Radiation by Point Sources

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Maxwell's equations

$$\nabla \cdot \mathbf{D} = \rho \quad (1)$$
$$\nabla \cdot \mathbf{B} = 0 \quad (2)$$
$$\nabla \times \mathbf{E} = -j\omega \mathbf{B} \quad (3)$$
$$\nabla \times \mathbf{H} = j\omega \mathbf{D} + \mathbf{J} \quad (4)$$

$$\mathbf{D} = \epsilon \mathbf{E} \quad (5)$$
$$\mathbf{B} = \mu \mathbf{H} \quad (6)$$
$$\nabla \cdot \mathbf{J} = -j\omega \rho \quad (7)$$

- Linear → one can make use of the theory of linear systems: LS's are fully characterized from the “impulse response”.
- Charge and current linked through continuity relation, a single source is required.
- Computation of fields due to an impulsive source tell the whole story.

Magnetic and Electric potentials

From (2) we can express the divergence-less $\mathbf{B}$ as the curl of an auxiliary potential $\mathbf{A}$ called magnetic vector potential:

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{8}$$

substituting this in Faraday law (3):

$$\nabla \times (\mathbf{E} + j\omega\mathbf{A}) = 0 \tag{9}$$

This curl-free vector can be expressed as the gradient of a scalar function Φ called electric potential:

$$\mathbf{E} + j\omega\mathbf{A} = -\nabla\Phi \tag{10}$$

Now substitute the last two equations into Ampere-Maxwell law (4):

$$\nabla \times \nabla \times \mathbf{A} - k^2\mathbf{A} = \mathbf{J} - j\omega\epsilon\nabla\Phi \tag{11}$$

$$\nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A} - k^2\mathbf{A} = \mathbf{J} - j\omega\epsilon\nabla\Phi \tag{12}$$

Magnetic and electric potentials (c'ed)

Only $\nabla \times \mathbf{A}$ was specified, there still remains $\nabla \cdot \mathbf{A}$ to be defined. Among the possible choices we have the so-called Lorentz-Lorenz gauge:

$$\nabla \cdot \mathbf{A} = -j\omega\epsilon\Phi \tag{13}$$

substituting this into (12) leads to the Helmholtz equation or complex wave equation, whose solutions are *wave potentials*:

$$\nabla^2\mathbf{A} + k^2\mathbf{A} = -\mathbf{J} \tag{14}$$

Fields can be obtained directly from $\mathbf{A}$:

$$\mathbf{E} = -j\omega\mu\mathbf{A} + \frac{1}{j\omega\epsilon}\nabla(\nabla \cdot \mathbf{A}) \tag{15}$$

$$\mathbf{H} = \frac{\nabla \times \mathbf{A}}{\mu} \tag{16}$$

Solution of Complex Wave Equation

(14) may be solved one component at a time for an impulse source located at the origin:

$$\nabla^2 A_\dagger + k^2 A_\dagger = 0 \qquad (r \neq 0) \tag{17}$$

where $\dagger$ may be e.g. x y or z. Note that the form of this equation and the source term implies a spherically symmetric solution $A_\dagger = f(r)$:

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{df}{dr}\right) + k^2 f = 0 \tag{18}$$

solutions to this equation are:

$$f = C\frac{e^{\pm jkr}}{r} \tag{19}$$

Enforcing Boundary Conditions

Solutions above are spherical waves, inwards for positive sign and outwards for negative. Clearly an inwards wave makes no sense as effect must move *away* from source. An additional argument is that when media is dissipative, $k = k_r - jk_i$ and only the outwards wave results in vanishing fields when $r \to \infty$, while the one going inwards diverges:

$$f = C\frac{e^{-jkr}}{r} \tag{20}$$

In order to compute C, we first write the wave equation valid in all points in space including the singular source term at the origin, which is a 3-D unit dirac delta function:

$$\nabla^2 A_\dagger + k^2 A_\dagger = -\delta(x)\delta(y)\delta(z) \tag{21}$$

Enforcing Boundary Conditions (c'ed)

We now compute the volume integral of the above on a sphere centered at the origin with radius ρ:

$$\iiint_{S_\rho} \left(\nabla^2 f + k^2 f\right) dV \quad = \quad -\iiint_{S_\rho} \delta(x)\delta(y)\delta(z) dV$$

The first integral at the left can be transformed into a surface integral using the divergence theorem, while the second vanishes as the $1/r$ singularity is cancelled out by the r^3 term of the volume. The third is identically -1 by the sampling property of the Dirac delta function:

$$\iint_{\partial S_\rho} \nabla f \cdot \mathbf{dS} \quad = \quad -1 \tag{22}$$

The constant C must be independent from k, for simplicity it can be set to 0:

$$\iint_{\partial S_\rho} -\frac{C}{\rho^2}\hat{r} \cdot \left(\hat{r}\rho^2 \sin\theta d\theta d\phi\right) \quad = \quad -1 \tag{23}$$

$$\Rightarrow C = \frac{1}{4\pi} \Rightarrow f = \frac{e^{-jkr}}{4\pi r} \tag{24}$$

Fields due to an infinitesimal current along z

For a current $\mathbf{J} = \hat{z}\delta(x)\delta(y)\delta(z)$ we have $\mathbf{A} = \frac{e^{-jkr}}{4\pi r}\hat{z}$ (the boundary condition at the origin makes $C = 0$ for the remaining Cartesian components as the zero source term implies zero right hand side integral). From (15):

$$E_r \quad = \quad 2\frac{e^{-jkr}}{4\pi r}\left(\frac{\eta}{r} + \frac{1}{j\omega\epsilon r^2}\right)\cos\theta \tag{25}$$

$$E_\theta \quad = \quad \frac{e^{-jkr}}{4\pi r}\left(j\omega\mu + \frac{\eta}{r} + \frac{1}{j\omega\epsilon r^2}\right)\sin\theta \tag{26}$$

$$H_\phi \quad = \quad \frac{e^{-jkr}}{4\pi r}\left(jk + \frac{1}{r}\right)\sin\theta \tag{27}$$

Radiation field is obtained far away from the source (only $1/r$ terms contribute to net outwards power flow):

$$E_\theta^{far} \quad = \quad \frac{e^{-jkr}}{4\pi r} jk\eta\sin\theta \tag{28}$$

$$H_\phi^{far} \quad = \quad \frac{e^{-jkr}}{4\pi r} jk\sin\theta \tag{29}$$

Arbitrary Source Distribution

Fields at observation point $\mathbf{r}$ due to an impulse source located at $\mathbf{r}'$ with arbitrary orientation and magnitude given by $\mathbf{J}$ are:

$$\mathbf{E}\left(\mathbf{r},\mathbf{r}',\mathbf{J}\right) = \frac{e^{-jkr}}{4\pi r} jk\eta \left[\mathbf{J}\left(-1+\frac{j}{kR}+\frac{1}{(kR)^2}\right)+ + \hat{\mathbf{R}}\left(\mathbf{J}\cdot\hat{\mathbf{R}}\right)\left(1-\frac{3j}{kR}-\frac{3}{(kR)^2}\right)\right] \tag{30}$$

$$\mathbf{H}\left(\mathbf{r},\mathbf{r}',\mathbf{J}\right) = \frac{e^{-jkR}}{4\pi R}\left(jk+\frac{1}{R}\right)\mathbf{J}\times\hat{\mathbf{R}} \tag{31}$$

$$\mathbf{R} = \mathbf{r}-\mathbf{r}' \tag{32}$$

Fields due to an arbitray source distribution are obtained through a convolution integral:

$$\aleph_{tot}(\mathbf{r}) = \iiint_V \aleph\left(\mathbf{r},\mathbf{r}',\mathbf{J}(\mathbf{r}')\right) dV' \tag{33}$$

where $\aleph$ may be $\mathbf{E}$ or $\mathbf{H}$.