UNIVERSITY OF MICHIGAN

# AERO 584, Homework 5

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①

$$\underline{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} r \\ \theta \end{bmatrix}$$

$$\frac{x_1}{r} = \cos(\theta) \qquad \frac{x_2}{r} = \sin(\theta) \qquad \tan\theta = \frac{x_2}{x_1}$$

$$\theta = \tan^{-1}\left(\frac{x_2}{x_1}\right)$$

DISTANCE

Also,

$$y_1^2 = x_1^2 + x_2^2 \Rightarrow y_1 = \pm\sqrt{x_1^2 + x_2^2}$$

So,

$$y = \begin{bmatrix} +\sqrt{x_1^2 + x_2^2} \\ \tan^{-1}\left(\frac{x_2}{x_1}\right) \end{bmatrix} = \begin{bmatrix} g_1(x) \\ g_2(x) \end{bmatrix}$$

$$\frac{\partial \underline{g}}{\partial \underline{x}} = \begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} \end{bmatrix}$$

$$\frac{\partial g_1}{\partial x_1} = \frac{1}{\cancel{2}}\left(x_1^2 + x_2^2\right)^{-1/2} \cancel{2} x_1 = \frac{x_1}{\sqrt{x_1^2 + x_2^2}}$$

$$\frac{\partial g_1}{\partial x_2} = \frac{x_2}{\sqrt{x_1^2 + x_2^2}}$$

PROB 4.2 pg#1

$$\frac{\partial g_2}{\partial x_1} = \left(\frac{1}{1+\left(\frac{x_2}{x_1}\right)^2}\right)\left(-x_1^{-2}x_2\right) = \frac{-x_2}{x_1^2+x_2^2}$$

$$\frac{\partial g_2}{\partial x_2} = \left(\frac{1}{1+\left(\frac{x_2}{x_1}\right)^2}\right)\left(\frac{1}{x_1}\right) = \frac{x_1^{\cancel{2}}}{\cancel{x_1^2}+x_2^2}\left(\frac{1}{\cancel{x_1}}\right)$$

THUS,

(2)
$$\frac{\partial g}{\partial \underline{x}} = \begin{bmatrix} \frac{x_1}{\sqrt{x_1^2+x_2^2}} & \frac{x_2}{\sqrt{x_1^2+x_2^2}} \\ \frac{-x_2}{x_1^2+x_2^2} & \frac{x_1}{x_1^2+x_2^2} \end{bmatrix}$$

(3) USE

NEWTON'S METHOD

$$x^{k+1} = x^k + \left(\frac{\partial g}{\partial x}\right)_{x^k}^{-\cancel{T}}\left(y - g(x^k, t)\right)$$

CHOOSE SOME INITIAL GUESS FOR $x^k$ AND ITERATIVELY APPLY THE ABOVE EQUATION. SHOULD WORK AS LONG AS JACOBIAN IS NONSINGULAR AND THE GUESS IS WITHIN THE NEIGHBORHOOD OF $x^*$.

(4) $y = g(x,t) + v$ $\quad v = N(0, R_v)$ ASSUME ZERO MEAN

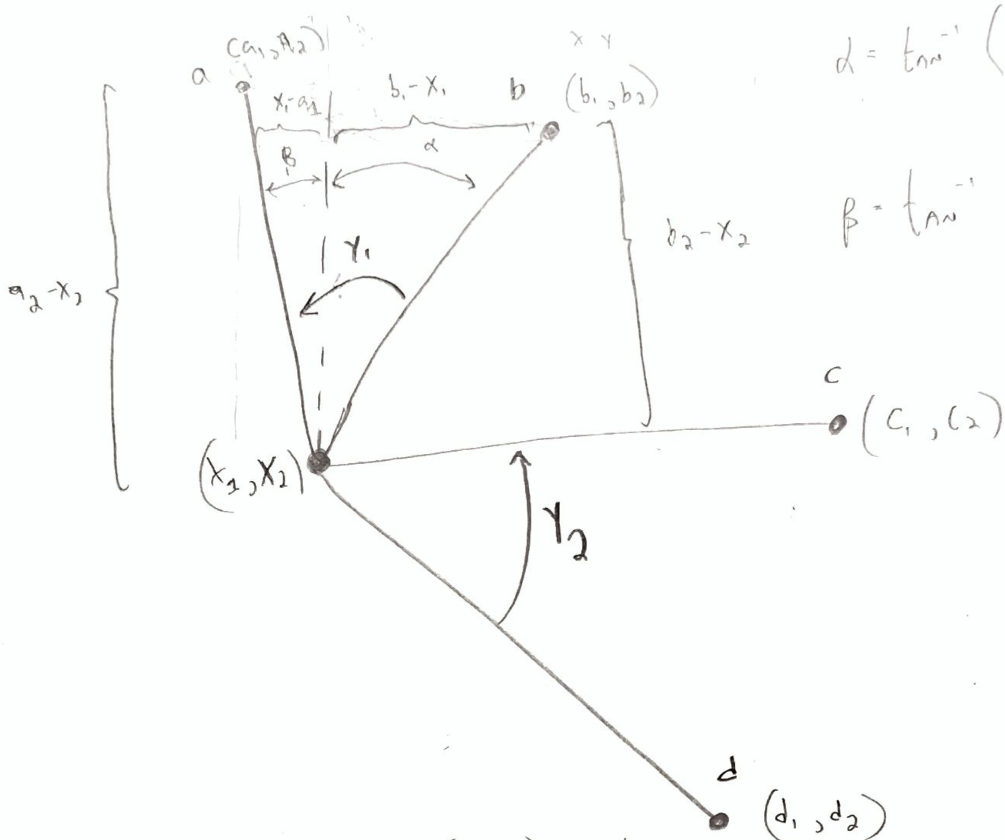
UNDER STANDARD ASSUMPTIONS (PAGE# 81)

$$\epsilon_x = \left(\frac{\partial g}{\partial x}\right)_x^{-T} v$$

LEADS TO,

$$R_{\epsilon x} = \left(\frac{\partial g}{\partial x}\right)_x^{-T} R_v \left(\frac{\partial g}{\partial x}\right)_x^{-1}$$

PROB 4.3 PS#1




$$\alpha = \tan^{-1}\left(\frac{b_1 - x_1}{b_2 - x_2}\right)$$

$$\beta = \tan^{-1}\left(\frac{x_1 - a_1}{a_2 - x_2}\right)$$

$$\gamma_1 = \alpha + \beta$$

① $$\gamma = \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix} = \begin{bmatrix} \tan^{-1}\left(\frac{b_1 - x_1}{b_2 - x_2}\right) + \tan^{-1}\left(\frac{x_1 - a_1}{a_2 - x_2}\right) \\ \tan^{-1}\left(\frac{d_1 - x_1}{d_2 - x_2}\right) + \tan^{-1}\left(\frac{x_1 - c_1}{c_2 - x_2}\right) \end{bmatrix} = \begin{bmatrix} g_1(\underline{x}) \\ g_2(\underline{x}) \end{bmatrix},$$

② $$\frac{(b_2 - x_2)^{\cancel{2}}}{(b_2 - x_2)^2 + (b_1 - x_1)^2}\left(\frac{-1}{\cancel{b_2 - x_2}}\right) = \frac{x_2 - b_2}{(b_2 - x_2)^2 + (b_1 - x_1)^2}$$

$$\frac{(a_2 - x_2)^{\cancel{2}}}{(a_2 - x_2)^2 + (x_1 - a_1)^2}\,\frac{1}{\cancel{(a_2 - x_2)}} = \frac{a_2 - x_2}{(a_2 - x_2)^2 + (x_1 - a_1)^2}$$

So, $$\frac{\partial g_1}{\partial x_1} = \frac{x_2 - b_2}{(b_2 - x_2)^2 + (b_1 - x_1)^2} + \frac{a_2 - x_2}{(a_2 - x_2)^2 + (x_1 - a_1)^2}$$

THEN,

$$\frac{\cancel{(b_2-x_2)^2}}{(b_2-x_2)^2+(b_1-x_1)^2}(b_1-x_1)(-1)(b_2-x_2)^{-2} = \frac{x_1-b_1}{(b_2-x_2)^2+(b_1-x_1)^2}$$

So,

$$\frac{\partial g_1}{\partial x_2} = \frac{x_1-b_1}{(b_2-x_2)^2+(b_1-x_1)^2} + \frac{x_1-a_1}{(a_2-x_2)^2+(a_1-x_1)^2}$$

Similarly,

$$\frac{\partial g_2}{\partial x_1} = \frac{x_2-d_2}{(d_2-x_2)^2+(d_1-x_1)^2} + \frac{c_2-x_2}{(c_2-x_2)^2+(x_1-c_1)^2}$$

$$\frac{\partial g_2}{\partial x_2} = \frac{x_1-d_1}{(d_2-x_2)^2+(d_1-x_1)^2} + \frac{x_1-c_1}{(c_2-x_2)^2+(c_1-x_1)^2}$$

(3) USE NEWTON'S METHOD

$$X^{k+1} = X^k + \left(\frac{\partial g}{\partial x}\right)^{-1}_{X^k}\left(Y - g(x^k, t)\right)$$

TAKE AN INITIAL GUESS FOR $X^k$ THEN UPDATE USING EQUATIONS.

(4) $Y = g(x,t) + v$ $\quad v = N(0, R_v)$

$$\epsilon_x = -\left(\frac{\partial g}{\partial x}\right)^{-1}_x v$$

$$R_{\epsilon x} = \left(\frac{\partial g}{\partial x}\right)^{-T}_x R_v \left(\frac{\partial g}{\partial x}\right)^{-1}_x$$

PROBLEM 4.4
Pg# 1

(1)

$$y_1 = r_1 - r_2$$

$$y_2 = r_3 - r_4$$

ASSUME POINTS ARE GIVEN AS IN 4.3. $(a_1, a_2)$ etc.

$$r_1 = \sqrt{(a_1 - x_1)^2 + (a_2 - x_2^2)} \qquad r_2 = \sqrt{(b_1 - x_1)^2 + (b_2 - x_2)^2}$$

$$r_3 = \sqrt{(c_1 - x_1)^2 + (c_2 - x_2)^2} \qquad r_4 = \sqrt{(d_1 - x_1)^2 + (d_2 - x_2)^2}$$

THUS,

$$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \sqrt{(a_1 - x_1)^2 + (a_2 - x_2)^2} - \sqrt{(b_1 - x_1)^2 + (b_2 - x_2)^2} \\ \sqrt{(c_1 - x_1)^2 + (c_2 - x_2)^2} - \sqrt{(d_1 - x_1)^2 + (d_2 - x_2)^2} \end{bmatrix}$$

(2)

$$\frac{\partial g_1}{\partial x_1} = \frac{1}{\cancel{2}}\left((a_1 - x_1)^2 + (a_2 - x_2)^2\right)^{-1/2} \cancel{2}(a_1 - x_1)(-1) + \frac{1}{\cancel{2}}\left((b_1 - x_1)^2 + (b_2 - x_2)^2\right)^{-1/2} \cancel{2}(b_1 - x_1)(+1)$$

$$\frac{\partial g_1}{\partial x_1} = \frac{x_1 - a_1}{\sqrt{(a_1 - x_1)^2 + (a_2 - x_2)^2}} + \frac{b_1 - x_1}{\sqrt{(b_1 - x_1)^2 + (b_2 - x_2)^2}}$$

$$\frac{\partial g_1}{\partial x_2} = \frac{x_2 - a_2}{\sqrt{(a_1 - x_1)^2 + (a_2 - x_2)^2}} + \frac{b_2 - x_2}{\sqrt{(b_1 - x_1)^2 + (b_2 - x_2)^2}}$$

$$\frac{\partial g_2}{\partial x_1} = \frac{x_1 - c_1}{\sqrt{(c_1 - x_1)^2 + (c_2 - x_2)^2}} + \frac{d_1 - x_1}{\sqrt{(d_1 - x_1)^2 + (d_2 - x_2)^2}}$$

PROB 4.4
pg #2

$$\frac{\partial g_2}{\partial x_2} = \frac{x_2 - c_1}{\sqrt{(c_2 - x_1)^2 + (c_2 - x_2)^2}} + \frac{d_2 - x_2}{\sqrt{(d_1 - x_1)^2 + (d_2 - x_2)^2}}$$

(3) USE NEWTONS METHOD

$$x^{k+1} = x_k + \left(\frac{\partial g}{\partial x}\right)^{-T}_{x^k} \left(y - g(x^k, t)\right)$$

GUESS FOR $x_k$ AND USE EQUATIONS

(4) $y = g(x,t) + v \qquad v = N(0, R_v)$

$$\epsilon_x = -\left(\frac{\partial g}{\partial x}\right)^{-T}_x v$$

$$R_{\epsilon x} = \left(\frac{\partial g}{\partial x}\right)^{-T}_x R_v \left(\frac{\partial g}{\partial x}\right)^{-1}_x$$

$\partial x_1 \qquad \sqrt{(d_1 - x_1)^2 + (d_2 - x_2)^2} \qquad \sqrt{(d_1 - x_1)^2 + (d_1 - x_2)^2}$

PROB 4.5

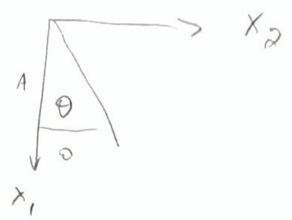
① $Y = \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix}$

FROM GEOMETRY & TRIG;

$$Y_1 = \sqrt{x_1^2 + x_2^2 + x_3^2}$$

$$Y_2 = \tan^{-1}\left(\frac{x_2}{x_1}\right)$$

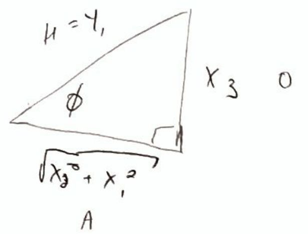
$$Y_3 = \sin^{-1}\left(\frac{x_3}{\sqrt{x_1^2 + x_2^2 + x_3^2}}\right)$$




$\tan\theta = \frac{x_2}{x_1}$

②

$$\frac{\partial g_1}{\partial x_1} = \frac{1}{\cancel{2}}\left(x_1^2 + x_2^2 + x_3^2\right)^{-1/2} \cancel{2}(x_1) = \frac{x_1}{\sqrt{x_1^2 + x_2^2 + x_3^2}}$$




$\sin\phi = \frac{x_3}{Y_1}$

$\phi = \sin^{-1}\left(\frac{x_3}{Y_1}\right)$

$$\frac{\partial g_1}{\partial x_2} = \frac{x_2}{\sqrt{x_1^2 + x_2^2 + x_3^2}}$$

$$\frac{\partial g_1}{\partial x_3} = \frac{x_3}{\sqrt{x_1^2 + x_2^2 + x_3^2}}$$

$$\frac{\partial g_2}{\partial x_1} = \frac{\cancel{x_1^2}}{x_1^2 + x_2^2} x_2 (-1)\cancel{(x_1^{-2})} = \frac{-x_2}{x_1^2 + x_2^2}$$

$$\frac{\partial g_2}{\partial x_2} = \frac{x_1^{\cancel{2}}}{x_1^2 + x_2^2} \frac{1}{\cancel{x_1}} = \frac{x_1}{x_1^2 + x_2^2}$$

$$\frac{\partial g_2}{\partial x_3} = 0$$

PROB #4.5
Pg #2

$$\frac{\partial g_3}{\partial x_1} = \frac{1}{\sqrt{1 - \frac{x_3^2}{x_1^2 + x_2^2 + x_3^2}}} \cdot (-\tfrac{1}{2}) X_3 (x_1^2 + x_2^2 + x_3^2)^{-3/2} (\cancel{2}) X_1$$

$$\frac{\partial g_3}{\partial x_1} = \frac{-X_3 X_1}{\sqrt{1 - \frac{x_3^2}{x_1^2 + x_2^2 + x_3^2}} \left(x_1^2 + x_2^2 + x_3^2\right)^{3/2}}$$

$$\frac{\partial g_3}{\partial x_2} = \frac{-X_3 X_2}{\sqrt{1 - \frac{x_3^2}{x_1^2 + x_2^2 + x_3^2}} \left(X_1^2 + X_2^2 + X_3^2\right)^{3/2}}$$

$$\frac{\partial g_3}{\partial x_3} = \frac{1}{\sqrt{1 - \frac{x_3^2}{x_1^2 + x_2^2 + x_3^2}}} \left[ \left( \frac{1}{\sqrt{X_1^2 + x_2^2 + x_3^2}} \right) + X_3 (-\tfrac{1}{\cancel{2}}) (x_1^2 + x_2^2 + x_3^2)^{-3/2} (\cancel{2}) X_3 \right]$$

$$\frac{\partial g_3}{\partial x_3} = \frac{1}{\sqrt{1 - \frac{x_3^2}{x_1^2 + x_2^2 + x_3^2}}} \left[ \frac{1}{\sqrt{x_1^2 + x_2^2 + x_3^2}} - \frac{x_3^2}{(X_1^2 + X_2^2 + X_3^2)^{3/2}} \right]$$

(3) USE NEWTONS METHOD

$$X_{K+1} = X_K + \left(\frac{\partial g}{\partial x}\right)_x^{-T} \left(Y - g(x^K, t)\right)$$

GUESS FOR $X_K$ AND USE EQUATIONS

(4) $Y = g(x, t), + V$ $\qquad V = N(0, R_V)$

$$E_x = -\left(\frac{\partial g}{\partial x}\right)_x^{-T} V \qquad \& \qquad R_{EX} = \left(\left(\frac{\partial g}{\partial x}\right)_x^{T} R_V \left(\frac{\partial g}{\partial x}\right)_x\right)^{-1}$$

PROBLEM 4.10

GIVEN

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u$$

$$Y = \begin{bmatrix} 0 & C_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

SHOW

if $(A_{22}, C_2)$ is OBSERVABLE & $A_{11}$ is ASY. STABLE, THEN AN ASY. OBSERVER CAN BE DESIGNED TO GET STATES EVEN THOUGH IT IS UNOBSERVABLE.

SOLUTION

$$\tilde{x}(t) = x(t) - \hat{x}(t)$$

$$\dot{\tilde{x}}(t) = A(t)\tilde{x}(t) + G(t)C(t)\tilde{x}(t)$$

$$\dot{\tilde{x}}(t) = [A(t) + G(t)C(t)]\tilde{x}(t)$$

$$= \left[\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} + \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} \begin{bmatrix} 0 & C_1 \end{bmatrix}\right] \tilde{x}(t)$$

$$= \begin{bmatrix} A_{11} & A_{12} + g_1 C_1 \\ 0 & A_{22} + g_2 C_1 \end{bmatrix} \tilde{x}(t)$$

# Problem 4.10, continued

Similar to the result that is seen in Problem 4.15, Part 3, even though the system is not observable, but it is detectable; meaning that even though the eigenvalues of the matrix $A+GC$ cannot all be placed using $G$, the ones that cannot be palaced are negative, so they the estimates will converge asymtotically to zero. For this case, because $A_1 1$ is asmptotically stable and the pair $(A_{22}, C_2)$ is completely observable the gain, $g_2$ can be designed so that $A_{22}+g_2C_2$ is stable and thus the states can be reconstructed.

# Problem 4.15, Part 1

Given the pulse input shown in the bottom of Fig. 0.1, the reponses for both the actual system and the symptotic observer can alse be seen in Fig. 0.1.

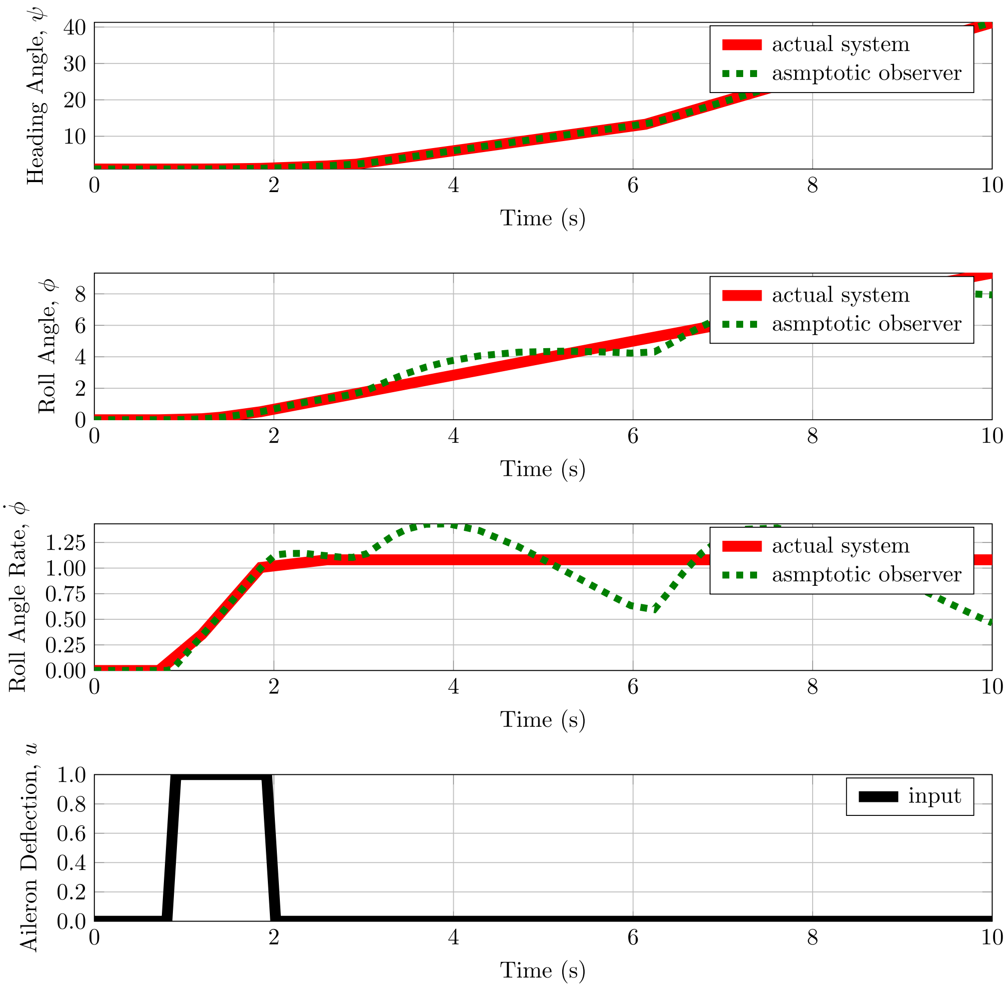



Figure 0.1: Actual system response compared to asmptotic observer shown for a pulse input. $g1 = -10$

It can be seen that while the heading angle is accurately reconstructed, both the roll angle and the roll angle rate are not. For the results in Fig. 0.2, $g1$ is decreased by an order of magnitude and it can be seen that all of the states are not accurately reconstructed.

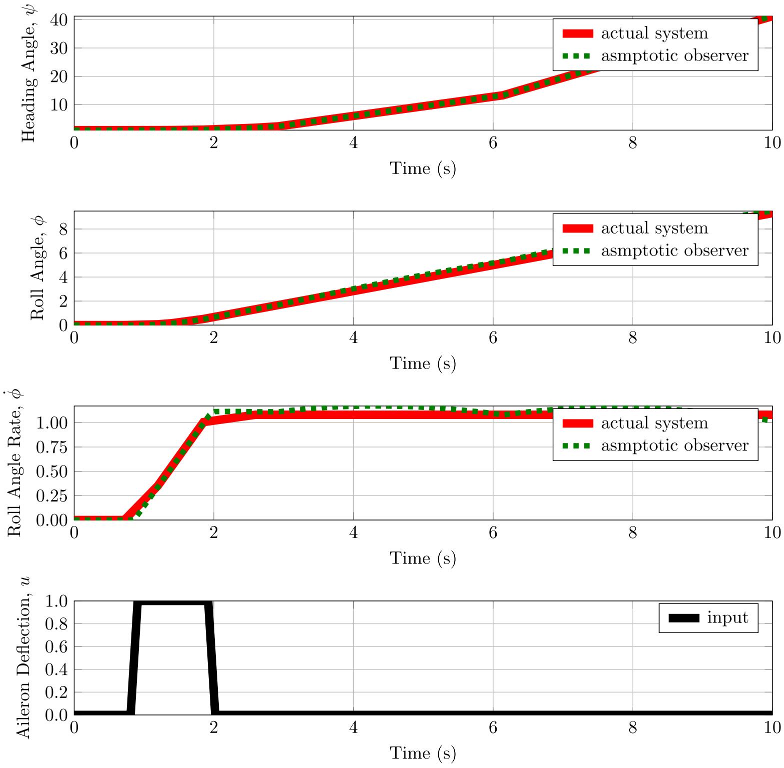



Figure 0.2: Actual system response compared to asmptotic observer shown for a pulse input, $g1 = -100$

## PROBLEM 4.15, PART 2

PROB 4.15, PART 2

$\dot{\tilde{x}} = [A + GC]\tilde{x}$ $\quad C = [1 \quad 1 \quad 0]$

$$A = \begin{bmatrix} 0 & K_1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad G = \begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix}$$

$$A + GC = A_c = \begin{bmatrix} 0 & K_1 + g_1 & 0 \\ 0 & g_2 & 1 \\ 0 & g_3 & 0 \end{bmatrix}$$

$$\det(\lambda I - A_c) = \begin{vmatrix} \lambda & k_1 + g_1 & 0 \\ 0 & \lambda - g_2 & 0 \\ 0 & 0 & \lambda \end{vmatrix}$$

$$= \lambda(\lambda - g_2)(\lambda) = \lambda^2(\lambda - g_2) \Rightarrow \lambda_{1,2} = 0 \ \& \ \lambda_3 = g_2$$

EXCLUDEDS ASYMTOTIC STABILITY OF ESTIMATION ERROR $\tilde{x}(t)$

$$\lim_{t \to \infty} \tilde{x}(t) \neq 0$$

$\hat{x}(t)$ DOES NOT CONVERGE TO $x(t)$!

UNABLE TO RECONSTRUCT SYSTEM STATE!

Next in Fig. 0.3 and Fig. 0.4 (longer simulation time), it can be seen that the roll angle is able to be reconstructed, but the observer does a poor job with the other two states.

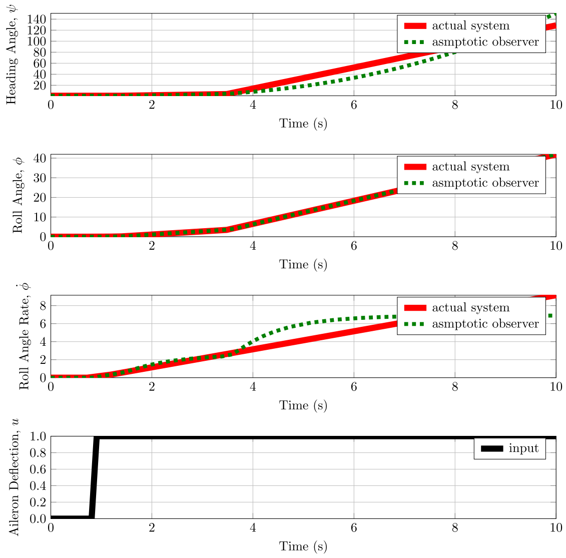



Figure 0.3: System response with asymtotic observer compared to actual system response; for a step input.

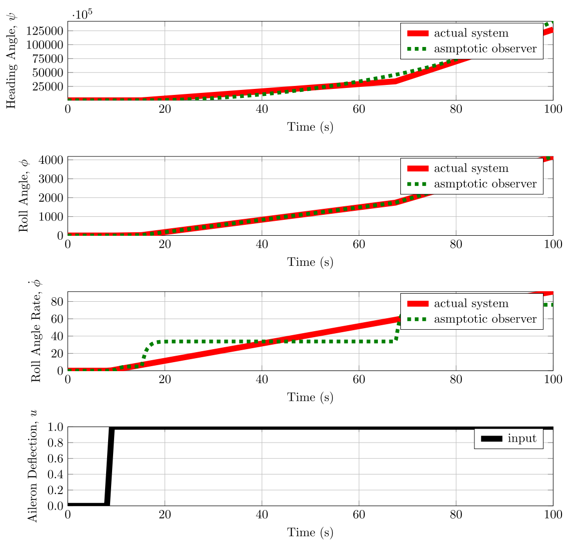



Figure 0.4: System response with asymtotic observer compared to actual system response; for a step input.

PROBLEM 4.15, PART 3

PROB 4.15, PART 3

pg #1

$$\dot{\psi} = -k\psi + k_1\phi \qquad k>0,\ k_1 \neq 0$$

$$\ddot{\phi} = k_2 u \qquad k_2 \neq 0$$

$$C = [\,0 \quad 1 \quad 0\,]$$

$$A = \begin{bmatrix} -k & k_1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\dot{\tilde{x}} = [A + GC]\tilde{x} \qquad A + GC = A_c = \begin{bmatrix} -k & k_1 + g_1 & 0 \\ 0 & g_2 & 1 \\ 0 & g_3 & 0 \end{bmatrix}$$

$$\det(\lambda I - A_c) = \begin{vmatrix} \lambda + k & k_1 + g_1 & 0 \\ 0 & \lambda - g_2 & 1 \\ 0 & g_3 & \lambda \end{vmatrix}$$

$$= (\lambda + k)\left[(\lambda - g_2)\lambda - g_3\right] = (\lambda + k)\left[\lambda^2 - g_2\lambda - g_3\right] = 0$$

$\lambda_1 = -k$ ✓

↳ NEGATIVE

$$\lambda_{2,3} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{g_2 \pm \sqrt{g_2^2 + 4g_3}}{2} \qquad +b = -g_2,\ c = -g_3$$

* CANNOT PLACE THIS E-VALUE, BUT IT IN THE LEFT HALF PLANE!

→ ERROR GOES TO ZERO ASMTOTICALLY ANYWAYS

PROB 4.15, PART #3
PG# 2

$$\lambda_2 = \frac{g_2 + \sqrt{g_2^2 + 4g_3}}{2} \qquad \lambda_3 = \frac{g_2 - \sqrt{g_2^2 + 4g_3}}{2}$$

→ CAN DESIGN $g$ SO $\lambda_2$ & $\lambda_3$ ARE NEGATIVE

// FOR OBSERVABILITY

$$\mathcal{O} = \begin{bmatrix} -C- \\ -CA- \\ -CA^2- \end{bmatrix} \qquad A^2 = \begin{bmatrix} -k & k_1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} -k & k_1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} k^2 & -kk_1 & k_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$CA^2 = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \qquad \mathcal{O} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

→ EVEN THOUGH THE $\text{Rank}(\mathcal{O}) = 2$

SYSTEM IS NOT OBSERVABLE IT IS DETECTABLE, SINCE WE CAN PLACE $\lambda_2$ & $\lambda_3$ & $\lambda_1$ IS ALWAYS NEGATIVE.

SO, THE SYSTEM CAN BE RECONSTRUCTED

TO ENSURE STABILITY ON ESTIMATION ERROR:

$$\lambda_2 < 0 \quad \& \quad \lambda_3 < 0$$

$$g_2 < -\sqrt{g_2^2 + 4g_3} \qquad g_2 < \sqrt{g_2^2 + 4g_3}$$

$$g_2^2 > g_2^2 + 4g_3$$

$$\boxed{0 > g_3}$$

ALSO $\boxed{g_2 < 0}$ & $\boxed{g_2^2 + 4g_3 > 0}$

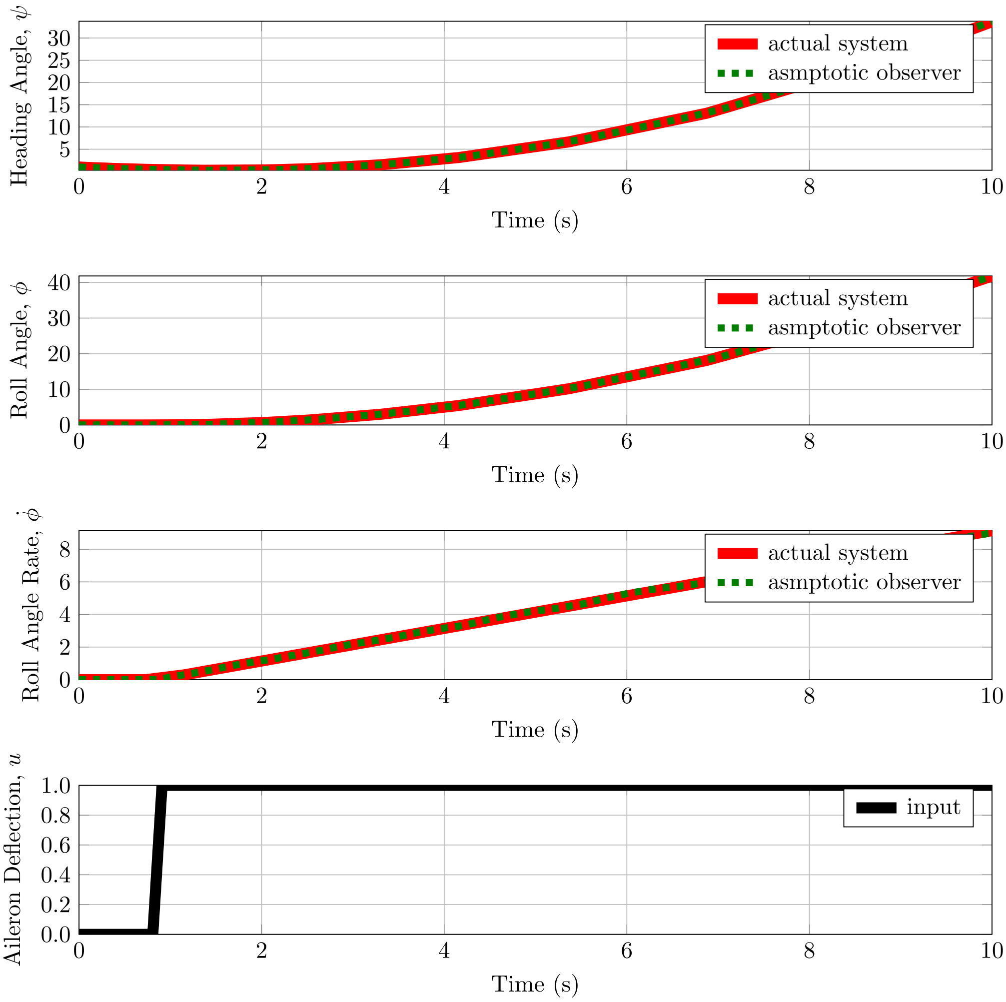



Figure 0.5: System response with asymtotic observer compared to actual system response; for a step input.

## Problem 4.16, Part 1

**Linear Gauss-Markov Model**

The standard model is:

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + w(t), t \geq t_0 \tag{0.1}$$
$$y(t) = C(t)x(t) + v(t) \tag{0.2}$$

For our time invariant case:

$$\dot{x}(t) = Ax(t) + Bu(t) + w(t), t \geq t_0 \tag{0.3}$$
$$y(t) = Cx(t) + v(t) \tag{0.4}$$

With,

$$A = \begin{bmatrix} 0 & -k_1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

$$w(t) = \begin{bmatrix} 0 \\ 0 \\ w_{\dot{\phi}}(t) \end{bmatrix}$$

$$v(t) = \begin{bmatrix} v_{\psi}(t) \\ 0 \\ 0 \end{bmatrix}$$

also,

$$v_{\psi}(t) = N(0, \sigma_v^2)$$

$$w_{\dot{\phi}}(t) = N(0, \sigma_w^2)$$

$$R_w = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sigma_w^2 \end{bmatrix}$$

$$R_v = \sigma_v^2$$

$$A^T = \begin{bmatrix} 0 & 0 & 0 \\ -k_1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$P = \begin{bmatrix} p_1 & p_2 & p_3 \\ p_2 & p_5 & p_6 \\ p_3 & p_6 & p_9 \end{bmatrix}$$

**Steady-State Covariance Matrix**

**Differential Riccati Equation** the optimal covariance matrix satisfies:

$$\begin{aligned}
&\dot{P}(t) = A(t)P(t) + P(t)A^T(t) - P(t)C(t)^T R_v^{-1}(t)C(t)P + R_w(t) \\
&P(t_0) = P_0
\end{aligned}$$

**Algebraic Riccati Equation** The steady-state covariance matrix, $P$ for our time-invariant problem is the solution to:

$$AP + PA^T - PC^T R_v^{-1} CP + R_w = 0$$

$$\begin{bmatrix} 0 & -k_1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_1 & p_2 & p_3 \\ p_2 & p_5 & p_6 \\ p_3 & p_6 & p_9 \end{bmatrix} + \begin{bmatrix} p_1 & p_2 & p_3 \\ p_2 & p_5 & p_6 \\ p_3 & p_6 & p_9 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ -k_1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \ldots$$

$$-\begin{bmatrix} p_1 & p_2 & p_3 \\ p_2 & p_5 & p_6 \\ p_3 & p_6 & p_9 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \frac{1}{\sigma_v^2} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_1 & p_2 & p_3 \\ p_2 & p_5 & p_6 \\ p_3 & p_6 & p_9 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sigma_w^2 \end{bmatrix} = 0$$

$$\begin{bmatrix} -k_1 p_2 & -k_1 p_5 & -k_1 p_6 \\ p_3 & p_6 & p_9 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} -p_2 k_1 & p_3 & 0 \\ -p_5 k_1 & p_6 & 0 \\ -k_1 p_6 & p_9 & \sigma_w^2 \end{bmatrix} - \frac{1}{\sigma_v^2} \begin{bmatrix} p_1^2 & p_1 p_2 & p_1 p_3 \\ p_1 p_2 & p_2^2 & p_2 p_3 \\ p_1 p_3 & p_2 p_3 & p_3^2 \end{bmatrix} = 0$$

$$\begin{bmatrix} -k_1 p_2 - p_2 k_1 - \frac{1}{\sigma_v^2} p_1^2 & -k_1 p_5 + p_3 - \frac{1}{\sigma_v^2} p_1 p_2 & -k_1 p_6 - \frac{1}{\sigma_v^2} p_1 p_3 \\ p_3 + -p_5 k_1 - \frac{1}{\sigma_v^2} p_1 p_2 & 2p_6 - \frac{1}{\sigma_v^2} p_2^2 & p_9 - \frac{1}{\sigma_v^2} p_2 p_3 \\ -k_1 p_6 - \frac{1}{\sigma_v^2} p_1 p_3 & p_9 - \frac{1}{\sigma_v^2} p_2 p_3 & \sigma_w^2 - \frac{1}{\sigma_v^2} p_3^2 \end{bmatrix} = 0$$

Then using julia,

```
using SymPy
@syms p1 p2 p3 p5 p6 p9 k1 sigma_w sigma_v
k1 = symbols("k1",nonzero=true, real=true)
p1,p2,p3,p4,p5,p6,p9,sigma_w,sigma_v = symbols("p1,p2,p3,p4,p5,p6,p9,sigma_w,sigma_v",real=true)
exs = [-k1*p2-p2*k1-1/sigma_v^2*p1^2, -k1*p5+p3-1/sigma_v^2*p1*p2,-k1*p6-1/sigma_v^2*p1*p3,p9-1/sigma_v^2*p2*p3,2*p6-1/sigma_v^2*p2^2,sigma_w^2-1/sigma_v^2*p3^2
d = solve(exs,[p1,p2,p3,p4,p5,p6,p9])
```

which gives,

```
Dict(p2=>0,p5=>sigma_v*sigma_w/k1,p9=>0,p6=>0,p3=>sigma_v*sigma_w,p1=>0)
```

So,

$$P = \begin{bmatrix} 0 & 0 & \sigma_w \sigma_v \\ 0 & \frac{\sigma_v \sigma_w}{k_1} & 0 \\ \sigma_w \sigma_v & 0 & 0 \end{bmatrix}$$

**Optimal Kalman Gain**

$$G = -PC^T R_v^{-1}$$

$$-\begin{bmatrix} 0 & 0 & \sigma_w \sigma_v \\ 0 & \frac{\sigma_v \sigma_w}{k_1} & 0 \\ \sigma_w \sigma_v & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \frac{1}{\sigma_v^2} = -\begin{bmatrix} 0 \\ 0 \\ \frac{\sigma_w}{\sigma_v} \end{bmatrix}$$

## Problem 4.16, Part 2

Yes. The stability analysis performed helps identify what the gains need to be for a stable system, this can help us to draw conculsions about what vaules the standard deviations should be for good performance.

## Problem 4.16, Part 3

In Fig. 0.6 the response can be seen for $\frac{\sigma_w}{\sigma_v} = .001 << 1$ and in Fig. 0.7 the response can be seen for $\frac{\sigma_w}{\sigma_v} = 100 >> 1$. Notice that in the first case, the response is stable and in the second case, the response is unstable.

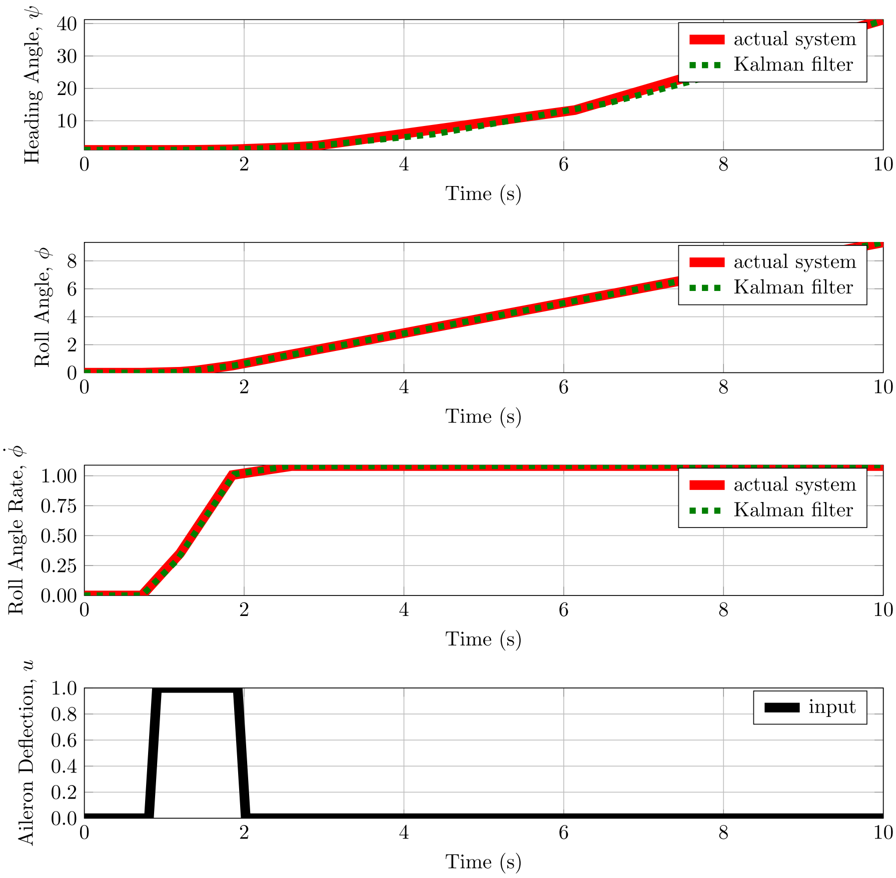



Figure 0.6: System response with Kalman-filter compared to actual system response; for a pulse input. $\frac{\sigma_w}{\sigma_v} = .001 << 1$

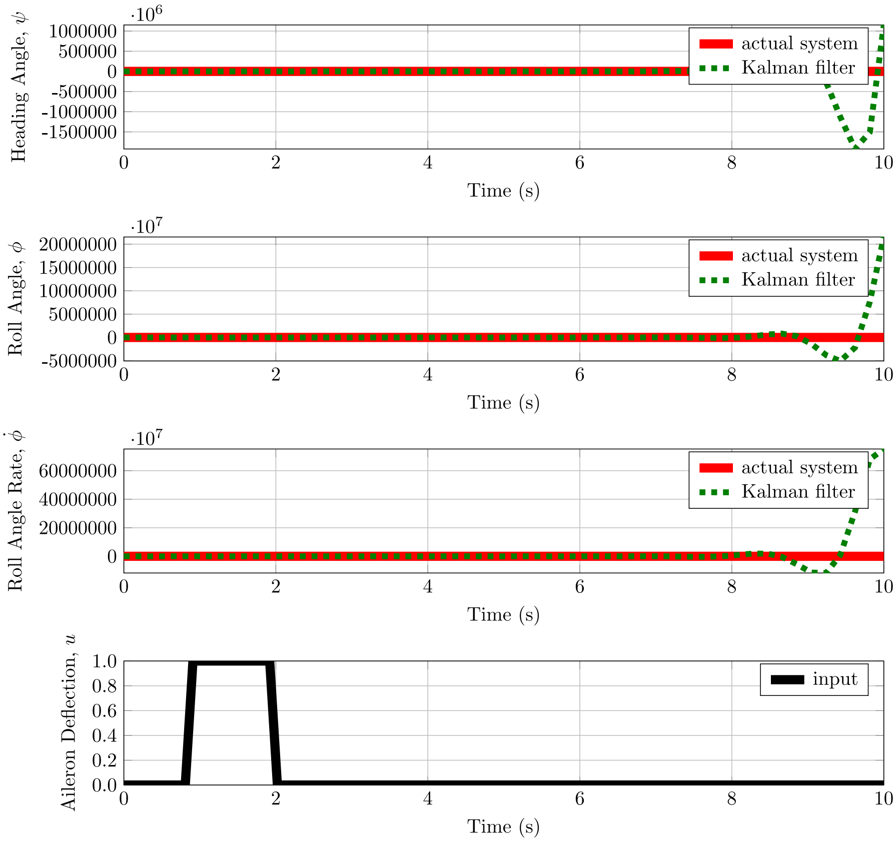



Figure 0.7: System response with Kalman-filter compared to actual system response; for a pulse input. $\frac{\sigma_w}{\sigma_v} = 100 >> 1$