Solow Model - Assumptions, Results and Steady-State Dynamics

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Introduction

The Solow model (Solow 1956) is an extremely simple economic model, which explores the relationship between capital and labour in a market economy. In doing so it provides results relating to the expected growth rates of a nation, the capital share of labour, labour wage growth and capital growth.

This essay will build up the model step-by-step, exploring the assumptions implied by each element of the model, their implications and the limitations of these results.

The Model

The while Solow's model describes output (Y) as a function of capital (K) and labour (L), it has become common to extend this to effective labour, modelling labour as a combination of L and a technology factor (A).

$$Y = F(K, AL)$$

and we define capital growth as

$$\dot{K} = sY - \delta K$$

with the exogenous rates for savings (s) and depreciation (δ).

This will model a closed market economy, as it has no trade. Additionally the capital increase is proportional to the savings rate, because the savings and investment are assumed to be equal.

Analytical model

In order to make use of this model, it is necessary to give it mathematical form, which will further encapsulate the behaviour we want to represent in this model. In this case we are expecting diminishing, but positive returns from both capital and effective labour. An effective way of representing this is as a Cobb-Douglas function.

$$Y = K^{\alpha}(AL)^{\beta}$$

Given the additional assumption that in order to double the output you will need to double both the capital stock and effective labour, we can conclude that $\alpha + \beta = 1$ therefore $\beta = 1 - \alpha$. This assumption implies the economy is large enough to already be considered efficient such that additional competition, and that the growth factors for A and L are exogeneous, so neither technological growth nor immigration rates will be effected by output.

$$L(t) = L(0)e^{nt}$$
$$A(t) = A(0)E^{gt}$$

from which we show that the growth rates (n and g) are defined as the

$$
\begin{aligned}
\frac{\dot{L}}{L} = \frac{dL(t)/dt}{L(t)} = \frac{nL(0)e^{nt}}{L(0)e^{nt}} = n \\
\frac{\dot{A}}{A} = \frac{dA(t)/dt}{A(t)} = \frac{nA(0)e^{nt}}{A(0)e^{gt}} = g
\end{aligned}
$$

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Capital Growth

More interesting than looking at the capital output for the economy is looking at the output per unit of effective labour. This can be achieved just by dividing through by AL, which gives

$$
\begin{aligned}
\frac{Y}{AL} &= F(\frac{K}{AL}, 1) \\
&= K^{\alpha}(AL)^{1-\alpha}(AL)^{-1} \\
&= \frac{K^{\alpha}}{AL^{\alpha}} = \left(\frac{K}{AL}\right)^{\alpha} \\
\Rightarrow y &= k^{\alpha}
\end{aligned}
$$

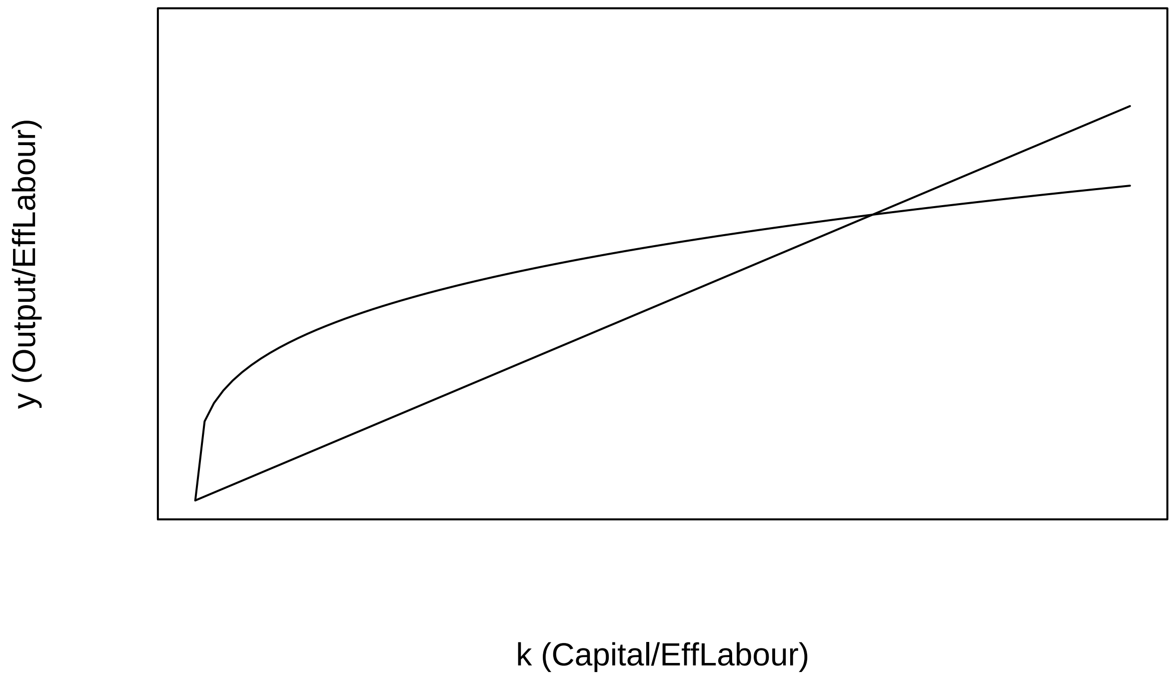
where $y \equiv \frac{Y}{AL}$, $k \equiv \frac{K}{AL}$. This then gives us capital growth of

$$\dot{k} = s \cdot f(k) - (n + g + \delta)k$$

If we then assume that capital per unit effective labour is constant (i.e. capital growth per worker is zero: $\dot{k} = 0$) we get

$$
\begin{aligned}
0 = s \cdot f(k) - (n + g + \delta)k \\
\Rightarrow s \cdot f(k) = (n + g + \delta)k
\end{aligned}
$$

Graphically this can be plotted as



which gives us an equilibrium point for labour share of output with respect to capital k^*.

We can also use this to see how the equilibrium would vary based upon differences in savings rates (k^* increases with savings rate), differences in labour or technology growth asset depreciation (k^* decreases with increased rates) between countries. This result also makes intuitive sense as putting more money into capital will cause there to be more capital. Similarly having faster increasing population or technology growth will reduce the reliance on capital.

Model Dynamics

From this equlibrium k^{**} we can look at the behaviour of the system when $k \neq k^*$. In the case that $k < k^*$ we find that $\dot{k}$ is then greater than 0, which means that over time $k \to k^*$. Similarly when $k > k^*$ the value of k will also decrease towards k^*.

The implication of the steady-state result is that given the other assumptions in the model, the assumption of $\dot{k} = 0$ is valid.

So given that the model converges, the question then becomes, at what rate does it converge. This is going to be defined by the growth rate of $frac\dot{k}k$, which we can estimate from $\dot{k} = s \cdot f(k) - (n + g + \delta)k$.

Log Linearization

Log linearization makes use of a Taylor approximation, to create a linear model near a known state to find the properties of the behaviour nearby. The first order Taylor approximation is defined as

$$f(x) \approx f(x_0) + f'(x) \mid_{x=x_0} (x - x_0)$$

which simply says, from a known state, if you move in the direction of the gradient of the curve then you are likely to be close to the curve (so long as you don't go to far).

While this is useful extremely close to the steady-state, for an exponential function a far closer approximation can be found by performing a log-linearization, differentiating with respect to log x. If we define $g(k) = \dot{k} =$

$s \cdot f(k) - (n+G+\delta)k$ then we can attempt to find $\frac{\delta g(k)}{\delta logk}$. Given $k^* = 0$ we find the log-linear approximation as

$$g(k) \approx \frac{\delta g(k)}{\delta logk}\bigg|_{k=k^*} (logk - logk^*)$$

In order to actually find this value we make use of the fact that $k = e^{logk}$, making the differential with respect to $logk$ feasible. Also given that $\frac{\delta e^{logx}}{\delta logx} = f'(x) \cdot x$ we get

$$g(k) \approx (sf'(k)k^* - (n+g+\delta)k^*) \cdot (logk - logk^*)$$

By substituting for s, we can find the capital growth as

$$\begin{aligned} g(k) &\approx \left(\frac{f'(k)k^*}{f(k^*)} - 1\right)(n+g+\delta)k^*(logk - logk^*) \\ g(k) &\approx (\alpha - 1)\,(n+g+\delta)k^*(logk - logk^*) \end{aligned}$$

As this is the capital share of income. This then allows us to identify the convergence rate (λ) of $logk$ approaching $logk^*$

$$\frac{\dot{k}}{k} \approx -(1-\alpha)(n+g+\delta)(logk - logk^*)$$

giving $\lambda = (1-\alpha)(n+g+\delta)$. As such we end up with convergence rates for k and y towards the steady-state of

$$\begin{aligned} \frac{\dot{k}}{k} &\approx -\lambda(logk - logk^*) \\ \frac{\dot{y}}{y} &\approx -\lambda(logy - logy^*) \end{aligned}$$

Per Capita GDP

Having this equlibrium allows us to also identify $k^* = \left(\frac{s}{n+g+\delta}\right)^{\frac{1}{1-\alpha}}$, $y^* = \left(\frac{s}{n+g+\delta}\right)^{\frac{\alpha}{1-\alpha}}$ and find that output per worker is

$$\frac{Y}{L} = A\left(\frac{s}{n+g+\delta}\right)^{\frac{\alpha}{1-\alpha}}$$

meaning that output per labourer will increase only with an increase in the technology factor (as all other variables are considered to be exogenous).

Given that per capita GDP is a common metric for comparing countries, this is an important result to see what makes the difference in countries outcomes. From this we can see that the only endogeneous variable within the model that will drive per capita growth is technology. This implies that, all else held equal, the outcome differentials between countries are driven primarily by technology.

Given the exponential growth of this variable, the model predicts that it will be the only significant differentiator in comparing per-capita outcomes for different nations.

Conclusion

Solow provides us with an extremely simple model that can be used to provide insight into the relationships between growth in labour, capital and development for an economy, however the many assumptions it makes limit its value in deep analysis. Many of the properties of the model, such as the consistency of labour/capital output ratio and steady growth rates are in keeping with trends.

It also provides helpful insight in comparing economies, and modeling their differences based on rates.

Problems can come from the assumption that capital can be measured independently of output. Given that capital units are not directly comparable, the inclination is to define their value in monetary terms. The paradox in this is that the market value of capital tends to be very closely related to its output capacity, meaning that comparisons of capital share of output are likely to express a cultural agreement on the valuation of the capital.

The exogeneity of savings, labour and technological growth rates rely on a stable society. In particular savings(/investment) rates being constant regardless of returns, but also constant technological growth regardless of how close an economy is to subsistance levels, and a single-type labour force.

However the results obtained by Mankiw et al (1992) show that the model does hold stand up to analysis given the addition of human capital to the model.

References

Mankiw, N. Gregory and Romer, David and Weil, David N. (1992) A Contribution to the Empirics of Economic Growth. The Quarterly Journal of Economics 107 (2): 407:437.

Robert M. Solow (1956) A Contribution to the Theory of Economic Growth. The Quarterly Journal of Economics 70 (1): 65:94.