Results in Physics 31 (2021) 104950

Contents lists available at ScienceDirect

Results in Physics

journal homepage: www.elsevier.com/locate/rinp

A new and general fractional Lagrangian approach: A capacitor microphone case study

A. Jajarmi [a], D. Baleanu [b,c,d,*], K. Zarghami Vahid [e], H. Mohammadi Pirouz [a], J.H. Asad [f]

[a] Department of Electrical Engineering, University of Bojnord, P.O. Box, 94531-1339, Bojnord, Iran
[b] Department of Mathematics, Faculty of Arts and Sciences, Cankaya University, 06530 Ankara, Turkey
[c] Institute of Space Sciences, P.O. Box, MG-23, R 76900, Magurele-Bucharest, Romania
[d] Department of Medical Research, China Medical University Hospital, China Medical University, Taichung, Taiwan
[e] Islamic Azad University, Tehran Science and Research Branch, Tehran, Iran
[f] Department of Physics, College of Sciences, Palestine Technical University, Tulkarm, Palestine

ARTICLE INFO

Keywords:
Fractional calculus
Euler–Lagrange equations
General kernel function
Capacitor microphone system
Matrix approximation technique

ABSTRACT

In this study, a new and general fractional formulation is presented to investigate the complex behaviors of a capacitor microphone dynamical system. Initially, for both displacement and electrical charge, the classical Euler–Lagrange equations are constructed by using the classical Lagrangian approach. Expanding this classical scheme in a general fractional framework provides the new fractional Euler–Lagrange equations in which non-integer order derivatives involve a general function as their kernel. Applying an appropriate matrix approximation technique changes the latter fractional formulation into a nonlinear algebraic system. Finally, the derived system is solved numerically with a discussion on its dynamical behaviors. According to the obtained results, various features of the capacitor microphone under study are discovered due to the flexibility in choosing the kernel, unlike the previous mathematical formalism.

Introduction

In order to demonstrate the dynamical behaviors of real-world systems, there are two principal approaches [1]. The first one, a force-based scheme, is the Newtonian method. Nevertheless, a number of issues might happen in the first approach because it requires to adjust all forces while they may not be clear. The next method applies energies, which is known as Lagrangian technique and introduced by the French Mathematician Joseph Louis Lagrange. Numerous significant dynamical systems like coupled and spring pendulums, Atwood's machine, etc., are described by means of this energy-based approach.

Fractional calculus (FC) is the science of non-integer order integral and differential operators together with their applications. In numerous studies such as control, mechanics, finance, and biology, the benefits of FC have been explored [2–9]. In addition, this calculus modifies the classical mechanics in a new way; for instance, non-conservative Lagrangian systems were investigated by a FC approach in the earlier study [10]. In that work, the fractional-order derivatives defined the conjugate momenta, and the Hamilton equations were formulated in a fractional sense. To study the path of Lévy flights, a fractional path integral technique was applied in [11]. Besides, the fractional quantum mechanics was developed in [12]. Then several relevant articles were published by numerous scientists who followed the aforesaid ideas [13,14]. Based on the mentioned investigations, the asymptotic behavior of a mechanical system can be discovered by means of the fractional Lagrangian approach. Then the new relations namely the fractional Euler–Lagrange equations (FELEs) are achieved by means of this technique. However, significant problems are required to be studied in this regard such as the expansion of effective numerical techniques to solve the FELEs; variational iteration method [15] and Adams–Bashforth–Moulton technique [16] are a few examples of these schemes.

Recently, a new type of fractional derivatives with a general kernel function was defined in a new study by Luchko and Yamamoto [17]. In this approach, numerous applications are covered by means of the flexibility in choosing the kernel function. In fact, changing the kernel causes various asymptotic behaviors and shows the hidden aspects of realistic phenomena in an appropriate, precise manner. Nonetheless, several practical cases should examine the benefits of this new approach. Moreover, efficient numerical and analytical methods are required to be expanded in order to find the solution of fractional differential equations (FDEs) related to the new general fractional operators. More to the point, the theoretical aspects of this general idea

* Corresponding author at: Department of Mathematics, Faculty of Arts and Sciences, Cankaya University, 06530 Ankara, Turkey.
E-mail address: dumitru@cankaya.edu.tr (D. Baleanu).

https://doi.org/10.1016/j.rinp.2021.104950
Received 9 September 2021; Received in revised form 20 October 2021; Accepted 25 October 2021
Available online 6 November 2021

should be explored to find out whether or not they can be extended and used in the other branches of sciences and engineering. These points encourage us to study the behavior of a capacitor microphone by utilizing its new general FELEs. The main contributions of present study are highlighted as follows:

- In the new equations, the fractional-order displacement and electrical charge are formulated in general sense. To the best of our knowledge, this is the first time that one utilizes a general fractional operator for the free motion of a capacitor microphone case study.
- A theoretical study is provided to derive the corresponding FELEs in general sense for the physical system under investigation.
- The derived equations are complicated to solve in practice since they include the left and right general fractional derivatives simultaneously. To overcome this issue, a matrix approximation scheme is prepared, which converts the general equations into a nonlinear algebraic system.
- It is apparent, based on the simulation results, that we could see various asymptotic behaviors for different kernel functions, a fact which is not available in the standard FC modeling.

Therefore, the general fractional Lagrangian approach provides a more flexible model than the standard FC modeling due to the use of general kernel function. This feature could help us to discover the hidden aspects of capacitor microphone under study better than the traditional fractional formulations. Consequently, we believe that the FELEs in general sense and their solution method presented in this paper are new and comprise quite different information than the corresponding standard fractional equations.

This paper is classified into the following sections. In Section "Symbols and preliminaries", some preliminaries related to the general fractional operators are presented. The capacitor microphone model in both classical and fractional frameworks are described in Section "Dynamical behaviors". An effective and new approximation method solves the derived FELEs in Section "Numerical method". Additionally, the numerical results are presented in Section "Simulation results", and the paper is closed by Section "Conclusions". Finally, the correctness of the formulas is proved in Appendix "Derivation of FELE in general sense".

Symbols and preliminaries

The general concept of fractional derivatives and integrals are introduced in this section by means of some symbols and preliminaries. Based on [17], the left-sided Riemann–Liouville (RL) and Caputo fractional derivatives are described, respectively, by the following relations

$$ {}_0\mathscr{D}_t^{\rho} y(t) = \frac{d}{dt}\int_0^t y(\omega)\mathcal{X}_L(t-\omega)d\omega, \tag{1}$$

$$ {}_0^C\mathscr{D}_t^{\rho} y(t) = \int_0^t \dot{y}(\omega)\mathcal{X}_L(t-\omega)d\omega, \tag{2}$$

where $y \in \mathbb{R}$ is an absolutely continuous function satisfying $\dot{y} \in L_1^{loc}(\mathbb{R}^+)$. Moreover, the nonnegative function $\mathcal{X}_L > 0$ is a locally integrable kernel, and ρ is the fractional order such that $0 < \rho < 1$. According to [17], the following relation exists between the general RL and Caputo fractional derivatives (1) and (2)

$$ {}_0^C\mathscr{D}_t^{\rho} y(t) = \frac{d}{dt}\int_0^t y(\omega)\mathcal{X}_L(t-\omega)d\omega - \mathcal{X}_L(t)y(0) = {}_0\mathscr{D}_t^{\rho} y(t) - \mathcal{X}_L(t)y(0). \tag{3}$$

Additionally, it is noticeable that the above operators are linear, *i.e.*,

$$ {}_0\mathscr{D}_t^{\rho}(c_1y_1(t)+c_2y_2(t)) = c_1\,{}_0\mathscr{D}_t^{\rho}y_1(t) + c_2\,{}_0\mathscr{D}_t^{\rho}y_2(t), \tag{4}$$

$$ {}_0^C\mathscr{D}_t^{\rho}(c_1y_1(t)+c_2y_2(t)) = c_1\,{}_0^C\mathscr{D}_t^{\rho}y_1(t) + c_2\,{}_0^C\mathscr{D}_t^{\rho}y_2(t). \tag{5}$$

In addition, if some appropriate conditions are satisfied by the kernel function $\mathcal{X}_L$ [17], an entirely monotone function $\mathscr{Z}_L$ can be defined such that

$$ \mathcal{X}_L(t) * \mathscr{Z}_L(t) = \int_0^{\infty} \mathcal{X}_L(t-\omega)\mathscr{Z}_L(\omega)d\omega = 1,\ t>0. \tag{6}$$

Then we can define the general RL fractional integral explained by

$$ {}_0\mathscr{I}_t^{\rho} y(t) = \int_0^t y(\omega)\mathscr{Z}_L(t-\omega)d\omega. \tag{7}$$

Additionally, for $y \in L_1^{loc}(\mathbb{R}^+)$ we have

$$ {}_0\mathscr{I}_t^{\rho}[{}_0\mathscr{D}_t^{\rho}y(t)] = y(t), \tag{8}$$

$$ {}_0\mathscr{I}_t^{\rho}[{}_0^C\mathscr{D}_t^{\rho}y(t)] = y(t) - y(0). \tag{9}$$

Similarly, the next equations describe the right-sided fractional operators in general sense

$$ {}_t\mathscr{D}_T^{\rho} y(t) = \frac{d}{dt}\int_t^T y(\omega)\mathcal{X}_R(\omega-t)d\omega, \tag{10}$$

$$ {}_t^C\mathscr{D}_T^{\rho} y(t) = \int_t^T \dot{y}(\omega)\mathcal{X}_R(\omega-t)d\omega, \tag{11}$$

$$ {}_t\mathscr{I}_T^{\rho} y(t) = \int_t^T y(\omega)\mathscr{Z}_R(\omega-t)d\omega. \tag{12}$$

Based on [18], there is another generalization, which coincides with the above explanations; thus, according to the results of [18], the aforesaid definitions satisfy the integration by parts.

Here, some special cases are considered according to the above new definitions. First, the kernel $\mathcal{X}_L(t) = \frac{t^{-\rho}}{\Gamma(1-\rho)}$, $0 < \rho < 1$, is chosen, so $\mathscr{Z}_L$ becomes $\mathscr{Z}_L(t) = \frac{t^{\rho-1}}{\Gamma(\rho)}$. Therefore, the conventional forms of RL and Caputo fractional derivatives as well as RL integral are obtained from Eqs. (1), (2), and (7), respectively [19]. Next, we can choose $\mathcal{X}_L(t) = \int_0^1 \frac{t^{-\rho}}{\Gamma(1-\rho)}dm(\rho)$, where m is a Borel measure on $[0,1]$, or $\mathcal{X}_L(t) = \sum_{k=1}^n a_k \frac{t^{-\rho_k}}{\Gamma(1-\rho_k)}$, $0 < \rho_1 < \cdots < \rho_n < 1$. In these cases, we recover the derivative of distributed order and the multi-term derivatives, respectively [17].

Dynamical behaviors

This section discusses the capacitor microphone dynamical behaviors in both the frameworks of classical calculus and the new general FC. In the system under consideration, the capacitance C depends on the displacement of the bottom (moving) plate. Here, the aforesaid displacement is denoted by $y(t)$. In addition, a spring and a damper with, respectively, $k_s > 0$ and $b_d > 0$ as their constants are attached to the moving plate from below [20]. This plate is also under a mechanical force $f(t)$ modeling the air pressure created by sound. Further, the external voltage source is presented by $v(t)$.

Classical equations

Based on the physical system under study, the following relation results in the potential energy

$$ U(t) = \frac{1}{2}k_s y^2(t) + \frac{1}{2\epsilon A_p}(y_d - y(t))q_c^2(t), \tag{13}$$

whereas the kinetic energy is calculated by

$$ K(t) = \frac{1}{2}m_p\dot{y}^2(t) + \frac{1}{2}l_i\dot{q}_c^2(t), \tag{14}$$

where l_i is the inductance, m_p is the mass of moving plate, A_p denotes the area of each plate, $y_d - y(t)$ represents the distance between two plates, $q_c(t)$ is the charge of capacitor, and ϵ is the air dielectric constant. As the half of dissipated energy, the power function $P(t)$ is also derived by

$$P(t) = \frac{1}{2}b_d\dot{y}^2(t) + \frac{1}{2}r_e\dot{q}_c^2(t), \tag{15}$$

in which the resistance is denoted by r_e. Now, we can formulate the classical Lagrangian according to the following relation

$$C_L(t) = K(t) - U(t) = \frac{1}{2}m_p\dot{y}^2(t) + \frac{1}{2}l_i\dot{q}_c^2(t) - \frac{1}{2}k_s y^2(t) - \frac{1}{2\epsilon A_p}(y_d - y(t))q_c^2(t), \tag{16}$$

specifying the equilibrium among no dissipative energy. The Lagrangian function (16) is a quantity that characterizes the state of capacitor microphone dynamical system in terms of the state variables $q_c(t)$ and $y(t)$ and their time derivatives. A justification for introducing the Lagrangian in the form above becomes apparent when the classical Euler–Lagrange equations (CELEs) describing the dynamical behaviors are determined by minimizing the time integral of the Lagrangian. Given that the variables $q_c(t)$ and $y(t)$ are the two degrees of freedom in the capacitor microphone system, we attain

$$\frac{d}{dt}\left(\frac{\partial C_L(t)}{\partial \dot{q}_c(t)}\right) - \frac{\partial C_L(t)}{\partial q_c(t)} + \frac{\partial P(t)}{\partial \dot{q}_c(t)} = v(t), \tag{17}$$

$$\frac{d}{dt}\left(\frac{\partial C_L(t)}{\partial \dot{y}(t)}\right) - \frac{\partial C_L(t)}{\partial y(t)} + \frac{\partial P(t)}{\partial \dot{y}(t)} = f(t), \tag{18}$$

which correspond to the CELEs. By means of replacing $P(t)$ and $C_L(t)$ from (15)–(16) into (17)–(18), we then obtain

$$l_i\ddot{q}_c(t) + r_e\dot{q}_c(t) + \frac{1}{\epsilon A_p}(y_d - y(t))q_c(t) = v(t), \tag{19}$$

$$m_p\ddot{y}(t) + b_d\dot{y}(t) + k_s y(t) - \frac{q_c^2(t)}{2\epsilon A_p} = f(t). \tag{20}$$

Fractional equations

According to the study [21], although numerous nature laws can be obtained from the calculus of variations theory, there are some exceptions such as nonconservative dynamical systems, which could not be described by means of conventional energy approach [10,22]. From the other point of view, various aspects of complex systems could be presented by fractional modeling, which possesses the effects of memory. In this regard, some valuable efforts such as the studies in [23,24] have been done to revise and discuss the fractional formulation of dynamics in nonconservative systems. Inspired by these statements, here we develop a general fractional Lagrangian approach through the generalization introduced in Section "Symbols and preliminaries". More specifically, the classical Lagrangian (16) is expanded into a general fractional Lagrangian by replacing the ordinary time derivatives with the general fractional operators described in Section "Symbols and preliminaries"; as a result, the CELEs (19)–(20) are then modified into a general fractional framework, namely the general FELEs. In the Appendix "Derivation of FELE in general sense", the correctness of the following formulas is also proved and investigated. To present the new idea, first we define the new general fractional Lagrangian through fractionalizing the relation (16) as follows

$$F_L(t) = \frac{1}{2}m_p({}^{C}_{0}\mathscr{D}^{\rho}_{t}y(t))^2 + \frac{1}{2}l_i({}^{C}_{0}\mathscr{D}^{\rho}_{t}q_c(t))^2 - \frac{1}{2}k_s y^2(t) - \frac{1}{2\epsilon A_p}(y_d - y(t))q_c^2(t), \tag{21}$$

in which the ordinary time derivatives in (16) have been replaced by the general Caputo fractional derivative (2). Following the same procedure for $P(t)$, we derive

$$P(t) = \frac{1}{2}b_d({}^{C}_{0}\mathscr{D}^{\rho}_{t}y(t))^2 + \frac{1}{2}r_e({}^{C}_{0}\mathscr{D}^{\rho}_{t}q_c(t))^2. \tag{22}$$

Then the FELEs are gained by

$$-\,{}_{t}\mathscr{D}^{\rho}_{T}\frac{\partial F_L(t)}{\partial {}^{C}_{0}\mathscr{D}^{\rho}_{t}q_c(t)} - \frac{\partial F_L(t)}{\partial q_c(t)} + \frac{\partial P(t)}{\partial {}^{C}_{0}\mathscr{D}^{\rho}_{t}q_c(t)} = v(t), \tag{23}$$

$$-\,{}_{t}\mathscr{D}^{\rho}_{T}\frac{\partial F_L(t)}{\partial {}^{C}_{0}\mathscr{D}^{\rho}_{t}y(t)} - \frac{\partial F_L(t)}{\partial y(t)} + \frac{\partial P(t)}{\partial {}^{C}_{0}\mathscr{D}^{\rho}_{t}y(t)} = f(t). \tag{24}$$

By means of the relations (21)–(24), we attain

$$-\,l_i\,{}_{t}\mathscr{D}^{\rho}_{T}[{}^{C}_{0}\mathscr{D}^{\rho}_{t}q_c(t)] + r_e\,{}^{C}_{0}\mathscr{D}^{\rho}_{t}q_c(t) + \frac{1}{\epsilon A_p}(y_d - y(t))q_c(t) = v(t), \tag{25}$$

$$-\,m_p\,{}_{t}\mathscr{D}^{\rho}_{T}[{}^{C}_{0}\mathscr{D}^{\rho}_{t}y(t)] + b_d\,{}^{C}_{0}\mathscr{D}^{\rho}_{t}y(t) + k_s y(t) - \frac{1}{2\epsilon A_p}q_c^2(t) = f(t). \tag{26}$$

With regard to the formulas (25)–(26) it is apparent that the FELEs (25)–(26) are reduced to the CELEs (19)–(20) as the fractional order $\rho \to 1$.

Deriving the general fractional Hamilton equations (FHEs) is our next aim in this section; then we compare them with the FELEs (25)–(26). To do so, the fractional Hamiltonian function is written by

$$F_H(t) = \mathscr{L}_{q_c}(t)\,{}^{C}_{0}\mathscr{D}^{\rho}_{t}q_c(t) + \mathscr{L}_{y}(t)\,{}^{C}_{0}\mathscr{D}^{\rho}_{t}y(t) - F_L(t), \tag{27}$$

where

$$\mathscr{L}_{q_c}(t) = \frac{\partial F_L(t)}{\partial {}^{C}_{0}\mathscr{D}^{\rho}_{t}q_c(t)} = l_i\,{}^{C}_{0}\mathscr{D}^{\rho}_{t}q_c(t),\ \mathscr{L}_{y}(t) = \frac{\partial F_L(t)}{\partial {}^{C}_{0}\mathscr{D}^{\rho}_{t}y(t)} = m_p\,{}^{C}_{0}\mathscr{D}^{\rho}_{t}y(t), \tag{28}$$

are the generalized momenta. Within the use of Eqs. (21) and (28), we can compute

$$\begin{aligned} F_H(t) = {} & l_i({}^{C}_{0}\mathscr{D}^{\rho}_{t}q_c(t))^2 + m_p({}^{C}_{0}\mathscr{D}^{\rho}_{t}y(t))^2 - \frac{1}{2}m_p({}^{C}_{0}\mathscr{D}^{\rho}_{t}y(t))^2 \\ & - \frac{1}{2}l_i({}^{C}_{0}\mathscr{D}^{\rho}_{t}q_c(t))^2 + \frac{1}{2}k_s y^2(t) + \frac{1}{2\epsilon A_p}(y_d - y(t))q_c^2(t). \end{aligned} \tag{29}$$

Then the FHEs are obtained from

$$\frac{\partial F_H(t)}{\partial q_c(t)} - {}_{t}\mathscr{D}^{\rho}_{T}\mathscr{L}_{q_c}(t) + \frac{\partial P(t)}{\partial {}^{C}_{0}\mathscr{D}^{\rho}_{t}q_c(t)} = v(t), \tag{30}$$

$$\frac{\partial F_H(t)}{\partial y(t)} - {}_{t}\mathscr{D}^{\rho}_{T}\mathscr{L}_{y}(t) + \frac{\partial P(t)}{\partial {}^{C}_{0}\mathscr{D}^{\rho}_{t}y(t)} = f(t), \tag{31}$$

which result

$$-\,l_i\,{}_{t}\mathscr{D}^{\rho}_{T}[{}^{C}_{0}\mathscr{D}^{\rho}_{t}q_c(t)] + r_e\,{}^{C}_{0}\mathscr{D}^{\rho}_{t}q_c(t) + \frac{1}{\epsilon A_p}(y_d - y(t))q_c(t) = v(t), \tag{32}$$

$$-\,m_p\,{}_{t}\mathscr{D}^{\rho}_{T}[{}^{C}_{0}\mathscr{D}^{\rho}_{t}y(t)] + b_d\,{}^{C}_{0}\mathscr{D}^{\rho}_{t}y(t) + k_s y(t) - \frac{1}{2\epsilon A_p}q_c^2(t) = f(t). \tag{33}$$

It is obvious that the FHEs (32)–(33) give the same results as the FELEs (25)–(26). Further, the CELEs (19)–(20) are recovered by the FHEs (32)–(33) when $\rho \to 1$.

Numerical method

The FELEs (25)–(26) or the FHEs (32)–(33) are solved numerically in this part by a suitable matrix approximation approach. To do so, first the changes of variables $x(t) = {}^{C}_{0}\mathscr{D}^{q}_{t}q_c(t)$ and $z(t) = {}^{C}_{0}\mathscr{D}^{q}_{t}y(t)$ define the new states $x(t)$ and $z(t)$, respectively. Following this procedure, we convert Eqs. (25)–(26) into

$${}^{C}_{0}\mathscr{D}^{\rho}_{t}q_c(t) = x(t), \tag{34}$$

$${}_{t}\mathscr{D}^{\rho}_{T}x(t) = \frac{1}{l_i}(r_e x(t) + \frac{1}{\epsilon A_p}(y_d - y(t))q_c(t) - v(t)), \tag{35}$$

$${}^{C}_{0}\mathscr{D}^{\rho}_{t}y(t) = z(t), \tag{36}$$

$${}_{t}\mathscr{D}^{\rho}_{T}z(t) = \frac{1}{m_p}(b_d z(t) + k_s y(t) - \frac{1}{2\epsilon A_p}q_c^2(t) - f(t)). \tag{37}$$

Then the integral operators (7) and (12) are applied to the Eqs. (34)–(37) in order to change the above-mentioned relations into the fractional integral equations

$$q_c(t) = q_c(0) + \int_0^t x(\tau)\mathscr{Z}_L(t-\omega)d\omega, \tag{38}$$

$$x(t)=\frac{1}{l_i}\int_t^T (r_e x(\omega)+\frac{1}{\epsilon A_p}(y_d-y(\omega))q_c(\omega)-v(\omega))\mathscr{Z}_R(\omega-t)d\omega, \tag{39}$$

$$y(t)=y(0)+\int_0^t z(\tau)\mathscr{Z}_L(t-\omega)d\omega, \tag{40}$$

$$z(t)=\frac{1}{m_p}\int_t^T (b_d z(\omega)+k_s y(\omega)-\frac{1}{2\epsilon A_p}q_c^2(\omega)-f(\omega))\mathscr{Z}_R(\omega-t)d\omega. \tag{41}$$

Considering the time step size $h=\frac{T-0}{M}$ with an arbitrary positive integer M, now we define a uniform partition on $[0,T]$ in which $t_k=0+kh$ $(0\le k\le M)$ displays the time at node k, and $q_{c,k},x_k,y_k,z_k$ show the numerical approximations of $q_c(t_k),x(t_k),y(t_k),z(t_k)$, respectively. Therefore, based on Eqs. (38)–(41) we have

$$\begin{aligned}
q_{c,k+1}&=q_{c,0}+\int_0^{t_{k+1}} x(\omega)\mathscr{Z}_L(t_{k+1}-\omega)d\omega\\
&=q_{c,0}+\sum_{j=0}^{k}\int_{t_j}^{t_{j+1}} x(\omega)\mathscr{Z}_L(t_{k+1}-\omega)d\omega\\
&\approx q_{c,0}+\sum_{j=0}^{k} x_j\underbrace{\int_{t_j}^{t_{j+1}}\mathscr{Z}_L(t_{k+1}-\omega)d\omega}_{a_{k+1,j}}\\
&\approx q_{c,0}+\sum_{j=0}^{k} a_{k+1,j}x_j,\ k=0,1,\dots,M-1,
\end{aligned} \tag{42}$$

$$\begin{aligned}
x_{k-1}&=\frac{1}{l_i}\int_{t_{k-1}}^{T}(r_e x(\omega)+\frac{1}{\epsilon A_p}(y_d-y(\omega))q_c(\omega)-v(\omega))\mathscr{Z}_R(\omega-t_{k-1})d\omega\\
&=\frac{1}{l_i}\sum_{j=k}^{M}\int_{t_{j-1}}^{t_j}(r_e x(\omega)+\frac{1}{\epsilon A_p}(y_d-y(\omega))q_c(\omega)-v(\omega))\mathscr{Z}_R(\omega-t_{k-1})d\omega\\
&\approx\frac{1}{l_i}\sum_{j=k}^{M} b_{k-1,j}(r_e x_j+\frac{1}{\epsilon A_p}(y_d-y_j)q_{c,j}-v(t_j)),\ k=1,\dots,M-1,M,
\end{aligned} \tag{43}$$

$$\begin{aligned}
y_{k+1}&=y_0+\int_0^{t_{k+1}} z(\omega)\mathscr{Z}_L(t_{k+1}-\omega)d\omega\\
&=y_0+\sum_{j=0}^{k}\int_{t_j}^{t_{j+1}} z(\omega)\mathscr{Z}_L(t_{k+1}-\omega)d\omega\\
&\approx y_0+\sum_{j=0}^{k} z_j\underbrace{\int_{t_j}^{t_{j+1}}\mathscr{Z}_L(t_{k+1}-\omega)d\omega}_{a_{k+1,j}}\\
&\approx y_0+\sum_{j=0}^{k} a_{k+1,j}z_j,\ k=0,1,\dots,M-1,
\end{aligned} \tag{44}$$

$$\begin{aligned}
z_{k-1}&=\frac{1}{m_p}\int_{t_{k-1}}^{T}(b_d z(\omega)+k_s y(\omega)-\frac{1}{2\epsilon A_p}q_c^2(\omega)-f(\omega))\mathscr{Z}_R(\omega-t_{k-1})d\omega\\
&=\frac{1}{m_p}\sum_{j=k}^{M}\int_{t_{j-1}}^{t_j}(b_d z(\omega)+k_s y(\omega)\\
&\quad-\frac{1}{2\epsilon A_p}q_c^2(\omega)-f(\omega))\mathscr{Z}_R(\omega-t_{k-1})d\omega\\
&\approx\frac{1}{m_p}\sum_{j=k}^{M} b_{k-1,j}(b_d z_j+k_s y_j\\
&\quad-\frac{1}{2\epsilon A_p}q_{c,j}^2-f(t_j)),\ k=1,\dots,M-1,M,
\end{aligned} \tag{45}$$

in which

$$b_{k-1,j}=\int_{t_{j-1}}^{t_j}\mathscr{Z}_R(\omega-t_{k-1})d\omega. \tag{46}$$

By defining the matrices

$$\mathbf{H_M}=\begin{bmatrix} a_{1,0} & 0 & \dots & 0\\ a_{2,0} & \ddots & \ddots & \vdots\\ \vdots & \ddots & \ddots & 0\\ a_{M,0} & \dots & a_{M,M-2} & a_{M,M-1}\end{bmatrix}, \tag{47}$$

$$\mathbf{F_M}=\begin{bmatrix} b_{0,1} & b_{0,2} & \dots & b_{0,M}\\ 0 & \ddots & \ddots & \vdots\\ \vdots & \ddots & \ddots & b_{M-2,M}\\ 0 & \dots & 0 & b_{M-1,M}\end{bmatrix}, \tag{48}$$

we can rewrite Eqs. (42)–(45) in the matrix form

$$\begin{bmatrix} q_{c,1}\\ \vdots\\ q_{c,M}\end{bmatrix}=\begin{bmatrix} q_{c,0}\\ \vdots\\ q_{c,0}\end{bmatrix}+\mathbf{H_M}\begin{bmatrix} x_0\\ \vdots\\ x_{M-1}\end{bmatrix}, \tag{49}$$

$$\begin{bmatrix} x_0\\ \vdots\\ x_{M-1}\end{bmatrix}=\frac{1}{l_i}\mathbf{F_M}\left(r_e\begin{bmatrix} x_1\\ \vdots\\ x_M\end{bmatrix}+\frac{1}{\epsilon A_p}\begin{bmatrix} (y_d-y_1)q_{c,1}\\ \vdots\\ (y_d-y_M)q_{c,M}\end{bmatrix}-\begin{bmatrix} v(t_1)\\ \vdots\\ v(t_M)\end{bmatrix}\right), \tag{50}$$

$$\begin{bmatrix} y_1\\ \vdots\\ y_M\end{bmatrix}=\begin{bmatrix} y_0\\ \vdots\\ y_0\end{bmatrix}+\mathbf{H_M}\begin{bmatrix} z_0\\ \vdots\\ z_{M-1}\end{bmatrix}, \tag{51}$$

$$\begin{bmatrix} z_0\\ \vdots\\ z_{M-1}\end{bmatrix}=\frac{1}{m_p}\mathbf{F_M}\left(b_d\begin{bmatrix} z_1\\ \vdots\\ z_M\end{bmatrix}+k_s\begin{bmatrix} y_1\\ \vdots\\ y_M\end{bmatrix}-\frac{1}{\epsilon A_p}\begin{bmatrix} q_{c,1}^2\\ \vdots\\ q_{c,M}^2\end{bmatrix}-\begin{bmatrix} f(t_1)\\ \vdots\\ f(t_M)\end{bmatrix}\right). \tag{52}$$

Now we employ the augmented matrices

$$\bar{\mathbf{H}}_{\mathbf{M}}=\begin{bmatrix} 0 & 0\\ \mathbf{H_M} & 0\end{bmatrix},\ \bar{\mathbf{F}}_{\mathbf{M}}=\begin{bmatrix} 0 & \mathbf{F_M}\\ 0 & 0\end{bmatrix}, \tag{53}$$

and the vectors

$$Q_c=\begin{bmatrix} q_{c,0}\\ \vdots\\ q_{c,M}\end{bmatrix},\ X=\begin{bmatrix} x_0\\ \vdots\\ x_M\end{bmatrix},\ Y=\begin{bmatrix} y_0\\ \vdots\\ y_M\end{bmatrix},\ Z=\begin{bmatrix} z_0\\ \vdots\\ z_M\end{bmatrix}, \tag{54}$$

$$Q_{c,0}=\begin{bmatrix} q_{c,0}\\ \vdots\\ q_{c,0}\end{bmatrix},\ Y_0=\begin{bmatrix} y_0\\ \vdots\\ y_0\end{bmatrix},\ V=\begin{bmatrix} v(t_0)\\ \vdots\\ v(t_M)\end{bmatrix},\ F=\begin{bmatrix} f(t_0)\\ \vdots\\ f(t_M)\end{bmatrix}, \tag{55}$$

$$F_X(Y,Q_c,V)=\frac{1}{\epsilon A_p}\begin{bmatrix} (y_d-y_0)q_{c,0}\\ \vdots\\ (y_d-y_M)q_{c,M}\end{bmatrix}-\begin{bmatrix} v(t_0)\\ \vdots\\ v(t_M)\end{bmatrix}, \tag{56}$$

$$F_Z(Q_c,F)=-\frac{1}{\epsilon A_p}\begin{bmatrix} q_{c,0}^2\\ \vdots\\ q_{c,M}^2\end{bmatrix}-\begin{bmatrix} f(t_0)\\ \vdots\\ f(t_M)\end{bmatrix}, \tag{57}$$

to compact the relations (49)–(52) as below

$$\begin{cases} Q_c=Q_{c,0}+\bar{\mathbf{H}}_{\mathbf{M}}X,\\ X=\frac{1}{l_i}\bar{\mathbf{F}}_{\mathbf{M}}(r_e X+F_X(Y,Q_c,V)),\\ Y=Y_0+\bar{\mathbf{H}}_{\mathbf{M}}Z,\\ Z=\frac{1}{m_p}\bar{\mathbf{F}}_{\mathbf{M}}(b_d Z+k_s Y+F_Z(Q_c,F)).\end{cases} \tag{58}$$

Rearranging Eq. (58), we get

$$\begin{cases} Q_c-\bar{\mathbf{H}}_{\mathbf{M}}X=Q_{c,0},\\ (\mathbf{I}-\frac{1}{l_i}\bar{\mathbf{F}}_{\mathbf{M}}r_e)X=\frac{1}{l_i}\bar{\mathbf{F}}_{\mathbf{M}}F_X(Y,Q_c,V),\\ Y-\bar{\mathbf{H}}_{\mathbf{M}}Z=Y_0,\\ -\frac{1}{m_p}\bar{\mathbf{F}}_{\mathbf{M}}k_s Y+(\mathbf{I}-\frac{1}{m_p}\bar{\mathbf{F}}_{\mathbf{M}}b_d)Z=\frac{1}{m_p}\bar{\mathbf{F}}_{\mathbf{M}}F_Z(Q_c,F),\end{cases} \tag{59}$$

where $\mathbf{I}$ denotes an $(M+1)\times(M+1)$ identity matrix. Eventually, we can write

$$\begin{bmatrix} \mathbf{I} & -\bar{\mathbf{H}}_{\mathbf{M}} & 0 & 0\\ 0 & \mathbf{I}-\frac{1}{l_i}\bar{\mathbf{F}}_{\mathbf{M}}r_e & 0 & 0\\ 0 & 0 & \mathbf{I} & -\bar{\mathbf{H}}_{\mathbf{M}}\\ 0 & 0 & -\frac{1}{m_p}\bar{\mathbf{F}}_{\mathbf{M}}k_s & \mathbf{I}-\frac{1}{m_p}\bar{\mathbf{F}}_{\mathbf{M}}b_d\end{bmatrix}\begin{bmatrix} Q_c\\ X\\ Y\\ Z\end{bmatrix}$$

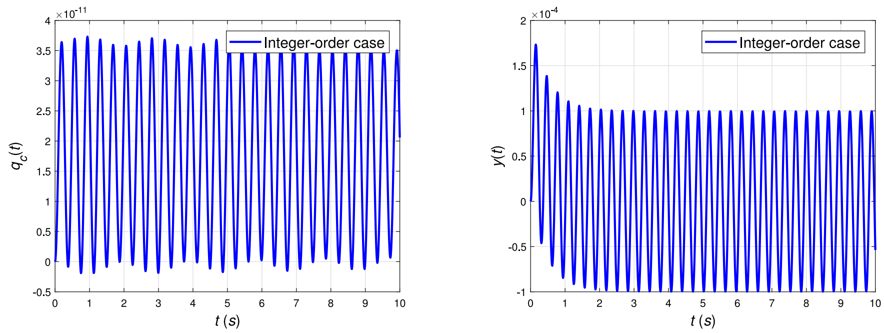

Fig. 1. The charge of capacitor and the displacement of moving plate under a sinusoidal mechanical force (integer-order model).

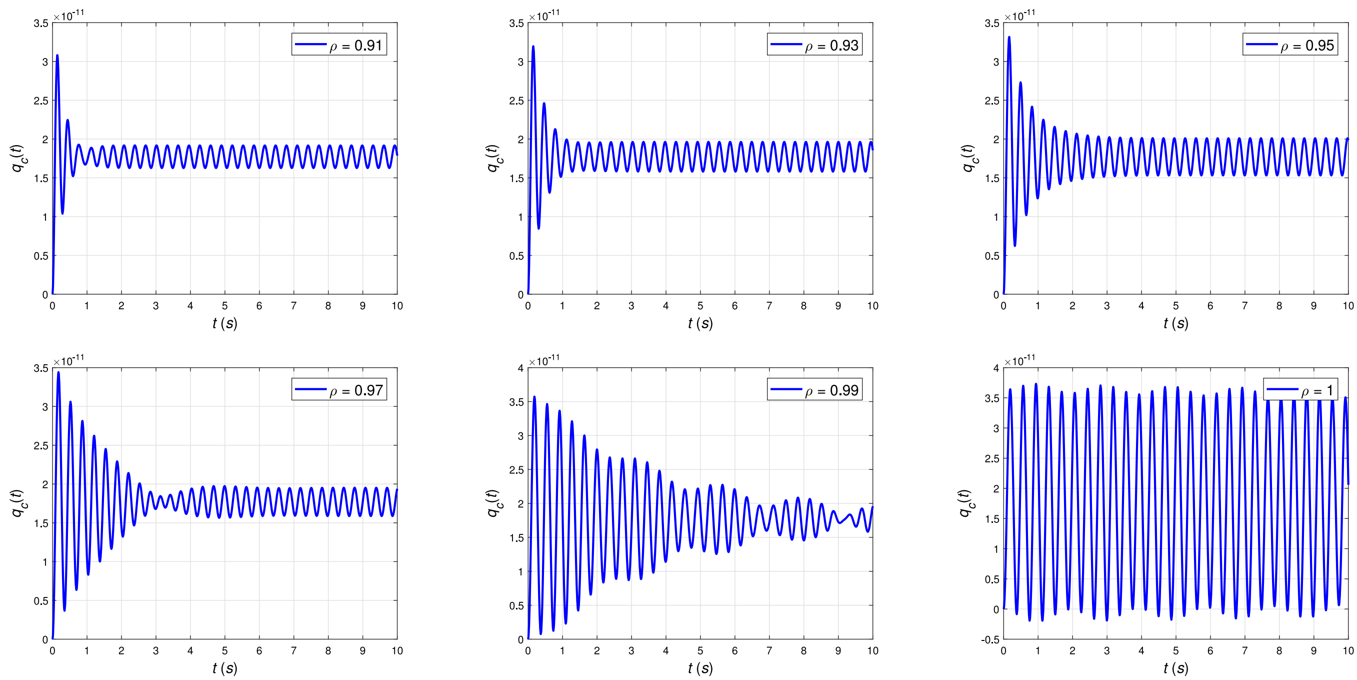

Fig. 2. The charge of capacitor under a sinusoidal mechanical force (fractional-order model with a power-law kernel).

$$= \begin{bmatrix} Q_{c,0} \\ \frac{1}{l_i}\bar{\mathbf{F}}_{\mathbf{M}} F_X(Y, Q_c, V) \\ Y_0 \\ \frac{1}{m_p}\bar{\mathbf{F}}_{\mathbf{M}} F_Z(Q_c, F) \end{bmatrix}, \tag{60}$$

which is a nonlinear system of algebraic equations.

Simulation results

The FELEs (25)–(26) are simulated in this part, which is according to the forced capacitor microphone under study by considering the special case of general fractional operators with a power-law kernel as introduced in Section "Symbols and preliminaries". Moreover, two cases are considered here in which $f(t) = 0.01\sin(20t)$ (periodic mechanical excitation) and $f(t) = 0.01$ (constant mechanical excitation). The relation $q_c(0) = y(0) = 0$ defines the initial setting; also, the other parameters are considered to be $r_e = 2\times 10^6$ Ω, $v = 1$ V, $l_i = 2\times 10^8$ H, $b_d = 5\ \frac{\text{N s}}{\text{m}}$, $k_s = 10\ \frac{\text{N}}{\text{m}}$, $m_p = 0.01$ kg, $A_p = 10^{-2}$ m^2, $\epsilon = 8.854\times 10^{-12}\ \frac{\text{F}}{\text{m}}$, $y_d = 0.005$ m, and $\rho = 0.91, 0.93, 0.95, 0.97, 0.99, 1$. Figs. 1–6 illustrate the simulation results. In Fig. 1, we show the charge of capacitor and the displacement of moving plate as the solutions of the CELEs (19)–(20) with the sinusoidal mechanical force $f(t) = 0.01\sin(20t)$. Under the same mechanical force, the corresponding responses of the FELEs (25)–(26) with a power-law kernel and different fractional orders are also depicted in Figs. 2 and 3 for the charge of capacitor and the displacement of moving plate, respectively. Fig. 4 displays the behaviors of electrical and mechanical variables for the integer-order model with the constant mechanical force $f(t) = 0.01$, while Figs. 5–6 show the corresponding variables for the fractional-order model with a power-law kernel and different fractional orders under the same constant mechanical force. According to the presented figures, we can come to the conclusion that the general relations (25)–(26) show various behaviors, *i.e.*, the characteristics of the responses like damped frequency, overshoot, rise time, settling time, etc., are entirely different. In addition, the numerical solution of FELEs goes to that of CELEs as ρ goes to 1. Consequently, a new valid system, which could exhibit various features of capacitor microphone dynamics, is prepared by means of the general fractional derivatives. Notice that these features are not available when we work with classic integer-order models. Additionally, with regard to the fact that the new general fractional operators prepare more flexible models, we achieve a remarkable benefit in better comprehension of complex dynamical systems.

Conclusions

In this research, a new and general fractional Lagrangian approach was investigated in order to estimate the complex behaviors of a capacitor microphone case study. The classical Lagrangian was first

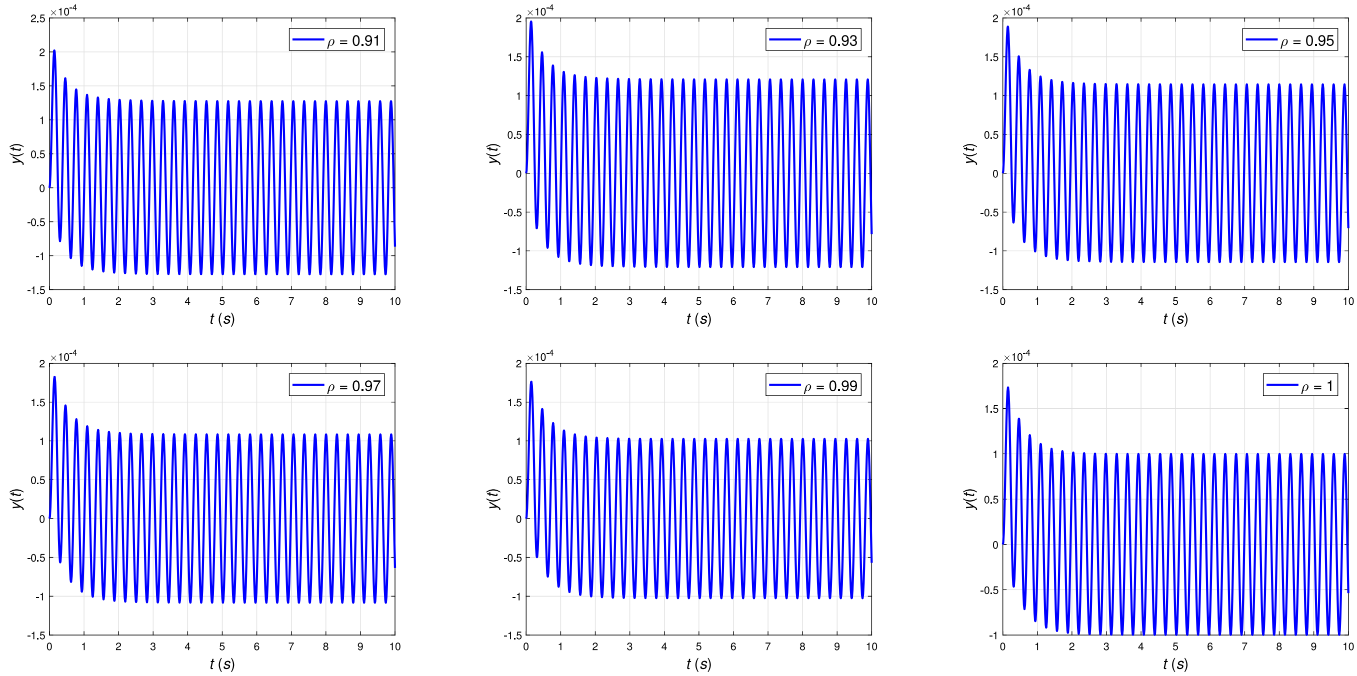

Fig. 3. The displacement of moving plate under a sinusoidal mechanical force (fractional-order model with a power-law kernel).

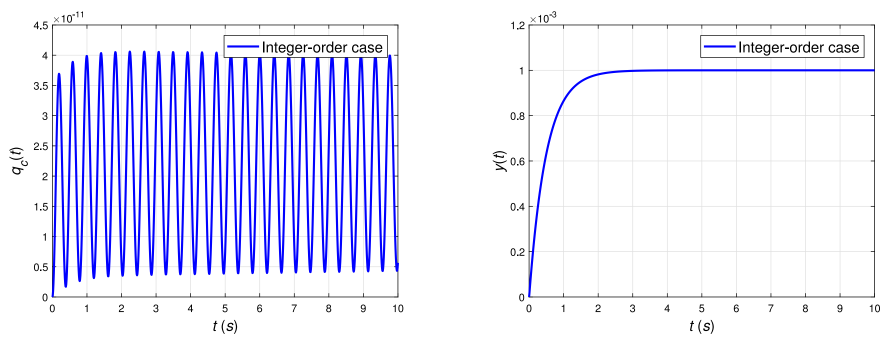

Fig. 4. The charge of capacitor and the displacement of moving plate under a constant mechanical force (integer-order model).

set up, and then the general fractional Lagrangian was introduced by using the newly developed general fractional derivatives instead of integer-order ones. Afterwards, the FELEs were formulated containing a general kernel function for the fractional operators. As the numerical part, an effective matrix approximation scheme was presented changing the latter general equations into a nonlinear algebraic system. Figs. 1–6 showed the simulation results, which indicated that the fractional dynamical behaviors in our case rely on the kernel function; in other words, changing the kernel function leads to various asymptotic behaviors. This approves the benefit of generalized fractional model to extract the hidden aspects of the system under study, whereas this feature is not accessible with classical fractional modeling. Eventually, there is an open problem if the general relations, derived in this paper, could be solved through other approximation techniques, an issue which can be taken as a clue to future works. Additionally, one can extend the idea of general FC for the other types of problems such as control, mathematical modeling, etc. [25–27]

CRediT authorship contribution statement

A. Jajarmi: Conceptualization, Writing - original draft. **D. Baleanu:** Supervision. **K. Zarghami Vahid:** Writing – review & editing. **H. Mohammadi Pirouz:** Software. **J.H. Asad:** Methodology, Validation.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgment

All authors read and approved the final manuscript.

This research was funded by PTUK- Scientific Research Fund 2021.

Appendix (Derivation of FELE in general sense)

Let $\mathcal{F}({}^{C}_{0}\mathcal{D}^{\rho}_{t}\omega(t), {}^{C}_{t}\mathcal{D}^{\rho}_{T}\omega(t), \omega(t), t)$ be a scalar function having continuous first and second partial derivatives with respect to all its argument. Following the concept of calculus of variations in fractional sense [21], we consider $\mathcal{F}({}^{C}_{0}\mathcal{D}^{\rho}_{t}\omega(t), {}^{C}_{t}\mathcal{D}^{\rho}_{T}\omega(t), \omega(t), t)$ as the general fractional Lagrangian function, including the left and right general Caputo fractional derivatives described in the Eqs. (2) and (11), respectively. Next, we construct the action function $S[\omega(t)]$ in the classical field as follows

$$S[\omega(t)] = \int_0^T \mathcal{F}({}^{C}_{t}\mathcal{D}^{\rho}_{T}\omega(t), {}_{0}^{C}\mathcal{D}^{\rho}_{t}\omega(t), \omega(t), t)dt. \tag{61}$$

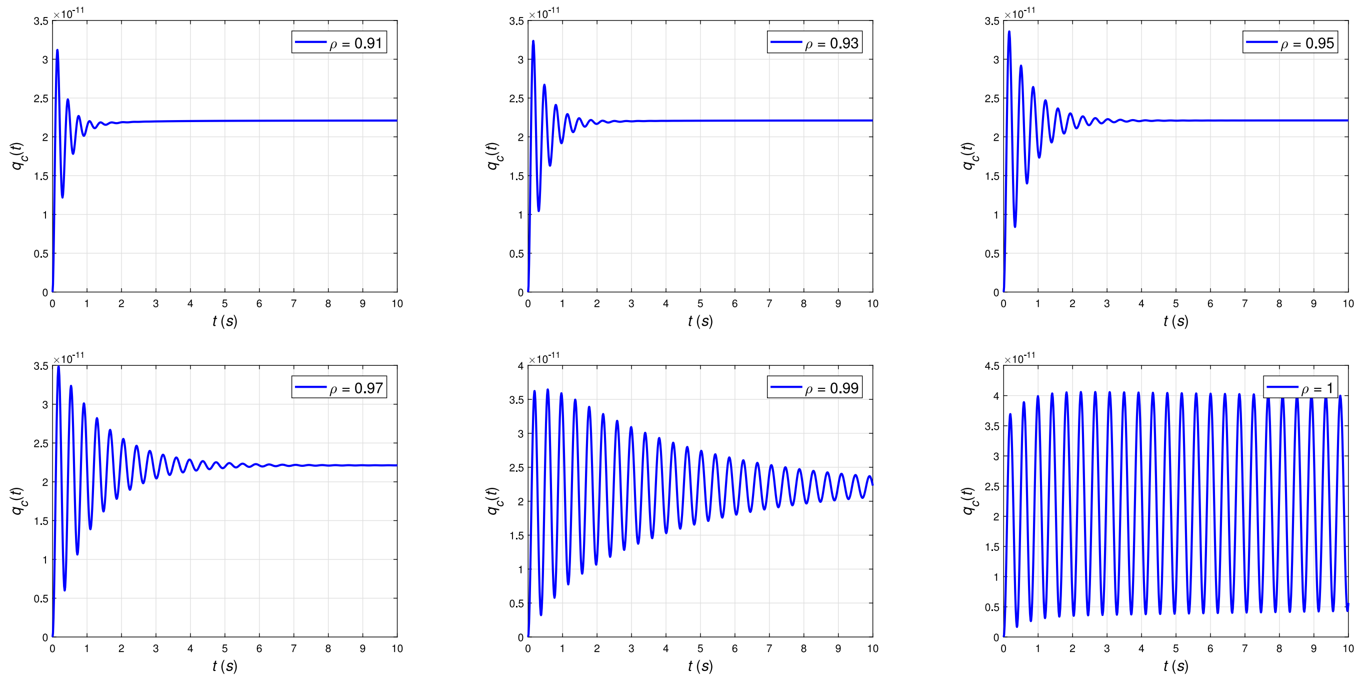

Fig. 5. The charge of capacitor under a constant mechanical force (fractional-order model with a power-law kernel).

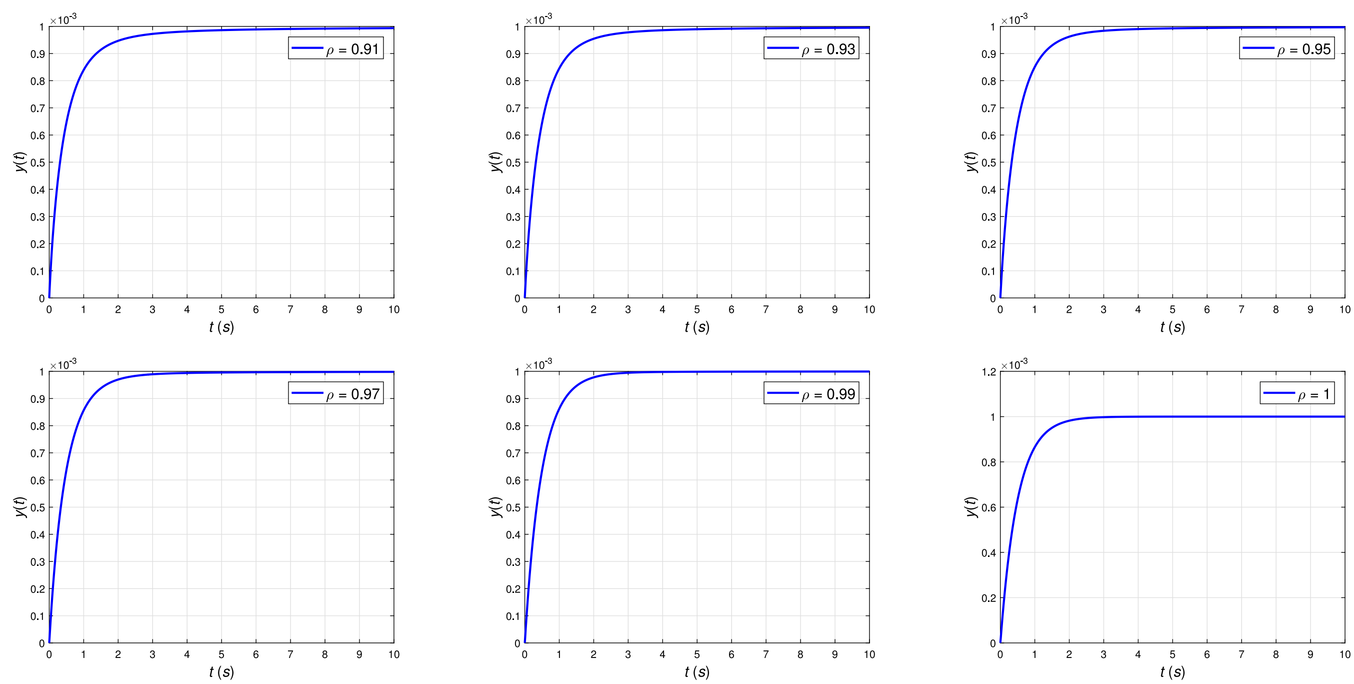

Fig. 6. The displacement of moving plate under a constant mechanical force (fractional-order model with a power-law kernel).

By gaining the extreme point of $\mathcal{S}[\omega(t)]$ as well as using integration by parts formulations in fractional framework, we can obtain the FELE based on the fractional Lagrangian mechanics [21]. For this purpose, let $\omega(t)$ fulfill the condition $(\omega(0), \omega(T)) = (\omega_0, \omega_T)$ and ${}^{C}_{0}\mathcal{D}^{\rho}_{t}\omega(t), {}^{C}_{t}\mathcal{D}^{\rho}_{T}\omega(t)$ be continuous for all $0 \leq t \leq T$. Then the following category of curves is built to discover the extreme point $\omega^*(t)$ of $\mathcal{S}[\omega(t)]$

$$\omega(t) = \omega^*(t) + \delta\kappa(t), \tag{62}$$

where $\kappa(0) = \kappa(T) = 0$, and $\delta \in \mathbb{R}$ is an arbitrary real number. Since the general derivatives ${}^{C}_{t}\mathcal{D}^{\rho}_{T}\omega(t)$ and ${}^{C}_{0}\mathcal{D}^{\rho}_{t}\omega(t)$ are linear operators (Section "Symbols and preliminaries"), we have

$${}^{C}_{0}\mathcal{D}^{\rho}_{t}\omega(t) = {}^{C}_{0}\mathcal{D}^{\rho}_{t}\omega^*(t) + \delta\, {}^{C}_{0}\mathcal{D}^{\rho}_{t}\kappa(t), \tag{63}$$

$${}^{C}_{t}\mathcal{D}^{\rho}_{T}\omega(t) = {}^{C}_{t}\mathcal{D}^{\rho}_{T}\omega^*(t) + \delta\, {}^{C}_{t}\mathcal{D}^{\rho}_{T}\kappa(t). \tag{64}$$

Now, the relations (62)–(64) are substituted into the action function $\mathcal{S}[\omega(t)]$; thus, for every $\kappa(t)$ we obtain

$$\mathcal{S} = \mathcal{S}(\delta) = \int_0^T \mathcal{F}\big({}^{C}_{0}\mathcal{D}^{\rho}_{t}\omega^*(t) + \delta\, {}^{C}_{0}\mathcal{D}^{\rho}_{t}\kappa(t), \\ {}^{C}_{t}\mathcal{D}^{\rho}_{T}\omega^*(t) + \delta\, {}^{C}_{t}\mathcal{D}^{\rho}_{T}\kappa(t), \omega^*(t) + \delta\kappa(t), t\big)dt. \tag{65}$$

The necessary condition to attain the extreme point $\omega^*(t)$ of $\mathcal{S}(\delta)$ is derived by

$$\frac{d\mathcal{S}}{d\delta} = \int_0^T \left[\frac{\partial \mathcal{F}}{\partial \omega(t)}\kappa(t) + \frac{\partial \mathcal{F}}{\partial {}_0^C\mathcal{D}_t^\rho \omega(t)} {}_0^C\mathcal{D}_t^\rho \kappa(t) + \frac{\partial \mathcal{F}}{\partial {}_t^C\mathcal{D}_T^\rho \omega(t)} {}_t^C\mathcal{D}_T^\rho \kappa(t) \right] dt = 0, \tag{66}$$

for all admissible $\kappa(t)$. The formulas of the fractional integration by parts are now applied by considering the boundary conditions $\kappa(0) = \kappa(T) = 0$

$$\int_0^T \frac{\partial \mathcal{F}}{\partial {}_0^C\mathcal{D}_t^\rho \omega(t)} {}_0^C\mathcal{D}_t^\rho \kappa(t) dt = \int_0^T [{}_t\mathcal{D}_T^\rho \frac{\partial \mathcal{F}}{\partial {}_0^C\mathcal{D}_t^\rho \omega(t)}]\kappa(t)dt, \tag{67}$$

$$\int_0^T \frac{\partial \mathcal{F}}{\partial {}_t^C\mathcal{D}_T^\rho \omega(t)} {}_t^C\mathcal{D}_T^\rho \kappa(t) dt = \int_0^T [{}_0\mathcal{D}_t^\rho \frac{\partial \mathcal{F}}{\partial {}_t^C\mathcal{D}_T^\rho \omega(t)}]\kappa(t)dt. \tag{68}$$

Substituting Eqs. (67)–(68) into (66), we prepare

$$\frac{d\mathcal{S}}{d\delta} = \int_0^T \left[\frac{\partial \mathcal{F}}{\partial \omega(t)} + {}_t\mathcal{D}_T^\rho \frac{\partial \mathcal{F}}{\partial {}_0^C\mathcal{D}_t^\rho \omega(t)} + {}_0\mathcal{D}_t^\rho \frac{\partial \mathcal{F}}{\partial {}_t^C\mathcal{D}_T^\rho \omega(t)} \right] \kappa(t) dt = 0. \tag{69}$$

Eventually, the FELE is obtained from

$$\frac{\partial \mathcal{F}}{\partial \omega(t)} + {}_t\mathcal{D}_T^\rho \frac{\partial \mathcal{F}}{\partial {}_0^C\mathcal{D}_t^\rho \omega(t)} + {}_0\mathcal{D}_t^\rho \frac{\partial \mathcal{F}}{\partial {}_t^C\mathcal{D}_T^\rho \omega(t)} = 0, \tag{70}$$

by taking into account the concept of calculus of variations and paying attention to the point that the function $\kappa(t)$ is arbitrary.

References

[1] Greiner W. Classical mechanics, systems of particles and hamiltonian dynamics. Berlin: Springer-Verlag; 2010.
[2] Akgül A. A novel method for a fractional derivative with non-local and non-singular kernel. Chaos Solitons Fractals 2018;114:478–82.
[3] Atangana A, Akgül A, Owolabi KM. Analysis of fractal fractional differential equations. Alexandria Eng J 2020;59(3):1117–34.
[4] Baleanu D, Zibaei S, Namjoo M, Jajarmi A. A nonstandard finite difference scheme for the modelling and nonidentical synchronization of a novel fractional chaotic system. Adv Difference Equ 2021;2021:308.
[5] Owolabi KM, Atangana A, Akgül A. Modelling and analysis of fractal-fractional partial differential equations: Application to reaction–diffusion model. Alexandria Eng J 2020;59(4):2477–90.
[6] Baleanu D, Sajjadi SS, Jajarmi A, Defterli O. On a nonlinear dynamical system with both chaotic and non-chaotic behaviours: a new fractional analysis and control. Adv Difference Equ 2021;2021:234.
[7] Akgül A, Ahmad S, Ullah A, Baleanu D, Akg ül EK. A novel method for analysing the fractal fractional integrator circuit. Alexandria Eng J 2021;60(4):3721–9.
[8] Atangana A, Akgül A. Can transfer function and bode diagram be obtained from sumudu transform. Alexandria Eng J 2020;59(4):1971–84.
[9] Baleanu D, Sajjadi SS, Asad JH, Jajarmi A, Estiri E, behaviours Hyperchaotic, control optimal. And synchronization of a nonautonomous cardiac conduction system. Adv Difference Equ 2021;2021:157.
[10] Riewe F. Nonconservative lagrangian and hamiltonian mechanics. Phys Rev E 1996;53:1890–9.
[11] Laskin N. Fractional quantum mechanics. Phys Rev E 2000;62:3135–45.
[12] Laskin N. Fractional quantum mechanics. Phys Rev E 2002;66:056108.
[13] Almeida R, Morgado ML. The Euler–Lagrange and Legendre equations for functionals involving distributed-order fractional derivatives. Appl Math Comput 2018;331:394–403.
[14] Baleanu D, Sajjadi SS, Jajarmi A, Defterli O, Asad JH. The fractional dynamics of a linear triatomic molecule. Rom Rep Phys 2021;73(1):105.
[15] Momani S, Odibat Z, Alawneh A. Variational iteration method for solving the space and time-fractional kdv equation. Numer Methods Partial Differential Equations 2008;24:262–71.
[16] Diethelm K, Ford NJ, Freed AD. A predictor–corrector approach for the numerical solution of fractional differential equations. Nonlinear Dynam 2002;29:3–22.
[17] Luchko Y, Yamamoto M. General time-fractional diffusion equation: some uniqueness and existence results for the initial–boundary-value problems. Fract Calc Appl Anal 2016;19(3):676–95.
[18] Agrawal OP. Generalized variational problems and Euler–Lagrange equations. Comput Math Appl 2010;59:1852–64.
[19] Podlubny I. Fractional differential equations: an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications. New York: Academic Press; 1999.
[20] van der Schaft AJ. Port-Hamiltonian systems: Network modeling and control of nonlinear physical systems. In: Irschik H, Schlacher K, editors. Advanced dynamics and control of structures and machines. (International centre for mechanical sciences (courses and lectures), Vienna: Springer; 2004.
[21] Agrawal OP. Formulation of Euler–Lagrange equations for fractional variational problems. J Math Anal Appl 2002;272(1):368–79.
[22] Riewe F. Mechanics with fractional derivatives. Phys Rev E 1996;55:3582–92.
[23] Dreisigmeyer DW, Young PM. Nonconservative Lagrangian mechanics: a generalized function approach. J Phys A: Math Gen 2003;36:8297–310.
[24] Dreisigmeyer DW, Young PM. Extending Bauer's corollary to fractional derivatives. J Phys A: Math Gen 2004;37:L117–21.
[25] Jajarmi A, Baleanu D, Zarghami Vahid K, Mobayen S. A general fractional formulation and tracking control for immunogenic tumor dynamics. Math Methods Appl Sci 2021. http://dx.doi.org/10.1002/mma.7804.
[26] Jajarmi A, Pariz N, Effati S, Kamyad AV. Infinite horizon optimal control for nonlinear interconnected large-scale dynamical systems with an application to optimal attitude control. Asian J Control 2012;14(5):1239–50.
[27] Effati S, Nik HS, Jajarmi A. Hyperchaos control of the hyperchaotic Chen system by optimal control design. Nonlinear Dynam 2013;73(1–2):499–508.