# hw10

March 26, 2021

## 0.1 This Jupyter notebook answers HW10 questions for PHY 981 Nuclear Structure.

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Date: 03/24/2021

### 0.1.1 HW10 question 3

Derive Eqs. 22.26 and 22.27.

Eq. 22.26

$$(a^\dagger_\lambda a_\lambda + a_\lambda a^\lambda_\lambda)|\alpha_1 \dots \alpha_n \lambda\rangle = |\alpha_1 \dots \alpha_n \lambda\rangle$$

We can go term-by-term applying the operators according to Eqs. 22.13 through 22.16.

Starting with the LHS:

$$(a^\dagger_\lambda a_\lambda + a_\lambda a^\dagger_\lambda)|\alpha_1 \dots \alpha_n \lambda\rangle = a^\dagger_\lambda a_\lambda|\alpha_1 \dots \alpha_n \lambda\rangle + a_\lambda a^\dagger_\lambda|\alpha_1 \dots \alpha_n \lambda\rangle$$

The first term on the RHS in the expression above evaluates as follow:

$$a^\dagger_\lambda a_\lambda|\alpha_1 \dots \alpha_n \lambda\rangle = (-1)^n a^\dagger_\lambda|\alpha_1 \dots \alpha_n\rangle = (-1)^{2n}|\alpha_1 \dots \alpha_n \lambda\rangle = |\alpha_1 \dots \alpha_n \lambda\rangle$$

The second term on the RHS vanishes by Eq 22.13. Therefore,

$$(a^\dagger_\lambda a_\lambda + a_\lambda a^\dagger_\lambda)|\alpha_1 \dots \alpha_n \lambda\rangle = |\alpha_1 \dots \alpha_n \lambda\rangle$$

Eq. 22.27

$$(a^\dagger_\lambda a_\lambda + a_\lambda a^\lambda_\lambda)|\alpha_1 \dots \alpha_n\rangle = |\alpha_1 \dots \alpha_n\rangle$$

Starting with the LHS again:

$$(a^\dagger_\lambda a_\lambda + a_\lambda a^\dagger_\lambda)|\alpha_1 \dots \alpha_n\rangle = a^\dagger_\lambda a_\lambda|\alpha_1 \dots \alpha_n\rangle + a_\lambda a^\dagger_\lambda|\alpha_1 \dots \alpha_n\rangle$$

The first term in the RHS vanishes according to Eq 22.16. The second term in the RHS evaluates to:

$$a_\lambda a_\lambda^\dagger |\alpha_1 \dots \alpha_n\rangle = (-1)^n a_\lambda |\alpha_1 \dots \alpha_n \lambda\rangle = (-1)^{2n} |\alpha_1 \dots \alpha_n\rangle = |\alpha_1 \dots \alpha_n\rangle$$

### 0.1.2 HW10 question 4

Use Eq. 22.31 to obtain an expression for $< C|\hat{F}|C >$.

Eq. 22.31 (second quantized one-body operator)

$$\hat{F} = \sum_{\alpha\beta} < \alpha|F|\beta > a_\alpha^\dagger a_\beta$$

Plug this operator expression into the matrix element:

$$< C|\hat{F}|C >= \sum_{\alpha\beta} < \alpha|F|\beta >< C|a_\alpha^\dagger a_\beta|C >$$

We switch to particle-hole formalism and apply Wick's theorem. Let the operator $N[a^\dagger a]$ be normal ordered with respect to normalized reference $|C >$. Let $N[\overline{a^\dagger a}]$ represent the Wick contraction. Then we evaluate the matrix element:

$$< C|a_\alpha^\dagger a_\beta|C >=< C|N[a_\alpha^\dagger a_\beta]|C > + < C|N[\overline{a_\alpha^\dagger a_\beta}]|C >= 0 + \chi(\alpha)\delta_{\alpha\beta}$$

The function $\chi(\alpha)$ returns 1 if the index $\alpha$ is occupied in $|C >$ and zero otherwise. The $\delta_{\alpha\beta}$ collapses an index. Therefore, we are left with a sum over diagonal occupied states labeled $i$.

Finally, we arrive at:

$$< C|\hat{F}|C >= \sum_i < i|F|i >$$

### 0.1.3 HW10 question 5

Use the second-quantization method to reduce the following many-particle matrix elements to a sum of single-particle matrix elements for the states (a,b,c,d). (1111) means they are all filled, etc. $< (1100)|\hat{F}|(1100) >$, $< (1100)|\hat{F}|(1010) >$, $< (1100)|\hat{F}|(0101) >$, $< (1100)|\hat{F}|(0011) >$.

First, let's use particle-hole Wick's theorem to derive the matrix element $< C|\hat{F}|C_i^a >$ where $|C_i^a >$ represents a one-body excitation relative to $|C >$.

$$< C|\hat{F}|C_i^a >=< C|\hat{F}a_a^\dagger a_i|C >= \sum_{\alpha\beta} < \alpha|F|\beta >< C|a_\alpha^\dagger a_\beta a_a^\dagger a_i|C >$$

$$< C|a_\alpha^\dagger a_\beta a_a^\dagger a_i|C >=< C|N[a_\alpha^\dagger a_\beta a_a^\dagger a_i]|C > + < C|N[\overline{a_\alpha^\dagger a_\beta a_a^\dagger} a_i]|C >= 0 + \chi(\alpha)\delta_{\alpha i}\pi(\beta)\delta_{\beta a}$$

The function $\pi(\beta)$ returns 1 if $\beta$ is unoccupied in the reference, and 0 otherwise. Two deltas collapse both indices in the sum, and we arrive at:

$$< C|\hat{F}|C_i^a >=< i|F|a >$$

The only nonzero matrix element occurs at the excited indices (i.e. the creation/annihilation pair that returns the excited state back to the reference).

We proceed by the result in question 4 and question 5. Let $|C >= |(1100) >$

$$< (1100)|\hat{F}|(1100) >=< a|F|a > + < b|F|b >$$

$$< (1100)|\hat{F}|(1010) >=< b|F|c >$$

$$< (1100)|\hat{F}|(0101) >=< a|F|d >$$

$$< (1100)|\hat{F}|(0011) >= 0$$

We could derive this last result from Wick's theorem (cannot fully contract two operators with four operators in the double excitation), or we can use the logic that a one-body operator cannot connect states that represent two particle excitations relative to the reference.

### 0.1.4 HW10 question 6

For the last problem replace $\hat{F}$ with the number operator of Eq. 22.33.

$$< (1100)|\hat{N}|(1100) >=< a|1|a > + < b|1|b >= 2$$

$$< (1100)|\hat{N}|(1010) >=< b|1|c >= 0$$

$$< (1100)|\hat{N}|(0101) >=< a|1|d >= 0$$

$$< (1100)|\hat{N}|(0011) >= 0$$

### 0.1.5 HW10 question 7

Calculate the B(E2) for the 1/2+ to 5/2+ transitions in 17F and 17O assuming these states are represented by $1s_{1/2}$ and $0d_{5/2}$. single-particle states outside of a 16O closed shell. Use hamonic-oscillator radial wavefunctions with $\hbar\omega = 14$ MeV. Use can use the rme program for the matrix elements. Compare to experiment.

The reduced transition probability is defined:

$$B(i \to f) = \frac{|\langle J_f||O(E2)||J_i\rangle|^2}{(2J_i + 1)}$$

If we treat the transition states as single-particle states outside of a closed shell, then the transition operator is a sum over a single nucleon. For the proton case,

$$O(E2) = r^2 Y^2_\mu(\hat{r})$$

so that the reduced transition probability is

$$B(i \to f) = \frac{|\langle J_f || r^2 Y^2_\mu(\hat{r}) || J_i \rangle|^2}{(2J_i + 1)}$$

We compute the matrix element $\langle J_f || r^2 Y \mu^2(\hat{r}) || J_i \rangle$ in a harmonic oscillator basis using `rme`. For the 1/2+ to 5/2+ transition, the inputs k1 and k2 to `rme` are 4 and 6 respectively. The output is in units of $b^2$, and we get

$$\langle 0d_{5/2} || r^2 Y^2_\mu(\hat{r}) || 1s_{1/2} \rangle = -2.18510(\frac{41.1}{14}) = -6.41483$$

Then $B(E2)$

$$B(E2) = \frac{(-6.41483)^2}{2 * 0.5 + 1} = 20.57502$$

### 0.1.6 HW10 question 8

What are all possible ($\ell$, S, T) values for each of the following:

a) $J^\pi = 3^-$ resonance of the two-neutron system

Parity restricts $\ell$=odd; For a two-neutron system, S=1 to satisfy antisymmetry. Therefore,

($\ell$=3,S=1,T=1)

b) $J^\pi = 2^-$ resonance of the two-nucleon system with $T_z = 1$

Parity restricts $\ell$=odd; We can generate $J$ states $J = \quad \ell - 1, \ell, \ell + 1$; Must have $\ell$+S+T=odd; $T_z = 1$ restricts $T = 1$

($\ell$=1,S=1,T=1); ($\ell$=3,S=1,T=1)

c) $J^\pi = 2^+$ resonance of the two-nucleon system with $T_z = 0$

Parity restricts $\ell$=even; We can generate $J$ states $J = \quad \ell - 1, \ell, \ell + 1$; Must have $\ell$+S+T=odd; $T_z = 0$ means $T = 0, 1$

($\ell$=2,S=0,T=1); ($\ell$=2,S=1,T=0)

### 0.1.7 HW10 question 9

For the configuration $(0f_{7/2})^2$ for two neutrons what are the allowed J values?

jj-coupling T=1 $\implies$ J=0,2,4,6; the wavefunction vanishes when J=odd and $k_1 = k_2$

### 0.1.8 HW10 question 10

For the configuration $(0f_{7/2}; 1p_{3/2})$ for two neutrons what are the allowed J values?

jj-coupling $\implies$ J=2,3,4,5

### 0.1.9 HW10 question 11

For the configuration $(0f_{7/2})^2$ for two nucleons what are the allowed combinations of (J, T) values?

$J = 0, 1, 2, 3, 4, 5, 6, 7$

$J + T$ must be odd (or else wavefunction vanishes)

Therefore, (J,T) = (0,1), (1,0), (2,1), (3,0), (4,1), (5,0), (6,1), (7,0)

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