Ising model on random graphs with non-limited range of interactions

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Abstract

The Ising model, renowned for its simplicity and effectiveness in capturing phase transitions, serves as a powerful tool to analyze the emergent properties of complex systems. The core objective of this research is to unravel the implications of non-limited interaction ranges in the context of random graphs. Traditional Ising models often assume a fixed range of interactions among neighboring spins. This work challenges that assumption by considering scenarios where interactions extend beyond the nearest neighbors, incorporating a broader and more realistic perspective on the interplay between spins.

1 Introduction

Historically, the Ising model has played a crucial role in the development of statistical mechanics and the understanding of critical phenomena. The starting point - one-dimensional Ising model - was solved exactly by Ernst Ising's advisor, Wilhelm Lenz, in 1920. However, the two-dimensional case remained unsolved for many years until the solution has been found independently by Lars Onsager in 1944, marking a significant breakthrough.

1.1 The Ising model

It serves as a basic model for understanding phase transitions in physical systems. In its simplest form, the model considers a lattice of discrete spins, that represent magnetic moments. These spins can be either "up" (+1) or "down" (-1), corresponding to the two possible magnetic states $s_{ij} \in \{-1, +1\}$.

The energy of the system is determined by the interactions between neighboring spins. The model assumes that each spin interacts with its nearest neighbors, and the total energy of the system depends on the alignment of these spins relative to each other.

The Hamiltonian H of such model can therefore be described as:

$$H = -\frac{1}{2} \sum_{\langle i,j \rangle} J_{ij} s_i s_j + \mu \sum_i h_i s_i \tag{1}$$

Depending on the sign of an interaction constant J, the evolution of the system differs. Hence, in case of $J > 0$ (ferromagnetic interaction), if neighboring spins are aligned in the same direction, the energy is lowered, and if they are aligned oppositely, the energy is increased. However, when $J < 0$, indicating an antiferromagnetic interaction, the dynamics of the model undergo a reversal.

1.2 Random graphs

Since then, the model has been successively extended and applied to various disciplines beyond physics. There exist various ways of extending the model.

One could for instance change the topography of the spin lattice. By employing the Ising model on random graphs, one can study e.g. how information spreads through a social network, shedding light on the mechanisms that govern the diffusion of ideas, opinions, or trends within these complex systems. This approach not only contributes to the understanding of social dynamics but also showcases the versatility of the Ising model in capturing the essence of diverse real-world scenarios.

TODO: exemplary graph figure

For this thesis, I have chosen to focus my analysis on an Erdős-Renyi graph $G_{N,p}$. Such network is constructed by choosing a number of edges for each node, with some probability described by a parameter p. Moreover, the average number of nearest neighbors for each node is given by the following equation:

$$\langle k \rangle = (N - 1) \cdot p \tag{2}$$

TODO: mean-field

TODO: general solution to mean field

2 Motivation

3 Literature Review

Several works were found that include similar fields of study. Simulation of Ising model on small-networks ...

4 Analytical mean-field approximation for critical temperature

For the purpose of a comparative analysis, we derived a formula that is based on the mean-field approximation was derived. To start with, we assume that interaction strength between spins undergoes exponential decay over longer distances, denoted by l. This decay can be represented as $\exp[-\alpha(l-1)]$. The parameter α was added for better control over the decay speed.

Since we consider a model with unlimited, long-range interactions, the sum of all neighbors at a given distance is equal to the total number of spins, N. We can approximate, that root spin

will have $\langle k \rangle = k$ neighbors and at each consecutive path length, $(k-1)$. Therefore, we can write such sum as

$$N = \sum_{l=1}^{l_{\max}} N_l = \sum_{l=1}^{l_{\max}} k(k-1)^{(l-1)}. \tag{3}$$

The formula (3) is a finite geometric series, which can be simplified into

$$N = k \sum_{l=1}^{l_{\max}} (k-1)^{(l-1)} = k \frac{(1-q^{l_{\max}})}{(1-q)}, \text{ where } q = k-1. \tag{4}$$

Then, by using the above, we can obtain a formula for maximum path length, $l_{\max}$, in the ER graph as

$$l_{\max} = \log_{(k-1)} \left[N \frac{(k-2)}{k} + 1 \right]. \tag{5}$$

Based on the general mean-field solution, the critical temperature, T_C, is given by

$$k_B T_C = J \sum_{l=1}^{l_{\max}} N_l \; e^{-\alpha(l-1)} = kJ \sum_{l=1}^{l_{\max}} \left[e^{-\alpha}(k-1) \right]^{(l-1)}, \tag{6}$$

which once more is a geometric series. Similar to (4), the equation (6) can be simplified, but this time with the common ratio of $q = e^{-\alpha}(k-1)$.

When we make $k_B = 1$, $J = 1$ and use (5), the final analytical mean-field formula for the critical temperature, T_C, is

$$T_C = \frac{1 - \left[e^{-\alpha}(k-1) \right]^{\log_{(k-1)} \left[N \frac{(k-2)}{k} + 1 \right]}}{1 - e^{-\alpha}(k-1)}. \tag{7}$$

The key point that emerges, following the above derivation, is that there are two distinct regions of behavior of the system. The sum in (6) converges only when $|q| < 1$, thus when $\exp[\alpha] > (k-1)$.

Therefore, we find the equation describing boundary separating those regions to be

$$e^{\alpha} = k - 1. \tag{8}$$

5 Simulation

5.1 Methodology

In order to perform the simulation, we used Metropolis algorithm. It consists of the following steps:

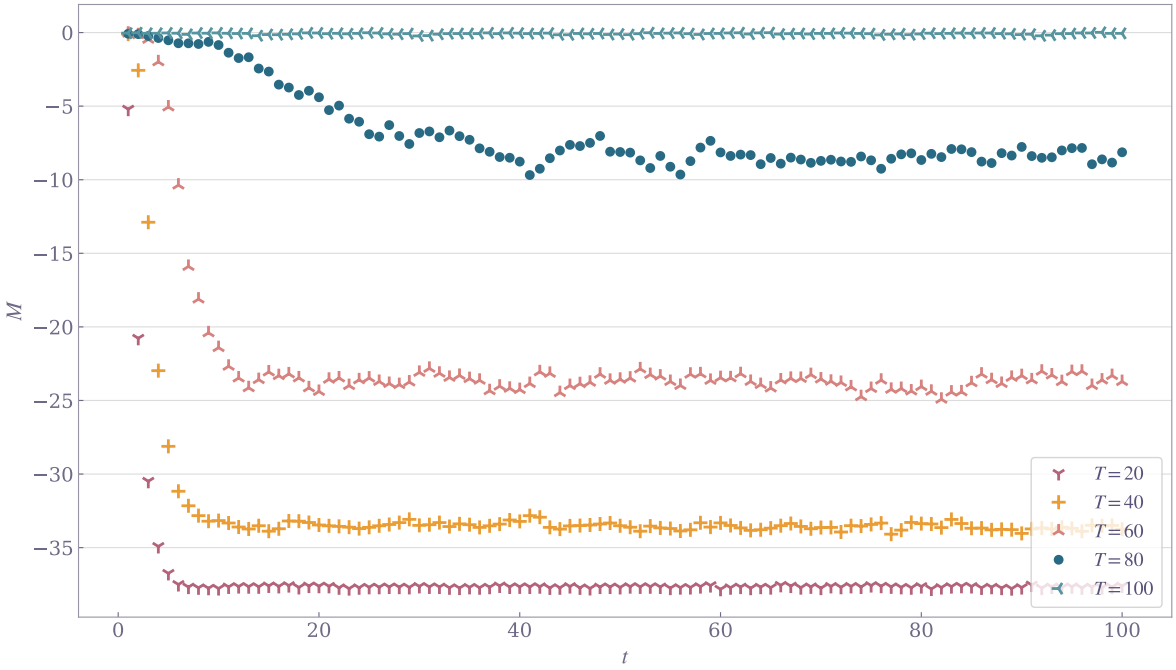
- Generate the ER graph with a probability parameter p and N spins, each initialized with a random value of either +1 or -1.
- The main loop:
  - Take a randomly selected spin s with a uniform probability $\mathbb{P}(s) = 1/N$.
  - Calculate the change in system's energy as $dE = 2s \sum_{\langle i,j \rangle}^{\text{neighbors}} J_{ij} s_j$
  - If $dE \leq 0$, we flip the spin, else if $dE > 0$, we flip only when $x < \exp[-dE/T]$ is satisfied (where x is a random number, drawn from range $[0, 1]$)
  - Continue to the next iteration until we reach the set number of steps.

During one time step (N iterations), each spin will be flipped, on average, at least once.

For the first 0-30 time steps, with temperatures lower than the critical T_C, the system tends towards equilibrium. For larger than T_C however, it typically oscillates around some value.

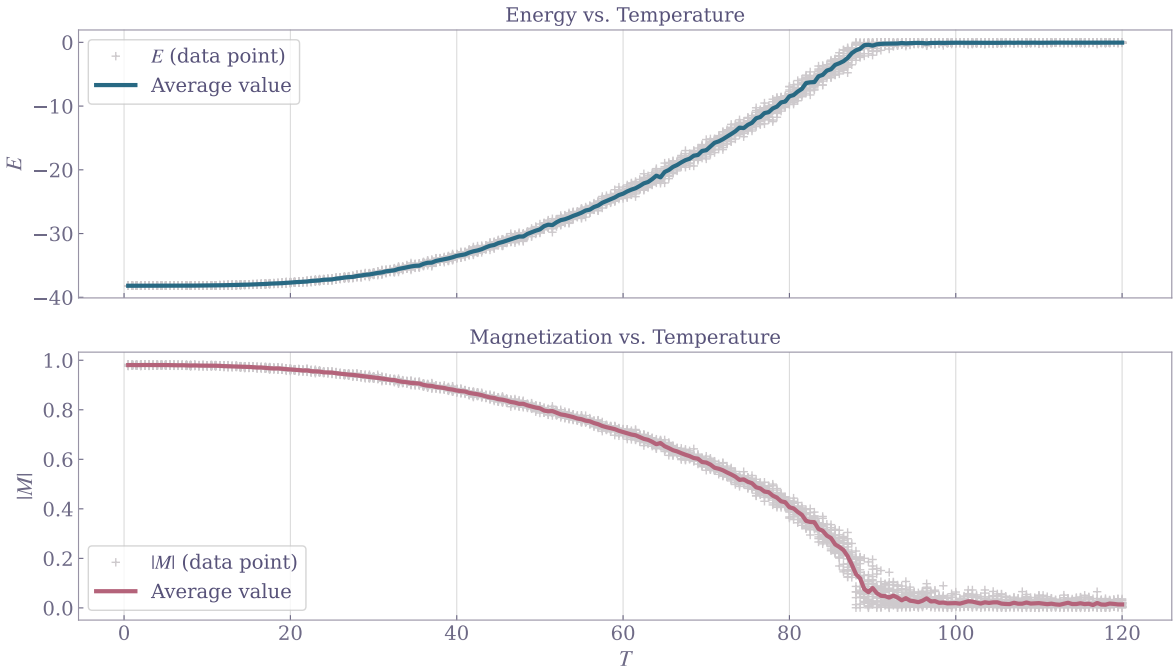
We show this in the figure 1 by looking at how magnetization changes with time step. In this case, for the highest temperature $T = 100$, it fluctuates around zero, indicating a state in which approximately half of the spins are pointing up and the other half are pointing down.

Figure 1: **Magnetization vs. time step**



The simulations were performed on random graphs of differing sizes and parameters. Typically, we chose $N = 100, 200, 400, 1000, 2000, 4000$ and 10000.

Figure 2: **Energy and Magnetization vs. time step**



5.2 Results and Discussion

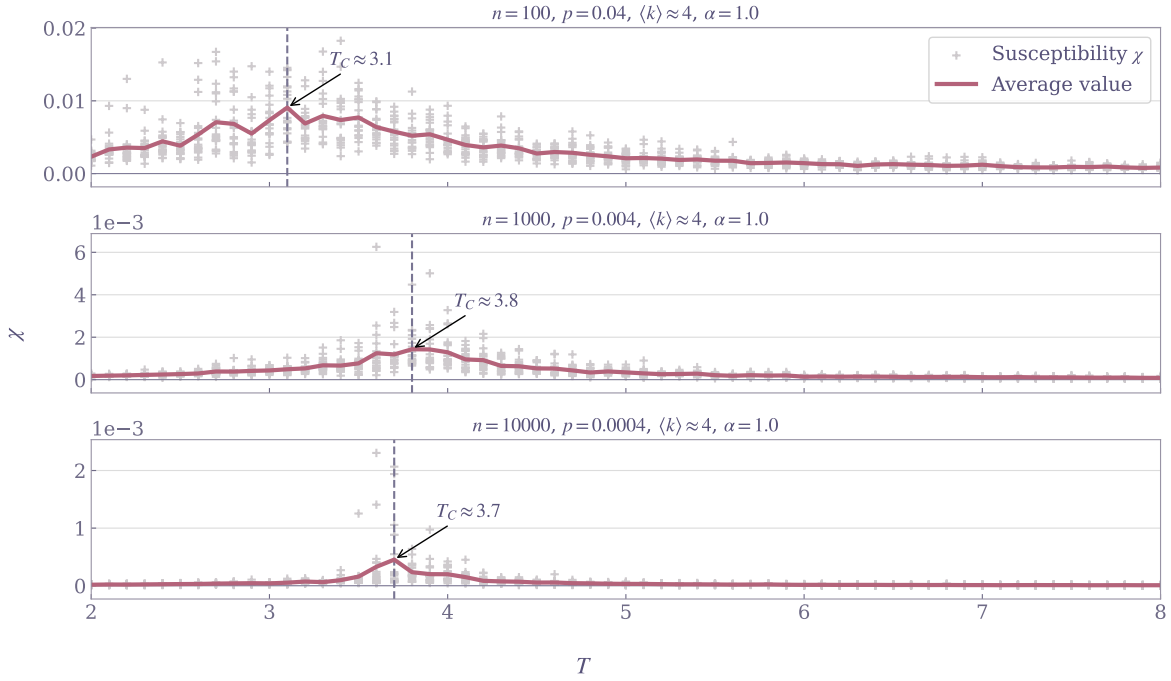
TODO: differences between J generation methods with k constant

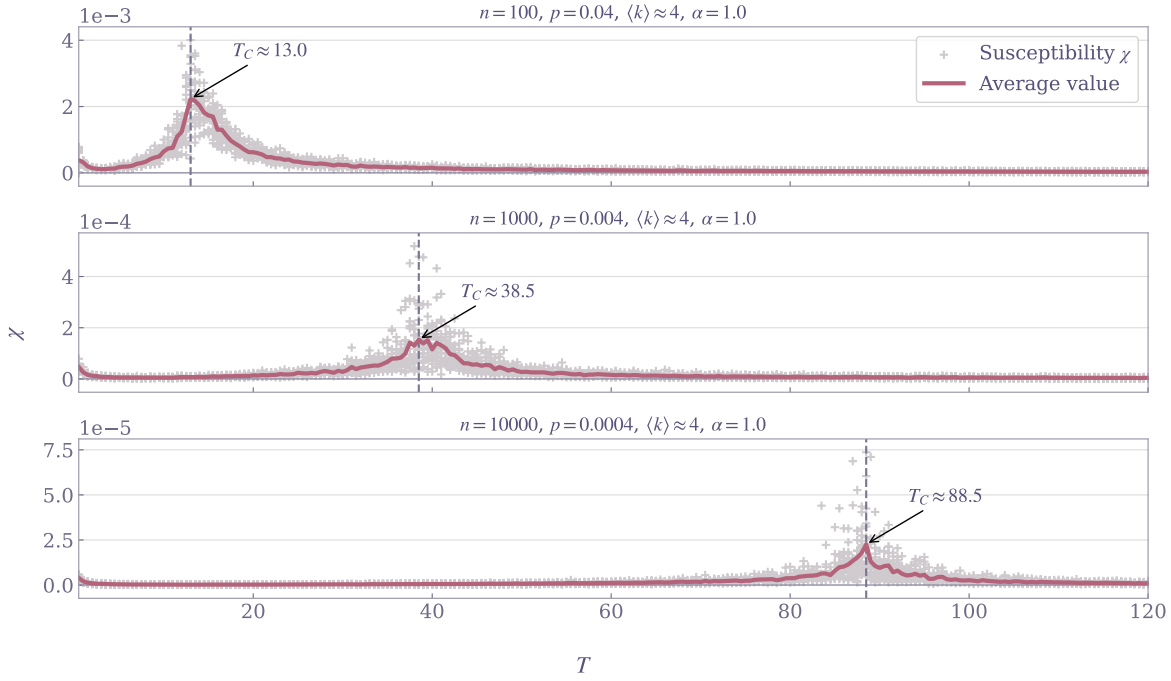
6 Conclusion

TODO: T_C vs. k with n kept constant

TODO: T_C vs. α for exponential decay

TODO: $e^{-\alpha k}$ vs. $1/k^{\alpha}$


$n = 100,\ p = 0.04,\ \langle k \rangle \approx 4,\ \alpha = 1.0$
$T_C \approx 3.1$
Susceptibility χ
Average value
$n = 1000,\ p = 0.004,\ \langle k \rangle \approx 4,\ \alpha = 1.0$
$T_C \approx 3.8$
$n = 10000,\ p = 0.0004,\ \langle k \rangle \approx 4,\ \alpha = 1.0$
$T_C \approx 3.7$
χ
T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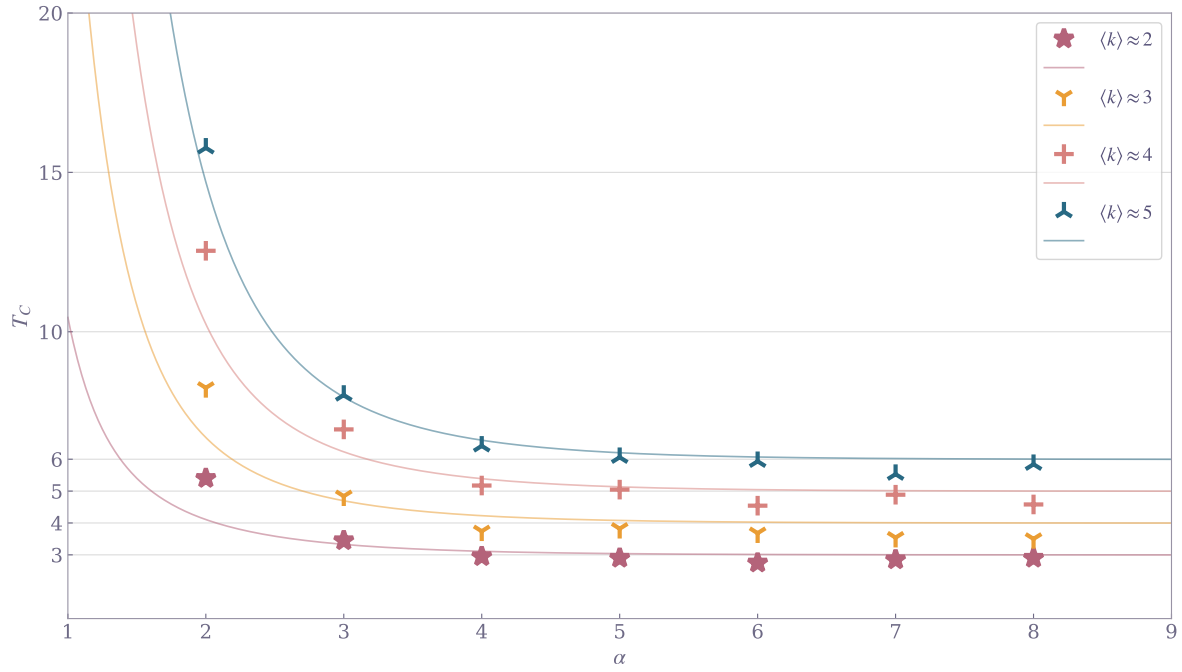
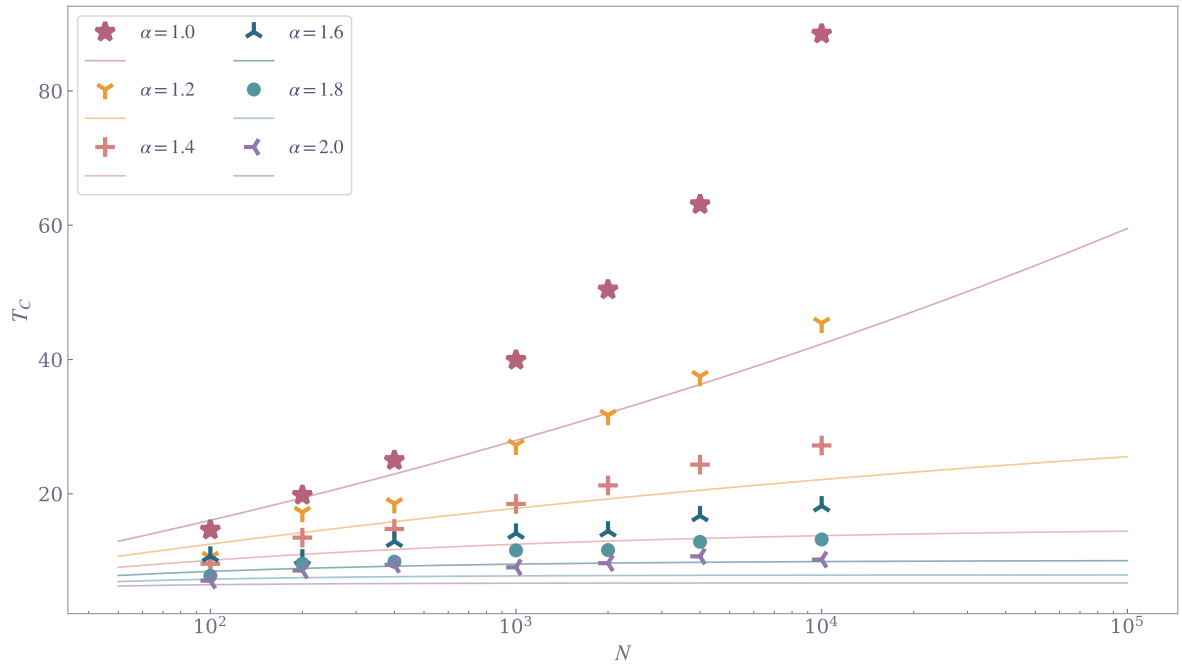
$T_C \approx 13.0$
Susceptibility χ
Average value
$n = 1000,\ p = 0.004,\ \langle k \rangle \approx 4,\ \alpha = 1.0$
$T_C \approx 38.5$
$n = 10000,\ p = 0.0004,\ \langle k \rangle \approx 4,\ \alpha = 1.0$
$T_C \approx 88.5$
χ
T


20
15
10
6
5
4
3
T_C
1
2
3
4
5
6
7
8
9
α
$\langle k \rangle \approx 2$
$\langle k \rangle \approx 3$
$\langle k \rangle \approx 4$
$\langle k \rangle \approx 5$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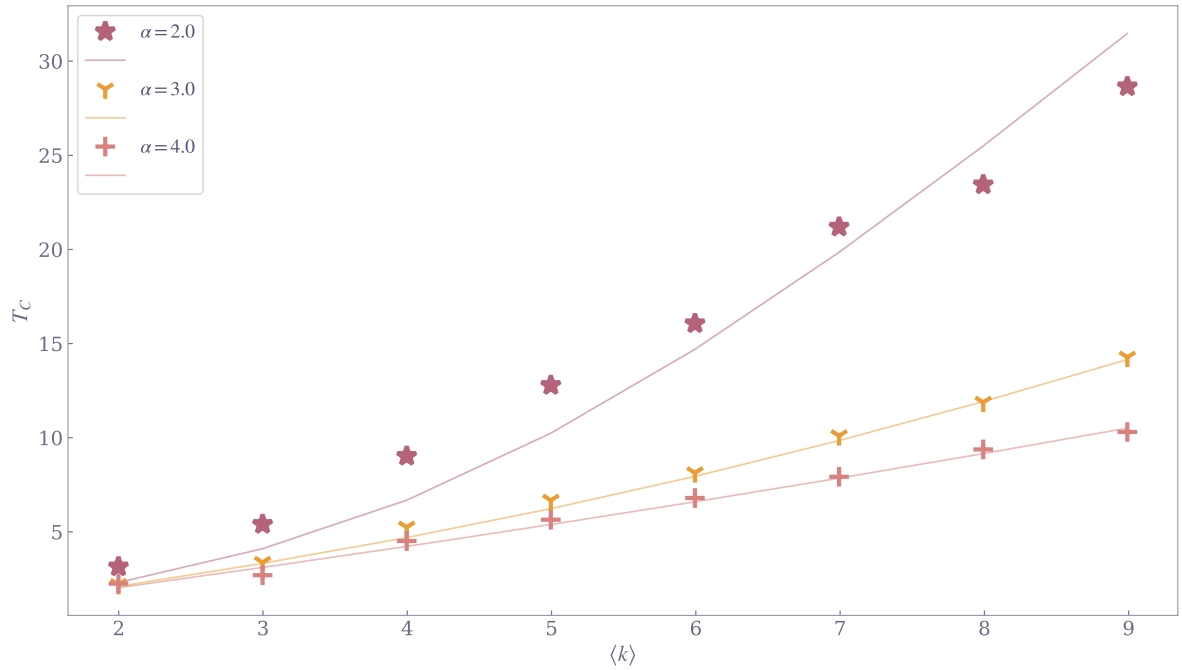
$\alpha = 1.2$
$\alpha = 1.4$
$\alpha = 1.6$
$\alpha = 1.8$
$\alpha = 2.0$
80
60
40
20
T_C
10^2
10^3
10^4
10^5
N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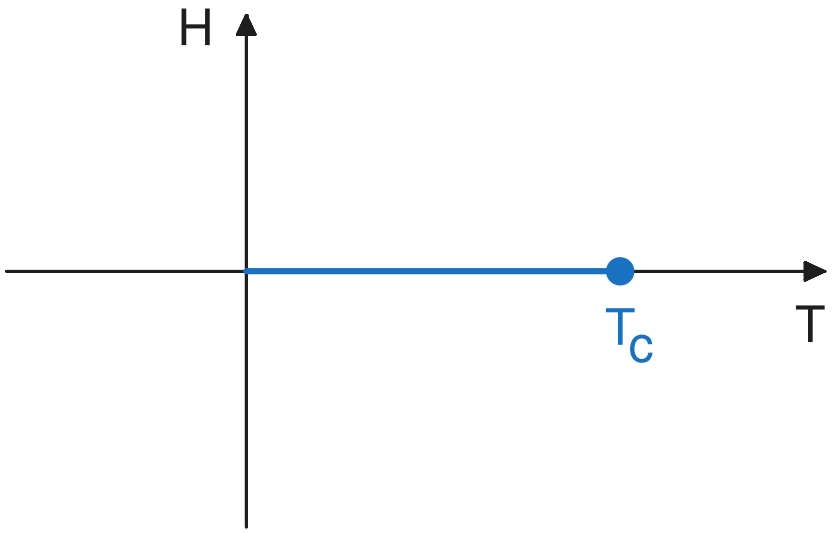
α = 3.0
α = 4.0
30
25
20
15
10
5
T_C
2
3
4
5
6
7
8
9
⟨k⟩


H
T_C
T