Chapter 1

Introduction

CHAPTER OUTLINE

1.1 NONLINEAR SYSTEMS

Nonlinear systems play a vital role in the control systems from an engineering point of view. This is due to the fact that in practice all plants are nonlinear in nature. This is the main reason for considering the nonlinear systems in our work. In mathematics, a nonlinear system does not satisfy the superposition principle, or its output is not directly proportional to its input. The best example to explain nonlinearity is obviously a saturation. This condition exists because it is impossible to deliver an infinite amount of energy to any real-world system.

In general, the state and output equations for nonlinear systems may be written as follows:

$$\begin{aligned} \dot{x}(t) &= f[x(t), u(t)], \\ y(t) &= g[x(t), u(t)]. \end{aligned} \tag{1.1}$$

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The Lorenz chaotic system is an example of a nonlinear system described as follows:

$$\begin{aligned}\dot{x}_1(t) =& -10x_1(t) + 10x_2(t) + u(t),\\ \dot{x}_2(t) =& 28x_1(t) - x_2(t) + x_1(t)x_3(t),\\ \dot{x}_3(t) =& x_1(t)x_2(t) - \frac{8}{3}x_3(t).\end{aligned} \tag{1.2}$$

Notice that because of the terms $x_1(t)x_3(t)$ and $x_1(t)x_2(t)$, system (1.2) is nonlinear in nature. In the sequel, the nonlinear discrete-time systems considered in this book are introduced.

1.2 NONLINEAR DISCRETE-TIME SYSTEMS

Nowadays we can see that almost all controllers are implemented using computers. Such controllers are known as digital controllers. Basically, the use of digital controllers has rapidly increased since the first idea of using digital computers as one of the components in control systems emerged somewhere in 1950. The detailed history of this development can be found in [1]. The main reason for this development is due to the advances in hardware; hence it provides the control engineer with more powerful, reliable, faster, and above all cheaper computers that could be implemented as process controllers. The another significant factor that drives the increase in development of digital controllers is the advantage of working with digital signals rather than continuous-time signals [2]. The aforementioned factors generally motivate us to deliver the research in the framework of discrete-time systems rather than continuous-time systems.

Generally, a closed-loop system of computer controlled systems can be illustrated by Fig. 1.1, where the output of the process $y(t)$ is a continuous-time signal. The measurements of the output signal are fed into an analog-to-digital (A-D) converter, where the continuous-time signal is converted into a digital signal, a sequence of measurements at sampling times t_k. At this point, if a digital measurement device is used, the A-D converter is no longer needed. This is true because the measurements are now taken at sampling times only. The computer interprets the converted output signals $y(t_k)$ as a sequence of numbers, and this sequence is then used by the control algorithm to compute a sequence of digital control signals $u(t_k)$. Notice that the process input is in continuous-time, and hence a digital-to-analog (D-A) converter is used to transform the signals into a continuous-time signal. It is important to highlight here that between the sampling instants the system is in open-loop mode. The system is synchronized by a real time clock in the computer. Consequently, the inter-sample behavior is very often an issue

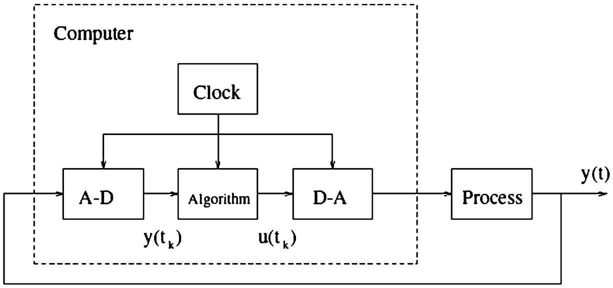


FIGURE 1.1 Schematic diagram of a computer-controlled system.

and should not be disregarded. However, in many applications it is sufficient to describe the dynamic behavior of the system at the sampling instants. At this stage, the interested signals are only at discrete time, and this system is classified as a discrete-time system [1,3]. We can now simply justify that if the dynamic of the process is in linear forms, then such a system is called a linear discrete-time system. Meanwhile, if the behavior of the process is nonlinear, then it is known as a nonlinear discrete-time system.

1.2.1 Discretization

Since systems in this world are naturally in continuous time, discretization shall be performed so that an approximated discrete-time system can be obtained. We further list the available methods in the discretization framework:

- **Euler's forward differentiation method and Euler's backward differentiation method**: The methods are based on approximations of the time derivatives of the differential equation. The forward method is commonly used in developing simple simulators, whereas the backward method is normally used in discretizing simple signal filters and industrial controllers. The forward differentiation method is somewhat less accurate than the backward one, but it is simpler to use. Particularly, with nonlinear models, the backward differentiation may give problems since it results in an impact equation for the output variable. In contrast, the forward differentiation method always gives an explicit equation to the solution.
- **Zero-Order Hold (ZOH) method:** Using this method, it is assumed that the system has a zero-order hold element on the input element of the system. This is the case where the physical system is controlled by a computer via a digital-to-analogue (D-A) converter. ZOH means that the physical input signal to the system is held fixed between the discrete points. Unfortunately, this method is relatively complicated to apply, and

in practice the computer tool MATLAB or LabVIEW can perform the job.

- **Tustin's method:** The discretization method is based on an integral approximation where the integral is interpreted as the area between the integrand and the time axis, and this area is approximated with trapezoids. It should be noted here that in Euler's method this area is approximated by a rectangle.
- **Tustin's method with frequency prewarping, or Bilinear transformation:** This method follows Tustin's method with a modification so that the original continuous-time system and the resulting discrete-time system have exactly the same frequency response at one or more specified frequencies.

It should be mentioned here that discretization methods are not the main focus of this book. The discretization is only applied in the simulation examples as to convert the continuous-time systems into discrete-time systems. To perform such a discretization, in this research work, the Euler method is used due to its simplicity.

1.2.2 Brief overview on the literature of nonlinear discrete-time systems

Due to the tremendous increase in digital control applications, a theory for discrete-time systems must be one of the important theories to be investigated especially for control design purposes. It is obvious that the desired performance may not be achieved if the controller design is based on the linearized model, and in many cases, it is not possible to control nonlinear systems from the linearized model. Besides, the linear control theory cannot be applied in the cases where a large dynamic range of process variables is possible, multiple operating points are required, the process is operating close to its limits, small actuators cause saturation, etc. [3]. The feedback linearization approach [4] also cannot be extended to handle a system with parametric uncertainties. This is the major drawback of the feedback linearization approach. With this knowledge, a significant amount of works can be found in the literature that attempts to provide a more general and less conservative result than the linearized approach. One of the popular approaches is obviously the backstepping control technique [5]. This approach is actually a combination of two popular theories, Lyapunov stability theory and the geometric method. By exploring the recursive design procedure the time-varying uncertainties and parameter uncertainties can be incorporated into the problem formulation. However, it is difficult to find a general class of Lyapunov functions that could ensure the stability of such systems. This is the main disadvantage of backstepping control techniques, and obviously this drawback is common to all approaches that use constructive procedures in developing Lyapunov candidates [6].

Besides the existence of feedback linearization and backstepping control techniques for stabilizing nonlinear systems, there is one more popular method available that is widely used in control system engineering and is called gain-scheduling [7]. The primary advantage of gain-scheduling for nonlinear control design is that it is usually possible to meet performance objectives over a wide range of operating conditions while still taking advantage of the wealth of tools and designers' experience from linear controller synthesis. From this gain-scheduling approach a more systematic control design technique is developed in the framework of linear parameter-varying (LPV) systems that guarantees stability and performance properties [8–10]. However it is important to highlight here that the stability and performance properties of the LPV systems only hold locally, and it is well known that the application of LPV control techniques always requires convertion of the nonlinear systems into their quasi-LPV forms. These are usually the main sources of conservatism of this gain-scheduling method. Another popular approach in this area is based on the Takagi–Sugeno fuzzy approach. It is well known that Takagi–Sugeno fuzzy models can be used to approximate nonlinear systems [11–25]. However, in the T–S fuzzy model, the premise variables are assumed to be bounded. In general, the premise variables are related to the state variables, which implies that the state variables have to be bounded. This is one of major drawbacks of the T–S fuzzy model approach.

Based on the above statements, it is clear that there is plenty of room available to conduct study on stabilizing nonlinear discrete-time systems, and a better methodology should be proposed to reduce the conservatism of the above-mentioned approaches. This motivates us to deliver the research in the framework of discrete-time systems, so that a less conservative approach can be proposed for stabilizing nonlinear discrete-time systems. Before ending this section, it is important to note that general nonlinear discrete-time systems are too complex, and hence in this research work we limit the scope by considering only the polynomial discrete-time systems. The reason of selecting the polynomial systems will be given in the following text.

1.3 POLYNOMIAL SYSTEMS

It is well known that a wide class of nonlinear systems can be exactly represented by polynomial systems, that is, Lorenz chaotic systems. Moreover, the polynomial system has an ability to approach any analytical of nonlinear systems. These advantages explain why the polynomial systems constitute an important class of nonlinear systems and have attracted considerable attention from control researchers to involve themselves in this area, especially on the stability analysis and controller synthesis of polynomial systems [26].

The polynomial systems are systems where the dynamic of the system is given in terms of polynomial functions or polynomial matrices. The general polynomial systems can be described as follows:

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t)), \\ y(t) &= g(x(t)), \end{aligned} \tag{1.3}$$

where $f(x(t), u(t))$ and $g(x(t))$ are in polynomial forms, and $x(t)$, $u(t)$, and $y(t)$ are respectively the states, input, and the measured output.

Meanwhile, in discrete-time, (1.3) can be written as follows:

$$\begin{aligned} x(k+1) &= f(x(k), u(k)), \\ y(k) &= g(x(k)), \end{aligned} \tag{1.4}$$

where $f(x(k), u(k))$ and $g(x(k))$ are in polynomial forms, and $x(k)$, $u(k)$, and $y(k)$ are respectively the states, input, and the measured output of the system at sampling time k. More precisely, the class of polynomial systems under consideration in this research work is described in terms of a state-dependent linear form as follows:

$$\begin{aligned} \dot{x}(t) =& A(x(t))x(t) + B(x(t))u(t), \\ y(t) =& C(x(t))x(t), \end{aligned} \tag{1.5}$$

where $x(t) \in R^n$ is the state vector, $u(t) \in R^m$ is the input, and $y(t)$ is the measured output. $A(x(t))$, $B(x(t))$, and $C(x(t))$ are polynomial matrices of appropriate dimensions. Notice that (1.5) looks similar to the general nonlinear system, except the matrices $A(x(t))$, $B(x(t))$, and $C(x(t))$ must be of polynomial forms.

In our work, the polynomial discrete-time system is described as follows:

$$\begin{aligned} x(k+1) =& A(x(k))x(k) + B(x(k))u(k), \\ y(k) =& C(x(k))x(k), \end{aligned} \tag{1.6}$$

where $x(k) \in R^n$ is the state vector, $u(k) \in R^m$ is the input, $y(k)$ is the measured output, and $A(x(k))$, $B(x(k))$, and $C(x(k))$ are polynomial matrices of appropriate dimensions.

We provide two examples of polynomial systems.

Example 1. A Lorenz Chaotic System

$$\begin{aligned}\dot{x}_1(t) =& -10x_1(t) + 10x_2(t) + u(t),\\ \dot{x}_2(t) =& 28x_1(t) - x_2(t) + x_1(t)x_3(t),\\ \dot{x}_3(t) =& x_1(t)x_2(t) - \frac{8}{3}x_3(t).\end{aligned} \tag{1.7}$$

Example 2. A Tunnel Diode Circuit

$$\begin{aligned}C\dot{x}_1(t) =& -0.002x_1(t) - 0.01x_1^3(t) + x_2(t),\\ L\dot{x}_2(t) =& -x_1 - Rx_2(t) + u(t),\end{aligned} \tag{1.8}$$

where C is a capacitor value, L is inductance, and R is a resistor.

We can see here that systems (1.7) and (1.8) are in fact nonlinear systems. These two examples illustrate the validity of the statement we claimed earlier that many nonlinear systems can be represented by polynomial forms. It is also important to stress here that in this research we do not focus on the method of discretizing the nonlinear systems to yield their discrete-time version, but our main focus is to perform the controller synthesis for polynomial discrete-time systems.

1.3.1 Recent work on polynomial systems

Our focus in this research work is on the controller synthesis of polynomial discrete-time systems. However, in this section, the results of polynomial continuous-time systems are also discussed. This is due to the fact that some results of the discrete-time systems are based or extended from the continuous-time systems. In this regard, we present the following recent development on the controller synthesis for polynomial systems. It is worth mentioning that we limit the results to the sum-of-squares (SOS) decomposition method and linear matrix inequalities (LMIs) only. This is because the SOS decomposition method will be considered in this research work, and it is in fact complementary to the LMI approach. A detailed description regarding the SOS decomposition method will be provided later in this chapter.

The controller synthesis or stabilization problem is an important area in the research of polynomial systems. Therefore, considerable attention has been devoted to this framework; for instance, see [28–33,36–39]. In the present work, numerous techniques have been proposed to address the controller design problem for polynomial systems. We now briefly describe the proposed techniques.

1. **Dissipation inequalities and SOS**. Dissipativity theory is known to be one of the most successful methods of analyzing and synthesizing the nonlinear control systems [27]. Mathematically speaking, this method is known as dissipation inequalities and has major advantages on the analysis and design of nonlinear systems. This might be due to the fact that the investigation of a possibly large number of differential equations given by the control system description is reduced to a small number of algebraic inequalities. Hence, the complexity of analysis and design is usually essentially reduced. In [28] the dissipation inequalities, together with the SOS programming, have been utilized to stabilize such polynomial systems. In particular, the authors represent their systems to be of descriptor systems or differential-algebraic systems where the functions are described by polynomial functions. They have managed to obtain affine dissipation inequalities by the proposed method, and hence the inequalities can be solved computationally via SOS programming. However, the process of achieving affine dissipation inequalities varies for different types of problem. This means that the proposed method might not work for other problems.
2. **Kronecker products and LMIs**. The stabilization of polynomial systems using Kronecker products method can be found in [29–31], where the polynomial systems have been simplified using the Kronecker product and power of vectors and matrices. Moreover, a new stability criterion for polynomial systems has been developed. A sufficient condition for the existence of the proposed controller is given in terms of LMIs. The proposed controller can be applied to high-order polynomial systems. This is the main advantage of this method. The strength of this approach comes from solid theoretical results on the Kronecker products and the power of vectors and matrices.
3. **Semitensor products**. The semitensor product of matrices is a generalization of the conventional matrix product in the case where the column number of the first factor matrix is not the same as the row number of the second factor matrix. A brief survey for the related material can be found in [32]. The advantage of this method is that general polynomial systems can be considered without any homogeneous assumption. In [32], a method to stabilize the polynomial systems has been developed. In the present paper, we first propose a sufficient condition for a polynomial to be positive definite. Then, the formula for the time derivation of a candidate polynomial Lyapunov function with respect to a polynomial systems is provided. Through the sufficient condition, the candidate of the Lyapunov function can be checked for positive definiteness and for negative definiteness of its derivative. A sufficient condition is given by a system of linear algebraic inequalities. However, using this method,

it is difficult to choose a suitable candidate for the Lyapunov function because there is no unique way to choose that Lyapunov function. An incorrect selection of the Lyapunov candidate leads linear algebraic inequalities that have no solution. This is the main challenge of applying this method in the framework of controller synthesis for polynomial systems.

4. **Theory of moments and SOS**. Interesting work on the polynomial stabilization that utilizes the theory of moments can be found in [33]. It has been known for a long time that the theory of moments is strongly related to and is in fact in duality with the theory of nonnegative polynomials and Hilbert's 17th problem on the representation of nonnegative polynomials [34]. In the light of this duality relationship, the authors of [33] study the problem of polynomial system stabilization. They managed to show that the global solution to the problem can be obtained in a less conservative way than the available approaches, and the solution can be solved easily by SDP [35]. However, to achieve a convex solution to the controller synthesis problem, the Hermite stability criterion is used rather than the Lyapunov stability theorem. In doing so, the controller matrix can be decoupled from the Lyapunov matrix, and the solvability conditions of the proposed controller are developed through a hierarchy of convex LMI relaxations. As stated by the authors, this methodology suffers from a large number of constraints in a PMI, which consequently leads to the need for reliable numerical software to handle the problem.
5. **Fuzzy method and SOS**. The T–S fuzzy model is well known to be good at approximating such nonlinear systems. Using this approach, [36] presented an SOS approach for modeling and control of nonlinear dynamical systems using polynomial fuzzy systems. A polynomial Lyapunov function has been proposed in this work rather than a quadratic Lyapunov function. Hence the result is more general and less conservative than available LMI-based approaches of T–S fuzzy modeling and control. Furthermore, a sufficient condition of the existence of a controller is given by polynomial matrix inequalities and formulated as SOS constraints. On the other hand, in [37] an improved sum-of-squares (SOS)-based stability analysis result is proposed for the polynomial fuzzy-model-based control system, formed by a polynomial fuzzy model and a polynomial fuzzy controller connected in a closed loop. Two cases, perfect and imperfect premise matching, are considered. Under the perfect premise matching, the polynomial fuzzy model and polynomial fuzzy controller share the same premise membership functions. When different sets of membership functions are employed, it falls into the case of imperfect premise matching. Based on the Lyapunov stability theory, improved SOS-based stability conditions are derived to

determine the system stability and facilitate the controller synthesis approach. The application of rge polynomial T–S fuzzy approach to the two-link robot arm can be found in [38]. Meanwhile, for static output control, a result can be found in [39]. However, in the T–S fuzzy model, the premise variables are assumed to be bounded. In general, the premise variables are related to the state variables, which implies that the state variables have to be bounded. This is one of the major drawbacks of the T–S fuzzy model approach.

6. **Lyapunov method and SOS**. This is a common method widely applied in the literature for stabilizing polynomial systems. This method is used in this research work, and therefore the complete literature of this framework is provided further.

1.3.1.1 On the literature on controller synthesis for polynomial systems: the Lyapunov method and SOS decomposition approach

It is well known that Lyapunov's stability theory [40] is one of the most fundamental pillars in control theory. Although this method was introduced more than hundred years ago, it remains popular among control researchers. This success is owed to its simplicity, generality, and usefulness. The Lyapunov stability is a method that was developed for analysis purposes. However, it has become of equal importance for control designs over the last decades [6–48]. The Lyapunov stability theory can be generalized as follows. Let us consider the problem of solving the stability for an equilibrium of a dynamical system $\dot{x} = f(x)$ using the Lyapunov function method. It is clear that to find a stability using the Lyapunov method, we need to find a positive definite Lyapunov function $V(x)$ defined in some region of the state space containing the equilibrium point whose derivative $\dot{V} = \frac{dv}{dx} f(x)$ is negative semidefinite along the system trajectories. Taking the linear case, for instance, $\dot{x} = Ax$, these conditions amount to finding a positive definite matrix P such that $A^T P + PA$ is negative definite [49]; then the associated Lyapunov function is given by $V(x) = x^T Px$. Meanwhile, for discrete-time systems $x(k+1) = f(x(k))$, we need to search for a positive definite Lyapunov function $V(x)$ defined in some region of the state space containing the equilibrium point whose difference of the Lyapunov function, $\Delta V = x^T(k+1)Px(k+1) - x^T(k)Px(k)$, is negative semidefinite along the system trajectories. The associated Lyapunov function is given by $V(x) = x^T Px$.

Since the SOS decomposition technique introduced about 10 years ago [50], the system analysis for polynomial systems can be performed more efficiently because it helps to answer many difficult questions on system analysis that were hard to answer before. The popularity of this method grew quickly among the community of control researchers because the algorithmic analysis of nonlinear systems can be delivered using the most popular Lyapunov method (as discussed earlier). Generally, the most interesting and important point that was never seen until recently is that the amount of proving the certificates of the Lyapunov function $V(x)$ and $-\dot{V}(x)$ can be reduced to the SOS [50]. Notice that, for small systems, the construction of the Lyapunov function can be done manually. The difficulty of this construction is solely dependent upon the analytical skills of the researcher. However, when the vector field of the system $f(x)$ and the Lyapunov function candidate $V(x)$ are in polynomial forms, then the Lyapunov conditions are essentially polynomial nonnegativity conditions, which can be NP-hard to test [65]. This is probably due to the lack of algorithmic constructions of Lyapunov functions. However, if these nonnegativity conditions are replaced by the SOS conditions, then not only testing the Lyapunov function conditions, but also constructing the Lyapunov function can be done effectively using SDP [50]. This is the main advantage of using the SOS decomposition approach because the solution is indeed tractable. We will further describe the details of the SOS decomposition method.

The recent results in the framework of state feedback control synthesis for polynomial systems which utilize SOS decomposition method can be referred in [51–54]. In particular, [51,52] propose the polynomial systems to be represented as a state-dependent linear form, and a state-dependent Lyapunov function is proposed to be in terms of polynomial vector fields. The introduction of a state-dependent Lyapunov function or parameter-dependent Lyapunov function arises due to the fact that a quadratic Lyapunov function is always inadequate to stabilize the polynomial systems. Furthermore, sufficient conditions to the problem are formulated as state-dependent LMIs and solved using the SOS-SDP-based programming method. It is well known that optimizing the control problem for polynomial systems is hard because the solution is always not jointly convex. In the present paper, such a nonconvexity is avoided by assuming the Lyapunov matrix $P(x)$ to be dependent upon the states $x(t)$ whose dynamics are not directly affected by the control input, i.e., the states whose corresponding rows in the input matrix $B(x)$ are zero. This, however, leads the result to be conservative. More recent and less conservative results can be found in [53]. In this paper, the effect of the nonlinear terms that exist in the problem formulation is described as an index, so that the control problem can be transformed into a tractable solution and can be possibly solved via SDP [35]. The optimization approach is proposed to find a zero optimum of this index and solved using SOS programming effectively. However, to render a convex solution, the authors follow the same assumption as made in [52]. An improved version of the aforementioned approach can be found in [54], where an additional matrix variable is introduced to decouple the Lyapunov matrix from the system matrices. Therefore, the controller design can be performed in a more relaxed way, and the proposed methodology can be extended to the robust control problem of polynomial systems. However, to obtain a convex solution, the nonconvex term is bounded from above. Therefore, the stability can only be guaranteed within the bound region.

Sometimes it is difficult to synthesize a controller that works globally. Besides, in a restricted region, local controllers often provide a better solution than global controllers. Some developments in this field can be found in [55, 56]. In [55], a rational Lyapunov function of states was used to synthesize the polynomial systems. The variation of states is bounded, and the domain of attraction was embedded in the specified region by the nonlinear vector. With this, the state feedback controller is established and formulated as a set of polynomial matrix inequalities and solved using any SOS programming. The coupling between system matrices and the Lyapunov matrix causes the results to be quite conservative in general. Hence, [56] relaxes this issue by introducing a slack variable matrix. In doing so, the Lyapunov matrix is decoupled from the system matrices. Now, the parameterization of the resulting controllers is independent of the Lyapunov matrix variables. This allows them to extend their result to construct robust controller for uncertain polynomial systems using state-dependent Lyapunov functions.

With the knowledge that the full-state variables are not always accessible in practical nonlinear systems and the dynamic output feedbacks result in high-order controllers, which may not be practical in industry, the static output feedback design attracts much attention among practitioners. Some developments of this area that utilized the SOS decomposition approach can be found in [57–60]. The systems discussed in [57] are represented in a state-dependent linear form. More precisely, the authors assumed that the control input matrix has some zero rows and the Lyapunov function only depends on states whose corresponding rows in control matrix are zeros, that is, the state dynamics are not directly affected by the control input. This assumption leads to the conservatism of the controller design. The latest results of this area can be found in [58–60], where an iterative algorithm based on SOS has been proposed to convert the nonconvex problem into a convex problem of polynomial system synthesis, so that it can be efficiently solved using SDP. The authors in [58–60] have managed to show that their approach is less conservative than the available approaches and provide more general results in this field. But the main disadvantage of this approach is the selection of the initial polynomial function $\epsilon(x)$, which is hard to choose because it is unknown.

The above-mentioned results are dedicated to solving the polynomial continuous-time systems. In regard to the polynomial discrete-time systems, there are only few results available, which utilize the SOS decomposition method in their approach. The first result is proposed in [61], where the authors employ a state-dependent polynomial Lyapunov function as their Lyapunov candidate. Then, some transformations are required to represent the system with introduction of new matrices (in polynomial). Furthermore, YALMIP and PENOPT for PENBMI [62,63] have been utilized to solve the problem. However, the main drawback of this approach is that the selected new matrices are not unique and hence difficult to choose. The most recent result was addressed by [64], where the nonconvex term is bounded from above; the optimization is carried out to find a zero optimum for the nonlinear term. Here, the problem was formulated as SOS and could be solved by using any SOS solver. By bounding the nonconvex terms the controller that resulted from this method can only guarantee the closed-loop stability within bounds. This is similar to the method proposed in [53,54]. Therefore, they share the same weaknesses as encountered in [53,54].

1.3.2 Sum-of-squares (SOS) decomposition

In this section, we give a brief overview of the SOS decomposition method. A more detailed description of the SOS decomposition method can be referred in [50].

Generally, proving the nonnegativity of multivariable functions is considered as one of the important aspects in control system engineering. This problem is similar to the problem of proving the nonnegativity of a Lyapunov function. If the nonlinear system is concerned, then it is hard to prove the nonnegativity of such systems. Basically, the problem is to prove that

$$F(x_1, \ldots, x_n) \geq 0, \quad x_1, \ldots, x_n \in \Re. \tag{1.9}$$

A great amount of research has been devoted to proving (1.9). However, up to now, there is no unique solution to the problem in (1.9).

Thus, some limit should be applied to the possible functions $F(\cdot)$, at the same time making the problem general enough to guarantee the applicability of the results. It has been shown in [50] that considering the case of polynomial functions is a good compromise for this issue.

Definition 1. [50] A form is a polynomial if all the monomials have the same degree $d := \sum_i \alpha_i$. In this case, the polynomial is homogeneous of degree d since it satisfies $f(\lambda x_1, \ldots, \lambda x_n) = \lambda^d f(x_1, \ldots, x_n)$.

It should be highlighted here that the general problem of testing global positivity of a polynomial function is NP-hard problem (when the degree is at least four) [65]. Therefore, a problem with a large number of variables will have unacceptable behavior for any method that guarantees obtaining the right answer in every possible instance. This is actually the main drawback of theoretically powerful methodologies such as the quantifier elimination approach [66,67].

The question now is: are there any conditions to guarantee the global positivity of a tested polynomial time function? This question underlines the existence of the SOS decomposition approach [50] as one condition to guarantee the global positivity of polynomial functions.

It is obvious that a necessary condition for a polynomial $F(x)$ to satisfy (1.9) is that the degree of the polynomial in the homogeneous case must be even. Hence, a simple sufficient condition for a real-valued function $F(x)$ to be positive everywhere is given by the existence of an SOS decomposition

$$F(x) = \sum_i {f_i}^2(x). \tag{1.10}$$

It can been seen that if $F(x)$ can be written as (1.10), then the nonnegativity of $F(x)$ can be guaranteed. It is stated in [50] that for the problem to make sense, some restriction on the class of functions f_i has to be again imposed. Otherwise, we need to always define f_1 to be the square root of F, but this results in a condition both useless and trivial.

It has been shown in [50] that $F(x)$ is an SOS polynomial if and only if there exists a positive definite matrix Q such that

$$F(x) = z^T Qz, \tag{1.11}$$

where $z(x)$ is the vector of all monomials of degree less than or equal to the half degree of $F(x)$. This is the idea given in [71], and it can be shown to be conservative in general. The main reason is that since the variables z_i are not independent, representation (1.11) may not be unique, and Q may be positive definite or positive semidefinite for some representations but not for others. Similar issues appear in the analysis of quasi-LPV systems; we refer to [72]. However, using identically satisfied constraints that relate the z_i variables among themselves, it is easily shown that there is a linear subspace of matrices Q that satisfy (1.11). If the intersection of this subspace with the positive semidefinite matrix cone is nonempty for the original function, then F is guaranteed to be SOS and therefore positive semidefinite. So, if F can indeed be written as the SOS of polynomials, then expanding in monomials will provide representation (1.11). The following example explains this concept.

Example 1. [50] Consider the quartic form in two variables

$$\begin{aligned} F(x_1,x_2) &= 2{x_1}^4 + 2{x_1}^3x_2 - {x_1}^2{x_2}^2 + 5{x_2}^4 \\ &= \begin{bmatrix} {x_1}^2 \\ {x_2}^2 \\ x_1x_2 \end{bmatrix}^T \begin{bmatrix} 2 & 0 & 1 \\ 0 & 5 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} {x_1}^2 \\ {x_2}^2 \\ x_1x_2 \end{bmatrix} \\ &= \begin{bmatrix} {x_1}^2 \\ {x_2}^2 \\ x_1x_2 \end{bmatrix}^T \begin{bmatrix} 2 & -\lambda & 1 \\ -\lambda & 5 & 0 \\ 1 & 0 & -1+2\lambda \end{bmatrix} \begin{bmatrix} {x_1}^2 \\ {x_2}^2 \\ x_1x_2 \end{bmatrix} \end{aligned} \tag{1.12}$$

and define $z_1; = {x_1}^2.\ z_2 := {x_2}^2,\ z_3 := x_1x_2$. Take, for instance, $\lambda = 3$. In this case,

$$Q = L^T L,\ L = \frac{1}{\sqrt{2}} \begin{bmatrix} 2 & -3 & 1 \\ 0 & 1 & 3 \end{bmatrix}. \tag{1.13}$$

Therefore we have the sum-of-squares decomposition

$$F(x_1,x_2) = \frac{1}{2}((2{x_1}^2 - 3{x_2}^2 + x_1x_2)^2 + ({x_2}^2 + 3x_1x_2)^2) \tag{1.14}$$

Parrilo [50] also observed that the existence of (1.11) can be cast as a semidefinite programming [35]. This is the most important property that distinguishes its from other approaches. This feature is proved to be critical in the application to many control-related problems. How does it works? Basically, by expanding the $z^T Qz$ and equating the coefficient of the resulting monomials to the ones in $F(x)$ we obtain a set of affine relations in the elements of Q. We know that, for $F(x)$ being an SOS is equivalent to $Q \geq 0$, then the problem of finding Q which proves that $F(x)$ is an SOS can certainly be cast as a semidefinite program.

Thus, although checking the nonnegativity of $F(x)$ is NP-hard when the degree in $F(x)$ is 4 as stated before, checking whether $F(x)$ can be written as SOS is definitely tractable; it can be formulated as a semidefinite program, which has worst-case polynomial time complexity as mentioned in the previous paragraph. Authors in [50] produced significance results in suggesting that the relaxation is not too conservative in general. It must be noted here that as the degree of $F(x)$ increases or the number of its variables increases, the computational complexity for testing whether $F(x)$ is an SOS significantly increases. Nonetheless, the complexity overload is still a polynomial function of these parameters.

In general, the conversion from SOS decomposition to the semidefinite programming can be manually done for small size instance or tailored for specific problem classes. However, such a conversion is cumbersome in general. Thus the software is absolutely necessary to aid in converting them. Specifically, the relaxation uses Gram matrix methods to efficiently transform the NP-hard problem into LMIs [49]. These can in turn be solved in polynomial time with semidefinite programming (SDP) [49,35]. To date, there exist several freely available toolboxes to formulate these problems in Matlab, for example, SOSTOOLS [73], YALMIP [74], CVX [75], and GLoptiPoly [76]. Whereas SOSTOOLS is specifically designed to address polynomial nonnegativity problems, the latter toolboxes have further functionality, such as modules to solve the dual of the SOS problem, the moment problem.

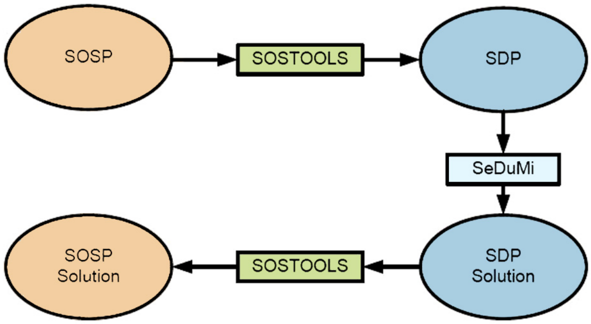
In this work, we use SOSTOOLS to perform this conversion for our problem formulation. Hence we will describe the working principles of this software in the following section.

Basically, the polynomial case is a well-analyzed problem, first studied by David Hilbert more than century ago [68]. He raised a very popular and important question in his famous list of twenty-three unsolved problems, which was presented at the International Congress of Mathematicians in Paris, 1900, dealing with the representation of a definite form as an SOS. Hilbert also noted that not every positive semidefinite polynomial (or form) is SOS. However, [69] has proved that the numerical examples seem to indicate that the gap between the SOS and nonnegativity polynomial is small. A complete characterization has been outlined by Hilbert in explaining when these two classes are equivalent. There are three cases in which the equality holds: 1. The case of two variables ($n = 2$); 2. The familiar case of quadratic form (i.e., $m = 2$); 3. A surprising case where $P_{3,4} = \sum_{3,4}$; refer [70] for detailed explanations.

Before ending this section, the following lemma is presented, which is useful for our main results later.

Lemma 1. *[52] Let $F(x)$ be an $N \times N$ symmetric polynomial matrix of degree $2d$ in $x \in R^n$. Furthermore, let $Z(x)$ be a column vector whose entries are all monomials in x with a degree no greater than d, and consider the following conditions:*

1. *$F(x) \geq 0$ for all $x \in R^n$;*
2. *$v^T F(x)v$ is an SOS, where $v \in R^N$;*
3. *There exists a positive semidefinite matrix Q such that $v^T F(x)v = (v \otimes Z(x))^T\, Q\, (v \otimes Z(x))$, with $\otimes$ denoting the Kronecker product.*



■ **FIGURE 1.2** Diagram depicting how SOS programs (SOSPs) are solved using SOSTOOLS.

It is clear that $F(x)$ being an SOS implies $F(x) \geq 0$, but the converse is generally not true. Furthermore, statement (2) and statement (3) are equivalent.

1.3.2.1 SOSTOOLS

SOSTOOLS is a free, third-party MATLAB toolbox specially designed to handle and solve the SOS programs. The techniques behind it are based on the SOS decomposition for multivariate polynomials, which can be efficiently computed using semidefinite programming frameworks. The availability of SOSTOOLS gives a great advantage to the researchers that involved in SOS polynomial frameworks. Moreover, the SOSTOOLS gives a new direction for solving many hard problems such as global, constrained, and Boolean optimization due to the fact that these technique provide a convex relaxation approach.

The working principles of SOSTOOLS are shown in Fig. 1.2. Basically, the SOSTOOLS automatically convert an SOS program (SOSP) into semidefinite programs (SDPs). Then, it calls the SDP solver and converts the SDP solution back to the solution of the SOS program. In this way the details of the reformulation are abstracted from the user, who can work at the polynomial object level. The user interface of SOSTOOLS has been designed to be simple, easy to use, and transparent while keeping a large degree of flexibility. The current version of SOSTOOLS uses either SeDuMi or SDPT3, both of which are free MATLAB add-ons, as the SDP solver. A detailed description of how SOSTOOLS works can be found in the SOSTOOLS user's guide [77].

Table 1.1 Example of Polynomial Systems.

System	Description
$\dot{x} = x^2$	Finite escape behavior
Lorenz system	Chaos
Brockett integrator	Discontinuous time
Van der Pol system	Limit cycles
Artstein circle	Nonsmooth control
MY conjecture	Global stability

1.4 RESEARCH MOTIVATION

This section provides the reasons that prompted us to conduct research in the framework of controller synthesis for polynomial discrete-time systems. The key motivations for this book come from several sources. The most general motivation comes from the fact that polynomial systems appear in a wide range of applications. This is due to the fact that many nonlinear systems can be modeled as, transformed into, or approximated by polynomial systems. The polynomial systems do not only exist in process control and systems biology, but also appear in many other fields of application, for instance, in mechatronic systems and laser physics; see [78,79]. A few well-known polynomial systems are captured in Table 1.1 (borrowed from [28]). From this table we can see that polynomial systems can show a very rich variety of dynamic behavior. On the other hand, the table also depicts that polynomial systems maybe in general very difficult to study. Therefore, this class of systems is considered in this paper.

The second motivation arises from the fact that state-dependent or parameter-dependent Lyapunov functions ares widely used in the framework of stability analysis and controller design for nonlinear systems. It has been shown recently that state-dependent Lyapunov functions provide a great advantage when dealing with the controller synthesis for polynomial systems [51–54]. This leads to our belief that the utilization of the state-dependent Lyapunov function method should also be effective in designing the controller for polynomial discrete-time systems.

However, with utilization of state-dependent Lyapunov functions, the controller design for polynomial discrete-time systems becomes very difficult. This is due to the fact that the relation between the Lyapunov function and the controller matrix is no longer jointly convex. This problem will be highlighted in detail in Chapter 2. In a continuous-time system, the aforementioned problem can be avoided by assuming that the Lyapunov matrix is only dependent upon the control input whose corresponding rows are zero [53]. Unfortunately, for discrete-time systems, although the same assumption is made, the problem still exists. A possible way to resolve this problem is given in [64], but the results suffer from some conservatism, and such a conservatism has been discussed earlier. In this book, we attempt to relax the problem by incorporating an integrator into the controller structures. In particular, we call this method the integrator method.

Due to the problem discussed before, only a few results are available in the area of controller synthesis in the context of polynomial discrete-time systems [61,64]. As our discussion in the literature shows, the results from both papers suffer from their own conservatism. This consequently motivates us to carry out work on polynomial discrete-time system stabilization. Hence a more general and less conservative result can be provided than the available approaches.

Furthermore, it is also necessary for us to consider the robust controller design for polynomial discrete-time systems because to date, to the authors' knowledge, no results have been presented in this framework that consider the SOS programming technique. For this context, the polytopic uncertainty and norm-bounded uncertainty will be considered because both of them commonly appear in the real world. Besides, the norm-bounded uncertainty is not fully studied in the area of polynomial systems. This is another motivation that leads us to consider the norm-bounded uncertainty in our research work.

The final and somewhat peripheral motivation is that many control design problems are normally formulated in terms of inequalities rather than simple equalities. Moreover, a lot of problems in control engineering can be formulated as polynomial matrix inequalities (PMIs) feasibility problems. Using the SOS decomposition method, such PMIs can be further formulated as SOS constraints [50]. The SOS inequalities framework provides a tractable method to solve the problem through an analytical solution. Furthermore, the advantage of formulating the problem in terms of the SOS inequalities is the availability of toolboxes compatible with MATLAB that are capable for solving the feasibility and optimization problems by interior-point methods. All of these toolboxes are actually SOS-SDP based, and the general concept regarding them has already been explained earlier.

1.5 CONTRIBUTION OF THE BOOK

The focus of this book is to establish novel methodologies for robust stabilization, control with disturbance attenuation, and filter design for a class of polynomial discrete-time systems. The polytopic uncertainties and norm-bounded uncertainties are considered, and the proposed controller is able to handle the appearance of such uncertainties.

The main contribution arises from the incorporation of an integrator into the controller structures. In doing so, a convex solution to the polynomial discrete-time system stabilization with the utilization of a state-dependent Lyapunov function can be obtained in a less conservative way than the available approaches. In light of this integrator method, the problems of robust control and robust H_∞ control for polynomial discrete-time systems are tackled. The integrator method is also applied to the filter design problem.

In this book, we first highlight the problem of the controller design for polynomial discrete-time systems when the state-dependent Lyapunov function is under consideration. Motivated by this problem, we propose a novel method in which an integrator is incorporated into the controller structures. Then, we show that the original systems with the proposed controller can be described in augmented forms. In addition, by choosing the Lyapunov matrix to be only dependent upon the original system's states, a convex solution to the robust control problem and robust H_∞ control problem for polynomial discrete-time systems can be rendered in a less conservative way than available approaches.

In light of the integrator method, we propose a novel methodology for designing a robust nonlinear controller in which the polytopic and norm-bounded uncertainties are under consideration. It should be noted here that, to date, no result is available in this framework that utilizes SOS programming for polynomial discrete-time systems. Furthermore, the interconnection between the nonlinear H_∞ control problem and the robust nonlinear H_∞ control problem is provided through a so-called *scaled* system. This allows us to efficiently solve the robust H_∞ control problem with the existence of norm-bounded uncertainties. Next, we show that by exploiting the integrator method a filter design methodology can also be established for polynomial discrete-time systems. Furthermore, by applying the integrator method the output feedback controller is developed for polynomial discrete-time systems with and without H_∞ performance and also with and without uncertainties.

Motivated by most of the existing control design methods, discrete-time fuzzy polynomial systems cannot guarantee their Lyapunov function to be a radially unbounded polynomial function, and hence the global stability cannot be ensured. This book also provides controller design methods for discrete-time fuzzy polynomial systems that guarantee a radially unbounded polynomial Lyapunov function that ensures the global stability.

Finally, to demonstrate the effectiveness and advantages of the proposed design methodologies of this book, some numerical examples are given. The simulation results also show that the proposed design methodologies can achieve the stability requirement or/and a prescribed performance index.

1.6 BOOK OUTLINE

Chapter 2 describes a nonlinear feedback controller design for polynomial discrete-time systems. In this chapter, the problems of designing a controller for polynomial discrete-time systems are first highlighted. Then, a novel integrator method for solving the problem is proposed. Furthermore, we show that the results can be directly extended to the robust control problem with either polytopic uncertainties or norm-bounded uncertainties. The existence of the proposed controller is given in terms of the solvability of polynomial matrix inequalities (PMIs), which are formulated as SOS constraints and can be solved by the recently developed SOS solvers. The effectiveness of the proposed method is confirmed through simulation examples.

Chapter 3 utilizes the integrator approach proposed in the previous chapter. We provide a less conservative design procedure in the framework of H_∞ control of polynomial discrete-time systems. This result is subsequently extended to the robust H_∞ control problem with the existence of the polytopic uncertainties and norm-bounded uncertainties. The attention here is to design a nonlinear feedback controller such that both stability and a prescribed disturbance attenuation for the closed-loop polynomial discrete-time system are achieved. Based on the SOS-based method, sufficient conditions for the existence of a nonlinear H_∞ controller are given in terms of the solvability of polynomial matrix inequalities (PMIs). Numerical examples are used to demonstrate the effectiveness of the proposed approach.

Chapter 4 deals with the problem of robust nonlinear filtering design for polynomial discrete-time systems. By utilizing the integrator method sufficient conditions for the existence of a robust nonlinear filter are provided in terms of SOS constraints. Numerical examples are given along with the theoretical presentation.

Chapter 5 aims at designing an H_∞ filter for polynomial discrete-time systems with and without uncertainties. The uncertainty under consideration in this chapter is described by integral functional constraints. The objective of an H_∞ nonlinear filtering problem is to design a dynamic nonlinear filter such that the gain from an exogenous input to an estimation error is minimized or guaranteed to be less or equal to a prescribed value for all admissible uncertainties. The effectiveness of the proposed method is confirmed through simulation examples.

Chapter 6 investigates the problem of nonlinear H_∞ static output feedback controller design for polynomial discrete-time systems. In this chapter, we address the problem of H_∞ control in which both stability and a prescribed H_∞ performance are required to be fulfilled. An integrator method is proposed to convert the nonconvex control problem into a convex control problem. Hence, the problem can be solved efficiently by SDP. Sufficient conditions for the existence of the controller are given in terms of the solvability conditions of SOS constraints. It is important to note that the resulting controller gains are in the rational matrix functions of the system output matrices and the additional augmented state. The results are then directly extended to the robust H_∞ output feedback control with polytopic uncertainty and norm-bounded uncertainty.

Most of the existing control design methods of discrete-time fuzzy polynomial systems cannot guarantee their Lyapunov function to be a radially unbounded polynomial function, and hence the global stability cannot be assured. Motivated by this drawback, **Chapter** 7 examines the problem of designing a controller for a class of discrete-time nonlinear systems that are represented by discrete-time polynomial fuzzy models. The design methods provided in this chapter guarantee radially unbounded polynomial Lyapunov functions, which ensures global stability. Furthermore, in all the existing methods in the literature, the Lyapunov function is assumed to be a function of the states whose corresponding rows in the control matrix are zero to avoid the nonconvexity problem. This assumption is relaxed in this chapter, where the polynomial Lyapunov function is state dependent without restriction of having zero rows in the control matrix. In addition, a convex solution is obtained by incorporating an integrator into the controller. Sufficient conditions of stability are derived in terms of polynomial matrix inequalities which are solved via SOSTOOLS solvers.

Chapter 8 examines the control problem of uncertain discrete-time polynomial fuzzy systems with H_∞ performance objective using an SOS approach. Uncertainties under consideration are described by norm-bounded constraints. The objective of a nonlinear H_∞ controller is to stabilize the closed-loop system globally and to ensure that the gain from an exogenous input to the regulated output is minimized or guaranteed to be less or equal to a prescribed value for all admissible uncertainties. The design conditions are provided in terms of SOS constraints that can be numerically solved via the SOSTOOLS. The proposed design method guarantees a radially unbounded polynomial Lyapunov function, which ensures global stability.

Concluding remarks are given, and suggestions for future research work are discussed in **Chapter 9**. Finally, some mathematical background knowledge used throughout this research is provided in the **Appendix A**.

REFERENCES

[1] J.K. Åström, B. Wittenmark, Computer Controlled Systems Theory and Design, Prentice Hall Inc., 1984.

[2] K. Ogata, Discrete-Time Control Systems, 2nd edition, Prentice Hall Inc., 1995.

[3] D. Nešić, Dead-Beat Control for Polynomial Systems, PhD dissertation, Australian National University, 1996.

[4] J.J. Slotine, W. Li, Applied Nonlinear Control, Prentice-Hall, Englewood Cliffs, NJ, 1991.

[5] M. Krstic, I. Kanellakopoulos, P.V. Kokopotic, Nonlinear and Adaptive Control Design, John Wiley and Sons, New York, NY, 1995.

[6] P.V. Kokotovic, M. Arcak, Constructive nonlinear control: a historical perspective, Automatica 37 (2001) 637–662.

[7] W.J. Rugh, J.S. Shamma, Research on gain scheduling, Automatica 36 (2000) 1401–1425.

[8] G. Becker, A. Packard, Robust performance of linear parametrically varying systems using parametrically-dependent linear feedback, Systems & Control Letters 23 (1994) 205–215.

[9] F. Wu, X.H. Yang, A. Packard, G. Becker, Induced L_2 norm control for LPV systems with bounded parameter variation rates, International Journal of Robust and Nonlinear Control 6 (1996) 983–998.

[10] P. Apkarian, R. Adams, Advanced gain-scheduling techniques for uncertain systems, IEEE Transactions on Control Systems Technology 6 (1997) 21–32.

[11] J.C. Lo, R. Adams, M.L. Lin, Robust H_∞ nonlinear control via fuzzy static output feedback, IEEE Transactions on Circuit and Systems. I 50 (2003) 1494–1502.

[12] D. Huang, S.K. Nguang, Static output feedback controller design for fuzzy systems: an ILMI approach, Journal of Information Sciences 177 (2007) 3005–3015.

[13] D. Huang, S.K. Nguang, Robust H_∞ static output feedback controller design for fuzzy systems: an ILMI approach, IEEE Transaction on Systems, Man and Cybernetics – Part B: Cybernetics 36 (2006) 216–222.

[14] S.K. Nguang, W. Assawinchaichote, P. Shi, Y. Shi, Robust H_∞ control design for uncertain fuzzy systems with Markovian jumps: an LMI approach, American Control Conference (2005) 1805–1810.

[15] S.K. Nguang, P. Zhang, S.X. Ding, Parity relation based fault estimation for nonlinear systems: an LMI approach, International Journal of Automation and Computing 4 (2) (2007) 164–168.

[16] W. Assawinchaichote, S.K. Nguang, P. Shi, Fuzzy Control and Filter Design for Uncertain Fuzzy Systems, Springer, 2006.

[17] S.K. Nguang, P. Shi, On designing filters for uncertain sampled-data nonlinear systems, Systems & Control Letters 41 (5) (2000) 305–316.

[18] S.K. Nguang, P. Shi, Stabilisation of a class of nonlinear time-delay systems using fuzzy models, in: Proceedings of the 39th IEEE Conference on Decision and Control, 2000, pp. 5–11.

[19] W. Assawinchaichote, S.K. Nguang, P. Shi, E.K. Boukas, H_∞ fuzzy state-feedback control design for nonlinear systems with stability constraints: an LMI approach, Mathematics and Computers in Simulation 78 (4) (2008) 514–531.

[20] S.K. Nguang, Comments on “Robust stabilization of uncertain input-delay systems by sliding mode control with delay compensation”, Automatica 37 (10) (2001) 1677.

[21] S.K. Nguang, P. Shi, H_∞ output feedback control of fuzzy system models under sampled measurements, Computers and Mathematics With Applications 46 (5) (2003) 705–717.

[22] J. Zhang, P. Shi, J. Qiu, S.K. Nguang, A novel observer-based output feedback controller design for discrete-time fuzzy systems, IEEE Transactions on Fuzzy Systems 23 (1) (2015) 223–229.

[23] S.K. Nguang, P. Shi, Delay-dependent H_∞ filtering for uncertain time delay nonlinear systems: an LMI approach, IET Control Theory & Applications 1 (1) (2007) 133–140.

[24] Y. Zhang, P. Shi, S.K. Nguang, H.R. Karimi, Observer-based finite-time fuzzy H_∞ control for discrete-time systems with stochastic jumps and time-delays, Signal Processing 97 (2014) 252–261.

[25] S. Chae, S.K. Nguang, SOS based robust H_∞ fuzzy dynamic output feedback control of nonlinear networked control systems, IEEE Transactions on Cybernetics 44 (7) (2014) 1204–1213.

[26] G. Chesi, LMI techniques for optimization over polynomials in control: a survey, IEEE Transaction on Automatic Control 55 (11) (2010) 2500–2510.

[27] J.C. Willems, Dissipative dynamical systems, part I: general theory, Archive for Rational Mechanics and Analysis 45 (1972) 321–351.

[28] C. Ebenbauer, F. Allgower, Analysis and design of polynomial control systems using dissipation inequalities and sum of squares, Journal of Computers and Chemical Engineering 30 (11) (2006) 1601–1614.

[29] R. Mtar, M.M. Belhaouane, M.F. Ghariani, H. Belkhiria, N.B. Braiek, An LMI criterion for the global stability analysis of nonlinear polynomial systems, Nonlinear Dynamics and Systems Theory 9 (2009) 171–183.

[30] M.M. Belhaouane, R. Mtar, H. Belkhiria, N.B. Braiek, Improved results on robust stability analysis and stabilization for a class of uncertain nonlinear systems, International Journal of Computers, Communications & Control 4 (4) (2009).

[31] M.M. Belhaouane, H. Belkhiria, N.B. Braiek, Improved Results on Robust Stability Analysis and Stabilization for a Class of Uncertain Nonlinear Systems, Mathematical Problems in Engineering, Hindawi Publishing, 2010.

[32] D. Cheng, H. Qi, Global stability and stabilization of polynomial systems, in: Proceedings of the 46th Conference on Decision and Control, 2007, pp. 1746–1751.

[33] D. Henrion, J.B. Lasserre, Convergent relaxations of polynomial matrix inequalities and static output feedback, IEEE Transaction on Automatic Control 51 (2) (2006).

[34] J.B. Lasserre, Global optimization with polynomials and the problem of moments, SIAM Journal of Optimal 11 (3) (2001).

[35] L. Vandenberghe, S.P. Boyd, Semidefinite programming, SIAM Review 38 (1) (1996) 49–95.

[36] K. Tanaka, A sum of squares approach to modeling and control of nonlinear dynamical systems with polynomial fuzzy systems, IEEE Transaction on Fuzzy Systems 17 (2009) 911–922.

[37] H.K. Lam, L.D. Seneviratne, Stability analysis of polynomial fuzzy-model-based control systems under perfect/imperfect premise matching, IET Control Theory & Applications 5 (15) (2011).

[38] L. Huang, C.Y. Cheng, G.R. Yu, Design of polynomial controllers for a two-link robot arm using sum-of-squares approachs, in: Proceedings of the 8th Asian Control Conference (ASCC), 2011.

[39] B.W. Sanjaya, B.R. Trilaksono, A. Syaichu-Rohman, Static output feedback control synthesis for nonlinear polynomial fuzzy systems using a sum of squares approach, in: Proceedings of the International Conference on Instrumentation, Communication, Information Technology and Biomedical Engineering, 2011.

[40] A.M. Lyapunov, The General Problem of the Stability of Motion, CRC Press, 1992, English Translation.

[41] F. Rasool, D. Huang, S.K. Nguang, Robust H_∞ output feedback control of networked control systems with multiple quantizers, Journal of the Franklin Institute 349 (3) (2012) 1153–1173.

[42] D. Huang, S.K. Nguang, Robust Control for Uncertain Networked Control Systems With Random Delays, Springer Science & Business Media, 2009.

[43] J. Zhang, A.K. Swain, S.K. Nguang, Robust sensor fault estimation scheme for satellite attitude control systems, Journal of the Franklin Institute 350 (9) (2013) 2581–2604.

[44] Z. Hou, J. Luo, P. Shi, S.K. Nguang, Stochastic stability of Ito differential equations with semi-Markovian jump parameters, IEEE Transactions on Automatic Control 51 (8) (2006) 1383–1387.

[45] F. Rasool, D. Huang, S.K. Nguang, Robust H_∞ output feedback control of discrete-time networked systems with limited information, Systems & Control Letters 60 (10) (2011) 845–853.

[46] S. Saat, S.K. Nguang, Nonlinear H_∞ output feedback control with integrator for polynomial discrete-time systems, International Journal of Robust and Nonlinear Control 25 (2015) 1051–1065.

[47] S. Chae, F. Rasool, S.K. Nguang, A. Swain, Robust mode delay-dependent H_∞ control of discrete-time systems with random communication delays, IET Control Theory & Applications 4 (6) (2010) 936–944.

[48] Y. Zhang, P. Shi, S.K. Nguang, Y. Song, Robust finite-time H_∞ control for uncertain discrete-time singular systems with Markovian jumps, IET Control Theory & Applications 8 (12) (2014) 1105–1111.

[49] S. Boyd, L. Ghaoui, E. Feron, V. Balakrishnan, Linear Matrix Inequalities in System and Control Theory, SIAM, 1994.

[50] P.A. Parrilo, Structured Semidefinite Programs and Semialgebraic Geometry Methods in Robustness and Optimization, PhD dissertation, California Inst. Technol., Pasadena, 2000.

[51] A. Papachristodoulou, S. Prajna, On the construction of Lyapunov functions using the sum of squares decomposition, in: Proceedings of the 41st IEEE Conference on Decision and Control (CDC), Las Vegas, 2002.

[52] S. Prajna, A. Papachristodoulou, F. Wu, Nonlinear control synthesis by sum of squares optimization: a Lyapunov-based approach, in: Proceedings of the 5th Asian Control Conference, 2004, pp. 157–165.

[53] H.J. Ma, G.H. Yang, Fault-tolerant control synthesis for a class of nonlinear systems: sum of squares optimization approach, International Journal of Robust and Nonlinear Control 19 (5) (2009) 591–610.
[54] D. Zhao, J. Wang, An improved H_∞ synthesis for parameter-dependent polynomial nonlinear systems using sos programming, in: American Control Conference, 2009, pp. 796–801.
[55] Q. Zheng, F. Wu, Regional stabilisation of polynomial nonlinear systems using rational Lyapunov functions, International Journal of Control 82 (9) (2009) 1605–1615.
[56] T. Jennawasin, T. Narikiyo, M. Kawanishi, An improved SOS-based stabilization condition for uncertain polynomial systems, in: SICE Annual Conference, 2010, pp. 3030–3034.
[57] D. Zhao, J. Wang, Robust static output feedback design for polynomial nonlinear systems, International Journal of Robust and Nonlinear Control (2009).
[58] S. Saat, M. Krug, S.K. Nguang, A nonlinear static output controller design for polynomial systems: an iterative sums of squares approach, in: 4th International Conference on Control and Mechatronics (ICOM), 2011, pp. 1–6.
[59] S. Saat, M. Krug, S.K. Nguang, Nonlinear H_∞ static output feedback controller design for polynomial systems: an iterative sums of squares approach, in: 6th IEEE Conference on Industrial Electronics and Applications (ICIEA), 2011, pp. 985–990.
[60] S.K. Nguang, S. Saat, M. Krug, Static output feedback controller design for uncertain polynomial systems: an iterative sums of squares approach, IET Control Theory & Applications 5 (9) (2011) 1079–1084.
[61] J. Xu, L. Xie, Y. Wang, Synthesis of discrete-time nonlinear systems: a SOS approach, in: American Control Conference, 2007, pp. 4829–4834.
[62] W. Tan, Nonlinear Control Analysis and Synthesis Using Sums-of-Squares Programming, PhD dissertation, University of California, Berkeley, USA, Spring 2006.
[63] M. Kocvara, M. Stingl, PENBMI User Manual, Tech. Rep., 1.2 edition, Jan. 2004.
[64] H.J. Ma, G.H. Yang, Fault tolerant H_∞ control for a class of nonlinear discrete-time systems: using sum of squares optimization, in: Proceeding of American Control Conference, 2008, pp. 1588–1593.
[65] K. Murty, S.N. Kabadi, Some NP-complete problems in quadratic and nonlinear programming, Mathematical Programming 39 (1987) 117–129.
[66] P. Dorato, W. Yang, C. Abdallah, Robust multiobjective feedback design by quantifier elimination, Journal of Symbolic Computation 24 (1997) 153–159.
[67] M. Jirstrand, Nonlinear control system design by quantifier elimination, Journal of Symbolic Computation 24 (1997) 137–152.
[68] B. Reznick, Uniform denominators in Hilbert's seventeen problem, Mathematische Zeitschrift 220 (1995) 75–97.
[69] P.A. Parrilo, B. Sturmfels, Minimizing polynomial functions, in: Workshop on Algorithm and Quantitative Aspects of Real Algebraic, in: Geometry in Mathematics and Computer Science, 2001.
[70] B. Reznick, Some concrete aspects of Hilbert's 17th problem, in: Contemporary Mathematics, vol. 253, American Mathematics Society, 2005, pp. 251–272.
[71] N.K. Bose, C.C. Li, A quadratic form representation of polynomials of several variables and its application, IEEE Transaction on Automatic Control 14 (1968) 447–448.
[72] Y. Huang, Nonlinear Optimal Control: An Enhanced Quasi-LPV Approach, PhD dissertation, California Inst. Technol., Pasadena, 1998.

[73] S. Prajna, A. Papachristodoulou, P.A. Parrilo, Introducing SOSTOOLS: a general purpose sum of squares programming solver, in: Conference on Decision and Control, vol. 1, 2002, pp. 741–746.

[74] J. Lofberg, YALMIP: a toolbox for modeling and optimization in Matlab, in: IEEE International Symposium on Computer Aided Control Systems Design, 2004, pp. 284–289.

[75] M. Grant, S. Boyd, Graph implementations for nonsmooth convex programs, in: Recent Advances in Learning and Control, in: Lecture Notes in Control and Information Sciences, Springer-Verlag Limited, 2008, pp. 95–110.

[76] D. Henrion, J.B. Lasserre, J. Lofberg, GloptiPoly 3: moments, optimization and semidefinite programming, Optimization Methods and Software 24 (4) (2009) 761–779.

[77] S. Prajna, A. Papachristodoulou, P. Seiler, SOSTOOLS: Sum of Squares Optimization Toolbox for MATLAB, User's Guide, 2004.

[78] S.L. Chen, C.T. Chen, Exact linearization of a voltage-controlled 3-pole active magnetic bearing system, IEEE Transaction on Control Systems Technology 10 (2002) 618–625.

[79] A.N. Pisarchik, B.F. Kuntsevich, Control of multistability in a directly modulated diode laser, IEEE Transaction on Quantum Electronics 38 (12) (2002) 1594–1598.