①

$$\underline{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} r \\ \theta \end{bmatrix}$$

$\frac{x_1}{r} = \cos(\theta)$ $\quad \frac{x_2}{r} = \sin(\theta)$ $\quad \tan\theta = \frac{x_2}{x_1}$

$\theta = \tan^{-1}\left(\frac{x_2}{x_1}\right)$

Also,

$y_1^2 = x_1^2 + x_2^2 \Rightarrow y_1 = \overset{\text{DISTANCE}}{\pm}\sqrt{x_1^2 + x_2^2}$

So,

$$y = \begin{bmatrix} +\sqrt{x_1^2 + x_2^2} \\ \tan^{-1}\left(\frac{x_2}{x_1}\right) \end{bmatrix} = \begin{bmatrix} g_1(x) \\ g_2(x) \end{bmatrix}$$

$$\frac{\partial \underline{g}}{\partial \underline{x}} = \begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} \end{bmatrix}$$

$$\frac{\partial g_1}{\partial x_1} = \frac{1}{\cancel{2}}\left(x_1^2 + x_2^2\right)^{-1/2} \cancel{2} x_1 = \frac{x_1}{\sqrt{x_1^2 + x_2^2}}$$

$$\frac{\partial g_1}{\partial x_2} = \frac{x_2}{\sqrt{x_1^2 + x_2^2}}$$

PROB 4.2 PG#1

$$\frac{\partial g_2}{\partial x_1} = \left(\frac{1}{1+\left(\frac{x_2}{x_1}\right)^2}\right)\left(-x_1^{-2}x_2\right) = \frac{-x_2}{x_1^2+x_2^2}$$

$$\frac{\partial g_2}{\partial x_2} = \left(\frac{1}{1+\left(\frac{x_2}{x_1}\right)^2}\right)\left(\frac{1}{x_1}\right) = \frac{\cancel{x_1^2}}{\cancel{x_1^2}+x_2^2}\left(\frac{1}{\cancel{x_1}}\right)$$

THUS,

(2)
$$\frac{\partial g}{\partial \underline{x}} = \begin{bmatrix} \dfrac{x_1}{\sqrt{x_1^2+x_2^2}} & \dfrac{x_2}{\sqrt{x_1^2+x_2^2}} \\[2ex] \dfrac{-x_2}{x_1^2+x_2^2} & \dfrac{x_1}{x_1^2+x_2^2} \end{bmatrix}$$

(3) USE

NEWTON'S METHOD

$$x^{k+1} = x^k + \left(\frac{\partial g}{\partial x}\right)^{-1}_{x^k}\left(y - g(x^k, t)\right)$$

CHOOSE SOME INITIAL GUESS FOR x^k AND ITERATIVELY APPLY THE ABOVE EQUATION. SHOULD WORK AS LONG AS JACOBIAN IS NONSINGULAR AND THE GUESS IS WITHIN THE NEIGHBORHOOD OF x^*.

(4) $$y = g(x,t) + v \qquad v = N(0, R_v)$$

ASSUME ZERO MEAN

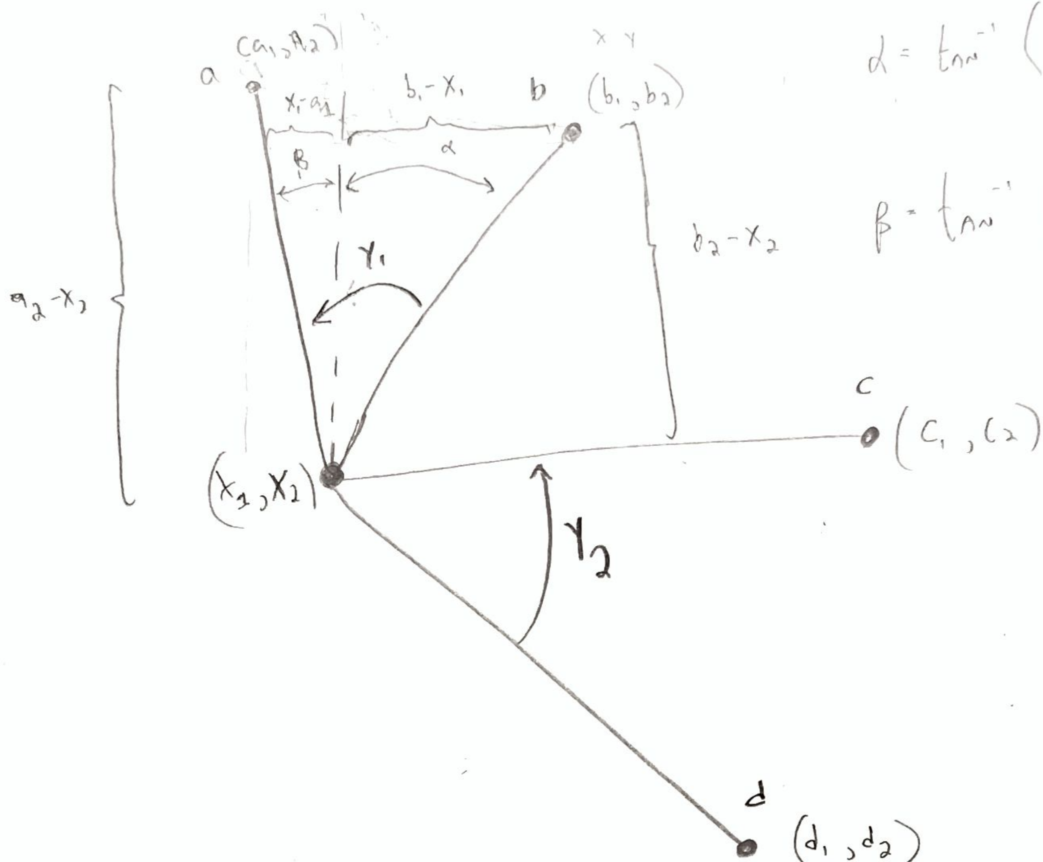
UNDER STANDARD ASSUMPTIONS (PAGE# 81)

$$\epsilon_x = \left(\frac{\partial g}{\partial x}\right)^{-T}_x v$$

LEADS TO,

$$R_{\epsilon x} = \left(\frac{\partial g}{\partial x}\right)^{-T}_x R_v \left(\frac{\partial g}{\partial x}\right)^{-1}_x$$

PROB 4.3 PS#1



$$\alpha = \tan^{-1}\left(\frac{b_1-x_1}{b_2-x_2}\right)$$

$$\beta = \tan^{-1}\left(\frac{x_1-a_1}{a_2-x_2}\right)$$

$$\gamma_1 = \alpha + \beta$$

① $$\gamma = \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix} = \begin{bmatrix} \tan^{-1}\left(\frac{b_1-x_1}{b_2-x_2}\right) + \tan^{-1}\left(\frac{x_1-a_1}{a_2-x_2}\right) \\ \tan^{-1}\left(\frac{d_1-x_1}{d_2-x_2}\right) + \tan^{-1}\left(\frac{x_1-c_1}{c_2-x_2}\right) \end{bmatrix} = \begin{bmatrix} g_1(x) \\ g_2(x) \end{bmatrix},$$

② $$\frac{(b_2-x_2)^{\cancel{2}}}{(b_2-x_2)^2 + (b_1-x_1)^2}\left(\frac{-1}{\cancel{b_2-x_2}}\right) = \frac{x_2-b_2}{(b_2-x_2)^2+(b_1-x_1)^2}$$

$$\frac{(a_2-x_2)^{\cancel{2}}}{(a_2-x_2)^2 + (x_1-a_1)^2}\,\frac{1}{\cancel{(a_2-x_2)}} = \frac{a_2-x_2}{(a_2-x_2)^2+(x_1-a_1)^2}$$

So, $$\frac{dg_1}{dx_1} = \frac{x_2-b_2}{(b_2-x_2)^2+(b_1-x_1)^2} + \frac{a_2-x_2}{(a_2-x_2)^2+(x_1-a_1)^2}$$

THEN,

$$\frac{(b_2-x_2)^2}{(b_2-x_2)^2+(b_1-x_1)^2}(b_1-x_1)(-1)(b_2-x_2)^{-2} = \frac{x_1-b_1}{(b_2-x_2)^2+(b_1-x_1)^2}$$

So,

$$\frac{\partial g_1}{\partial x_2} = \frac{x_1-b_1}{(b_2-x_2)^2+(b_1-x_1)^2} + \frac{x_1-a_1}{(a_2-x_2)^2+(a_1-x_1)^2}$$

SIMILARLY,

$$\frac{\partial g_2}{\partial x_1} = \frac{x_2-d_2}{(d_2-x_2)^2+(d_1-x_1)^2} + \frac{c_2-x_2}{(c_2-x_2)^2+(x_1-c_1)^2}$$

$$\frac{\partial g_2}{\partial x_2} = \frac{x_1-d_1}{(d_2-x_2)^2+(d_1-x_1)^2} + \frac{x_1-c_1}{(c_2-x_2)^2+(c_1-x_1)^2}$$

(3) USE NEWTON'S METHOD

$$x^{k+1} = x^k + \left(\frac{\partial g}{\partial x}\right)^{-1}_{x^k}\left(Y - g(x^k, t)\right)$$

TAKE AN INITIAL GUESS FOR x^k THEN UPDATE USING EQUATIONS.

(4) $Y = g(x,t) + v$ $\quad v = N(0, R_v)$

$$\epsilon_x = -\left(\frac{\partial g}{\partial x}\right)^{-1}_x v$$

$$R_{\epsilon x} = \left(\left(\frac{\partial g}{\partial x}\right)^{-T}_x R_v \left(\frac{\partial g}{\partial x}\right)_x\right)^{-1}$$

PROBLEM 4.4
Pg# 1

① $y_1 = r_1 - r_2$

$y_2 = r_3 - r_4$

ASSUME points ARE GIVEN AS IN 4.3. (a_1, a_2) etc.

$$r_1 = \sqrt{(a_1 - x_1)^2 + (a_2 - x_2)^2} \qquad r_2 = \sqrt{(b_1 - x_1)^2 + (b_2 - x_2)^2}$$

$$r_3 = \sqrt{(c_1 - x_1)^2 + (c_2 - x_2)^2} \qquad r_4 = \sqrt{(d_1 - x_1)^2 + (d_2 - x_2)^2}$$

THUS

$$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \sqrt{(a_1 - x_1)^2 + (a_2 - x_2)^2} - \sqrt{(b_1 - x_1)^2 + (b_2 - x_2)^2} \\ \sqrt{(c_1 - x_1)^2 + (c_2 - x_2)^2} - \sqrt{(d_1 - x_1)^2 + (d_2 - x_2)^2} \end{bmatrix}$$

②

$$\frac{\partial g_1}{\partial x_1} = \frac{1}{\cancel{2}}\left((a_1 - x_1)^2 + (a_2 - x_2)^2\right)^{-1/2} \cancel{2}(a_1 - x_1)(-1) + \frac{1}{2}\left((b_1 - x_1)^2 + (b_2 - x_2)^2\right)^{-1/2} \cancel{2}(b_1 - x_1)(+1)$$

$$\frac{\partial g_1}{\partial x_1} = \frac{x_1 - a_1}{\sqrt{(a_1 - x_1)^2 + (a_2 - x_2)^2}} + \frac{b_1 - x_1}{\sqrt{(b_1 - x_1)^2 + (b_2 - x_2)^2}}$$

$$\frac{\partial g_1}{\partial x_2} = \frac{x_2 - a_2}{\sqrt{(a_1 - x_1)^2 + (a_2 - x_2)^2}} + \frac{b_2 - x_2}{\sqrt{(b_1 - x_1)^2 + (b_2 - x_2)^2}}$$

$$\frac{\partial g_2}{\partial x_1} = \frac{x_1 - c_1}{\sqrt{(c_1 - x_1)^2 + (c_2 - x_2)^2}} + \frac{d_1 - x_1}{\sqrt{(d_1 - x_1)^2 + (d_2 - x_2)^2}}$$

PROB U.4
PG #2

$$\frac{\partial g_2}{\partial x_2} = \frac{x_2 - c_1}{\sqrt{(c_2 - x_1)^2 + (c_2 - x_2)^2}} + \frac{d_2 - x_2}{\sqrt{(d_1 - x_1)^2 + (d_2 - x_2)^2}}$$

③ USE NEWTONS METHOD

$$x^{k+1} = x_k + \left(\frac{\partial g}{\partial x}\right)^{-T}_{x^k} \left(y - g(x^k, t)\right)$$

GUESS FOR x_k AND USE EQUATIONS

④ $y = g(x,t) + v \qquad v = N(0, R_v)$

$$\epsilon_x = -\left(\frac{\partial g}{\partial x}\right)^{-T}_{x} v$$

$$R_{\epsilon x} = \left(\frac{\partial g}{\partial x}\right)^{-T}_{x} R_v \left(\frac{\partial g}{\partial x}\right)^{-1}_{x}$$

∂x_1 $\sqrt{(d_1 - x_1)^2 + (d_2 - x_2)^2}$ $\sqrt{(d_1 - x_1)^2 + (d_1 - x_2)^2}$

PROB 4.5

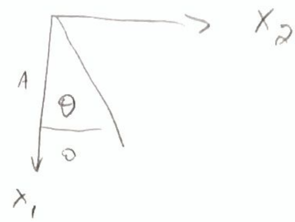
① $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

FROM GEOMETRY & TRIG ;

$$Y_1 = \sqrt{x_1^2 + x_2^2 + x_3^2}$$

$$Y_2 = \tan^{-1}\left(\frac{x_2}{x_1}\right)$$

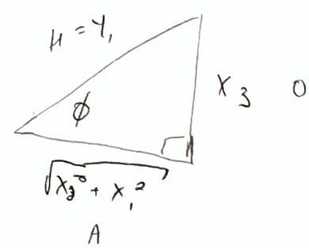
$$Y_3 = \sin^{-1}\left(\frac{x_3}{\sqrt{x_1^2 + x_2^2 + x_3^2}}\right)$$



$\tan\theta = \frac{x_2}{x_1}$

②

$$\frac{\partial g_1}{\partial x_1} = \frac{1}{\cancel{2}}\left(x_1^2 + x_2^2 + x_3^2\right)^{-1/2} \cancel{2}(x_1) = \frac{x_1}{\sqrt{x_1^2 + x_2^2 + x_3^2}}$$



$\sin\phi = \frac{x_3}{Y_1}$

$\phi = \sin^{-1}\left(\frac{x_3}{Y_1}\right)$

$$\frac{\partial g_1}{\partial x_2} = \frac{x_2}{\sqrt{x_1^2 + x_2^2 + x_3^2}}$$

$$\frac{\partial g_1}{\partial x_3} = \frac{x_3}{\sqrt{x_1^2 + x_2^2 + x_3^2}}$$

$$\frac{\partial g_2}{\partial x_1} = \frac{\cancel{x_1^2}}{x_1^2 + x_2^2} x_2 (-1)\cancel{\left(x_1^{-2}\right)} = \frac{-x_2}{x_1^2 + x_2^2}$$

$$\frac{\partial g_2}{\partial x_2} = \frac{x_1^{\cancel{2}}}{x_1^2 + x_2^2} \frac{1}{\cancel{x_1}} = \frac{x_1}{x_1^2 + x_2^2}$$

$$\frac{\partial g_2}{\partial x_3} = 0$$

PROB #4.5
Pg #2

$$\frac{\partial g_3}{\partial x_1} = \frac{1}{\sqrt{1 - \frac{x_3^2}{x_1^2+x_2^2+x_3^2}}}\,(-\tfrac{1}{2})X_3\left(x_1^2+x_2^2+x_3^2\right)^{-3/2}(\cancel{2})X_1$$

$$\frac{\partial g_3}{\partial x_1} = \frac{-X_3X_1}{\sqrt{1-\frac{X_3^2}{X_1^2+X_2^2+X_3^2}}\left(X_1^2+X_2^2+X_3^2\right)^{3/2}}$$

$$\frac{\partial g_3}{\partial x_2} = \frac{-X_3X_2}{\sqrt{1-\frac{X_3^2}{X_1^2+X_2^2+X_3^2}}\left(X_1^2+X_2^2+X_3^2\right)^{3/2}}$$

$$\frac{\partial g_3}{\partial x_3} = \frac{1}{\sqrt{1-\frac{X_3^2}{X_1^2+X_2^2+X_3^2}}}\left[\left(\frac{1}{\sqrt{X_1^2+X_2^2+X_3^2}}\right) + X_3\left(-\tfrac{1}{\cancel{2}}\right)\left(X_1^2+X_2^2+X_3^2\right)^{-3/2}(\cancel{2})X_3\right]$$

$$\frac{\partial g_3}{\partial x_3} = \frac{1}{\sqrt{1-\frac{X_3^2}{X_1^2+X_2^2+X_3^2}}}\left[\frac{1}{\sqrt{X_1^2+X_2^2+X_3^2}} - \frac{X_3^2}{\left(X_1^2+X_2^2+X_3^2\right)^{3/2}}\right]$$

③ USE NEWTONS METHOD

$$X_{K+1} = X_K + \left(\frac{\partial g}{\partial x}\right)_x^{-T}\left(Y - g(x^k,t)\right)$$

GUESS FOR X_K AND USE EQUATIONS

④ $Y = g(x,t)_j + V$ $\quad V = N(0,R_V)$

$$E_x = -\left(\frac{\partial g}{\partial x}\right)_x^{-T} V$$

$$\& \quad R_{EX} = \left(\frac{\partial g}{\partial x}\right)_x^{T} R_V \left(\frac{\partial g}{\partial x}\right)_x^{-1}$$

<u>GIVEN</u>

$$\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u$$

$$Y = \begin{bmatrix} 0 & C_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

<u>SHOW</u>

if (A_{22}, C_2) is OBSERVABLE & A_{11} is ASY. STABLE, THEN AN ASY. OBSERVER CAN BE DESIGNED TO GET STATES EVEN THOUGH IT IS UNOBSERVABLE.

<u>SOLUTION</u>

$$\tilde{X}(t) = X(t) - \hat{X}(t)$$

$$\dot{\tilde{X}}(t) = A(t)\tilde{X}(T) + G(t)C(t)\tilde{X}(t)$$

$$\dot{\tilde{X}}(t) = \left[A(t) + G(t)C(t)\right]\tilde{X}(t)$$

$$= \left[\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} + \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} \begin{bmatrix} 0 & C_1 \end{bmatrix}\right] \tilde{X}(t)$$

$$= \begin{bmatrix} A_{11} & A_{12} + g_1 C_1 \\ 0 & A_{22} + g_2 C_1 \end{bmatrix} \tilde{X}(t)$$