AE 584 Exam 1 Fall 2017

Math

Binomial Expansions

$$(a^2 - b^2) = (a-b)(a+b)$$
$$(a^3 - b^3) = (a-b)(a^2 + ab + b^2)$$
$$(a^4 - b^4) = (a-b)(a+b)(a^2 + b^2)$$

Exponentials

$$e^a e^b = e^{a+b}$$
$$(e^x)^n = e^{2x}$$

Matricies

$$(A+B)^T = A^T + B^T$$
$$(AB)^T = B^T A^T$$
$$AB^T = BA^T$$
$$(A^T)^{-1} = (A^{-1})^T$$

Square Matricies

$$det(A^T) = det(A)$$

Also, if A is square, then its eigenvalues are equal to the eigenvalues of its transpose. Additionally, if the matrix is also differentiable and nonsingular

$$\frac{d}{dt}(P^{-1}(t) = -P^{-1}(t)\dot{P}(t)P^{-1}(t)$$

which can be found with; $\frac{d}{dt}(P(t)P^{-1}(t)) = \frac{d}{dt}(I) = 0$

Symetric Matricies

$$A^T = A$$

Skew-Symetric Matricies

$$A^T = -A$$

Inverting a $2x2$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$
$$A^{-1} = \frac{1}{|A|}\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}.$$

Determinant of a $3x3$ matrix

$$|B| = det(B) = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

$$= a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

Trig Identities

$$sin(2a) = 2cos(a)sin(a)$$
$$cos(2a) = 1 - 2sin^2(a) = 2cos^2(a) - 1$$

Leibniz Integral Rule

States that the derivative of the integral is (for constant limits of integration):

$$\frac{d}{dt}(\int_a^b f(x,t)\,dt) = \int_a^b \frac{\partial}{\partial x} f(x,t)\,dt$$

States that the derivative of the integral is (for limits of integration that are not constant)

$$\frac{d}{dt}(\int_{f(a)}^{f(b)} f(x,t)\,dt) =$$
$$= f(x,b(x))\frac{d}{dx}b(x) - f(x,a(x))\frac{d}{dx}a(x) + \int_{f(a)}^{f(b)} \frac{\partial}{\partial x} f(x,t)\,dt$$

Note: t is a variable in the integration limit, so the above formula must be used to derive $\dot{P}(t)$.

Integrals

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}$$
$$\int \frac{1}{x}\,dx = \ln|x|$$
$$\int u\dot{v}\,dx = uv - \int vdu$$
$$\int e^x\,dx = e^x$$
$$\int a^x\,dx = \frac{1}{\ln a}a^x$$
$$\int \ln x\,dx = x\ln x - x$$
$$\int \sin x\,dx = -\cos x$$
$$\int \cos x\,dx = \sin x$$
$$\int \tan x\,dx = \ln|\sec x|$$
$$\int \sec x\,dx = \ln|\sec x + \tan x|$$
$$\int \sec^2 x\,dx = \tan x$$
$$\int \sec(x)\tan(x)\,dx = \sec x$$

Polar Coordinates (θ, r)

$$dxdy = rdrd\theta$$

u Substitution

don't forget to change the limits of integration!

Derivatives

$$\frac{d}{dt}(tan^-1(x)) = \frac{1}{1+x^2}$$

Laplace X-Forms

$$
\begin{aligned}
f(t) &= \mathcal{L}[f(t)] = F(s) \\
1 &= \frac{1}{s} \\
\delta(t) &= 1 \\
\delta(t-t_0) &= e^{-st_0} \\
f'(t) &= sF(s) - f(0) \\
f^n(t) &= s^nF(s) - s^{(n-1)}f(0) - \cdots - f^{(n-1)}(0) \\
t^n(n=0,1,2,\dots) &= \frac{n!}{s^{n+1}} \\
\sin kt &= \frac{k}{s^2+k^2} \\
\cos kt &= \frac{s}{s^2+k^2} \\
e^{at} &= \frac{1}{s-a} \\
t^ne^{at} &= \frac{n!}{(s-a)^{n+1}} \\
e^{at}\sin kt &= \frac{k}{(s-a)^2+k^2} \\
e^{at}\cos kt &= \frac{s-a}{(s-a)^2+k^2} \\
t\sin kt &= \frac{2ks}{(s^2+k^2)^2} \\
t\cos kt &= \frac{s^2-k^2}{(s^2+k^2)^2}
\end{aligned}
$$

First translation theorom:

$$\mathcal{L}\left[e^{at}f(t)\right] = F(s-a)$$

Delta Dirac Function

$$
\begin{aligned}
\int_{-\infty}^{\infty} f(x)\delta(x-a)\,dx &= f(a) \\
\int_{-\infty+\epsilon}^{\infty+\epsilon} f(x)\delta(x-a)\,dx &= f(a), \epsilon > 0 \\
\delta(x-a) &= 0
\end{aligned}
$$

Linearization

Given some nonlinear system of the form:

$$
\begin{aligned}
\dot{x}(t) &= f(x(t), u(t), t) \\
y(t) &= g(x(t), t)
\end{aligned}
$$

A linearization can be performed about nominal trajectories $x^0(t)$, $u^0(t)$, and $u^0(t)$ (shorthand is 0) by defining the jacobian of w.r.t. the states evaluated at the nominal trajectories (0) as:

$$A(t) = \frac{\partial f}{\partial x}\Big|_0 = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}\Bigg|_0$$

As well as the jacobian of w.r.t. the controls evaluated at the nominal trajectories (0) as:

$$B(t) = \frac{\partial f}{\partial u}\Big|_0 = \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} & \cdots & \frac{\partial f_1}{\partial u_m} \\ \frac{\partial f_2}{\partial u_1} & \frac{\partial f_2}{\partial u_2} & \cdots & \frac{\partial f_2}{\partial u_m} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial u_1} & \frac{\partial f_n}{\partial u_2} & \cdots & \frac{\partial f_n}{\partial u_m} \end{bmatrix}\Bigg|_0$$

Finally, the output equation may need to be linearized about 0 as well

$$C(t) = \frac{\partial g}{\partial x}\Big|_0 = \begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} & \cdots & \frac{\partial g_1}{\partial x_n} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} & \cdots & \frac{\partial g_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial g_n}{\partial x_2} & \cdots & \frac{\partial g_n}{\partial x_n} \end{bmatrix}\Bigg|_0$$

Then the perterbation variables are defined as:

$$
\begin{aligned}
\delta x(t) &= x(t) - x^0(t) \\
\delta u(t) &= u(t) - u^0(t) \\
\delta y(t) &= y(t) - y^0(t)
\end{aligned}
$$

The final linearized system is:

$$
\begin{aligned}
\delta\dot{x}(t) &= A(t)\delta x(t) + B(t)\delta u(t) \\
\delta y(t) &= C(t)\delta x(t)
\end{aligned}
$$

State Transition Matrix, $\Phi(t, t_0)$

Without intut to the system

$$
\begin{aligned}
\dot{x}(t) &= A(t)x(t) \\
x(t) &= \Phi(t,t_0)x_0 \\
\Phi(t,t_0) &= x(t)x(t_0)^-1
\end{aligned}
$$

With input to the system

The system is

$$
\begin{aligned}
\dot{x}(t) &= A(t)x(t) + B(t)u(t) \\
y(t) &= C(t)x(t) \\
x(t_0) &= x_0
\end{aligned} \tag{1}
$$

where the solution is,

$$x(t) = \Phi(t,t_0)x_0 + \int_{t_0}^{t} \Phi(t,\tau)B(\tau)u(\tau)\,d\tau \tag{2}$$

$$y(t) = C(t)\Phi(t,t_0)x_0 + \int_{t_0}^{t} \underbrace{C(t)\Phi(t,\tau)B(\tau)}_{(G(t,\tau))}u(\tau)\,d\tau$$

The above two equations are called the **variation of constants formulas**. They contain two terms, the first term is the **free response** which is due to x_0 and the second term is the **forced resonse** due to the input $u(t)$. Additionally, the **impulse response is** defined as

$$G(t,\tau) = C(t)\Phi(t,t_0)B(\tau), \tau \le t \tag{3}$$

Finding Φ: LTI Systems A

The basic equation is:

$$\Phi(t,\tau) = e^{A(t-\tau)}$$

Which can be calculated using:

$$\mathcal{L}^{-1}\left[(sI-A)^{-1}\right] = e^{At}$$

After this, τ must be added in, which can be done with by taking the inverse of $\Phi(\tau, 0)$ to get $\Phi(0,\tau)$ and then multiplying by $\Phi(t,0)$ as:

$$\Phi(t,0)\Phi(0,\tau) = \Phi(t,\tau)$$

Finding Φ: LTV Systems A(t)

The STM is the unique solution to

$$\frac{\partial}{\partial t}(\Phi(t,t_0)) = A(t)\Phi(t,t_0)$$

with inital conditions $\Phi(t_0,t_0) = I$.
To solve:

- multiply the above matrices out
- take the Laplace Transform of each element in the matrix
- solve the algebreic equation for each $\Phi_{i,i}(s)$
- take the inverse Laplace transfrom to find $\Phi_{i,i}(t)$

Properties of STM

$$
\begin{aligned}
\Phi(t_0,t_0) &= I \\
\Phi(t,t_0)^-1 &= \Phi(t_0,t) \\
\Phi(t,t_0) &= \Phi(t,t_1)\Phi(t_1,t_0)
\end{aligned}
$$

Deterministic Asymtotic Observers

Stability

The system Eqn. 1 is stable if, given $x(t_0) = x_0$, $x(t)$ (Eqn. 2) is bounded is bounded $\forall t \geq t_0$. In this case,

$$lim_{t\to\infty} x(t)$$

may not go to zero. Stability can also be determined by looking at each (i,j) component of the STM as:

$$|\Phi_{ij}(t,t_0)| \leq k < \infty, \forall t_0 \leq t$$

Asymtotic Stability

The system Eqn. 1 is asymtotically stable if, given $x(t_0) = x_0$ $x(t)$, $x(t)$ (Eqn. 2) decays to zero, that is:

$$lim_{t\to\infty} x(t) = 0$$

Unstable Systems

BIBO Stabile Systems

The system, Eqn. 1, is BIBO stable if when $x_0 = 0$, the forced output response $y(y)$ to every bounded input $u(t)$ is bounded. This can be determined as:

$$\int_{-\infty}^{t} |G_{ij}(t,\tau)| d\tau \leq k < infty$$

where, $G(t,\tau)$ was defined in Eqn. 3 and the above equation requires that $G(t,\tau)$ is "absolutely integrable."

Observabilty

Can we estimate a unique $x(t_0) = x_0$, given $u(t)$ and $y(t)$ over the time interval $[t_0, t_1]$? If we have x_0 we can solve

$$x(t) = \Phi(t,t_0)x_0 + \int_{t_0}^{t} \Phi(t,\tau)B(\tau)u(\tau)\, d\tau$$

The state is unobservable for the unforced system ($u(t) = 0$), if:

$$y(t) = C(t)\Phi(t,t_0)x_0 = 0$$

LTI Systems A

The observability matrix for a LTI system is:

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

if the $rank(\mathcal{O}) = n$ then the system is observable. The unobservable states are in the null space of the observability matrix, i.e. $\mathcal{O}x_0 = 0$

LTV Systems A(t)

Observability Gramian

$$M(t_0,t_1) = \int_{t_0}^{t_1} \Phi^T(t,t_0)C^T(t)C(t)\Phi^T(t_0,t)dt$$

a state $x_0 = x(t_0)$ unobservable at time t_0 iff

$$M(t_0,t_1)x_0 = 0, \forall t_1 > t_0$$

so, the unobservable states are in the null-space of the Observability Gramian. If the only solution that lives in the null-space is the zero vector $x(t_0) = 0$, then the system is completely observable. Also, note that ther is no need to carry out the complete integral to see that the Observability Gramian will have unobservable states in its null space, i.e. integration does not change the form of the matrix.

Controlability

A system is controllable if we can find a $u(t)$ that drives the state $x(t)$ from x_0 in finite time t_f.

LTI Systems A

The observability matrix for a LTI system is:

$$\mathcal{C} = \begin{bmatrix} B & BA & \ldots & BA^{n-1} \end{bmatrix}$$

if the $rank(\mathcal{C}) = n$ then the system is controllable. Recall that the rank of a matrix is the number of linearly independent columns.

LTV Systems A(t)

Controllability Gramian

$$W(t_0,t_1) = \int_{t_0}^{t_1} \Phi^T(t,t_0)B(t)B^T(t)\Phi^T(t_0,t)dt$$

this matrix is always symetric and positive definite. The system is completly controllable if there exists $t_1 > t_0$: $W(t_0,t_1) > 0$. If the system is controllable at t_0, then one control that drives the state to the origin is:

$$u_0(t) = -B^T(t)\Phi^T(t_0,t)W^{-1}(t_0,t_1)x_0$$

Duality

Given a LTV system as

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t) \\ y(t) &= C(t)x(t) \end{aligned}$$

its dual system is

$$\begin{aligned} \dot{x}(t) &= -A^T(t)x(t) + C^T(t)u(t) \\ y(t) &= B^T(t)x(t) \end{aligned}$$

The controllable (or uncontrollable) states of one system are the observable (or unobservable) states of the other system.

Random Vectors

If x is a random vector with a PDF $f(x)$ and $g: \mathbb{R}_n \to \mathbb{R}_m$ is a function of x

Expected Value

$$E[g(x)] = \int_{\mathbb{R}_n} g(x)f(x)\, dx \in \mathbb{R}_m$$

Mean Value

$$\bar{x} = E[x] = \int_{\mathbb{R}_n} xf(x)\, dx \in \mathbb{R}_n$$

Covariance

$$\begin{aligned} P_{xx} &= E[(x-\bar{x})(x-\bar{x})^T] \\ &= \int_{\mathbb{R}_n} (x-\bar{x})(x-\bar{x})^T f(x)\, dx \in \mathbb{R}_{nxn} \end{aligned}$$

Note: the covariance matrix is symmetric as well as positive semidefinite, so

$$\forall v \in \mathbb{R}_n, v^T P_x v \geq 0$$

Thus,

$$\begin{aligned} v^T P_x v &= \int_{R_n} v^T(x-\bar{x})(x-\bar{x})^T v f(x)\, dx \\ &= \int_{R_n} (x-\bar{x})^2 f(x)\, dx \geq 0 \end{aligned}$$

Cross-Covariance

If $x1$ and $x2$ are subvectors of x

$$\begin{aligned} P_{x_1x_2} &= E[(x_1-\bar{x_1})(x_2-\bar{x_2})^T] \\ &= \int_{\mathbb{R}_n} (x_1-\bar{x_1})(x_2-\bar{x_2})^T f(x)\, dx \in \mathbb{R}_{n_1xn_2} \end{aligned}$$

Variance of Error

$$P^+ = ((P^-)^{-1} + C^T R^{-1} C)^{-1}$$

Probability Density Functions

The relative likelihood that a random variable x will take on vaules on a given interval.

$$Area = P(a \leq x \leq b) = \int_a^b f_x(x)dx$$

Properties of PDF

If you integrate over the PDF over the entire range then it must equal 1 and the PDF must always be greater than 0.

$$\int_{\mathbb{R}} f_x dx = 1 \forall x, f_x(x) \geq 0$$

Uniform PDF

For a PDF uniformly distributed over $[a,b]$, $f(x) = constant = c$. c can then be determined by $\int_a^b c dx = 1$ which results in $f(x) = \frac{1}{b-a}$.

Marginal PDF

Given the joint PDF $f_x(x) = f_x(x_1, x_2)$, the marginal PDF of x_1 is:

$$f_{x_1}(x_1) = \int_{-\infty}^{\infty} f_x(x_1, x_2) dx_2$$

Characteristic Function, $\phi_x(s)$

$\phi_x(s)$ is useful to compute the PDF for x and the Gaussian distribution. The expected value can be used to calculate the characteristic function as:

$$\phi_x(s) = E[e^{jx^T s}] = \int_{\mathbb{R}_n} e^{jx^T s} f(x)\, dx$$

where $j^2 = -1$ and s is a complex vector of order n
The statistal properties of x are equivalently specified by PDF $f(x)$ or by the characteristic function $\phi_x(s)$.

Inversion of, $\phi_x(s)$

Similar to a Fourier Transform $\phi_x(s)$ can be put back into the time domain with:

$$f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}_n} e^{jx^T s} \phi_x(s), ds$$

Independence

If the following equations are true, then the PDF's are independent.

$$f(x, y) = f_x(x) f_y(y)$$
$$\phi_{xy}(s, r) = \phi_x(s)\phi_y(r)$$

If x and y are independent, the conditional density function and conditional mean satisfy:

$$f(x|y) = \frac{f(x,y)}{f_y(y)} = \frac{f_x(x)f_y(y)}{f_y(y)} = f_x(x)$$
$$E[x|y] = \int_{\mathbb{R}_n} x f_x(x)\, dx = E[x] = \bar{x}$$

Correlation

If x and y are independent, then they are uncorrelated. But, if they are uncorrelated, they may not be independent!
If x and y are uncorrelated, then they satisfy:

$$E[xy^T] = E[x]E[y^T]$$

If x and y are uncorrelated, the cross-covariances must be zero:

$$\begin{aligned} P_{xy} &= E[(x - \bar{x})(y - \bar{y})^t] \\ &= E[xy^T - x\bar{y}^T - \bar{x}y + \bar{x}\bar{y}^T] \\ &= E[xy^T] - E[x]\bar{y}^T - \bar{x}E[y^T] + \bar{x}\bar{y}^T \\ &= E[xy^T] - \bar{x}\bar{y}^T - \bar{x}\bar{y}^T + \bar{x}\bar{y}^T \\ &= E[xy^T] - \bar{x}\bar{y}^T = 0 \end{aligned}$$

Gaussian Distribution

Why model the probablity densit function as a Gaussian (or normal) distribution:

- provides a good statistical model for many natural phenomina
- computationally tractable because the statistical properties are described completely by first (mean, $\bar{x}$) and second (variance, P) moments
- normality is preserved through linear transforms (both static and dynamic)

Characteristic Function, for a Gaussian vector

A random vector $x \in \mathbb{R}_n$ is Gaussian distributed or normal if the characteristic function has the form:

$$\phi_x(s) = e^{j\bar{x}^T s - \frac{1}{2} s^T P s}$$

where $s \in C_n$, $\bar{x} = E[x]$, and $P = [(x - \bar{x})(x - \bar{x})^T]$.

- For such a random vector, we use the notation $x = N(\bar{x}, P)$.
- Two vectors x and y are jointly Gaussian distributed if (x^T, y^T) is Gaussian.
- When a random vector x is Gaussian and it's covariance matrix P_x is nonsingular, it's PDF can be evaluated with:

$$f(x) = \frac{e^{-\frac{1}{2}(x-\bar{x})^T P_x^{-1}(x-\bar{x})}}{\sqrt{(2\pi)^n det(P_x)}}$$

If P_x is singular, then the above equation will not work, but the characteristic function can define the Gaussian distribution indirectly.

Random Process

In a random process, we are looking at a family of random vectors $(x(t), t \in I)$ indexed by time.

Mean Value Function

$$\bar{x}(t) = E[x(t)], t \in I$$

Covariance Kernal

$$P(t, \tau) = E[(x(t) - \bar{x}(t))(x(t) - \bar{x}(t))^T], t \in I$$

Cross-Covariance Kernal

For two random processes $x(t)$ and $y(t)$

$$P_{xy}(t, \tau) = E[(x(t) - \bar{x}(t))(y(t) - \bar{y}(t))^T], t \in I$$

Covariance Matrix $= P(t, t)$

Which satisfies:

$$P(t) = E[x(t)x^T(t)] - \bar{x}(t)\bar{x}^T(t)$$

Cross-Covariance Matrix $= P_{xy}(t, t)$

Which satisfies:

$$P_{xy}(t) = E[x(t)y^T(t)] - \bar{x}(t)\bar{y}^T(t)$$

Independence

todo..

Correlation

todo..

Gauss-Markov Process

This section combines the ideas of **Gaussian distribution** and **random process**

- a random process is $x(t)$ is **Gaussian** if all of the vectors $x_1(t)$,...$x_n(t)$ are jointly Gaussian
- a Gaussian random process is **white** if the vectors $x(t_1)$, ...$x(t_m)$ are independent, otherwise it is **colored**
- for a **Gaussian** and **white** process, the **covariance kernal** satisfies

$$P(t, \tau) = 0, t \neq \tau$$
$$P(t, \tau) = Q(t)\delta(t - \tau)$$

- a random process is **Markov** if

$$f(x(t_m)|x(t_{m-1}), ..., x(t_1)) = f(x(t_m)|x(t_{m-1}))$$

- a random process is **Gauss-Markov** if it is both **Gauss** and **Markov**

Linear Gauss-Markov Models

The standard model is:

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + w(t), t \geq t_0 \quad (4)$$
$$y(t) = C(t)x(t) + v(t) \quad (5)$$

Standard Assumptions

1. the intial condition $x(t_0)$ is Gaussian

$$x(t_0) = N(\bar{x}(t_0), P(t_0))$$

2. the disturbance $w(t)$ is a zero-mean, Gaussian, white process that is independent of $x(t_0)$

$$E[w(t)] = 0$$
$$E[w(t)w^T(\tau)] = \underbrace{R_w(t)\delta(t - \tau)}_{covariance\ kernal}$$
$$E[w(t)(x(t_0 - \bar{x}(t_0))^T] = 0$$

EX: in the case of a $2X2$ system, if there is some covariance σ_w given for the second state variable then

$$R_w(t) = \begin{bmatrix} 0 & 0 \\ 0 & \sigma_w \end{bmatrix}$$

3. the measurment noinse $v(t)$ is a zero-mean, Gaussian, white process that is independent of $x(t_0)$

$$E[v(t)] = 0$$
$$E[v(t)v^T(\tau)] = R_v(t)\delta(t - \tau)$$
$$E[v(t)(x(t_0 - \bar{x}(t_0))^T] = 0$$

4. the processes $v(t)$ and $w(t)$ are uncorrelated

$$E[w(t)v^T(t)] = 0$$

With the standard model and assumptions, the process $x(t)$ is **Markov**.

Proof that $x(t)$ is Markov

Recall the variation of consants formula:

$$x(t) = \Phi(t,t_0)x(t_0) + \int_{t_0}^{t} \Phi(t,\tau)B(\tau)u(\tau)d\tau + ...$$
$$+ \int_{t_0}^{t} \Phi(t,\tau)w(\tau)d\tau$$

Then,

$$x(t_m) = \Phi(t_m,t_{m-1})x(t_{m-1}) + ...$$
$$+ \int_{t_{m-1}}^{t_m} \Phi(t_m,\tau)(B(\tau)u(\tau) + w(\tau))d\tau$$

Notice that the results does not depend on $x(\tau)$, $\tau < t_{m-1}$.

Mean Value Functions, for Eqn. 5

Also refered to as the expected value.

Mean Value Function, $\bar{x}(t)$

$$\bar{x}(t) = \Phi(t,t_0)\bar{x}(t_0) + \int_{t_0}^{t} \Phi(t,\tau)B(\tau)u(\tau)d\tau$$

Mean Value Function, $\bar{y}(t)$

$$\bar{y}(t) = E[C(t)x(t) + v(t)] = C(t)\bar{x}(t)$$

Covariance Matrixes, for Eqn. 5

Covariance Matrix, $P_x(t) = P(t)$

Using $x(t) - \bar{x}(t)$ we can define:

$$P(t) = \Phi(t,t_0)P(t_0)\Phi^T(t,t_0) + \int_{t_0}^{t} \Phi(t,\tau)R_w(\tau)\Phi^T(t,\tau)d\tau$$

The above equation also satisfies the Lyapunov Equation.

Covariance Matrix, $P_y(t)$

For the output $y(t)$:

$$P_y(t) = C(t)P(t)C^T(t) + R_v(t)$$

Lyapunov Equation

with the standard model and assumptions, the covariance matrix satisfies:

$$\dot{P}(t) = A(t)P(t) + P(t)A^T(t) + R_w(t)$$

Which can be derived using the **Leibniz Integral Rule**.

Steady State Covariance Matrix

To find the steady state covariance matrix:

- set $\dot{P}(t) = 0$
- then solve for P_{ss}
  - this will require a computer!

Covariance Kernals, for Eqn. 5

Covariance Kernal, $P_x(t,\tau)$

For the state $x(t)$:

$$P_x(t,\tau) = \Phi(t,t_0)P(t_0)\Phi^T(t,t_0) + \int_{t_0}^{t} \Phi(t,\sigma)R_w(\sigma)\Phi^T(t,\sigma)d\sigma$$

Covariance Kernal, $P_y(t,\tau)$

For the output $y(t)$:

$$P_y(t,\tau) = C(t)\Phi(t,t_0)P(t_0)\Phi^T(t,t_0)C^T(\tau) +$$
$$\int_{t_0}^{t} C(t)\Phi(t,\sigma)R_w(\sigma)\Phi^T(t,\sigma)C^T(\tau)d\sigma + R_v(t)\delta(t-\tau)$$

Estimation

After evaluating $\frac{f(x,y)}{f_y(y)}$, the result is PDF of a Guassian vector defined as:

$$f(x|y) = \frac{e^{-\frac{1}{2}(x - E^T[x|y])^T P_{x|y}^{-1}(x - E[x|y])}}{\sqrt{(2\pi)^n det(P_{x|y})}}$$

with a mean and covariance defined as follows:

Mean, $\hat{x}^+ = E[x|y]$

$$E[x|y] = \bar{x} + P_{xy}P_y^{-1}(y - \bar{y})$$

Covariance, $P^+ = P_{x|y}$

$$P_{x|y} = P_x + P_{xy}P_y^{-1}P_{yx}$$

The basic estimation procedure is:

1. determine $\hat{x}^- = \bar{x}$ which is an estimate of the state
   - based off of state equations
   - affected by state uncertainty, $w(t)$
2. collect measurments from output equation $y(t)$
3. update estimate of state ($\hat{x}^-$) based off of new info from $y(t)$
   - based off of output equation
   - affected by sensor uncertainty, $v(t)$

Batch Process

Measurments are incorporated simultaneously.

$$\hat{x}^+ = \hat{x}^- + k(z - C\hat{x}^-)$$
$$P^+ = P^-C^T(CP^-C^T + R)^{-1}$$

where, $k = P^-C^T(CP^-C^T + R)^{-1}$ or in an algebraically equivalent form,

$$\hat{x}^+ = (P^+(P^-)^{-1})\hat{x}^- + (P^+C^TR^{-1})z$$
$$P^+ = ((P^-)^{-1} + C^TR^{-1}C)^{-1}$$

where R is the covariance of the sensor measurment from $y(t)$ also note: $z = y$

Sequential Process

Measurments are incorporated recursively.