

# A Compact Introduction to Fractional Calculus

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## Contents

### 1 Fractional Derivatives

Fractional calculus (FC) is now an efficient tool for problems in science and engineering [?, ?, ?, ?, ?, ?, ?, ?]. The term "fractional" is kept for the historical reasons — it is a misnomer since the order can be real [?, ?].

Historical survey of the development of FC starting from the letter by Gottfried Leibniz to Guillaume l'Hôpital (1695) including contributions by Joseph Liouville, Bernhard Riemann, Niels Abel, Grünwald, Aleksey Letnikov, Gerasimov, Marcel Riesz, Magnus Mittag-Leffler, Paul Lévy, Raoul Nigmatullin, Yuri Rabotnov, Arthur Erdélyi and others during the XIX and XX centuries could be found in refs. [?, ?, ?, ?, ?].

Lorenzo & Hartley [?] analysed the minimal set of criteria for the generalized calculus formulated by B. Ross: *analyticity*: if  $f(z)$  is an analytic function of the complex variable  $z$ , the derivative  $D_z^\nu f(z)$  is an analytic function of  $z$  and  $\nu$ ; *backward compatibility*: the operation  $D_z^\nu f(z)$  must produce the same result as ordinary differentiation when  $\nu = n$  is a positive integer; the operation  $D_z^\nu f(z)$  must produce the same result as ordinary  $n$ -fold integration when  $n$  is a negative integer;  $D_z^\nu f(z)$  must vanish along with its  $n - 1$  derivatives at  $x = c$ ; *zero property*: the operation of order zero leaves the function unchanged  $D_z^0 f(z) = f(x)$ ; *linearity*: the fractional operators must be linear  $D_z^\nu [af(x) + bg(x)] = aD_z^\nu f(x) + D_z^\nu g(x)$ ; *composition (index law)*: the law of exponents for integration of arbitrary order  $D_z^\nu D_z^\mu f(x) = D_z^{\nu+\mu} f(x)$ .

The fractional derivatives are based on the extension of the repeated integration and are defined either by the continuation of the fractional integral to the negative order or by the integer order derivatives of the fractional integrals [?]. There is no unique definition [?, ?, ?] (and notation is not standardized [?, ?]).

There are several kinds of definitions of the fractional derivatives (Riemann-Liouville, Caputo, Grünfeld-Letnikov, Riesz, Weyl, Marchaud, Caputo-Fabrizio, Yang, Chen, He and others) that are not equivalent [?, ?]. The initial conditions for the Caputo derivative are expressed in terms of the initial values of the integer order derivatives; for

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the zero initial conditions Riemann-Liouville, Caputo and Grünwald-Letnikov derivatives coincide [?]. Most of the fractional derivatives are defined through the fractional integral thus derivatives inherent some non-local behaviour [?]. The following relation is valid for all types of the fractional derivatives [?]

$$\frac{d^{\alpha+\beta}}{dt^{\alpha+\beta}} = \frac{d^\alpha}{dt^\alpha} \frac{d^\beta}{dt^\beta} = \frac{d^\beta}{dt^\beta} \frac{d^\alpha}{dt^\alpha}.$$

The most frequently used are the Riemann-Liouville (e.g., in the calculus), the Caputo (e.g., in the physics, the numerical computations) and Grünwald-Letnikov (e.g., in the signal processing, the engineering) fractional derivatives [?, ?].

Grigoletto & de Oliveira [?] considered the generalization of the fundamental theorem of calculus — Fundamental Theorem of Fractional Calculus (FTFC) for the cases of the Riemann-Liouville, Liouville, Caputo, Weyl and Riesz derivatives.

Baleanu & Fernandez [?] considered the possible classification of the fractional operators into broad classes under some restrictions and criteria. In particularity, many operators could be considered as the special cases of [?]

$${}_c^A I_x^{\alpha,\beta} \int_c^x (x-t)^{\alpha-1} A((x-t)^\beta) f(t) dt, \quad A(z) = \sum_{k=0}^{\infty} a_k z^k$$

where  $c$  is a constant often taken as zero or  $\infty$ ,  $\alpha$  and  $\beta$  are complex parameters with positive real parts, and  $A(z)$  is a general analytic function whose coefficients  $a_k \in C$  are permitted to depend on  $\alpha$  and  $\beta$ ,  $x$  as a real variable larger than  $c$ .

## 1.1 Riemann-Liouville Fractional Integral

Both the Riemann-Liouville (RL) and the Caputo fractional derivatives are based on the RL fractional integral that for any  $\alpha > 0$  is defined as [?, ?]

$$J_{a^+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt, \quad (1)$$

If  $\alpha = 0$ ,  $J_{a^+}^0 = I$ ,  $I$  is the identity operator. Here

$$\Gamma(\alpha) = \int_0^\infty \exp(-u) u^{\alpha-1} du$$

is the Euler Gamma function. This integral exists if  $f(t)$  is the locally integrable function and for  $t \rightarrow 0$  behaves like  $O(t^{-\nu})$  with  $\nu < \alpha$ . To get the strict mathematical rigor it is possible to use the framework of the Lebesgue spaces of the summable functions or the Sobolev spaces of the generalized functions [?].

The RL integral is a generalization of Cauchy's formula for an n-fold integral

$$\int_a^x dx_1 \int_a^{x_1} dx_2 \cdots \int_a^{x_{n-1}} dx_n = \frac{1}{(n-1)!} \int_a^x (x-t)^{n-1} dt$$

using the relation

$$(n-1)! = \prod_{k=1}^{n-1} k = \Gamma(n).$$

The equation (??) is *left-sided* RL integral. The *right-sided* RL integral is written as

$$J_{b^-}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t) dt. \quad (2)$$

The RL integral is a case of the convolution integral of the Volterra type [?]

$$K * f(x) = \int_a^b k(x-t) f(t) dt.$$

The RL integral has the *semi-group* property (also called *additivity law* [?]):

$$J_{a^+}^\alpha J_{a^+}^\beta f(x) = J_{a^+}^{\alpha+\beta} f(x), \quad \alpha > 0, \quad \beta > 0$$

which implies the *commutative* property [?]:  $J_{a^+}^\beta J_{a^+}^\alpha = J_{a^+}^\alpha J_{a^+}^\beta$ .

The RL fractional integral coincides with the classical definition in the case  $\alpha \in \mathcal{N}$ . The fractional integration improves the smoothness of functions [?].

Sometimes the RL integral could be expressed via the elementary functions, e.g.,

$$J_{a^+}^\alpha (x-a)^\mu = \frac{\Gamma(\mu+1)}{\Gamma(\alpha+\mu+1)} (x-a)^{\alpha+\mu}.$$

A particular case of the RL fractional integrals is the Liouville fractional integrals [?] that is obtained by transitions  $a \rightarrow -\infty$  and  $b \rightarrow \infty$  in equations (??) and (??) as

$$J_+^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^x (x-t)^{\alpha-1} f(t) dt, \quad J_-^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1} f(t) dt.$$

## 1.2 Riemann-Liouville Fractional Derivative

The left and the right Riemann-Liouville fractional derivatives are defined as [?]

$$D_{a^+}^\alpha [f(x)] = \begin{cases} \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x (x-t)^{-\alpha} f(t) dt, & \alpha \in (0, 1) \\ \frac{df(x)}{dt}, & \alpha = 1 \end{cases} \quad (3)$$

and

$$D_{b^-}^\alpha [f(x)] = \begin{cases} \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_x^b (t-x)^{-\alpha} f(t) dt, & \alpha \in (0, 1) \\ \frac{df(x)}{dt}, & \alpha = 1 \end{cases} \quad (4)$$

Operator  $D_{a^+}^\alpha$  is left-inverse meaning that  $D_{a^+}^\alpha J_{a^+}^\alpha = I$ ,  $I$  is the identity operator. Thus  $D_{a^+}^\alpha J_{a^+}^\alpha f = f$  but the unconditional semigroup property of fractional differentiation in the RL sense does not hold: Diethelm [?] gives examples where  $D_{a^+}^{\alpha_1} D_{a^+}^{\alpha_2} f = D_{a^+}^{\alpha_2} D_{a^+}^{\alpha_1} f \neq D_{a^+}^{\alpha_1 + \alpha_2} f$  and  $D_{a^+}^{\alpha_1} D_{a^+}^{\alpha_2} f \neq D_{a^+}^{\alpha_2} D_{a^+}^{\alpha_1} f = D_{a^+}^{\alpha_1 + \alpha_2} f$ .

Atangana & Secer [?] presented tables of the RL derivatives of the trigonometric and some special functions. The fractional RL derivative of the power function is

$$D_{a^+}^\alpha t^\nu = \frac{\Gamma(1+\nu)}{\Gamma(1+\nu-\alpha)} t^{\nu-\alpha}$$

and, particular, the derivative of a constant  $D_{a^+}^\alpha 1 = t^{-\alpha}/\Gamma(1-\alpha)$ .

Since the fractional RL derivative of a constant is not zero, thus the magnitude of the fractional derivative changes with adding of the constant.

Jumarie [?] suggested a modification to remove this drawback. He started with a fractional derivative (*F-derivative*)

$$f^\alpha(x) = \lim_{h \rightarrow 0} \frac{\Delta^\alpha f(x)}{h^\alpha}$$

based on the fractional difference  $\Delta^\alpha f(x)$  of order  $\alpha$ ,  $\alpha \in \mathbb{R}$ ,  $0 < \alpha \leq 1$ .

Jumarie proposed the modification of the fractional RL derivative

$$\frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x (x-t)^{-\alpha} (f(t) - f(0)) dt.$$

### 1.2.1 Leibniz' formula

The classical Leibnitz' formula for the first-order derivative (i.e. when  $n \in \mathbb{N}$ ) is

$$D^n[f(x)g(x)] = \sum_{k=0}^n \binom{n}{k} [D^k g(x) D^{n-k} f(x)]$$

where  $f(x)$  and  $g(x)$  are the  $n$ -time differentiable functions.

The fractional derivatives violate the classical Leibnitz' rule [?, ?]. Generalization of the Leibnitz' formula was developed by Osler [?, ?].

The Leibniz' formula for the differentiation of the product of the functions for the RL operators for the functions that are analytic on  $(a-h, a+h)$  is written as [?]

$$D_{a^+}^n [fg](x) = \sum_{k=0}^{\lfloor n \rfloor} \binom{n}{k} (D_{a^+}^k f)(x) (D_{a^+}^{n-k} g)(x) + \sum_{k=\lfloor n \rfloor + 1}^{\infty} \binom{n}{k} (D_{a^+}^k f)(x) (J_{a^+}^{n-k} g)(x).$$

where  $\lfloor \cdot \rfloor$  denotes the floor function. Jumarie studied the Leibniz' formula for the differentiation of the product of the non-differentiable functions [?].

### 1.2.2 Fa  di Bruno formula (the chain rule)

For the functions  $f$  and  $g$  with a sufficient number of the derivatives and  $n \in \mathbb{N}$  [?, ?, ?]

$$D^n[g(f(\cdot))](x) = \sum (D^k g) \prod_{m=1}^n (D^m f(x))^{b_m}$$

where the sum is over all partitions of  $\{1, 2, \dots, n\}$  and for each partition  $k$  is its number of the blocks and  $b_j$  is the number of the blocks with exactly  $j$  elements.

Tarasov [?] analysed the simplified chain rules suggested by Jumarie [?, ?, ?] and found that these simplifications are not universally valid.

### 1.2.3 Fractional Taylor expansion

The fractional Taylor expansion is written as [?, ?, ?]

$$\begin{aligned} f(x) &= \frac{(x-a)^{n-m}}{\Gamma(n-m+1)} \lim_{z \rightarrow a^+} J_a^{m-n} f(z) + \\ &\sum_{k=1}^{m-1} \frac{(x-a)^{k+n-m}}{\Gamma(k+n-m+1)} \lim_{z \rightarrow a^+} D^k J_a^{m-n} f(z) + J_a^n D_a^n f(x). \end{aligned}$$

### 1.2.4 Symmetrised space derivative

Vermeersch & Shakouri [?] formulated the symmetrised space derivatives of the fractional order between 1 and 2 and between 0 and 1:

- $1 < \alpha < 2$ . The symmetrised space derivative of the function  $g(x)$  that is integrable over the entire real axis is

$$\frac{\partial^\alpha g}{\partial|x|^\alpha} = \frac{\partial}{\partial x} \left[ w_\alpha \star \frac{\partial g}{\partial x} \right]$$

where  $\star$  denotes the convolution and  $w_\alpha$  is an unknown kernel function with the Fourier image found to be  $W_\alpha = 1/|\xi|^{2-\alpha}$ . The Fourier inversion yields

$$w_\alpha = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\exp(j\xi x) d\xi}{|\xi|^{2-\alpha}} = \frac{|x|^{-(\alpha-1)}}{2\Gamma(2-\alpha) \cos[(2-\alpha)\frac{\pi}{2}]}.$$

Thus

$$\frac{\partial^\alpha g}{\partial|x|^\alpha} = \frac{1}{2\Gamma(2-\alpha) \cos[(2-\alpha)\frac{\pi}{2}]} \frac{\partial}{\partial x} \int_{-\infty}^{\infty} \frac{\frac{\partial g}{\partial x}(x') dx'}{|x-x'|^{\alpha-1}}.$$

- $0 < \alpha < 1$ . The symmetrised space derivative of the function  $g(x)$  is

$$\frac{\partial^\alpha g}{\partial|x|^\alpha} = w_\alpha \star \frac{\partial g}{\partial x}.$$

The Fourier image of the kernel function  $w_\alpha$  is  $W_\alpha = j \cdot \text{sgn}(\xi)/|\xi|^{1-\alpha}$ . Performing the Fourier inversion, the authors get finally

$$\frac{\partial^\alpha g}{\partial|x|^\alpha} = \frac{-1}{2\Gamma(1-\alpha)\cos[(1-\alpha)\frac{\pi}{2}]} \frac{\partial}{\partial x} \int_{-\infty}^{\infty} \frac{-\text{sgn}(x) \cdot \frac{\partial g}{\partial x}(x') dx'}{|x-x'|^\alpha}.$$

In the case  $\alpha = 1/2$  the fractional integrals and derivatives are also called *semi-integrals* and *semi-derivatives* [?].

### 1.3 Caputo Fractional Derivative

The fractional derivatives in the Caputo sense on the left ( ${}_C D_{a+}^\alpha$ ) and on the right ( ${}_C D_{b-}^\alpha$ ) are defined via the RL fractional integral [?]  $({}_C D_{a+}^\alpha f) = (J_{a+}^{n-\alpha} f^{(n)})(x)$  and  $(-1)^n ({}_C D_{b-}^\alpha f) = (J_{b-}^{n-\alpha} f^{(n)})(x)$ . It was introduced independently in 1948 by M. Caputo and by A.N. Gerasimov [?]; later by Dzherbashyan & Nersesian [?].

The major difference of the Caputo fractional derivative is that the derivative act first on the function and after the integral is evaluated while in the RL approach the derivative act on the integral.

The Caputo fractional derivative is defined as [?]

$$D_{\star}^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-x)^{-\alpha} \frac{df(x)}{dt} dt, & \alpha \in (0, 1) \\ \frac{df(x)}{dt}, & \alpha = 1 \end{cases} \quad (5)$$

The definition of the Caputo derivative (??) is more restrictive than of the RL one (??, ??) since it requires the absolute integrability of the derivative  $df(x)/dt$  [?].

The Caputo fractional derivative can be considered as the regularization in the time origin for the RL derivative [?]

$$D_{\star}^\alpha f(t) = D^\alpha f(t) - f(0^+) \frac{t^{-\alpha}}{\Gamma(1-\alpha)}$$

and satisfies the property of being zero when applied to a constant.

Yuan & Agrawal and Singh & Chatterjee suggested the alternative definitions of the Caputo fractional derivative [?]. The first approach is based on the introduction of the auxiliary bivariate function  $\phi : (0, \infty) \times [a, b] \rightarrow \mathcal{R}$  as

$$\phi(w, x) = (-1)^{\lfloor n \rfloor} \frac{2 \sin \pi n}{\pi} w^{2n-2\lceil n \rceil+1} \int_a^x f^{(\lceil n \rceil)}(\tau) e^{-(x-\tau)w^2} d\tau$$

where  $\lceil \cdot \rceil$  denote the ceiling function, and, finally

$$D_{\star a}^n f(x) = \int_0^\infty \phi(w, x) dw.$$

The second approach is based on expressing the fractional derivative of the given function in the form of the integral over  $(0, \infty)$  with the integrand that can be obtained as the solution of the first-order initial value problem

$$\frac{\partial \phi^*(w, x)}{\partial x} = -w^{\frac{1}{n-\lceil n \rceil-1}} \phi^*(w, x) + \frac{(-1)^{\lfloor n \rfloor} \sin \pi n}{\pi(n - \lceil n \rceil - 1)} f^{(\lceil n \rceil)}(x)$$

with the initial condition  $\phi^*(w, a) = 0$ . Thus

$$\begin{aligned} \phi^*(w, x) &= \frac{(-1)^{\lfloor n \rfloor} \sin \pi n}{\pi(n - \lceil n \rceil - 1)} \int_0^x f^{(\lceil n \rceil)}(\tau) \exp(-(x - \tau) w^{\frac{1}{n-\lceil n \rceil-1}}) d\tau \\ D_{\star a}^n f(x) &= \int_0^\infty \phi^*(w, x) dw. \end{aligned}$$

## 1.4 Matrix Approach

Operations of the fractional calculus can be expressed by matrix [?, ?]. E.g., the left-sided RL or Caputo derivative can be approximated in the nodes in the equidistant discretization net with the help of the upper triangular strip matrix  $B_n^{(\alpha)}$  as [?]

$$\begin{bmatrix} v_n^{(\alpha)} & v_{n-1}^{(\alpha)} & \dots & v_1^{(\alpha)} & v_0^{(\alpha)} \end{bmatrix}^T = B_n^{(\alpha)} \begin{bmatrix} v_n & v_{n-1} & \dots & v_1 & v_0 \end{bmatrix}^T$$

where

$$B_n^{(\alpha)} = \frac{1}{\tau^\alpha} \begin{bmatrix} \omega_0^{(\alpha)} & \omega_0^{(\alpha)} & \dots & \dots & \omega_{n-1}^{(\alpha)} & \omega_n^{(\alpha)} \\ 0 & \omega_0^{(\alpha)} & \omega_0^{(\alpha)} & \dots & \dots & \omega_{n-1}^{(\alpha)} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \dots & \omega_0^{(\alpha)} \end{bmatrix}$$

Similarly, the right-hand RL or Caputo fractional derivative can be approximated with the help of the corresponding lower triangular strip matrix.

## 1.5 Caputo & Fabrizio Fractional Derivatives

Caputo & Fabrizio [?, ?] proposed the fractional derivatives without the singular kernel [?] by replacing the kernel  $(t - \tau)^{-\alpha}$  with the function  $\exp(-\alpha/(1 - \alpha))$  that does not have singularity for  $t = \tau$  in the definition of the Caputo derivative and replacing the factor  $1/\Gamma(1 - \alpha)$  with  $M(\alpha)/(1 - \alpha)$ .

E.g., the fractional time derivative for  $\alpha \in [0, 1]$  and function  $f \in L^1(-\infty, b)$  is

$$\mathcal{D}_t^\alpha f(t) = \frac{\alpha M(\alpha)}{1 - \alpha} \int_{-\infty}^t (f(t) - f(\tau)) \exp\left[-\frac{\alpha(t - \tau)}{1 - \alpha}\right] d\tau$$

where  $M(\alpha)$  is a normalization function such as  $M(0) = M(1) = 1$ .

## 1.6 GC & GRL derivatives

Zhao & Luo [?] suggested to divide the fractional derivative with different — singular and non-singular — kernels (e.g., RL, Caputo, Caputo-Fabrizio, Atangana-Baleanu [?])<sup>1</sup> with the kernel

$$k(x, \alpha) = E_\alpha \left( -\frac{\alpha}{1-\alpha} x \right),$$

Atangana-Gomez [?] with the kernel

$$k(x, \alpha) = \exp \left( -\frac{\alpha}{1-\alpha} x^2 \right)$$

derivative with the stretched exponential kernel [?] (that is useful in the study of the water diffusion in the human brain using the magnetic resonance imaging [?])

$$k(x, \alpha) = \exp \left( -\frac{\alpha}{1-\alpha} x^\beta \right), \quad \beta > 0, \quad \beta \neq 1$$

into two classes — GC (general, Caputo sense) and GRL (general, RL) derivatives that obeys the the principles formulated by V. Volterra in his "general laws of heredity" [?]: the linearity principle, the invariance principle, the fading memory principle, the compatibility principle. The compatibility principle requires the validity of two limits:  $D_\alpha f(x) \rightarrow f(x)$  when  $\alpha \rightarrow 0$  and  $D_\alpha f(x) \rightarrow f'(x)$  when  $\alpha \rightarrow 1$ .

The principle of *nonlocality* was suggested by Tarasov [?].

### 1.6.1 GC derivatives

Zhao & Luo [?] defined the GC derivative by

$$D_{a,\alpha}^{GC} f(x) = N(\alpha) \int_a^x k(x-t, \alpha) \frac{df(t)}{dt} dt$$

The fading memory principle requires that the remote time and position has less effect:  $k(x-t, \alpha)$  decreases when  $x$  increases and  $k(x-t, \alpha) \rightarrow 0$  when  $x \rightarrow \infty$ .

The compatibility principle requires that  $N(\alpha)/k(x, \alpha) \rightarrow 1$  when  $\alpha \rightarrow 0$  and  $N(\alpha)/k(x, \alpha) \rightarrow \delta(x)$  when  $\alpha \rightarrow 1$ .

### 1.6.2 GRL derivatives

$$D_{a,\alpha}^{RL} f(x) = \frac{d}{dx} N(\alpha) \int_a^x k(x-t, \alpha) f(t) dt.$$

The restrictions on  $k(x-t, \alpha)$  and  $N(\alpha)$  are the same.

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<sup>1</sup>The equation with the Atangana-Baleanu operator is related to the derivatives of distributed order [?].

## 1.7 Marchaud-Hadamard Fractional Derivatives

Marchaud's approach is based on the analytic continuation of the fractional integrals to the negative orders using the Hadamard's finite parts of the divergent integrals (Hadamard's idea is to ignore the unbounded contribution to the integral and to assign the value of the remaining — finite — expression [?]).

The Marchaud fractional derivative with the lower limit  $a$  is

$$(M_{a+}^{\alpha}f)(x) = \frac{f(x)}{\Gamma(1-\alpha)(x-a)^{\alpha}} + \frac{\alpha}{\Gamma(1-\alpha)} \int_a^x \frac{f(x)-f(y)}{(x-y)^{\alpha+1}} dy$$

and with the upper limit  $b$  is

$$(M_{b-}^{\alpha}f)(x) = \frac{f(x)}{\Gamma(1-\alpha)(b-x)^{\alpha}} + \frac{\alpha}{\Gamma(1-\alpha)} \int_x^b \frac{f(x)-f(y)}{(x-y)^{\alpha+1}} dy.$$

Marchaud's method is to extend the RL integral to  $\alpha < 0$

$$(J_{+}^{-\alpha}f)(x) = \frac{1}{\Gamma(-\alpha)} \int_0^{\infty} y^{-\alpha-1} f(x-y) dy \quad (6)$$

and to subtract the divergent part of the integral in (??)

$$\int_{\epsilon}^{\infty} y^{-\alpha-1} f(x-y) dy = \frac{f(x)}{\alpha \epsilon^{\alpha}}$$

to get finally

$$(M_{+}^{\alpha}f)(x) = \lim_{\epsilon \rightarrow 0+} \frac{1}{\Gamma(-\alpha)} \int_{\epsilon}^{\infty} \frac{f(x)-f(y)}{y^{\alpha+1}} dy. \quad (7)$$

There are two approaches to extend the definition (??) to the case  $\alpha > 1$  [?]:

1. To apply (??) to the  $n$ th derivative  $d^n f / dx^n$  for  $n < \alpha < n+1$ .
2. To consider  $f(x-y) - f(x)$  as the first-order difference and to generalize to  $n$ th order difference (difference quotient)

$$(\Delta_h^n f)(x) = (\mathcal{I} - T_h)^n f(x) = \sum_{k=0}^n (-1)^k \binom{n}{k} f(x-kh) \quad (8)$$

where  $\mathcal{I}$  is the identity operator and  $T_h = f(x-h)$  is the translation operator.

Thus Marchaud fractional derivative for  $0 < \alpha < n$  is written as

$$(M_{+}^{\alpha}f)(x) = \lim_{\epsilon \rightarrow 0+} \frac{1}{C_{\alpha,n}} \int_{\epsilon}^{\infty} \frac{\Delta_y^n f(x)}{y^{\alpha+1}} dy$$

where

$$C_{\alpha,n} = \int_0^\infty \frac{(1-e^{-y})^n}{y^{\alpha+1}}.$$

## 1.8 Grünwald - Letnikov Derivative

The approach suggested independently by Grünwald in 1867 and Letnikov [?] in 1868 is based on the use the limits of the difference quotients (??) similar to the classical definition of the derivatives for  $n \in \mathcal{N}$ ,  $f \in C^n[a, b]$ ,  $a < x \leq b$

$$\tilde{D}^n f(x) = \lim_{h \rightarrow 0} \frac{\Delta_h^n f(x)}{h^n}$$

and extension to the case of the arbitrary  $n$ .

Since  $\binom{n}{k} = 0$  if  $n \in \mathcal{N}$  and  $n < k$  the expression (??) is equivalent to

$$(\Delta_h^n f)(x) = \sum_{k=0}^{\infty} (-1)^k \binom{n}{k} f(x - kh). \quad (9)$$

The series (??) is uniformly convergent for any bounded function if  $n > 0$  [?].

The use of (??) introduce two problems [?]: the function  $f$  needs to be defined on  $(\infty, b]$ ; the function  $f$  should be such that the series converges.

These problems are solved by the introduction a new function  $f^*$

$$f^* = \begin{cases} f(x) & x \in [a, b] \\ 0 & x \in (-\infty, a) \end{cases}$$

that is used instead of the original  $f$ .

It is also assumed that in the tending to zero  $h$  takes only the Grünwald-Letnikov fractional derivative of order  $n$  defined as

$$\tilde{D}_a^n = \lim_{N \rightarrow \infty} \frac{\Delta_{h_N}^n f(x)}{h_N^n} = \lim_{N \rightarrow \infty} \sum_{k=0}^N (-1)^k \binom{n}{k} f(x - kh_N). \quad (10)$$

The Grünwald-Letnikov derivative is called *pointwise* or *strong* depending on whether the limit is taken pointwise or in the norm of a suitable Banach space [?].

The binomial coefficient can be generalized to the non-integer arguments

$$(-1)^j \binom{q}{j} = (-1)^j \frac{\Gamma(q+1)}{\Gamma(j+1)\Gamma(q-j+1)} = \frac{\Gamma(j-q)}{\Gamma(-q)\Gamma(j+1)}.$$

The (left-sided) Grünwald-Letnikov derivative could be written as ( $nh = x - a$ )

$$\tilde{D}_a^\alpha f(x) = \lim_{h \rightarrow 0} \frac{1}{h^\alpha} \sum_{k=0}^{\lfloor n \rfloor} (-1)^k \frac{\Gamma(\alpha+1)f(x-kh)}{\Gamma(k+1)\Gamma(\alpha-k+1)}$$

and right-sided ( $nh = b - x$ ) as

$$\tilde{D}_b^\alpha f(x) = \lim_{h \rightarrow 0} \frac{1}{h^\alpha} \sum_{k=0}^{\lfloor n \rfloor} (-1)^k \frac{\Gamma(\alpha+1)f(x+kh)}{\Gamma(k+1)\Gamma(\alpha-k+1)}.$$

The Grünwald-Letnikov integral of the order  $n$  of the function  $f$  is written as

$$\tilde{J}_a^n f(x) = \frac{1}{\Gamma(n)} \lim_{N \rightarrow \infty} h_N^n \sum_{k=0}^N \frac{\Gamma(n+k)}{\Gamma(k+1)} f(x - kh_N).$$

## 1.9 Riesz Fractional Operators

The fractional integral of the order  $\alpha$  in the Riesz sense (also known as the Riesz potential) is defined by the Fourier convolution product

$$(\mathcal{I}^\alpha f)(x) = \int_{\mathbb{R}^n} K_\alpha(x - \xi) f(\xi) d\xi,$$

where  $\operatorname{Re}(\alpha) > 0$ . The Riesz kernel

$$K_\alpha = \frac{1}{\gamma_n(\alpha)} \begin{cases} \|x\|^{\alpha-n}, & \alpha - n \neq 0, 2, \dots, \\ \|x\|^{\alpha-n} \ln\left(\frac{1}{\|x\|}\right), & \alpha - n = 0, 2, \dots \end{cases}$$

where  $\gamma_n(\alpha)$  is defined by

$$\frac{\gamma_n(\alpha)}{2^\alpha \pi^{\frac{n}{2}} \Gamma(\alpha/2)} = \begin{cases} \left[ \Gamma\left(\frac{n-\alpha}{2}\right) \right]^{-1}, & \alpha - n \neq 0, 2, \dots, \\ (-1)^{\frac{n-\alpha}{2}} 2^{-1} \Gamma\left(\frac{\alpha-n}{2}\right), & \alpha - n = 0, 2, \dots \end{cases}$$

The Riesz fractional integral is [?]

$$(\mathcal{I}^\alpha f)(x) = \frac{\Gamma\left(\frac{1-\alpha}{2}\right)}{2^\alpha \pi^{\frac{n}{2}} \Gamma(\alpha/2)} \int_{-\infty}^{\infty} f(\xi) |x - \xi|^{\alpha-1} d\xi.$$

The Riesz fractional derivative is [?]

$$D^\alpha [f(x)] = -\frac{1}{2 \cos(\alpha\pi/2)} \frac{1}{\Gamma(\alpha)}$$

$$\frac{d^n}{dx^n} \left[ \int_{-\infty}^x (x - \xi)^{n-\alpha_n-1} f(\xi) d\xi + \int_x^{\infty} (x - \xi)^{n-\alpha_n-1} f(\xi) d\xi \right].$$

The Riesz derivative is the generalization of the Laplace operator [?]

$$(-\Delta)^{\frac{\alpha}{2}} = -\frac{1}{2 \cos(\alpha\pi/2)} \left[ \frac{d^\alpha}{dx^\alpha} + \frac{d^\alpha}{d(-x)^\alpha} \right], \quad \alpha \neq 1.$$

The Riesz derivative could be expressed in terms of the Marchaud derivative

$$D^\alpha[f(x)] = -\frac{1}{2 \cos(\alpha\pi/2)} [(M_+^\alpha f)(x) + (M_-^\alpha f)].$$

The related Riesz-Feller derivative [?] has an additional parameter - "skewness"  $\theta$

$$D_\theta^\alpha f(x) = \frac{\Gamma(1+\alpha)}{\pi} \times \\ \left[ \sin\left[(\alpha+\theta)\frac{\pi}{2}\right] \int_0^\infty \frac{f(x+\xi)f(x)}{\xi^{1+\alpha}} d\xi + \sin\left[(\alpha-\theta)\frac{\pi}{2}\right] \int_0^\infty \frac{f(x-\xi)f(x)}{\xi^{1+\alpha}} d\xi \right].$$

The allowed region of the parameters  $\alpha$  and  $\theta$  turn out to be a diamond in the plane  $\{\alpha, \theta\}$  with the vertices in the points  $(0,0), (1,1), (2,0), (1,-1)$  called the "Feller-Takayasu diamond" [?, ?].

## 1.10 Weyl Fractional Derivative

The Weyl derivative is based on the generalization of the differentiation in the Fourier space [?] — the integer derivative of the  $n$ th order  $(ik)^n$  of the absolutely integrable function on  $[-\pi, \pi]$  presented as the Fourier series is extended to the noninteger  $n$ .

The Weyl fractional derivative is defined as [?]

$$D_\pm^\alpha = \begin{cases} \pm \frac{d}{dx} [I_\pm^{1-\alpha} f(x)] & 0 < \alpha < 1, \\ \frac{d^2}{dx^2} [I_\pm^{2-\alpha} f(x)] & 1 < \alpha < 2, \end{cases}$$

where the Weyl fractional integrals are ( $\mu > 0$ )

$$I_+^\mu = \frac{1}{\Gamma(\mu)} \int_{-\infty}^x (x-\chi)^{\mu-1} f(\chi) d\chi.$$

## 1.11 Erdélye-Kober Fractional Operators

The Erdélye-Kober integral for a well-behaved function  $\phi(t)$  is defined as [?, ?]

$$I_\eta^{\gamma,\mu} \phi(t) = \frac{\eta}{\Gamma(\mu)} t^{-\eta(\mu+\gamma)} \int_0^t \tau^{\eta(\gamma+1)-1} (t^\eta - \tau^\eta)^{\mu-1} \phi(\tau) d\tau,$$

where  $\mu > 0, \eta > 0, \gamma \in \mathcal{R}$ .

In the special case  $\gamma = 0, \eta = 1$  the Erdélye-Kober fractional integral is related to the RL fractional integral of the order  $\mu$  as

$$I_1^{0,\mu} \phi(t) = \frac{t^{-\mu}}{\Gamma(\mu)} \int_0^t (t-\tau)^{\mu-1} \phi(\tau) d\tau = t^{-\mu} J^\mu \phi(t).$$

The Erdélye-Kober fractional derivative for  $n - 1 < \mu < n$ ,  $n \in \mathcal{N}$  is defined as

$$D_{\eta}^{\gamma,\mu} \phi(t) = \prod_{j=1}^n \left( \gamma + j + \frac{1}{\eta} t \frac{d}{dt} \right) (I_{\eta}^{\gamma+\mu,n-\mu} \phi(t)).$$

The Erdélye-Kober fractional derivative reduces to the identity operator when  $\mu = 0$

$$D_{\eta}^{\gamma,0} \phi(t) = \phi(t)$$

and for  $\eta = 1$  and  $\gamma = -\mu$  is related to the RL fractional derivative as

$$D_{\eta}^{\gamma,\mu} \phi(t) = t^{\mu} D_{RL}^{\mu} \phi(t).$$

## 1.12 Interpretation of Fractional Integral and Derivatives

The integer-order and integrals have a clear physical and geometrical interpretation that simplify their use in practice. The numerous different interpretations of the fractional derivatives and integrals have been proposed [?]: the probabilistic [?, ?, ?], geometric [?, ?, ?, ?], physical interpretations [?, ?, ?, ?, ?].

However, as noted Podlubny [?], "since the appearance of the idea of differentiation and of arbitrary (not necessary integer) order there was not any acceptable geometric and physical interpretation of these operations for more than 300 years".

Teneiro Machado [?] wrote the Günwald-Letnikov derivative of  $x(t)$  as

$$D^{\alpha}[x(t)] = \lim_{h \rightarrow 0} \left[ \frac{1}{h^{\alpha}} \sum_{k=0}^{\infty} \gamma(\alpha, k) x(t - kh) \right], \quad \gamma(\alpha, k) = (-1)^k \frac{\Gamma(\alpha + 1)}{k! \Gamma(\alpha - k + 1)}$$

where  $h$  is the time increment. The author noted that

$$\gamma(\alpha, 0) = 1, \quad - \sum_{k=0}^{\infty} \gamma(\alpha, k) = 1$$

thus the "present" (P) is constituted by  $x(t)$  the probability 1 while the totality of the "past/future" (PF) is constituted by the samples  $x(t), x(t-h), x(t-2h), \dots$ ; each sample is weighted with a probability  $-\gamma(\alpha, k)$ .

Nigmatullin [?, ?] interpreted the fractional integral in terms of the fractal Cantor set. The author considered the evolution of the state of the physical system

$$J(t) = \int_0^t K(t, \tau) f(\tau) d\tau$$

where the memory function  $K(t, \tau) f(\tau)$  accounts for the loss of some states of the system; the fractional index of integration equals the fractal dimension of the Cantor set.

Podlubny [?] and Podlubny et al. [?] suggested the geometrical interpretation of the left-sided (equation (??)) and right-sided (equation (??)) RL integrals and of the RL

(equations (??) - (??)) and the Caputo (equation (??)) derivatives, as well as of the Riesz potential that is the sum of the left-sided and right-sided RL fractional integrals

$$R_b^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \left[ \int_a^x (x-t)^{\alpha-1} f(t) dt + \int_x^b (t-x)^{\alpha-1} f(t) dt \right]$$

and of the Feller potential

$$\Phi^\alpha f(x) = c J_{a^+}^\alpha f(x) + d J_{b^-}^\alpha f(x).$$

The geometric interpretation by Podlubny is based on adding the third dimension

$$g_x(t) = \frac{1}{\Gamma(\alpha+1)} [x^\alpha + (x-t)^\alpha]$$

to the pair (t, f(x)) and considering the three-dimensional line (t, g\_x(t), f(t)) as the top edge of the "fence" that gives shadow on the wall in the (g,f) plane.

Tarasov [?] proposed the "informatic" ("computer science") interpretation of the RL and the Caputo derivatives of the non-integer orders using the reconstructions from the infinite sequence of the derivatives of the integer orders. Such reconstructions are based on the Kotel'nikov theorem (also known as the sampling theorem) proved by Vladimir Kotel'nikov in 1933 and also by Claude Shannon 1949: under the certain restrictive conditions, function f(t) can be restored from its samples f[n] = f(nT) according to the Whittaker-Shannon interpolation formula. The author stressed that infinity of the sequences of the integer derivatives plays a fundamental role in representation of the fractional derivatives that describe nonlocality and memory.

Gómez-Aguilar et al. [?] analysed the Caputo differentiation using the RC circuit for which the fractional version of the Ohm's law and Kirchhoff's law are written as

$$v(t) = \frac{1}{\sigma^{1-\gamma}} \frac{d^\gamma q}{dt^\gamma}, \quad R \frac{dq}{dt} + \frac{1}{C} q(t) = v(t)$$

where  $q$  is the electric charge,  $v$  is the voltage,  $R$  is the resistance of the conductor,  $C$  is the capacitance. The parameter  $\sigma$  is introduced in order to be consistent with the dimensionality; it characterizes the fractional structures (the components that show the intermediate behaviour between conservative (capacitor) and dissipative (resistor) [?]. The authors derived the fractional differential equation for the RC circuit

$$\frac{d^\gamma}{dt^\gamma} + \frac{1}{\tau_\gamma} q(t) = \frac{C}{\tau_\gamma} v(t), \quad \tau_\gamma = \frac{RC}{\sigma^{1-\gamma}}$$

where  $\tau_\gamma$  is the time constant. Gómez-Aguilar et al. claimed that the differentiation is related to the memory effects that reflect the intrinsic dissipation in the system.

Sierociuk et al. [?] used the RC network to model the fractional order diffusion based on the analogy between the heat and electrical conduction. The authors showed that the equations for the capacitor and for the resistor in the transmission line could be used to get the diffusion equation; the loss of heat was represented by the additional resistors connected parallel to capacitors.

Carpinteri et al. [?] considered the mechanical interpretation of the Marchaud fractional derivative using the body springs connecting the non-adjacent points of the body with the stiffness decaying with the distance between the material points.

### 1.13 Local Fractional Derivatives

The fractional derivatives are nonlocal. Several researches introduced the local variants [?] that are useful for study of the pointwise behaviour of the fractal and multifractal functions that describe, e.g., the stress and deformation patterns in materials exhibiting the fractal-like microstructure [?] or the velocity field of turbulent fluid [?].

Kolwankar & Gangal [?, ?, ?, ?] defined the derivative via the RL derivative as

$$\mathfrak{D}^q f(y) = \lim_{x \rightarrow y} \frac{D^q(f(x) - f(y))}{(x - y)^q}$$

if the limit exists and finite.

The local fractional Taylor formula is written as [?]

$$f(x) = \sum_{i=0}^n \frac{f^{(n)}(y)}{\Gamma(1+n)} (x-y)^n \frac{\mathfrak{D}^\alpha}{\Gamma(n!+\alpha)} (x-y)^\alpha + R_\alpha(x,y).$$

Yang et al. [?, ?] used similar definition

$$\mathfrak{D}^{(k)} f(\tau) = \lim_{\tau \rightarrow \tau_0} \frac{f(\tau) - f(\tau_0)}{\tau^k - \tau_0^k}.$$

Chen et al. [?] proposed the local derivatives based on the integrals of the difference-quotient (IDQ) or the singular of difference-quotient (SIDQ). For example, the right SIDQ local derivative is

$$\mathfrak{D}^\alpha f(y) = \frac{1}{\Gamma(1-\alpha)} \lim_{h \rightarrow 0_+} \int_0^1 (1-t)^{-\alpha} \frac{f(th+y) - f(y)}{h^\alpha} dt.$$

The local fractional derivative is essentially the *fractal* derivative [?, ?]. In contrast to the purely analytical approach of the fractional calculus, the fractal calculus follows the physical-geometric approach; to avoid confusion it is suggested to call the latter the *scaled* calculus [?].

The fractal ("Hausdorff") derivative on the time fractal is defined as [?]

$$\frac{\partial f}{\partial t^\sigma} = \lim_{t_B \rightarrow t_A} \frac{f(t_B) - f(t_A)}{(t_B)^\sigma - (t_A)^\sigma}$$

where  $\sigma$  is the fractal dimension of time.

A more general definition is formulated as [?, ?]

$$\frac{\partial^\tau f}{\partial t^\sigma} = \lim_{t_B \rightarrow t_A} \frac{f^\tau(t_B) - f^\tau(t_A)}{(t_B)^\sigma - (t_A)^\sigma}$$

Since the fractal derivative is the local operator, the numerical solution of the fractal derivative equations can be performed by the standard numerical techniques for the integer-order differential equations [?]. The similar properties have the so called "conformable" fractional derivatives.

### 1.13.1 "Conformable" Fractional Derivative

Most fractional derivatives do not have the desirable properties [?, ?, ?]: the derivative of a constant is not zero; they do not obey the product rule  $D^\alpha(fg) = fD^\alpha(g) + gD^\alpha f$ ; they do not obey the quotient rule  $D^\alpha(f/g) = (gD^\alpha(f) - fD^\alpha(g))/g^2$ ; they do not obey the chain rule  $D^\alpha(fg) = f^\alpha(g(t))g^\alpha(t)$ ; they do not obey in general  $D^\alpha D^\beta f = D^{\alpha+\beta} f$ .

Khalil et al.[?] and Katugampola [?, ?] suggested the so called "conformable" limit based [?] derivatives

$$D^\alpha f(t) = \lim_{\epsilon \rightarrow 0} \frac{f(t + \epsilon t^{1-\alpha}) - f(t)}{\epsilon}, \quad 0 < \alpha < 1,$$

and

$$D^\alpha f(t) = \lim_{\epsilon \rightarrow 0} \frac{f(te^{\epsilon t^{-\alpha}}) - f(t)}{\epsilon}, \quad 0 < \alpha < 1.$$

Since the conformable derivative is the extension of the classical derivative definition, this derivative obeys the product rule, the quotient rule, the linearity property, the zero derivative for the constant and are valid for some extensions of the classical calculus such as the Rolle's Theorem or Mean Value Theorem [?].

## 2 Tempered Fractional Calculus

Sabzikar et al. [?] suggested a variant of the fractional calculus where power laws are tempered by the exponential factor. The random walks model with the exponentially tempered power law jumps converges to a tempered stable motion [?]. This *tempered* fractional diffusion is useful in the geophysical [?, ?] and financial [?] problems.

The authors considered two intervals for the parameter  $\alpha$ :

- $0 < \alpha < 1$ . The *tempered* fractional derivative  $\partial_x^{\alpha,\lambda}$  is defined as the function with the Fourier transform  $[(\lambda + ik)^\alpha - \lambda^\alpha]\hat{f}(k)$  that in real space is written as

$$\partial_x^{\alpha,\lambda} f(x) = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty (f(x) - f(x-y)) e^{-\lambda y} y^{-\alpha-1} dy.$$

The negative *tempered* fractional derivative  $\partial_{-x}^{\alpha,\lambda}$  is defined as the function with the Fourier transform  $[(\lambda - ik)^\alpha - \lambda^\alpha]\hat{f}(k)$  that in real space is written as

$$\partial_{-x}^{\alpha,\lambda} f(x) = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty (f(x) - f(x+y)) e^{-\lambda y} y^{-\alpha-1} dy.$$

- $1 < \alpha < 2$ . The *tempered* fractional derivative  $\partial_x^{\alpha,\lambda}$  is defined as the function with the Fourier transform  $[(\lambda + ik)^\alpha - \lambda^\alpha - ik\alpha\lambda^{\alpha-1}]\hat{f}(k)$  that in real space is

$$\partial_x^{\alpha,\lambda} f(x) = \frac{\alpha(1-\alpha)}{\Gamma(2-\alpha)} \int_0^\infty (f(x-y) - f(x) + y f'(x)) e^{-\lambda y} y^{-\alpha-1} dy.$$

The negative *tempered* fractional derivative  $\partial_x^{\alpha,\lambda}$  is defined as the function with the Fourier transform  $[(\lambda - ik)^\alpha - \lambda^\alpha + ik\alpha\lambda^{\alpha-1}]\hat{f}(k)$  that in real space is

$$\partial_x^{\alpha,\lambda} f(x) = \frac{\alpha(1-\alpha)}{\Gamma(2-\alpha)} \int_0^\infty (f(x+y) - f(x) - yf'(x)) e^{-\lambda y} y^{-\alpha-1} dy.$$

Sabzikar et al. introduced the positive tempered integral as

$$\mathfrak{I}_+^{\alpha,\lambda} f(x) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^x f(u)(x-u)^{\alpha-1} e^{-\lambda(x-u)} du$$

and the negative tempered integral as

$$\mathfrak{I}_-^{\alpha,\lambda} f(x) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^x f(u)(u-x)^{\alpha-1} e^{-\lambda(u-x)} du$$

called the RL tempered integrals since for  $\lambda = 0$  they reduce to the usual RL integrals.

The authors defined the RL tempered fractional derivatives  $\mathcal{D}_\pm^{\alpha,\lambda}$  as functions with the Fourier transform  $(\lambda \pm ik)^\alpha \hat{f}(k)$  that can be expressed

$$\mathcal{D}_\pm^{\alpha,\lambda} f(x) = \begin{cases} \partial_{\pm x}^{\alpha,\lambda} f(x) + \lambda^\alpha f(x) & 0 < \alpha < 1 \\ \partial_{\pm x}^{\alpha,\lambda} f(x) + \lambda^\alpha f(x) \pm \alpha\lambda^{\alpha-1} f'(x) & 1 < \alpha < 2. \end{cases}$$

Evidently, integration and differentiation are the inverse operators:

$$\mathcal{D}_\pm^{\alpha,\lambda} \mathfrak{I}_\pm^{\alpha,\lambda} f(x) = f(x), \quad \mathfrak{I}_\pm^{\alpha,\lambda} \mathcal{D}_\pm^{\alpha,\lambda} f(x) = f(x).$$

The integration and differentiation operators have the semigroup property

$$\mathfrak{I}_\pm^{\alpha,\lambda} \mathfrak{I}_\pm^{\beta,\lambda} f = \mathfrak{I}_\pm^{\alpha+\beta,\lambda} f, \quad \mathcal{D}_\pm^{\alpha,\lambda} \mathcal{D}_\pm^{\beta,\lambda} f = \mathcal{D}_\pm^{\alpha+\beta,\lambda} f.$$

### 3 Fractional Differential Equations

Generally, the fractal media could not be considered as continuous media. The use of the non-integer dimensional spaces [?] is necessary to describe a fractal medium by the continuous models [?]. The fractional differential equations [?, ?, ?, ?] are non-local (i.e. could incorporate the effects of the memory and the spatial correlations) and could be formulated in the distinct but mathematically equivalent forms. Mainardi et al. [?] compared the fractional extensions of the standard Cauchy problem

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2}, \quad x \in \mathbf{R}, \quad t \in \mathbf{R}_0^+, \quad \mathbf{u}(\mathbf{x}, \mathbf{0}^+) = \mathbf{u}_0(\mathbf{x}). \quad (11)$$

The fundamental solution (or Green function) of (??), i.e. the solution subjected to the initial condition  $u_0(x) = \delta(x)$ , is the Gaussian probability density function

$$u(x,t) = \frac{1}{2\sqrt{\pi}} t^{-1/2} e^{-x^2/(4t)}.$$

The Green function has the scaling property  $u(x, t) = t^{1/2}U(x/t^{1/2})$ ,  $U(x)$  is the reduced Green function.

The Cauchy problem (??) is equivalent to the integro-differential equation

$$u(x, t) = u_0(x) + \int_0^t \left[ \frac{\partial^2 u(x, \tau)}{\partial x^2} \right] d\tau$$

where the initial condition is incorporated.

The fractional diffusion equation could be written with the use of the RL derivative  $D^{1-\beta}$  ( $\beta$  is the real number  $0 < \beta < 1$ )

$$\frac{\partial u(x, t)}{\partial t} = D^{1-\beta} \frac{\partial^2 u(x, t)}{\partial x^2} \quad (12)$$

or the Caputo derivative  $D_\star^\beta$

$$D_\star^\beta u(x, t) = \frac{\partial^2 u(x, t)}{\partial x^2}. \quad (13)$$

The equations (??) and (??) are equivalent to the equation based on the use of the RL fractional integral of the order  $\beta$

$$u(x, t) = u_0(x) + J^\beta \left[ \frac{\partial^2 u(x, \tau)}{\partial x^2} \right]. \quad (14)$$

Note that the equation (??) could be obtained by differentiating (??), the equation (??) can be derived by the fractional integration of (??).

The equation (??) was studied by Metzler et al. [?] and by Saichev & Zaslavsky [?], the equation (??) by Gorenflo et al. [?, ?] and by Mainardi [?, ?], the integrodifferential equation (??) by Schneider & Wyss [?] using the Mellin transform.

Mainardi et al. [?] search for the fundamental solution of the equation (??) by applying the sequence of the Fourier

$$\mathcal{F}\{v(x); k\} = \hat{v}(k) = \int_{-\infty}^{\infty} e^{ikx} v(x) dx, \quad k \in \mathbf{R}$$

and the Laplace

$$\mathcal{L}\{w(t); s\} = \tilde{w}(s) = \int_0^{\infty} e^{-st} w(t) dt, \quad s \in \mathbf{C}$$

transforms. Thus the Green function in the Fourier-Laplace domain is determined by

$$\hat{u}(k, s) = \frac{s^{\beta-1}}{s^\beta + k^2}, \quad 0 < \beta \leq 1, \quad \mathcal{R}(s) > 0, \quad k \in \mathbf{R}. \quad (15)$$

There are two strategies to determine the Green function in the space-time domain  $u(x, t)$  related to the order in performing inversions in the expression (??) [?]: 1) Invert the Fourier transform to get  $\tilde{u}(x, s)$  and then invert the Laplace transform [?, ?] or 2) invert the Laplace transform to get  $\hat{u}(k, t)$  and then invert the Fourier transform [?, ?].

Nieto [?] studied the linear fractional differential equation with the spatial RL derivative for initial or periodic boundary conditions and derived the maximum principle using the properties of the Mittag-Leffler functions. Compte [?] and West et al. [?] studied the equation for the hyperdiffusion (Lévy-flight diffusion)

$$\frac{\partial P}{\partial t} = D(-\Delta)^{\frac{\gamma}{2}}$$

where the fractional  $n$ -dimensional Laplace operator  $(-\Delta)^{\frac{\gamma}{2}}$  is defined by its Fourier transform with respect to the spatial variable [?]

$$\mathcal{F}[(-\Delta)^{\frac{\gamma}{2}} g(x)] = |\omega|^{\gamma} \mathcal{F}[g(x)].$$

Luchko [?] derived the maximum principle for the initial-boundary-value problem for the time-fractional diffusion equation with Caputo derivative over the open bounded domain  $G \times (0, T)$ ,  $G \subset R^n$ .

The equation could be subjected to the complex transformation [?, ?, ?]  $s = x^S / \Gamma(1 + \alpha)$  to convert to a partial differential equation<sup>2</sup> For example, the heat conduction equation ( $\alpha$  is the fractal dimension of the fractal medium)

$$\frac{\partial T}{\partial t} = \frac{\partial^\alpha}{\partial x^\alpha} \left( \lambda \frac{\partial^\alpha T}{\partial x^\alpha} \right)$$

is converted into the equation

$$\frac{\partial T}{\partial t} = \frac{\partial}{\partial s} \left( \lambda \frac{\partial T}{\partial s} \right).$$

That could be further transformed by introduction of the Boltzmann variable [?]  $\chi = s/\sqrt{t} = x^\alpha / \sqrt{t} \Gamma(1 + \alpha)$  into the ordinary differential equation

$$\frac{d}{d\chi} \left( \lambda \frac{dT}{d\chi} \right) + \frac{\chi}{2} \frac{dT}{d\chi}.$$

For the general fractional differential equation in the Jumarie's modification of the RL derivatives

$$f(u, u_t^\alpha, u_x^\beta, u_y^\gamma, u_z^\lambda, u_t^{2\alpha}, u_x^{2\beta}, u_y^{2\gamma}, u_z^{2\lambda}, \dots) = 0$$

---

<sup>2</sup>Such transformation is possible in the multidimensional case if the variables  $t, x, y, z$  obey the following constraint [?] ( $q, p, k, l$  are constants)

$$\xi = \frac{qt^\alpha}{\Gamma(1 + \alpha)} + \frac{px^\beta}{\Gamma(1 + \beta)} + \frac{ky^\gamma}{\Gamma(1 + \gamma)} + \frac{lz^\lambda}{\Gamma(1 + \lambda)}.$$

He & Li [?] suggested the following transforms

$$s = \frac{qt^\alpha}{\Gamma(1+\alpha)}, \quad X = \frac{px^\beta}{\Gamma(1+\beta)}, \quad Y = \frac{ky^\gamma}{\Gamma(1+\gamma)}, \quad Z = \frac{lz^\lambda}{\Gamma(1+\lambda)}$$

thus converting the fractional derivatives into classical derivatives

$$\frac{\partial^\alpha u}{\partial t^\alpha} = q \frac{\partial u}{\partial s}, \quad \frac{\partial^\beta u}{\partial x^\beta} = p \frac{\partial u}{\partial X}, \quad \frac{\partial^\gamma u}{\partial y^\gamma} = k \frac{\partial u}{\partial Y}, \quad \frac{\partial^\lambda u}{\partial z^\lambda} = l \frac{\partial u}{\partial Z}.$$

Babusci et al. [?] discussed relations between the differential equations and the theories of the pseudo-operators [?, ?] and the generalized integral transforms.

### 3.1 Distributed order differential equations

The distributed order differential equations (DODE) are a special class of the fractional differential equations [?, ?, ?, ?, ?, ?, ?, ?, ?]. Chechkin et al. [?] discussed the natural and the modified forms of DODEs and noted that the latter in combination with the continuity equation and the retarded linear response equation for the flux exhibiting memory of the processes at the previous times admits a thermodynamic interpretation. DODEs are used to describe the accelerating subdiffusion, decelerating superdiffusion or transformation of the anomalous behaviour at the short times into the normal behaviour at the long times. For example, Metzler & Klafter [?] considered the DODE for the description of the ultraslow diffusion with the logarithmic time dependence  $\langle x^2(t) \rangle \propto \log^k t$  including the so called Sinai diffusion ( $k = 4$ ).

The concept of the distributed order differentiation is close to the variable order fractional operators that are useful for the study of the viscoelasticity, the reaction kinetics of proteins, the electrorheological fluids, the damage modelling [?, ?, ?].

There are two approaches to the formulation of the distributed order differential equations: 1) direct — a new variable does not assigned; 2) Independent variable approach — the order is considered as a function of some independent variable.

Mainardi et al. [?] studied the fractional diffusion equation of distributed order

$$\int_0^1 b(\beta) [D^\beta u(x, t)] d\beta = \frac{\partial^2 u(x, t)}{\partial x^2}, \quad b(\beta) \geq 0, \quad \int_0^1 b(\beta) d\beta = 1 \quad (16)$$

with  $x \in \mathbf{R}$ ,  $t \geq 0$  and the initial condition  $u(x, 0^+) = \delta(x)$ . The weight function  $b(\beta)$  is called the order-density. The authors used the Fourier and Laplace transforms to get the fundamental solution similar to a single-order case (??)

$$\left[ \int_0^1 b(\beta) s^\beta d\beta \right] \hat{u}(k, s) - \int_0^1 b(\beta) s^{\beta-1} d\beta = -k^2 \hat{u}(k, s)$$

and

$$\hat{u}(k, s) = \frac{B(s)/s}{B(s) + k^2}, \quad k \in \mathbf{R}, \quad \mathbf{B}(\mathbf{s}) = \int_0^1 \mathbf{b}(\beta) \mathbf{s}^\beta d\beta. \quad (17)$$

In the case of small  $k$  the equation (??) can be approximated as

$$\hat{u}(k, s) = \frac{1}{s} \left( 1 - \frac{k^2}{B(s)} + \dots \right)$$

and the second moment is written as

$$\tilde{\mu}_2(s) = -\frac{\partial^2}{\partial k^2} \hat{u}(k=0, s) = \frac{2}{B(s)}. \quad (18)$$

The special case of DODEs are the double-order fractional equations [?]

$$b(\beta) = b_1\delta(\beta - \beta_1) + b_2\delta(\beta - \beta_2), \quad 0 < \beta_1 < \beta_2 \leq 1, \quad \beta_1 > 0, \beta_2 > 0, \quad \beta_1 + \beta_2 = 1.$$

Asymptotic behaviour of  $\mu_2(t)$  follows from (??) for cases of the slow diffusion (the power-law growth,  $b(\beta) = b_1\delta(\beta - \beta_1) + b_2\delta(\beta - \beta_2)$ ) where  $\tilde{\mu}_2(s) = 2/(b_1s^{\beta_1+1} + b_2s^{\beta_2+1})$  and the ultra-slow diffusion (the logarithmic growth,  $b(\beta) = 1$ ) with  $\tilde{\mu}_2(s) = 2\ln s/s(s-1)$ .

The distributed order equations allow to describe the more complex media. The time-fractional diffusion equation of the distributed order (??) is potentially more flexible to represent the local phenomena while the space-fractional diffusion equation of the distributive order is more suited to represent the variations in space [?].

## 3.2 Special Functions

There are special functions related to the differential equation similar to the classical case (such as e.g., the Bessel and the cylindrical functions, the classical orthogonal polynomials, Airy functions etc.) [?]. The most important functions in the fractional calculus are the Mittag-Leffler function [?], the H-functions [?, ?, ?], the Wright functions [?, ?], the generalized Lommel-Write functions [?]. The Mittag-Leffler function is even called the "Queen"-function of the fractional calculus [?].

### 3.2.1 Mittag-Leffler Functions

The eigenfunction of the RL derivatives are the solutions of the equation [?]

$$D_{0+}^\alpha[f(x)] = \lambda f(x)$$

where  $\lambda$  is the eigenvalue. The eigenfunctions are  $f(x) = x^{1-\alpha} E_{\alpha,\alpha}(\lambda x^\alpha)$  where

$$E_{\alpha,\beta} = \sum_{k=0}^{\infty} \frac{x^K}{\Gamma(\alpha k + \beta)} \quad (19)$$

is the generalized Mittag-Leffler function (also called the Wiman's function [?]).

The more general eigenvalue equation for derivatives of the orders  $\alpha$  and  $\beta$  is

$$D_{0+}^{\alpha,\beta}[f(x)] = \lambda f(x)$$

The solution is [?]  $f(x) = x^{(1-\beta)(1-\alpha)} E_{\alpha,\alpha+\beta}(\lambda x^\alpha)$ .

The special case is the equation  $D_{0+}^{\alpha,1}[f(x)] = \lambda f(x)$  with eigenfunction  $f(x) = E_\alpha(\lambda x^\alpha)$ .

The one-parameter Mittag-Leffler function is the particular case of (??) for  $\beta = \alpha$ . Evidently [?],

$$E_{0,1}(x) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(1)} = \sum_{k=0}^{\infty} x^k = \frac{1}{1-x}, \quad E_1(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} = \exp(x).$$

There are other special cases such as [?]  $E_2(-x^2) = E_{2,1}(-x^2) = \cos(x)$ ;  $E_2(x^2) = E_{2,1}(x^2) = \cosh(x)$ ; for  $x > 0$   $E_{1/2}(x^{1/2}) = E_{1/2}(x^{1/2}) = (1 + \text{erf}(x)) \exp(x^2)$ ; for  $x \in C$  and  $r \in N$

$$E_{1,r} = \frac{1}{x^{r-1}} \left( \exp(x) - \sum_{k=0}^{r-2} \frac{x^k}{k!} \right);$$

$E_3(x) = 1/2[e^{x^{1/3}} + 2e^{-1/2x^{1/3}} \cos(\sqrt{3}/2x^{1/3})]$ ;  $E_4(x) = 1/2[\cos(x^{1/4}) + \cosh(x^{1/4})]$   
where  $\text{erf}(x) = 2/\sqrt{\pi} \int_0^x \exp(-t^2) dt$ .

The Mittag-Leffler function  $E_1$  satisfies the functional relation [?, ?]  $E_1(x-y) = E_1(x)/E_1(y)$  and the relation between two Mittag-Leffler functions with different parameters  $E_{n_1,n_2}(x) = xE_{n_1,n_1+n_2}(x) + 1/\Gamma(n_2)$ . Note that the frequently used relation  $E_\alpha(a(t+s)^\alpha) = E_\alpha(at^\alpha)E_\alpha(as^\alpha)$ ,  $t, s \geq 1$  is valid only if  $\alpha = 0$  or  $\alpha = 1$  [?].

Asymptotic expansions and integral representations of the Mittag-Leffler functions could be found in the papers [?, ?, ?].

Prabhakar [?] suggested the extension

$$E_{\alpha,\beta}^\gamma(x) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\alpha n + \beta)} \frac{x^n}{n!}, \quad \text{Re}(\alpha) > 0, \text{Re}(\beta) > 0$$

$(\gamma)_n$  is the Pochhammer symbol [?]  $(\gamma)_0 = 1$ ,  $(\gamma)_n = \gamma(\gamma + 1)(\gamma + 2) \dots (\gamma + n - 1)$ .

The extension to the multi-index Mittag-Leffler functions [?, ?]

$$E_{(\frac{1}{\rho_1}),(\mu_i)}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\mu_1 + k/\rho_1) \dots \Gamma(\mu_m + k/\rho_m)}$$

is performed by replacing of the indices  $\alpha = 1/\rho$  and  $\beta = \mu$  by two sets of multi-indices  $\alpha \rightarrow (1/\rho_1, 1/\rho_2, \dots, 1/\rho_m)$  and  $\beta \rightarrow (\mu_1, \mu_2, \dots, \mu_m)$ .

There are a couple of related functions [?]

- Barret's function

$$U(x, \lambda) = \sum_{k=1}^{\infty} \frac{\lambda^{k-1} x^{k\alpha i}}{\Gamma(k\alpha - i + 1)};$$

- Rabotnov's (fractional exponential) function [?, ?]

$$\mathcal{E}_\alpha(\beta, x) = x^\alpha \sum_{n=0}^{\infty} \frac{\beta^n x^{n(\alpha+1)}}{\Gamma((n+1)(1+\alpha))}.$$

### 3.2.2 H Functions

The H-function of order  $(m, n, p, q) \in \mathcal{N}^4$  is defined via the Mellin-Barnes type contour integral [?, ?]

$$H_{p,q}^{m,n}(z) = \frac{1}{2\pi i} \int_{\mathcal{L}} \mathcal{H}_{p,q}^{m,n} z^s ds$$

where  $z^s = \exp[s(\ln|z| + i\arg z)]$ ,

$$\begin{aligned} \mathcal{H}_{p,q}^{m,n} &= \frac{A(s)B(s)}{C(s)D(s)}, & A(s) &= \prod_{j=1}^m \Gamma(b_j - \beta_j s), & B(s) &= \prod_{j=1}^n \Gamma(1 - a_j + \alpha_j s), \\ C(s) &= \prod_{j=m+1}^q \Gamma(1 - b_j + \beta_j s), & D(s) &= \prod_{j=n+1}^p \Gamma(a_j - \alpha_j s). \end{aligned}$$

Here  $m, n, p, q$  are integers satisfying  $0 \leq n \leq p$ ,  $1 \leq m \leq q$ ,  $m^2 + n^2 \neq 0$ ,  $a_j (j = 1, \dots, p)$ ,  $b_j (j = 1, \dots, q)$  are complex numbers.

The integration contour  $\mathcal{L}$  could be chosen in different ways:

- $\mathcal{L} = \mathcal{L}_{-i\infty, i\infty}$  chosen to go from  $-i\infty$  to  $i\infty$  leaving to the right all poles of  $\mathcal{P}(A)$  of the functions  $\Gamma$  in  $A(s)$  and to the left all poles of  $\mathcal{P}(B)$  of the functions  $\Gamma$  in  $B(s)$ ;
- $\mathcal{L} = \mathcal{L}_{i\infty}$  is a loop beginning and ending at  $+\infty$  and encircling in the negative direction all the poles of  $\mathcal{P}(A)$ ;
- $\mathcal{L} = \mathcal{L}_{-i\infty}$  is a loop beginning and ending at  $-\infty$  and encircling in the negative direction all the poles of  $\mathcal{P}(B)$ .

### 3.2.3 Wright Functions

The Write function is defined by the series representation that is convergent in the whole  $z$ -complex plane [?, ?, ?, ?]

$$W_{\lambda,\mu}(z) = \sum_{n=0}^{\infty} \frac{z^n}{n! \Gamma(\lambda n + \mu)}, \quad \lambda > -1, \mu \in \mathbb{C}.$$

The integral representation of the Write function is written as

$$W_{\lambda,\mu}(z) = \frac{1}{2\pi i} \int_{Ha} e^{\sigma+z\sigma^{-\lambda}} \frac{d\sigma}{\sigma^\mu}$$

where  $Ha$  is the Hankel path (a loop that starts from  $-\infty$  along the lower side of the negative real axis, encircles the circular area the origin with radius  $\epsilon \rightarrow 0$  in the positive sense, and ends at  $-\infty$  along the upper side of the negative real axis).

There are Write-type auxiliary functions  $F_\nu(z) = W_{-\nu,0}(z)$ ,  $M_\nu(z) = W_{-\nu,1-\nu}(z)$ , where  $0 < \nu < 1$ ; these functions are related  $F_\nu(z) = \nu z M_\nu(z)$ .

The series representations of the *auxiliary* functions are

$$F_\nu(z) = \sum_{n=1}^{\infty} \frac{(-z)^n}{n! \Gamma(-\nu n)} = \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{(-z)^{n-1}}{n!} \Gamma(\nu n + 1) \sin(\pi \nu n)$$

and

$$M_\nu(z) = F_\nu(z) = \sum_{n=0}^{\infty} \frac{(-z)^n}{n! \Gamma[-\nu n + (1 - \nu)]} = \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{(-z)^{n-1}}{(n-1)!} \Gamma(\nu n) \sin(\pi \nu n).$$

## 4 Solution of Fractional Differential Equations

### 4.1 Analytical Methods

Numerous approximate analytical methods are known:

- the Adomian decomposition method (ADM) [?];
- the combined Laplace-Adomian method (CLAM) [?];
- the variational iteration method (VIM) [?, ?, ?, ?, ?]<sup>3</sup> and its local (LVIM) [?, ?, ?, ?] and fractional (using the fractional order Lagrange multipliers) [?, ?] variants;
- the homotopy perturbation method (HPM)[?, ?, ?, ?, ?, ?] and its modification [?] and local fractional variant (LFHPM) [?];
- the differential transformation method [?, ?, ?];
- the heat-balance integral method (HBIM)[?, ?, ?];
- the fractional complex transform method (FCTM) [?, ?, ?, ?, ?];
- the local fractional Fourier series method (FSM) [?, ?];
- the modified simple equation method [?, ?];
- the method of images (limited to special spatial symmetries);
- the Mellin integral transform method [?];
- the local fractional decomposition method (LFDM) [?];
- the fractional sub-equation method [?, ?];

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<sup>3</sup>VIM includes three steps to determine the variational iteration formula:

1. establishing the correction functional;
2. identifying the Lagrange multipliers;
3. determining the initial iteration.

The second step is the crucial one [?].

- the Sumudu transform methods [?, ?] and its variant — the local fractional homotopy perturbation Sumudu transform method [?];
- the theta-method [?];
- the Picard successive approximation method (PSAM) [?, ?];
- the local Laplace transforms.

Frequently analytical methods are variants of perturbation methods [?]). For example, He [?, ?, ?] based his method to solve the general equation  $A(u) - f(r) = 0$  with the general differential operator  $A$  divided into linear  $L$  and nonlinear  $N$  parts  $L(u) + N(u) - f(r) = 0$  on the approach of Liao [?] (who used the two-parameter family of equations) by considering the one-parameter family  $(1-p)L(u) + pN(u) = 0$ .

He constructed the homotopy  $v(r, p) : \Omega \times [0, 1] \rightarrow R$  that satisfies

$$H(v, p) = (1-p)[L(v) - L(v_0)] + p[A(v) - f(r)] = 0$$

where the homotopy parameter  $p \in [0, 1]$ ,  $v_0$  is the initial approximation.

Evidently,  $H(v, 0) = L(v) - L(v_0) = 0$  and  $H(v, 1) = [A(v) - f(r)] = 0$ .

In topology,  $L(v) - L(v_0)$  is called deformation. The homotopy parameter  $p$  is considered as a small parameter and the solution is written as a series

$$v = v_0 + pv_1 + p^2v_2 + p^3v_3 + \dots$$

and when  $p \rightarrow 1$

$$u = \lim_{p \rightarrow 1} v = v_0 + v_1 + v_2 + v_3 + \dots$$

The Adomian decomposition method (ADM) [?, ?] does not use linearization, perturbation or the Green's functions. The accuracy of the approximate analytical solutions can be verified by the direct substitution.

The initial value is written as  $Lu + Ru + Nu = g$  where  $L$  is the linear operator to be inverted,  $R$  is the linear remainder operator and  $N$  is the nonlinear operator. Thus  $L^{-1}Lu = u - \Phi$ ,  $\Phi$  incorporates the initial values.

The solution and nonlinear term are decomposed into series

$$u = \sum_{n=0}^{\infty} u_n, \quad Nu = \sum_{n=0}^{\infty} A_n$$

where  $A_n$  are the Adomian polynomials for  $Nu = f(u)$  are [?]

$$A_n = \frac{1}{n!} \frac{\partial^n}{\partial \lambda^n} f \left( \sum_{k=0}^{\infty} u_k \lambda^k \right), \quad n = 0, 1, 2, \dots$$

Finally

$$\sum_{n=0}^{\infty} u_n = \Phi + L^{-1}g - L^{-1} \left[ R \sum_{n=0}^{\infty} u_n + \sum_{n=0}^{\infty} A_n \right].$$

The nonlinear term  $Nu(x, t)$  can be also decomposed [?] as

$$Nu = \sum_{n=0}^{\infty} p^n H_n$$

where He's polynomials are [?, ?]

$$H_n(u_0, u_1, \dots, u_n) = \frac{1}{n!} \frac{\partial^n}{\partial p^n} \left[ N \left( \sum_{i=0}^n p^i u_i \right) \right].$$

The fractional sub-equation method includes several steps [?]:

- Transformation of the nonlinear fractional equation in two variables  $x$  and  $t$   $D_t^\alpha u, D_x^\alpha u, \dots = 0$ ,  $0 < \alpha \leq 1$ ,  $D_t^\alpha u$  and  $D_x^\alpha u$  are Jumarie modification of the RL derivatives, using the travelling wave transformation  $u(x, t) = u(\xi)$ ,  $\xi = x + ct$ , where  $c$  is a constant to be determined, to the equation

$$P(u, cu', u', cD_\xi^\alpha u, D_\xi^\alpha u, \dots) = 0. \quad (20)$$

- The solution of the equation (??) is assumed to have the form

$$u(\xi) = \sum_{i=-n}^{-1} a_i \phi^i + a_0 + \sum_{i=1}^n a_i \phi^i,$$

where  $a_i (i = -n, -n+1, \dots, n-1, n)$  are constants to be determined,  $\phi = \phi(\xi)$  are functions that satisfy the following Riccati equation  $D_\xi^\alpha \phi(\xi) = \sigma \phi^2(\xi)$ ,  $\sigma$  is a constant.

- Formulation of a set of overdetermined nonlinear algebraic equations for  $c$  and  $a_i (i = -n, -n+1, \dots, n-1, n)$  [?].

## 4.2 Numerical Methods

Diethelm et al. [?] listed the requirement to the numerical methods that should be convergent, consistent of some reasonable order  $h^p$ , stable, reasonably inexpensive to run, reasonably easy to program.

Numerous methods are used in practice: finite difference, finite elements, radial basis functions, spectral methods, meshfree methods. The numerical methods for the fractional differential equations usually are constructed by the modification of the methods for the ordinary differential equations but require significantly more computation time and storage. The approximation of the fractional derivative needs the computation of the convolution integral that requires to sample and multiply the behaviour of two functions over the whole of the interval of integration leading to the operation count of  $O(n^2)$  where  $n$  is number of sampling points [?].

The reduction of the computational efforts is related to the fading memory property of the fractional derivatives that allows to restrict the integration interval — using the

*short memory principle* [?, ?] (also *fixed memory principle* [?] and *logarithmic memory principle* [?]), and using adaptive time stepping and basis selection [?].

Numerous methods are used to solve the fractional differential equations in practice: the finite difference [?, ?] (both the explicit, e.g. Euler [?] and the implicit [?, ?], e.g., the Crank-Nicolson [?, ?] or the alternating direction implicit [?, ?] schemes, compact schemes [?, ?, ?, ?]), the finite elements [?, ?, ?, ?, ?, ?] (including least squares FEM [?], Galerkin FEM [?, ?], discontinuous Galerkin FEM [?]), the spectral methods [?, ?, ?, ?], the meshfree methods [?, ?] (including the radial basis functions methods that exploit cubic  $\phi = r^3$ , Gaussian  $\phi = \exp(-r^2/c^2)$ , multiquadratics  $\phi = \sqrt{c^2 + r^2}$  or inverse multiquadratics  $\phi = 1/\sqrt{c^2 + r^2}$  functions [?, ?, ?]), Legendre wavelet collocation method [?].

Bahuguna et al. [?], Hanert [?], Deng et al. [?] and Deng & Li [?], Ford & Simpson [?], Ford & Connolly [?], Momani et al. [?] reported the results of the comparison of several numerical methods.

## References

- [1] ABBAASBANDY, S. The application of homotopy analysis method to nonlinear equations arising in heat transfer. *Phys. Lett. A* 360 (2006), 109–113.
- [2] ABDELJAWAD, T. On conformable fractional calculus. arXiv: 1402.6892 [math.DS], 2014.
- [3] ADOMIAN, G. *Solving Frontier Problems of Physics: The Decomposition Method*. Kluver Academic Publishers, Boston, 1994.
- [4] AGUILAR, J. F. G., AND HERNÁNDEZ. Space-time fractional diffusion-advection equation with Caputo derivative. *Abstr. Appl. Anal.* 2014 (2014), 283019.
- [5] AHMAD, J., MOHYUD-DIN, S. T., SRIVASTAVA, H. M., AND YANG, X.-J. Analytic solutions of the Helmholtz and Laplace equations by using local fractional derivative operators. *Wave Wavelets Fractals Adv. Anal.* 1 (2015), 22–26.
- [6] ALLAHVIRANLOO, T., KIANI, N. A., AND MOTAMEDI, N. Solving fuzzy differential equations by differential transformation method. *Inform. Sci.* 170 (2009), 956–966.
- [7] ANDERSON, D. R., AND ULNESS, D. J. Properties of the Katugampola fractional derivative with potential applications n quantum mechanics. *J. Math. Phys.* 56 (2015).
- [8] ASLEFALLAH, M., ROSTAMY, D., AND HOSSEINKHANI, K. Solving time-fractional differential diffusion equation by theta-method. *Int. J. Adv. Appl. Math. Mech.* 2 (2014), 1–8.
- [9] ASLEFALLAH, M., AND SHIVANIAN, E. Nonlinear fractional integro-differential reaction-diffusion equation via radial basis functions. *Eur. Phys. J. Phys.* 130 (2015), 47.

- [10] ATANGANA, A. Fractal-fractional differentiation and integration: connecting fractal calculus and fractional calculus to predict complex systems. *Chaos, Solit. Fract.* 102 (2017), 396–406.
- [11] ATANGANA, A., AND BALEANU, D. New fractional derivatives with nonlocal and non-singular kernel: theory and application to heat transfer model. *Thermal Sci.* 20 (2016), 763–769.
- [12] ATANGANA, A., AND GOMEZ-AGUILAR, J. A new derivative with normal distribution kernel: theory, methods and applications. *Phys. A Stat. Mech. Appl.* 476 (2017), 1–14.
- [13] ATANGANA, A., AND SECER, A. A note on fractional derivatives of some special functions. *Abstr. Appl. Anal.* 2013 (2013), 279681.
- [14] BABUSCI, D., DATTOLE, G., AND QUATTROMINI, M. Relativistic equations with fractional and pseudodifferential operators. *Phys. Rev. A* 83 (2011), 062109.
- [15] BAHUGUNA, D., UJLAYAN, A., AND PANDEY, D. N. A comparative study of numerical methods for solving an integro-differential equation. *Comput. Math. Appl.* 57 (2009), 1485–1493.
- [16] BALEANU, D., AND FERNANDEZ, A. On fractional operators and their classifications. *Mathematics* 7 (2019), 830.
- [17] BEN ADDA, F. Geometric interpretation of the differentiability and gradient of real order. *Compt. Rend. - Series I - Math.* 326 (1997), 931–934.
- [18] BEN ADDA, F. Geometric interpretation of the fractional derivative. *J. Fract. Calc.* 11 (1997), 21–51.
- [19] BENNETT, K. M., HYDE, J. S., AND SCHMAINDA, K. M. Water diffusion heterogeneity index in the human brain is insensitive to the orientation of applied magnetic field gradient. *Magn. Resonan. Med.* 56 (2006), 235–239.
- [20] BENSON, D. A., MEERSCHAERT, M. M., AND REVIELLE, J. Fractional calculus in hydrological modelling: A numerical perspective. *Adv. Water Resour.* 51 (2013), 479–497.
- [21] BOSIakov, S., AND ROGOSIN, S. Analytical modeling of the viscoelastic behavior of periodontal ligament with using Rabotnov's fractional exponential function. In *Lect. Notes Electr. Eng.* (2015), pp. 156–167.
- [22] BRUNNER, H., LING, L., AND YAMAMOTO, M. Numerical simulation of 2D fractional subdiffusion problems. *J. Comput. Phys.* 229 (2010), 6613–6622.
- [23] CANOTO, C., HUSSAINI, M. Y., QUARTERONI, A., AND ZANG, T. A. *Spectral Methods. Fundamentals in Single Domains*. Springer, Berlin, 2006.
- [24] CAPUTO, M. Distributed order differential equations modelling dielectric induction and diffusion. *Fract. Calc. Appl. Anal.* 4 (2001), 421–442.

- [25] CAPUTO, M. Diffusion with space memory modelle with distributed order space fractional differential equations. *Ann. Geophys.* 46 (2003), 223–234.
- [26] CAPUTO, M., AND FABRIZIO, M. A new definition of fractional derivative without singular kernel. *Progr. Fract. Differ. Appl.* 1 (2015), 73–85.
- [27] CAPUTO, M., AND FABRIZIO, M. Applications of new time and spatial fractional derivatives with exponential kernels. *Progr. Fract. Differ. Appl.* (2016).
- [28] CARPIN, A., AND MAINARDI, F. *Fractals and Fractional Calculus in Continuum Mechanics*. Springer, Wien, N. Y., 1997.
- [29] CARPINTERI, A., CORNETTI, P., SAPORA, A., DI PAOLA, M., AND ZINGALES, M. Fractional calculus in solid mechanics: local versus non-local approach. *Phys. Scr.* T136 (2009), 014003.
- [30] CARR, P., GEMAN, H., MADAN, D. B., AND YOR, M. Stochastic volatility for Lévy processes. *Math. Finance* 13 (2003), 345–382.
- [31] CHAKRABARTY, A., AND MEERSCHAERT, M. M. Tempered stable laws as random walk limits. *Stat. Probab. Lett.* 81 (2011).
- [32] CHECHKIN, A. V., GONCHAR, V. Y., GORENFLO, R., KORABEL, N., AND SOKOLOV, I. M. Generalized fractional diffusion equations for accelerating subdiffusion and truncated Lévy flights. *Phys. Rev. E* 78 (2008), 021111.
- [33] CHECHKIN, A. V., GORENFLO, R., AND SOKOLOV, I. M. Retarding subdiffusion and accelerating superdiffusion governed by distributed-order fractional diffusion equation. *Phys. Rev. E* 66 (2002), 046129.
- [34] CHECHKIN, A. V., GORENFLO, R., AND SOKOLOV, I. M. Fractional diffusion in inhomogeneous media. *Phys. Rev. E* 38 (2005), L679–L684.
- [35] CHECHKIN, A. V., GORENFLO, R., SOKOLOV, I. M., AND GONCHCHAR, V. Y. Distributed order time fractional diffusion equation. *Fract. Calc. Appl. Anal.* 6 (2003), 259–279.
- [36] CHEN, W. Time-space fabric underlying anomalous diffusion. *Chaos, Solitons, Fractals* 28 (2006), 923–929.
- [37] CHEN, W., AND LIANG, Y. New methodologies in fractional and fractal derivatives modeling. *Chaos, Solitons Fract.* 102 (2017), 72–77.
- [38] CHEN, W., SUN, H., ZHANG, X., AND KOROŠAK, D. Anomalous diffusion by fractal and fractional derivatives. *Comput. Math. Appl.* 59 (2010), 1754–1758.
- [39] CHEN, Y., YAN, Y., AND ZHANG, K. On the local fractional derivative. *J. Math. Anal. Appl.* 362 (2010), 17–33.
- [40] CIOC, R. Physical and geometrical interpretation of Grunwald-Letnikov differ-integrals: Measurement of path and acceleration. *Fract. Calcul. Appl. Anal.* 19 (2016), 161–172.

- [41] COIMBRA, C. F. M. Mechanics wuth variable-order differential operators. *Ann. Phys.* 12 (2003), 692–703.
- [42] COMpte, A. Stochastic foundations of fractional dynamics. *Phys. Rev. E* 53 (1996), 4191–4193.
- [43] DE OLIVEIRA, E. C., AND TENEIRO MACHADO, J. A. A review of definitions for fractional derivatives and integral. *Math. Probl. Eng.* 2014 (2014), 238459.
- [44] DELKHOSH, M. Introduction of derivatives and integrals of fractional order and its applications. *Appl. Math. Phys.* 1 (2013), 103–119.
- [45] DEMIRAY, S. T., BULUT, H., AND BELGASEM, F. Sumudu transform methods for analytical solution of fractional type ordinary differential equations. *Math. Probl. Eng.* 2015 (2015), 131690.
- [46] DENG, W. Short memory principle and predicor-corrector approach for fractional differential equations. *J. Comp. Appl. Math.* 206 (2007), 1774–188.
- [47] DENG, W., AND LI, C. Numerical schemes for fractional ordinary differential euations. In *Numerical Modelling* (2012), P. Miidla, Ed., pp. 356–34.
- [48] DENG, W., SINGE, V. P., AND BENGTSSON, L. Numerical solution of fractional advection-dispersion equation. *J. Hydraulic Eng.* 130 (2004), 422–431.
- [49] DENG, W., ZHAO, L. J., AND WU, Y. J. Efficient algorithm for solving the fractional ordinary differential equations. *Appl. Math. Comp.* 269 (2015), 196–216.
- [50] DIETHELM, K. *The Analysis of Fractional Differential Equations. An Application-Oriented Exposition Using Differential Operators of Caputo Type.* Springer, Berlin-Helderberg, 2010.
- [51] DOHA, E. H., BHRAWY, A. H., AND EZZ-ELDIEN, S. S. A Chebyshev spectral method based on operational matrix fr initial and boundary vale problems of fractional order. *Comput. Math. Appl.* 62 (2011), 2364–2373.
- [52] DU, R., GAO, G., AND SUN, Z. A compact difference scheme for the fractional diffusion-wave equations. *Appl. Math. Model.* 34 (2010), 2998–3007.
- [53] DUAN, J.-S. Time- and space-fractional partial differential equations. *J. Math. Phys.* 46 (2005), 013504.
- [54] DUAN, J.-S., RACH, R., BALENU, D., AND WAZWAZ, A.-M. A review of the Adomian decomposition method and its applications to fractional differential equations. *Commun. Frac. Calc.* 3 (2012), 73–99.
- [55] DUAN, Y. A note on the meshless method using radial basis functions. *Comp. Math. Appl.* 55 (2008), 66–75.
- [56] DZHERBASHYAN, M. M., AND NERSESIAN, A. D. Fractional derivatives and the Cauchy problem for differential equations of fractional order (in Russian). *Izv. Acad. Nauk Armjanskoy SSR, Matematika* 3 (1968), 3–29.

- [57] E., T. V. Interpretation of fractional derivatives as reconstruction from sequence of integer derivatives. *Fundamenta Informat.* 151 (2017), 431–442.
- [58] ESMAELI, S., AND SHAMSI, M. A pseudo-spectral scheme for the approximate solution of a family of fractional differential equations. *Commun. Nonlinear Sci. Numer. Simulat.* 16 (2011), 3646–3654.
- [59] FARAZ, N., KHAN, Y., JAFARI, H., YILDRIM, A., AND MADANI, M. Fractional variational iteration method via modified Riemann-Liouville derivative. *J. King Saud Univ. - Sci.* 23 (2011), 413–417.
- [60] FERNANDEZ, A., ÖZARSLAN, M. A., AND BALEANU, D. On fractional calculus with general analytic kernels. *Appl. Math. Comput.* 354 (2019), 248–265.
- [61] FIX, G. J., AND ROOP, J. P. Least square finite-element solution of a fractional order two-point boundary value problem. *Comput. Math. Appl.* 48 (2004), 1017–1033.
- [62] FORD, N. J., AND CONNOLLY, J. A. Comparison of numerical methods for fractional differential equations. *Comm. Pure Appl. Anal.* 5 (2006), 289–307.
- [63] FORD, N. J., AND SIMPSON, A. C. The numerical solution of fractional differential equations: speed versus accuracy. *Numer. Algor.* 26 (2001), 333–346.
- [64] GAO, G., AND SUN, Z. A compact difference scheme for the fractional sub-diffusion equations. *J. Comput. Phys.* 230 (2011), 586–595.
- [65] GERASIMOV, A. N. Generalization of the linear deformation laws and applications to the problems of inner friction (in Russian). *Appl. Math. Mech.* 12 (1948), 529–539.
- [66] GHAZANFARI, B., AND EBRAHIMI, P. Differential transformation method for solving fuzzy fractonal heat equations. *Int. J. Math. Model. Comput.* 5 (2015), 81–89.
- [67] GHAZIZADEH, H. R., MAEREFAT, M., AND AZIMI, A. Explicit and implicit finite difference schemes for fractional Cattaneo equation. *J. Comput. Phys.* 229 (2010), 7042–7057.
- [68] GHORBANI, A. Beyond (A)domian polynomials: He polynomials. *Chaos Soliton. Fract.* 39 (2009), 1486–1492.
- [69] GOLOVIZNIN, V. M., KONDRATENKO, P. S., MATVEEV, L. V., KOROTKIN, I. A., AND DRANIKOV, I. L. *Anomalous Diffusion of Radionuclides in Strongly Nonuniform Geological Formations* (in Russian). Nauka, Moscow, 2010.
- [70] GÓMEZ-AGUILAR, J. F., RAZO-HERNANDEZ, R., AND GRANADOS-LIEBERMA, D. A physical interpretation of fractional calculus in observables terms: analysis of fractional time constant and the transitory response. *Revista Mexicana de Fisica* 60 (2014), 32–38.

- [71] GORENFLO, R., LOUTSCHKO, J., AND LUCHKO, Y. Computation of the Mittag-Leffler function and its derivatives. *Fract. Calcul. Appl. Anal.* 5 (2002), 491–518.
- [72] GORENFLO, R., LUCHKO, Y., AND MAINARDI, F. Analytical properties and applications of the Wright function. *Fract. Calcul. Appl. Anal.* 2 (1999), 383–414.
- [73] GORENFLO, R., LUCHKO, Y., AND MAINARDI, F. Wright functions as scale-invariant solutions of diffusion-wave equation. *J. Comput. Appl. Math.* 118 (2000), 175–191.
- [74] GORENFLO, R., LUCHKO, Y., AND ROGOSIN, S. V. Mittag-Leffler type functions, notes on growth properties and distribution of zeros. Tech. Rep. A04-97, Freie Universität Berlin, 1997.
- [75] GORENFLO, R., AND MAINARDI, F. Integral and differential equations of fractional order. In *Fractals and Fractional Calculus in Continuum Mechanics* (Wien, New York, 1997), A. Carpinteri and F. Mainardi, Eds., Springer.
- [76] GORENFLO, R., AND MAINARDI, F. Fractional diffusion processes: probability distributions and continuos time random walk. In *Processes with Long Range Correlations* (Berlin, 2003), G. Rangarajan and M. Ding, Eds., Springer, pp. 148–166.
- [77] GORENFLO, R., AND MAINARDI, F. Simply and multiply scaled diffusion limits for continuous time random walks. *J. Phys. Conf. Series* 7 (2005), 1–16.
- [78] GORENFLO, R., MAINARDI, F., MORETTI, D., PAGNINI, G., AND PARADISO, P. Fractional diffusion: probability disiributions and random walk models. *Physica A* 305 (2002), 106–112.
- [79] GORENFLO, R., MAINARDI, F., AND SRIVASTAVA, H. M. Special functions in fractional relaxation-oscillation and fractional diffusion-wave phenomena. In *Proc. VIII Int. Colloq. Differ. Equat.* (1997), D. Bainov, Ed., pp. 195–202.
- [80] GORENFLO, R., AND RUTMAN, R. On ultraslow and intermediate processes. In *Transform Metods and Special Functions* (Singapore, 1995), D. Rusev, I. Dimovski, and V. Kiryakova, Eds., pp. 171–183.
- [81] GORENFO, R., ISKENDEROV, A., AND LUCHKO, Y. Mapping between soltions of fractional diffusion-wave equations. *Fract. Calc. Appl. Anal.* 3 (2000), 75.
- [82] GRIGOLETTO, E. C., AND DE OLIVEIRA, E. C. Fractional versions of the fundamental theorem of calculus. *Appl. Math.* 4 (2013), 23–33.
- [83] GUO, S., MEI, L., LI, Y., AND SUN, Y. The improved fractional sub-equation method and its applications to the space-time fractional differential equations in fluid mechanics. *Phys. Lett. A* 376 (2012), 407–411.
- [84] HANERT, E. A comparison of three Eulerian numerical methods for fractional-order transport models. *Environ. Fluid Mech.* 10 (2010), 7–20.

- [85] HANERT, E. On the numerical solution of space-time fractional diffusion models. *Comput. Fluids* 46 (2011), 33–39.
- [86] HAUBOLD, H. J., MATHAI, A. M., AND SAXENA, R. K. Mittag-Leffler functions and their applications. *J. Appl. Math.* 2011 (2011), 298628.
- [87] HE, J.-H. Variational iteration approach to nonlinear problems and its applications. *Int. J. Non-Linear Mech.* 20 (1998), 30–31.
- [88] HE, J.-H. Homotopy perturbation technique. *Comp. Meth. Appl. Mech.* 178 (1999), 257–262.
- [89] HE, J.-H. New interpretation of homotopy perturbation method. *Int. J. Modern Phys. B* 20 (2006), 2561–2568.
- [90] HE, J.-H. A tutorial review on fractal spacetime and fractional calculus. *Int. J. Theor. Phys.* 53 (2014), 3698–3718.
- [91] HE, J.-H. Recent development of the homotopy perturbation method. *Topolog. Meth. Nonlinear Anal.* 31 (2016), 205–209.
- [92] HE, J.-H. Fractal calculus and its geometrical explanation. *Results Phys.* 10 (2018), 272–276.
- [93] HE, J.-H., ELAGAN, S. K., AND LI, Z. B. Geometrical explanation of the fractional complex transform and derivative chain rule for fractional calculus. *Phys. Lett. A* 376 (2012), 257–259.
- [94] HE, J.-H., AND LI, Z.-B. Converting fractional differential equations into partial differential equations. *Thermal Sci.* 16 (2012), 331–334.
- [95] HE, J. H., AND LIU, F. Local fractional iterative method for fractal heat transfer in silk cocoon hierarchy. *Nonlinear Sci. Lett. A* 4 (2013), 15–20.
- [96] HEYDARI, M. H., MAALEK GHAINI, F. M., AND HOOSHMANDASL, M. R. Legendre wavelet method for numerical solution of time-fractional heat equation. *Wavelets Lin. Algeb.* 1 (2014), 15–24.
- [97] HEYMANS, N., AND PODLUBNY, I. Physical interpretation of initial conditions for fractional differential equations with Riemann-Liouville fractional derivatives. *Rheolog. Acta* 45 (2006), 765–772.
- [98] HILFER, R. Fractional diffusion based on Riemann-Liouville fractional derivatives. arXiv:cond-mat/0006427 [cond-mat.stat-mech], 2000.
- [99] HILFER, R. Threefold introduction to fractional derivatives. In *Anomalous Transport. Foundations and Applications* (2008), R. Klages, G. Radons, and I. M. Sokolov, Eds., Wiley-VCH, pp. 17–74.
- [100] HILFER, R. Mathematical and physical interpretations of fractional derivatives and integrals. In *In Handbook of Fractional Calculus with Applications* (Berlin, 2019), A. Kochubei and Y. Luchko, Eds., vol. 1, de Gruyter, pp. 47–85.

- [101] HRISTOV, J. Heat-balance integral to fractional (half-time) heat diffusion sub-model. *Thermal Sci.* 14 (2010), 291–316.
- [102] HRISTOV, J. Approximate solutions to fractional sub-diffusion equations: The heat-balance integral method. *Europ. Phys. J. — Special Topics* 193 (2011), 229–243.
- [103] HRISTOV, J. Transient flow of a generalized second grade fluid due to a constant surface shear stress: an approximate integral-balance solution. *Int. Rev. Chem. Eng.* 3 (2011), 802–809.
- [104] HUANG, Q., HUANG, G., AND ZHAN, H. A finite elements solution for the fractional advection-dispersion equation. *Adv. Water Res.* 31 (2008), 1578–1589.
- [105] IYIOLA, O. S., AND NWAEEZE, E. R. Some new results on the new conformable fractional calculus with application using D’Alambert approach. *Progr. Fract. Differ. Appl.* 2 (2016), 1–7.
- [106] JAWAD, A. J. W., PETCOVIC, M. D., AND BISWAS, A. Modified simple equation method for nonlinear evolution equations. *Appl. Math. Comput.* 217 (2010), 869–877.
- [107] JIANG, Y., AND MA, J. High-order finite element methods fr time-fractional partial differential equations. *J. Comp. Appl. Math.* 235 (2011), 3285–3290.
- [108] JIN, B., LAZAROV, R., LIU, Y., AND ZHOU, Z. the Galerkin finite element method for a multi-term time-fractional diffusion equation. *J. Comput. Phys* 281 (2015), 825–843.
- [109] JONEIDI, A. A., GANJI, D. D., AND BABAELAHI, M. Differential transformation method to detrmine the efficiency of convective straight fins with temperature dependent thermal conductivity. *Int. Comm. Heat Mass Transfer* 36 (2009), 757–762.
- [110] JUMARIE, G. Modified Riemann-Liouville derivative and fractional Taylor series of nondifferentiable functions further results. *Comput. Math. Appl.* 51 (2006), 1376–1376.
- [111] JUMARIE, G. Lagrangian mechanics of farctional order, Hamilton-Jacobi fractional PDE and Taylor’s series of nondifferentiable functions. *Chaos Solitons Fractals* 32 (2007), 969–987.
- [112] JUMARIE, G. The Minlowski’s space-time is consistent with differential geometry of fractional order. *Phys. Lett.* 363 (2007).
- [113] JUMARIE, G. The Leibniz rule for fractional derivatives holds with non-differentiable functions. *Math. Stat.* 1 (2013), 50–52.
- [114] JUMARIE, G. On the derivative chain-rules in fractional calculus via fractional difference and their application to systems modelling. *Open. Phys.* 11 (2013), 617–633.

- [115] K., D., FORD, J. M., FORD, N. J., AND WEILBEER, M. Pitfalls in fast solvers for fractional differential equations. *J. Comp. Appl. Math.* 186 (2006), 482–503.
- [116] KATUGAMPOLA, U. N. New approach to a generalized fractional derivative. *Math. Anal. Appl.* 6 (2014), 1–15.
- [117] KATUGAMPOLA, U. N. A new fractional derivative with classical properties. arXiv: 1410.6535 [math.CA], 2014.
- [118] KHALIL, R., AL HORANI, M., YOUSEF, A., AND SABABHEH, M. A new definition of fractional derivative. *J. Comput. Appl. Math.* 264 (2014), 65–70.
- [119] KHAN, Y., FARAZ, N., YILDIRIM, A., AND WU, Q. Fractional varoational iteration method for fractional initial-boundary value problems arising in the application of nonlinear science. *Comput. Math. Appl.* 62 (2011), 2273–2278.
- [120] KILBAS, A. A., SAIGO, M., AND TRUJILLO, J. J. On the generalized Wright function. *Fract. Calcul. Appl. Anal.* 5 (2002), 437–460.
- [121] KILBAS, A. A., SRIVASTAVE, H. M., AND TRUJILLO, J. J. *Theory and Applications of Fractional Differential Equations*. North Holland, 2006.
- [122] KIRYAKOVA, V. *Generalized fractional calculus and applications*. Longman, Harlow, 1994.
- [123] KIRYAKOVA, V. Multiple (multiindex) Mittag-Leffler functions and relations to generalized fractional calculus. *J. Comput. Appl. Math.* 118 (2000), 241–259.
- [124] KIRYAKOVA, V. Multiindex Mittag-Leffler functions as an important class of special functions of fractional calculus. *Comput. Math. Appl.* 59 (2010), 1885–1895.
- [125] KOCHUBEI, A. I. Distributed order calculus and equations of ultraslow diffusion. *J. Math. Anal. Appl.* 340 (2008), 252–281.
- [126] KOLWANKAR, K. M. Local fractional calculus: a review. arXiv:1307.0739, 2013.
- [127] KOLWANKAR, K. M., AND GANGAL, A. D. Hölder exponents of irregular functions and local fractional derivatives. *Pramana J. Phys.* 48 (1997), 49–68.
- [128] KOLWANKAR, K. M., AND GANGAL, A. D. Local fractional derivatives and fractal functions of several variables. arXiv: physics/9801010 [math-ph], 1998.
- [129] KOLWANKAR, K. M., AND GANGAL, A. D. Local fractional Fokker-Plank equation. *Phys. Rev. Lett.* 80 (1998), 214–217.
- [130] LEBEDEV, N. N. *Special Functions and their Applications*. Prentice-Hall, 1965.
- [131] LESNIC, D., AND ELLIOT, L. The decomposition approach to inverse heat conduction. *J. Math. Anal. Appl.* 232 (1999), 82–98.

- [132] LETNIKOV, A. V. Theory of differentiation of fractional order. *Math. Sb.* 3 (1868), 1–7.
- [133] LI, C., AND DENG, W. Remarks on fractional derivatives. *Appl. Math. Comput.* 187 (2007), 777–784.
- [134] LI, C., QIAN, D., AND CHWN, Y. Q. On Riemann-Liouville and Caputo derivatives. *Discr. Dynam. Nature Soc.* 2011 (2011), 562494.
- [135] LI, C., AND ZENG, F. Finite difference methods for fractional differential rquations. *Int. J. Bifurc. Chaos* 22 (2012), 1230014.
- [136] LI, Z.-B. An extended fractional complex transform. *J. Nonlinear Sci. Numer. Simul.* 11 (2010), S0335–S0337.
- [137] LI, Z.-B., AND HE, J.-H. Fractional complex transform for fracial differential equations. *Math. Comput. Appl.* 15 (2010), 970–973.
- [138] LI, Z.-B., ZHU, W.-H., AND HE, J.-H. Exact solutions of time-fractional heat conduction equation by the extended fractional complex transform. *Thermal Sci* 16 (2012), 335–338.
- [139] LIAO, S. J. Numerically solving nonlinear problems by the homotopy analysis method. *Comp. Mech.* 20 (1997), 530–540.
- [140] LIU, F.-J., LI, Z.-B., ZHANG, S., AND LIU, H.-Y. He’s fractional derivative for heat conduction in a fractal medium arising in silkworm cocon hierarchy. *Thermal Sci.* 19 (2015), 1155–1159.
- [141] LORENZO, C. F., AND HARTLEY, T. T. Initialization, conceptualization, and application in the generalized fractional calculus. NASA/TP-1998-208415, 1998.
- [142] LOSADA, J., AND NIETO, J. J. Properties of a new fractionalmderivative without singular kernel. *Prog. Fract. Differ. Appl.* 1 (2015), 87–92.
- [143] LUCHKO, Y. Operational rules for a mixed operator of the Erdélyi-Kober type. *Frac. Calc. Appl. Anal.* 7 (2004), 339–364.
- [144] LUCHKO, Y. Maximum principle for the generalized time-fractional diffusion equation. *J. Math. Anal. Appl.* 351 (2009), 218–223.
- [145] LUCHKO, Y., AND KIRYAKOVA, V. The Mellin integral transform in fractional calculus. *Fract. Calc. Appl. Anal.* 16 (2013), 405–430.
- [146] MAINARDI, F. Fractional relaxation-oscillation and fractional diffusion-wave phenomena. *Chaos Solitons Fract.* 7 (1996), 1461–1477.
- [147] MAINARDI, F. Fractional calculus: some basic problems in continuum and statistical mechanics. In *Fractals and Fractional Calculus in Continuum Mechanics* (Wien and N.Y., 1997), A. Carpinteri and F. Mainardi, Eds., Springer, pp. 231–248.

- [148] MAINARDI, F. Application of integral transforms in fractional diffusion processes. *Integral Transf. Special Funct.* 15 (2004), 477–484.
- [149] MAINARDI, F., AND GORENFLO, R. On mittag-leffler function in fractional evaluation processes. *J. Comput. Appl. Math.* 118 (2000), 283–299.
- [150] MAINARDI, F., AND GORENFLO, R. Time-fractional derivatives in relaxation processes: a tutorial survey. *Int. J. Theor. Appl.* 10 (2007), 269–308.
- [151] MAINARDI, F., LUCHKO, Y., AND PAGNINI, G. The fundamental solution of the space-time fractional diffusion equation. *Fract. Calc. Appl. Anal.* 4 (2001), 153–192.
- [152] MAINARDI, F., MURA, A., AND PAGNINI, G. The M-Wright function in time-fractional diffusion processes: a tutorial survey. *Int. J. Differ. Equations* 2010 (2010), 104505.
- [153] MAINARDI, F., MURA, A., PAGNINI, G., AND GORENFLO, R. Time-fractional diffusion of distributed order. arXiv: 0701132 [cond-mat.stat-mech], 2007.
- [154] MAINARDI, F., AND PAGNINI, G. The Wright functions as soluition of the time-fractional diffusion equation. *Appl.Math. Comput.* 141 (2003), 51–62.
- [155] MAINARDI, F., PAGNINI, G., AND GORENFLO, R. Some aspects of fractional diffusion equatins of single and distributed order. *Appl. Math. Comput.* 187 (2007), 295–305.
- [156] MAINARDI, F., PAGNINI, G., AND SAXENA, R. K. The Fox H functions in fractional diffusion. *J. Comput. Appl. Math.* 178 (2005), 321–331.
- [157] MEERSCHAERT, M. M., ZHANG, Y., AND BAEUMER, B. Tempered anomalous diffusions in heterogeneous systems. *Geophys. Res. Lett.* 35 (2008), L17403–L17407.
- [158] METZLER, R., GLÖCKLE, W. G., AND NONNENMACHER, T. F. Fractional model equation for anomalous diffusion. *Physica A* 211 (1994), 13–24.
- [159] METZLER, R., AND KLAFTER, J. The restaurant at the end of the random walk: recent developments in the description of anomalous transport by fractional dynamics. *J. Phys. A: Math. Gen.* 37 (2004), R161–R208.
- [160] MILLER, K., AND ROSS, B. *An introduction to the fractional calculus and fractional differential equations*. John Wiley and Sons, New York, 1993.
- [161] MOMANI, S., ODIBAT, Z., AND HASHIM, I. Algorithms for nonlonear fractional partial differentialequations: a selection of numerical methods. *Topol. Meth. Nonlin. Anal.* 31 (2008), 211–226.
- [162] MOSHREFI-TORBATI, M., AND HAMMOND, J. K. Physical and geometrical interpretation of fractional operators. *J. Franklin Institute* 335 (1998), 1077–1086.

- [163] MURIO, D. A. Implicit finite difference approximation for time fractional diffusion equations. *Comput. Math. Appl.* 56 (2008), 1138–1145.
- [164] MUSTAPHA, K. A. Superconvergent discontinuous Galerkin method for Volterra integro-differential equations. *Math. Comput.* 82 (2013), 1987–2005.
- [165] NABER, M. Distributed order fractional sub-diffusion. *Fractals* 12 (2004), 23–32.
- [166] NAKHUSHEV, A. M. *Fractional Calculus and Applications*. Fismatlit, Moscow, 2003.
- [167] NAKHUSHEVA, V. A. *Differential Equations of Mathematical Models of Nonlocal Processes (in Russian)*. Nauka, Moscow, 2006.
- [168] NIETO, J. J. Maximum principle for fractional differential equations derived from Mittag-Leffler functions. *Appl. Math. Lett.* 23 (2010), 1248–1251.
- [169] NIGMATULLIN, R. R. A fractional integral and its physical interpretation. *Theor. Math. Phys.* 90 (1992), 242–251.
- [170] NIGMATULLIN, R. R., AND LE MEHAUTE, A. Is there geometrical/physical meaning of the fractional integral with complex exponent. *J. Non-Crystalline Solids* 351 (2005), 2888–2899.
- [171] NOOR, M. A., AND MOHYUD-DIN, S. T. Variational iteration method for solving higher-order nonlinear boundary value problems using He's polynomials. *Int. J. Nonlinear Sci. Numer. Simul.* 9 (2008), 141–156.
- [172] ODIBAT, Z. M., AND MOMANI, S. An algorithm for the numerical solution of differential equations of fractional order. *J. Appl. Math. Informatics* 26 (2008), 15–27.
- [173] ODIBAT, Z. M., AND SHAWAGFEH, N. T. Generalized Taylor's formula. *Appl. Math. Comput.* 186 (2007), 285–294.
- [174] OLDHAM, K., AND SPANIER, J. *The fractional calculus; theory and applications of differentiation and integration to arbitrary order*. Academic Press, New York, 1974.
- [175] ONTOOLAN, J., BORRES, M., PATAK, A., AND MAGLASANG, G. Review of fractal and fractal derivatives in relation to the physics of fractals. *UVJ. Res.* (2013), 219–228.
- [176] OSLER, T. J. Leibniz rule for fractional derivatives generalized and application to infinite series. *SIAM J. Appl. Math.* 18 (1970), 658–674.
- [177] OSLER, T. J. A correction to Leibniz rule for fractional derivatives. *SIAM J. Math. Anal.* 4 (1973), 456–459.
- [178] PAGNINI, G. Erdélyi-Kober fractional diffusion. *Frac. Calc. Appl. Anal.* 15 (2012), 117–127.

- [179] PENG, J., AND LI, K. A note on property of the Mittag-Leffler function. *J. Math. Anal. Appl.* 370 (2010), 635–638.
- [180] PIRET, C., AND HANERT, E. A radial basis function method for fractional diffusion. *J. Comput. Phys* 238 (2013), 71–78.
- [181] PODLUBNY, I. *Fractional Differential Equations*. Academic Press, 1998.
- [182] PODLUBNY, I. Matrix approach to discrete fractional calculus. *Frac. Calc. Appl. Anal.* 3 (2000), 359–386.
- [183] PODLUBNY, I. Geometric and physical interpretation of fractional integration and fractional differentiation. *Frac. Calc. Appl. Anal.* 5 (2002), 36–386.
- [184] PODLUBNY, I., CHECHKIN, A., CHEN, Y. Q., AND VINAGRE, B. M. J. Matrix approach to discrete fractional calculus II: Partial differential equations. *J. Comput. Phys.* 228 (2009), 3137–3153.
- [185] PODLUBNY, I., DESPOTOVIC, V., SKOVRALEK, T., AND McNAUGHTON, B. H. Shadows on the walls: Geometric interpretation of fractional integration. *The J. Online Math. Appl.* 7 (2007), 1664.
- [186] PRABHAKAR, T. R. A singular integral equation with a generalized Mittag-Leffler function in the kernel. *Yokohama Math.* 19 (1971), 7–15.
- [187] PRIETO, A. I., DE ROMERO, S. S., AND SRIVASTAVA, H. M. Some fractional-calculus results involving the generalized Lommel-Wright and related functions. *Appl. Math. Lett.* 20 (2007), 17–22.
- [188] QI, H. T., XU, H. Y., AND GUO, X. W. The generalized Taylor’s formula. *Appl. Math. Comput.* 186 (2007), 286–293.
- [189] RABOTNOV, Y. N. *Elements of Hereditary Mechanics of Solids (in Russian)*. Nauka, Moscow, 1977.
- [190] RAFTARI, B., AND VAJRAVELU, K. Homotopy analysis method for MHD viscoelastic fluid flow and heat transfer in a channel with a stretching wall. *Comm. Nonlin. Sci. Numer. Simul.* 17 (2012), 4149–4162.
- [191] RAHIMY, M. Applications of fractional differential equations. *Appl. Math. Sci.* 4 (2010), 2453–2461.
- [192] ROOP, J. P. Computational aspects of FEM approximation of fractional advection diffusion equations on bounded domains in  $\mathbb{R}^2$ . *J. Comput. appl. Math.* 193 (2006), 243–268.
- [193] ROSS, B. The development of fractional calculus 1695-1990. *Historia Math.* 4 (1977), 75–89.
- [194] RUTMAN, R. S. On physical interpretations of fractional integration and differentiation. *Theor. Math. Phys.* 105 (1995), 1509–1519.

- [195] SABZIKARA, F., MEERSCHAERTA, M. M., AND CHEN, J. Tempered fractional calculus. *J. Comp. Phys.* 293 (2015), 14–28.
- [196] SAICHEV, A., AND ZASLAVSKY, G. Fractional kinetic equations: solutions and applications. *Chaos* 7 (1997), 753–764.
- [197] SAMKO, A. G. Fractional integration and differentiatin of variable order. *Anal. Math.* 21 (1995), 213–236.
- [198] SAMKO, S. G., KILBAS, A. A., AND MARICHEV, O. I. *Fractional integrals and derivatives. Theory and applications*. Gordon and Breach Science Publishers, 1993.
- [199] SCHERER, R., KALLA, S. L., TANG, Y., AND HUANG, J. The Grünwald-Letnikov method for fractional differential equations. *Comput. Math. Appl.* 62 (2011), 902–917.
- [200] SCHNEIDER, W. R., AND WYSS, W. Fractional diffusion and wave equations. *J. Math. Phys.* 30 (1989), 134–144.
- [201] SHIRZADI, A., LING, L., AND ABBASBANDY, S. Meshless simulations of the two-dimensional fractional-time convection-diffusion-reaction equations. *Eng. Anal. Boundar. Elem.* 36 (2012), 1522–1527.
- [202] SHUBIN, M. A. *Pseudo Differential Operators and Spectral Theory*. Springer, Berlin, 1987.
- [203] SHUKLA, A. K., AND PRAJAPATI, J. C. On a generalization of Mittag-Lefler function and its application. *J. Math. Anal. Appl.* 336 (2007), 797–811.
- [204] SIEROCIUK, D., SKOVRANEK, T., MACIAS, M., PODLUBNY, I., PETRA, I., DZIELINSKI, A., AND ZIUBINSKY, P. Diffusion process modeling by using fractional-order models. *Appl. Math. Comp.* 257 (2015), 2–11.
- [205] SINGH, J., GUPTA, P. K., AND RAI, K. N. Homotopy perturbation method to space-time fractional solidification in a finite slab. *Appl. Math. Model.* 35 (2011), 1937–1945.
- [206] SINGH, S. J., AND CHATTERJEE, A. Galerkin projections and finite elements for fractional order derivatives. *Nonlinear Dyn.* 45 (2006), 183–206.
- [207] SOKOLOV, I. M., KLAFTER, J., AND BLUMEN, A. Fractional kinetics. *Phys. Todsay* (2002), 48–54.
- [208] STANISLAVSKY, A. A. Probabilistic interpretation of the integral of fractional order. *Theor. Math. Phys.* 138 (2004), 418–431.
- [209] SU, W.-H., YANG, X.-J., JAFAN, H., AND BALEANU, D. Fractional complex transform method for wave equations on Cantor sets within local fractional differential operator. *Adv. Diff. Equat.* 2013 (2013), 97.

- [210] SUN, X. G., HAO, X., ZHANG, Y., AND BALEANU, D. Relaxation and diffusion models with non-singular kernels. *Phys. A Stat. Mech. Appl.* 468 (2017), 590–596.
- [211] TADJERAN, C., AND MEERSCHAERT, M. M. A second-order accurate numerical method for the two-dimensional fractional diffusion equation. *J. Comput. Phys* 220 (2007), 813–823.
- [212] TADJERAN, C., MEERSCHAERT, M. M., AND SCHEEFFLER, H.-P. A second-order accurate numerical approximation for the fractional diffusion equation. *J. Comput. Phys* 213 (2006), 205–213.
- [213] TARASOV, V. No nonlocality, no farctional derivative. *Nonlinear Sci. Numer. Simulat.* 62 (2018), 157–163.
- [214] TARASOV, V. E. *Fractional Dynamics: Applications of Fractional Calculus to Dynamics of Particles, Fields and Media (Nonlinear Physical Science)*. Springer, 2011.
- [215] TARASOV, V. E. No violation of the Lebniz rule. *Commun. Nonlinear Sci. Numer. Simul.* 18 (2013), 2945–2948.
- [216] TARASOV, V. E. Heat transfer in fractal materials. *Int. J. Heat Mass Transfer* 93 (2016), 427–430.
- [217] TARASOV, V. E. Leibniz rule and fractional derivative of power functions. *J. Comput. Nonlinear Dynam.* 11 (2016), 031014.
- [218] TARASOV, V. E. On chain rule for fractional derivatives. *Comm. Nonlin. Sci. Num. Simul.* 30 (2016), 1–4.
- [219] TATEISHI, A. A., RIBEIRO, H. V., AND LENZI, E. K. The role of fractional time-derivative operators on anomalous diffusion. *Front. Phys.* 5 (2017), 52.
- [220] TAYLOR, M. E. *Pseudo Differential Operators*. Princeton University Press, Princeton, 1981.
- [221] TENEIRO MACHADO, J. A. Fractional derivatives: Probability interpretation and frequency response of rational approximations. *Comm. Nonlin. Sci. Num. Sim.* 14 (2009), 3492–3497.
- [222] TENEIRO MACHADO, J. A. A probabilistic interpretation of the fractional-order differentiation. *Fract. Calcul. Appl. Anal.* 6 (2009), 73–80.
- [223] TENEIRO MACHADO, J. A., KIRYAKOVA, V., AND MAINARDI, F. A poster about the recent history of fractional calculus. *Fract. Calculus Appl. Anal.* 13 (2010), 329–334.
- [224] TENEIRO MACHADO, J. A., KIRYAKOVA, V., AND MAINARDI, F. Recent history of fractional calculus. *Commun. Nonlin. Sci. Numer. Simul.* 16 (2011), 1140–1153.

- [225] VALERIO, D., MACHADO, J. T., AND KIRYAKOVA, V. Some pioneers of the applications of fractional calculus. *Fract. Calculus Appl. Anal.* 17 (2011), 552–578.
- [226] VAN DYKE, M. *Perturbation Methods in Fluid Mechanics*. Parabolic Press, Standford, 1975.
- [227] VERMEERSCH, B., AND SHAKOURI, A. Spatiotemporal flux memory to nondiffusive transport. arXiv: 1412.8571 [cond-mat.stat-mech], 2014.
- [228] VOLTERRA, V. *Theory of Functional and of Integral and Integro-Differential Equations*. Blackie and Son Ltd., London and Glasgow, 1930.
- [229] VONG, S., LYU, P., CHEN, X., AND LEI, S., L. High order finite difference method for time-space fractional differential equations with Caputo and Riemann-Liouville derivatives. *Numer. Algor.* 72 (2016), 195–210.
- [230] VONG, S., AND WANG, Z. A high order compact scheme for the fractional Fokker-Planck equation. *Appl. Math. Lett.* 43 (2015), 38–43.
- [231] WATUGALA, G. K. The Sumudu transform for functions of two variables. *Math. Eng. Industr.* 8 (2002), 293–302.
- [232] WAZWAZ, A.-M., AND MEHANNA, M. S. The combined Laplace-Adomian method for handling singular integral equation of heat transfer. *Int. J. Nonlinear Sci.* 10 (2010), 248–252.
- [233] WEI, C., AND WANG, H. Solutions of the heat-conduction model described by fractional Emden-Fowler type equation. *Thermal Sci.* 21 (2017), S113–S120.
- [234] WEI, S., CHEN, W., AND HON, Y.-C. Implicit local radial basis function method for solving two-dimensional time fractional diffusion equations. *Thermal Sci.* 19 (2015), S59–S67.
- [235] WEST, B. J., GRIGOLINI, P., METZLER, R., AND NONNENMACHER, T. F. Fractional diffusion and Lévy stable processes. *Phys. Rev. E* 55 (1997), 99–106.
- [236] WU, G. C. Applications of the variational iteration method to fractional diffusion equation: local versus nonlocal ones. *Int. Rev. Chem. Eng.* 4 (2012), 505–510.
- [237] WU, G. C., AND LEE, E. W. M. Fractional variational iteration method and its applications. *Phys. Lett. A* 374 (2010), 2506–2509.
- [238] YAN, L.-M. Modified homotopy perturbation method coupled with Laplace transform for fractional heat transfer and porous media equations. *Thermal Sci.* 17 (2013), 1409–1414.
- [239] YANG, A. M., CATTANI, C., JAFARI, H., AND YANG, X. J. Analytical solutions of the one-dimensional heat equations arising in fractal transient conduction with local fractional derivatives. *Abstract Appl. Anal.* 2013 (2013), 462535.

- [240] YANG, A.-M., ZHANG, C., JAFARI, H., CATTANI, C., AND JIAO, Y. Picard successive approximation method for solving differential equations in fractal heat transfer with local fractional derivative. *Abstr. Appl. Anal.* 2014 (2014), 395710.
- [241] YANG, X. J. *Advanced Local Fractional Calculus and its Applications*. World Science Publisher, N. Y., 2012.
- [242] YANG, X.-J. Generalized local fractional Taylor's formula with local farctional derivative. *J. Expert Systems* 1 (2012), 1–5.
- [243] YANG, X. J. Picard's approximation method for solving a class of local fractional Volterra integral equations. *Adv. Intell. Transport. Syst.* 1 (2012), 67–70.
- [244] YANG, X. J., AND BALEANU, D. Fractal heat conduction problem solved by local fractional variation method. *Thermal Sci.* 17 (2013), 625–628.
- [245] YANG, X.-J., BALEANU, D., AND HE, J.-H. Transport equations in fractal porous media within fractional complex transform method. *Proc. Romanian Acad.* 14 (2013), 287–292.
- [246] YANG, X.-J., BALEANU, D., AND SRIVASTAVA, H. M. *Local Fractional Integral Transforms and their Applications*. Academic Press, 2015.
- [247] YANG, X.-J., SRIVASTAVA, H. M., AND CATTANI, C. Local fractional homotopy perturbation method for solving fractal partial differential equations arising in mathematical physics. *Rom. Rep. Phys.* 67 (2015), 752–761.
- [248] YANG, X.-J., AND ZHANG, F. R. Local fractional variational iteration method and its algorithms. *Adv. Comput. Math. Appl.* 1 (2012), 139–145.
- [249] YANG, X.-J., ZHANG, Z.-Z., TENEIRO MACHADO, J. A., AND BALEANU, D. On local fractional operators view of computational complexity. Diffusion and relaxation defined on Cantor sets. *Thermal Sci.* 20 (2016), S755–S767.
- [250] YANG, Y.-J., BALEANU, D., AND YANG, X.-J. Analysis of fractal wave equations by local fractional Fourier series method. *Adv. Math. Phys.* 2013 (2013), 632309.
- [251] YILDRIM, A., AND S., M. Series solutions of a fractional oscillator by means of the homotopy perturbation method. *Int. J. Comp. Math.* 87 (2010), 1072–1082.
- [252] YOUNIS, M. A new approach for the exact solution of nonlinear equations of fractional order via modified simple equation method. *Appl. Math.* 5 (2014), 1927–1932.
- [253] ZASLAVSKY, G. M. Chaos, fractional kinetics and anomalous transport. *Phys. Reports* 371 (2002), 461–580.
- [254] ZASLAVSKY, G. M. *Hamiltonian Chaos and Fractional Dynamics*. OUP, 2008.
- [255] ZAYERNOURI, M., AND KARNIADAKIS, G. M. Exponentially accurate spectral and spectral element methods for fractional ODEs. *J. Comput. Phys.* 257 (2014), 460–480.

- [256] ZENG, F., LI, C., LIU, F., AND TURNER, I. Numerical algorithms for time-fractional subdiffusion equations with second-order accuracy. *SIAM J. SCI. Comp.* 37 (2015), A55–A78.
- [257] ZHAI, S., AND FENG, X. Investigations on several compact ADI methods for the 2D time fractional diffusion equation. *Numer. Heat Transfer Fundam.* 69 (2016), 364–376.
- [258] ZHAI, S., WENG, Z., FENG, X., AND YUAN, J. Investigations on several high-order adi methods for time-space fractional diffusion equation. *Numer. Algor.* 82 (2019), 69–106.
- [259] ZHANG, G., AND LI, B. Thermal conductivity of nanotubes revisited: Effects of chirality, isotope impurity, tube length, and temperature. arXiv:cond-mat/0501194[cond-mat.mtrl-sci], 2006.
- [260] ZHANG, S., AND ZHANG, H. Q. Fractional sub-equation method and its applications to nonlinear fractional PDE. *Phys. Lett. A* 375 (2011), 1069–1073.
- [261] ZHANG, Y. Solving initial-boundary value problems for local fractional differential equation by local fractional Fourier series method. *Abstr. Appl. Anal.* 2014 (2014), 912464.
- [262] ZHANG, Y., CATTANI, C., AND YANG, X.-J. Local fractional homotopy perturbation method for solving non-homogeneous heat conduction equations in fractal domains. *Entropy* 17 (2015), 6753–6764.
- [263] ZHANG, Y., MEERSCHAERT, M. M., AND PACKMAN, A. I. Linking fluvial bed sediment transport across scales. *Geophys. Res. Lett.* 39 (2012), L20404–L20406.
- [264] ZHAO, D., AND LUO, M. Representations of acting processes and memory effects: general fractional derivative and its application to theory of heat conduction with finite wave speeds. *Appl. Math. Comput.* 346 (2019), 531–544.
- [265] ZHAO, D., SINGH, J., KUMAR, D., RATHORE, S., AND YANG, X.-J. An efficient computational technique for local fractional heat conduction equations in fractal media. *J. Nonlin. Sci. Appl.* 10 (2017), 1478–1486.
- [266] ZHAO, Y., CAI, Y.-G., AND YANG, X.-J. A local fractional derivative with applications to fractal relaxation and diffusion phenomena. *Thermal Sci.* 20 (2016), S723–S727.
- [267] ZHU, P., AND XIE, S. ADI finite element method for 2D nonlinear time fractional reaction-subdiffusion equation. *Am. J. Comput. Math.* 6 (2016), 336–356.