A review of the Adomian decomposition method and its applications to fractional differential equations

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Received 4 Aug 2012; accepted 18 Aug 2012

Abstract

In this article we review the Adomian decomposition method (ADM) and its modifications including different modified and parametrized recursion schemes, the multistage ADM for initial value problems as well as the multistage ADM for boundary value problems, new developments of the method and its applications to linear or nonlinear and ordinary or partial differential equations, including fractional differential equations.

Keywords: Adomian decomposition method; Adomian polynomials; Nonlinear differential equations; Fractional differential equations; Solution continuation

1 Introduction

The Adomian decomposition method (ADM) [1–14] is a well-known systematic method for practical solution of linear or nonlinear and deterministic or stochastic operator equations, including ordinary differential equations (ODEs), partial differential equations (PDEs), integral equations, integro-differential equations, etc. The ADM is a powerful technique, which provides efficient algorithms for analytic approximate solutions and numeric simulations for real-world applications in the applied sciences and engineering. It permits us to solve both nonlinear initial value problems (IVPs) and boundary value problems (BVPs) [7, 15–38] without unphysical restrictive assumptions such as required by linearization, perturbation, ad hoc assumptions, guessing the initial term or a set of basis functions, and so forth. Furthermore the ADM does not require the use of Green's functions, which would complicate such analytic calculations since Green's functions are not easily determined in most cases. The accuracy of the analytic approximate solutions obtained can be verified by direct substitution. Advantages of the ADM over Picard's iterated method were demonstrated in [39]. More advantages of the ADM over the variational iteration method were presented in [40, 41]. A key notion is the Adomian polynomials, which are tailored to the particular nonlinearity to solve nonlinear operator equations.

Adomian and co-workers have solved nonlinear differential equations for a wide class of nonlinearities, including product [42], polynomial [43], exponential [44], trigonometric [45], hyperbolic [46], composite [47], negative-power [48], radical [49] and even decimal-power nonlinearities [50]. We find that the ADM solves nonlinear operator equations for any analytic nonlinearity, providing us with

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an easily computable, readily verifiable, and rapidly convergent sequence of analytic approximate functions.

Let us first recall the basic principles of the ADM using an IVP for a nonlinear ODE in the form

$$Lu+Ru+Nu=g, \tag{1}$$

where g is the system input and u is the system output, and where L is the linear operator to be inverted, which usually is just the highest order differential operator, R is the linear remainder operator, and N is the nonlinear operator, which is assumed to be analytic. We remark that this choice of the linear operator is designed to yield an easily invertible operator with resulting trivial integrations. Furthermore we emphasize that the choice for L and concomitantly its inverse L^{-1} are determined by the particular equation to be solved, hence the choice is nonunique, e.g. for cases of differential equations with singular coefficients, we choose a different form for the linear operator [51–53].

Generally we choose $L=\frac{d^p}{dx^p}(\cdot)$ for pth-order differential equations and thus its inverse L^{-1} follows as the p-fold definite integration operator from x_0 to x. We have $L^{-1}Lu=u-\Phi$, where Φ incorporates the initial values as $\Phi=\sum_{\nu=0}^{p-1}\beta_\nu\frac{(x-x_0)^\nu}{\nu!}$.

Applying the inverse linear operator L^{-1} to both sides of Eq. (1) gives

$$u=\gamma(x)-L^{-1}[Ru+Nu], \tag{2}$$

where $\gamma(x)=\Phi+L^{-1}g$.

The ADM decomposes the solution into a series

$$u=\sum_{n=0}^{\infty}u_n, \tag{3}$$

and then decomposes the nonlinear term Nu into a series

$$Nu=\sum_{n=0}^{\infty}A_n, \tag{4}$$

where the A_n, depending on $u_0,u_1,\cdots,u_n$, are called the Adomian polynomials, and are obtained for the nonlinearity $Nu=f(u)$ by the definitional formula [1]

$$A_n=\frac{1}{n!}\frac{\partial^n}{\partial\lambda^n}\left[f\left(\sum_{k=0}^{\infty}u_k\lambda^k\right)\right]_{\lambda=0}, \quad n=0,1,2,\cdots, \tag{5}$$

where λ is a grouping parameter of convenience.

We list the formulas of the first several Adomian polynomials for the one-variable simple analytic nonlinearity $Nu=f(u(x))$ from A_0 through A_5, inclusively, for convenient reference as

$$\begin{aligned}
A_0 &= f(u_0),\\
A_1 &= f'(u_0)u_1,\\
A_2 &= f'(u_0)u_2+f''(u_0)\tfrac{u_1^2}{2!},\\
A_3 &= f'(u_0)u_3+f''(u_0)u_1u_2+f^{(3)}(u_0)\tfrac{u_1^3}{3!},\\
A_4 &= f'(u_0)u_4+f''(u_0)\left(\tfrac{u_2^2}{2!}+u_1u_3\right)+f^{(3)}(u_0)\tfrac{u_1^2u_2}{2!}+f^{(4)}(u_0)\tfrac{u_1^4}{4!},\\
A_5 &= f'(u_0)u_5+f''(u_0)(u_2u_3+u_1u_4)+f^{(3)}(u_0)\left(\tfrac{u_1u_2^2}{2!}+\tfrac{u_1^2u_3}{2!}\right)+f^{(4)}(u_0)\tfrac{u_1^3u_2}{3!}+f^{(5)}(u_0)\tfrac{u_1^5}{5!}.
\end{aligned}$$

Upon substitution of the Adomian decomposition series for the solution $u(x)$ and the series of Adomian polynomials tailored to the nonlinearity Nu from Eqs. (3) and (4) into Eq. (2), we have

$$\sum_{n=0}^{\infty} u_n = \gamma(x) - L^{-1}\left[R\sum_{n=0}^{\infty} u_n + \sum_{n=0}^{\infty} A_n\right]. \tag{6}$$

The solution components $u_n(x)$ may be determined by one of several advantageous recursion schemes, which differ from one another by the choice of the initial solution component $u_0(x)$, beginning with the classic Adomian recursion scheme

$$\begin{aligned} &u_0(x) = \gamma(x), \\ &u_{n+1}(x) = -L^{-1}[Ru_n + A_n],\ n \geq 0, \end{aligned} \tag{7}$$

where Adomian has chosen the initial solution component as $u_0 = \gamma(x)$. The n-term approximation of the solution is

$$\varphi_n(x) = \sum_{k=0}^{n-1} u_k(x). \tag{8}$$

By various partitions of the original initial term and then delaying the contribution of its remainder by different algorithms, we can design alternate recursion schemes, such as the Adomian-Rach [18, 19], Wazwaz [54], Wazwaz-El-Sayed [55], Duan [56], and Duan-Rach [37, 57] modified recursion schemes for different computational advantages.

Several investigators including Cherruault and co-workers [58–61], among others [62–65], have previously proved convergence of the Adomian decomposition series and the series of the Adomian polynomials. For example, Cherruault and Adomian [59] have proved convergence of the decomposition series without appealing to the fixed point theorem, which is too restrictive for most physical and engineering applications. Furthermore, Abdelrazec and Pelinovsky [65] have recently published a rigorous proof of convergence for the ADM under the aegis of the Cauchy-Kovalevskaya theorem for IVPs. A key concept is that the Adomian decomposition series is a computationally advantageous rearrangement of the Banach-space analog of the Taylor expansion series about the initial solution component function, which permits solution by recursion. A remarkable measure of success of the ADM is demonstrated by its widespread adoption and many adaptations to enhance computability for specific purposes, such as the various modified recursion schemes. The choice of decomposition is nonunique, which provides a valuable advantage to the analyst, permitting the freedom to design modified recursion schemes for ease of computation in realistic systems.

Our discussion is organized as follows. In the next section, we review the computation of the Adomian polynomials. In Section 3, we consider the Rach-Adomian-Meyers modified decomposition method. In Sections 4 and 5, solutions of BVPs for ODEs and PDEs by the ADM are reviewed, respectively. In Section 6, we discuss expansion of the effective region of convergence and the concept of multistage decomposition. In Section 7, we review various applications of the ADM and its modifications in solving fractional differential equations (FDEs). Section 8 summarizes our conclusions.

2 The single-variable and multivariable Adomian polynomials

In computational practice for the Adomian polynomials in Eq. (5), we truncate the decomposition series after $n = M$ for some finite M, since the higher order solution components $u_n(x)$ for $n > M$ do not contribute to the calculation of the $A_n(x)$ for $n \leq M$ in practice, thus

$$A_n(x) = \frac{1}{n!}\frac{\partial^n}{\partial\lambda^n} f\left(\sum_{k=0}^{M} \lambda^k u_k(x)\right)\Bigg|_{\lambda=0},\ 0 \leq n \leq M, \tag{9}$$

from which we can calculate the first $M+1$ Adomian polynomials from $A_0(x)$ through $A_M(x)$, inclusively, such that $A_n(x)=A_n(u_0(x),u_1(x),\ldots,u_n(x))$ for $n=0,1,\ldots,M$.

Other algorithms for the Adomian polynomials have been developed by Adomian and Rach [1, 66], Rach [64, 67], Wazwaz [11, 68], Abdelwahid [69] and several others [56, 70–78] to efficiently generate the Adomian polynomials quickly and to high orders, including the single variable and multivariable cases. For example, a convenient formula for the Adomian polynomials is Rach's Rule, which reads (Page 16 in [6] and Page 51 in [7])

$$A_n=\sum_{k=1}^{n} f^{(k)}(u_0)C_n^k,\ n\geq 1, \tag{10}$$

where the coefficients C_n^k are the sums of all possible products of k components from u_1, u_2, $\cdots$, u_{n-k+1}, whose subscripts sum to n, divided by the factorial of the number of repeated subscripts [67].

New, more efficient algorithms and subroutines in MATHEMATICA for fast generation of the one-variable and multi-variable Adomian polynomials to high orders have been provided by Duan in [56, 75, 76] and Duan and Guo in [77, 78]. For the case of the one-variable Adomian polynomials, we list Duan's Corollary 1 algorithm [56] and Corollary 3 algorithm [76] as follows.

Corollary 1 algorithm [56]: For $n\geq 1$,

$$C_n^1=u_n. \tag{11}$$

For $n\geq 2$ and $\left[\frac{n}{2}\right]<k\leq n$,

$$C_n^k=C_{n-1}^{k-1}|_{p_1\to p_1+1}. \tag{12}$$

For $n\geq 4$ and $2\leq k\leq \left[\frac{n}{2}\right]$,

$$C_n^k=C_{n-1}^{k-1}|_{p_1\to p_1+1}+C_{n-k}^k|_{u_j\to u_{j+1}}. \tag{13}$$

Here $p_1\to p_1+1$ stands for replacing $\frac{u_1^{p_1}}{p_1!}$ by $\frac{u_1^{p_1+1}}{(p_1+1)!}$, where $p_1\geq 0$.

Corollary 3 algorithm [76]: For $n\geq 1$,

$$C_n^1=u_n. \tag{14}$$

For $2\leq k\leq n$,

$$C_n^k=\frac{1}{n}\sum_{j=0}^{n-k}(j+1)u_{j+1}C_{n-1-j}^{k-1}. \tag{15}$$

Then the Adomian polynomials are given by the formula $A_n=\sum_{k=1}^{n} f^{(k)}(u_0)C_n^k$.

Neither one of these two algorithms involves the differentiation operator; furthermore we point out that the second algorithm requires only the operations of addition and multiplication, which is eminently convenient for computer algebra systems. We present the MATHEMATICA code generating the single-variable Adomian polynomials based on the second algorithm in the Appendix.

In [6] Adomian introduced the concept of the accelerated Adomian polynomials $\hat{A}_n$. In [79] Adomian and Rach presented two new kinds of modified Adomian polynomials $\bar{A}_n$ and $\bar{\bar{A}}_n$. Rach [64] gave a new definition of the Adomian polynomials, in which different classes of the Adomian polynomials were defined within the same premise. Duan [80] presented new recurrence algorithms for these nonclassic Adomian polynomials. Generalized forms of the Adomian polynomials were also proposed by Duan [81].

The multivariable Adomian polynomials are used for decomposing multivariable nonlinear functions occurring in either single nonlinear nth-order differential equations with multi-order differential nonlinearities $f(u,u',u'',...,u^{(n-1)})$ or in systems of coupled nonlinear differential equations with multivariable nonlinearities. We suppose f is an m-ary analytic function $f(u_1,\cdots,u_m)$, where the u_k, for

$1 \le k \le m$, are the unknown functions to be determined. We decompose the solutions $u_i, i=1,2,\cdots,m$, and the nonlinear function $f(u_1,\ \cdots,\ u_m)$ as

$$u_i = \sum_{j=0}^{\infty} u_{i,j},\ i=1,2,\cdots,m, \tag{16}$$

and

$$f(u_1,\cdots,u_m) = \sum_{n=0}^{\infty} A_n, \tag{17}$$

where the multivariable Adomian polynomials A_n depend on the solution components

$$u_{1,0}, u_{1,1}\cdots,u_{1,n}; u_{2,0}, u_{2,1},\cdots,u_{2,n};\cdots; u_{m,0}, u_{m,1},\cdots,u_{m,n},$$

and are defined via the parametrization [1]

$$A_n = \frac{1}{n!}\frac{\partial^n}{\partial \lambda^n} f\left(\sum_{j=0}^{\infty} u_{1,j}\lambda^j,\cdots,\sum_{j=0}^{\infty} u_{m,j}\lambda^j\right)\Bigg|_{\lambda=0}, \tag{18}$$

where λ is a grouping parameter of convenience.

The first m-variable Adomian polynomial A_0 is

$$A_0 = f(\mathbf{u_0}),\ \text{where } \mathbf{u_0} = (u_{1,0},\cdots,u_{m,0}). \tag{19}$$

Another expression for the multivariable Adomian polynomials [71] is

$$A_n \;=\; \sum_{\sum_{j=1}^{n} j\sum_{i=1}^{m} p_{ij}=n} f^{(p_{1*},\cdots,p_{m*})}(\mathbf{u_0})\prod_{i=1}^{m}\prod_{j=1}^{n}\frac{{u_{i,j}}^{p_{ij}}}{p_{ij}!}, \tag{20}$$

where $p_{i*} = \sum_{j=1}^{n} p_{ij}$ and

$$f^{(p_{1*},\cdots,p_{m*})}(\mathbf{u_0}) = \frac{\partial^{p_{1*}+\cdots+p_{m*}} f(\mathbf{u_0})}{\partial u_{1,0}^{p_{1*}}\cdots \partial u_{m,0}^{p_{m*}}}, \tag{21}$$

denotes the mixed partial derivatives.

Equivalently, we have [75]

$$A_n = \sum_{k=1}^{n}\sum_{P\in\mathcal{S}_{m,n}^k} f^{(p_{1*},\cdots,p_{m*})}(\mathbf{u_0})\prod_{i=1}^{m}\prod_{j=1}^{n-k+1}\frac{{u_{i,j}}^{p_{ij}}}{p_{ij}!}, \tag{22}$$

where every matrix in $\mathcal{S}_{m,n}^k$ is $m\times(n-k+1)$ and $P=(p_{ij})\in\mathcal{S}_{m,n}^k$ if and only if

$$\sum_{j=1}^{n-k+1} j\sum_{i=1}^{m} p_{ij} = n \quad\text{and}\quad \sum_{j=1}^{n-k+1}\sum_{i=1}^{m} p_{ij} = k, \tag{23}$$

and where $p_{i*} = \sum_{j=1}^{n-k+1} p_{ij}$.

Next we recall the algorithm given in [75] to efficiently generate the multivariable Adomian polynomials.

Corollary 1 algorithm [75]:

(i) Define the sets $T_{m,n}^k$ as follows, for $n\ge 1$,

$$T_{m,n}^1 = \{f^{(1,0,\cdots,0)}(\mathbf{u_0})u_{1,n}, f^{(0,1,\cdots,0)}(\mathbf{u_0})u_{2,n},\cdots,f^{(0,0,\cdots,1)}(\mathbf{u_0})u_{m,n}\}, \tag{24}$$

for $n \geq 2$ and $2 \leq k \leq \left[\frac{n}{2}\right]$,

$$T_{m,n}^{k} = \cup_{i=1}^{m} T_{m,n-1}^{k-1}\Big|_{p_{i1}\to p_{i1}+1, p_{i*}\to p_{i*}+1} \bigcup T_{m,n-k}^{k}\Big|_{u_{i,j}\to u_{i,j+1}}, \tag{25}$$

and for $n \geq 2$ and $\left[\frac{n}{2}\right] < k \leq n$,

$$T_{m,n}^{k} = \cup_{i=1}^{m} T_{m,n-1}^{k-1}\Big|_{p_{i1}\to p_{i1}+1, p_{i*}\to p_{i*}+1}, \tag{26}$$

where $u_{i,j} \to u_{i,j+1}$ denotes replacing all $u_{i,j}$ by $u_{i,j+1}$ in all objects of $T_{m,n-k}^{k}$, while $p_{i1} \to p_{i1}+1$, $p_{i*} \to p_{i*}+1$ stands for replacing $\frac{u_{i,1}^{p_{i1}}}{p_{i1}!}$ by $\frac{u_{i,1}^{p_{i1}+1}}{(p_{i1}+1)!}$ and p_{i*} by $p_{i*}+1$ for the index i in all objects of $T_{m,n-1}^{k-1}$.

(ii) The Adomian polynomials are obtained as

$$A_n = \sum_{k=1}^{n} \mathbf{S}\left(T_{m,n}^{k}\right), \tag{27}$$

where $\mathbf{S}\left(T_{m,n}^{k}\right)$ denotes the sum of all the objects of the set $T_{m,n}^{k}$.

Note also that this algorithm does not involve the differentiation operator. We present the MATHEMATICA code generating the multivariable Adomian polynomials based on this algorithm in the Appendix.

In addition, the m-variable Adomian polynomials A_n satisfy the recurrence rule [76]:

$$A_n = \frac{1}{n}\sum_{i=1}^{m}\sum_{k=0}^{n-1}(k+1)u_{i,k+1}\frac{\partial}{\partial u_{i,0}}A_{n-1-k}, n \geq 1, \tag{28}$$

and

$$A_n = \frac{1}{n}\sum_{i=1}^{m}\sum_{k=0}^{n-1}(k+1)u_{i,k+1}\frac{\partial}{\partial u_{i,k}}A_{n-1}, n \geq 1. \tag{29}$$

For a detailed proof, see [76]. We list the two-variable Adomian polynomials A_1 through A_4 for the abstract analytic function $f(u,v)$, with the decompositions $u = \sum_{n=0}^{\infty} u_n$ and $v = \sum_{n=0}^{\infty} v_n$ and where $f^{(p,q)}$ stands for $f^{(p,q)}(u_0, v_0)$, as

$$\begin{aligned}
A_1 &= u_1 f^{(1,0)} + v_1 f^{(0,1)},\\
A_2 &= u_2 f^{(1,0)} + v_2 f^{(0,1)} + \frac{u_1^2}{2} f^{(2,0)} + \frac{v_1^2}{2} f^{(0,2)} + u_1 v_1 f^{(1,1)},\\
A_3 &= u_3 f^{(1,0)} + v_3 f^{(0,1)} + u_1 u_2 f^{(2,0)} + v_1 v_2 f^{(0,2)} + (u_1 v_2 + u_2 v_1) f^{(1,1)} + \frac{u_1^3}{6} f^{(3,0)} + \frac{v_1^3}{6} f^{(0,3)}\\
&\quad + \frac{u_1^2 v_1}{2} f^{(2,1)} + \frac{u_1 v_1^2}{2} f^{(1,2)},\\
A_4 &= u_4 f^{(1,0)} + v_4 f^{(0,1)} + (u_1 u_3 + \frac{u_2^2}{2}) f^{(2,0)} + (v_1 v_3 + \frac{v_2^2}{2}) f^{(0,2)} + (u_1 v_3 + u_2 v_2 + u_3 v_1) f^{(1,1)}\\
&\quad + \frac{u_1^2 u_2}{2} f^{(3,0)} + \frac{v_1^2 v_2}{2} f^{(0,3)} + (u_1 u_2 v_1 + \frac{u_1^2 v_2}{2}) f^{(2,1)} + (u_1 v_1 v_2 + \frac{u_2 v_1^2}{2}) f^{(1,2)} + \frac{u_1^4}{4!} f^{(4,0)}\\
&\quad + \frac{u_1^3 v_1}{6} f^{(3,1)} + \frac{u_1^2 v_1^2}{4} f^{(2,2)} + \frac{u_1 v_1^3}{6} f^{(1,3)} + \frac{v_1^4}{4!} f^{(0,4)}.
\end{aligned}$$

For other new algorithms and their corresponding MATHEMATICA subroutines for fast generation of the Adomian polynomials, see [56, 75–78].

3 The Rach-Adomian-Meyers modified decomposition method and its generalization

In 1992, Rach, Adomian and Meyers [82] proposed a modified decomposition method based on the nonlinear transformation of series by the Adomian–Rach theorem [83, 84]:

$$\text{If}\quad u(x)=\sum_{n=0}^{\infty}a_n(x-x_0)^n,\quad \text{then}\quad f(u(x))=\sum_{n=0}^{\infty}A_n(x-x_0)^n, \tag{30}$$

where the $A_n = A_n(a_0, a_1, \ldots, a_n)$ are the Adomian polynomials in terms of the solution coefficients. The Rach-Adomian-Meyers modified decomposition method combines the power series solution and the Adomian–Rach theorem, and has been efficiently applied to solve various nonlinear models [7, 51, 85–89].

Higher-order numerical one-step methods based on the Rach-Adomian-Meyers modified decomposition method were developed by Adomian et al [90] and Duan and Rach [36, 91]. We observe that one of the difficulties in applying explicit Runge-Kutta one-step methods is that there is no general procedure to generate higher-order numeric methods. In [36] the numeric scheme we designed permits a straightforward universal procedure to generate higher-order numeric methods at will such as a 12th-order or 24th-order one-step method, if need be. Another key advantage is that we can easily estimate the maximum step-size prior to computing data sets representing the discretized solution, because we can approximate the radius of convergence from the solution approximants unlike the Runge-Kutta approach with its intrinsic linearization between computed data points. Also Duan and Rach [36] proposed new variable step-size, variable order and variable step-size, variable order algorithms for automatic step-size control to increase the computational efficiency and reduce the computational costs even further for critical engineering models.

The multivariable version of the Adomian-Rach theorem is: If $u_j(x) = \sum_{n=0}^{\infty} a_{j,n}(x-x_0)^n$, for $1 \le j \le m$, then

$$f\Big(\sum_{n=0}^{\infty}a_{1,n}(x-x_0)^n,\sum_{n=0}^{\infty}a_{2,n}(x-x_0)^n,\ldots,\sum_{n=0}^{\infty}a_{m,n}(x-x_0)^n\Big)=\sum_{n=0}^{\infty}A_n(x-x_0)^n, \tag{31}$$

where f is an m-ary analytic function, and where the

$$A_n = A_n(a_{1,0}, a_{1,1}, \ldots, a_{1,n}; a_{2,0}, a_{2,1}, \ldots, a_{2,n}; \ldots; a_{m,0}, a_{m,1}, \ldots, a_{m,n})$$

are the m-variable Adomian polynomials in terms of the solution coefficients [7, 66, 71, 75, 76, 83, 84].

For fractional differential equations, the solution usually involves a generalized power series [92–95] in the form of

$$u(x)=\sum_{n=0}^{\infty}a_n(x-x_0)^{n\lambda}, \tag{32}$$

where λ is a real number.

If we regard $(x-x_0)^{\lambda}$ as a single parameter, then by the Adomian-Rach theorem (30), we develop the generalized Adomian-Rach theorem,

$$f\Big(\sum_{n=0}^{\infty}a_n(x-x_0)^{n\lambda}\Big)=\sum_{n=0}^{\infty}A_n(x-x_0)^{n\lambda}, \tag{33}$$

where $A_n = A_n(a_0, a_1, \ldots, a_n)$ are the Adomian polynomials in terms of the solution coefficients.

The multivariable version of the generalized Adomian-Rach theorem is: If $u_j(x) = \sum_{n=0}^{\infty} a_{j,n}(x-x_0)^{n\lambda}$, for $1 \le j \le m$, them

$$f(\sum_{n=0}^{\infty} a_{1,n}(x-x_0)^{n\lambda}, \sum_{n=0}^{\infty} a_{2,n}(x-x_0)^{n\lambda}, \ldots, \sum_{n=0}^{\infty} a_{m,n}(x-x_0)^{n\lambda}) = \sum_{n=0}^{\infty} A_n(x-x_0)^{n\lambda}, \tag{34}$$

where f is an m-ary analytic function, and where the

$$A_n = A_n(a_{1,0}, a_{1,1}, \ldots, a_{1,n}; a_{2,0}, a_{2,1}, \ldots, a_{2,n}; \ldots; a_{m,0}, a_{m,1}, \ldots, a_{m,n})$$

are the m-variable Adomian polynomials in terms of the solution coefficients [7, 66, 71, 75, 76, 83, 84].

In [96] the generalized Adomian-Rach theorem has been used to solve nonlinear FDEs. We note that the fractional differential transform method [97] or the power series method in [92] for the solution of FDEs also seeks the generalized power series solutions, whereas these two later techniques treat linear FDEs or nonlinear FDEs but only for the case of quadratic nonlinearities. By the ADM with the Adomian polynomials and their fast, efficient generation algorithms, we can readily treat any analytic nonlinearity.

4 BVPs for nonlinear ODEs

Several different resolution techniques for solving BVPs for nonlinear ODEs by using the ADM were considered by Adomian and Rach [18, 19, 88], Adomian [7], Bigi and Riganti [15], Wazwaz [11, 20, 21, 98–102], Ebadi and Rashedi [24], Tatari and Dehghan [25], Dehghan and Tatari [26], Jang [27], Ebaid [28, 29], Al-Hayani [30] et al.

By the recursion scheme with undetermined coefficients for a nonlinear BVP, solution of a sequence of nonlinear algebraic equations in the undetermined coefficients is often involved [11, 20, 21, 25, 26, 98–102]. First we obtain the n-term approximations with the undetermined coefficients, then we match the unused boundary conditions. Next we solve the resulting sequence of nonlinear algebraic equations, or more complex transcendental equations, as the case may be. Usually this approach is fraught with spurious multiple roots for the undetermined coefficients with the subsequent necessity to devise an algorithm to discard the unphysical roots. This method has been frequently used in combination with the Padé approximants in order to more quickly obtain improved estimates.

We now demonstrate this method for the following nonlinear BVP,

Example 1. Consider the nonlinear BVP,

$$u''(x) = e^{u(x)}, \ 0 \le x \le 1, \tag{35}$$

$$u(0) = 0, \ u(1) = 0. \tag{36}$$

The exact solution for this problem [41, 103] is

$$u^*(x) = 2\ln(C \sec\frac{C(2x-1)}{4}) - \ln(2), \tag{37}$$

where C satisfies $C\sec\frac{C}{4} = \sqrt{2}$, hence, to 16 significant figures, $C = 1.336055694906108$.

Applying the inverse linear operator $L^{-1}(\cdot) = \int_0^x \int_0^x (\cdot) dx dx$ to both sides of Eq. (35) and using the boundary value specified at $x = 0$ yield

$$u(x) = Bx + L^{-1}e^{u(x)}, \tag{38}$$

where $B = u'(0)$ is retained as an undetermined coefficient.

Next we decompose the solution $u(x)$ and the nonlinearity $e^{u(x)}$,

$$u(x)=\sum_{n=0}^{\infty}u_n(x),\ e^{u(x)}=\sum_{n=0}^{\infty}A_n,$$

where the A_n are the Adomian polynomials for the nonlinearity $Nu(x)=e^{u(x)}$,

$$\begin{aligned}
A_0 &= e^{u_0},\\
A_1 &= e^{u_0}u_1,\\
A_2 &= e^{u_0}\left(\frac{u_1^2}{2}+u_2\right),\\
A_3 &= e^{u_0}\left(\frac{u_1^3}{6}+u_1u_2+u_3\right),\\
A_4 &= e^{u_0}\left(\frac{u_1^4}{24}+\frac{1}{2}u_1^2u_2+\frac{u_2^2}{2}+u_1u_3+u_4\right),\\
&\cdots.
\end{aligned}$$

If we use the standard recursion scheme

$$\begin{aligned}
u_0 &= Bx,\\
u_n &= L^{-1}A_{n-1},\ n=1,2,\ldots,
\end{aligned}$$

we calculate the solution components as

$$\begin{aligned}
u_1 &= \frac{-1+e^{Bx}-Bx}{B^2},\\
u_2 &= \frac{-5+e^{2Bx}-2Bx+e^{Bx}(4-4Bx)}{4B^4},\\
&\ldots,
\end{aligned}$$

and the approximate solutions as

$$\begin{aligned}
\varphi_1(x) &= Bx,\\
\varphi_2(x) &= Bx+\frac{-1+e^{Bx}-Bx}{B^2},\\
\varphi_3(x) &= Bx+\frac{-1+e^{Bx}-Bx}{B^2}+\frac{-5+e^{2Bx}-2Bx+e^{Bx}(4-4Bx)}{4B^4},\\
&\cdots.
\end{aligned}$$

By matching the $\varphi_n(x)$ at $x=1$, we can then determine the value of B as a convergent sequence of approximate values. If we match $\varphi_2(x)$ at $x=1$, then we need to solve the equation $B+\frac{-1-B+e^B}{B^2}=0$. By the native command 'FindRoot' in MATHEMATICA, we obtain $B=-0.4347754841$, and thus the approximate solution is

$$\varphi_2(x)=-5.29017+5.29017e^{-0.434775x}+1.86526x.$$

If we match $\varphi_3(x)$ at $x=1$, then we need to solve the equation

$$B+\frac{-1-B+e^B}{B^2}+\frac{-5-2B+(4-4B)e^B+e^{2B}}{4B^4}=0.$$

Solving this transcendental equation yields $B=-0.4604486353$ and thus the approximate solution is

$$\begin{aligned}
\varphi_3(x) &= -32.5257+5.5618e^{-0.920897x}+26.9639e^{-0.460449x}+6.83319x\\
&\quad +10.2437e^{-0.460449x}x.
\end{aligned}$$

If we match $\varphi_4(x)$ at $x=1$, then we obtain, by a similar manner, $B=-0.4632279437$ and thus the approximate solution is

$$\begin{aligned}\varphi_4(x) &= -217.363+8.43432e^{-1.38968x}+56.0355e^{-0.926456x}+152.893e^{-0.463228x} \\ &\quad +30.1678x+23.4421e^{-0.926456x}x+80.3867e^{-0.463228x}x+10.859e^{-0.463228x}x^2.\end{aligned}$$

We note that the true value of B is $B=u'(0)=-0.4636325917\ldots$.

If we use an alternate recursion scheme such as

$$\begin{aligned} u_0 &= 0, \\ u_1 &= Bx+L^{-1}A_0, \\ u_n &= L^{-1}A_{n-1},\ n=2,3,\ldots, \end{aligned}$$

we calculate instead the sequence of approximate solutions

$$\begin{aligned} \varphi_1(x) &= 0, \\ \varphi_2(x) &= Bx+\frac{x^2}{2}, \\ \varphi_3(x) &= Bx+\frac{x^2}{2}+\frac{Bx^3}{6}+\frac{x^4}{24}, \\ \varphi_4(x) &= Bx+\frac{x^2}{2}+\frac{Bx^3}{6}+\frac{x^4}{24}+\frac{B^2x^4}{24}+\frac{Bx^5}{30}+\frac{x^6}{180}, \\ \varphi_5(x) &= Bx+\frac{x^2}{2}+\frac{Bx^3}{6}+\frac{x^4}{24}+\frac{B^2x^4}{24}+\frac{Bx^5}{30}+\frac{B^3x^5}{120}+\frac{x^6}{180} \\ &\quad +\frac{11B^2x^6}{720}+\frac{17Bx^7}{2520}+\frac{17x^8}{20160}, \\ &\quad \ldots. \end{aligned}$$

By matching the approximate solutions incorporating the undetermined coefficients at $x=1$, we can then determine the sequence of the approximate solutions without any undetemined coefficients. By $\varphi_2(x)$, we have $B=-0.5$ and thus

$$\varphi_2(x)=-\frac{x}{2}+\frac{x^2}{2}.$$

By $\varphi_3(x)$, we have $\frac{13}{24}+\frac{7B}{6}=0$, hence $B=-0.4642857143$ and thus

$$\varphi_3(x)=-\frac{13x}{28}+\frac{x^2}{2}-\frac{13x^3}{168}+\frac{x^4}{24}.$$

By $\varphi_4(x)$, we solve $\frac{197}{360}+\frac{6B}{5}+\frac{B^2}{24}=0$, which yields two roots -0.463477239 and -28.33652276. The latter is unreasonable in comparison with the computed values from the other matching equations.

If such a recursion scheme were to contain more than one undetermined parameter such as the case of fourth-order BVPs, then the matching equations would become a system of multivariable nonlinear, algebraic or transcendental, equations.

We note that the Padé approximants technique [7, 104, 105] is sometimes used in conjunction with the method of undetermined coefficients [11, 98] in order to more quickly obtain improved estimates.

In order to avoid solving such nonlinear algebraic equations, Adomian and Rach [18, 19, 88] proposed the double decomposition method, and Ebadi and Rashedi [24], Jang [27] and Ebaid [28] introduced different modified inverse linear operators. Al-Hayani [30] instead used Green's functions to treat two-point higher order BVPs. Recently Duan and Rach [37] have proposed a new modification of the ADM for solving BVPs for higher order nonlinear differential equations. We note that Aly, Ebaid and Rach [106] devised a specialized inverse linear operator to solve two-point nonlinear BVPs with Neumann boundary conditions.

Next we review the Adomian-Rach modified recursion scheme for BVPs [18, 19, 88], alias the double decomposition method, by solving the nonlinear BVP,

$$Lu = Nu + g(x),\ a < x < b, \tag{39}$$

$$u(a) = \alpha,\ u(b) = \beta. \tag{40}$$

In the double decomposition method, the inverse linear operator L^{-1} is taken as a two-fold indefinite integration for the case of second-order differential equations, i.e.

$$L^{-1}(\cdot) = C_0 + C_1 x + I_x^2(\cdot), \tag{41}$$

where C_0 and C_1 are constants of integration, which are called the matching coefficients, and where $I_x^2(\cdot) = \int\int(\cdot)dxdx$ denotes pure two-fold integration. Applying the operator $L^{-1}(\cdot)$ to both sides of Eq. (39) yields the nonlinear integral equation

$$u(x) = C_0 + C_1 x + I_x^2 Nu + I_x^2 g(x),$$

where C_0 and C_1 are arbitrary constants of integration to be determined in the sequel.

The double decomposition method decomposes the solution $u(x)$, the nonlinearity Nu, and the constants C_0 and C_1:

$$u(x) = \sum_{n=0}^{\infty} u_n(x),\ Nu = \sum_{n=0}^{\infty} A_n,\ C_0 = \sum_{n=0}^{\infty} C_{0,n},\ C_1 = \sum_{n=0}^{\infty} C_{1,n}.$$

Upon substitution of these series into Eq. (42), we design the recursion scheme

$$\begin{aligned} u_0 &= C_{0,0} + C_{1,0}x + I_x^2 g(x), \\ u_n &= C_{0,n} + C_{1,n}x + I_x^2 A_{n-1},\ n = 1, 2, \ldots. \end{aligned}$$

The constants $C_{0,k}$ and $C_{1,k}$ are determined by matching each partial sum $\varphi_n(x)$ to the boundary values for the case of Dirichlet boundary conditions. That is, matching $\varphi_1(x) = u_0$ to the boundary values determines the values of $C_{0,0}$ and $C_{1,0}$, matching $\varphi_2(x) = u_0 + u_1$ to the boundary values determines the values of $C_{0,1}$ and $C_{1,1}$, $\ldots$, and matching $\varphi_{n+1}(x) = \sum_{k=0}^{n} u_k$ to the boundary values determines the values of $C_{0,n}$ and $C_{1,n}$.

For the nonlinear BVP in Example 1, the Adomian-Rach modified recursion scheme is

$$\begin{aligned} u_0 &= C_{0,0} + C_{1,0}x, \\ u_n &= C_{0,n} + C_{1,n}x + I_x^2 A_{n-1},\ n \geq 1, \end{aligned}$$

where the A_n are the Adomian polynomials for the nonlinearity $Nu = e^u$, which were previously given.

Matching $\varphi_1(x) = u_0$ to the boundary values determines that $C_{0,0} = C_{1,0} = 0$. Thus $u_0 = 0$, and $u_1 = C_{0,1} + C_{1,1}x + \frac{x^2}{2}$.

Matching $\varphi_2(x) = u_0 + u_1$ to the boundary values determines $C_{0,1} = 0$ and $C_{1,1} = -1/2$. Thus $u_1 = (x^2 - x)/2$ and $u_2 = C_{0,2} + C_{1,2}x - \frac{x^3}{12} + \frac{x^4}{24}$.

Matching $\varphi_3(x) = u_0 + u_1 + u_2$ to the boundary values determines that $C_{0,2} = 0$ and $C_{1,2} = 1/24$. Thus $u_2 = \frac{x}{24} - \frac{x^3}{12} + \frac{x^4}{24}$, and so on.

The Duan-Rach modified recursion scheme for BVPs [37] transforms the original nonlinear BVP into an equivalent nonlinear Fredholm-Volterra integral equation for the solution before designing the recursion scheme to calculate the solution components. Thus Duan and Rach [37] develop a modified recursion scheme excluding all undetermined coefficients when computing successive solution components. Furthermore the new modified recursion scheme can be parametrized in order to achieve simple-to-integrate series, faster rates of convergence and extended regions of convergence [56], in particular for such cases when one of the boundary points lies outside the interval of convergence of the usual decomposition series [37]. For Example 1, we derive the integral equation [37]

$$u(x) = L^{-1}e^{u(x)} - x[L^{-1}e^{u(x)}]_{x=1}, \tag{42}$$

before designing the new recursion scheme, where $L^{-1}(\cdot) = \int_0^x \int_0^x (\cdot)dxdx$. For different examples of higher order equations, see [37].

5 BVPs for nonlinear PDEs

The partial solution concept of the ADM can be applied to efficiently solve nonlinear PDEs, which states that the temporal partial solution for its set of initial conditions, or the t-partial solution, and the various spatial partial solutions for their respective sets of boundary conditions, such as the x-partial solution, are all identically equal, which was discovered by Adomian and Rach [107, 108] and proven by Wazwaz [109].

Usually we solve for the partial solution in the coordinate, for which we most accurately know the values of its auxiliary conditions.

For example, the following nonlinear second-order homogeneous PDE is considered in [37],

$$\theta_{xx}(x,t) + \frac{\varepsilon}{1+\varepsilon\theta(x,t)}(\theta_x(x,t))^2 - K^2\frac{\theta(x,t)}{1+\varepsilon\theta(x,t)} - \frac{\theta_t(x,t)}{1+\varepsilon\theta(x,t)} = 0, \tag{43}$$

$$\theta_x(0,t) = 0,\ \theta(1,t) = 1 + S\cos(Bt). \tag{44}$$

where K depends on the physical properties and design parameters, and where $\theta(x,t)$ is defined on the domain $(x,t) \in [0,1] \times [0,\infty)$ and subject to a mixed set of homogeneous Neumann and inhomogeneous Dirichlet boundary conditions, including a sinusoidally varying boundary value. The physical variable and parameters are θ, x, t, ε, K, S and B, which represent the dimensionless temperature, distance, time, thermal conductivity parameter, fin parameter, amplitude of oscillation and frequency of oscillation, respectively. The interested reader is referred to [110] for further details in regard to the derivation and design limitations of this engineering model. For this particular nonlinear BVP, approximations of the x-partial solution have been solved in [37].

We can also choose to average partial solutions for matching at the boundaries, when the values of more than one set of boundary conditions are known accurately, for sake of computational advantages [2–4, 6, 7, 13, 111–121].

For example, Patel and Serrano [121] considered the BVP for multidimensional linear and nonlinear groundwater equations, including the following nonlinear model,

$$\frac{\partial}{\partial x}\left(Kh\frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial y}\left(Kh\frac{\partial h}{\partial y}\right) = -R_g,\ 0 \le x \le l_x,\ 0 \le y \le l_y, \tag{45}$$

$$h(0,y) = f_1(y),\ h(l_x,y) = f_2(y),\ h(x,0) = f_3(x),\ h(x,l_y) = f_4(x), \tag{46}$$

where h is the hydraulic head, R_g is mean monthly recharge from rainfall, K is a constant, l_x and l_y are the aquifer horizontal dimensions in the x and y direction, respectively, and the functions f_1, f_2, f_3 and f_4 represent the mean elevation of the water at the boundaries.

Define the operators $L_x = \frac{\partial^2}{\partial x^2}$ and $L_y = \frac{\partial^2}{\partial y^2}$. Eq. (45) is rewritten as

$$L_x h + L_y h = Nh, \tag{47}$$

where $Nh=-\frac{1}{h}[\frac{R_g}{K}+(\frac{\partial h}{\partial x})^2+(\frac{\partial h}{\partial y})^2]$.

In [121], the inverse operators L_x^{-1} and L_y^{-1} are defined as the corresponding two-fold indefinite integrals with respect to x and y, respectively. The x-partial approximate solution and the y-partial approximate solution are obtained by using the double decomposition method, respectively. The average of the two partial approximate solutions by stages yields the improved solution.

The more general class of PDEs has been solved by using the ADM by different approaches [122–124]. Here we review the weighted algorithm proposed by Shidfar and Garshasbi [123] for a class of evolutionary PDEs having nonlinear advection, diffusion and reaction terms,

$$u_t+[f(u)]_x=[g(u)]_{xx}+h(u),\ (x,t)\in[0,1]\times[0,T], \tag{48}$$

subject to the following conditions

$$u(x,0)=\varphi(x),\ 0<x<1, \tag{49}$$
$$u(0,t)=p(t),\ u_x(1,t)=q(t),\ 0<t<T. \tag{50}$$

Using the linear operators $L_{xx}=\frac{\partial^2}{\partial x^2}$, $L_x=\frac{\partial}{\partial x}$ and $L_t=\frac{\partial}{\partial t}$, Eq. (48) becomes

$$L_tu+L_x[f(u)]=L_{xx}[g(u)]+h(u). \tag{51}$$

Applying the inverse linear operator $L_t^{-1}(\cdot)=\int_0^t(\cdot)dt$ to both sides of Eq. (51) yields

$$u(x,t)=\varphi(x)+L_t^{-1}[L_{xx}[g(u)]-L_x[f(u)]+h(u)]. \tag{52}$$

The components of the t-partial solution $u=\sum_{n=0}^{\infty}\hat{u}_n(x,t)$, for which the n-term approximation is $\hat{S}_n(x,t)=\sum_{k=0}^{n-1}\hat{u}_k(x,t)$, are determined

$$\hat{u}_0=\varphi(x), \tag{53}$$
$$\hat{u}_{n+1}=L_t^{-1}[L_{xx}(A_n)-L_x(B_n)+C_n],\ n\geq 0, \tag{54}$$

where A_n,B_n,C_n are the Adomian polynomials for the nonlinearities $g(u),f(u),h(u)$, respectively.

On the other hand, rewriting Eq. (51) as

$$L_{xx}u=L_tu+L_x[f(u)]-L_{xx}[g(u)-u]-h(u), \tag{55}$$

and applying the inverse linear operator $L_{xx}^{-1}(\cdot)=\int_0^x dx\int_1^x(\cdot)dx$ to both sides of Eq. (55), we construct the components of the x-partial solution $u=\sum_{n=0}^{\infty}\tilde{u}_n(x,t)$,

$$\tilde{u}_0=p(t)+xq(t), \tag{56}$$
$$\tilde{u}_{n+1}=L_{xx}^{-1}[L_t\tilde{u}_n+L_x(B_n)-L_{xx}(D_n)-C_n],\ n\geq 0, \tag{57}$$

where D_n is the Adomian polynomial for the function $G(u)=g(u)-u$, and Adomian polynomials B_n,D_n and C_n are obtained subject to $\tilde{u}_n,n\geq 0$. Denote the n-term approximation of the x-partial solution as $\tilde{S}_n(x,t)=\sum_{k=0}^{n-1}\tilde{u}_k(x,t)$.

The weighted algorithm constructs the solution as $u(x,t)=\sum_{n=0}^{\infty}u_n(x,t)$, where

$$u_n(x,t)=\alpha_n\hat{u}_n(x,t)+(1-\alpha_n)\tilde{u}_n(x,t),\ 0\leq\alpha_n\leq 1. \tag{58}$$

The n-term approximation is

$$S_n(x,t)=\alpha_n\hat{S}_n(x,t)+(1-\alpha_n)\tilde{S}_n(x,t). \tag{59}$$

Shidfar and Garshasbi [123] gave the best values for the α_n as

$$\alpha_n = \frac{\beta_{3n}^2}{\beta_{1n}^2 + \beta_{2n}^2 + \beta_{3n}^2},\ n \geq 0, \tag{60}$$

where

$$\beta_{1n} = \|\hat{S}_n(0,t) - p(t)\|,\ \beta_{2n} = \|\frac{\partial}{\partial x}\hat{S}_n(1,t) - q(t)\|,\ \beta_{3n} = \|\tilde{S}_n(x,0) - \varphi(x)\|, \tag{61}$$

and where $\|\cdot\|$ denotes the L^2-norm.

6 Expansion of the effective region of convergence, the multistage decomposition and its numeric schemes

We remark that the domain of the convergence for the decomposition series solution, like other series solutions, may not always be sufficiently large for engineering purposes. But we can readily handle this problem by means of one of several common convergence acceleration techniques [125], such as the diagonal Padé approximants [7, 11, 104, 126] or the iterated Shanks transform [7, 127, 128]. Rach and Duan [104] presented the combined solution of the near-field and far-field approximations by the Adomian and asymptotic decomposition methods, where the Padé approximants technique was used as necessary. In the ADM, Duan's parametrized recursion scheme [37, 56, 57] was proposed in order to obtain decomposition solutions with large effective regions of convergence.

Next we consider the multistage ADM and its numeric schemes for IVPs [35, 36, 90, 91, 129–136]. In [36] Duan and Rach have considered one-step numeric algorithms for IVPs for ODEs based on the ADM and the Rach-Adomian-Meyers modified decomposition method, respectively. Therein algorithms are parametrized by the order of the method, so that we can easily and rapidly generate a higher-order algorithm for numerical integration, such as a 12th-order or 24th-order one-step routine, if need be, for increased precision, and so forth, whereas it is extremely difficult and quite tedious to develop even an 8th-order Runge-Kutta algorithm. In fact, there is no common algorithm to generate higher orders of explicit Runge-Kutta formulas. Also new variable step-size, variable order and variable step-size, variable order algorithms for automatic step-size control were proposed to increase the computational efficiency and reduce the computational costs even further for critical engineering models.

In [91] higher-order numeric schemes based on the Wazwaz-El-Sayed modified Adomian decomposition method were proposed. Therein the van der Pol equation with sinusoidal input and a product nonlinearity was solved by using a 20th-order numeric modified ADM solution so as to demonstrate a large effective region of convergence for this approach. Also the Rössler system of equations was solved by using a 20th-order numeric modified ADM solution, and the Rössler attractor in phase space was reproduced using the new numeric algorithms.

Ghosh and Roy [137] considered the numeric-analytic form of the ADM for two-point BVPs. The multistage ADM for BVPs with Robin boundary conditions has been considered in [138], where a multistage ADM for BVPs has been developed through partitioning the domain into two, or more, subdomains, where a separate series in each subdomain using Duan-Rach modified recursion scheme for nonlinear BVPs [37] was computed. The sub-solutions are combined by applying the conditions of continuity at the interior boundary points in analogy to the multistage ADM for IVPs.

It has been shown that how the multistage ADM for BVPs can easily treat nonlinear examples when the original series diverges over the specified domain [138]. Another aim of the multistage ADM for BVPs is to solve nonlinear Neumann BVPs relying upon the key concept of converting the original BVP into two sub-BVPs, where each is subject to a mixed set of Neumann and Dirichlet boundary conditions [138]. In [139] the multistage ADM for BVPs has been shown to be practical for computing positive solutions of homogeneous nonlinear BVPs.

In contrast to the multistage ADM for IVPs, which is a solution extrapolation technique, we emphasize that the multistage ADM for BVPs [138, 139] is a solution interpolation technique.

7 Application to FDEs

The fractional calculus approach provides a powerful tool for the description of memory and hereditary properties of various materials and processes [92, 140–152]. It has been applied to many fields in science and engineering, such as viscoelasticity, anomalous diffusion, fluid mechanics, biology, chemistry, acoustics, control theory, etc. Thus FDEs [92, 141, 142, 144–150, 152–156], a class of integro-differential equations with singularities, occur naturally.

The theorem of existence and uniqueness of solutions for fractional ODEs has been presented in [92, 144, 157, 158]. For linear FDEs, the integral transform methods, including the Laplace, Fourier and Mellin transforms [92, 140–144, 153, 159–164] are usually used to obtain analytic solutions. The ADM has been used to efficiently solve linear or nonlinear and ordinary or partial FDEs [126, 165–180]. Wazwaz [181] solved this class of equations by using the ADM in the form of weakly singular Volterra-type integral equations of the second kind. A numeric scheme solving the FDEs based on the ADM was designed by Li and Wang [182]. The ADM-Padé approximants technique was also used for solving FDEs [126]. The Rach-Adomian-Meyers modified decomposition method was generalized to solve nonlinear FDEs [96]. There are several other analytic and numeric methods for the nonlinear FDEs [92, 140, 144, 148, 152, 183–195].

Next we review the definitions of the Riemann-Liouville and Caputo fractional derivatives [92, 140–145, 152].

Let $f(t)$ be piecewise continuous on $(t_0,+\infty)$ and integrable on any finite subinterval of $(t_0,+\infty)$ (this class of functions is denoted by the symbol $\mathfrak{C}$). Then for $t > t_0$, the Riemann-Liouville fractional integral of $f(t)$ of order β is defined as

$$ {}_{t_0}J_t^{\beta} f(t) = \int_{t_0}^{t} \frac{(t-\tau)^{\beta-1}}{\Gamma(\beta)} f(\tau)d\tau, \tag{62} $$

where β is a positive real number, and $\Gamma(\cdot)$ is Euler's gamma function. For complementarity, we define ${}_{t_0}J_t^0 f(t) = f(t)$.

The fractional integral satisfies the following equalities,

$$ {}_{t_0}J_t^{\beta}\, {}_{t_0}J_t^{\mu} f(t) = {}_{t_0}J_t^{\beta+\mu} f(t),\ \beta \geq 0,\ \mu \geq 0, \tag{63} $$

$$ {}_{t_0}J_t^{\nu}(t-t_0)^{\mu} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\nu+1)}(t-t_0)^{\mu+\nu},\ \nu \geq 0,\ \mu > -1. \tag{64} $$

Let $f(t)$ be a function of class $\mathfrak{C}$ and α be a positive real number satisfying $m-1 < \alpha \leq m$ and $m \in \mathbb{N}^+$, where $\mathbb{N}^+$ is the set of positive integers. Then the Riemann-Liouville fractional derivative of $f(t)$ of order α is defined, when it exists, as

$$ {}_{t_0}\mathbf{D}_t^{\alpha} f(t) = \frac{d^m}{dt^m}\left({}_{t_0}J_t^{m-\alpha} f(t)\right),\ t > t_0. \tag{65} $$

For complementarity, we define ${}_{t_0}\mathbf{D}_t^0 f(t) = f(t)$.

For power functions, the following equality holds

$$ {}_{t_0}\mathbf{D}_t^{\alpha}(t-t_0)^{\mu} \quad = \quad \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)}(t-t_0)^{\mu-\alpha}, \tag{66} $$

where $\mu > -1,\ 0 \le m-1 < \alpha \le m$ and $t > t_0$.

Let α be a positive real number, $m-1 < \alpha \le m$ and $m \in \mathbb{N}^+$, and let $f^{(m)}(t)$ exist and be a function of class $\mathfrak{C}$. Then the Caputo fractional derivative of $f(t)$ of order α [92, 96, 143] is defined as

$$\begin{aligned} {}_{t_0}D_t^{\alpha} f(t) &= {}_{t_0}J_t^{m-\alpha} f^{(m)}(t) \\ &= {}_{t_0}\mathbf{D}_t{}^{\alpha} \left[f(t) - \sum_{k=0}^{m-1} \frac{(x-x_0)^k}{k!} f^{(k)}(t_0^+) \right],\ t > t_0. \end{aligned} \tag{67}$$

For complementarity, we define ${}_{t_0}D_t^0 f(t) = f(t)$.

For the Caputo fractional derivative of a polynomial function, the following equality holds

$${}_{t_0}D_t^{\alpha}(a_0 t^{m-1} + a_1 t^{m-2} + \cdots + a_{m-1}) = 0,\ m-1 < \alpha \le m. \tag{68}$$

Moreover, the α-order integral of the α-order Caputo fractional derivative requires the knowledge of the initial values of the function and its integer order derivatives just as in the case of the usual integer order case, thus

$${}_{t_0}J_t^{\alpha}\ {}_{t_0}D_t^{\alpha} f(t) = f(t) - \sum_{k=0}^{m-1} f^{(k)}(t_0^+) \frac{(t-t_0)^k}{k!},\ m-1 < \alpha \le m. \tag{69}$$

Furthermore for $\beta > \alpha > 0$ and $m-1 < \alpha \le m$, we have

$${}_{t_0}J_t^{\beta}\ {}_{t_0}D_t^{\alpha} f(t) = {}_{t_0}J_t^{\beta-\alpha} f(t) - \sum_{k=0}^{m-1} f^{(k)}(t_0^+) \frac{(t-t_0)^{k+\beta-\alpha}}{\Gamma(k+1+\beta-\alpha)}. \tag{70}$$

We remark that either the Laplace transform or the same-order integral of the Riemann-Liouville fractional derivative would also involve the initial values of fractional derivatives [92]. This limits its practical applicability by the absence of any physical interpretation of this type of initial values. Instead we model a FDE with the Caputo fractional derivatives associated with the initial or boundary conditions in the traditional form.

For the Caputo fractional derivative of the power function $(t–t_0)^{\mu}$, $\mu > 0$, if $0 \le m–1 < \alpha \le m < \mu+1$, then we have

$${}_{t_0}D_t^{\alpha}(t-t_0)^{\mu} = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)}(t-t_0)^{\mu-\alpha},\ t > t_0. \tag{71}$$

Denote ${}_0J_t^{\lambda}$ and ${}_0D_t^{\lambda}$ as J_t^{λ} and D_t^{λ} for short, respectively. First we solve a FDE by the Rach-Adomian-Meyers modified decomposition method.

Example 2. Consider the IVP for the nonlinear fractional ODE with a quartic nonlinearity

$$D_t^{\lambda} u(t) + u^4(t) = 1, \tag{72}$$

$$u(0) = C_0, \tag{73}$$

where λ is a real number that satisfies $0 < \lambda \le 1$.

When $C_0 = 0$ and $\lambda = 0.5$, the IVP, which governs the radiation of heat from a semi-infinite solid having a constant heat source, was previously solved by using the ADM in the form of weakly singular Volterra-type integral equation of the second kind [181].

We decompose the solution as $u(t) = \sum_{n=0}^{\infty} a_n t^{\lambda n}$. Then decompose the nonlinearity $Nu(t) = u^4(t)$ as $Nu(t) = \sum_{n=0}^{\infty} A_n t^{\lambda n}$, where the Adomian polynomials are

$$A_0 = a_0^4,\ A_1 = 4a_0^3 a_1,\ A_2 = 4a_0^3 a_2 + 6a_0^2 a_1^2,\ \ldots,$$

in terms of the solution coefficients. For the case of quartic nonlinearities, we have

$$A_n = \sum_{l=0}^{n}\sum_{s=0}^{l}\sum_{k=0}^{s} a_{n-l}a_{l-s}a_{s-k}a_k. \tag{74}$$

Substituting the decompositions of the solution and nonlinearity into Eq. (72), we design the following modified recursion scheme for the solution coefficients

$$\begin{aligned} a_0 &= C_0, & (75)\\ a_1 &= \frac{1-A_0}{\Gamma(\lambda+1)}, & (76)\\ a_{n+1} &= -\frac{\Gamma(n\lambda+1)}{\Gamma(n\lambda+\lambda+1)}A_n,\ n\geq 1. & (77)\end{aligned}$$

For the case $C_0=0$ and $\lambda=1/2$, we find that $a_n=0$ for $n\neq 4k+1$. We list the 21-term approximation $\varphi_{21}(t;1/2)=\sum_{n=0}^{20} a_n t^{n/2}$ as

$$\varphi_{21}(t;1/2) = \frac{2\sqrt{t}}{\sqrt{\pi}} - \frac{256t^{5/2}}{15\pi^{5/2}} + \frac{2097152t^{9/2}}{4725\pi^{9/2}} - \frac{205084688384t^{13/2}}{14189175\pi^{13/2}} + \frac{40259102036134461 44t^{17/2}}{7761123995625\pi^{17/2}}. \tag{78}$$

The radius of convergence can be estimated by computing the sequence of values

$$\rho_{4n+1}(1/2) = \left(\frac{1}{\sqrt[4n+1]{|a_{4n+1}|}}\right)^2.$$

When $n=19$, we calculate $\rho_{77}(1/2)\approx 0.52$. In Eq. (78) substituting $t^{1/2}$ with s, calculating the [10/10] Padé approximant about s by MATHEMATICA, then replacing s by $t^{1/2}$ we obtain

$$\text{Padé}[\varphi_{21}(t;1/2)] = \frac{\frac{2\sqrt{t}}{\sqrt{\pi}} + \frac{2388220672t^{5/2}}{29456427\pi^{5/2}} + \frac{21569223000064t^{9/2}}{46393872525\pi^{9/2}}}{1+\frac{2409119744t^2}{49094045\pi^2} + \frac{1531998896128t^4}{3568759425\pi^4}}. \tag{79}$$

In Fig. 1, we plot the curves of $\varphi_{21}(t;1/2)$ and Padé$[\varphi_{21}(t;1/2)]$, which demonstrates that the Padé approximant expands the region of convergence for the approximant solution $\varphi_{21}(t;1/2)$.

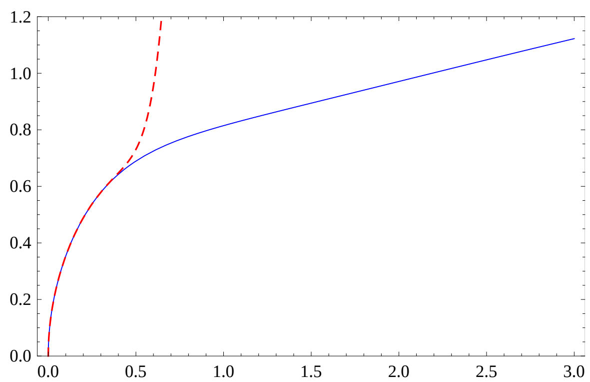

Fig. 1. The curves of the approximant solution $\varphi_{21}(t;1/2)$ (dashed line) and its Padé approximant Padé$[\varphi_{21}(t;1/2)]$ (solid line).

A rational order FDE with constant coefficients

$${}_{t_0}D_t^{\frac{q}{p}}u(t) + \alpha_1 \cdot {}_{t_0}D_t^{\frac{q-1}{p}}u(t) + \ldots \alpha_{q-1}\cdot {}_{t_0}D_t^{\frac{1}{p}}u(t) + \alpha_q u(t) + \alpha_{q+1}f(u(t)) = g(t), \tag{80}$$

where p,q are positive integers, $p \geq 2$, f is an analytic nonlinear function and $g(t)$ is the system input function that can be written in the form of a generalized power series $g(t) = \sum_{n=0}^{\infty} g_n (t-t_0)^{n/p}$, always has the solution in the generalized power series form [96] as

$$u(t) = \sum_{n=0}^{\infty} a_n (t-t_0)^{n/p}. \tag{81}$$

The ADM can be used to treat more general cases, including multi-order FDEs such as

$$D_t^{\mu} u(t) + D_t^{\beta} u(t) + N u(t) = g(t), \tag{82}$$

$$u(0) = C_0,\ u'(0) = C_1, \tag{83}$$

where μ and β are real numbers satisfying $1 < \mu \leq 2$ and $0 < \beta \leq 1$, $Nu(t)$ is an analytic nonlinearity, $g(t)$ is a specified polynomial function such that the fractional integral can be easily calculated. If not, we can use the power series expansion of $g(t)$ in the Wazwaz-El-Sayed modified recursion scheme [55].

Applying the fractional integral operator J_t^{μ} to both sides of Eq. (82) yields

$$u(t) = C_0 + C_1 t + J_t^{\mu} g(t) + \frac{C_0 t^{\mu-\beta}}{\Gamma(\mu-\beta+1)} - J_t^{\mu-\beta} u(t) - J_t^{\mu} N u(t). \tag{84}$$

We decompose the solution as $u(t) = \sum_{n=0}^{\infty} u_n$ and the nonlinearity as $Nu(t) = \sum_{n=0}^{\infty} A_n$. For a complicated nonlinearity we usually use the modified recursion scheme

$$u_0 = C_0, \tag{85}$$

$$u_1 = C_1 t + J_t^{\mu} g(t) + \frac{C_0 t^{\mu-\beta}}{\Gamma(\mu-\beta+1)} - J_t^{\mu-\beta} u_0 - J_t^{\mu} A_0, \tag{86}$$

$$u_{n+1} = -J_t^{\mu-\beta} u_n - J_t^{\mu} A_n,\ n \geq 1, \tag{87}$$

to calculate the solution components for the sake of trivial integrations .

Jafari and Daftardar-Gejji [172, 196] considered a system of nonlinear FDEs and the nonlinear fractional BVPs using the ADM, including the fractional planar Bratu-type problem

$$D_x^{\alpha} u(x) + \mu e^{u(x)} = 0,\ 1 < \alpha \leq 2,\ 0 \leq x \leq 1, \tag{88}$$

$$u(0) = u(1) = 0. \tag{89}$$

Linear and nonlinear fractional PDEs have also been solved using the ADM by Momani [197, 198], Momani and Odibat [199], Odibat and Momani [200], El-Sayed and Gaber [201], El-Wakil et al [202], Jafari and Daftardar-Gejji [203], Jafari et al [204], Wang [205], Chen and An [206], et al. Here we list the coupled Burgers equations with time- and space-fractional derivatives considered by Chen and An [206],

$$D_t^{\alpha} u = L_{2x} u + 2u D_x^{\alpha} u - L_x(uv),\ 0 < \alpha \leq 1, \tag{90}$$

$$D_t^{\beta} u = L_{2x} v + 2v D_x^{\beta} v - L_x(uv),\ 0 < \beta \leq 1, \tag{91}$$

subject to the initial conditions

$$u(x,0) = f(x),\ v(x,0) = g(x), \tag{92}$$

where we have adopted the notation $L_{nx} = \frac{\partial^n}{\partial x^n}$.

Conclusions

We have presented a contemporary review of the ADM and discussed its utility and advantages for solving linear or nonlinear and deterministic or stochastic operator equations without any restrictive assumptions, including ODEs, PDEs, integral equations and integro-differential equations for IVPs or BVPs. The ADM is the method of choice for solving nonlinear differential equations with a wide class of analytic nonlinearities including product, polynomial, exponential, trigonometric, hyperbolic, composite, negative-power, radical and decimal-power nonlinearities. Furthermore the ADM has been shown to be a reliable method for the solution of nonlinear fractional ODEs and PDEs for both IVPs and BVPs.

In the introduction, we reviewed the basic principles of the ADM, including decomposition of the solution and decomposition of the nonlinearities into the Adomian polynomials, various modified and parametrized recursion schemes and convergence of the Adomian decomposition series. In Section 2, we considered several recent algorithms for computer generation of the Adomian polynomials quickly and to high orders, including the single-variable Adomian polynomials for solving single nonlinear differential equations with simple nonlinearities and the multivariable Adomian polynomials for solving either single nonlinear differential equations with multi-order differential nonlinearities or systems of coupled nonlinear differential equations with multivariable nonlinearities. We featured several recent algorithms by Duan for the efficient computer generation of the Adomian polynomials. Indeed Duan's MATHEMATICA subroutines have been timed to the fastest on record in generating the Adomian polynomials [76]. In Section 3, we recalled the Rach-Adomian-Meyers modified decomposition method and its generalization for nonlinear FDEs for single variable and multivariable nonlinearities. In the next two sections, we reviewed the solution of BVPs for nonlinear ODEs and PDEs by the ADM for different techniques to approximate the constants of integration, including the standard recursion scheme with undetermined coefficients, the Adomian-Rach modified recursion scheme, which explicitly decomposes the constants of integration, also known as the double decomposition method, and the Duan-Rach modified recursion scheme, for which we first derive the equivalent nonlinear Fredholm-Volterra integral equation for the solution before designing the recursion scheme without any undetermined coefficients. For nonlinear PDEs, we reviewed the concept of the equality of partial solutions and also the concepts of equal- and unequal-weighted averages of partial solutions. In Section 6, we discussed several techniques for expansion of the region of convergence of the computed Adomian decomposition series for nonlinear differential equations, including well-known nonlinear sequence transformations such as the diagonal Padé approximants and the iterated Shanks transform. Next we considered solution continuation by various numeric schemes derived from the ADM and the Rach-Adomian-Meyers modified decomposition method. Furthermore we also reviewed the multistage ADM for nonlinear IVPs, which is a solution extrapolation technique, and the newer multistage ADM for nonlinear BVPs, which is a solution interpolation technique, and discuss its most important advantages. In the next to the last section, we reviewed the theory and common practice of solution of linear and nonlinear FDEs, including the usual concepts of fractional derivatives, fractional integrals, etc. Then we reviewed the application of the ADM and the modified decomposition methods for solution of nonlinear FDEs.

In summary, the ADM is a powerful and efficient technique for the solution of nonlinear ordinary, partial and fractional differential equations. It provides the analyst with an easily computable, readily verifiable and rapidly convergent sequence of analytic approximate functions for the solution.

Appendix. MATHEMATICA code for generating the Adomian polynomials

(i) Generating the single-variable Adomian polynomials based on Corollary 3 algorithm [76]

```
AP[f_, M_] := Module[{c, n, k, j, der},
Table[c[n, k], {n, 1, M}, {k, 1, n}];
der = Table[D[f[Subscript[u, 0]], {Subscript[u, 0], k}], {k, 1, M}];
A[0] = f[Subscript[u, 0]];
For[n = 1, n <= M, n++, c[n, 1] = Subscript[u, n];
For[k = 2, k <= n, k++,
c[n, k] = Expand[1/n* Sum[(j + 1)*Subscript[u, j+1]* c[n-1-j, k-1],
{j, 0, n-k}]] ];
A[n] = Take[der, n].Table[c[n, k], {k, 1, n}] ];
Table[A[n], {n, 0, M}] ]
```

(ii) Generating the multivariable Adomian polynomials based on Corollary 1 algorithm [75]

```
APmulti[f_, m_, M_] := Module[{i, j, r},
Subscript[u, 0] = Table[Subscript[u, i, 0], {i, 1, m}];
A[0] = f@@Subscript[u, 0]; Table[T[i, j], {i, 1, M}, {j, 1, i}];
se = Table[_, {m}] /. List -> Sequence;
For[r = 1, r <= M, r++,
T[r, 1]=Table[Subscript[u, i, r]*D[f@@Subscript[u, 0],
Subscript[u, i, 0]], {i, 1, m}];
For[k = 2, k <= r, k++,
T[r, k]=Union[Flatten[Table[D[Map[#*Subscript[u, i, 1]/
(Exponent[#, Subscript[u, i, 1]]+1)&,
T[r-1, k-1]], Subscript[u, i, 0]], {i, 1, m}]]]];
For[k = 2, k <= Floor[r/2], k++,
T[r, k] = T[r, k] \[Union](T[r - k, k] /.
Flatten[Table[Subscript[u, i, j] -> Subscript[u, i, j+1], {i,1,m},
{j, 1, r-2*k+1}]])];
A[r] = Sum[Total[T[r, k] ], {k, 1, r}];
If[EvenQ[r], Do[T[r/2, k] =., {k, 1, r/2}]] ];
Table[A[r], {r, 0, M}]]
```

Acknowledgements

This work was supported in part by the Innovation Program of Shanghai Municipal Education Commission (No. 11YZ225).

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