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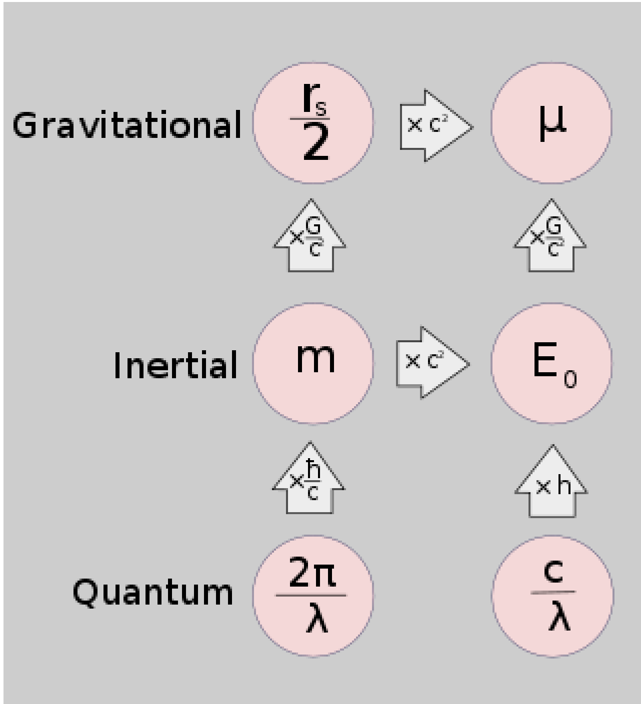
Schwarzschild radius

The **Schwarzschild radius** (sometimes historically referred to as the **gravitational radius**) is a physical parameter that shows up in the Schwarzschild solution to Einstein's field equations, corresponding to the radius defining the event horizon of a Schwarzschild black hole. It is a characteristic radius associated with every quantity of mass. The *Schwarzschild radius* was named after the German astronomer Karl Schwarzschild, who calculated this exact solution for the theory of general relativity in 1916.

The Schwarzschild radius is given as

$$r_s = \frac{2GM}{c^2}$$

where G is the gravitational constant, M is the object mass, and c is the speed of light.[1]



The relation between properties of mass and their associated physical constants. Every massive object is believed to exhibit all five properties. However, due to extremely large or extremely small constants, it is generally impossible to verify more than two or three properties for any object.

- The Schwarzschild radius (r_s) represents the ability of mass to cause curvature in space and time.
- The standard gravitational parameter (μ) represents the ability of a massive body to exert Newtonian gravitational forces on other bodies.
- Inertial mass (m) represents the Newtonian response of mass to forces.
- Rest energy (E_0) represents the ability of mass to be converted into other forms of energy
- The Compton wavelength (λ) represents the quantum response of mass to local geometry

Contents

History

In 1916, Karl Schwarzschild obtained the exact solution[2][3] to Einstein's field equations for the gravitational field outside a non-rotating, spherically symmetric body with mass M (see Schwarzschild metric). The solution contained terms of the form $1 - r_s/r$ and $\frac{1}{1 - r_s/r}$, which become singular at $r = 0$ and $r = r_s$ respectively. The r_s has come to be known as the *Schwarzschild radius*.

The physical significance of these singularities was debated for decades. It was found that the one at $r = r_s$ is a coordinate singularity, meaning that it can be removed by a change of coordinates, while the one at $r = 0$ is physical, and cannot be removed.[4] The Schwarzschild radius is nonetheless a physically relevant quantity, as noted above and below

This expression had previously been calculated, using Newtonian mechanics, as the radius of a spherically symmetric body at which the escape velocity was equal to the speed of light. It had been identified in the 18th century by John Michell[5] and by 19th century astronomers such as Pierre-Simon Laplace[6]

Parameters

The Schwarzschild radius of an object is proportional to the mass. Accordingly, the Sun has a Schwarzschild radius of approximately 3.0 km (1.9 mi), whereas Earth's is only about 9 mm (0.35 in) and the Moon's is about 0.1 mm (0.0039 in). The observable universe's mass has a Schwarzschild radius of approximately 13.7 billion light-years.[7][8]

Object	Mass: M	Schwarzschild radius: $\frac{2GM}{c^2}$	Schwarzschild density: $\frac{3c^6}{32\pi G^3 M^2}$
Observable universe[7]	8.8×10^{52} kg	1.3×10^{26} m (13.7 billion ly)	9.5×10^{-27} kg/m^3
Milky Way	1.6×10^{42} kg	2.4×10^{15} m (~0.25 ly)	0.000029 kg/m^3
SMBH in NGC 4889	4.2×10^{40} kg	6.2×10^{13} m	0.042 kg/m^3
SMBH in Andromeda Galaxy[9]	3.4×10^{38} kg	5.0×10^{11} m	640 kg/m^3
Sagittarius A* (SMBH)	8.2×10^{36} kg	1.2×10^{10} m	1.1×10^{6} kg/m^3
Sun	1.99×10^{30} kg	2.95×10^{3} m	1.84×10^{19} kg/m^3
Jupiter	1.90×10^{27} kg	2.82 meters	2.02×10^{25} kg/m^3
Earth	5.97×10^{24} kg	8.87×10^{-3} m	2.04×10^{30} kg/m^3
Moon	7.35×10^{22} kg	1.09×10^{-4} m	1.35×10^{34} kg/m^3
Human	70 kilograms	1.04×10^{-25} m	1.49×10^{76} kg/m^3
Big Mac	0.215 kilograms	3.19×10^{-28} m	1.58×10^{81} kg/m^3
Planck mass	2.18×10^{-8} kg	3.23×10^{-35} m	1.54×10^{95} kg/m^3

Derivation

Black hole classification by Schwarzschild radius

Any object whose radius is smaller than its Schwarzschild radius is called a black hole. The surface at the Schwarzschild radius acts as an event horizon in a non-rotating body (a rotating black hole operates slightly differently). Neither light nor particles can escape through this surface from the region inside, hence the name "black hole".

Black holes can be classified based on their Schwarzschild radius, or equivalently, by their density. As the radius is linearly related to mass, while the enclosed volume corresponds to the third power of the radius, small black holes are therefore much more dense than large ones. The volume enclosed in the event horizon of the most massive black holes has an average density lower than main sequence stars.

Supermassive black hole

A supermassive black hole (SMBH) is the largest type of black hole, though there are few official criteria on how such an object is considered so, on the order of hundreds of thousands to billions of solar masses. (Supermassive black holes up to 21 billion (2.1×10^{10}) $M_\odot$ have been detected, such as NGC 4889.)[10] Unlike stellar mass black holes, supermassive black holes have comparatively low average densities. (Note that a black hole is a spherical region in space that surrounds the singularity at its center; it is not the singularity itself.) With that in mind, the average density of a supermassive black hole can be less than the density of water.

The Schwarzschild radius of a body is proportional to its mass and therefore to its volume, assuming that the body has a constant mass-density.[11] In contrast, the physical radius of the body is proportional to the cube root of its volume. Therefore, as the body accumulates matter at a given fixed density (in this example, 997 kg/m^3, the density of water), its Schwarzschild radius will increase more quickly than its physical radius. When a body of this density has grown to around 136 million solar masses (1.36×10^8) $M_\odot$, its physical radius would be overtaken by its Schwarzschild radius, and thus it would form a supermassive black hole.

It is thought that supermassive black holes like these do not form immediately from the singular collapse of a cluster of stars. Instead they may begin life as smaller, stellar-sized black holes and grow larger by the accretion of matter, or even of other black holes.

The Schwarzschild radius of the supermassive black hole at the Galactic Center is approximately 12 million kilometres.[12]

Stellar black hole

Stellar black holes have much greater average densities than supermassive black holes. If one accumulates matter at nuclear density (the density of the nucleus of an atom, about 10^{18} kg/m^3; neutron stars also reach this density), such an accumulation would fall within its own Schwarzschild radius at about 3 $M_\odot$ and thus would be a stellar black hole.

Primordial black hole

A small mass has an extremely small Schwarzschild radius. A mass similar to Mount Everest[13][note 1] has a Schwarzschild radius much smaller than a nanometre.[note 2] Its average density at that size would be so high that no known mechanism could form such extremely compact objects. Such black holes might possibly be formed in an early stage of the evolution of the universe, just after th Big Bang, when densities were extremely high. Therefore, these hypothetical miniature black holes are called primordial black holes.

Other uses

In gravitational time dilation

Gravitational time dilation near a large, slowly rotating, nearly spherical body, such as the Earth or Sun can be reasonably approximated using the Schwarzschild radius as follows:

$$\frac{t_r}{t} = \sqrt{1 - \frac{r_s}{r}}$$

where:

- t_r is the elapsed time for an observer at radial coordinate r within the gravitational field;
- t is the elapsed time for an observer distant from the massive object (and therefore outside of the gravitational field);
- r is the radial coordinate of the observer (which is analogous to the classical distance from the center of the object);
- r_s is the Schwarzschild radius.

The results of the Pound–Rebka experiment in 1959 were found to be consistent with predictions made by general relativity. By measuring Earth's gravitational time dilation, this experiment indirectly measured Earth's Schwarzschild radius.

In Newtonian gravitational fields

The Newtonian gravitational field near a large, slowly rotating, nearly spherical body can be reasonably approximated using the Schwarzschild radius as follows:

$$mg = \frac{GMm}{r^2} \quad \Rightarrow \quad gr^2 = GM$$

and

$$r_\mathrm{s} = \frac{2GM}{c^2} \quad \Rightarrow \quad r_\mathrm{s}c^2 = 2GM$$

Therefore, on dividing above by below:

$$\frac{g}{r_\mathrm{s}}\left(\frac{r}{c}\right)^2 = \frac{1}{2}$$

where:

- g is the gravitational acceleration at radial coordinate r;
- r_s is the Schwarzschild radius of the gravitating central body;
- r is the radial coordinate;
- c is the speed of light in vacuum.

On the surface of the Earth:

$$\frac{9.80665\ \mathrm{m/s^2}}{8.870056\ \mathrm{mm}}\left(\frac{6375416\ \mathrm{m}}{299792458\ \mathrm{m/s}}\right)^2 = \left(1105.59\ \mathrm{s^{-2}}\right)\left(0.0212661\ \mathrm{s}\right)^2 = \frac{1}{2}.$$

In Keplerian orbits

For all circular orbits around a given central body:

$$\frac{mv^2}{r} = \text{Centripetal force} = \text{Gravitational force} = \frac{GMm}{r^2}$$

Therefore,

$$rv^2 = GM,$$

but

$r_s c^2 = 2GM$ (derived above)

Therefore,

$$\frac{r}{r_s}\left(\frac{v}{c}\right)^2 = \frac{1}{2}$$

where:

- r is the orbit radius;
- r_S is the Schwarzschild radius of the gravitating central body;
- v is the orbital speed;
- c is the speed of light in vacuum.

This equality can be generalized to elliptic orbits as follows:

$$\frac{a}{r_s}\left(\frac{2\pi a}{cT}\right)^2 = \frac{1}{2}$$

where:

- a is the semi-major axis;
- T is the orbital period.

For the Earth, as a planet orbiting the Sun:

$$\frac{1\,\mathrm{AU}}{2953.25\,\mathrm{m}}\left(\frac{2\pi\,\mathrm{AU}}{\text{light year}}\right)^2 = (50655379.7)\left(9.8714403\times 10^{-9}\right) = \frac{1}{2}.$$

Relativistic circular orbits and the photon sphere

The Keplerian equation for circular orbits can be generalized to the relativistic equation for circular orbits by accounting for time dilation in the velocity term:

$$\frac{r}{r_s}\left(\frac{v}{c}\sqrt{1-\frac{r_s}{r}}\right)^2 = \frac{1}{2}$$

$$\frac{r}{r_s}\left(\frac{v}{c}\right)^2\left(1-\frac{r_s}{r}\right) = \frac{1}{2}$$

$$\left(\frac{v}{c}\right)^2\left(\frac{r}{r_s}-1\right) = \frac{1}{2}.$$

This final equation indicates that an object orbiting at the speed of light would have an orbital radius of 1.5 times the Schwarzschild radius. This is a special orbit known as the photon sphere.

Schwarzschild radius for Planck mass

For the Planck mass $m_\mathrm{P} = \sqrt{\hbar c/G}$, the Schwarzschild radius $r_\mathrm{S} = 2l_\mathrm{P}$ and the Compton wavelength $\lambda_\mathrm{C} = 2\pi l_\mathrm{P}$ are of the same order as the Planck length $l_\mathrm{P} = \sqrt{\hbar G/c^3}$.

See also

- Black hole, a general survey
- Chandrasekhar limit, a second requirement for black hole formation
- John Michell

Classification of black holes by type:

- Static or Schwarzschild black hole
- Rotating or Kerr black hole
- Charged black hole or Newman black hole and Kerr-Newman black hole

A classification of black holes by mass:

- Micro black hole and extra-dimensional black hole
- Primordial black hole, a hypothetical leftover of the Big Bang
- Stellar black hole, which could either be a static black hole or a rotating black hole
- Supermassive black hole, which could also either be a static black hole or a rotating black hole
- Visible universe, if its density is the critical density, as a hypothetical black hole

Notes

1. Using these values,[13] one can calculate a mass estimate of 6.3715e14 kg.
2. One can calculate the Schwarzschild radius: 2 × 6.6738e-11 m^3 kg^{-1} s^{-2} × 6.3715e14 kg / (299 792 458 m s^{-1})^2 = 9.46e-13 m, or 9.46e-4 nm.

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