PROBLEM #1

PART A:

GIVEN:

$$\underline{\dot{X}}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \underline{X}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} a \qquad X(t_0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \underline{X}(t)$$

WHERE, $\underline{X}(t) = \begin{bmatrix} r(t) \\ V(t) \end{bmatrix}$

DETERMINE:

USE METHOD OF ADJOINTS TO EVALUATE THE POSITION OF VEHICLE AT FINAL TIME

ASSUMPTIONS:

$t_0 = 0$

SOLUTION:

$$\underline{\dot{P}}(t) = -A^T(t)\,\underline{P}(t)$$

$$\underline{P}(t_f) = C^T(t_f)$$

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad -A^T = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 \end{bmatrix} \qquad C^T(t_f) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

PROB 1, PS # 2

$$\dot{p}(t) = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} p(t)$$

$\dot{p}_1(t) = 0 \qquad p_1(t_f) = 1 \quad \to \quad p_1(t) = 1$

$\dot{p}_2(t) = -p_1(t) \qquad p_2(t_f) = 0$

$$\frac{dp_2(t)}{dt} = -1 \quad \to \quad \int_{p_2(t_f)}^{p_2(t)} dp_2(t) = -\int_{t_f}^{t} dt$$

$$p_2(t) - \cancelto{0}{p_2(t_f)} = -[t - t_f]$$

$$p_2(t) = t_f - t$$

So, $\underline{p}(t) = \begin{bmatrix} 1 \\ t_f - t \end{bmatrix}$ & $p(\tau) = \begin{bmatrix} 1 \\ t_f - \tau \end{bmatrix}$

FORWARD INTEGRATION,

$$y(t_f) = p^T(t_0)\cancelto{0}{x(t_0)} + \int_{t_0}^{t_f} p^T(\tau)\, B(\tau)\, u(\tau)\, d\tau$$

$$= \int_{t_0}^{t_f} \begin{bmatrix} 1 & t_f - \tau \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} a \, d\tau = a \int_{t_0}^{t_f} (t_f - \tau)\, d\tau$$

$= a t_f^2 - \frac{a t_f^2}{2}$ So, $\boxed{y(t_f) = \frac{a t_f^2}{2}}$

PART B:

FOR AN LTI SYSTEM,

$$\Phi(t,\tau) = e^{A(t-\tau)} \qquad \mathcal{L}^{-1}\left\{(sI-A)^{-1}\right\} = e^{At}$$

$$(sI-A)^{-1} = \begin{bmatrix} s & -1 \\ 0 & s \end{bmatrix}^{-1} = \frac{1}{s^2}\begin{bmatrix} s & 1 \\ 0 & s \end{bmatrix} = \begin{bmatrix} \frac{1}{s} & \frac{1}{s^2} \\ 0 & \frac{1}{s} \end{bmatrix}$$

So,

$$\Phi(t,0) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

THEN,

$$y(t_f) = C(t_f)\Phi(t,0)\cancel{x_0}^{\,0} + \int_{t_0}^{t_f} C(t_f)\Phi(\tau,0)\,B(\tau)u(\tau)\,d\tau$$

$$* \quad [1 \;\; 0]\begin{bmatrix} 1 & \tau \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} a = [1 \;\; \tau]\begin{bmatrix} 0 \\ 1 \end{bmatrix} a = \tau a$$

$$y(t_f) = \int_{t_0=0}^{t_f} \tau a \, d\tau = \frac{a\tau^2}{2}\Bigg|_0^{t_f} = \frac{a t_f^2}{2} \quad \checkmark$$

$\Rightarrow$ MATCHES RESULT FROM PART A!

PROBLEM #2

PART A:

ASSUME $t_0 = 0$

$$\frac{d\psi}{dt} = \frac{V}{R}$$

$$\int_{\psi(0)}^{\psi(t)} d\psi = \int_0^t \frac{V}{R}\,dt$$

$$\psi(t) = \frac{V}{R}t \qquad \psi(t_f) \Rightarrow \frac{V}{R}t_f = \pi$$

So, THE AIRCRAFT HAS A 180° AT:

$$\boxed{t_f = \frac{R}{V}\pi}$$

$$\frac{dX}{dt} = V\cos\left(\frac{V}{R}t\right)$$

$$\int_0^{X(t)} dx = \int_0^t V\cos\left(\frac{V}{R}\tau\right)d\tau$$

$$X(t) = V\int_0^t \cos(u)\,du$$

$$u = \frac{V}{R}\tau$$

LIMITS

$t=0 \Rightarrow u=0$

$t=t \Rightarrow u=\frac{V}{R}t$

$$du = \frac{V}{R}d\tau \Rightarrow d\tau = \frac{R}{V}du$$

So, $$X(t) = \cancel{V}\int_0^{V/R\,t} \cos(u)\,\frac{R}{\cancel{V}}\,du$$

$$= R\sin(u)\Big|_0^{\frac{V}{R}t}$$

$$= R\sin\left(\frac{V}{R}t\right)$$

THUS, AT $t_f = \frac{R}{V}\pi$:

$$X(t_f) = R\sin\left(\cancel{\frac{V}{R}}\,\cancel{\frac{R}{V}}\,\pi\right)$$

$$X(t_f) = R\sin(\pi) = 0$$

SIMILARLY FOR Y,

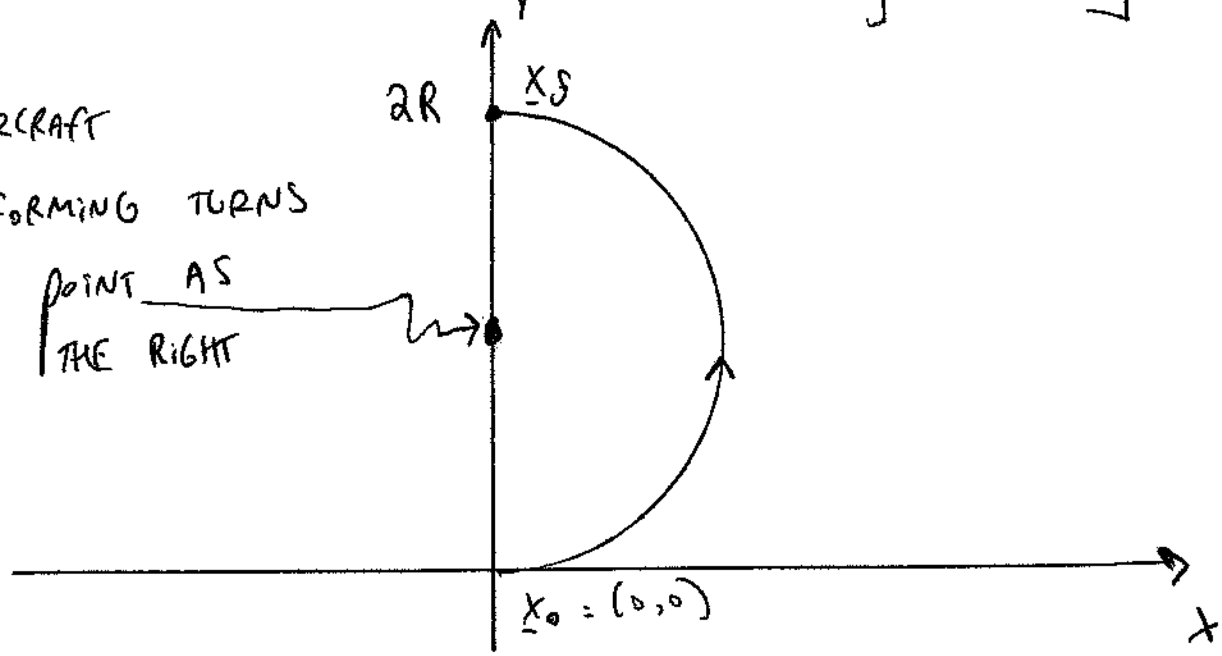
$$\int_0^{Y(t)} dY = \int_0^t V\sin\left(\frac{V}{R}t\right)dt \Rightarrow Y(t) = -R\left[\cos\left(\frac{V}{R}t\right) - 1\right]$$

$$Y(t_f) = -R\Big[\cos(\pi) - \cos(0)\Big] = -R[-1-1]$$

SO, AT t_f: $Y(t_f) = 2R$

THUS, THE POSITION AT t_f IS: $\begin{bmatrix} X(t_f) \\ Y(t_f) \end{bmatrix} = \begin{bmatrix} 0 \\ 2R \end{bmatrix}$

THE AIRCRAFT IS PERFORMING TURNS AROUND A POINT AS SHOWN TO THE RIGHT



// PART B:

GIVEN: R^0 & V^0

FIND: $\delta X(t)$ & $\delta Y(t)$ IN TERMS OF δR & δV

P2, P#3

$$\left.\frac{\partial x(t)}{\partial R}\right|_0 = V_0 \sin\left(\frac{V_0}{R_0}t\right) V_0 t R_0^{-2}$$

$$\left.\frac{\partial x(t)}{\partial V}\right|_0 = \cos\left(\frac{V_0}{R_0}t\right) - \frac{V_0}{R_0}\sin\left(\frac{V_0}{R_0}t\right) t$$

$$\left.\frac{\partial y(t)}{\partial R}\right|_0 = -\frac{V_0^2}{R_0^2}\cos\left(\frac{V_0}{R_0}t\right) t$$

$$\left.\frac{\partial y(t)}{\partial V}\right|_0 = \sin\left(\frac{V_0}{R_0}t\right) + \frac{V_0}{R_0}\cos\left(\frac{V_0}{R_0}t\right) t$$

THEN,

$$\begin{bmatrix} \delta x(t) \\ \delta y(t) \end{bmatrix} = \begin{bmatrix} \left.\frac{\partial x(t)}{\partial R}\right|_0 & \left.\frac{\partial x(t)}{\partial V}\right|_0 \\ \left.\frac{\partial y(t)}{\partial R}\right|_0 & \left.\frac{\partial y(t)}{\partial V}\right|_0 \end{bmatrix} \begin{bmatrix} \delta R \\ \delta V \end{bmatrix}$$

So,

$$\underbrace{\begin{bmatrix} \delta x(t) \\ \delta y(t) \end{bmatrix}}_{\delta X} = \underbrace{\begin{bmatrix} \frac{V_0^2}{R_0^2}\sin\left(\frac{V_0}{R_0}t\right) t & \cos\left(\frac{V_0}{R_0}t\right) - \frac{V_0}{R_0}\sin\left(\frac{V_0}{R_0}t\right) t \\ -\frac{V_0^2}{R_0^2}\cos\left(\frac{V_0}{R_0}t\right) t & \sin\left(\frac{V_0}{R_0}t\right) + \frac{V_0}{R_0}\cos\left(\frac{V_0}{R_0}t\right) t \end{bmatrix}}_{A(t)} \underbrace{\begin{bmatrix} \delta R \\ \delta V \end{bmatrix}}_{\delta Q}$$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

// PART C: GIVEN THE SYSTEM FROM PART B: P2, PS#4

$$\delta X(t) = A(t)\delta Q$$

THE COVARIANCE CAN BE DEFINED AS:

$$P_X(t) = A(t)\, P\, A(t)^T$$

WHERE THE DIAGONAL TERMS IN $P_X(t)$ WILL BE THE VARIANCES OF $\delta x(t)$ & $\delta y(t)$.

$$P_X = \begin{bmatrix} P_{X11} & P_{X12} \\ P_{X21} & P_{X22} \end{bmatrix}$$

// TERM BY TERM:

$$A(t)\, P\, A^T(t) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} \sigma_{RR} & \sigma_{RV} \\ \sigma_{VR} & \sigma_{VV} \end{bmatrix} \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}$$

$$= \begin{bmatrix} (a_{11}\sigma_{RR} + a_{12}\sigma_{VR}) & (a_{11}\sigma_{RV} + a_{12}\sigma_{VV}) \\ (a_{21}\sigma_{RR} + a_{22}\sigma_{VR}) & (a_{21}\sigma_{RV} + a_{22}\sigma_{VV}) \end{bmatrix} \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}$$

$$= \begin{bmatrix} \underbrace{\left(a_{11}(a_{11}\sigma_{RR} + a_{12}\sigma_{VR}) + a_{12}(a_{11}\sigma_{RV} + a_{12}\sigma_{VV})\right)}_{P_{X11}} & P_{X12} \\ P_{X21} & \underbrace{\left(a_{21}(a_{21}\sigma_{RR} + a_{22}\sigma_{VR}) + a_{22}(a_{21}\sigma_{RV} + a_{22}\sigma_{VV})\right)}_{P_{X22}} \end{bmatrix}$$

$$P_{X11} = a_{11}^2\sigma_{RR} + a_{11}a_{12}\sigma_{VR} + a_{12}a_{11}\sigma_{RV} + a_{12}^2\sigma_{VV}$$

$$P_{X22} = a_{21}^2\sigma_{RR} + a_{21}a_{22}\sigma_{VR} + a_{22}a_{21}\sigma_{RV} + a_{22}^2\sigma_{VV}$$

THE VARIANCE OF $\delta x(t)$ IS:

$$P_{x11} = \left[\frac{V_0^2}{R_0^2}\sin\left(\frac{V_0}{R_0}t\right)t\right]^2 \sigma_{RR} + \left[\frac{V_0^2}{R_0^2}\sin\left(\frac{V_0}{R_0}t\right)t\right]\left[\cos\left(\frac{V_0}{R_0}t\right) - \frac{V_0}{R_0}\sin\left(\frac{V_0}{R_0}t\right)t\right]\left[\sigma_{VR}+\sigma_{RV}\right]$$

$$+ \left[\cos\left(\frac{V_0}{R_0}t\right) - \frac{V_0}{R_0}\sin\left(\frac{V_0}{R_0}t\right)t\right]^2 \sigma_{VV}$$

THE VARIANCE OF $\delta y(t)$ IS:

$$P_{x22} = \left[\frac{-V_0^2}{R_0^2}\cos\left(\frac{V_0}{R_0}t\right)t\right]^2 \sigma_{RR} + \left[\frac{-V_0^2}{R_0^2}\cos\left(\frac{V_0}{R_0}t\right)t\right]\left[\sin\left(\frac{V_0}{R_0}t\right) + \frac{V_0}{R_0}\cos\left(\frac{V_0}{R_0}t\right)t\right]\left[\sigma_{VR}+\sigma_{RV}\right]$$

$$+ \left[\sin\left(\frac{V_0}{R_0}t\right) + \frac{V_0}{R_0}\cos\left(\frac{V_0}{R_0}t\right)t\right]^2 \sigma_{VV}$$

NOTE: COVARIANCE MATRICES ARE SYMETRIC SO, σ_{VR} SHOULD EQUAL σ_{RV}, BUT IT IS LEFT AS SEPERATE.

PART D:

$$\left.\begin{aligned} \dot{X}(t) &= f(x,u,t) + w(t) \\ Y(t) &= g(x,t) + v(t) \end{aligned}\right\} \text{ NONLINEAR MARKOV MODEL}$$

ASSUMPTIONS:

1. $V(t)$ & $w(t)$ ARE GAUSSIAN WHITENOISE PROCESSES REPRESENTING DISTURBANCES
2. OBSERVATION ERROR $\tilde{X}(t) = X(t) - \hat{X}(t)$ IS SMALL, SO NONLINEAR TERMS CAN BE NEGLECTED IN OBSERVATION ERROR DYNAMICS

TO SOLVE THIS PROBLEM WE WILL USE AN EXTENDED KALMAN FILTER.

FIRST WE DEFINE THE STATE AS $X(t) = \begin{bmatrix} x(t) \\ y(t) \\ \omega(t) \end{bmatrix}$, WHERE $\omega = \frac{V}{R}$ IS ADDED. THE SYSTEM IS THEN:

$$\dot{X}_1(t) = V\cos(\omega t) + w_1(t)$$

$$\dot{X}_2(t) = V\sin(\omega t) + w_2(t)$$

$$\dot{X}_3(t) = \omega + w_3(t)$$

$R = \sqrt{x^2 + y^2}$
- DON'T KNOW R IN PART (d)
- NEED TO ESTIMATE IT VIA $\hat{x}$ & $\hat{y}$

WITH OUTPUT EQUATIONS:

$$Y(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} (x^2 + y^2)^{1/2}\sin(\omega t) + v_1(t) \\ (x^2 + y^2)^{1/2}(1 - \cos(\omega t)) + v_2(t) \end{bmatrix}$$

WHERE, $w(t) = \begin{bmatrix} w_1(t) \\ w_2(t) \\ w_3(t) \end{bmatrix}$ & $v(t) = \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix}$

NEXT WE NEED TO LINEARIZE OUR SYSTEM DYNAMICS AROUND $\hat{x}$ (THE ESTIMATE OF THE STATE):

$$\dot{X}(t) = f(x,u,t) + w(t) \cong f(\hat{x},u,t) + \left(\frac{\partial f}{\partial x}\right)\Bigg|_{\hat{x}} \overbrace{(X - \hat{X})}^{\tilde{X}\ \text{(ERROR)}} + w(t)$$

WHERE,

$$\frac{\partial f}{\partial x}\Bigg|_{\hat{x}} = \begin{bmatrix} \frac{\partial \dot{X}_1}{\partial x} & \frac{\partial \dot{X}_1}{\partial y} & \frac{\partial \dot{X}_1}{\partial \omega} \\ \frac{\partial \dot{X}_2}{\partial x} & \frac{\partial \dot{X}_2}{\partial y} & \frac{\partial \dot{X}_2}{\partial \omega} \\ \frac{\partial \dot{X}_3}{\partial x} & \frac{\partial \dot{X}_3}{\partial y} & \frac{\partial \dot{X}_3}{\partial \omega} \end{bmatrix}\Bigg|_{\hat{x}}$$

WHERE, $\hat{X} = \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{\omega} \end{bmatrix}$

P2, PS #7

$$\frac{\partial \dot{x}_1}{\partial x} = 0, \quad \frac{\partial \dot{x}_1}{\partial y} = 0, \quad \frac{\partial \dot{x}_1}{\partial \omega} = -V\sin(\omega t)\,t$$

$$\frac{\partial \dot{x}_2}{\partial x} = 0, \quad \frac{\partial \dot{x}_2}{\partial y} = 0, \quad \frac{\partial \dot{x}_2}{\partial \omega} = +V\cos(\omega t)\,t$$

$$\frac{\partial \dot{x}_3}{\partial x} = 0, \quad \frac{\partial \dot{x}_3}{\partial y} = 0, \quad \frac{\partial \dot{x}_3}{\partial \omega} = 1$$

So,

$$\left(\frac{\partial f}{\partial x}\right)\Bigg|_{\hat{x}} = \begin{bmatrix} 0 & 0 & -V\sin(\hat{\omega}t)\,t \\ 0 & 0 & V\cos(\hat{\omega}t)\,t \\ 0 & 0 & 1 \end{bmatrix}$$

THEN OUR SYSTEM DYNAMICS ARE:

$$\dot{\tilde{x}}(t) = \begin{bmatrix} V\cos(\hat{\omega}t) \\ V\sin(\hat{\omega}t) \\ \hat{\omega} \end{bmatrix} + \begin{bmatrix} 0 & 0 & -V\sin(\hat{\omega}t)t \\ 0 & 0 & V\cos(\hat{\omega}t)t \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{\omega} \end{bmatrix} + \begin{bmatrix} w_1(t) \\ w_2(t) \\ w_3(t) \end{bmatrix}$$

NEXT A LINEARIZED OBSERVER WILL BE OBTAINED OF THE FORM:

$$\dot{\hat{x}}(t) = f(\hat{x}, u, t) - G(t)\left(\frac{\partial g}{\partial x}\right)\Bigg|_{\hat{x}} \tilde{x} - G(t)\,v(t)$$

WHERE,

$$\left(\frac{\partial g}{\partial x}\right)\Bigg|_{\hat{x}} = \left[\begin{matrix} \frac{\partial g_1}{\partial x} & \frac{\partial g_1}{\partial y} & \frac{\partial g_1}{\partial w} \\ \frac{\partial g_2}{\partial x} & \frac{\partial g_2}{\partial y} & \frac{\partial g_2}{\partial w} \end{matrix}\right]\Bigg|_{\hat{x}}$$

$$\frac{\partial g_1}{\partial x} = \frac{1}{\cancel{2}}\left(x^2+y^2\right)^{-1/2}(\cancel{2})X \sin(wt)$$

$$\frac{\partial g_1}{\partial y} = \frac{1}{\cancel{2}}\left(x^2+y^2\right)^{-1/2}(\cancel{2})y \sin(wt)$$

$$\frac{\partial g_1}{\partial w} = \left(x^2+y^2\right)^{1/2}\cos(wt)\,t$$

$$\frac{\partial g_2}{\partial x} = \frac{1}{\cancel{2}}\left(x^2+y^2\right)^{-1/2}(\cancel{2})(x)\left(1-\cos(wt)\right)$$

$$\frac{\partial g_2}{\partial y} = \frac{1}{\cancel{2}}\left(x^2+y^2\right)^{-1/2}(\cancel{2})(y)\left(1-\cos(wt)\right)$$

$$\frac{\partial g_2}{\partial w} = \left(x^2+y^2\right)^{1/2}\sin(wt)\,t$$

THE <u>NON LINEAR</u> ESTIMATOR FOR ALL OF THE STATES INCLUDING W IS THEN GIVEN AS

$$\dot{\hat{X}}(t) = \begin{bmatrix} V\cos(\hat{\omega}t) \\ V\sin(\hat{\omega}t) \\ \hat{\omega} \end{bmatrix} - \quad \cdots$$

$$G(t)\begin{bmatrix} \frac{\hat{x}\sin(\hat{\omega}t)}{(\hat{x}^2+\hat{y}^2)^{1/2}} & \frac{\hat{y}\sin(\hat{\omega}t)}{(\hat{x}^2+\hat{y}^2)^{1/2}} & (\hat{x}^2+\hat{y}^2)^{1/2}\cos(\hat{\omega}t)t \\ \frac{\hat{x}(1-\cos(\hat{\omega}t))}{(\hat{x}^2+\hat{y}^2)^{1/2}} & \frac{\hat{y}(1-\cos(\hat{\omega}t))}{(\hat{x}^2+\hat{y}^2)^{1/2}} & (\hat{x}^2+\hat{y}^2)^{1/2}\sin(\hat{\omega}t)t \end{bmatrix}\tilde{X}(t) - \quad \cdots$$

$$G(t)\,V(t)$$

WHERE ~~A~~GAIN, $\hat{X}(t)$ IS THE ERROR $\left(X(t)-\hat{X}(t)\right)$, AND THE SOLUTION FOR $X(t)$ COMES FROM THE SOLUTION FOR THE LINEARIZE SYSTEM DYNAMICS (DERIVED PREVIOUSLY). ALSO, $G(t)$ IS THE GAIN. WHICH IS CHOSEN TO BE THE KALMAN GAIN.

$$G(t) = -P_{\tilde{X}}(t)\,C^T(t)\,R_v^{-1}(t)$$

WHERE.

$$\dot{P}_{\tilde{X}}(t) = A(t)P_{\tilde{X}}(t) + P_{\tilde{X}}(t)A^T(t) - P_{\tilde{X}}(t)C^T(t)R_v^{-1}(t)C(t)P_{\tilde{X}}(t) + R_w(t)$$

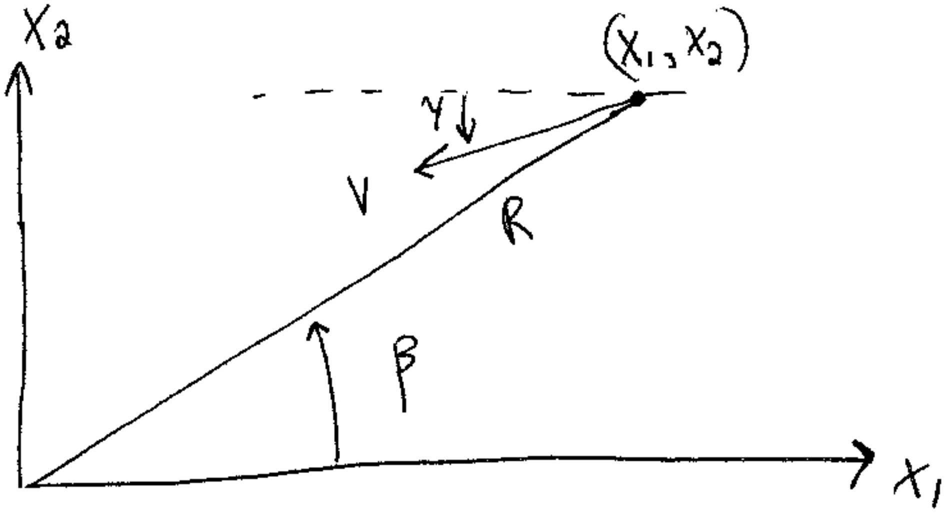
$$P_{\hat{X}}(t_0) = P_{\tilde{X}0} \qquad // \text{ SOLVE USING COMPUTER!}$$

ESTIMATE FOR W WILL BE $\hat{\omega}$.

PROBLEM #3

$$\dot{X}_1 = -V\cos\gamma$$
$$\dot{X}_2 = -V\sin\gamma$$
$$\gamma = X_2$$

$$X_1 = R\cos\beta$$
$$X_2 = R\sin\beta$$



// PART A:

$$R = \sqrt{x_1^2 + x_2^2}$$

$$\frac{dR}{dt} = \frac{1}{\cancel{2}}(x_1^2 + x_2^2)^{-1/2}(\cancel{2})X_2\dot{X}_2 + \frac{1}{\cancel{2}}(x_1^2 + x_2^2)^{-1/2}\cancel{2}X_1\dot{X}_1$$

$$\frac{dR}{dt} = \frac{X_1\dot{X}_1 + X_2\dot{X}_2}{R}$$

$$\frac{dR}{dt} = \frac{-V\cancel{R}\cos\beta\cos\gamma - V\cancel{R}\sin\beta\sin\gamma}{\cancel{R}}$$

ASIDE:
$$\cos A\cos B + \sin A\sin B = \cos(A-B)$$

$$\frac{dR}{dt} = -V\cos(\beta - \gamma)$$

ASIDE:
$$\frac{d}{dx}\left(\tan^{-1}(x)\right) = \frac{1}{1+x^2}$$

$$\beta = \tan^{-1}\left(\frac{X_2}{X_1}\right)$$

PROB3, pg#2

$$\frac{d\beta}{dt} = \left[\frac{1}{1+\left(\frac{x_2}{x_1}\right)^2}\right]\left[\frac{\dot{x}_2}{x_1} + x_2 x_1^{-2}(-1)\dot{x}_1\right]$$

$$= \left[\frac{1}{1+\left(\frac{x_2}{x_1}\right)^2}\right]\frac{\dot{x}_2 x_1 - \dot{x}_1 x_2}{x_1^2}$$

$$= \left[\frac{\cancel{x_1^2}}{\underbrace{x_1^2 + x_2^2}_{R^2}}\right]\frac{\dot{x}_2 x_1 - \dot{x}_1 x_2}{\cancel{x_1^2}} = \frac{-VR\sin\gamma\cos\beta + VR\cos\gamma\sin\beta}{R^2}$$

$$\frac{d\beta}{dt} = \frac{+V\cancel{R}\sin(\beta-\gamma)}{R^{\cancel{2}}}$$

ASIDE:

$\sin A\cos B - \sin B\cos A = \sin(A-B)$

THE EQUATIONS OF MOTION ARE:

$$\boxed{\begin{aligned}\dot{R} &= -V\cos(\beta-\gamma)\\ \dot{\beta} &= \frac{V\sin(\beta-\gamma)}{R}\end{aligned}}$$

// PART B:

$$\underline{\delta\dot{x}(t)} = \begin{bmatrix}\delta\dot{R}(t)\\ \delta\dot{\beta}(t)\end{bmatrix} = \begin{bmatrix}\left.\frac{\partial\dot{R}(t)}{\partial R}\right|_0 & \left.\frac{\partial\dot{R}(t)}{\partial\beta}\right|_0\\ \left.\frac{\partial\dot{\beta}(t)}{\partial R}\right|_0 & \left.\frac{\partial\dot{\beta}(t)}{\partial\beta}\right|_0\end{bmatrix}\begin{bmatrix}\delta R\\ \delta\beta\end{bmatrix}$$

$$\left.\frac{\partial\dot{R}}{\partial R}\right|_0 = 0 \qquad \left.\frac{\partial\dot{R}}{\partial\beta}\right|_0 = V_0\sin(\beta_0-\gamma_0)$$

$$\left.\frac{\partial\dot{\beta}}{\partial R}\right|_0 = V_0\sin(\beta_0-\gamma_0)R_0^{-2}(-1) = -\frac{V_0\sin(\beta_0-\gamma_0)}{R_0^2}$$

$$\left.\frac{\partial \dot{\beta}}{\partial \beta}\right|_0 = \frac{V_0}{R_0}\cos(\beta_0 - \gamma_0)$$

THUS, THE LINEARIZED SYSTEM IS:

$$\boxed{\begin{bmatrix} \delta \dot{R}(t) \\ \delta \dot{\beta}(t) \end{bmatrix} = \begin{bmatrix} 0 & V_0 \sin(\beta_0 - \gamma_0) \\ \frac{-V_0 \sin(\beta_0 - \gamma_0)}{R_0^2} & \frac{V_0 \cos(\beta_0 - \gamma_0)}{R_0} \end{bmatrix} \begin{bmatrix} \delta R \\ \delta \beta \end{bmatrix} \qquad \delta y(t) = [\sin\beta_0 \quad R_0 \cos\beta_0] \begin{bmatrix} \delta R \\ \delta \beta \end{bmatrix}}$$

PART C:

FIRST LINEARIZE THE OUPUT EQUATION $y = x_2$ AROUND NOMINAL TRAJECTORY (MAY BE PART OF PART B)

$$\delta y(t) = \left[\left.\frac{\partial y}{\partial R}\right|_0 \quad \left.\frac{\partial y}{\partial \beta}\right|_0 \right] \begin{bmatrix} \delta R \\ \delta \beta \end{bmatrix} \qquad y = R \sin\beta$$

$$\left.\frac{\partial y}{\partial R}\right|_0 = \sin\beta_0 \qquad \left.\frac{\partial y}{\partial \beta}\right|_0 = R_0 \cos\beta_0$$

So, $\delta y(t) = [\sin\beta_0 \quad R_0 \cos\beta_0] \begin{bmatrix} \delta R \\ \delta \beta \end{bmatrix}$ (*)

NEXT DEFINE A STATE OBSERVER SYSTEM AS:

$$\dot{\hat{x}}(t) = A(t)\hat{x}(t) + B(t)u(t) + G(t)\left(\underbrace{C(t)\hat{x}(t)}_{\text{ESTIMATE}} - \underbrace{y(t)}_{\text{ACTUAL}}\right)$$

$$\hat{x}(t_0) = \hat{x}_0$$

THEN THE ESTIMATION ERROR CAN BE DEFINED AS:

$$\tilde{X}(t) = X(t) - \hat{X}(t)$$

GOVERNED BY THE DYNAMIC EQUATION:

$$G = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}$$

$$\dot{\tilde{X}}(t) = \big(A(t) + G(t)\,C(t)\big)\,\tilde{X}(t)$$

IN THIS PROBLEM, THE ESTIMATION ERROR EQUATIONS ARE:

$$\boxed{\dot{\tilde{X}}(t) = \left[\underbrace{\begin{bmatrix} 0 & V_0 \sin(\beta_0-\gamma_0) \\ \dfrac{-V_0 \sin(\beta_0-\gamma_0)}{R_0^2} & \dfrac{V_0 \cos(\beta_0-\gamma_0)}{R_0} \end{bmatrix}}_{A} + \underbrace{\begin{bmatrix} g_1 \\ g_2 \end{bmatrix}}_{G} \underbrace{\begin{bmatrix} \sin\beta_0 & R_0\cos\beta_0 \end{bmatrix}}_{C}\right] \tilde{X}(t)}$$

THE GAINS OF THE G MATRIX SHOULD BE SELECTED SUCH THAT THE EIGENVALUES λ_i OF THE $(A+GC)$ MATRIX HAVE STRICTLY NEGATIVE REAL PARTS.

$Re(\lambda_i) < 0 \quad \forall i = 1, 2$; THIS WILL MAKE THE ESTIMATION ERROR BE ASYMPTOTICALLY STABLE.