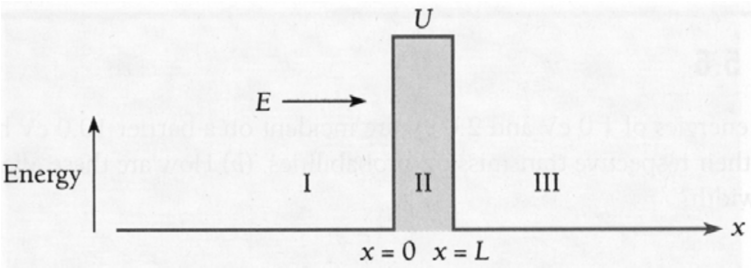
Tunnel Effect:

- particle with kinetic energy E strikes a barrier with height $U_0 > E$ and width L



- classically the particle cannot overcome the barrier

- quantum mechanically the particle can penetrated the barrier and appear on the other side

- then it is said to have **tunneled** through the barrier

examples:

- emission of alpha particles from radioactive nuclei by tunneling through the binding potential barrier
- tunneling of electrons from one metal to another through an oxide film
- tunneling in a more complex systems described by a generalized coordinate varying in some potential

approximate result:

- the transmission coefficient T is the probability of a particle incident from the left (region I) to be tunneling through the barrier (region II) and continue to travel to the right (region III)

$$T = e^{-2k_2 L} \quad \text{with} \quad k_2 = \frac{\sqrt{2m(U-E)}}{\hbar}$$

- depends exponentially on width of barrier L and the difference between the particle kinetic energy and the barrier height $(U_0-E)^{1/2}$ and mass of the particle $m^{1/2}$

example:

- An electron with kinetic energy $E = 1$ eV tunnels through a barrier with $U_0 = 10$ eV and width $L = 0.5$ nm. What is the transmission probability?

$$T = 1.1 \; 10^{-7}$$

- the probability is small, even for a light particle and a thin barrier
- but it can be experimentally observed and used in devices

sketch of calculation of tunnel rate:

- Schrödinger equation outside of barrier (regions I and III)

$$\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi_I + E\psi_I = 0$$

same for ψ_{III}

- has solutions

$$\psi_I = A e^{ik_1 x} + B e^{-ik_1 x}$$

$$\psi_{III} = F e^{ik_1 x} + G e^{-ik_1 x}$$

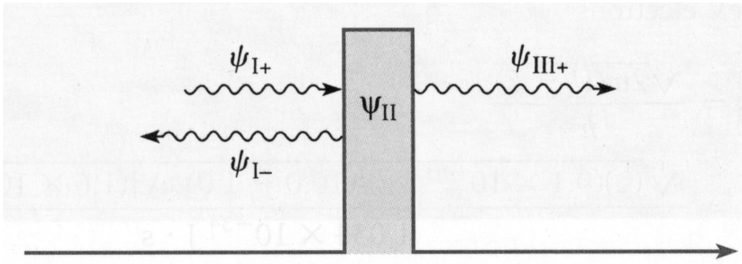
with $k_1 = \frac{\sqrt{2mE}}{\hbar} = \frac{p}{\hbar} = \frac{2\pi}{\lambda}$

- incoming wave

$$\psi_{I+} = A e^{ik_1 x}$$

reflected wave

$$\psi_{I-} = B e^{-ik_1 x}$$



- transmitted wave

$$\psi_{III+} = F e^{ik_1 x}$$

- incoming flux of particles with group velocity v_{I+}

$$S = |\psi_{I+}|^2 v_{I+}$$

transmission:

- probability T

$$T = \frac{|\psi_{III+}|^2 v_{III+}}{|\psi_{I+}|^2 v_{I+}} = \frac{FF^*}{AA^*}\frac{v_{III+}}{v_{I+}}$$

- ratio of flux of transmitted particles to incident particles

barrier region:

- Schrödinger equation

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi_{II} + (U-E)\psi_{II} = 0$$

- solution for $U > E$

$$\psi_{II} = C e^{-k_2 x} + D e^{k_2 x} \quad \text{with } k_2 = \frac{\sqrt{2m(U-E)}}{\hbar}$$

- exponentially decaying or increasing wave (no oscillations)

- does not describe a moving particle

- but probability in barrier region is non-zero boundary conditions: - at left edge of well ($x = 0$)

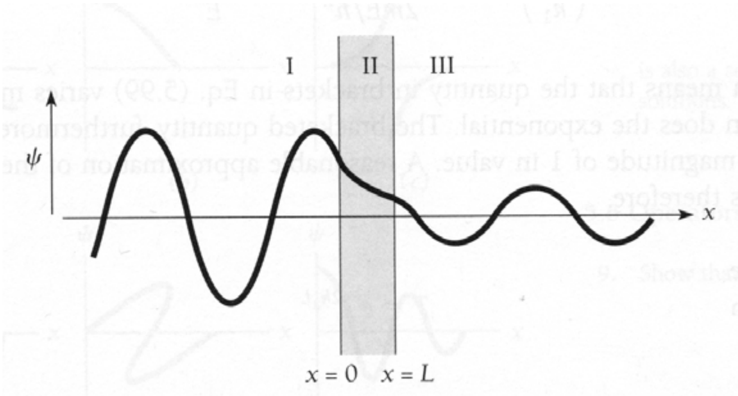
$$\psi_{I} = \psi_{II} \quad ; \quad \frac{\partial \psi_{I}}{\partial x} = \frac{\partial \psi_{II}}{\partial x}$$

- at right edge of well ($x = L$)

$$\psi_{II} = \psi_{III} \quad ; \quad \frac{\partial \psi_{II}}{\partial x} = \frac{\partial \psi_{III}}{\partial x}$$

- solve the four equations for the four coefficients and express them relative to A ($|A|^2$ is proportional to incoming flux)

- solution



transmission coefficient: - find A/F from set of boundary condition equations

$$\frac{A}{F} = \left[\frac{1}{2} + \frac{i}{4}\left(\frac{k_2}{k_1} - \frac{k_1}{k_2}\right)\right] e^{(ik_1+k_2)L} + \left[\frac{1}{2} - \frac{i}{4}\left(\frac{k_2}{k_1} - \frac{k_1}{k_2}\right)\right] e^{(ik_1-k_2)L}$$

simplify: - assume barrier U to be high relative to particle energy E

$$\frac{k_2}{k_1} > \frac{k_1}{k_2} \Rightarrow \frac{k_2}{k_1} - \frac{k_1}{k_2} \approx \frac{k_2}{k_1}$$

simplify: - assume barrier to be wide ($k_2 L > 1$)

$$e^{k_2 L} \gg e^{-k_2 L}$$

- therefore

$$\frac{A}{F} = \left(\frac{1}{2} + \frac{ik_2}{4k_1}\right) e^{(ik_1+k_2)L}$$

transmission coefficient:

$$T = \frac{AA^*}{FF^*}\,\frac{V_{III+}}{V_{I+}}$$

$$= \frac{16}{4+\left(\frac{k_2}{k_1}\right)^2}\, e^{-2k_2 L} \qquad \text{with } \left(\frac{k_2}{k_1}\right)^2 = \frac{U-E}{E}$$

$$k_2 = \frac{\sqrt{2m(U-E)}}{\hbar}$$

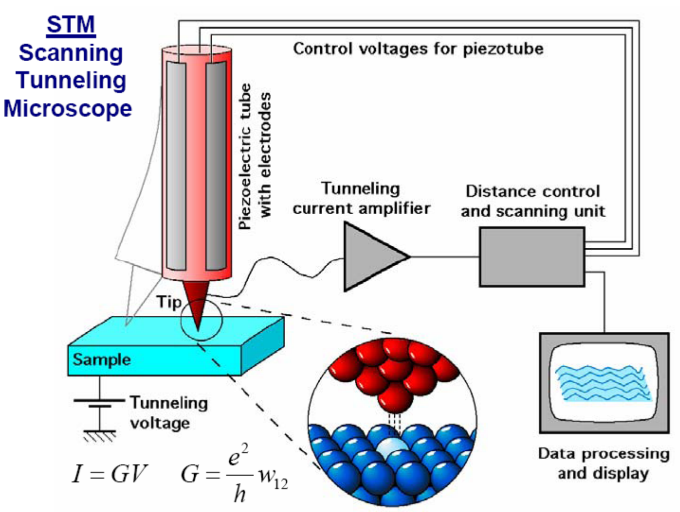
$$\approx e^{-2k_2 L} \qquad k_1 = \frac{\sqrt{2mE}}{\hbar}$$

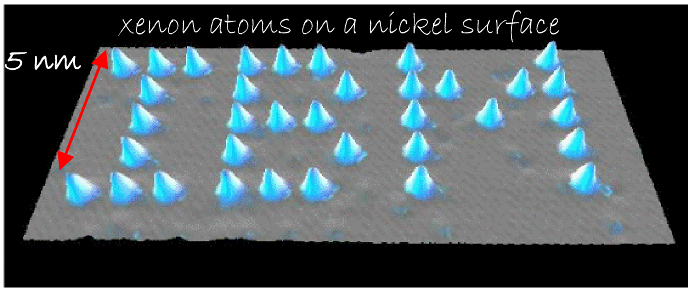
- T is exponentially sensitive to width of barrier

- T can be measured in terms of a particle flow (e.g. an electrical current) through a tunnel barrier

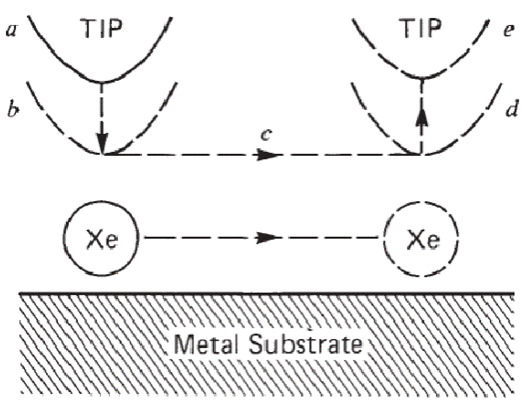
- makes this effect a great tool for measuring barrier thicknesses or distances for example in microscopy applications

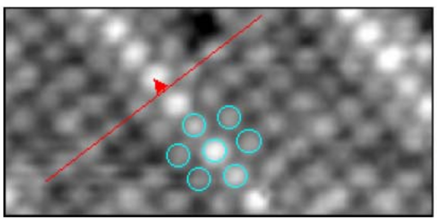
Scanning Tunneling Microscope (STM)

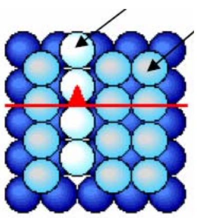




moving individual atoms around one by one





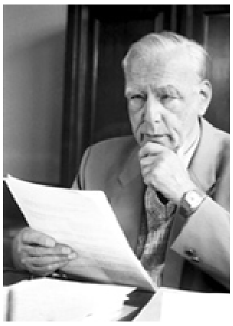


D.M. Eigler, E.K. Schweizer. **Positioning single atoms with a STM.** *Nature 344, 524-526 (1990)*

Nobel Prize in Physics (1986)

"for his fundamental work in electron optics, and for the design of the first electron microscope"

"for their design of the scanning tunneling microscope"



Ernst Ruska

1/2 of the prize

Federal Republic of Germany

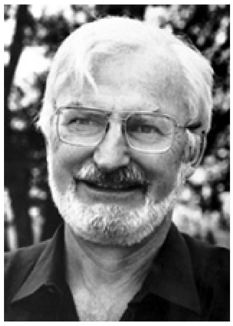
Fritz-Haber-Institut der Max-Planck-Gesellschaft
Berlin, Federal Republic of Germany

Gerd Binnig

1/4 of the prize

Federal Republic of Germany

IBM Zurich Research Laboratory
Rüschlikon, Switzerland



Heinrich Rohrer

1/4 of the prize

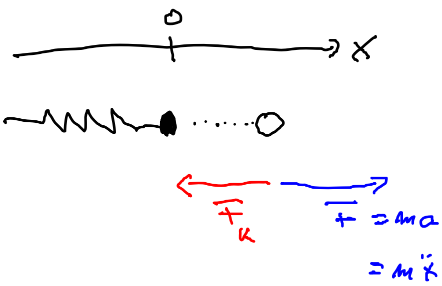
Switzerland

IBM Zurich Research Laboratory
Rüschlikon, Switzerland

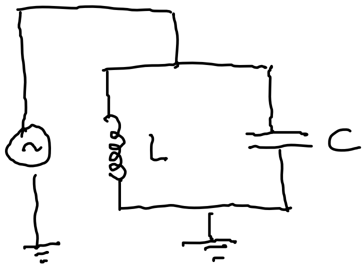
Quantum Harmonic Oscillator

general properties:
- oscillation around an equilibrium position
- at a single frequency
- linear restoring force

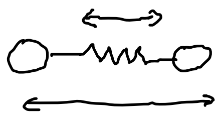
examples:
- mechanical oscillator, e.g. mass on a spring
- electrical oscillator, e.g. LC-circuit
- diatomic molecules
- lattice vibrations of a crystal



mass on a spring

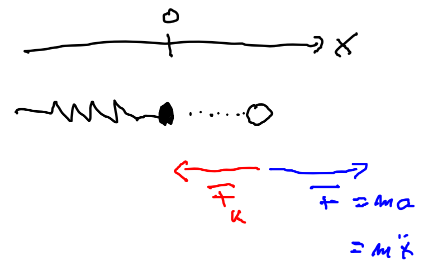


electrical oscillator



diatomic molecule

equation of motion:



- **linear restoring force** is a prerequisite for harmonic motion
- Hooke's law

$$F_k = -kx$$

- equation of motion for harmonic oscillator

$$m\frac{\partial^2 x}{\partial t^2} + kx = 0 \quad \text{for } x(t)$$

- a general solution

$$x(t) = A\cos(2\pi\nu t + \phi)$$

- **oscillator frequency** $\nu = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$

note:

- in many physical systems the restoring force is not strictly linear in the oscillation coordinate for large amplitude oscillations
- for small oscillation amplitudes however, the harmonic oscillator is usually a good approximation
- Taylor expansion of any force about the equilibrium position

$$F(x) = F_{x=0} + \frac{\partial F}{\partial x}\Big|_{x=0} x + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}\Big|_{x=0} x^2 + \frac{1}{6}\frac{\partial^3 F}{\partial x^3}\Big|_{x=0} x^3 \ldots$$

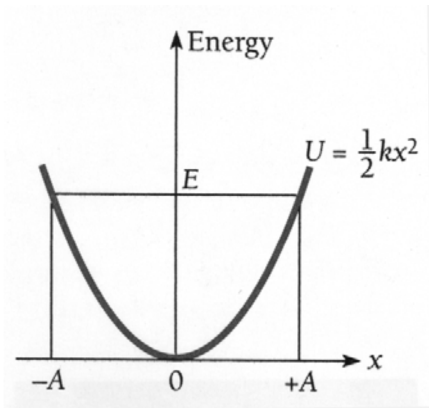
$$= \sum_i \frac{1}{i!}\frac{\partial^i}{\partial x^i}F(x)\Big|_{x=x_0}(x-x_0)^i$$

potential:

- potential associated with Hooke's law

$$U = \int_0^x -F(x)\,dx = \int_0^x kx\,dx = \frac{1}{2}kx^2$$

- U is used when solving the Schrödinger equation for a harmonic oscillator



expectations:

- only a discrete set of energies will be allowed for the oscillator
- the lowest allowed energy will not be $E = 0$ but will have some finite value $E = E_0$
- there will be a finite probability for the particle to penetrate into the walls of the potential well

Schrödinger equation for the harmonic oscillator:

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi + \frac{1}{2}kx^2\psi = E\psi$$

Solving the harmonic oscillator Schrödinger equation:

rewrite:

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{2m}{\hbar^2}\left(E - \frac{1}{2}kx^2\right)\psi = 0$$

normalize:

$$y = \left(\frac{1}{\hbar}\sqrt{km}\right)^{1/2} x = \sqrt{\frac{2\pi m \nu}{\hbar}}\, x$$

$$\alpha = \frac{2E}{\hbar}\sqrt{\frac{m}{k}} = \frac{2E}{h\nu}$$

with $\nu = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$

- these are dimensionless units for the coordinate y and the energy α

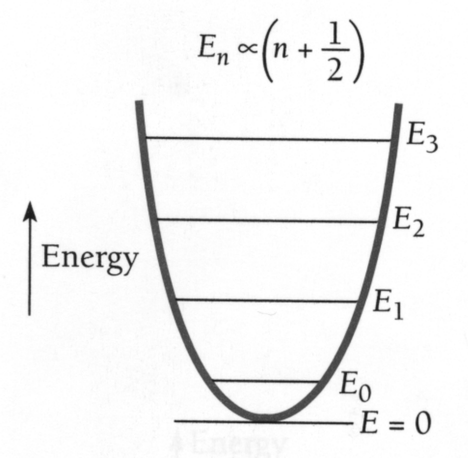
- the Schrödinger equation thus is given by

$$\frac{\partial^2 \psi}{\partial y^2} + (\alpha - y^2)\psi = 0$$

normalization condition for the solution wave functions ψ:

$$\int_{-\infty}^{\infty} |\psi|^2 \, dx = 1$$

energy quantization:



- condition on α for normalization

$$\alpha = 2n + 1 = \frac{2E}{h\nu} \qquad \text{for } n = 0, 1, 2, 3, \ldots$$

- **energy levels of the harmonic oscillator**

$$E_n = h\nu\left(n + \frac{1}{2}\right) \qquad n = 0, 1, 2, 3 \ldots$$

- equidistant energy levels

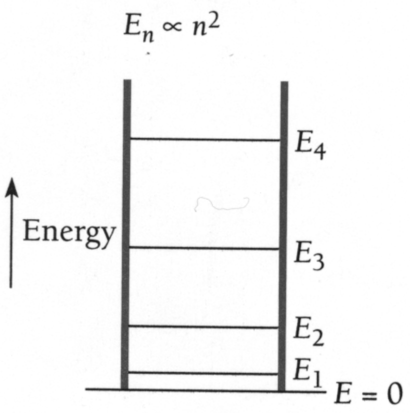
$$\Delta E = E_{n+1} - E_n = h\nu$$

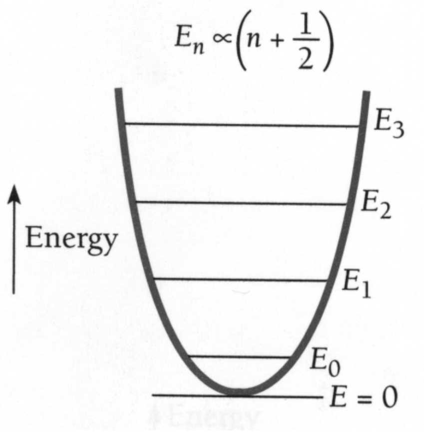
- this is a distinct feature of the harmonic oscillator

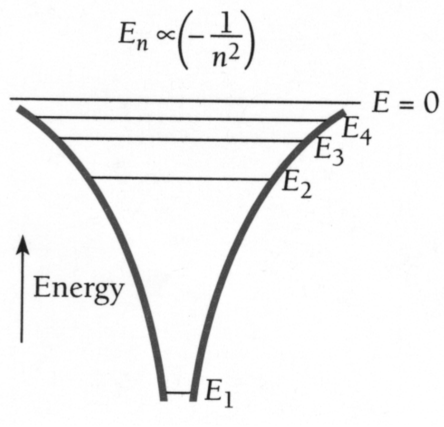
- **zero point energy** ($n = 0$, lowest possible energy of the harmonic oscillator)

$$E_0 = \frac{1}{2}h\nu$$

energy levels in different systems:







constant potential
particle in a box

x^2 potential
harmonic oscillator

$1/r$ - potential
Hydrogen atom

harmonic oscillator wave functions:

$$\psi_n = \left(\frac{2m\nu}{\hbar}\right)^{1/4} \left(2^n n!\right)^{-1/2} H_n(y)\, e^{-y^2/2}$$

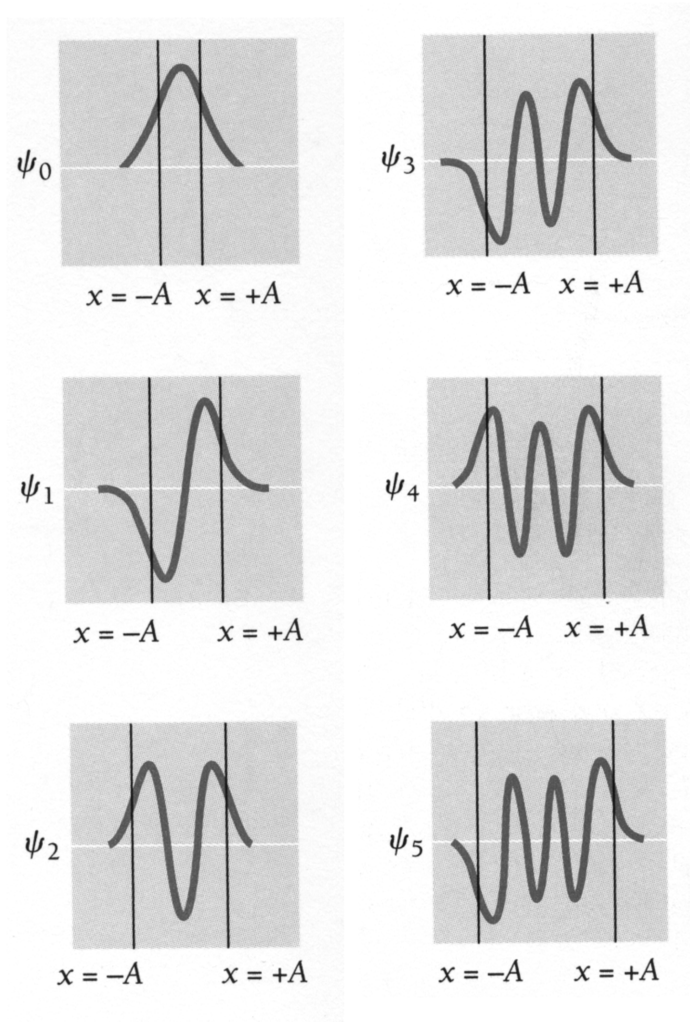
for $n = 0, 1, 2, 3, \ldots$

- with Hermite polynomials H_n

n	$H_n(y)$	α_n	E_n
0	1	1	$\frac{1}{2} h\nu$
1	$2y$	3	$\frac{3}{2} h\nu$
2	$4y^2 - 2$	5	$\frac{5}{2} h\nu$
3	$8y^3 - 12y$	7	$\frac{7}{2} h\nu$
4	$16y^4 - 48y^2 + 12$	9	$\frac{9}{2} h\nu$
5	$32y^5 - 160y^3 + 120y$	11	$\frac{11}{2} h\nu$

- the classical maximum oscillation amplitude is indicated in the plot by vertical black lines

- the particle enters into the classically forbidden regions of amplitudes

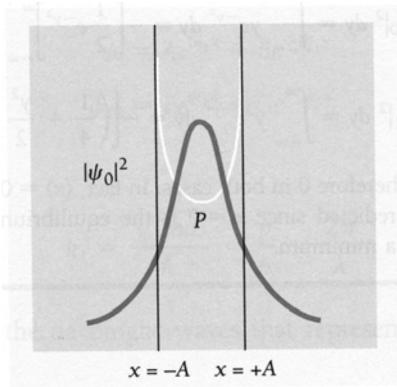


comparison of classical to quantum probability densities of position

classical: - largest probability density at the turning points ($x = \pm a$) of the oscillation

quantum: - in the ground state ($n = 0$) $|\psi|^2$ is largest at theequilibrium position ($x = 0$)

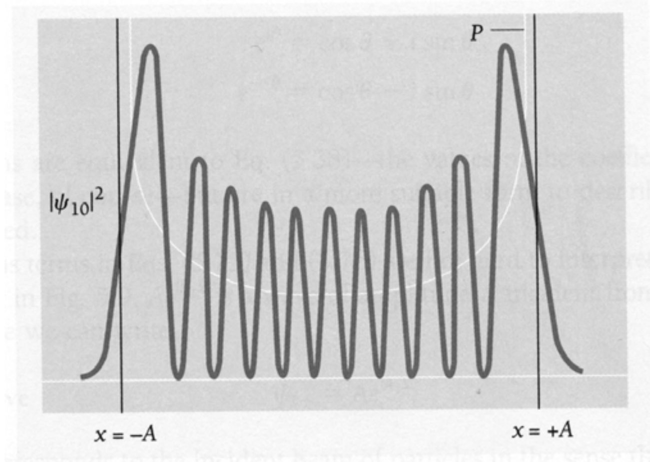
- for increasing n the quantum probability density approaches the classical one



- $n = 10$

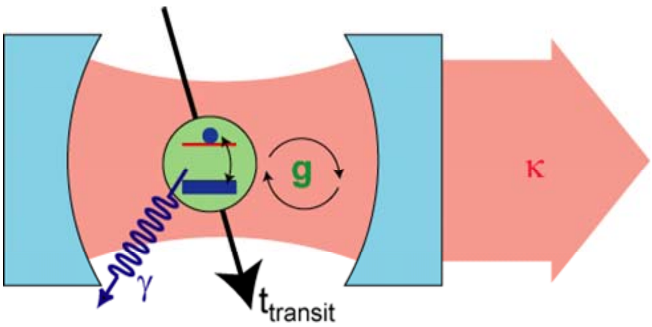
- the probability for the quantum oscillator to be at amplitudes larger then $\pm a$ decreases for increasing n

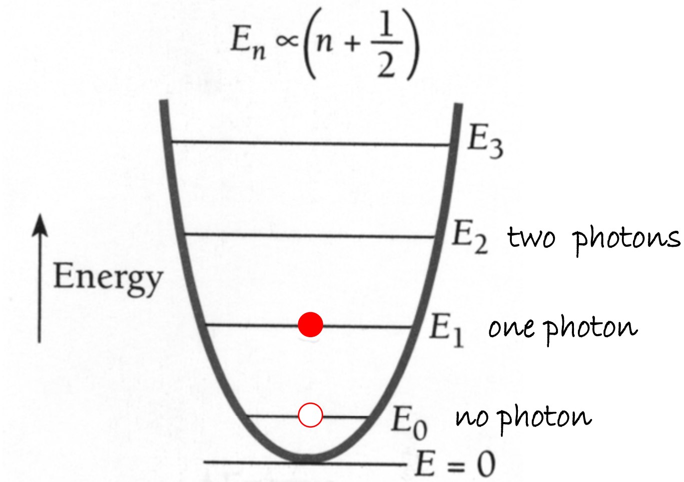
- this is an example of the correspondence principle for large n



Quantum Harmonic Oscillators

Cavity Quantum Electrodynamics (Cavity QED)



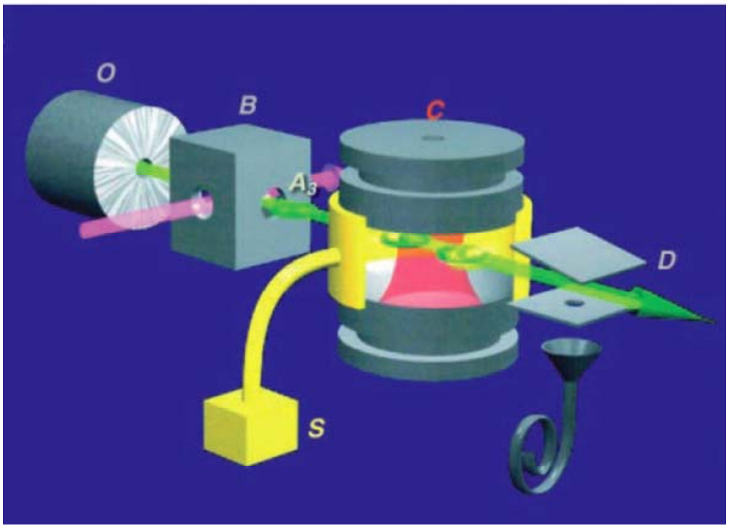


- atom (green) as a source and probe for single photons
- mirrors (blue) to contain photon in a cavity (a photon box)
- standing electromagnetic wave with a single photon

Review: J. M. Raimond, M. Brune, and S. Haroche *Rev. Mod. Phys.* **73**, 565 (2001)

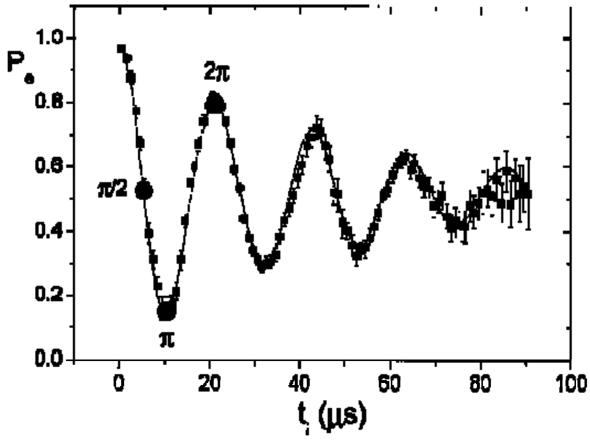
Cavity QED

experimental setup:



- O: oven as a source of atoms
- B: LASER preparation stage for atoms
- C: cavity (photon box)
- D: atom detector

one result:

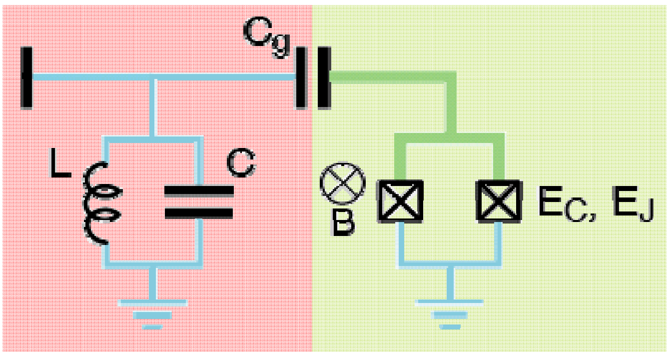


- measurement of probability for atom to be in excited state P_e versus the time t_i spend in cavity
- atom probes quantum state (number of photons) in the cavity

Review: J. M. Raimond, M. Brune, and S. Haroche *Rev. Mod. Phys.* **73**, 565 (2001)

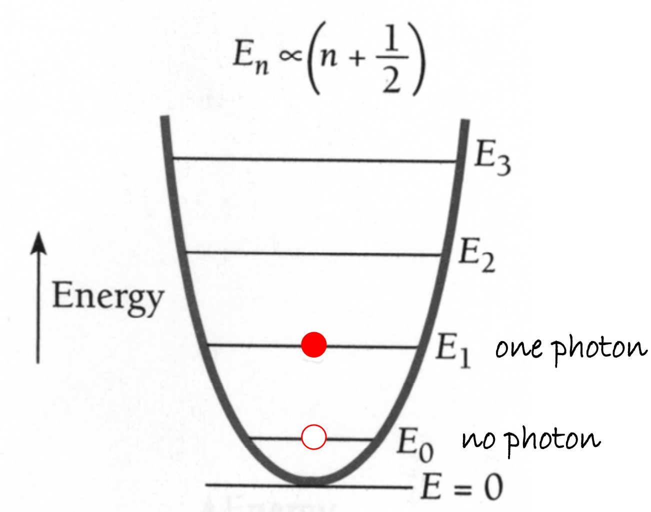
Quantum HO in Electrical Circuits

sketch of electrical circuit:



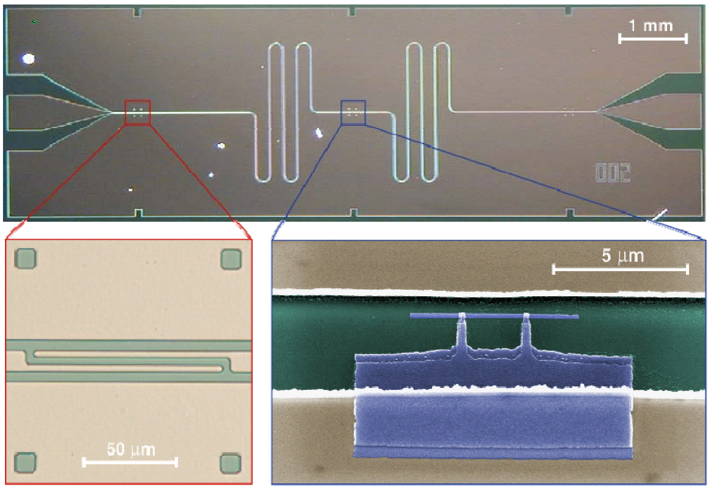
- electrical harmonic LC-oscillator
- inductor L
- capacitor C

- electrical artificial atom
- many non-equidistantly spaced energy levels



Experiment: Quantum HO in a Circuit

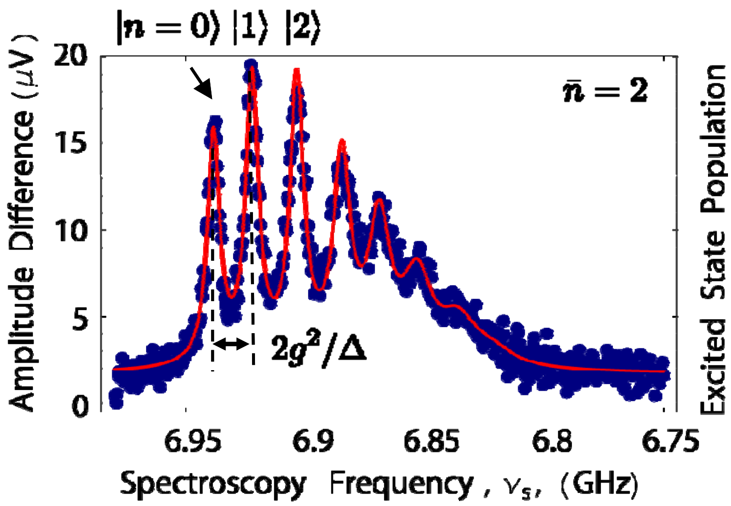
the LC-oscillator integrated circuit



- artificial atom (blue)
- LC oscillator (grey)

A. Wallraff, D. Schuster, ..., S. Girvin, and R. J. Schoelkopf, *Nature (London)* **431**, 162 (2004)

one result: seeing individual photons:



- spectrum of artificial atom
- one line each for 1, 2, 3, ... photons
- intensity of lines proportional to photon probability

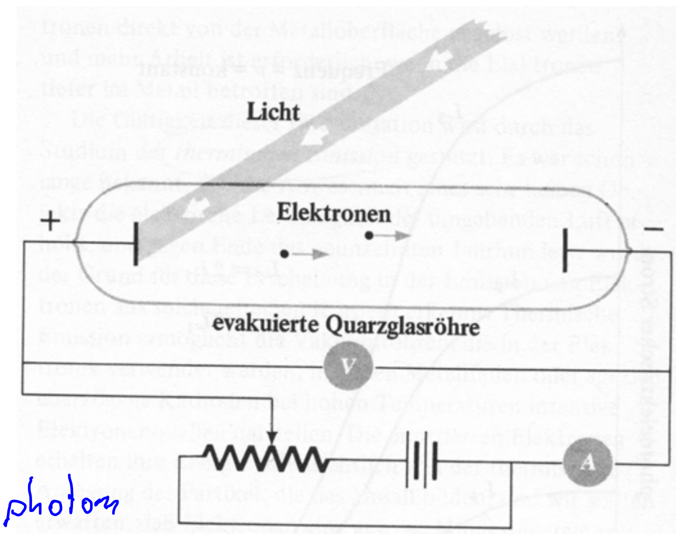
Photoelectric effect:

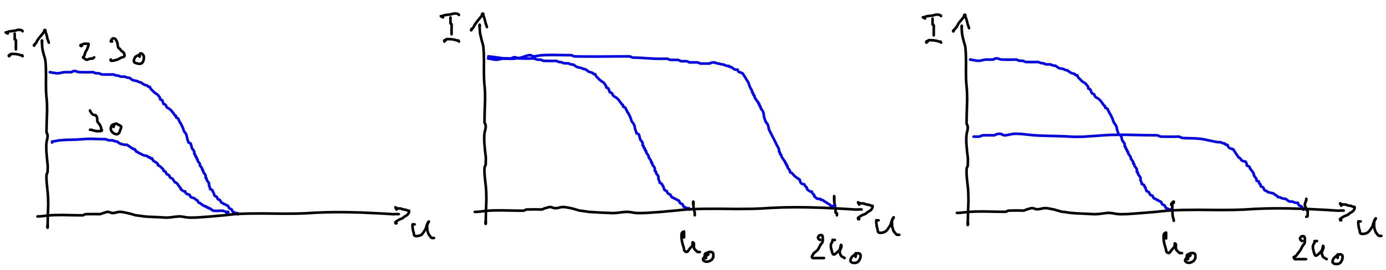
$$h\nu_0 = \phi + E_{kin} = \phi + eU_0$$

Intensity $$J = \frac{E}{\Delta t\, \Delta A} = \frac{n h\nu}{\Delta t\, \Delta A}$$

two limits: A) $h\nu \sim \phi \Rightarrow 1\, e^-$ per photon

B) $h\nu \gg \phi \Rightarrow \frac{h\nu}{\phi}\, e^-$ per photon





constant frequency different intensities

?

constant intensity different frequencies

?