Mechanics Research Communications 48 (2013) 42–45

Contents lists available at SciVerse ScienceDirect

Mechanics Research Communications

journal homepage: www.elsevier.com/locate/mechrescom

Numerical solution of composite left and right fractional Caputo derivative models for granular heat flow

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ARTICLE INFO

Article history:
Received 22 July 2012
Received in revised form 7 December 2012
Available online 19 December 2012

Keywords:
Caputo derivatives
Fractional ordinary differential equations
Granular matter
Heat transfer
Numerical solutions

ABSTRACT

In this paper we propose a numerical scheme based on a fractional trapezoidal method for solution of a fractional equation with composition of the left and right Caputo derivatives. The numerical results are compared with analytical solutions. We have illustrated the convergence of our scheme. Finally, we show an application of the considered equation.

1. Introduction

In recent years fractional differential equations have proven to be very useful tools for modelling many phenomena in physics, bioengineering, heat transfer, mechanics and many other areas, e.g. Blaszczyk et al. (2011), Hilfer (2000), Leszczynski (2004, 2011), Leszczynski and Blaszczyk (2011), Luchko and Punzi (2011), Magin (2006), Mainardi et al. (2000), Qi and Xu (2007), Scalas et al. (2000).

In this paper a fractional differential equation containing composition of left and right Caputo derivatives in a finite time interval is considered. The fractional operator in this type of equation can contain the left and right fractional derivatives in any sense (Caputo or Riemann–Liouville) simultaneously.

Some forms of this problem have been studied by many authors (Agrawal, 2002, 2004, 2010; Agrawal et al., 2011; Klimek, 2001, 2002; Odzijewicz et al., 2012; Riewe, 1996, 1997). This form of fractional operator makes it difficult to find an analytical solution of the considered equation. Some analytical results can be found in papers (Baleanu and Trujillo, 2008; Klimek, 2009, 2010), where a fixed point theorem was used. This solution has a complex form, i.e. contains a series of alternately left and right fractional integrals. Using the Mellin transform, Klimek (2008) obtain an analytical solution which was represented by a series of special functions. In both cases the analytical results are very difficult for practical calculations.

Numerical solutions of the equations containing specific compositions of fractional derivatives, i.e. the arbitrary form of Riemann–Liouville (left or right) composed with the arbitrary form of Caputo (also left or right) are studied in Blaszczyk and Ciesielski (2010), Blaszczyk et al. (2011), Leszczynski and Blaszczyk (2011).

Some problems in numerical solutions of an equation with a composed form of fractional derivatives (the left and right Caputo operators), were considered by Blaszczyk et al. (2011). This was limited by the fractional order to $\alpha \in (0, 1)$. In this paper we extend previous studies to a new numerical scheme based on a fractional trapezoidal method (Odibat, 2006, 2009) suitable for solution of fractional differential equations where the fractional operator is a composition of left and right Caputo derivatives of order $\alpha \in (1, 2)$.

2. Formulation of the problem

Let us recall definitions of fractional operators. The fractional integrals of order $\alpha \in R_+$ are defined as follows (Kilbas et al., 2006; Podlubny, 1999; Samko et al., 1993)

$$I_{0+}^{\alpha} f(t) := \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)\, ds}{(t-s)^{1-\alpha}} \quad t > 0, \tag{1}$$

$$I_{b-}^{\alpha} f(t) := \frac{1}{\Gamma(\alpha)} \int_t^b \frac{f(s)\, ds}{(s-t)^{1-\alpha}} \quad t < b. \tag{2}$$

The operators I_{0+}^{α} and I_{b-}^{α} are known as the left and right Riemann–Liouville fractional integral, respectively.

Using the above fractional integrals one can define fractional derivatives. The left Caputo fractional derivative for order $\alpha \in R$

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0093-6413/$ – see front matter © 2012 Elsevier Ltd. All rights reserved.
http://dx.doi.org/10.1016/j.mechrescom.2012.12.004

looks as follows (Kilbas et al., 2006; Podlubny, 1999; Samko et al., 1993)

$$ {}^{C}D^{\alpha}_{0+}f(t) := I^{n-\alpha}_{0+}D^{n}f(t), \tag{3} $$

and the right Caputo fractional derivative is defined as (Kilbas et al., 2006; Podlubny, 1999; Samko et al., 1993)

$$ {}^{C}D^{\alpha}_{b-}f(t) := I^{n-\alpha}_{b-}(-D)^{n}f(t), \tag{4} $$

where $n = [\alpha] + 1$ and $D^n := (d^n/dt^n)$ represents the classical derivative of order n.

We consider an ordinary fractional differential equation with composition of left and right Caputo derivatives in the following form

$$ {}^{C}D^{\alpha}_{b-}\,{}^{C}D^{\alpha}_{0+}f(t) + \lambda f(t) = 0, \quad t \in [0, b]. \tag{5} $$

We assume that order $\alpha \in (1, 2)$, then Eq. (5) are supplemented by the following boundary conditions

$$ f(0) = F1, \quad f(b) = F2, \quad Df(0) = G1, \quad Df(b) = G2. \tag{6} $$

We propose numerical schemes for a system containing Eq. (5) and conditions (6).

3. Numerical solution

In this section we present numerical schemes based on a fractional trapezoidal method (Leszczynski, 2011; Odibat, 2006, 2009) for Eq. (5) for fractional order $\alpha \in (1, 2)$.

We introduce a homogeneous grid $0 = t_0 < t_1 < \cdots < t_i < t_{i+1} < \cdots < t_{N-1} < t_N = b$ with time-step $h = t_{i+1} - t_i$. Additionally in calculations we denote $f_i = f(t_i)$.

Now we discretise the fractional derivatives in Eq. (5). Using results from the works (Leszczynski, 2011; Odibat, 2006, 2009) we obtain the discrete form for the left Caputo operator as

$$ {}^{C}D^{\alpha}_{0+}f(t)\Big|_{t=t_i} \cong \frac{h^{-\alpha}}{\Gamma(4-\alpha)} \Bigg\{ f_{i-1} - 2f_i + f_{i+1} + [-(i+\alpha-3)i^{2-\alpha} + (i-1)^{3-\alpha}](f_{-1} - 2f_0 + f_1) + \sum_{j=1}^{i-1} [(i-j+1)^{3-\alpha} - 2(i-j)^{3-\alpha} + (i-j-1)^{3-\alpha}](f_{j-1} - 2f_j + f_{j+1}) \Bigg\} = h^{-\alpha} \sum_{j=-1}^{i+1} f_j v(i,j), \tag{7} $$

where

$$ v(i,j) = \frac{1}{\Gamma(4-\alpha)} \begin{cases} (i-1)^{3-\alpha} - (i+\alpha-3)i^{2-\alpha} & \textit{for} \ \ j=-1 \\ 2\alpha - 3 & \textit{for} \ \ j = 0 \wedge i = 1 \\ i^{3-\alpha} - 4(i-1)^{3-\alpha} + (i-2)^{3-\alpha} + 2(i+\alpha-3)i^{2-\alpha} & \textit{for} \ \ j = 0 \wedge i \neq 1 \\ 2-\alpha & \textit{for} \ \ j = 1 \wedge i = 1 \\ 6 - (\alpha-1)2^{2-\alpha} - 2\cdot 2^{3-\alpha} & \textit{for} \ \ j = 1 \wedge i = 2 \\ -2i^{3-\alpha} + 6(i-1)^{3-\alpha} - 4(i-2)^{3-\alpha} & \\ +(i-3)^{3-\alpha} & \\ -(i+\alpha-3)i^{2-\alpha} & \textit{for} \ \ j = 1 \wedge i \neq 1, 2 \\ (i-j+2)^{3-\alpha} - 4(i-j+1)^{3-\alpha} & \\ +6(i-j)^{3-\alpha} - 4(i-j-1)^{3-\alpha} & \\ (i-j-2)^{3-\alpha} & \textit{for} \ \ j = 2, \ldots, i-2 \\ 3^{3-\alpha} - 4\cdot 2^{3-\alpha} + 6 & \textit{for} \ \ j = i-1 \\ 2^{3-\alpha} - 4 & \textit{for} \ \ j = i \\ 1 & \text{for} \ \ j = i+1 \end{cases} \tag{8} $$

Substituting $g(t) = {}^{C}D^{\alpha}_{0+}f(t)$ and denoting $g_i = g(t_i)$ we get a discrete form of the right Caputo derivative

$$ {}^{C}D^{\alpha}_{b-}\,{}^{C}D^{\alpha}_{0+}f(t)\Big|_{t=t_i} = {}^{C}D^{\alpha}_{b-}\,g(t)\Big|_{t=t_i} \cong \frac{h^{-\alpha}}{\Gamma(4-\alpha)} \Bigg\{ g_{i-1} - 2g_i + g_{i+1} + [(N-i-1)^{3-\alpha} - (N-i+\alpha-3)(N-i)^{2-\alpha}] \times (g_{N-1} - 2g_N + g_{N+1}) + \sum_{j=i+1}^{N-1} [(i-j+1)^{3-\alpha} - 2(i-j)^{3-\alpha} + (i-j-1)^{3-\alpha}] \times (g_{j-1} - 2g_j + g_{j+1}) \Bigg\} = h^{-\alpha} \sum_{j=i-1}^{N+1} g_j v(N-i, N-j). \tag{9} $$

Next, substituting the discrete form of the derivative (7) into (9), the following form is obtained

$$ {}^{C}D^{\alpha}_{b-}\,{}^{C}D^{\alpha}_{0+}f(t)\Big|_{t=t_i} \cong h^{-2\alpha} \sum_{j=i-1}^{N+1} \left[v(N-i, N-j) \sum_{k=-1}^{j+1} v(j,k) f_k \right]. \tag{10} $$

And finally, for order $\alpha \in (1, 2)$ we have the following discrete form of Eq. (5)

$$ h^{-2\alpha} \sum_{j=i-1}^{N+1} \left[v(N-i, N-j) \sum_{k=-1}^{j+1} v(j,k) f_k \right] + \lambda f_i = 0, \tag{11} $$

for $i = 2, \ldots, N-2$.

Table 1
Errors generated in scheme (11) for $\lambda = 0$.

h	$\alpha = 1.2$		$\alpha = 1.4$		$\alpha = 1.6$	
	Error	EOC	Error	EOC	Error	EOC
1/40	4.51e−3		4.61e−3		5.35e−3	
1/80	2.30e−3	0.97	2.30e−3	1.00	2.82e−3	0.92
1/160	1.18e−3	0.96	1.16e−3	0.99	1.49e−3	0.91
1/320	6.05e−4	0.96	5.86e−4	0.99	7.96e−4	0.91
1/640	3.10e−4	0.96	2.96e−4	0.99	4.26e−4	0.90
1/1280	1.59e−4	0.97	1.49e−4	0.99	2.29e−4	0.90

4. Results

In this section we present numerical results for order $\alpha \in (1, 2)$. We assume $b = 1$ and then we analyse the errors and convergence of our scheme (11).

In order to estimate the error of our method we compare numerical results with the analytical ones. The error formula is determined by the following form

$$error[N] = \frac{(1/2)\left|f(t_0) - f_0\right| + (1/2)\left|f(t_N) - f_N\right| + \sum_{i=1}^{N-1}\left|f(t_i) - f_i\right|}{N}. \tag{12}$$

We calculate experimental estimation of the convergence row (EOC). Thus we have

$$EOC = \log_2\left(\frac{error[N]}{error[2N]}\right). \tag{13}$$

Let us introduce boundary-value conditions as

$$f(0) = 0, \quad Df(0) = 0, \quad f(1) = 1, \quad Df(1) = 0. \tag{14}$$

We consider a case when $\lambda = 0$, then the analytical solution of Eq. (5) has the following form (Blaszczyk et al., 2011; Klimek, 2009)

$$f(t) = c_0 t^{\alpha} + c_1 t^{\alpha+1}. \tag{15}$$

Table 1 shows errors generated by the numerical scheme (11) being dependent on the fractional order α and time-step of calculations h. Analysing values of EOC in Table 1 one can observe that the convergence of our numerical schemes is $O(h)$ and does not depend on parameter α.

Next, we calculated some examples for different values of α and λ in order to show how numerical solutions of Eq. (5) behave.

Fig. 1 presents comparison between the numerical results for $\lambda = -1$ and for different values of the parameter $\alpha \in \{1.95; 1.9; 1.85; 1.8\}$ and the solution generated by the following ordinary differential equation

$$D^4 f(t) + \lambda f(t) = 0. \tag{16}$$

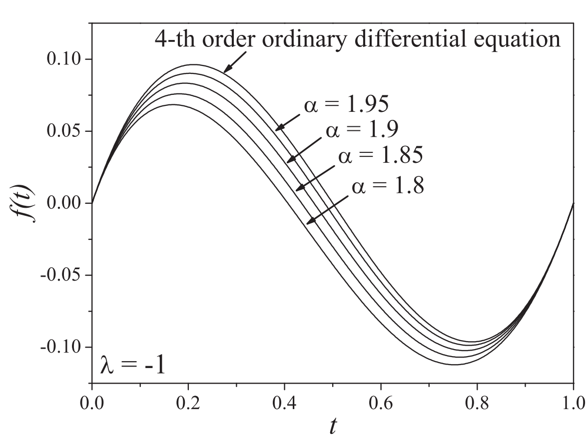

Fig. 1. Numerical solutions of Eq. (5) for boundary conditions (14) and $\lambda = -1$.

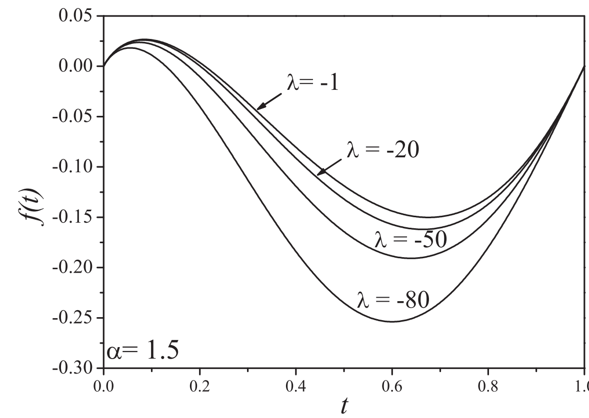

Fig. 2. Numerical solutions of Eq. (5) for boundary conditions (14) and $\alpha = 1.5$.

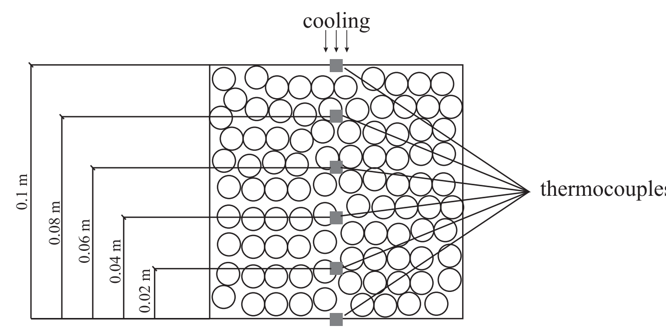

Fig. 3. Experimental setup.

Our results show that the solution of Eq. (5) approaches the solution of the 4th order ordinary differential equation when order $\alpha \to 2^-$.

Results for different values of the parameter $\lambda \in \{-1; -20; -50; -80\}$ and for order $\alpha = 1.5$ are show by Fig. 2. The calculations were performed for time-step $h = 1/500$.

5. Comparison with experimental data

In this section we present an application of the considered equation for modelling heat transfer in a granular layer.

In Eq. (5), we put $T(x)$ into $f(t)$, and we have

$$^{C}D^{\alpha}_{b-}\,{}^{C}D^{\alpha}_{0+}T(x) + \lambda T(x) = 0, \quad x \in [0, b], \tag{17}$$

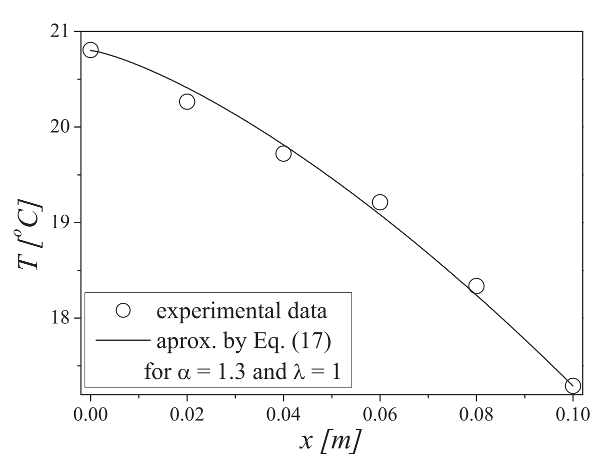

Fig. 4. Comparison the numerical solution of Eq. (17) with experimental data.

where $T(x)$ represents a function of temperature in the granular bed and $b = 0.1\ m$ is the height of the granular layer. Next we compare numerical solution of Eq. (17) with the experimental data.

Fig. 3 shows experimental setup. A granular material with initial temperature 21° C was cooled externally. Six thermocouples were located inside the granular material in order to register temperatures of the two-phase medium, air-particle medium.

Fig. 4 presents the comparison of experimental data with numerical solutions of Eq. (17). An analysis of the results presented in Fig. 4 shows that the temperature profile is nonlinear. Additionally, we observe a good agreement between the experimental temperature profile and numerical solution of Eq. (17).

6. Conclusions

In this paper a numerical solution of the fractional differential equation containing the composition of left and right Caputo derivatives of order $\alpha \in (1, 2)$ was presented. The numerical scheme was based on the fractional trapezoidal method. We showed that the convergence row of our numerical scheme was $O(h)$ and does not depend on parameter α.

Our studies show that this equation provides a good description of a steady state temperature profile in a granular medium.

Acknowledgements

Scientific work was supported by the National Centre for Research and Development, as Strategic Project PS/E/2/ 66420/10 “Advanced Technologies for Energy Generation: Oxy-combustion Technology for PC and FBC Boilers with CO2 Capture” and by the Czestochowa University of Technology Grant Number BS/MN 1-105-301/12/P. The supports are gratefully acknowledged.

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