Let t be a point of T, $\overline{\mathrm{S}}$ the strict localization of S at a geometric point over t and S′ the completion of $\overline{\mathrm{S}}$, with closed point t' ; then $\overline{\mathrm{S}}$ is excellent according to (EGA IV 7.9.5), so S′ is a complete excellent strictly local scheme. Given a scheme U over X (resp. an S-morphism f, resp. etc.), we will denote by U′ (resp. f', resp. etc.) the scheme $\mathrm{U} \times_{\mathrm{S}} \mathrm{S}'$ (resp. the morphism $f_{(\mathrm{S}')}$, resp. etc.). We have the cartesian square

$$\begin{tikzcd} \mathrm{U}' \arrow[r, "h"] \arrow[d, "f'"'] & \mathrm{U} \arrow[d, "f"] \\ \mathrm{S}' \arrow[r, "g"] & \mathrm{S}, \end{tikzcd}$$

in which the morphism g is regular (EGA IV 7.8.2). Let us show that it suffices to prove that (for any point $t \in \mathrm{T}$) the following two properties are equivalent : **198**

$(\mathrm{i})_t$ For any scheme U_1 étale over U, affine over S, letting $f_1 \colon \mathrm{U}_1 \to \mathrm{S}$, we have

$$\operatorname{prof}_{t'}(\mathrm{R}_! f'_1(\mathrm{F}'_1)) \geqslant n.$$

$(\mathrm{ii})_t$ For any point u' of U', we have

$$\operatorname{prof}_{u'}(\mathrm{F}') \geqslant n - \delta_{t'}(u').$$

It suffices to prove the following lemma :

Lemma 4.2.1. — *We have* (i) $\Leftrightarrow$ $(\mathrm{i})_t$ *for all* $t \in \mathrm{T}$ *and* (ii) $\Leftrightarrow$ $(\mathrm{ii})_t$ *for all* $t \in \mathrm{T}$.

(i) $\Leftrightarrow$ $(\mathrm{i})_t$ for all $t \in \mathrm{T}$. Indeed (i) is equivalent to saying that, for any scheme U_1 étale over U, affine over S, we have

$$\operatorname{prof}_{\mathrm{T}}(\mathrm{R}_! f_1(\mathrm{F}_1)) \geqslant n \,;$$

now according to 1.8

$$\operatorname{prof}_{\mathrm{T}}(\mathrm{R}_! f_1(\mathrm{F}_1)) = \inf_{t \in \mathrm{T}} \operatorname{prof}_t(\mathrm{R}_! f_1(\mathrm{F}_1)).$$

Since $g^*(\mathrm{R}_! f_1(\mathrm{F}_1)) \simeq \mathrm{R}_! f'_1(\mathrm{F}'_1)$ (SGA 4 XVII), we have according to 1.16

$$\operatorname{prof}_t(\mathrm{R}_! f_1(\mathrm{F}_1)) = \operatorname{prof}_{t'}(\mathrm{R}_! f'_1(\mathrm{F}'_1)),$$

199 so (i) is equivalent to saying that we have, for all $t \in \mathrm{T}$, $\operatorname{prof}_{t'}(\mathrm{R}_! f'_1(\mathrm{F}'_1)) \geqslant n$, which is none other than $(\mathrm{i})_t$.

$(\mathrm{ii})_t$ for all $t \in \mathrm{T} \Rightarrow$ (ii). Indeed let $u \in \mathrm{U}$; we must show the relation

$$\operatorname{prof}_u(\mathrm{F}) \geqslant n - \delta_{\mathrm{T}}(u),$$

where $\delta_{\mathrm{T}}(u) = \inf_{t \in \mathrm{T} \cap \overline{\{s\}}} \delta_t(u)$ (cf. 2.2) ; we are thus reduced to showing that we have, for all $t \in \mathrm{T} \cap \overline{\{s\}}$

$$\operatorname{prof}_u(\mathrm{F}) \geqslant n - \delta_t(u).$$

Let u' be a point of U' such that we have $h(u') = u$ and $\delta_{t'}(u') = \delta_t(u)$ (cf. 2.1.1). Since h is locally acyclic (SGA 4 XIX 4.1), it follows from 1.16 and from the fact that u' is a generic point of U'_u that we have

$$\operatorname{prof}_{u'}(\mathrm{F}') = \operatorname{prof}_u(\mathrm{F}).$$

But we have according to $(\mathrm{ii})_t$ $\operatorname{prof}_u(\mathrm{F}) = \operatorname{prof}_{u'}(\mathrm{F}') \geqslant n - \delta_t(u)$, which proves (ii).

(ii) $\Rightarrow$ $(\mathrm{ii})_t$ for all t. With the notation of 2.2.1, for any point u' of U', we have thanks to 1.16

$$\operatorname{prof}_{u'}(\mathrm{F}') \geqslant \operatorname{prof}_u(\mathrm{F}) + 2 \dim h_{u'}^{-1}(u) \geqslant \operatorname{prof}_u(\mathrm{F}) + \dim h_{u'}^{-1}(u).$$

Taking into account 2.2.1 and (ii), we obtain

$$\operatorname{prof}_{u'}(\mathrm{F}') \geqslant n - \delta_{\mathrm{T}}(u) + \dim h_{u'}^{-1}(u) \geqslant n - \delta_{t'}(u'),$$

200 which is none other than $(\mathrm{ii})_t$.

2°) $(\mathrm{ii})_t \Leftrightarrow (\mathrm{i})_t$. We immediately reduce to the case where F has bounded degrees, by truncating F at a sufficiently high rank. We can realize S' as a closed subscheme of a complete, regular, hence excellent local scheme; it then follows from (SGA 5 I 3.4.3) that there exists a dualizing complex K on S′ and that $\mathrm{R}^! f'(\mathrm{K}) = \mathrm{K}'$ is a dualizing complex on U′. We will choose K such that we have $\delta^{\mathrm{K}}(t') = 0$ (for the definition of $\delta^{\mathrm{K}}(t')$, cf. 2.3), and will denote by $\mathrm{D}\,\mathrm{F}'$ the dual of F′ with respect to K′. We can reformulate the hypothesis $(\mathrm{ii})_t$ as follows :

Lemma 4.2.2. — *Let u' be a point of* U′ *; then the following conditions are equivalent :*

(i) *We have* $\mathrm{prof}_{u'}(\mathrm{F}') \geqslant n - \delta_{t'}(u')$.

(ii) *We have* $\big(\underline{\mathrm{H}}^q(\mathrm{D}\,\mathrm{F}')\big)_{\overline{u}'} = 0$ *for* $q > -n - \delta_{t'}(u')$ ($\overline{u}'$ *geometric point over* u').

Let $\overline{\mathrm{U}}'$ be the strict localization of U′ at $\overline{u}'$ and $\overline{\mathrm{F}}'$ the inverse image of F by the morphism $\overline{\mathrm{U}}' \to \mathrm{U}'$. The relation $\mathrm{prof}_{u'}(\mathrm{F}') \geqslant n - \delta_{t'}(u')$ is equivalent by definition to the following :

$$(*) \qquad \mathrm{H}^i_{\overline{u}'}(\overline{\mathrm{F}}') = 0 \quad \text{for } i > n - \delta_{t'}(u').$$

Let $\mathrm{D}\big(\mathrm{H}^i_{\overline{u}'}(\overline{\mathrm{F}}')\big)$ be the dual of the abelian group $\mathrm{H}^i_{\overline{u}'}(\overline{\mathrm{F}}')$ with respect to $\mathbf{Z}/m\mathbf{Z}$. According to 2.3.2, $\mathrm{K}'[-2\delta_{t'}(u')] = \mathrm{K}''$ satisfies $\delta^{\mathrm{K}''}(u') = 0$; since F′ has constructible cohomology, we have $\overline{\mathrm{D}\,\mathrm{F}'} = \mathrm{D}(\overline{\mathrm{F}}')$ and the local duality theorem (SGA 5 I 4.5.3) then shows that we have 201

$$\mathrm{D}\big(\mathrm{H}^i_{\overline{u}'}(\overline{\mathrm{F}}')\big) \simeq \big(\mathrm{H}^{-i-2\delta_{t'}(u')}(\mathrm{D}\,\mathrm{F}')\big)_{\overline{u}'}.$$

So $(*)$ is equivalent to the relation

$$(**) \qquad \big(\underline{\mathrm{H}}^q(\mathrm{D}\,\mathrm{F}')\big)_{\overline{u}'} = 0 \quad \text{for } q > -n - \delta_{t'}(u').$$

We are now in a position to prove the theorem. The relation $(\mathrm{ii})_t$ is equivalent to the relation $(**)$. Let $\mathrm{G}^q = \underline{\mathrm{H}}^q(\mathrm{D}\,\mathrm{F}')$; the affine Lefschetz theorem (3.1) implies in particular that, for any scheme U_1 étale over U, affine over S, we have

$$\mathrm{H}^p(\mathrm{U}'_1, \mathrm{G}^q) = 0 \quad \text{for } p > \delta(\mathrm{G}^q),$$

where $\delta(\mathrm{G}^q)$ is the upper bound of the $\delta_{t'}(u')$ for the u' such that we have $\mathrm{G}^q_{\overline{u}'} \neq 0$; according to $(**)$ we have $\delta(\mathrm{G}^q) \leqslant -n-q$, so $(\mathrm{ii})_t$ implies the relation

$$\mathrm{H}^p(\mathrm{U}'_1, \underline{\mathrm{H}}^q(\mathrm{D}\,\mathrm{F}')) = 0 \quad \text{for } p > -q-n.$$

Taking into account the hypercohomology spectral sequence of the functor ”sections on U'_1” with respect to the complex $\mathrm{D}\,\mathrm{F}'$:

$$\mathrm{E}_2^{pq} = \mathrm{H}^p(\mathrm{U}'_1, \underline{\mathrm{H}}^q(\mathrm{D}\,\mathrm{F}')) \Longrightarrow \mathrm{H}^*(\mathrm{U}'_1, \mathrm{D}\,\mathrm{F}'),$$

we obtain the relation

$$(***) \qquad \mathrm{H}^i(\mathrm{U}'_1, \mathrm{D}\,\mathrm{F}') = 0 \quad \text{for } i > -n.$$

Conversely suppose the preceding relation is verified, for any U_1 étale over U, affine over S. Let us apply proposition 3.6 by replacing S by $\overline{\mathrm{S}}'$; the hypotheses of 3.6 concerning $\overline{\mathrm{S}}$ are satisfied, because, for any affine scheme $\overline{\mathrm{U}}_1$ étale over $\overline{\mathrm{U}}$, we can find a scheme over $\overline{\mathrm{U}}_1$ which comes by inverse image from a scheme étale over U, affine 202 over S ; as for the hypotheses concerning F, they are satisfied thanks to 2.3.3. We thus have, for any point u' of U' such that $\big(\underline{\mathrm{H}}^q(\mathrm{D}\,\mathrm{F}')\big)_{\overline{u}'} \neq 0$:

$$\delta_{t'}(u') \leqslant -n-q,$$

which is none other than the relation $(**)$; we have thus proven the equivalence

$$(\mathrm{ii})_t \Longleftrightarrow (***).$$

We will transform the relation $(***)$; we first have

$$\mathrm{H}^i(\mathrm{U}'_1, \mathrm{D}\,\mathrm{F}') = \big(\underline{\mathrm{H}}^i(\mathrm{R}\,f'_{1*}(\mathrm{D}\,\mathrm{F}'_1))\big)_{t'} \ ;$$

but according to (SGA 5 I 1.12), there exists a canonical isomorphism

$$\mathrm{R}\,f'_{1*}(\mathrm{D}\,\mathrm{F}'_1) \simeq \mathrm{D}\left(\mathrm{R}_!\,f'_1(\mathrm{F}'_1)\right),$$

where $\mathrm{D}\left(\mathrm{R}_!\,f'_1(\mathrm{F}'_1)\right)$ denotes the dual of $\mathrm{R}_!\,f'_1(\mathrm{F}'_1)$ with respect to K. We thus see that $(\mathrm{ii})_t$ is equivalent to

$$\big(\underline{\mathrm{H}}^i(\mathrm{D}(\mathrm{R}_!\,f'_1(\mathrm{F}'_1)))\big)_{t'} = 0 \quad \text{for } i > -n.$$

203 Applying again the local duality theorem (SGA 5 I 4.5.3), but this time to the point t', we find that

$$\big(\underline{\mathrm{H}}^i(\mathrm{D}(\mathrm{R}_!\,f'_1(\mathrm{F}'_1)))\big)_{t'} \simeq \mathrm{D}\left(\mathrm{H}^{-i}_{t'}(\mathrm{R}_!\,f'_1(\mathrm{F}'_1))\right),$$

and finally $(\mathrm{ii})_t$ is equivalent to the relation

$$\mathrm{H}^i_{t'}(\mathrm{R}_!\,f'_1(\mathrm{F}'_1)) = 0 \quad \text{for } i < n,$$

that is $\mathrm{prof}_{t'}(\mathrm{R}_!\,f'_1(\mathrm{F}'_1)) \geqslant n$, which completes the proof of the theorem.

Remark 4.2.3. — The reasoning is simplified quite considerably when we suppose that S admits (at least locally) a dualizing complex (for example is locally immersible in a regular scheme). This avoids the recourse to a completion (the passage to the case where S is strictly local being immediate), to 2.3.3 and to the rather unpleasant technical statement 3.6, which can then be replaced by the more pleasant reference 3.7.

Corollary 4.3. — *Let* S *be an excellent scheme of characteristic zero and* $f\colon \mathrm{U} \to \mathrm{S}$ *a separated morphism of finite type, such that* U *is a* union of $c+1$ open sets, affine over S. *Let* F *be a complex of sheaves of* $\mathbf{Z}/m\mathbf{Z}$*-modules, with degrees bounded below and constructible cohomology,* n *an integer and* T *a closed subset of* S. *Suppose that, for any point* $u \in \mathrm{U}$*, we have*

$$\mathrm{prof}_u(\mathrm{F}) \geqslant n - \delta_{\mathrm{T}}(u).$$

204 *Then we have*

$$\mathrm{prof}_{\mathrm{T}}(\mathrm{R}_!\,f(\mathrm{F})) \geqslant n - c.$$

Let indeed U_j, $0 \leqslant j \leqslant c$, be a covering of U by open sets U_j, affine over S. Resuming the notation of the proof of 4.2, we have, for all j,

$$\mathrm{H}^i(\mathrm{U}'_j, \underline{\mathrm{H}}^q(\mathrm{D}\,\mathrm{F}')) = 0 \quad \text{for } i > -n.$$

Using the spectral sequence which relates the cohomology of U to that of the covering formed by the U_j (SGA 4 V 2.4), the preceding relation shows that we have

$$\mathrm{H}^i(\mathrm{U}', \underline{\mathrm{H}}^q(\mathrm{D}\,\mathrm{F}')) = 0 \quad \text{for } i > -n + c.$$

The corollary then follows from the end of the proof of 4.2.

Corollary 4.4. — *Let* S *be an excellent scheme of characteristic zero,* $g\colon \mathrm{X} \to \mathrm{S}$ *a morphism,* U *an open set of* X*, union of* $c+1$ *affine open sets over* S, Y *a closed subscheme with underlying space* $\mathrm{X} - \mathrm{U}$ *and* $j\colon \mathrm{Y} \to \mathrm{X}$ *the natural morphism. Let* F *be a complex of sheaves of* $\mathbf{Z}/m\mathbf{Z}$*-modules on* X*, with degrees bounded below and constructible cohomology,* T *a closed subset of* S *and* n *an integer. Suppose that, for any point* u *of* U*, we have*

$$\mathrm{prof}_u(\mathrm{F}) \geqslant n - \delta_{\mathrm{T}}(u).$$

Then the canonical morphism 205

$$\mathrm{H}^i_{g^{-1}(\mathrm{T})!}(\mathrm{X}/\mathrm{S}, \mathrm{F}) \longrightarrow \mathrm{H}^i_{(g^{-1}(\mathrm{T})\cap \mathrm{Y})!}(\mathrm{Y}/\mathrm{S}, j^*\mathrm{F})$$

is bijective for $i < n - c - 1$*, injective for* $i = n - c - 1$.

This follows immediately from 4.1 and 4.3.

***Corollary 4.5* (Local Lefschetz Theorem)**. — *Let* S *be an excellent hensilian local scheme of characteristic zero,* t *the closed point of* S, X *a scheme proper over* $\mathrm{S}' = \mathrm{S} - \{t\}$ *and* U *an open set of* X*, union of* $c+1$ *affine open sets. Let* Y *be a closed subscheme of* X*, with underlying space* $\mathrm{X} - \mathrm{U}$, $j\colon \mathrm{Y} \to \mathrm{X}$ *the canonical morphism,* F *a complex of sheaves of* $\mathbf{Z}/m\mathbf{Z}$*-modules on* X*, with degrees bounded below and constructible cohomology, and* n *an integer. Suppose that, for any point* u *of* U*, we have*

$$\mathrm{prof}_u(\mathrm{F}) \geqslant n - \delta'_t(u), \quad \textit{where } \delta'_t(u) = \delta_t(u) - 1.$$

Then the canonical morphism

$$\mathrm{H}^i(\mathrm{X}, \mathrm{F}) \longrightarrow \mathrm{H}^i(\mathrm{Y}, j^*\mathrm{F})$$

is bijective for $i < n - c - 1$*, injective for* $i = n - c - 1$.

Let $f\colon \mathrm{U} \to \mathrm{S}$ be the canonical morphism; it follows from 4.2, applied by replacing n by $n+1$, that we have

$$\mathrm{prof}_t(\mathrm{R}_!\, f(\mathrm{F}_{|\mathrm{U}})) \geqslant n + 1 - c.$$

The preceding relation shows that the canonical morphism 206

$$\mathrm{H}^i(\mathrm{S}, \mathrm{R}_!\, f(\mathrm{F}_{|\mathrm{U}})) \longrightarrow \mathrm{H}^i(\mathrm{S}', \mathrm{R}_!\, f(\mathrm{F}_{|\mathrm{U}}))$$

is bijective for $i < n - c$, injective for $i = n - c$. Since $\mathrm{R}_! f(\mathrm{F}_{|\mathrm{U}})$ is zero outside of S', we have $\mathrm{H}^i(\mathrm{S}, \mathrm{R}_! f(\mathrm{F}_{|\mathrm{U}})) \simeq \big(\underline{\mathrm{H}}^i(\mathrm{R}_! f(\mathrm{F}_{|\mathrm{U}}))\big)_t = 0$, and consequently

$$(*) \qquad \mathrm{H}^i(\mathrm{S}', \mathrm{R}_! f(\mathrm{F}_{|\mathrm{U}})) = 0 \quad \text{for } i < n - c.$$

Let $g\colon \mathrm{X} \to \mathrm{S}'$, $h\colon \mathrm{Y} \to \mathrm{S}'$, $f'\colon \mathrm{U} \to \mathrm{S}'$ be the canonical morphisms. It follows from the distinguished triangle

$$\begin{tikzcd} & \mathrm{R}\, h_*(j^*\mathrm{F}) \arrow[dl] & \\ \mathrm{R}_! f'(\mathrm{F}_{|\mathrm{U}}) \arrow[rr] & & \mathrm{R}\, g_*(\mathrm{F}) \arrow[ul] \end{tikzcd}$$

that the condition $(*)$ is equivalent to the fact that the morphism

$$\mathrm{H}^i(\mathrm{S}', \mathrm{R}\, g_*(\mathrm{F})) \longrightarrow \mathrm{H}^i(\mathrm{S}', \mathrm{R}\, h_*(j^*\mathrm{F}))$$

is bijective for $i < n - c - 1$, injective for $i = n - c - 1$. Since this morphism is canonically identified with the morphism

$$\mathrm{H}^i(\mathrm{X}, \mathrm{F}) \longrightarrow \mathrm{H}^i(\mathrm{Y}, j^*\mathrm{F}),$$

the conclusion follows immediately.

207 ***Corollary 4.6* (Global Lefschetz Theorem)**. — *Let* S *be the spectrum of a field,* X *a scheme proper over* S *and* U *an open set of* X *which is a union of* $c+1$ *affine open sets. Let* Y *be a closed subscheme of* X*, with underlying space* $\mathrm{X} - \mathrm{U}$*,* $j\colon \mathrm{Y} \to \mathrm{X}$ *the canonical morphism,* F *a complex of sheaves of* $\mathbf{Z}/m\mathbf{Z}$*-modules on* X*, with degrees bounded below and constructible cohomology and* n *an integer. Suppose that, for any point* u *of* U*, we have*

$$\mathrm{prof}_u(\mathrm{F}) \geqslant n - \dim\big(\overline{\{u\}}\big).$$

Then the canonical morphism

$$\mathrm{H}^i(\mathrm{X}, \mathrm{F}) \longrightarrow \mathrm{H}^i(\mathrm{Y}, j^*\mathrm{F})$$

is bijective for $i < n - c - 1$*, injective for* $i = n - c - 1$*.*

More generally, if $g\colon \mathrm{X} \to \mathrm{S}$ *is a separated morphism of finite type, the hypotheses on* S*,* U*,* Y*,* F *being the same as before, then the canonical morphism*

$$\mathrm{H}^i_!(\mathrm{X}/\mathrm{S}, \mathrm{F}) \longrightarrow \mathrm{H}^i_!(\mathrm{Y}/\mathrm{S}, j^*\mathrm{F})$$

(where $\mathrm{H}^i_!$ *denotes the cohomology with proper support, that is* $\mathrm{H}^i_!(\mathrm{X}/\mathrm{S}, \mathrm{F}) = \mathrm{H}^i(\mathrm{S}, \mathrm{R}_! g(\mathrm{F})))$ *is bijective for* $i < n - c - 1$*, injective for* $i = n - c - 1$*.*

The corollary is a particular case of 4.4, with $\mathrm{T} = \mathrm{S}$.

Here is a partial converse to 4.3 :

208 ***Proposition 4.7***. — *Let* S *be a noetherian scheme,* $f\colon \mathrm{U} \to \mathrm{S}$ *a morphism of finite type. Suppose there exists a dualizing complex* K *on* S *and that* $\mathrm{R}^! f(\mathrm{K})$ *is a dualizing*

complex on U. *Let* T *be a closed subset of* S *and* c *an integer. Then the following conditions are equivalent :*

(i) *For any complex of sheaves of* $\mathbf{Z}/m\mathbf{Z}$*-modules* F *on* U, *with degrees bounded below and constructible cohomology, and for any integer* n *such that we have, for any point* u *of* U,

$$\mathrm{prof}_u(\mathrm{F}) \geqslant n - \delta_{\mathrm{T}}(u),$$

we have

$$\mathrm{prof}_{\mathrm{T}}(\mathrm{R}_! f(\mathrm{F})) \geqslant n - c.$$

(ii) *For any sheaf of* $\mathbf{Z}/m\mathbf{Z}$*-modules* G *on* U, *constructible, and for any point* $t \in \mathrm{T}$, *we have*

$$\big(\mathrm{R}^p f_*(\mathrm{G})\big)_{\overline{t}} = 0 \quad \textit{for } p > \delta(\mathrm{G}, f, t) + c$$

(*recall from* (SGA 4 XIX 6.0) *that* $\delta(\mathrm{G}, f, t) = \sup\{\delta_t(u) | t \in \overline{\{u\}}$ *and* $\mathrm{G}_{\overline{u}} \neq 0\}$).

N.B. Condition (ii) is satisfied by virtue of 3.1 if f is separated and if U is, locally on S for the étale topology, a union of $c + 1$ affine open sets over S, so 4.7 contains 4.3[(*)].

We can obviously suppose that S is local and that T is the closed point t of S. The proof of (ii) $\Rightarrow$ (i) is essentially identical to part 2°) of the proof of 4.2. Let us quickly 209
show that (i) $\Rightarrow$ (ii). The local duality theorem (SGA 5 I 4.3.2) applied to D G shows that

$$\mathrm{D}\left(\mathrm{H}^i_{\overline{u}}(\mathrm{D}\,\mathrm{G})\right) \simeq \big(\underline{\mathrm{H}}^{-i-2\delta_t(u)}(\mathrm{G})\big)_{\overline{u}}.$$

As G is reduced to degree 0, we thus have $\mathrm{H}^i_{\overline{u}}(\mathrm{D}\,\mathrm{G}) = 0$ except perhaps for $i = -2\delta_t(u)$; more precisely

$$\mathrm{prof}_u(\mathrm{D}\,\mathrm{G}) = \begin{cases} -2\delta_t(u) & \text{if } \mathrm{G}_{\overline{u}} \neq 0, \\ \infty & \text{if } \mathrm{G}_{\overline{u}} = 0. \end{cases}$$

It follows that we have, for any $u \in \mathrm{U}$:

$$\mathrm{prof}_u(\mathrm{D}\,\mathrm{G}) \geqslant -n - \delta_t(u).$$

It then follows from hypothesis (i) that we have $\mathrm{prof}_t(\mathrm{R}_! f(\mathrm{D}\,\mathrm{G})) \geqslant -n - c$. We transform this relation using the isomorphism $\mathrm{R}_! f(\mathrm{D}\,\mathrm{G}) \simeq \mathrm{D}(\mathrm{R} f_*(\mathrm{G}))$ (SGA 5 I 1.12) and by applying the local duality theorem at the point t ; we thus obtain

$$\big(\underline{\mathrm{H}}^i(\mathrm{R} f_*(\mathrm{G}))\big)_{\overline{t}} = 0 \quad \text{for } i > n + c,$$

which is none other than (ii).

4.8. The hypotheses being those of 4.4 with g proper (resp. 4.5, resp. 4.6 with g proper), if V is an open neighborhood of Y in X, the morphism 210

[(*)] At least in the case where S locally admits a dualizing complex, e.g. S is locally immersible in a regular scheme.

$$\mathrm{H}^i(\mathrm{V}, \mathrm{F}) \longrightarrow \mathrm{H}^i(\mathrm{Y}, j^*\mathrm{F})$$

is bijective for $i < n - c - 1$, injective for $i = n - c - 1$. If $\iota\colon \mathrm{V} \to \mathrm{X}$ is the canonical morphism, it suffices indeed to see this to apply 4.4 (resp. 4.5, resp. 4.6) to the complex $\mathrm{R}\,\iota_*(\mathrm{F}_{|\mathrm{V}})$. One can ask the question of whether the preceding morphism is bijective if $i = n - c - 1$, injective for $i = n - c$. It is obviously sufficient that the hypotheses be verified when we replace n by $n + 1$; the following proposition shows that a little less is sufficient.

Proposition 4.9. — *Let* S *be an excellent local scheme of characteristic zero, with closed point* t (*resp. in addition to the preceding conditions we suppose* S *hensilian*), $f\colon \mathrm{X} \to \mathrm{S}$ *a scheme proper over* S (*resp. proper over* $\mathrm{S} - \{t\}$) *and* U *an open set of* X *which is a union of* $c + 1$ *affine open sets. Let* Y *be a closed subscheme of* X, *with underlying space* $\mathrm{X} - \mathrm{U}$, $j\colon \mathrm{Y} \to \mathrm{X}$ *the canonical morphism,* F *a complex of sheaves of* $\mathbf{Z}/m\mathbf{Z}$-*modules on* X, *with degrees bounded below and constructible cohomology, and* n *an integer. We suppose that we have, for any point* u *of* U,

$$\mathrm{prof}_u(\mathrm{F}) \geqslant \inf(n-1, n-\delta_t(u)) \quad (\textit{resp.}\quad \mathrm{prof}_u(\mathrm{F}) \geqslant \inf(n-1, n+1-\delta_t(u))).$$

211 *Then for any open neighborhood* V *of* Y *in* X, *the canonical morphism*

$$\mathrm{H}^i_{f^{-1}(t)}(\mathrm{V}, \mathrm{F}) \longrightarrow \mathrm{H}^i_{f^{-1}(t)\cap\mathrm{Y}}(\mathrm{Y}, j^*\mathrm{F}) \quad (\textit{resp.}\quad \mathrm{H}^i(\mathrm{V}, \mathrm{F}) \longrightarrow \mathrm{H}^i(\mathrm{Y}, j^*\mathrm{F}))$$

is bijective for $i < n - c - 2$ *and injective for* $i = n - c - 2$. *Furthermore, there exists an open neighborhood* V_0 *of* Y *in* X, *such that, for any other such* V *with* $\mathrm{V} \subset \mathrm{V}_0$, *the canonical morphism*

$$\mathrm{H}^i_{f^{-1}(t)\cap\mathrm{V}}(\mathrm{V}, \mathrm{F}) \longrightarrow \mathrm{H}^i_{f^{-1}(t)\cap\mathrm{Y}}(\mathrm{Y}, j^*\mathrm{F}) \quad (\textit{resp.}\quad \mathrm{H}^i(\mathrm{V}, \mathrm{F}) \longrightarrow \mathrm{H}^i(\mathrm{Y}, j^*\mathrm{F}))$$

is bijective for $i < n - c - 1$, *injective for* $i = n - c - 1$.

Démonstration. — Let us set for simplicity $\delta'_t(u) = \delta_t(u)$ (resp. $\delta'_t(u) = \delta_t(u) - 1$). We deduce from 4.8 the first assertion of 4.9, because the hypotheses of 4.4 (resp. 4.5) are verified when we replace n by $n - 1$. They are also verified for n itself, except at the points u such that $\delta'_t(u) = 0$. Now, for a $u \in \mathrm{U}$, to say that we have $\delta'_t(u) = 0$ is equivalent to saying that u is a *closed point of* U_t (resp. a *closed point of* X). Let E be the set of points of U such that $\delta'_t(u) = 0$; let us show that, *for all points* $u \in \mathrm{E}$, *except for a finite number*, we have $\mathrm{prof}_u(\mathrm{F}) \geqslant n$. Let $\overline{\mathrm{S}}$ be the strict localization of S at t, S' the completion of $\overline{\mathrm{S}}$, with closed point t', and let us consider the cartesian square

$$\begin{tikzcd} \mathrm{U}' \arrow[r, "h"] \arrow[d, "f'"'] & \mathrm{U} \arrow[d, "f"] \\ \mathrm{S}' \arrow[r, "g"] & \mathrm{S}. \end{tikzcd}$$

212 The depth hypotheses at the points of U are preserved when we replace U by U' and F by the inverse image F' of F on U'. Let indeed $u' \in \mathrm{U}'$ and $u = h(u')$. If $u \notin \mathrm{E}$,

we have the relation $\operatorname{prof}_u(\mathrm{F}) \geqslant n - \delta'_t(u)$, and it follows from 4.2.1 that this implies the relation $\operatorname{prof}_{u'}(\mathrm{F}') \geqslant n - \delta'_t(u')$. If $u \in \mathrm{E}$, u' is a closed point of U'_t (resp. a closed point of $\mathrm{X}' = \mathrm{X} \times_{\mathrm{S}} \mathrm{S}'$), and, since the fiber U'_u of h at u is of dimension zero and h is regular, it follows from 1.16 that we have $\operatorname{prof}_{u'}(\mathrm{F}') = \operatorname{prof}_u(\mathrm{F}) \geqslant n - 1$.

Let then K be a dualizing complex on S', normalized at the closed point t', and $\mathrm{D}\,\mathrm{F}'$ the dual of F' with respect to $\mathrm{R}^! f(\mathrm{K})$. According to 4.2.2, the étale depth hypotheses at the points of U' translate into the relations :

$$\big(\underline{\mathrm{H}}^q(\mathrm{D}\,\mathrm{F}')\big)_{\overline{u}'} = 0 \quad \text{for } q > -n - \delta'_t(u') \quad (\text{resp. } q > -n - 2 - \delta'_{t'}(u')),$$

if u' is not a point of $\mathrm{E}' = h^{-1}(\mathrm{E})$,

$$\big(\underline{\mathrm{H}}^q(\mathrm{D}\,\mathrm{F}')\big)_{\overline{u}'} = 0 \quad \text{for } q > -(n-1) \quad (\text{resp. } q > -n - 1), \text{ if } u' \in \mathrm{E}'.$$

Let $\mathrm{G} = \underline{\mathrm{H}}^{-(n-1)}(\mathrm{D}\,\mathrm{F}')$ (resp. $\mathrm{G} = \underline{\mathrm{H}}^{-n-1}(\mathrm{D}\,\mathrm{F}')$) ; since G is a constructible sheaf, the set of points at which the geometric fiber is non-zero is a constructible set (SGA 4 IX 2.4 (iv)) ; now by hypothesis this set is contained in the set E' of closed points of $\mathrm{U}'_{t'}$ (resp. of points of U' closed in X') ; it therefore follows from 4.9.1 below that this set is reduced to a finite number of points. Applying 4.2.2, we see that, for all points of E', except a finite number, we have $\operatorname{prof}_{u'}(\mathrm{F}') \geqslant n$. It indeed follows by 1.16 that, for all points of E except a finite number, we have 213

$$\operatorname{prof}_u(\mathrm{F}) \geqslant n.$$

Let V be an open neighborhood of Y in X, contained in the complement in X of the finite set of points u of E for which we have $\operatorname{prof}_u(\mathrm{F}) = n - 1$. If $\iota\colon \mathrm{V} \to \mathrm{X}$ is the canonical immersion, let

$$\mathrm{F}_1 = \mathrm{R}\,\iota_*(\mathrm{F}_{|\mathrm{V}}) \,;$$

then F_1 is a complex of sheaves on X, with constructible cohomology (SGA 4 XIX 5.1) and with degrees bounded below. We will see that, for any point u of U, the complex F_1 verifies the relation

$$(*) \qquad \operatorname{prof}_u(\mathrm{F}_1) \geqslant n - \delta'_t(u).$$

If $u \in \mathrm{U} \cap \mathrm{V}$, we have $\operatorname{prof}_u(\mathrm{F}_1) = \operatorname{prof}_u(\mathrm{F})$, and the relation $(*)$ is verified by hypothesis on the points of U which do not belong to E ; for the latter, it is also verified by the choice of V. Finally, if $u \in \mathrm{U}$ and $u \notin \mathrm{V}$, we have according to 1.6 g) $\operatorname{prof}_u(\mathrm{F}_1) = \infty$. We then apply 4.4 (resp. 4.5) by replacing F by F_1 ; we obtain the announced result, taking into account the fact that we have, for any i :

$$\mathrm{H}^i_{f^{-1}(t)}(\mathrm{X}, \mathrm{R}\,\iota_*(\mathrm{F}_{|\mathrm{V}})) \simeq \mathrm{H}^i_{f^{-1}(t)\cap\mathrm{V}}(\mathrm{V}, \mathrm{F}) \quad (\text{resp. } \mathrm{H}^i(\mathrm{X}, \mathrm{R}\,\iota_*(\mathrm{F}_{|\mathrm{V}})) \simeq \mathrm{H}^i(\mathrm{V}, \mathrm{F}).$$

Lemma 4.9.1. — *A constructible set* E *contained in the set of closed points of a noetherian scheme* X *is reduced to a finite number of points.* 214

Indeed, E is a finite union of sets of the form $\mathrm{U} \cap \complement\mathrm{V}$, where U and V are open sets of X ; by hypothesis all points of $\mathrm{U} \cap \complement\mathrm{V}$ are maximal points of this set, so they are finite in number.

5. Geometric depth

To apply 4.2 and its corollaries in practice, we need a convenient criterion that allows to verify the hypotheses of étale depth at the points of U. We will give such a criterion, using the local Lefschetz theorem 4.5.

5.1. Let A be a noetherian local ring ; when we speak of the étale depth of A, it will be the depth at the closed point. We will introduce a notion of "geometric depth of A", and use 4.5 to compare it to the étale depth prof ét(A).

Proposition 5.2. — *Let* A *be a noetherian local ring ; suppose that* A *is isomorphic to a quotient of a regular local ring* B *by an ideal* I (*this is true for example when* A *is complete, by virtue of Cohen's theorem* (EGA 0_{IV} 19.8.8)). *Let q be the minimal number of generators of* I *; then the number* $\dim(\mathrm{B}) - q$ *is independent of the choice of* B.

The minimal number of generators of I is also equal to the rank of the k-vector space $\mathrm{I} \otimes_{\mathrm{B}} k$, where k denotes the residue field of A. We immediately reduce to the case where A is complete, because we have $\widehat{\mathrm{A}} \simeq \widehat{\mathrm{B}}/\widehat{\mathrm{I}}$ with $\dim \widehat{\mathrm{B}} = \dim \mathrm{B}$ and
215 $\mathrm{rg}_k(\mathrm{I} \otimes_{\mathrm{B}} k) = \mathrm{rg}_k(\widehat{\mathrm{I}} \otimes_{\widehat{\mathrm{B}}} k)$; for the same reason we can suppose that the rings B are complete. Let B and B′ be two complete regular local rings, $f \colon \mathrm{B} \to \mathrm{A}$, $f' \colon \mathrm{B}' \to \mathrm{A}$ two surjective homomorphisms and $\mathrm{I} = \mathrm{Ker}(f)$, $\mathrm{I}' = \mathrm{Ker}(f')$. We must show that

$$\dim \mathrm{B} - \mathrm{rg}_k(\mathrm{I} \otimes_{\mathrm{B}} k) = \dim \mathrm{B}' - \mathrm{rg}_k(\mathrm{I}' \otimes_{\mathrm{B}'} k).$$

Let us first place ourselves in the case where we have a factorization of the form

$$\begin{tikzcd} \mathrm{B} \arrow[rr, "f"] \arrow[dr, "g"'] & & \mathrm{A} \\ & \mathrm{B}' \arrow[ur, "f'"'] & \end{tikzcd}$$

with g surjective. Let $\mathrm{J} = \mathrm{Ker}(g)$; then $\mathrm{J} \subset \mathrm{I}$ and $\mathrm{I}/\mathrm{J} = \mathrm{I}'$. Since B′ is regular, $\dim(\mathrm{B}') = \dim(\mathrm{B}) - \mathrm{rg}_k(\mathrm{J} \otimes_{\mathrm{B}} k)$ and J is generated by elements forming part of a regular system of parameters of B. It follows that we have the exact sequence

$$0 \longrightarrow \mathrm{J} \otimes_{\mathrm{B}} k \longrightarrow \mathrm{I} \otimes_{\mathrm{B}} k \longrightarrow \mathrm{J}/\mathrm{I} \otimes_{\mathrm{B}'} k \longrightarrow 0,$$

and consequently

$$\dim \mathrm{B} - \mathrm{rg}_k(\mathrm{I} \otimes_{\mathrm{B}} k) = \dim \mathrm{B} - \mathrm{rg}_k(\mathrm{J} \otimes_{\mathrm{B}} k) - \mathrm{rg}_k(\mathrm{J}/\mathrm{I} \otimes_{\mathrm{B}'} k) = \dim \mathrm{B}' - \mathrm{rg}_k(\mathrm{I}' \otimes_{\mathrm{B}'} k).$$

The general case reduces to the preceding one ; to see this, it suffices to show that we can find a complete regular local ring B'' and surjective homomorphisms $g\colon B'' \to B$ and $g'\colon B'' \to B'$, making the diagram commutative 216

$$(*)\qquad \begin{tikzcd} & B \arrow[dr, "f"] & \\ B'' \arrow[ur, "g"] \arrow[dr, "g'"'] & & A \\ & B'. \arrow[ur, "f'"'] & \end{tikzcd}$$

Now, if W is a Cohen ring with residue field k, we have a local morphism $W \to A$ which lifts to B and B' (EGA IV 19.8.6), so that we have the commutative diagram

$$\begin{tikzcd} & B \arrow[dr] & \\ W \arrow[ur] \arrow[dr] & & A \\ & B'. \arrow[ur] & \end{tikzcd}$$

We can find integers n and n' and surjective morphisms $h\colon W[\![T_1,\dots,T_n]\!] \to B$ and $h'\colon W[\![T'_1,\dots,T'_{n'}]\!] \to B'$ which are W-algebra morphisms (EGA 0_{IV} 19.8.8) ; if we then set $B'' = W[\![T_1,\dots,T_n,T'_1,\dots,T'_{n'}]\!]$ and if we define g and g' as W-algebra morphisms such that

$$g(T_i) = h(T_i), \quad g(T'_i) = b_i, \quad g'(T_i) = b'_i, \quad g'(T'_i) = h'(T'_i),$$

where b_i (resp. b'_i) is an element of B (resp. of B') which lifts $(f' \circ h')(T'_i)$ (resp. $(f \circ h)(T_i)$), the diagram $(*)$ is indeed commutative.

Proposition 5.2 justifies the following definition :

Definition 5.3. — Let A be a noetherian local ring, $\widehat{A}$ its completion, which is therefore isomorphic to the quotient of a complete regular local ring B by an ideal I ; if q is the minimal number of generators of I, we call the *geometric depth* of A the number 217

$$\operatorname{prof\,géom}(A) = \dim B - q.$$

Proposition 5.4. — *Let* A *be a noetherian local ring. Then we have*

$$\operatorname{prof\,géom}(A) \leqslant \dim A,$$

and we have equality if and only if A *is a complete intersection.*

We can suppose A is complete. Let then $A = B/I$, where B is a complete regular local ring and I an ideal of B. If $(x_1,\dots,x_q)$ is a minimal system of generators of I, we have $\dim A \geqslant \dim B - q$, and saying that $\dim A = \dim B - q$ is equivalent to saying that $(x_1,\dots,x_q)$ is part of a system of parameters of B (EGA 0_{IV} 16.3.7) ; the proposition follows immediately.

Proposition 5.5. — *Let* A *and* A′ *be noetherian local rings,* $f\colon \mathrm{A}\to\mathrm{A}'$ *a local homomorphism. Suppose that* f *is flat and that, denoting by* k *the residue field of* A, $\mathrm{A}'\otimes_{\mathrm{A}} k$ *is a field, a separable extension of* k*. Then we have*

$$\text{prof géom}(\mathrm{A}) = \text{prof géom}(\mathrm{A}').$$

218 Up to replacing A and A′ by their completions, we can suppose A and A′ are complete (it follows from (EGA 0_{III} 10.2.1) that the flatness hypothesis is preserved and this is obvious for the other hypotheses). Let then $\mathrm{A}=\mathrm{B}/\mathrm{I}$, where B is a regular local ring and I an ideal of B. Since A′ is formally smooth over A (EGA 0_{IV} 19.8.2), it follows from (EGA 0_{IV} 19.7.2) that we can find a complete noetherian local ring B′ and a local homomorphism $\mathrm{B}\to\mathrm{B}'$, such that B′ is a flat B-module and we have $\mathrm{B}'\otimes_{\mathrm{B}}\mathrm{A}\simeq\mathrm{A}'$. We thus have $\mathrm{A}'\simeq\mathrm{B}'/\mathrm{IB}'$; moreover the ring B′ is regular; indeed, if $\mathfrak{m}$ is the maximal ideal of B, $\mathfrak{m}\mathrm{B}'$ is the maximal ideal of B′, and, since $\mathfrak{m}$ is generated by a regular sequence by definition of "regular", $\mathfrak{m}\mathrm{B}'$ is generated by a B′-regular sequence (EGA 0_{IV} 15.1.14). Since we obviously have $\dim\mathrm{B}=\dim\mathrm{B}'$, and since I and IB′ have the same minimal number of generators, the assertion follows.

Theorem 5.6(6). — *Let* A *be an excellent local ring of characteristic zero. Then we have*

$$\text{prof ét}(\mathrm{A}) \geqslant \text{prof géom}(\mathrm{A}).$$

We can suppose A is strictly local and complete, since the geometric depth and the étale depth are preserved by passing to the strict henselization and to the completion according to 5.5 and 1.16. Let $\mathrm{A}\simeq\mathrm{B}/\mathrm{I}$, where B is a complete regular local ring, and
219 let $(f_1,\dots,f_q)$ be a minimal system of generators of the ideal I. We thus have

$$\pi = \text{prof géom}(\mathrm{A}) = \dim\mathrm{B} - q.$$

Let us consider the closed immersion

$$\mathrm{Y}=\operatorname{Spec}\mathrm{A}\longrightarrow\mathrm{X}=\operatorname{Spec}\mathrm{B},$$

and let $\mathrm{U}=\mathrm{X}-\mathrm{Y}=\bigcup_{1\leqslant i\leqslant q}\mathrm{X}_{f_i}$. If a denotes the closed point of X, we must show that, for any prime number p, we have

$$\mathrm{H}^i_a(\mathrm{Y},\mathbf{Z}/p\mathbf{Z})=0\quad\text{ for } i<\pi.$$

Since B is excellent regular, we have $\text{prof ét}(\mathrm{B})=2\dim\mathrm{X}$ (cf. 1.10) and consequently $\mathrm{H}^i_a(\mathrm{X},\mathbf{Z}/p\mathbf{Z})=0$ for $i<2\dim\mathrm{X}$. It therefore suffices, to prove the theorem, to prove that the morphism

$$(*)\qquad \mathrm{H}^i_a(\mathrm{X},\mathbf{Z}/p\mathbf{Z})\longrightarrow\mathrm{H}^i_a(\mathrm{Y},\mathbf{Z}/p\mathbf{Z})$$

(6) N.D.E. : Illusie has since shown the inequality $\mathrm{prof}_x(\mathbf{Z}/\ell^\nu\mathbf{Z})\geqslant \text{prof géom}_x(\mathrm{X}/\mathrm{S})-\delta(x)+1$ for a point x of X, a scheme of finite type over a trait S of residual characteristic prime to ℓ, and $\nu\geqslant 1$. If S has characteristic zero, this is a consequence of theorem 5.6; see (Illusie L., "Perversity and variation", *Manuscripta Math.* **112** (2003), p. 271-295).

is bijective for $i < \pi$. We apply for this the local Lefschetz theorem 4.5 with $n = \pi + q - 1$, $c = q - 1$ so $n - c = \pi$. Note that $U = X - Y$ is the union of the q affine open sets X_{f_i}. Let us show that we have, for any point u of U :

$$\text{prof ét}_u(X) \geqslant \pi + q - 1 - \dim\left(\overline{\{u\}}\right) = \dim \mathscr{O}_{X,u}$$

(where $\overline{\{u\}}$ denotes the closure of u *in* $X - \overline{\{a\}}$). Indeed it follows from 1.10 that we have

$$\text{prof ét}_u(X) = 2 \dim \mathscr{O}_{X,u} \geqslant \dim \mathscr{O}_{X,u}.$$

By using 4.5, we see that $(*)$ is bijective for $i < \pi$, which completes the proof of the theorem. 220

Corollary 5.7. — *Let* S *be the spectrum of a field of characteristic zero* (*resp. an excellent hensilian local scheme of characteristic zero*), $f\colon X \to S$ *a scheme proper over* S (*resp. over* $S - \{s\}$)*. Let* U *be a union of* $c + 1$ *open sets of* X*, affine,* Y *a closed subscheme with underlying space* $X - U$*,* n *and* m *integers* > 0*. We suppose that, for any point* u *of* U*, we have*

$$\text{prof géom}(\mathscr{O}_{X,u}) \geqslant n - \dim\left(\overline{\{u\}}\right)$$

($\overline{\{u\}}$ *closure of* u *in* X)*. Then the canonical morphism*

$$H^i(X, \mathbf{Z}/m\mathbf{Z}) \longrightarrow H^i(Y, \mathbf{Z}/m\mathbf{Z})$$

is bijective for $i < n - c - 1$*, injective for* $i = n - c - 1$*.*

We apply 4.5 and 4.6. The hypotheses of étale depth at the points of U are verified because we have according to 5.6

$$\text{prof ét}_u(X) \geqslant \text{prof géom}(\mathscr{O}_{X,u}) \geqslant n - \dim\left(\overline{\{u\}}\right).$$

6. Open questions

6.1. We can ask if the implication (ii) $\Rightarrow$ (i) of 4.2 is valid more generally for torsion sheaves F, not necessarily annihilated by a given integer m and not necessarily constructible. In the case where S is not of characteristic zero, it seems possible that 221
this implication remains valid, even for sheaves of p-torsion (p the residual characteristic). Finally, it is also not clear that the hypothesis that S is excellent cannot be lifted.

6.2. Let X be a scheme proper over a field k or the complement of the closed point of a hensilian local scheme, and $j : Y \to X$ a closed subscheme of X, whose complement U is affine. Then, if F is a sheaf of sets on X or a not necessarily commutative sheaf of groups, the statements 4.5 or 4.6 and 4.9 still make sense for such an F, provided we restrict to small values of n. If u is a point of U, we denote by $\overline{u}$ a geometric point over u, by $X(\overline{u})$ the strict localization of X at $\overline{u}$ and by $F_{\overline{u}}$ the fiber of F at $\overline{u}$. Then, possibly making certain hypotheses on X and on F, for example by supposing X is excellent (possibly of characteristic zero, or of equal characteristic using resolution of singularities) and F is ind-finite (or even **L**-ind-finite with **L** prime to the characteristic of X), we would like to prove the following statements :

a) Let F be a sheaf of sets (resp. a sheaf of groups) and suppose that, for any point u of U, we have

$$\mathrm{F}_{\overline{u}} \longrightarrow \mathrm{H}^0(\mathrm{X}(\overline{u}) - \overline{u}, \mathrm{F}) \text{ injective if } \dim(\overline{\{u\}}) \leqslant 1$$

(that is, for such a u, we have $\mathrm{prof}_u(\mathrm{F}) \geqslant 1$). Then, as V runs through the set of open
222 neighborhoods of Y, the canonical morphism

$$\varinjlim_{\mathrm{V}} \mathrm{H}^0(\mathrm{V}, \mathrm{F}) \longrightarrow \mathrm{H}^0(\mathrm{Y}, j^*\mathrm{F})$$

is bijective (resp. we have the preceding conclusion and furthermore the morphism $\varinjlim_{\mathrm{V}} \mathrm{H}^1(\mathrm{V}, \mathrm{F}) \to \mathrm{H}^1(\mathrm{Y}, j^*\mathrm{F})$ is injective). If F is constructible, we can replace the $\varinjlim$ by the cohomology of V for V "small enough".

b) Let F be a sheaf of sets (resp. a sheaf of groups) and suppose that, for any point u of U, we have $\mathrm{prof}_u(\mathrm{F}) \geqslant 2 - \dim(\overline{\{u\}})$, which also translates into the relations

$$\mathrm{F}_{\overline{u}} \longrightarrow \mathrm{H}^0(\mathrm{X}(\overline{u}) - \overline{u}, \mathrm{F}) \text{ is bijective if } \dim(\overline{\{u\}}) = 0$$
$$\mathrm{F}_{\overline{u}} \longrightarrow \mathrm{H}^0(\mathrm{X}(\overline{u}) - \overline{u}, \mathrm{F}) \text{ is injective if } \dim(\overline{\{u\}}) = 1.$$

Then the canonical morphism

$$\mathrm{H}^0(\mathrm{X}, \mathrm{F}) \longrightarrow \mathrm{H}^0(\mathrm{Y}, j^*\mathrm{F})$$

is bijective (resp. we have the preceding conclusion and furthermore the morphism $\mathrm{H}^1(\mathrm{X}, \mathrm{F}) \to \mathrm{H}^1(\mathrm{Y}, j^*\mathrm{F})$ is injective).

c) Let F be an ind-finite sheaf of groups. Suppose that, for any point u of U, we have

$$\mathrm{F}_{\overline{u}} \longrightarrow \mathrm{H}^0(\mathrm{X}(\overline{u}) - \overline{u}, \mathrm{F}) \text{ bijective if } \dim(\overline{\{u\}}) = 0 \text{ or } 1,$$
$$\mathrm{F}_{\overline{u}} \longrightarrow \mathrm{H}^0(\mathrm{X}(\overline{u}) - \overline{u}, \mathrm{F}) \text{ injective if } \dim(\overline{\{u\}}) = 2.$$

223 Then, as V runs through the set of open neighborhoods of Y, the canonical morphisms

$$\varinjlim_{\mathrm{V}} \mathrm{H}^0(\mathrm{V}, \mathrm{F}) \longrightarrow \mathrm{H}^0(\mathrm{Y}, j^*\mathrm{F}) \text{ and } \varinjlim_{\mathrm{V}} \mathrm{H}^1(\mathrm{V}, \mathrm{F}) \longrightarrow \mathrm{H}^1(\mathrm{Y}, j^*\mathrm{F})$$

are bijective. If F is constructible, we can replace the $\varinjlim$ by the cohomology of V for V small enough.

d) Let F be a sheaf of groups. Suppose that, for any point u of U, we have $\mathrm{prof}_u(\mathrm{F}) \geqslant 3 - \dim(\overline{\{u\}})$, which also translates into the conditions

$$\mathrm{F}_{\overline{u}} \longrightarrow \mathrm{H}^0(\mathrm{X}(\overline{u}) - \overline{u}, \mathrm{F}) \text{ bijective, and } \mathrm{H}^1(\mathrm{X}(\overline{u}) - \overline{u}, \mathrm{F}) = 0 \text{ if } \dim(\overline{\{u\}}) = 0,$$
$$\mathrm{F}_{\overline{u}} \longrightarrow \mathrm{H}^0(\mathrm{X}(\overline{u}) - \overline{u}, \mathrm{F}) \text{ bijective if } \dim(\overline{\{u\}}) = 1,$$
$$\mathrm{F}_{\overline{u}} \longrightarrow \mathrm{H}^0(\mathrm{X}(\overline{u}) - \overline{u}, \mathrm{F}) \text{ injective if } \dim(\overline{\{u\}}) = 2.$$

Then the canonical morphisms

$$H^0(X, F) \longrightarrow H^0(Y, j^*F) \text{ and } H^1(X, F) \longrightarrow H^1(Y, j^*F)$$

are bijective.

As an indication in favor of these statements(7), let us mention XIII 2.1, X 3.4 and XII 3.5. Let us mention that, thanks to the argument of 4.8 and 4.9, statement a) (resp. c)) would follow from b) (resp. d)).

6.3. It would follow from d) the following statement analogous to 5.6 : if A is a noetherian local ring (possibly excellent) and if we have $\operatorname{prof\,géom}(A) \geqslant 3$, then we have $\operatorname{prof\,hop}(A) \geqslant 3$. To see this, we realize $Y' = \operatorname{Spec} A$ as a closed subscheme of a regular local scheme $X' = \operatorname{Spec} B$, whose complement is a union of q affine open sets, with the relation $\dim B - q = \operatorname{prof\,géom}(A)$. We have, for any point x of X', if $n = \dim B$, $\operatorname{prof\,hop}_x(X) \geqslant \inf(3, n - \dim(\overline{\{x\}}))$ (cf. 1.11) and we deduce from d) that this implies $\operatorname{prof\,hop}_y(Y') \geqslant \inf(3, n - q - \dim(\overline{\{y\}}))$, for any point y of Y'. The result is then obtained by taking for y the closed point of Y'. 224

6.4. A variant of 4.2, at least of the implication (ii) $\Rightarrow$ (i), should still be valid in the complex analytic case, provided we work with "analytically constructible" sheaves (cf. XIII) ; the proof would be analogous to that of 4.2, using the duality theory of J.-L. Verdier. Note on the other hand that, for the complex analytic analogue of the non-commutative variants mentioned in 6.2, we do not even have a method of attack for the statements concerning the fundamental group suggested by the results of exposés X, XII, XIII, recalled at the end of 6.2. The methods of the Seminar seem indeed irremediably linked to the case of *finite* coverings (which can be studied in terms of coherent sheaves of algebras).

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(7) N.D.E. : all the statements of 6.2, apart from the constructible variants, have been proven by Mrs. Raynaud ; see (Raynaud M., "Théorèmes de Lefschetz en cohomologie des faisceaux cohérents et en cohomologie étale. Application au groupe fondamental", *Ann. Sci. Éc. Norm. Sup.* (*4*) **7** (1974), p. 29–52, corollary III.1.3).

INDEX OF NOTATIONS

TERMINOLOGICAL INDEX