

## Digital implementation of fractional order PID controller and its application\*

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**Abstract:** A new discretization scheme is proposed for the design of a fractional order PID controller. In the design of a fractional order controller the interest is mainly focused on the s-domain, but there exists a difficult problem in the s-domain that needs to be solved, i. e. how to calculate fractional derivatives and integrals efficiently and quickly. Our scheme adopts the time domain that is well suited for Z-transform analysis and digital implementation. The main idea of the scheme is based on the definition of Grünwald-Letnicov fractional calculus. In this case some limited terms of the definition are taken so that it is much easier and faster to calculate fractional derivatives and integrals in the time domain or z-domain without loss much of the precision. Its effectiveness is illustrated by discretization of half-order fractional differential and integral operators compared with that of the analytical scheme. An example of designing fractional order digital controllers is included for illustration, in which different fractional order PID controllers are designed for the control of a nonlinear dynamic system containing one of the four different kinds of nonlinear blocks: saturation, deadzone, hysteresis, and relay.

**Key words:** fractional calculus, fractional order PID controller, discretization, Z-transform, nonlinear system.

### 1. INTRODUCTION

Fractional calculus has a history of 300 years. It is a subject that involves the theory of integrals and derivatives of arbitrary order. For three centuries the theory of fractional calculus has developed mainly as a pure theoretical field of mathematics useful for mathematicians. But in recent years its applications have been seen in many fields such as, viscoelasticity, acoustics, rheology, polymeric chemistry, fractals, control, and many other fields<sup>[1~4]</sup>.

Applying fractional calculus in the control field is just a recent focus of interest. Matignon proved the stability results as well as the theory of the controllability and observability of finite-dimensional linear fractional differential systems<sup>[5~8]</sup>. Oustaloup first proposed the concept of fractional PID controller and successfully used it in the CRONE control<sup>[9~12]</sup>. Podlubny also put forth another example of fractional order controller  $P^{\alpha}I^{\beta}D^{\gamma}$ <sup>[3,13]</sup>, an extension of classical integer order PID controller. In fact a fractional order system can include a fractional order controller or fractional order controlled plant or both. However, in real applications system models may have been built by using traditional integer order differentiation, so

it's reasonable for us to focus our interest on how to design a fractional order controller.

There exists one difficult problem of using fractional calculus in system control, e. g., how to solve fractional order systems quickly and efficiently. Because it is much more complicated and time-consuming to solve fractional order systems than that of integer order systems, some researchers propose Laplace transform method which is very common in system control. However, this method is not good for digital implementation. It should be pointed out that a band-limit implementation of a fractional order system (FOS) is more important in practice, i. e., the finite-dimensional approximation of FOS should be done in a proper range of frequency of practical interest<sup>[10,14]</sup>. Here we present an easy and efficient scheme to avoid the problem mentioned above. Our scheme can be called the Truncated Grünwald-Letnicov Scheme(TGLS) for it is based on the definition of Grünwald-Letnicov fractional derivative/integration, which has the form of the limit of an infinite series. The idea is simple, i. e., using finite terms of the series to approximate fractional derivative, then we can design a controller directly in the time domain, which makes it well suited for Z-transform analysis and dig-

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ital implementation. One point we should emphasize is that in control applications, the Grünwald-Letnicov or Riemann-Liouville fractional derivative/integration definition are equivalent for a wide class of functions important for applications<sup>[3]</sup>. For this reason, this literature does not distinguish between them.

## 2. BASIC DEFINITIONS OF FRACTIONAL CALCULUS AND SOME OF ITS PROPERTIES

There are several different definitions of fractional differentiation. One of the most famous one is Riemann–Liouville fractional differentiation, which can be written as

$$_a D_t^p x(t) = \frac{1}{\Gamma(n-p)} \frac{d^n}{dt^n} \int_a^t (t-\tau)^{n-p-1} x(\tau) d\tau, \quad (n-1 \leq p < n) \quad (1)$$

Where  $\Gamma(\cdot)$  refers to Euler's gamma function. Its Laplace transform is

$$L\{_a D_t^p x(t)\} = s^p X(s) - \sum_{k=0}^{n-1} s^k [{}_a D_t^{p-k-1} x(t)]_{t=a} \quad (2)$$

where  $X(s)$  is the Laplace transform of  $x(t)$ .

The definition of Riemann-Liouville fractional integration can be written as

$$_a D_t^{-p} x(t) = \frac{1}{\Gamma(p)} \int_a^t (t-\tau)^{p-1} x(\tau) d\tau, \quad p > 0 \quad (3)$$

Its Laplace transform is

$$L\{_a D_t^{-p} x(t)\} = s^{-p} X(s) \quad (4)$$

Grünwald-Letnicov fractional differentiation/integral definition has the form

$$D^p x(t) = \lim_{h \rightarrow 0} \left[ \frac{1}{h^p} \sum_{k=0}^{\infty} w_k^p x(t-kh) \right] \quad (5)$$

$$w_k^p = \frac{(-1)^k \Gamma(p+1)}{k! \Gamma(p-k+1)} \quad (6)$$

When  $p > 0$ , Eq. (5) is called Grünwald-Letnicov fractional differentiation; When  $p < 0$ , it is called Grünwald-Letnicov fractional integration.

The Riemann-Liouville fractional differentiation/integration of the power function  $x(t)$ ,  $x(t) = (t-a)^q$ , and  $q$  being real, is

$$_a D_t^p t^q = \frac{\Gamma(q+1)(t-a)^{q-p}}{\Gamma(q+1-p)} \quad (7)$$

## 3. FRACTIONAL ORDER PID CONTROLLER

Fractional order PID controller or  $PI^\alpha D^\beta$ -controller is a generalization of the classical PID-controller because it involves an integrator of order  $\alpha$  and differentiator

of order  $\beta$ , both of which can be real. The transfer function of such a controller has the form

$$G_c(s) = \frac{U(s)}{E(s)} = K_p + \frac{K_1}{s^\alpha} + K_D s^\beta \quad (\alpha, \beta > 0) \quad (8)$$

The equation for the fractional order  $PI^\alpha D^\beta$ -controller's output in the time domain is

$$u(t) = K_p e(t) + K_1 D^{-\alpha} e(t) + K_D D^\beta e(t) \quad (9)$$

$D^{-\alpha}$  and  $D^\beta$  can refer to Riemann-Liouville or Grünwald-Letnicov fractional integral and fractional differential operator respectively. Taking  $\alpha = 1$  and  $\beta = 1$ , we obtain a classical PID-controller.  $\alpha = 1$  and  $\beta = 0$  give a PI-controller.  $\alpha = 0$  and  $\beta = 1$  give a PD-controller.  $\alpha = 0$  and  $\beta = 0$  give a gain. All these classical types of PID-controllers are particular cases of the fractional  $PI^\alpha D^\beta$ -controller. However, the latter is more flexible and gives an opportunity to better adjust the dynamical properties of a fractional order control system.

## 4. TWO SCHEMES OF DISCRETIZATION OF FRACTIONAL DERIVATIVES AND INTEGRALS: LIS and TGLS

### 4.1 LIS

Interpolating  $x(k-1)$  and  $x(k)$  in the interval  $0 \leq t \leq T$  results

$$x(t) = [x(k) - x(k-1)] \frac{t}{T} + x(k-1) \quad (10)$$

From formula (7) one can obtain its fractional integral of order  $\alpha$  with the result

$$D^{-\alpha} x(t) = \frac{x(k) - x(k-1)}{\Gamma(2+\alpha)} \cdot \frac{t^{\alpha+1}}{T} + \frac{x(k-1)}{\Gamma(1+\alpha)} t^\alpha \quad (11)$$

For  $t = T$  the time and z-domain formulae are

$$D^{-\alpha} x = \frac{T^\alpha}{\Gamma(2+\alpha)} [x(k) + \alpha \cdot x(k-1)] \quad (12)$$

$$Z\{D^{-\alpha} x\} = \frac{T^\alpha}{\Gamma(2+\alpha)} [1 + \alpha \cdot z^{-1}] \cdot X(z) \quad (13)$$

For  $\alpha = -1, 0, 1$  these expressions correspond to the differential (D), proportional(P) and integral(I) actions respectively, and the following holds

$$H_D(z) = \frac{1}{T}(1 - z^{-1}), H_P(z) = 1, \\ H_I(z) = \frac{T}{2}(1 + z^{-1}) \quad (14)$$

Then Eq. (13) can be considered as the sum of PD

actions with gains  $K_p, K_D$  or the sum of PI actions with gains  $K_p, K_I$  respectively. For these two cases we obtain

$$Z\{D^{-\alpha}x\} = K_p \cdot H_p(z) + K_D \cdot H_D(z) \quad (15)$$

$$Z\{D^{-\alpha}x\} = K_p \cdot H_p(z) + K_I \cdot H_I(z) \quad (16)$$

which lead to

$$K_p = \frac{(1+\alpha)T^\alpha}{\Gamma(2+\alpha)}, K_D = -\frac{\alpha T^{\alpha+1}}{\Gamma(2+\alpha)} \quad (17)$$

$$K_p = \frac{(1-\alpha)T^\alpha}{\Gamma(2+\alpha)}, K_I = \frac{2\alpha T^{\alpha-1}}{\Gamma(2+\alpha)} \quad (18)$$

Clearly if  $-1 < \alpha < 0$  keeps, then  $K_D > 0, K_I < 0$  holds and if  $0 < \alpha < 1$  keeps, then  $K_D < 0, K_I > 0$  holds. Hence, we can approximate fractional order PID controllers by classical PD or PI controllers. Actually LIS can be regarded as a special case of TGLS, see 4.2.

#### 4.2 TGLS

TGLS is based on the definition of Grünwald-Letnikov fractional differentiation/integration, see formula (5) and (6). Therefore, for a discrete-time control algorithm with sampling period  $T$ , formula (5) can be approximated through a  $n$ -term truncated series, resulting the following equations in the discrete-time and z-domains

$$D^p x(nT) \approx \frac{1}{T^p} \sum_{k=0}^n w_k^p x(nT - kh) \quad (19)$$

$$Z(D^p x(nT)) \approx \left\{ \frac{1}{T^p} \sum_{k=0}^n w_k^p z^{-k} \right\} X(z) \quad (20)$$

Where  $X(z)$  is the Z-transform of  $x(nT)$ . In fact formula (5) has only a precision of  $O(h)^{[3]}$ . For higher order approximations one can refer to Ref. [15]. Clearly LIS can be regarded as a special case of TGLS when  $n = 1$ , in this case fractional order PID controller has almost the same structure of a classical

PD or PI controller. The approximate  $z$ -transform of Eq. (9) is

$$U(z) \approx K_p E(z) + K_I T^\alpha \sum_{k=0}^n w_k^{-\alpha} z^{-k} E(z) + \\ K_D T^{-\beta} \sum_{k=0}^n w_k^\beta z^{-k} E(z) \quad (21)$$

We can also define a generating function  $\omega^p(z^{-1})$ , whose coefficients of Taylor expansion at zero coincides with  $w_k^p$ , see formula (6). The following holds

$$\omega^p(z^{-1}) = (1 - z^{-1})^p = \\ \lim_{n \rightarrow \infty} \sum_{k=0}^n \frac{(-1)^k \Gamma(p+1)}{k! \Gamma(p-k+1)} z^{-k} = \lim_{n \rightarrow \infty} \sum_{k=0}^n w_k^p z^{-k} \quad (22)$$

As an example, now we discretize the half-order differential operator  $s^{0.5}$  and the half-order integral operator  $s^{-0.5}$ , both sampled at 0.001s, and both using the coefficients of the generating function  $\omega^{0.5}(z^{-1})$  and  $\omega^{-0.5}(z^{-1})$  respectively. Assuming  $G_{7d}(z)$  and  $G_{7i}(z)$  are the digital implementation of operator  $s^{0.5}$  and  $s^{-0.5}$  respectively. First we can easily obtain the weight coefficients  $w_k^{0.5}$  and  $w_k^{-0.5}$  from formula (22) with  $p$  replaced by 0.5 and -0.5 respectively. Then assuming  $n = 7$ , we can get the first seven terms of  $G_{7d}(z)$  and  $G_{7i}(z)$ , see the terms in the brace of formula (20).

$$G_{7d}(z) = 31.6228 - 15.8114z^{-1} - 3.9528z^{-2} - \\ 1.9764z^{-3} - 1.2353z^{-4} - 0.8647z^{-5} - \\ 0.6485z^{-6} - 0.5095z^{-7} \quad (23)$$

$$G_{7i}(z) = 0.0316 + 0.0158z^{-1} + 0.0119z^{-2} + \\ 0.0099z^{-3} + 0.0086z^{-4} + 0.0078z^{-5} + \\ 0.0071z^{-6} + 0.0066z^{-7} \quad (24)$$

Figures 1 and 2 present the continuous Bode diagrams and two different discrete Bode diagrams of  $s^{0.5}$  and  $s^{-0.5}$  respectively.

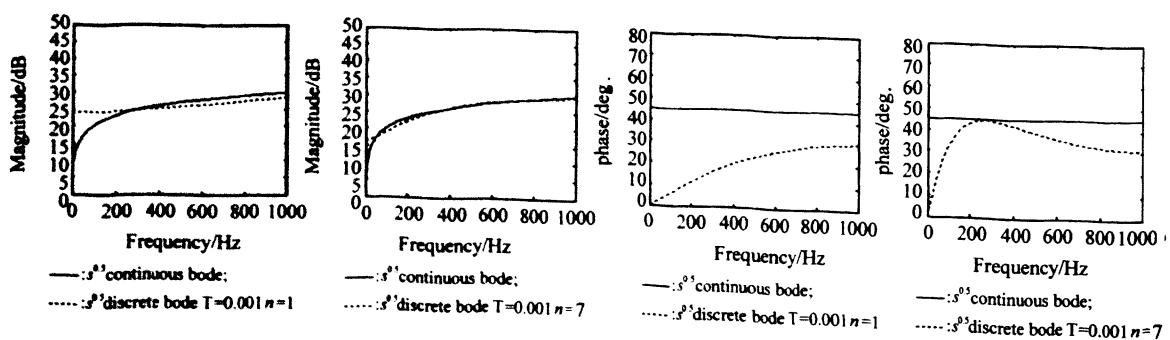
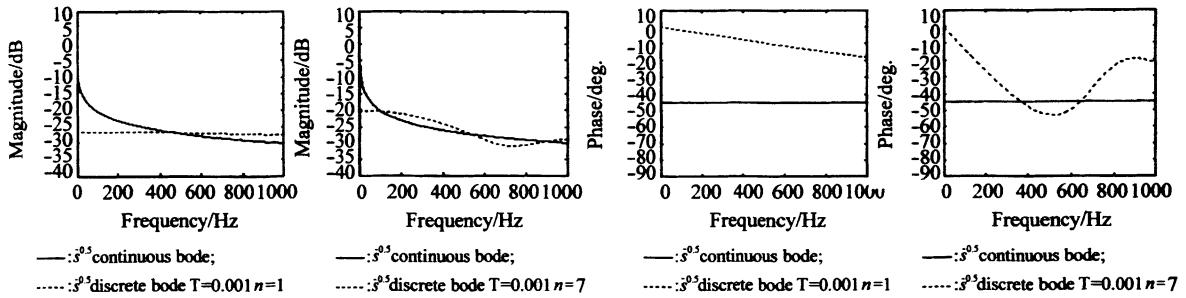


Fig. 1 Continuous and discrete Bode diagrams of  $s^{0.5}$  at  $T=0.001s$

Fig. 2 Continuous and discrete Bode diagrams of  $s^{-0.5}$  at  $T = 0.001$  s

Both Figs. show clearly that with the increase of the approximate terms, both the discrete logarithmic magnitude properties and the discrete phase properties tend to that of the continuous Bode diagrams. One can also compensate the phase error in the low frequency or high frequency range of the discretized transfer function via multiplying it by the compensator  $z^{0.5r}$  ( $r$  is a parameter determined by experiments). Clearly  $z^{0.5r}$  does not affect the magnitude fit of the discretized transfer function. From Figs. 1 and 2, one can find when a large number of terms are taken, TGLS will have a much higher approximate precision than that of LIS.

## 5. SIMULATION EXAMPLE

We now present an example to design suitable fractional order digital PID controllers for a given control system by the discretization scheme discussed in section 4. Figure 3 shows the architecture of the whole closed-loop system. Figure 4 depicts the four different characteristics of the nonlinear block in Fig. 3: saturation, deadzone, hysteresis, and relay.

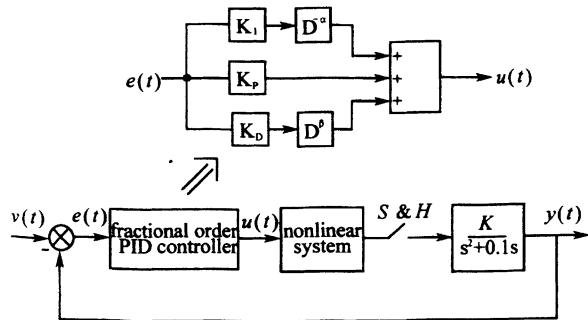


Fig. 3 The architecture of the closed-loop control system

The transfer function of the fractional order PID controller has the form of formula (9), and we use formula (21) for its digital implementation. In all cases we assume that the initial values of  $K_p$ ,  $K_I$ ,  $K_D$ ,  $\alpha$  and  $\beta$

remain the same with  $K_p = 2$ ,  $K_I = 0.1$ ,  $K_D = 3.5$ ,  $\alpha = 0.5$ ,  $\beta = 0.5$ . In all cases we also assume the sample and hold time  $T = 0.1$  s, gain  $K = 1$ , and unit step signal as the input signal.

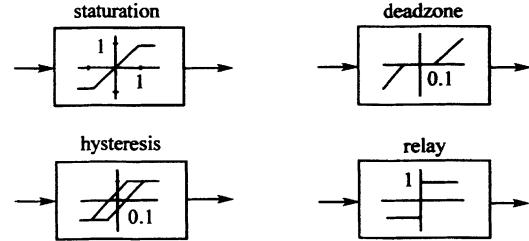
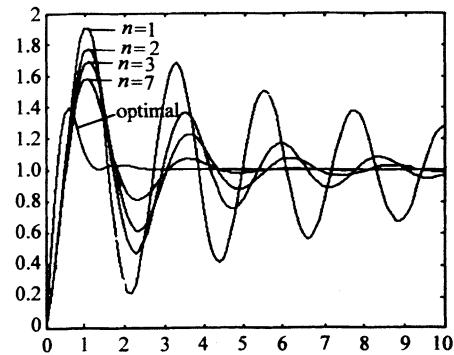


Fig. 4 The four different nonlinear phenomena of the nonlinear system block: saturation, deadzone, hysteresis, and relay

Figure 5 shows the linear system response (i.e. without the nonlinear block) for a unit step input and  $n$ -th order ( $1 \leq n \leq 7$ ) approximation to the fractional order PID controller. It is obvious the higher the order of the approximation the better the response.

Fig. 5 Linear system response (i.e. without the nonlinear block) for a  $n$ -th order series approximation to the fractional order PID controller and its optimal system response with  $n = 7$  and  $K_p = 2$ ,  $K_I = 0.1$ ,  $K_D = 3.5$ ,  $\alpha = 0.5$ , and  $\beta = 0.5$ , as the initial values

From section 4 we know that when the order  $n = 1$ , the fractional order PID controller can be regarded as a classical PD controller. In this case the TGLS is much better than classical PD like scheme if enough terms are taken.

The robustness of the TGLS over classical PD like scheme is highlighted in the presence of a nonlinear phenomenon. Figure. 6 shows the PD like scheme is much more sensitive to the saturation effect than that of TGLS. The same is true when we look at Figs. 7,8 and 9. Hence the 7-th order approximations to the fractional order PID controllers are robust for a large range of nonlinear phenomena.

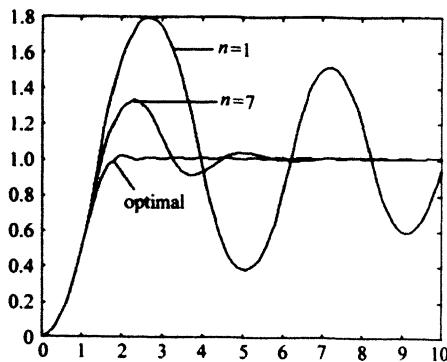


Fig. 6 System response with saturation as a nonlinear block for a  $n$ -th order series approximation to the fractional order PID controller and its optimal system response with  $n = 7$  and  $K_P = 2, K_I = 0.1, K_D = 3.5, \alpha = 0.5$ , and  $\beta = 0.5$ , as the initial values.

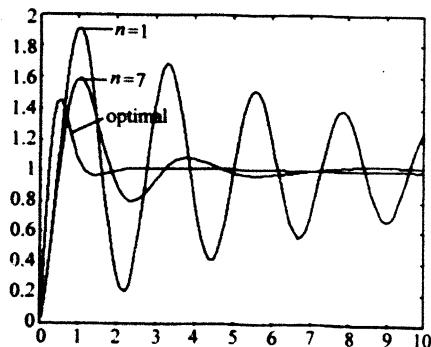


Fig. 7 System response with deadzone as a nonlinear block for a  $n$ -th order series approximation to the fractional order PID controller and its optimal system response with  $n = 7$  and  $K_P = 2, K_I = 0.1, K_D = 3.5, \alpha = 0.5$ , and  $\beta = 0.5$ , as the initial values

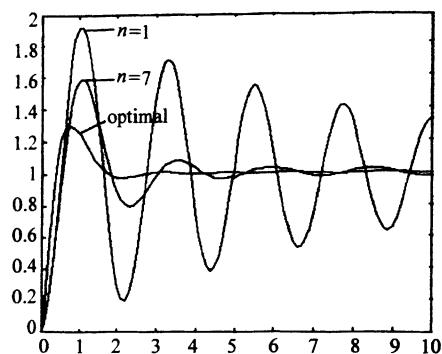


Fig. 8 System response with hysteresis as a nonlinear block for a  $n$ -th order series approximation to the fractional order PID controller and its optimal system response with  $n = 7$  and  $K_P = 2, K_I = 0.1, K_D = 3.5, \alpha = 0.5$ , and  $\beta = 0.5$ , as the initial values

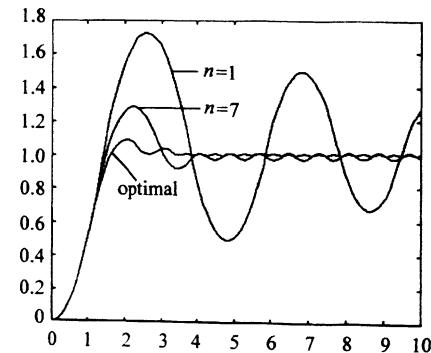


Fig. 9 System response with relay as a nonlinear block for a  $n$ -th order series approximation to the fractional order PID controller and its optimal system response with  $n = 7$  and  $K_P = 2, K_I = 0.1, K_D = 3.5, \alpha = 0.5$ , and  $\beta = 0.5$ , as the initial values

We can also choose the most suitable parameters, such as  $K_P, K_I, K_D, \alpha$  and  $\beta$ , by minimizing the ItAE performance criterion, which is defined by function  $Q$

$$Q = \int_0^t |e(t)| dt \quad (25)$$

where  $e(t)$  is the error function with respect to time, see Fig. 3. In this example we choose difference Quasi-Newton method as the optimization scheme<sup>[16]</sup>. During the process of the optimization we select 7-th order approximations to the fractional order PID controllers. Table 1 gives some of the system's performance criteria of both before and after the optimization. All the unit step response plots af-

ter the optimization are also plotted in Figs. 5, 6, 7, 8, and 9. Clearly the dynamic performances of the system after the optimization are much better than that before the optimization. One point we should emphasize is that in this case the optimization process

is strongly dependent on the initial values of such parameters as  $K_P$ ,  $K_I$ ,  $K_D$ ,  $\alpha$  and  $\beta$ . When one chooses these parameters with different initial values, the optimization outcome is also different, i. e. this minimization problem is local.

**Table 1** The system's performance criteria of both before and after the optimization with the 7-th order approximations to the fractional order PID controllers

		$K_P$	$K_I$	$K_D$	$\alpha$	$\beta$	$Q$	$\sigma\%$	$t_s/\text{sec}$
linear	initial	2	0.1	3.5	0.5	0.5	1.277	57	2.30
	final	3.392	0.489	5.665	0.424	0.757	0.199	39	0.89
saturation	initial	2	0.1	3.5	0.5	0.5	1.864	32	2.82
	final	7.478	4.999	15.495	0.669	0.752	0.603	2	1.37
deadzone	initial	2	0.1	3.5	0.5	0.5	2.127	58	2.50
	final	4.813	3.630	5.906	1.000	0.794	0.388	44	0.87
hysteresis	initial	2	0.1	3.5	0.5	0.5	1.753	58	2.31
	final	4.455	1.853	3.403	0.613	1.000	0.385	29	1.10
relay	initial	2	0.1	3.5	0.5	0.5	1.96	29	2.69
	final	3.214	0.324	5.526	0.438	0.653	0.863	9	1.34

## 6. CONCLUSION

In the field of automatic control fractional order PID controller is a new tool in the area of system design and control. Not like those schemes mentioned by Oustaloup and Podlubny, which restrict their applications to the frequency domain, our scheme adopts the time domain, which makes it well suited for Z-transform analysis and digital implementation. Illustrative examples show that our scheme is especially suitable for the design and digital implementation of a fractional order PID controller. This makes a big step towards its applications in the control applications. Future work will include finding higher precision approximations to fractional order PID controllers.

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