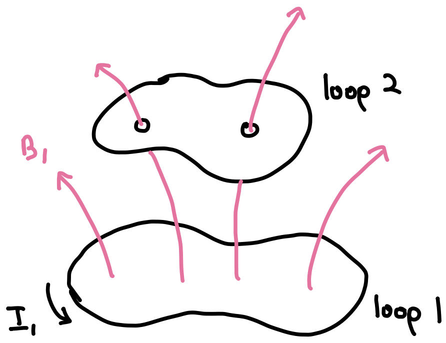
Mutual Inductance

have two loops of wires close to each other

run current I_1 through loop 1 $\longrightarrow$ get magnetic field, B_1

(from Biot-Savart Law / Ampere-Maxwell Law)

$$\vec{B} = \frac{\mu_0}{4\pi} I \oint \frac{d\vec{l}_1 \times \hat{\mathscr{r}}}{\mathscr{r}^2} \qquad \vec{\nabla} \times \vec{B} = \mu_0 \left(\vec{J} + \epsilon_0 \frac{\partial \vec{E}}{\partial t} \right)$$



some of the B_1 field lines pass through loop 2 $\longrightarrow$ get magnetic flux, Φ_2

$$\Phi_2 = \int \vec{B}_1 \cdot d\vec{A}_2$$

from Biot-Savart Law, we see $B \propto I$, so we can write

$$\boxed{\Phi_{21} = M_{21} \cdot I_1}$$

where M is mutual inductance

$\rightarrow$ const. of proportionality b/w flux & current

$$M_{21} \equiv \frac{\Phi_{21}}{I_1}$$

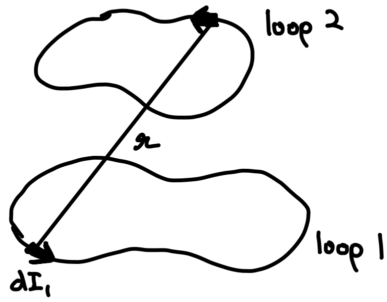
Φ_{21} ← flux in loop 2 due to loop 1

I_1 ← current in loop 1

using vector potential, $\vec{A}$, can derive Neumann formula for M $\qquad \vec{B} = \vec{\nabla} \times \vec{A}, \quad \vec{A} = \frac{\mu_0 I}{4\pi} \oint \frac{d\vec{l}}{\mathscr{r}}$

$$\boxed{M_{21} = \frac{\mu_0}{4\pi} \oint \oint \frac{d\vec{l}_1 \cdot d\vec{l}_2}{\mathscr{r}}}$$

(double line integral)



note: M is a purely geometrical quantity

depends on size, shape & relative positions of loops

now if current in loop 1 is varied, by Faraday's Law, we get an induced emf in loop 2

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\mathcal{E}_2 = -\frac{d\Phi_2}{dt} = -M_{21} \frac{dI_1}{dt}$$

a time varying current will also induce an emf in its own source loop, so we can similarily write

$$L_1 \equiv \frac{\Phi_1}{I_1}$$

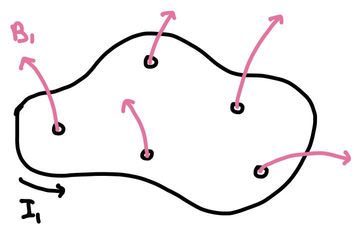
Φ_1 ← flux through loop 1

I_1 ← current in loop 1

$$\boxed{\Phi_1 = L_1 I_1}$$

where L is self inductance, or just inductance

$\longrightarrow$ const. of proportionality b/w own flux & current



note: L also depends on the geometry (size & shape) of the loop again, using Faraday's Law & considering loop 1 in isolation, we derive the familiar voltage-current relation for an inductor

$$\mathcal{E} = -\frac{d\Phi_1}{dt} = -L_1\frac{dI_1}{dt}$$

also mutually couples back to loop 1

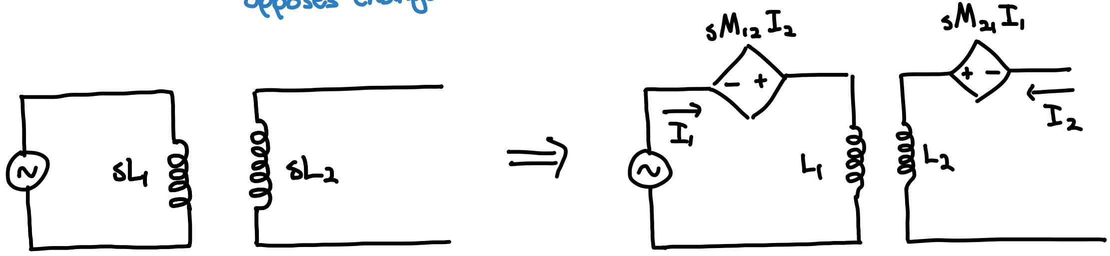
so now, since the time-varying induced emf in loop 2 creates a time-varying circuit, we must consider both mutual & self inductance

$$V_2(t) = M_{21}\frac{dI_1}{dt} - L_2\frac{dI_2}{dt} \xrightarrow{\mathcal{L}} sM_{21}I_1 - sL_2I_2$$

self inductance opposes change

model as voltage source

model as load



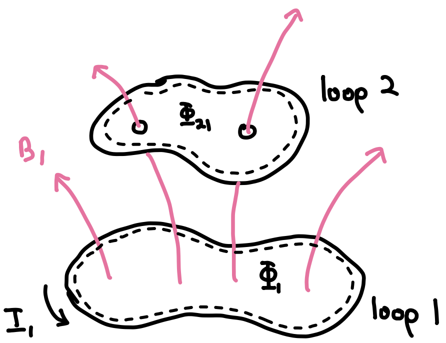
so clearly M quantifies the voltage induced in the other circuit, however, the double line integral in the Neumann formula doesn't give an intuitive understanding of M

so instead, we can define the coupling coefficient, k:

$$\boxed{k \equiv \frac{\Phi_{21}}{\Phi_1}}$$

flux through loop 2 from loop 1

flux from loop 1



physically, this represents the portion of flux from loop 1 that goes through loop 2, and so $0 \leq k \leq 1$ $k=1 \Rightarrow$ perfect coupling

to understand how this relates to mutual inductance, we can write

$$M_{21}\cdot M_{12} = \frac{\Phi_{21}}{I_1}\cdot\frac{\Phi_{12}}{I_2}$$

$$= \frac{k_{21}\Phi_1}{I_1}\cdot\frac{k_{12}\Phi_2}{I_2}$$

$$= k_{21}L_1\cdot k_{12}L_2$$

by symmetry of the Neumann equation & of the system

$$M_{21} = M_{12} = M \qquad k_{21} = k_{12} = k$$

hand-wavy symmetry/reciprocity argument
more rigourous way to relate M & k is through energy arguments
↓
write energy in matrix form, argue inductance matrix must have determinant ≥ 0 for non-neg. energy value

∴ we can write

$$M^2 = k^2 L_1 L_2 \implies \boxed{M = k\sqrt{L_1 L_2}}$$

from this, we can clearly see that the mutual inductance, & ∴ the voltage induced in loop 2, depends on the proportion of flux lines going through loop 2 from loop 1, & the self inductances of each loop

we can also look at the power/energy in this two-loop-system

$p(t) \rightarrow$ instantaneous power

we know $P = I \cdot V$

$$p(t) = V_1(t) I_1(t) + V_2(t) I_2(t)$$

$$= \left(L_1 \frac{dI_1}{dt} + M\frac{dI_2}{dt}\right) I_1 + \left(L_2 \frac{dI_2}{dt} + M\frac{dI_1}{dt}\right) I_2$$

assuming all add together for simplicity, would really have to look at relative orientations

$$\frac{dW}{dt} = L_1 I_1 \frac{dI_1}{dt} + L_2 I_2 \frac{dI_2}{dt} + M\underbrace{\left(I_1 \frac{dI_2}{dt} + I_2 \frac{dI_1}{dt}\right)}_{\frac{d}{dt}(I_1 I_2)}$$

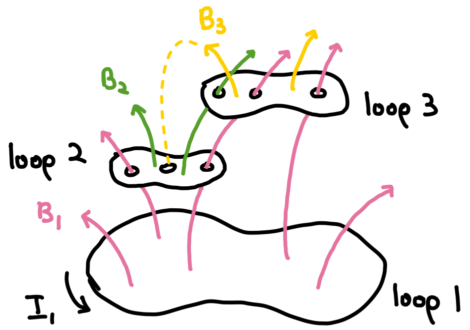
$$\implies \boxed{W = \underbrace{\tfrac{1}{2}L_1 I_1^2 + \tfrac{1}{2}L_2 I_2^2}_{\text{energy in "self" fields}} + \underbrace{M I_1 I_2}_{\text{energy in shared field}}}$$

integrate & assume 0 energy at 0 current
$\rightarrow$ const. of int. is 0

Now, if we bring a third loop into the system, it couples w/ both existing loops, so we get

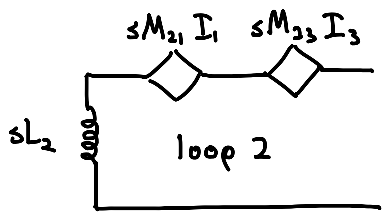
$$V_2 = sM_{21} I_1 + sL_2 I_2 + sM_{23} I_3$$

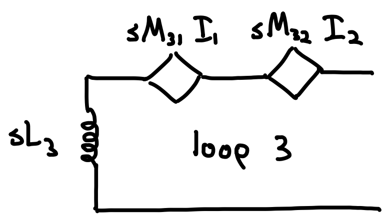
$$V_3 = sM_{31} I_1 + sM_{32} I_2 + sL_3 I_3$$



where $M_{23} = M_{32} = k_{23}\sqrt{L_2 L_3}$

in circuit form, this would look like





directions of current & polarities of voltages become hard to track
↓
practically, assume all add together, then if calculate & negative, indicates opp. dir.

and we have energy

$$W = \tfrac{1}{2}L_1 I_1^2 + \tfrac{1}{2}L_2 I_2^2 + \tfrac{1}{2}L_3 I_3^2 + M_{12} I_1 I_2 + M_{13} I_1 I_2 + M_{23} I_2 I_3$$

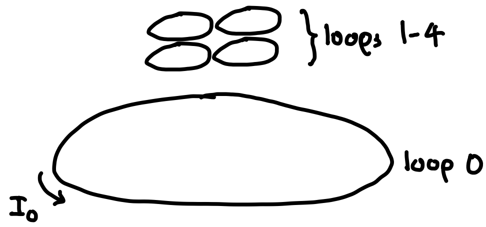
when generalizing to n loops, it is useful to use matrix notation

$$\begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ \vdots \\ V_n \end{bmatrix} = s \underbrace{\begin{bmatrix} L_1 & M_{12} & M_{13} & \cdots & M_{1n} \\ M_{21} & L_2 & M_{23} & \cdots & M_{2n} \\ M_{31} & M_{32} & L_3 & \cdots & M_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ M_{n1} & M_{n2} & M_{n3} & \cdots & L_n \end{bmatrix}}_{\text{symmetric matrix, } M_{ij} = M_{ji} = k_{ij}\sqrt{L_i L_j}} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ \vdots \\ I_n \end{bmatrix}$$

denoting this inductance matrix as $\overleftrightarrow{L}$, we can write the energy as

$$W = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \overleftrightarrow{L}_{ij} I_i I_j$$

Wireless Drone Analysis:



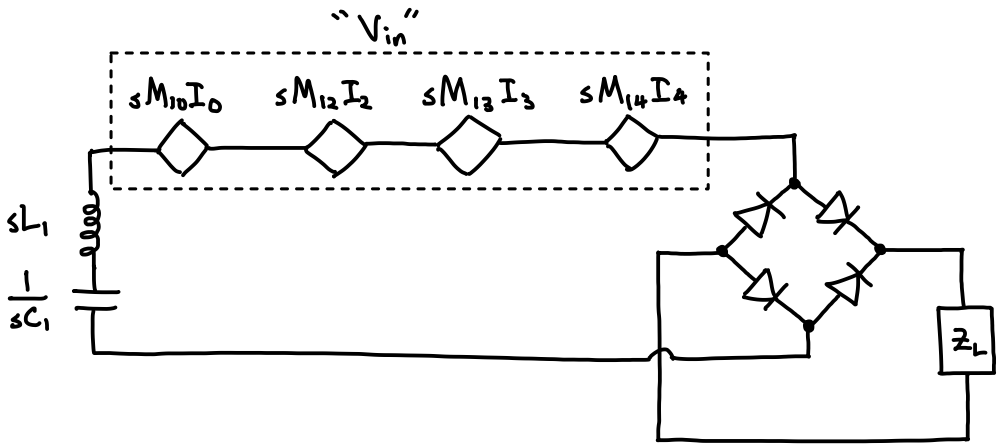
the wireless drone can be thought of as a 5-loop system

- 1 transmitter coil (TX) — denote as loop 0
- 4 receiver coils (RX's) — denote as loops 1, 2, 3, 4

note: the choice of 4 RX's was somewhat arbitrary & could potentially be tweaked as a design paramater

consistent w/ existing quadcopter drone designs

looking at loop 1, though the analysis is symmetric b/w all RX loops, we get the following circuit & voltage relation:



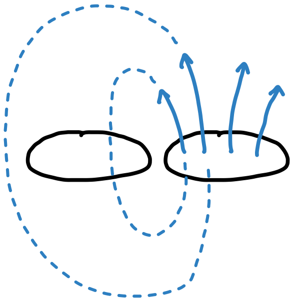
we can think of the combined induced voltage from mutual inductance as the "input" voltage into the RX circuit

$$V_{in,1} = s\left(M_{10}I_0 + M_{12}I_2 + M_{13}I_3 + M_{14}I_4\right)$$

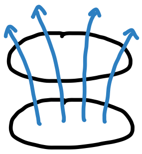
given that $M_{ij} = k_{ij}\sqrt{L_i L_j}$ & k_{ij} represents the portion of flux from loop i through loop j, it intuitively makes sense to say that

$$k_{coplanar} \ll k_{coaxial} \implies M_{coplanar} \ll M_{coaxial}$$

Coplanar:



Coaxial:



in terms of the drone, this translates to

$$M_{ij} \ll M_{i0} \quad \text{for } i,j \neq 0 \qquad M_{12}, M_{13}, M_{14} \ll M_{10}$$

additionally, since the current in TX, I_0, is being amplified & the currents in the RX's merely power small motors, we might also say

$$I_0 \gg I_1, I_2, I_3, I_4$$

more of a guess... haven't actually measured

thus, it seems reasonable to neglect the coupling b/w RX coils & say

$$V_{in,i} \approx sM_{i0}I_0 \quad \text{for } i = 1,2,3,4 \qquad V_{in,1} \approx sM_{10}I_0$$

this relationship also shows us why mutual inductance, specifically b/w the TX & RX coils, is an important system parameter as it directly relates to the voltage driving the motors that drive the propellers, or in a general sense, the wireless power transfer

$M \rightarrow$ WPT

using the relation $M_{ij} = k_{ij}\sqrt{L_i L_j}$ we can understand how M behaves in our system & how we might tune it to meet our goals

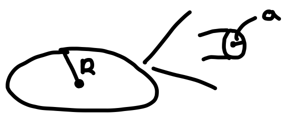
clearly, we see two parameters that affect M:

(1) inductance, L (2) coupling coefficient, k

Inductance: for a circular loop, the inductance L is given by

$$L \approx \mu_0 R\left[\ln\left(\frac{8R}{a}\right) - 2\right]$$

R: radius of loop
a: radius of wire



the important takeaway here is that $L \sim R \ln R$, so

$\uparrow$ loop radius $\implies$ $\uparrow$ WPT

<u>note:</u> the choice for circular loops is also seemingly arbitrary other geometries can lead to higher inductances & changes in the shape of the B-field well, which may be more desirable, so this might be something to look into

not included in this equation is how L scales w/ the number of windings, N:

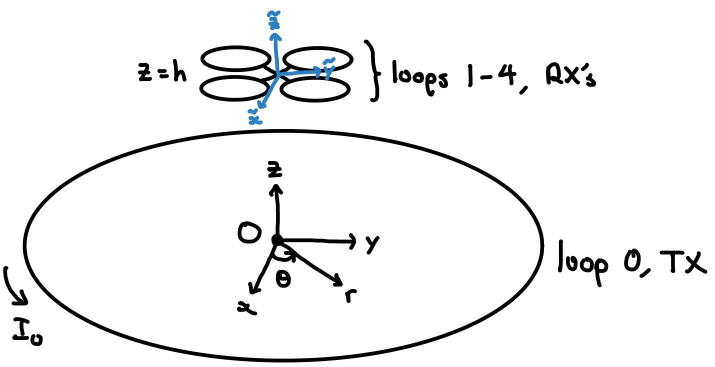
$$L \sim N^2$$

to get an intuitive understanding of this, we can look at magnetic flux since $L \equiv \frac{\Phi}{I}$

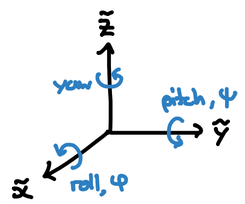
$$\Phi = \int \vec{B} \cdot d\vec{A} \quad \begin{cases} B \sim I \text{ from Biot-Savart} \\ N \text{ loops} = N \cdot I \text{ current, } B \sim N \\ \text{now counting } N \text{ areas, } A \sim N \end{cases} \Bigg\} \; \Phi \sim N^2 \rightarrow L \sim N^2 \; \square$$

we can then say that $M_{ij} \sim N_i N_j$

<u>Coupling Coefficient:</u> first, we define coordinates to parameterize the positions & orientations of the RX coils



the position of an RX coil is given by cylindrical coords. (r, θ, z) defined w/ origin at the centre of the TX coil



the orientation is given by roll, pitch & yaw, which can be defined as rotations about $\tilde{x}, \tilde{y}, \tilde{z}$, respectively, where the tilde coordinate system is defined w/ an origin at the centre of the drone w/ axes parallel to the one defined in the TX coil

moves w/ drone but stays aligned w/ ref. coord. sys.

given that the drone is rotational symmetric about $\tilde{z}$, we can discard yaw & assign roll $\rightarrow \varphi$, pitch $\rightarrow \psi$

we can now establish that the drone's position & orientation is given by the following quintuple: $(r, \theta, z, \varphi, \psi)$

given the definition of the coupling coefficient k, it is clearly dependent on all 5 of these values

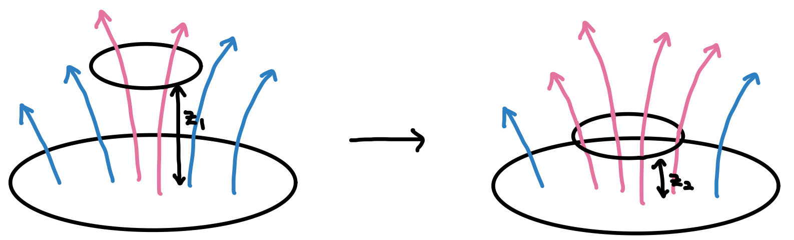
$$k \longrightarrow k(r, \theta, z, \varphi, \psi)$$

this k-value encodes the spatial response of the drone to the magnitude & direction of the B-field, i.e. the restoring nature of the well shape

<u>note:</u> the B-field & its spatial magnitude & direction, i.e. $\vec{B}(r, \theta, z)$, can be manipulated to derive the required response in the drone via k

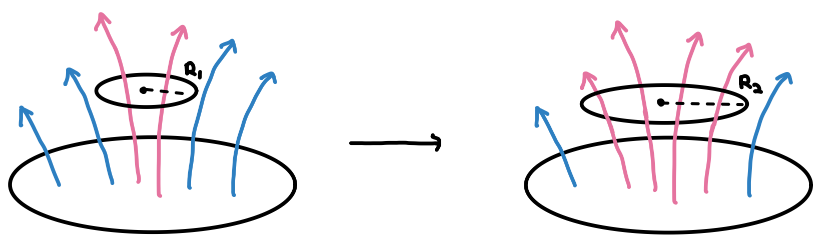
other ways k affects the WPT of the drone:

<u>Ex:</u> intuitively, increasing z decreases the proportion of flux going through the RX coils, leading to a lower WPT



$k_{z_2} > k_{z_1}$

<u>Ex:</u> though not included in the quintuple, we also see that increasing the radius of an RX coil increases the coupling coefficient k, & thus the WPT



$k_{R_2} > k_{R_1}$