AROUND $\mathrm{Hom}(\mathbb{G}_a, \mathbb{G}_m[1])$ IN CHARACTERISTIC ZERO

In this document we work always over $\mathbb{Q}$ and if there is a topology we will always assume the descendable topology. We will someitmes refer to symmetric monoidal presentable stable categories as 1-rings, and objects in the opposite category as 1-affine schemes. The $\mathbb{E}_\infty$ group scheme $\widehat{\mathbb{G}_m}$ (which can be identified with $\widehat{\mathbb{G}_a}$ via the exponential map) can be extended to a functor on $\mathbb{Q}$-linear symmetric monoidal categories. That it, for any $\mathbb{Q}$-linear symmetric monoidal category T, we define $\widehat{\mathbb{G}_m}(T)$ to be

$$\mathrm{Hom}_{\mathrm{CAlg}(\mathrm{Pr}^L_{\mathrm{St}})}(\mathrm{QCoh}(\widehat{\mathbb{G}_\mathrm{m}}), T)$$

The $\mathbb{E}_\infty$-group structure on $\widehat{\mathbb{G}_m}$ induces a bialgebra structure on $\mathrm{QCoh}(\widehat{\mathbb{G}_\mathrm{m}})$ and thus an $\mathbb{E}_\infty$-group structure on $\widehat{\mathbb{G}_m}(T)$.

Using the fact that $\mathrm{QCoh}(\widehat{\mathbb{G}_\mathrm{m}})$ is an idempotent $\mathrm{QCoh}(\mathbb{G}_\mathrm{m})$-algebra, which is itself an idempotent $\mathrm{QCoh}(\mathbb{A}^1)$-algebra, we see that $\widehat{\mathbb{G}_m}(T)$ is simply the subspace[1] of x in $\Omega^\infty(\mathrm{End}_T(\mathbf{1}_T))$ such that $\mathrm{colim}(\mathbf{1}_T \xrightarrow{x-1} \mathbf{1}_T \dots) \cong 0$ (or if R is an $\mathbb{E}_\infty$-ring, this is equivalent to the image of $x-1$ in $\pi_0(R)$ being nilpotent). For an $\mathbb{E}_\infty$-ring R, $\widehat{\mathbb{G}_m}(R)$ only depends on the connective cover of R.

Because $(i_0)_*\mathbb{Q} \in \mathrm{QCoh}(\widehat{\mathbb{G}_\mathrm{m}})$ is compact, the dual category $\mathrm{QCoh}(\widehat{\mathbb{G}_\mathrm{m}})^\vee$ is the module category over a commutative cocommutative bialgebra. Duality theory in Elliptic I Proposition 3.8.5 implies that for symmetric monoidal $\mathbb{Q}$-linear presentable stable categories T,

$$\mathrm{Hom}_{\mathrm{CMon}(T\text{-}\mathrm{Alg}(\mathrm{Pr}^L_{\mathrm{St}})^{\mathrm{op}})}((\widehat{\mathbb{G}_m})_T, \mathrm{Pic}^\dagger_T) \cong \mathrm{Hom}_{\mathrm{CAlg}(\mathrm{Pr}^L_{\mathrm{St}})}(\mathrm{QCoh}(\widehat{\mathbb{G}_\mathrm{m}})^\vee, T) \tag{0.1}$$

where $(\widehat{\mathbb{G}_m})_T$ is the bialgebra category $\mathrm{QCoh}(\widehat{\mathbb{G}_\mathrm{m}}) \otimes T$ and $\mathrm{Pic}^\dagger_T$ is the bialgebra category corepresenting the functor

$$C \mapsto (C^{\cong})^\times$$

sending a symmetric monoidal T-linear presentable stable category to its $\mathbb{E}_\infty$-monoid of units. The bialgebra category is the categorified group algebra on the sphere spectrum over T, i.e. $T[\Omega^\infty \mathbb{S}]$.

Therefore to identify the bialgebra $\mathrm{End}_{\mathrm{QCoh}(\widehat{\mathbb{G}_\mathrm{m}})^\vee}(\mathbf{1})$, we can look at $\mathrm{Hom}_{\mathrm{CMon}(R\text{-}\mathrm{Alg}(\mathrm{Pr}^L_{\mathrm{St}})^{\mathrm{op}})}((\widehat{\mathbb{G}_m})_R, \mathrm{Pic}^\dagger_R)$ for $\mathbb{E}_\infty$-rings R over $\mathbb{Q}$. The map $* \to \widehat{\mathbb{G}_m}$ of sheaves on nonconnective $\mathbb{Q}$-algebras is a descendable cover[2]. Additionally, the Hopf algebra dual of $\mathbb{Q}^{S^1} \cong \mathrm{Sym}_{\mathbb{Q}}(\mathbb{Q}[-1])$ is[3] $\mathbb{Q}[S^1]$ which corepresents the functor $\Omega\mathbb{G}_m$. These two facts combine to show

$$\begin{aligned} \mathrm{Hom}_{\mathrm{CMon}(R\text{-}\mathrm{Alg}(\mathrm{Pr}^L_{\mathrm{St}})^{\mathrm{op}})}((\widehat{\mathbb{G}_m})_R, \mathrm{Pic}^\dagger_R) &\cong \mathrm{Hom}_{\mathrm{Shv}(\mathsf{ncAff}_{/\mathrm{Spec}\,R}, \mathrm{Sp}^{\mathrm{cn}})}((\tau_{\geq 1}\mathbb{G}_m)^\#, \mathrm{Pic}^\dagger) \\ &\cong \mathrm{Hom}_{\mathrm{Shv}(\mathsf{ncAff}_{/\mathrm{Spec}\,R}, \mathrm{Sp}^{\mathrm{cn}})}(\Omega\mathbb{G}_m, \mathbb{G}_m) \\ &\cong \tau_{\geq 0}(R[1]) \end{aligned} \tag{0.2}$$

Hence we deduce that

$$\mathrm{QCoh}(\widehat{\mathbb{G}_\mathrm{m}})^\vee \cong \mathrm{QCoh}(\mathrm{Sym}_{\mathbb{Q}}(\mathbb{Q}[-1])) \cong \mathrm{QCoh}(\mathbb{G}_\mathrm{a}[1]) \tag{0.3}$$

From this, we see that

$$\mathrm{Hom}_{\mathrm{CMon}(T\text{-}\mathrm{Alg}(\mathrm{Pr}^L_{\mathrm{St}})^{\mathrm{op}})}((\mathbb{G}_a[1])_T, \mathrm{Pic}^\dagger_T) \cong \mathrm{Hom}_{\mathrm{CAlg}(\mathrm{Pr}^L_{\mathrm{St}})}(\mathrm{QCoh}(\widehat{\mathbb{G}_\mathrm{m}}), T) \cong \widehat{\mathbb{G}_m}(T) \tag{0.4}$$

[1]meaning the inclusion is a (-1)-truncated map

[2]the same proof as Propsition 0.2 below shows that it is even descendable as 1-affine schemes

[3]the previous map is induced from the natural map $B\mathbb{Z} \to B\mathbb{G}_a$

for all symmetric monoidal $\mathbb{Q}$-linear presentable stable categories T (where $\mathbb{G}_a[1]$ corepresents the functor whose value on T is $\tau_{\geq 0}(\mathrm{End}_T\,\mathbf{1}[1])$). After the isomorphism (the topology (on 1-affine schemes) is that of descent for module categories–which we also call the descendable topology),

$$\mathrm{Hom}_{\mathrm{CMon}(T\text{-}\mathrm{Alg}(\mathrm{Pr}^L_{\mathrm{St}})^{\mathrm{op}})}((\mathbb{G}_a[1])_T, \mathrm{Pic}^\dagger_T) \cong \mathrm{Hom}_{\mathrm{Shv}(T\text{-}\mathrm{Alg}(\mathrm{Pr}^L_{\mathrm{St}})^{\mathrm{op}}, \mathrm{Sp}^{\mathrm{cn}})}(\mathbb{G}_a[1], \mathrm{Pic}^\dagger)$$

note that there are two interpretations of $\mathbb{G}_a[1]$, the functor $\tau_{\geq 0}(\mathrm{End}_T\,\mathbf{1}[1])$ or the sheafification of the functor $\tau_{\geq 0}\,\mathrm{End}_T\,\mathbf{1}[1]$ but they agree because elements of $\Omega^\infty(\mathrm{End}_T\,\mathbf{1}[1])$ vanish descendable-locally. We conclude that

Proposition 0.1. *For any* 1*-ring (symmetric monoidal presentable stable category) T of characteristic zero, we have the isomorphism of connective spectra*

$$\mathrm{Hom}_{\mathrm{CMon}(1\mathsf{Aff}_{/\mathrm{Spec}\,T})}(\mathbb{G}_a, \mathbb{G}_m) \cong \widehat{\mathbb{G}_m}(T)$$

where $1\mathsf{Aff}$ *means the opposite category of* 1*-rings.*

An $\mathbb{E}_\infty$-monoid in formal qcqs algebraic spaces (which we require to be completion of qcqs algebraic space at the complement of a quasi-compact open) G induces a cocommutative and commutative bialgebra QCoh(G) in presentable stable categories, which is dualizable as an underlying presentable stable category. The category of quasicoherent sheaves QCoh(X) on a formal qcqs algebraic space is self-dual using the usual duality data for qcqs schemes (Fourier-Mukai functors with diagonal sheaf) except we replace pushforward and pullback functors f_* and f^* with $\Gamma_{Z'} f_* i_Z$ and $\Gamma_Z f^* i_{Z'}$ (i.e. precomposing with torsion incarnation and postcomposing with pullback to the formal scheme). We denote these functors by $\tilde{f}_*$ and $\tilde{f}^*$ below. Duality in $\mathrm{Pr}^L_{\mathrm{St}}$ swaps these pushfoward and pullback functors (altered as above in the formal case) if one identifies QCoh(X) and $\mathrm{QCoh(X)}^\vee$ as above. Hence, the algebra structure on $\mathrm{QCoh(G)}^\vee$ is the convolution tensor product on QCoh(G) induced by pushforward along $G \times G \to G$ and the coalgebra structure is induced by the pushfoward along $G \to G \times G$.

Proposition 0.2.

$$\mathrm{QCoh}(\mathbb{G}_\mathrm{a})^\vee\text{-}\mathrm{Mod} \cong \lim(\mathrm{QCoh}(\mathbb{Q})\text{-}\mathrm{Mod} \rightrightarrows \mathrm{QCoh}(\mathbb{Q}) \otimes_{\mathrm{QCoh}(\mathbb{G}_\mathrm{a})^\vee} \mathrm{QCoh}(\mathbb{Q})\text{-}\mathrm{Mod} \overset{\rightarrow}{\rightrightarrows} \ldots)$$

where Mod *means modules in* $\mathrm{Pr}^L_{\mathrm{St}}$.

Proof. Applying HA Corollary 4.7.5.3, it suffices to show that the functor

$$\_ \otimes_{\mathrm{QCoh}(\mathbb{G}_\mathrm{a})^\vee} \mathbb{Q} : \mathrm{QCoh}(\mathbb{G}_\mathrm{a})^\vee\text{-}\mathrm{Mod} \to \mathbb{Q}\text{-}\mathrm{Mod} \tag{0.5}$$

preserves limits and is conservative.

The base-change isomorphism for the pullback diagram

$$\begin{tikzcd} G \times G \arrow[r, "\pi_1"] \arrow[d, "\mu"] & G \arrow[d] \\ G \arrow[r] & * \end{tikzcd} \tag{0.6}$$

implies that the pullback functor $\mathrm{QCoh}(\mathbb{Q}) \to \mathrm{QCoh}(\mathbb{G}_\mathrm{a})$ is $\mathrm{QCoh}(\mathbb{G}_\mathrm{a})^\vee$-linear (as this is identified with $\mathrm{QCoh}(\mathbb{G}_\mathrm{a})$ with the convolution product). Hence the unit in $\mathrm{QCoh}(\mathbb{Q})$ is $\mathrm{QCoh}(\mathbb{G}_\mathrm{a})^\vee$-atomic and $\mathrm{QCoh}(\mathbb{Q})$ is $\mathrm{QCoh}(\mathbb{G}_\mathrm{a})^\vee$-dualizable and thus (0.5) preserves limits.

Conservativity follows from the fact the limit diagram (in $\mathrm{QCoh}(\mathbb{G}_\mathrm{a})^\vee\text{-}\mathrm{Mod}$)

$$\mathrm{QCoh}(\mathbb{G}_\mathrm{a})^\vee \cong \lim(\mathrm{QCoh}(\mathbb{Q}) \rightrightarrows \mathrm{QCoh}(\mathbb{Q}) \otimes_{\mathrm{QCoh}(\mathbb{G}_\mathrm{a})^\vee} \mathrm{QCoh}(\mathbb{Q}) \overset{\rightarrow}{\rightrightarrows} \ldots)$$

is preserved under tensoring as all nondegenerate transition functors admit linear left adjoints. ■

We can interpret Proposition 0.2 as showing that $\mathrm{QCoh}(\mathbb{G}_\mathrm{a})^\vee \to \mathrm{QCoh}(\mathbb{Q})$ is descendable (where a map of symmetric monoidal stable categories is descendable if there's descent for module categories). Duality theory in the form of Elliptic I Proposition 3.8.5 implies that for symmetric monoidal $\mathbb{Q}$-linear presentable stable categories T,

$$\mathrm{Hom}_{\mathrm{CMon}(T\text{-}\mathrm{Alg}(\mathrm{Pr}^L_{\mathrm{St}})^{\mathrm{op}})}((\mathbb{G}_a)_T, \mathrm{Pic}^\dagger_T) \cong \mathrm{Hom}_{\mathrm{CAlg}(\mathrm{Pr}^L_{\mathrm{St}})}(\mathrm{QCoh}(\mathbb{G}_\mathrm{a})^\vee, T) \tag{0.7}$$

Proposition 0.2 implies that the right hand side is the sheafification of its 1-connective cover. As

$$\Omega\,\mathrm{Hom}_{\mathrm{CAlg}(\mathrm{Pr}^L_{\mathrm{St}})}(\mathrm{QCoh}(\mathbb{G}_\mathrm{a})^\vee, T) \cong \mathrm{Hom}_{\mathrm{Shv}(T\text{-}\mathrm{Alg}(\mathrm{Pr}^L_{\mathrm{St}})^{\mathrm{op}},\mathrm{Sp}^{\mathrm{cn}})}((\mathbb{G}_a[1])_T, \mathrm{Pic}^\dagger_T)$$

we conclude from (0.3) and (0.4) that

$$\Omega\,\mathrm{Hom}_{\mathrm{CAlg}(\mathrm{Pr}^L_{\mathrm{St}})}(\mathrm{QCoh}(\mathbb{G}_\mathrm{a})^\vee, T) \cong \mathrm{Hom}_{\mathrm{CAlg}(\mathrm{Pr}^L_{\mathrm{St}})}(\mathrm{QCoh}(\widehat{\mathbb{G}_\mathrm{m}}), T)$$

and therefore

Proposition 0.3. *For any characteristic zero* 1*-ring* T*,*

$$\mathrm{Hom}_{\mathrm{CAlg}(\mathrm{Pr}^L_{\mathrm{St}})}(\mathrm{QCoh}(\mathbb{G}_\mathrm{a})^\vee, T) \cong (\widehat{\mathbb{G}_m}[1])^\#(T)$$

which implies that $\mathrm{QCoh}(\mathbb{G}_\mathrm{a})^\vee \cong \mathrm{QCoh}(\mathrm{B}\widehat{\mathbb{G}_\mathrm{m}})$ *and (after Proposition 0.2)* $(B\widehat{\mathbb{G}_m})^\#$ *is* 1*-affine*[4].

Corollary 0.4. *For any characteristic zero connective ring* R*,*

$$\mathrm{Hom}_{\mathrm{Shv}(\mathsf{Aff}_{/R},\mathrm{Sp}^{\mathrm{cn}})}(\mathbb{G}_a, (\mathbb{G}_m[1])^\#) \cong (B\widehat{\mathbb{G}_m})^\#_{1-desc}(R) \cong \mathrm{Hom}_{\mathrm{CAlg}(\mathrm{Pr}^L_{\mathrm{St}})}(\mathrm{QCoh}(\mathrm{B}\widehat{\mathbb{G}_\mathrm{m}}), \mathrm{QCoh}(\mathrm{R}))$$

where the sheafification is over R*-linear* 1*-affine schemes with the descendable topology.*

Proof. Direct consequence of the sequence

$$(\mathbb{G}_m[1])^\# \to \mathrm{Pic}^\dagger \to (\mathbb{Z})^\#$$

(which can be seen from Postnikov truncation in Zariski topology) and the fact that all group maps from $\mathbb{G}_a$ to $\mathbb{Z}$ are 0. ∎

Let $\mathbb{G}_{a,dR}$ be defined (on 1-affine schemes) to be the fibre $(B\widehat{\mathbb{G}_a})^\# \to (B\mathbb{G}_a)^\#$. It is corepresentable by the symmetric monoidal category $\mathrm{QCoh}(\mathbb{Q}) \otimes_{\mathrm{QCoh}(\mathrm{B}\widehat{\mathbb{G}_\mathrm{a}})} \mathrm{QCoh}(\mathrm{B}\mathbb{G}_\mathrm{a})$. Because $\widehat{\mathbb{G}_a} \to \mathbb{G}_a$ is (-1)-truncated, $\mathbb{G}_{a,dR}$ is valued in discrete abelian groups for any symmetric monoidal category of characteristic zero. If R is a connective $\mathbb{Q}$-algebra, we have the short exact sequence of abelian groups

$$0 \to R_{red} \to \mathbb{G}_{a,dR}(R) \to H^1_{1-desc}(\mathrm{Spec}\, R, \widehat{\mathbb{G}_a}) \to 0 \tag{0.8}$$

The flatness of $\mathbb{G}_a$ and nilcompleteness of $(B\mathbb{G}_m)^\#$ implies (with the Breen-Deligne resolution) that

$$\mathrm{Hom}_{\mathrm{Shv}(\mathsf{Aff}_{/R},\mathrm{Sp}^{\mathrm{cn}})}(\mathbb{G}_a, (\mathbb{G}_m[1])^\#) \cong (B\widehat{\mathbb{G}_a})^\#_{1-desc}(R)$$

is nilcomplete. It is also infinitesimally cohesive from the Breen-Deligne resolution and the fact that $\mathbb{G}_a$ is flat.

Let u be an element of $\mathrm{Hom}_{\mathrm{Shv}(\mathsf{Aff}_{/R},\mathrm{Sp}^{\mathrm{cn}})}(\mathbb{G}_a, (\mathbb{G}_m[1])^\#)$. The cotangent complex of $\mathrm{Eq}(u, 0)$ (the space of trivializations) at a R-algebra S (and a trivialization ϕ of u over S) exists and is the S-module corepresenting the functor (on connective S-modules)

$$M \mapsto \mathrm{fib}(\widehat{\mathbb{G}_m}(S \oplus M) \to \widehat{\mathbb{G}_m}(S))$$

i.e. the trivial module S. Thus we obtain

$$(B\widehat{\mathbb{G}_m})^\#_{1-desc}(R) \to (B\widehat{\mathbb{G}_m})^\#_{1-desc}(\tau_{\le n} R)$$

[4] where the sheafification is in the descendable topology on 1-affine schemes

is $n+1$-connective. Thus (0.8) only depends on $\pi_0(R)$ (where we identified $\widehat{\mathbb{G}_a}$ and $\widehat{\mathbb{G}_m}$ with the exponential map).

We thus assume R is discrete from now on. Now, by the Breen-Deligne resolution and the fact that $\mathrm{Hom}_{\mathrm{Shv}(\mathsf{Aff}_{/R}, \mathrm{Sp}^{\mathrm{cn}})}(\mathbb{G}_a, (\mathbb{G}_m[1])^{\#})$ can be computed in the étale topos, we know that it commutes with filtered colimits in R. As $\mathrm{Ext}^1(\mathbb{G}_a, \mathbb{G}_m)$ (in any topology) is nil-invariant by deformation theory, we see that it only depends on R_{red} for any connective $\mathbb{Q}$-algebra R. So we may also assume R reduced as well as discrete.[5]

We claim that $\mathbb{G}_{a,dR}(R) \cong R^{awn}$ where R^{awn} is the absolute weak normalization of R (which agrees with the seminormalization of R because we are in characteristic zero). Lemma 0CN8 and Lemma 0EUR of Stacks project imply that the functor $R \mapsto R^{awn}$ preserves filtered colimits. Hence it suffices to show this statement for finite-type $\mathbb{Q}$-algebras, as long as we show it functorially.

$\mathbb{G}_{a,dR}$ is an h-sheaf on the site of underived(!) finite-type affine schemes over $\mathbb{Q}$ because any h-cover is descendable and $\mathbb{G}_{a,dR}(R)$only depends on the classical reduced part of R. Hence, there's a map from $(R_{\mathrm{red}})^{\#}_h$ to $\mathbb{G}_{a,dR}$. The natural transformation $\mathbb{G}_{a,dR}(R) \to \mathbb{G}_{a,dR}(R^{awn})$ is pointwise an isomorphism by h-descent along $R \to R^{awn}$. Now, we are done because the map $R \to \mathbb{G}_{a,dR}(R)$ is an isomorphism when R is absolute weakly normal (= seminormal because we are in characteristic zero) as $\mathrm{Ext}^1(\mathbb{G}_a, \mathbb{G}_m)$ vanishes (by the Breen-Deligne resolution and the $\mathbb{A}^1$-invariance of Pic as a 1-truncated space on seminormal characteristic zero schemes).

[5]Compare with Lemma 6.4 of Ribeiro-Rosengarten