Embedded Control Laboratory

Report of Lab Exercise 1

Computational Fluid Dynamics

Modelling Flow in Pipes

WS 2022/23

Group 8

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Brief Introduction

Mathematical Modelling

Modeling refers to representing a real-world problem or situation in mathematical terms so it can be analyzed for a better understanding, and can also be used to predict future behaviors of the system. After the real-world variables are mapped to each other, various approximations, assumptions, and mathematical tools (such as the Taylor Series or the Fourier Transform) are used to simplify it. Most of the functions that are the result of mapping these problems are non-linear in nature. The simplification is often to linearize these functions.

While modeling any real-world problem into a mathematical one, we must consider the following elements:

1. Variables of interest (e.g. flow rate, pressure)
2. Known variables (e.g. dimensions, density)
3. Fundamental laws/models to build on (e.g. Laws of Conservation)

   OR/AND

   Assumptions

4. Hidden variables (e.g. convective acceleration)

Going through these in a step-wise manner will result in an accurate initial mathematical model, which can then be simplified for further analysis.

Navier Stokes Equations

The Navier Stokes Equations are a set of equations that describe how the velocity field of a fluid evolves over time. Compressible liquids require conservation of energy formulation. These equations are the extensions of Newton's second law of motion.

1. Mass Conservation Equation

$$\frac{\partial \rho}{\partial t} \; +.\,(\rho u) \; = \; 0 \tag{1}$$

2. Momentum Conservation Equation :

$$\rho(\frac{\partial u}{\partial t} + \; (u.\nabla)u) \; = \; -\,\nabla P \; + \; \nabla.\tau \; + \; F \tag{2}$$

where ρ = density of fluid, t = time, and u = flow rate of fluid, P = Pressure, τ = stress tensor, F = external forces (often gravity).

The gradient operator ∇ is of the form:

in the cartesian coordinate system, and,

$$\nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}) \tag{3}$$

in the cylindrical coordinate system

$$\nabla = (\frac{\partial}{\partial r}, \frac{1}{r}\frac{\partial}{\partial \theta}, \frac{\partial}{\partial z}) \tag{4}$$

Assumptions, and their implications

In order to simplify these equations, we make the following assumptions:

1. Steady Flow
   The flow parameters like velocity and pressure do not change with time. Mathematically, for all variables:

$$\frac{\partial}{\partial t} = 0 \tag{5}$$

2. Incompressible Flow
   This means that the fluid density is constant. Mathematically,

$$\rho = constant \tag{6}$$

   And

$$\nabla . u = 0 \tag{7}$$

3. Newtonian Fluid
   This means that the fluid has a constant viscosity μ, i.e. the relationship between the shear stress and shear rate is linear. So, mathematically,

$$\mu = constant \tag{8}$$

   Also, when we combine assumption numbers 2 and 3, for an incompressible Newtonian fluid, the stress tensor in the equation is given by:

$$\nabla.\tau = \mu\nabla^2 u \tag{9}$$

4. Axisymmetric Flow
   This means that the properties of fluid flow remain invariant around the axis. Mathematically,

$$\frac{\partial}{\partial\theta} = 0 \tag{10}$$

   And

$$u_\theta = 0 \tag{11}$$

5. Fully Developed Flow
   This means that the layers of the fluid at the boundaries are stationary. Or, mathematically,

$$\frac{\partial u_z}{\partial z} = 0 \tag{12}$$

   But

$$\frac{\partial P}{\partial z} \neq 0 \tag{13}$$

6. No radial velocity
   There is no radial velocity in the fluid. Mathematically,

$$u_r = 0 \tag{14}$$

Boundary Conditions

Boundary conditions are essential specifications that define the behavior of a solution within a specific region or at the boundaries of a problem domain. These conditions provide information about the system under consideration and are crucial for determining a unique solution. Boundary conditions are often applied to differential equations that describe physical phenomena.

For the Navier Stokes equation in our problem statement, the following boundary conditions have been defined.

- At r = 0, $\frac{\partial u_z}{\partial r} = 0$

  This condition means that the radial velocity gradient is zero at the center of the coordinate system. This condition is sometimes referred to as "no-slip" or "zero-gradient" at the center.

- At r = R, $u_z = 0$

  This condition indicates that the axial (z-direction) velocity is zero at the radial boundary of the cylindrical system,

- At z = 0, $u_z = 15$

  This condition indicates that the axial (z-direction) velocity at the plane z=0 is equal to 15.

- At z = L, $u_z = 15$

  This condition indicates that the axial (z-direction) velocity at the plane z=L is equal to 15.

Parameters:

• Pipe Diameter D = 0.1 m

• Pipe Length L = 1 m

• Viscosity μ = 0.01 kg/m.s

• Constant pressure gradient ∂p/∂z = −100 Pa/m

• Choose N = 100 (or more) grid points

• Base viscosity μ_0 = 0.05 kg/m.s

• Temperature expansion coefficient α = 0.08 1/K

• Reference temperature T0 = 273 K

• β = 5 K/m2

• γ = 0.25

TASK 1

Problem Statement:

Model steady-state and full-developed axial flow of incompressible Newtonian fluid in cylindrical pipe assuming axisymmetric flow with no radial velocity.

• Formulate the PDE by simplifying the Navier Stokes equations for cylindrical coordinates.
• Convert the PDE into difference equation.

Authors Approach

Applying the assumptions (5) to (14) and simplifying the equations:

1. Mass Conservation Equation

$$\frac{\partial \rho}{\partial t} + \nabla . (\rho u) = 0 \tag{15}$$

According to assumption (2), $\rho = constant$. So, this equation reduces to:

$$\nabla . u = 0 \tag{16}$$

Solving further,

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} = 0 \tag{17}$$

2. Momentum Conservation Equation

$$\rho(\frac{\partial u}{\partial t} + (u.\nabla)u) = -\nabla P + \nabla.\tau + F \tag{18}$$

using assumption (1) and (3),

$$\rho((u.\nabla)u) = -\nabla P + \mu\nabla^2 u + F \tag{19}$$

Solving this equation further,

$$\rho((u.\nabla)u) = \rho \begin{bmatrix} u_x\frac{\partial u_x}{\partial x} + u_y\frac{\partial u_x}{\partial y} + u_z\frac{\partial u_x}{\partial z} \\ u_x\frac{\partial u_y}{\partial x} + u_y\frac{\partial u_y}{\partial y} + u_z\frac{\partial u_y}{\partial z} \\ u_x\frac{\partial u_z}{\partial x} + u_y\frac{\partial u_z}{\partial y} + u_z\frac{\partial u_z}{\partial z} \end{bmatrix} \tag{20}$$

$$-\nabla P = -\begin{bmatrix} \frac{\partial P}{\partial x} \\ \frac{\partial P}{\partial y} \\ \frac{\partial P}{\partial z} \end{bmatrix} \tag{21}$$

$$\mu\nabla^2 u = \mu \begin{bmatrix} \frac{\partial^2 u_x}{\partial_x^2} + \frac{\partial^2 u_x}{\partial y^2}\frac{+\partial^2 u_x}{\partial z^2} \\ \frac{\partial^2 u_y}{\partial_x^2} + \frac{\partial^2 u_y}{\partial y^2}\frac{+\partial^2 u_y}{\partial z^2} \\ \frac{\partial^2 u_z}{\partial_x^2} + \frac{\partial^2 u_z}{\partial y^2}\frac{+\partial^2 u_z}{\partial z^2} \end{bmatrix} \tag{22}$$

putting all these together,

$$\rho \begin{bmatrix} u_x\frac{\partial u_x}{\partial x} + u_y\frac{\partial u_x}{\partial y} + u_z\frac{\partial u_x}{\partial z} \\ u_x\frac{\partial u_y}{\partial x} + u_y\frac{\partial u_y}{\partial y} + u_z\frac{\partial u_y}{\partial z} \\ u_x\frac{\partial u_z}{\partial x} + u_y\frac{\partial u_z}{\partial y} + u_z\frac{\partial u_z}{\partial z} \end{bmatrix} = -\begin{bmatrix} \frac{\partial P}{\partial x} \\ \frac{\partial P}{\partial y} \\ \frac{\partial P}{\partial z} \end{bmatrix} + \mu \begin{bmatrix} \frac{\partial^2 u_x}{\partial_x^2} + \frac{\partial^2 u_x}{\partial y^2}\frac{+\partial^2 u_x}{\partial z^2} \\ \frac{\partial^2 u_y}{\partial_x^2} + \frac{\partial^2 u_y}{\partial y^2}\frac{+\partial^2 u_y}{\partial z^2} \\ \frac{\partial^2 u_z}{\partial_x^2} + \frac{\partial^2 u_z}{\partial y^2}\frac{+\partial^2 u_z}{\partial z^2} \end{bmatrix} \tag{23}$$

Further solving these, we can extract three equations from this equation,

$$\rho\frac{\partial u_x}{\partial t} + \rho u_x\frac{\partial u_x}{\partial x} + \rho u_y\frac{\partial u_x}{\partial y} + \rho u_z\frac{\partial u_x}{\partial y} = -\frac{\partial P}{\partial x} + \mu\frac{\partial^2 u_x}{\partial x^2} + \mu\frac{\partial^2 u_x}{\partial y^2} + \mu\frac{\partial^2 u_x}{\partial z^2} \tag{24}$$

$$\rho\frac{\partial u_y}{\partial t} + \rho u_x\frac{\partial u_y}{\partial x} + \rho u_y\frac{\partial u_y}{\partial y} + \rho u_z\frac{\partial u_y}{\partial y} = -\frac{\partial P}{\partial x} + \mu\frac{\partial^2 u_y}{\partial x^2} + \mu\frac{\partial^2 u_y}{\partial y^2} + \mu\frac{\partial^2 u_y}{\partial z^2} \tag{25}$$

$$\rho\frac{\partial u_z}{\partial t} + \rho u_x\frac{\partial u_z}{\partial x} + \rho u_y\frac{\partial u_z}{\partial y} + \rho u_z\frac{\partial u_z}{\partial y} = -\frac{\partial P}{\partial x} + \mu\frac{\partial^2 u_z}{\partial x^2} + \mu\frac{\partial^2 u_z}{\partial y^2} + \mu\frac{\partial^2 u_z}{\partial z^2} \tag{26}$$

Converting these equations to cylindrical coordinates, we get the following four final equations:

1. Flow of Fluid in cylindrical coordinates

$$\frac{1}{r}\frac{\partial(ru_r)}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} = 0 \tag{27}$$

2. Radial component, u_r

$$\rho\left(\frac{\partial u_r}{\partial t} + u_r\frac{\partial u_r}{\partial r} + u_\theta\frac{1}{r}\frac{\partial u_r}{\partial \theta} + u_z\frac{\partial u_r}{\partial z}\right) = -\frac{\partial P}{\partial r} + \mu\left(\nabla^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}\right) + F_r \tag{28}$$

3. Azimuthal component, u_θ

$$\rho\left(\frac{\partial u_\theta}{\partial t} + u_r\frac{\partial u_\theta}{\partial r} + u_\theta\frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + u_z\frac{\partial u_\theta}{\partial z}\right) = -\frac{1}{r}\frac{\partial P}{\partial r} + \mu\left(\nabla^2 u_\theta - \frac{u_\theta}{r^2} + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta}\right) + F_\theta \tag{29}$$

4. Axial component, u_z

$$\rho\left(\frac{\partial u_z}{\partial t} + u_r\frac{\partial u_z}{\partial r} + u_\theta\frac{1}{r}\frac{\partial u_z}{\partial \theta} + u_z\frac{\partial u_z}{\partial z}\right) = -\frac{\partial P}{\partial z} + \mu\nabla^2 u_z + F_z \tag{30}$$

In these equations, ∇^2 for each component in cylindrical coordinates will be:

For u_r :

$$\nabla^2 u_r = \frac{\partial^2 u_r}{\partial r^2} + \frac{1}{r}\frac{\partial u_r}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u_r}{\partial \theta^2} + \frac{\partial^2 u_r}{\partial z^2} \tag{31}$$

For u_θ :

$$\nabla^2 u_\theta = \frac{\partial^2 u_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial u_\theta}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial \theta^2} + \frac{\partial^2 u_\theta}{\partial z^2} \tag{32}$$

For u_z :

$$\nabla^2 u_z = \frac{\partial^2 u_z}{\partial r^2} + \frac{1}{r}\frac{\partial u_z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u_z}{\partial \theta^2} + \frac{\partial^2 u_z}{\partial z^2} \tag{33}$$

Because the problem that will be solved is time invariant flow (steady state) several assumptions can be made to simplifies the Equations 27-30

Incompressible Flow : $Density\ \rho\ is\ constant$

Steady State:

$$\frac{\partial}{\partial t} = 0 \tag{34}$$

Axisymmetric Flow :

$$\frac{\partial}{\partial \theta} = 0 \tag{35}$$

and

$$u_\theta = 0 \tag{36}$$

Fully Developed:

$$\frac{\partial u_z}{\partial z} = 0 \tag{37}$$

but

$$\frac{\partial p}{\partial z} \neq 0 \tag{38}$$

No Radial Velocity:

$$u_r = 0 \tag{39}$$

Newtonian Fluid: μ $is\ constant$

No outside force acting on the fluid $F = 0$

Substituting Equations 39 to Eq 31, Equation 31 becomes:

$$\nabla^2 u_r = \frac{\partial^2 0}{\partial r^2} + \frac{1}{r}\frac{\partial 0}{\partial r} + \frac{1}{r^2}\frac{\partial^2 0}{\partial \theta^2} + \frac{\partial^2 0}{\partial z^2} \tag{40}$$

$$\nabla^2 u_r = 0 \tag{41}$$

Substituting Equations 35 and 36 to Equaton 32, Equation 32 becomes:

$$\nabla^2 u_\theta = \frac{\partial^2 u_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial u_\theta}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial \theta^2} + \frac{\partial^2 u_\theta}{\partial z^2} \tag{42}$$

$$\nabla^2 u_\theta = 0 \tag{43}$$

Substituting Assumptions 37 to 39 to Equation 33, Equation 33 becomes:

$$\nabla^2 u_z = \frac{\partial^2 u_z}{\partial r^2} + \frac{1}{r}\frac{\partial u_z}{\partial r} + \frac{1}{r^2}0 + 0 \tag{44}$$

$$\nabla^2 u_z = \frac{\partial^2 u_z}{\partial r^2} + \frac{1}{r}\frac{\partial u_z}{\partial r} \tag{45}$$

Upon simplifying $\nabla^2 u_r$, $\nabla^2 u_\theta$, and $\nabla^2 u_z$, and all of the assumptions and substitute it to Radial Component u_r, Azimuthal Component u_θ and Axial Component u_z,

Radial Component:

$$\rho(0 + 0\frac{\partial 0}{\partial r} + 0\frac{1}{r}\frac{\partial 0}{\partial \theta} + u_z\frac{\partial 0}{\partial z}) = -\frac{\partial P}{\partial r} + \mu(0 - \frac{0}{r^2} - \frac{2}{r^2}\frac{\partial 0}{\partial \theta}) + 0 \tag{46}$$

$$\rho(0) = -\frac{\partial P}{\partial r}(0) \tag{47}$$

Azimuthal Component:

$$\rho(0 + 0\frac{\partial 0}{\partial r} + 0\frac{1}{r}\frac{\partial 0}{\partial \theta} + u_z\frac{\partial 0}{\partial z}) = -\frac{1}{r}\frac{\partial P}{\partial r} + \mu(0 - \frac{\partial 0}{r^2} + \frac{2}{r^2}\frac{\partial 0}{\partial \theta}) + 0 \tag{48}$$

$$\rho(0) = -\frac{\partial P}{\partial r}(0) \tag{49}$$

Axial Component:

$$\rho(0 + 0\frac{\partial u_z}{\partial r} + 0\frac{1}{r}0 + u_z 0) = -\frac{\partial P}{\partial z} + \mu(\frac{\partial^2 u_z}{\partial r^2} + \frac{1}{r}\frac{\partial u_z}{\partial r}) + 0 \tag{50}$$

$$\frac{\partial P}{\partial r} = \mu(\frac{\partial^2 u_z}{\partial r^2} + \frac{1}{r}\frac{\partial u_z}{\partial r}) \tag{51}$$

Therefore the model for steady state and fully developed axial flow of incompressible Newtonian Fluid in cylindrical pipe assuming axisymmetric flow with no radial velocity is as follows:

$$\frac{\partial P}{\partial r} = \mu(\frac{\partial^2 u_z}{\partial r^2} + \frac{1}{r}\frac{\partial u_z}{\partial r}) \tag{52}$$

There are many methods by which the solutions of a differential equation can be found. These Finite Difference Methods. By using Finite Difference Methods can help to discretize a continuous system by sampling it. Sampling steps can be created at each discrete step within boundary conditions that have been defined, which for this problem the boundary conditions are as follows:

- At r = 0, $\frac{\partial u_z}{\partial r} = 0$
- At r = R, $u_z = 0$
- At z = 0, $u_z = 15$
- At z = L, $u_z = 15$

So, approximations of the derivatives by using a finite difference method are as follows:

$$\frac{\partial^2 u_z}{\partial r^2} = \frac{u_{z,i+1} - 2u_{z,i} + u_{z,i-1}}{\Delta r^2} \tag{53}$$

$$\frac{\partial u_z}{\partial r} = \frac{u_{z,i+1} - u_{z,i}}{\Delta r} \tag{54}$$

By using approximations of finite difference method into equation (52), the equation changed from continuous form to discrete form.

$$\frac{\partial P}{\partial z} = \mu\left(\frac{u_{z,i+1} - 2u_{z,i} + u_{z,i-1}}{\Delta r^2} + \frac{1}{r}\frac{u_{z,i+1} - u_{z,i}}{\Delta r}\right) \tag{55}$$

TASK 2

Problem Statement:

Write a Python program to solve the system of linear equations obtained from difference equations to solve for u_z as a function of r, and plot the u_z as a function of r to visualize the velocity profile in the pipe for the boundary condition and parameters given.

Differential equation:

$$\frac{\partial^2 u_z}{\partial r^2} + \frac{\partial u_z}{\partial r} = P \tag{56}$$

Where P is Continuous : Numerical partial differential equation

Authors Approach

Generally, to solve the equation at r > 0 where r $\in \mathbb{R}$ Equation 55 will be used, however because some of the boundary conditions have been defined, when r = radius of the pipe

$$u_z = 0 \tag{57}$$

therefore, the equation 52 becomes:

$$u_{z,i} = 0 \tag{58}$$

and when r = 0,

$$\frac{\partial u_z}{\partial r} = 0 \tag{59}$$

therefore, the equation 52 becomes:

$$\frac{\partial p}{\partial z} = \mu \frac{\partial^2 u_z}{\partial r^2} \tag{60}$$

By implementing finite difference method,

$$\frac{\partial p}{\partial z} = \mu\left(\frac{u_{z,i+1} - 2u_{z,i} + u_{z,i-1}}{\Delta r^2}\right) \tag{61}$$

Upon simplifying the equation 55 when its not in the defined boundary conditions, the equation 55 becomes

$$\frac{\partial p}{\partial z}\frac{\Delta r^2}{\mu} = u_{z,i+1} - 2u_{z,i} + u_{z,i-1} + \frac{\Delta r}{r}(u_{z,i+1} - u_{z,i}) \tag{62}$$

The system where the calculation will be used upon has some properties which are:

- $Pipe\ Radius\ (R) = 0.01$ m
- $Pipe\ Length\ (L) = 1$ m
- $Constant\ Pressure\ Gradient\ (\frac{\partial p}{\partial z}) = -100$ Pa/m
- $Fluid\ Viscosity\ (\mu) = 0.01$ kg/m.s

Because the Flow of fluid in Axial Component is dependent on radius of the pipe, 101 sample points on the radius have been chosen with radius interval = 0.0005m, therefore:

$\Delta r = 0.0005$ m

Upon simplifying Equation 55 becomes:

$$\frac{\partial p}{\partial z}\frac{\Delta r^2}{\mu} = \left(1+\frac{\Delta r}{r}\right)u_{z,i+1} - \left(2+\frac{\Delta r}{r}\right)u_{z,i} + u_{z,i-1} \tag{63}$$

Upon substituting the system properties Equation 62 becomes:

$$-100\frac{0.0005^2}{0.01} = u_{z,i+1} - 2u_{z,i} + u_{z,i-1} + \frac{0.0005}{r}(u_{z,i+1} - u_{z,i}) \tag{64}$$

$$-2.5x10^{-1} = \left(1+\frac{0.0005}{r}\right)u_{z,i+1} - \left(2+\frac{0.0005}{r}\right)u_{z,i} + u_{z,i-1} \tag{65}$$

and equations for the boundary conditions become:

when r = 0

$$-2.5x10^{-1} = u_{z,\,i+1} - 2\,u_{z,\,i} + u_{z,\,i-1} \tag{66}$$

when r = Radius of pipe

$$u_{z,\,i} = 0 \tag{67}$$

With 101 unknowns and 101 equations, all of the unknown can be found.

Linear Solver function (numpy.linalg.solve) from Numpy library will be used to get the value of u_z in every defined point. In order to use the linear solver function, two parameters are needed which are A Matrix and B Matrix which will be configured as followed:

$$A[NxN].\,x[N] = B[N]$$

The authors chose to start by initializing the matrix of A and B with a set of values, then readjusting the value of A and B according to Equation 64, Equation 65, and Equation 66.

After readjusting the values of Matrix A and Matrix B, then ran the matrixes through linear solver function a plot of u_z is made using a library in python, which is called matplotlib. The output of the graph is as followed:

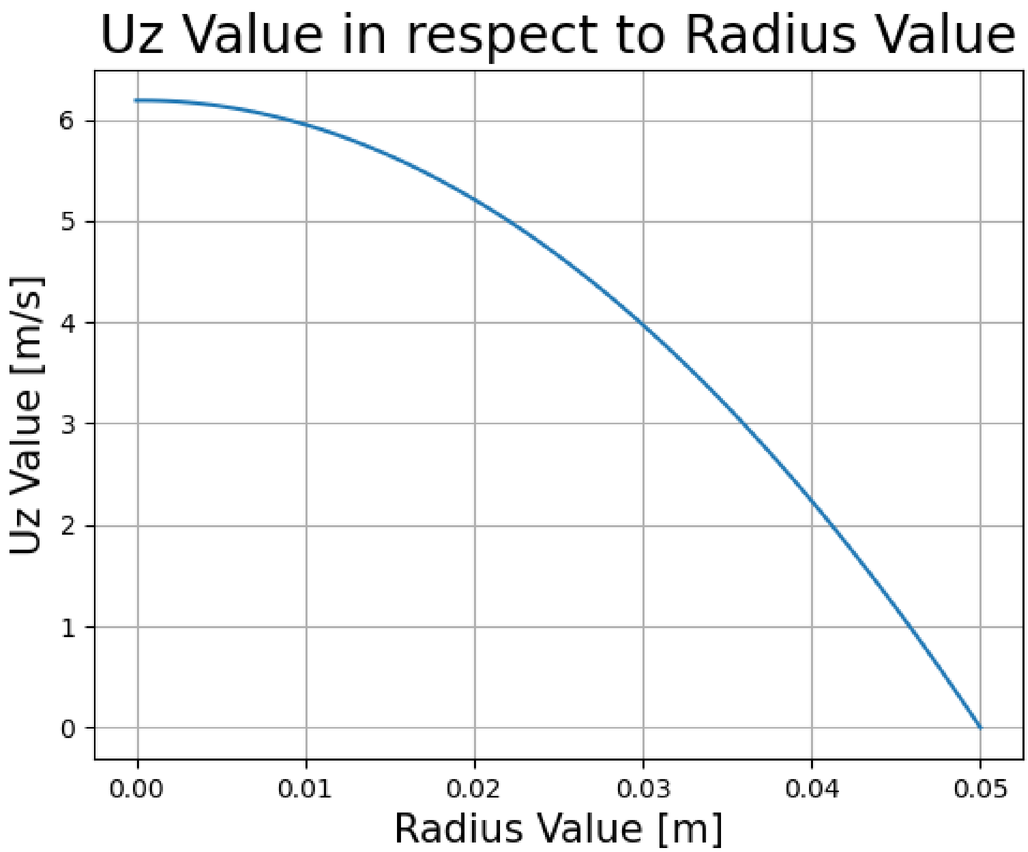

Figure 1. Plot of u_z in the system where viscosity constant value of 0.01 kg/m.s

Task 2 Conclusion

The system behaved like a quadratic function because the fluid layers experience shear stress as they slide over each other. Viscosity is the property of a fluid that resists this shearing motion. The fluid layers near the pipe wall experience higher shear stress due to their close proximity to the stationary wall, causing them to move more slowly.

In laminar flow conditions, the velocity profile takes on a parabolic shape. The maximum velocity occurs at the center of the pipe, where the fluid layers experience the least resistance to motion. The velocity decreases smoothly as you move from the center towards the pipe walls.

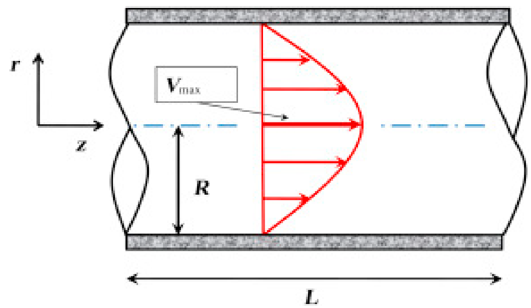

Figure 2. An overview of Hagen-Poiseuille Flow

This phenomenon is described by the Hagen–Poiseuille equation for laminar flow which explains that there are differences in pressure of a fluid between two ends of a pipe. Which the equations is as followed:

$$\Delta p = \frac{8\mu L Q}{\pi r^2} \quad (68)$$

Where:

- Δp: Pressure Difference
- μ: Fluid Viscosity
- L: Pipe Length
- Q: Volumetric Flow Rate
- r: Pipe Radius

Hagen-Poiseuille Equation stated that the pressure of the fluid is decreased proportional to the value of squared radius or area of the pipe.

Also this phenomenon can be explained by the conservation of mass equation and fluid continuity equation which are:

$$m' = \rho A V \quad (69)$$

$$A_1 V_1 = A_2 V_2 \; where \; AV \; = constant \tag{70}$$

Where:

- m': Pressure Difference
- ρ: Fluid Density
- A: Area of the Pipe
- V: Velocity of the fluid

As the Area increases, velocity of the fluid decreases. The differences of V_1 and V_2 will be reflected as a quadratic function because Area of a pipe is a quadratic function.

TASK 3

Problem Statement:

Write a Python program to model flow in pipe, where viscosity of the fluid is temperature dependent given and the temperature varies with radius as given below.

$$\mu(T) = \mu_0(1 + \alpha(T - T_0)) \tag{71}$$

$$T(r) = T_0 + \beta r^2 \tag{72}$$

Authors Approach

In Task 1 and 2, the viscosity of the fluid was taken to be constant. Here, however, the viscosity is changing with temperature, as is defined by the equations above.

Simplifying the equations for this task,

$$\mu = 0.05(1 + 0.08(T - 273) \tag{73}$$

Now, using the second equation given for T,

$$\mu = 0.05(1 + 0.08(273 + 5 * r^2 - 273) \tag{74}$$

$$\mu = 0.05(1 + 0.08(5r^2) \tag{75}$$

$$\mu = 0.05 + 0.02r^2 \tag{76}$$

The expression for μ obtained here can be substituted in our original equation which is,

$$\frac{\partial P}{\partial z} = \mu\left(\frac{\partial^2 u_z}{\partial} + \frac{1}{r} \frac{\partial u_z}{\partial r} \right) \tag{77}$$

So, combining these two equations,

$$\frac{\partial P}{\partial z} = (0.05 + 0.02r^2)\left(\frac{\partial^2 u_z}{\partial} + \frac{1}{r} \frac{\partial u_z}{\partial r} \right) \tag{78}$$

Solving this further,

$$\frac{\partial P}{\partial z} = 0.05\frac{\partial^2 u_z}{\partial} + \frac{0.05}{r} \frac{\partial u_z}{\partial r} + 0.02r^2 \frac{\partial^2 u_z}{\partial} + 0.02r \frac{\partial u_z}{\partial r} \tag{79}$$

$$\frac{\partial P}{\partial z} = 0.05\left(\frac{u_{z,i+1} - 2u_{z,i} + u_{z,i-1}}{\Delta r^2} + \frac{1}{r}\frac{u_{z,i+1} - u_{z,i}}{\Delta r}\right) + 0.02r^2\left(\frac{u_{z,i+1} - 2u_{z,i} + u_{z,i-1}}{\Delta r^2}\right) + 0.02r\left(\frac{u_{z,i+1} - u_{z,i}}{\Delta r}\right) \tag{80}$$

$$\frac{\partial P}{\partial z}\Delta r^2 = \frac{1}{20}u_{z,i+1} - \frac{2}{20}u_{z,i} + \frac{1}{20}u_{z,i-1} + \frac{\Delta r}{20r}u_{z,i+1} - \frac{\Delta r}{20r}u_{z,i} + \frac{r^2}{50}u_{z,i+1} - \frac{2r^2}{50}u_{z,i} + \frac{r^2}{50}u_{z,i} + \frac{r^2}{50}u_{z,i+1} - \frac{2r^2}{50}u_{z,i} + \frac{r^2}{50}u_{z,i} + \frac{\Delta r.r}{50}u_{z,i+1} - \frac{\Delta r.r}{50}u_{z,i} \tag{81}$$

Simplifying into the general form, and taking $\Delta r = 0.0005$,

$$\frac{\partial P}{\partial z}\Delta r^2 = \left(\frac{2r^3+0.01r^2+5r+0.025}{100r}\right)u_{z,i+1} - \left(\frac{4r^3+0.01r^2+10r+0.025}{100r}\right)u_{z,i} + \left(\frac{2r^2+5}{100}\right)u_{z,i-1} \tag{82}$$

This is a linear equation which can now be solved using matrices in Python by changing the coefficients in the program that was created for Task 2 using Numpy Linear Solver and plotted using the same library that is used in Task 2 which the output graph is as followed:

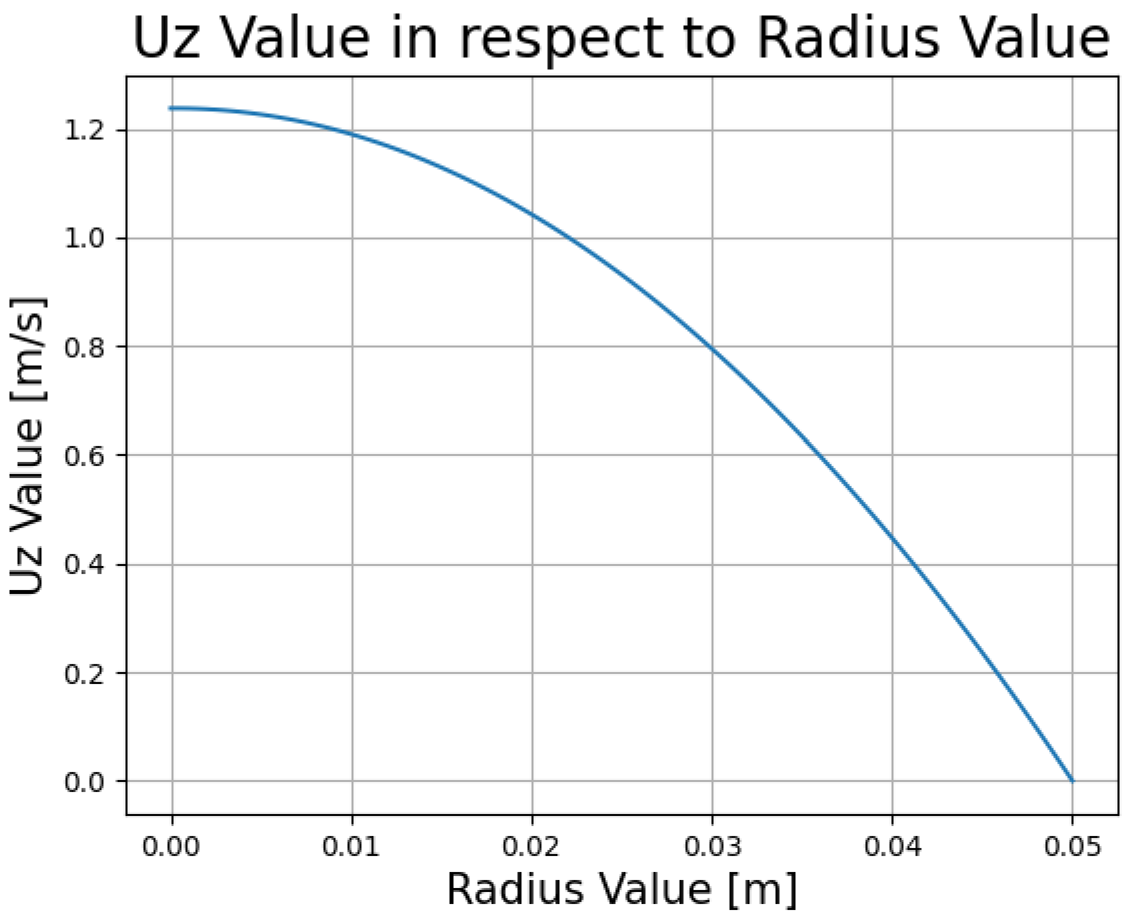

Figure 3. Plot of u_z in the system where viscosity is changing according to the temperature

Task 3 Conclusion

The characteristics of the system remain the same as the previous task where there is no change in viscosity in respect to temperature changes. This behavior can be explained by the Arrhenius Equation for Newtonian fluid.

$$\eta \ = \eta_0 e^{\frac{E}{RT}} \tag{83}$$

Where:

- η : Viscosity at temperature T
- η_0: Viscosity at reference temperature
- E : Activation energy for flow
- R : Universal gas constant
- T : Absolute Temperature

Although Arrhenius Equation for Newtonian Fluid takes form in exponential value which means it doesn't have linear characteristics, but over small changes in the temperature, usually linear approximation using Taylor Series is taken for this problem. Which the equation becomes:

$$\eta(T) \ = \eta_0(1 \ + \ A(T \ - \ T_0)) \tag{84}$$

Where:

- η_0: the viscosity at a reference temperature (T_0)
- A : coefficient that depends on the fluid and temperature range.

In addition to that, according to given Equation (Equation 72) the value of T is changing with respect to radius squared. Hence after substituting and simplifying the new viscosity value to equation 77, overall system characteristics remain as a quadratic function, hence explaining the quadratic characteristics of the graph.

As the initial viscosity value (initial viscosity will be the maximum velocity because as the radius increases, the velocity will also increases which according to equation 69, will resulted in slower velocity) in task 3 is 5 times viscosity value in task 2, explains the maximum velocity changes is 5 times slower than maximum velocity in task 2 because the system behaves linearly.

TASK 4

Problem Statement:

Consider the model for the flow of an incompressible Newtonian fluid in a pipe where the velocity profile u_z varies with both radial (r) and axial (z) coordinates, according to the equations given.

• Write a Python program to solve the system of linear equations obtained by converting the equation into difference equations.

• Create a 3D plot of velocity profile as a function of both radial (r) and axial (z) coordinates.

Authors Approach

The differences between Task 2 and Task 3 with Task 4 is the task 2 and task 3 only take radius of the pipe as the consideration for the u_z, however on the Task 4 length of the pipe is also take into consideration for the value of u_z. Therefore the equation for u_z varies with both radial and axial coordinates is:

$$\frac{\partial p}{\partial z} = \mu \left(\frac{\partial^2 u_z}{\partial r^2} + \frac{1}{r}\frac{\partial u_z}{\partial r}\right) + \gamma \frac{\partial^2 u_z}{\partial z^2} \tag{85}$$

Boundary conditions for this problems is as followed:

- At $r = 0$, $\frac{\partial u_z}{\partial r} = 0$
- At $r = R$, $u_z = 0$
- At $z = 0$, $u_z = 15$
- At $z = L$, $u_z = 15$

$$\frac{\partial p}{\partial z} = \mu \frac{u_{z,j,i+1} - 2u_{z,j,i} + u_{z,j,i-1}}{\Delta r^2} + \frac{1}{r}\left(\frac{u_{z,j,i+1} - u_{z,j,i}}{\Delta r}\right) + \gamma \frac{u_{z,j+1,i} - 2u_{z,j,i} + u_{z,j-1,i}}{\Delta z^2} \tag{86}$$

To simplify writing the equation, the author chose to substituting u_z into the notation:

$$u_{z,j,i+1} = a \tag{87}$$

$$u_{z,j,i} = b \tag{88}$$

$$u_{z,j,i-1} = c \tag{89}$$

$$u_{z,j+1,i} = d \tag{90}$$

$$u_{z,j-1,i} = e \tag{91}$$

Therefore the Equation (86) becomes:

$$\frac{\partial p}{\partial z}\frac{1}{\mu} = \frac{1}{\Delta r^2} a + \frac{1}{r\Delta r} a - \frac{2}{\Delta r^2} b - \frac{1}{r\Delta r} b - \frac{2\gamma}{\mu\,\Delta z^2} b + \frac{1}{\Delta r^2} c + \frac{\gamma}{\mu\,\Delta z^2} d + \frac{\gamma}{\mu\,\Delta z^2} e \tag{92}$$

$$\frac{\partial p}{\partial z}\frac{1}{\mu} = \left(\frac{1}{\Delta r^2} + \frac{1}{r\Delta r}\right) a - \left(\frac{2}{\Delta r^2} + \frac{1}{r\Delta r} + \frac{2\gamma}{\mu\,\Delta z^2}\right) b + \frac{1}{\Delta r^2} c + \frac{\gamma}{\mu\,\Delta z^2} d + \frac{\gamma}{\mu\,\Delta z^2} e \tag{93}$$

By inputting the value of the boundary conditions:

At $r = 0$, $\frac{\partial u_z}{\partial r} = 0$

$$\frac{\partial p}{\partial z} = \mu \frac{u_{z,j,i+1} - 2u_{z,j,i} + u_{z,j,i-1}}{\Delta r^2} + \gamma \frac{u_{z,j+1,i} - 2u_{z,j,i} + u_{z,j-1,i}}{\Delta z^2} \tag{94}$$

At $r = R$, $u_z = 0$

$$0 = u_{z,j,i} \tag{95}$$

At $z = 0$, $u_z = 15$

$$15 = u_{z,j,i} \tag{96}$$

At $z = L$, $u_z = 15$

$$15 = u_{z,j,i} \tag{97}$$

After calculating the equation for the boundary conditions and normal conditions, matrices of A and B are made:

$$A[10201x10201].x[101x101] = B[1x10201]$$

However due to the limitation of linear solver can only solve a 1 dimensional matrix, the authors chose to convert the matrices into:

$$A[10201x10201].x[1x10201] = B[1x10201]$$

Then, a mapping function (idx) is created which will map the value of the index i,j in the x matrices, where if x[i][j] the function is as followed:

$$idx = i + j * Number\ of\ Samples\ in\ radius \tag{98}$$

After reconfiguring matrices, the authors ran the matrices through the linear solver function which then the output value of the linear solver will be remapped back again to the previous format x[i][j]

With a help of function called inverseIdx where the calculation is as followed:

$$i = idx \ \% \ Number\ of\ Samples\ in\ radius \tag{99}$$

$$j = int\ (\ idx\ /\ Number\ of\ Samples\ in\ radius) \tag{100}$$

After remapping the output of calculated u_z into 2 dimensional matrix, the authors plotted the graph of u_z which can be seen in Figure 3

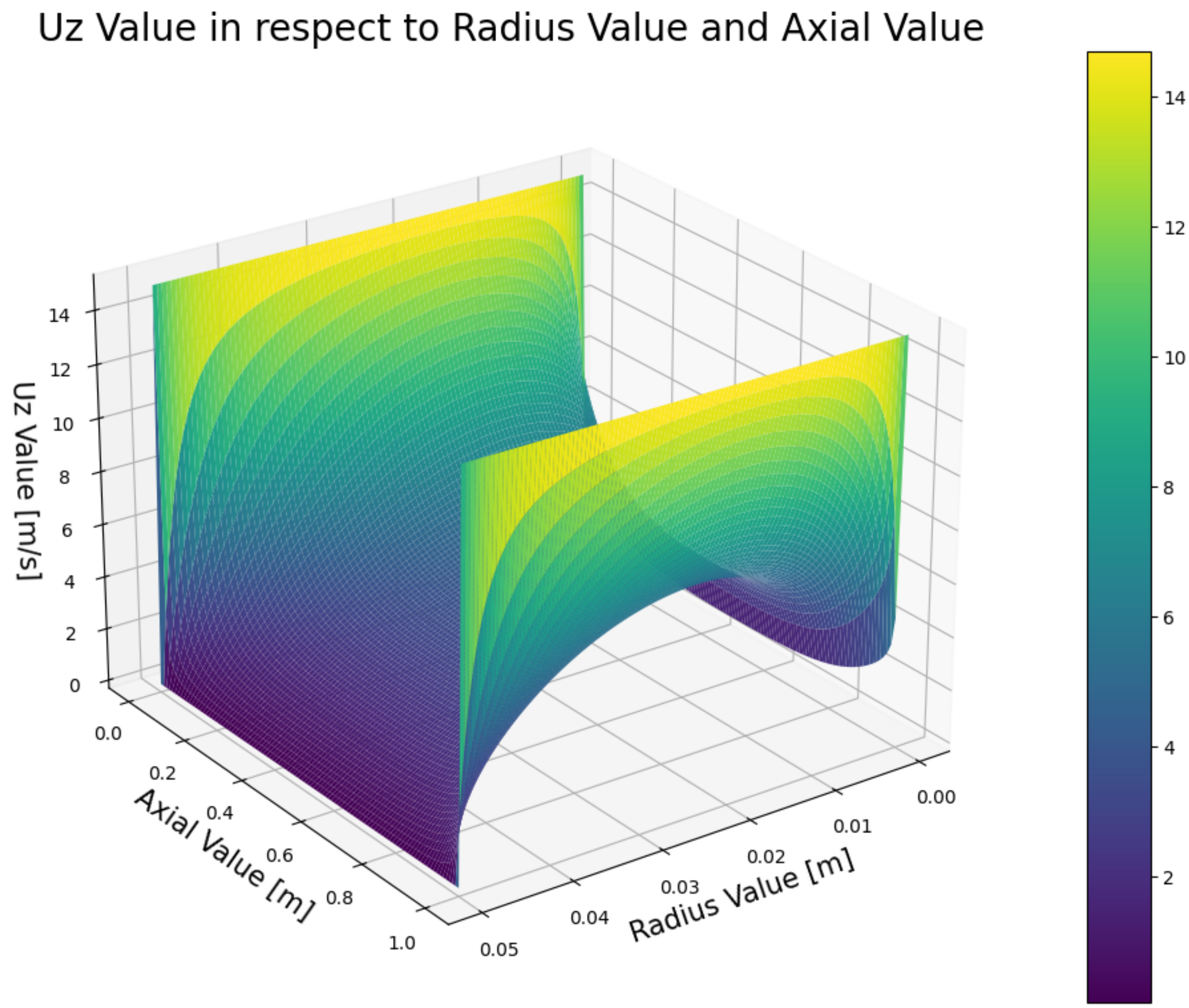

Figure 4. 3D Plot of the u_z where u_z is changing in respect to position axially and radially

Task 4 Conclusion

The authors believed that as the system is assumed as a fully developed flow which means there velocity of the fluid remains relatively constant along the length of pipe, hence Boundary Conditions of Axial value remains 15 m/s. As the graph shows that it has similar characteristics to the velocity of task 2 and task 3, if Figure 4 is seen perpendicular to the radius value.

This explains that the characteristics of the system affected by axial value is only a little.

Observations and Future Scope

Task 1

The simplification of Navier-Stokes equations were for cylindrical coordinates made for the axisymmetric flow, and fully developed flow in steady state and the neglect of radial velocity helped in the modelling of the Partial Differential equations. The finite difference method made it easier to graphically represent the equations, and take a step towards finding a solution. This model has many applications in the fluid transportation in pipelines and as future scope, can be simplified using other mathematical tools to find more accurate solutions.

Task 2

Since the final equation after applying all assumptions was a second order differential equation, a parabolic graph was expected, which was confirmed on being plotted. A second order differential equation can be solved in many ways, but here, it was chosen to be linearised using the finite difference method. After linearization, the numpy linear solver functin in python was used to solve it, which gave an accurate plot for this equation. The script is very dynamic, and by changing the values of parameters, or redefining the boundary conditions we could get the solutions of any number of scenarios. In this way, a kind of tool is developed, which can give the solution of the Navier Stokes equation with some accuracy under any parameters and boundary conditions.

Task 3

While the initial assumption of the equation that viscosity is constant made it quite simple to break down and resolve into solvable components, the dependency of viscosity in this task was solved with ease by just changing the coefficients of the original equation. This is a great stepping stone into the future scope of including other quantities that could depend on changing environmental conditions. Such as, in the next step, it could be considered how the radius of the pipe might change with the temperature, and so on.

This task also demonstrates the versatility of the python script, and how, with small changes to the program, we could make the problem closer to the real world situations while moving towards a plausible solution.

Task 4

This task was a great demonstration of the things described in the section above. It delivered on the scope of how we could take a changing velocity profile into consideration and make changes in the script to accommodate that. It also gave an insight into what tools could be used to plot a three dimensional graph using python's matplotlib projection 3d.

Even though it was a slightly tougher equation to get an end to, the visualisation was rich with information about how all the variables vary with each other, and also of the boundary conditions, and why they are necessary.