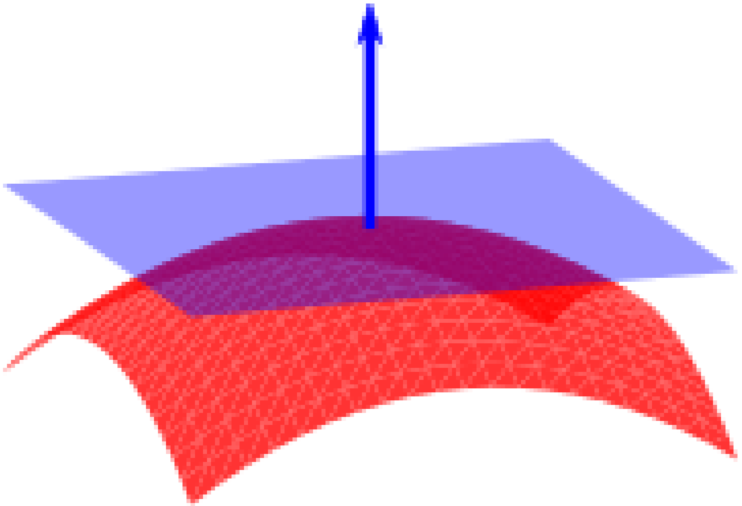
Calcolo della normale

Normale ad una superficie piana: un vettore tridimensionale perpendicolare a quella superficie.

Per un poligono (come un triangolo), la normale alla superficie può essere calcolata come il vettore prodotto vettoriale di due lati non paralleli del poligono.

Una normale ad una superficie **non piana** nel punto *P* su quella superficie è un vettore perpendicolare al piano tangente a quella superficie in *P*.



Superficie in forma implicita

Se la superficie è in forma implicita, cioè della forma S(x,y,z)=0, allora una normale alla superficie nel punto $P_0 = (x_0, y_0, z_0,$ è data dal vettore gradiente della superficie S calcolato in P_0, cioè il vettore normale è dato da $\nabla S(x_0, y_0, z_0)$, dove il gradiente ∇S ha per componenti

$$\nabla S(x \quad , y \quad , z \quad) = \begin{bmatrix} \frac{\partial S}{\partial x} \\ \frac{\partial S}{\partial y} \\ \frac{\partial S}{\partial z} \end{bmatrix}$$

Superficie in forma parametrica

Se una superficie S è parametrizzata da un sistema di coordinate curvilinee $\boldsymbol{S}(\boldsymbol{\theta}, \boldsymbol{\varphi})$ con $\boldsymbol{\theta}, \boldsymbol{\varphi}$ numeri reali, per calcolare il versore della normale in un suo punto P si procede nel seguente modo:

- si calcolano le derivate parziali $\frac{\partial \boldsymbol{S}(\boldsymbol{\theta},\boldsymbol{\varphi})}{\partial \boldsymbol{\theta}}$ e $\frac{\partial \boldsymbol{S}(\boldsymbol{\theta},\boldsymbol{\varphi})}{\partial \boldsymbol{\varphi}}$ e si normalizzano: rappresentano i versori che individuano il piano tangente alla curva nel punto $\boldsymbol{P}(\boldsymbol{\theta},\boldsymbol{\varphi})$.
- $\overline{\boldsymbol{S}_{\boldsymbol{\theta}}} = \frac{\frac{\partial \boldsymbol{S}(\boldsymbol{\theta},\boldsymbol{\varphi})}{\partial \boldsymbol{\theta}}}{\left\|\frac{\partial \boldsymbol{S}(\boldsymbol{\theta},\boldsymbol{\varphi})}{\partial \boldsymbol{\theta}}\right\|}$ ed $\overline{\boldsymbol{S}_{\boldsymbol{\varphi}}} = \frac{\frac{\partial \boldsymbol{S}(\boldsymbol{\theta},\boldsymbol{\varphi})}{\partial \boldsymbol{\varphi}}}{\left\|\frac{\partial \boldsymbol{S}(\boldsymbol{\theta},\boldsymbol{\varphi})}{\partial \boldsymbol{\varphi}}\right\|}$
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- allora il versore della normale è dato dal prodotto vettoriale $\overline{\boldsymbol{S}_{\boldsymbol{\theta}}} \times \overline{\boldsymbol{S}_{\boldsymbol{\varphi}}}$

Calcoliamo il versore della normale ad un punto P appartenente ad una sfera espressa in coordinate sferiche:

$$S(\theta, \varphi) = \begin{cases} r \cos(\theta) \sin(\varphi) \\ \mathrm{r} \cos(\varphi) \\ r \sin(\theta) \sin(\varphi) \end{cases}$$

$$\frac{\partial S(\theta,\varphi)}{\partial \theta} = \begin{cases} -r \sin(\theta) \sin(\varphi) \\ 0 \\ r \cos(\theta) \sin(\varphi) \end{cases}$$

$$\left\| \frac{\partial S(\theta,\varphi)}{\partial \theta} \right\| = \sqrt{r^2 \sin^2(\theta) \sin^2(\varphi) + r^2 \cos^2(\theta) \sin^2(\varphi)} =$$

$$= \sqrt{r^2 \sin^2(\varphi)(\sin^2(\theta) + \cos^2(\theta)} = \mathrm{r} \sin(\varphi)$$

$$\overline{\boldsymbol{S_\theta}} = \frac{\frac{\partial S(\theta,\varphi)}{\partial\theta}}{\left\|\frac{\partial S(\theta,\varphi)}{\partial\theta}\right\|} = \begin{bmatrix} -\sin(\theta) \\ 0 \\ \cos(\theta) \end{bmatrix}$$

$$\frac{\partial S(\theta,\varphi)}{\partial\varphi} = \begin{cases} r\ \cos(\theta)\cos(\varphi) \\ -\mathrm{r}\sin(\varphi) \\ r\ \sin(\theta)\cos(\varphi) \end{cases}$$

$$\left\|\frac{\partial S(\theta,\varphi)}{\partial\varphi}\right\| = \sqrt{r^2\ \cos^2(\theta)\cos^2(\varphi) + r^2 sin^2(\varphi) + \ \sin^2(\theta)\cos^2(\varphi)} =$$

$$= \sqrt{r^2\ \cos^2(\varphi)(\cos^2(\theta) + \sin^2(\theta)) + r^2 sin^2(\varphi)} = r$$

$$\overline{\boldsymbol{S_{\varphi}}} = \frac{\frac{\partial S(\theta,\varphi)}{\partial \varphi}}{\left\|\frac{\partial S(\theta,\varphi)}{\partial \varphi}\right\|} = \begin{bmatrix} \cos(\theta)\cos(\varphi) \\ -\sin(\varphi) \\ \sin(\theta)\cos(\varphi) \end{bmatrix}$$

Calcoliamo adesso il prodotto vettoriale dei due versori $\overline{\boldsymbol{S_{\theta}}} \times \overline{\boldsymbol{S_{\varphi}}} = \begin{bmatrix} -\sin(\theta) \\ 0 \\ \cos(\theta) \end{bmatrix} \times$

$$\begin{bmatrix} \cos(\theta)\cos(\varphi) \\ -\sin(\varphi) \\ \sin(\theta)\cos(\varphi) \end{bmatrix}$$

$$\begin{bmatrix} -\sin(\theta) & 0 & \cos(\theta) \\ \cos(\theta)\cos(\varphi) & -\sin(\varphi) & \sin(\theta)\cos(\varphi) \end{bmatrix}$$

Il prodotto vettoriale $\overline{\boldsymbol{S_{\theta}}} \times \overline{\boldsymbol{S_{\varphi}}}$ **ha per componenti:**

$$\overline{\boldsymbol{S_\theta}} \times \overline{\boldsymbol{S_\varphi}} = \begin{bmatrix} \sin(\varphi)\cos(\theta) \\ -(-\sin^2(\theta)\cos(\varphi) - \cos^2(\theta)\cos(\varphi)) \\ \sin(\theta)\sin(\varphi) \end{bmatrix}$$

$$\overline{\boldsymbol{S_\theta}} \times \overline{\boldsymbol{S_\varphi}} = \begin{bmatrix} \sin(\varphi)\cos(\theta) \\ \cos(\varphi) \\ \sin(\theta)\sin(\varphi) \end{bmatrix}$$

Calcoliamo il versore della normale ad un punto P appartenente ad un cilindro espresso in coordinate sferiche:

$$S(h,\theta) = \begin{cases} r\cos(\theta) \\ \text{h} \\ r\ \sin(\theta) \end{cases}$$

$$\frac{\partial S(\theta,h)}{\partial h} = \begin{cases} 0 \\ 1 \\ 0 \end{cases}$$

$$\overline{\boldsymbol{S_h}} = \frac{\frac{\partial S(\theta,h)}{\partial h}}{\left\|\frac{\partial S(\theta,h)}{\partial h}\right\|} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\frac{\partial S(\theta,\varphi)}{\partial \theta} = \begin{cases} -r\sin(\theta) \\ 0 \\ r\ \cos(\theta) \end{cases}$$

$$\left\|\frac{\partial S(\theta,\varphi)}{\partial\theta}\right\| = \sqrt{r^2\ \sin^2(\theta) \quad + r^2 \cos^2(\theta)} = r$$

$$\overline{\boldsymbol{S_\theta}} = \frac{\frac{\partial S(\theta,\varphi)}{\partial\theta}}{\left\|\frac{\partial S(\theta,\varphi)}{\partial\theta}\right\|} = \begin{bmatrix} -\sin(\theta) \\ 0 \\ \cos(\theta) \end{bmatrix}$$

Calcoliamo adesso il prodotto vettoriale dei due versori $\overline{\boldsymbol{S_h}} \times \overline{\boldsymbol{S_\theta}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \times \begin{bmatrix} -\sin(\theta) \\ 0 \\ \cos(\theta) \end{bmatrix}$

$$\begin{bmatrix} 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix}$$

Il prodotto vettoriale $\overline{\boldsymbol{S_h}} \times \overline{\boldsymbol{S_\theta}}$ **ha per componenti:**

$$\overline{\boldsymbol{S_h}} \times \overline{\boldsymbol{S_\theta}} = \begin{bmatrix} \cos(\theta) \\ \mathbf{0} \\ \sin(\theta) \end{bmatrix}$$

Calcoliamo il versore della normale ad un punto P appartenente ad un cono espresso in coordinate sferiche:

$$S(r,\theta) = \begin{cases} r\cos(\theta) \\ \mathrm{r} \\ r\ \sin(\theta) \end{cases}$$

$$\frac{\partial S(r,\theta)}{\partial r} = \begin{cases} \cos(\theta) \\ 1 \\ \sin(\theta) \end{cases}$$

$$\left\| \frac{\partial S(r,\theta)}{\partial r} \right\| = \sqrt{\cos^2(\theta) \quad + 1 + \sin^2(\theta)} = \sqrt{2}$$

$$\overline{\boldsymbol{S_r}} = \frac{\frac{\partial S(r,\theta)}{\partial r}}{\left\|\frac{\partial S(r,\theta)}{\partial r}\right\|} = \begin{bmatrix} \frac{\cos(\theta)}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ \frac{\sin(\theta)}{\sqrt{2}} \end{bmatrix}$$

$$S(r,\theta) = \begin{cases} r\cos(\theta) \\ \mathrm{r} \\ r\ \sin(\theta) \end{cases}$$

$$\frac{\partial S(r,\theta)}{\partial \theta} = \begin{cases} -r\sin(\theta) \\ 0 \\ r\ \cos(\theta) \end{cases}$$

$$\left\|\frac{\partial S(r,\theta)}{\partial\theta}\right\| = \sqrt{r^2\ \sin^2(\theta) \quad + r^2\cos^2(\theta)}=r$$

$$\overline{\boldsymbol{S_\theta}} = \frac{\frac{\partial S(\theta,r)}{\partial\theta}}{\left\|\frac{\partial S(\theta,r)}{\partial\theta}\right\|} = \begin{bmatrix} -\sin(\theta) \\ 0 \\ \cos(\theta) \end{bmatrix}$$

Calcoliamo adesso il prodotto vettoriale dei due versori $\overline{\boldsymbol{S_r}} \times \overline{\boldsymbol{S}_\theta} = \begin{bmatrix} \frac{\cos(\theta)}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ \frac{\sin(\theta)}{\sqrt{2}} \end{bmatrix} \times \begin{bmatrix} -\sin(\theta) \\ 0 \\ \cos(\theta) \end{bmatrix}$

$$\begin{bmatrix} \frac{\cos(\theta)}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{\sin(\theta)}{\sqrt{2}} \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix}$$

Il prodotto vettoriale $\overline{\boldsymbol{S_r}} \times \overline{\boldsymbol{S_\theta}}$ **ha per componenti:**

$$\overline{\boldsymbol{S_r}} \times \overline{\boldsymbol{S_\theta}} = \begin{bmatrix} \frac{\cos(\theta)}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \\ \frac{\sin(\theta)}{\sqrt{2}} \end{bmatrix}$$

Calcoliamo il versore della normale ad un punto P appartenente ad un toro espresso in coordinate sferiche:

$$S(\varphi,\theta)=\begin{cases}(R+r\cos(\varphi))\cos(\theta)\\ \mathrm{r}\sin(\varphi)\\ (R+r\cos(\varphi))\sin(\theta)\end{cases}$$

$$\frac{\partial S(\varphi,\theta)}{\partial\varphi}=\begin{cases}-\mathrm{r}\sin(\varphi)\cos(\theta)\\ \mathrm{r}\cos(\varphi)\\ -r\ \sin(\varphi)\sin(\theta)\end{cases}$$

$$\left\|\frac{\partial S(\varphi,\theta)}{\partial\varphi}\right\|=\sqrt{r^2\ \sin^2(\varphi)\cos^2(\theta)+r^2cos^2(\varphi)+r^2\ \sin^2(\varphi)\sin^2(\theta)}=$$

$$=\sqrt{r^2\ \sin^2(\varphi)(\cos^2(\theta)+\sin^2(\theta))+r^2cos^2(\varphi)}=r$$

$$\overline{\boldsymbol{S_\varphi}} = \frac{\frac{\partial S(\theta,\varphi)}{\partial \varphi}}{\left\|\frac{\partial S(\theta,\varphi)}{\partial \varphi}\right\|} = \begin{bmatrix} -\sin(\varphi)\cos(\theta) \\ \cos(\varphi) \\ -\sin(\varphi)\sin(\theta) \end{bmatrix}$$

$$S(\varphi,\theta) = \begin{cases} (R + r\cos(\varphi))\cos(\theta) \\ \mathrm{r}\sin(\varphi) \\ (R + r\cos(\varphi))\sin(\theta) \end{cases}$$

$$\frac{\partial S(\varphi,\theta)}{\partial \theta} = \begin{cases} -(R + r\cos(\varphi))\ \sin(\theta) \\ 0 \\ (R + r\cos(\varphi))\ \cos(\theta) \end{cases}$$

$$\left\|\frac{\partial S(\varphi,\theta)}{\partial \theta}\right\| = \sqrt{(R + r\cos(\varphi))^2\ \sin^2(\theta) \quad + (R + r\cos(\varphi))^2\ \cos^2(\theta)} =$$

$= \sqrt{(R + r\cos(\varphi))^2} = R + r\cos(\varphi)$

$$\overline{\boldsymbol{S_\theta}} = \frac{\frac{\partial S(\varphi,\theta)}{\partial \theta}}{\left\|\frac{\partial S(\varphi,\theta)}{\partial \theta}\right\|} = \begin{bmatrix} -\sin(\theta) \\ 0 \\ \cos(\theta) \end{bmatrix}$$

Calcoliamo adesso il prodotto vettoriale dei due versori $\overline{\boldsymbol{S_\varphi}} \times \overline{\boldsymbol{S_\theta}} = \begin{bmatrix} -\sin(\varphi)\cos(\theta) \\ \cos(\varphi) \\ -\sin(\varphi)\sin(\theta) \end{bmatrix} \times \begin{bmatrix} -\sin(\theta) \\ 0 \\ \cos(\theta) \end{bmatrix}$

$$\begin{bmatrix} -\sin(\varphi)\cos(\theta) & \cos(\varphi) & -\sin(\varphi)\sin(\theta) \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix}$$

Il prodotto vettoriale $\overline{\boldsymbol{S_\varphi}} \times \overline{\boldsymbol{S_\theta}}$ **ha per componenti:**

$$\overline{\boldsymbol{S_\varphi}} \times \overline{\boldsymbol{S_\theta}} == \begin{bmatrix} \cos(\varphi)\cos(\theta) \\ -(-\cos^2(\theta)\sin(\varphi) - \sin^2(\theta)\sin(\varphi)) \\ -\sin(\theta)\cos(\varphi) \end{bmatrix}$$

$$\overline{\boldsymbol{S_\varphi}} \times \overline{\boldsymbol{S_\theta}} == \begin{bmatrix} \cos(\varphi)\cos(\theta) \\ \sin(\varphi) \\ \sin(\theta)\cos(\varphi) \end{bmatrix}$$