# A survay about Fractional Order Calculus.

Here we will review derivations of fractional calculus.

# I.   INTRODUCTION

# II.   FRACTIONAL ORDER INTEGRATION

## A.   Useful relations

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt \tag{1}$$

$$B(\alpha, \beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1} dt \tag{2}$$

$$B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \tag{3}$$

$$\alpha\Gamma(\alpha) = \Gamma(\alpha+1), \Gamma(0) = \infty, \Gamma(1) = 1 \tag{4}$$

$$(\alpha+n)\cdots\alpha\Gamma(\alpha) = \Gamma(\alpha+n+1) \tag{5}$$

## B.   Cachy formula

$$_a\mathbf{I}_t^n[F] = \int_a^t d\tau_1 \int_a^{\tau_1} d\tau_2 \int_a^{\tau_2} d\tau_3 \cdots d\tau_{n-1} \int_a^{\tau_{n-1}} F(\tau_n) d\tau_n \tag{6}$$

$$
\begin{aligned}
_a\mathbf{I}_t^n[F] &= \int_a^t \underbrace{d\tau_1}_{d\nu} \underbrace{_a\mathbf{I}_{\tau_1}^{n-1}[F]}_{u} = \tau_1 {}_a I_{\tau_1}^{n-1}[F]|_a^t - \int_a^t \tau_1 d\tau_1 {}_a \mathbf{I}_{\tau_1}^{n-2}[F] \\
&= t \times {}_a\mathbf{I}_t^{n-1}[F] - \int_a^t \tau_1 d\tau_1 {}_a\mathbf{I}_{\tau_1}^{n-2}[F] = \int_a^t t d\tau_1 {}_a\mathbf{I}_{\tau_1}^{n-2}[F] - \int_a^t \tau_1 d\tau_1 {}_a\mathbf{I}_{\tau_1}^{n-2}[F] \\
&= \int_a^t \underbrace{d\tau_1(t-\tau_1)}_{d\nu} \underbrace{_a\mathbf{I}_{\tau_1}^{n-2}[F]}_{u}
\end{aligned} \tag{7}
$$

$$
\begin{aligned}
_a\mathbf{I}_t^n[F] &= \underbrace{-\frac{1}{2}(t-\tau_1)^2 {}_a I_{\tau_1}^{n-2}[F]|_a^t}_{=0} + \frac{1}{2}\int_a^t (t-\tau_1)^2 d\tau_1 {}_a\mathbf{I}_{\tau_1}^{n-3}[F] \\
&= \frac{1}{2}\int_a^t (t-\tau_1)^2 d\tau_1 {}_a\mathbf{I}_{\tau_1}^{n-3}[F] \\
&\vdots \\
&= \frac{1}{(n-1)!}\int_a^t (t-\tau_1)^{n-1} F(\tau_1) d\tau_1
\end{aligned} \tag{8}
$$

$$_a\mathbf{I}_t^n[F] = \frac{1}{\Gamma(n)}\int_a^t (t-\tau_1)^{n-1} F(\tau_1) d\tau_1 \tag{9}$$

## C. Arbitrary order integration

Based on Cachy integration formula for repeated integrations one may extend the integer order integration to arbitrary order integration as follows,

$$
{}_a\mathbf{I}_t^\alpha[F] = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} F(\tau) d\tau \tag{10}
$$

## III. FRACTIONAL ORDER DIFFERENTATION

### A. Reimann-Liouvile FOD

$$
{}_a^{\mathbf{RL}}\mathbf{D}_t^\alpha \mathbf{F}(t) = \frac{d^m}{dt^m} {}_a\mathbf{I}_t^{m-\alpha}[\mathbf{F}] = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_a^t (t-\tau)^{m-\alpha-1} \mathbf{F}(\tau) d\tau \tag{11}
$$

where $m - 1 \preceq \alpha \preceq m$.

### B. Caputo FOD

$$
{}_a^{\mathbf{C}}\mathbf{D}_t^\alpha \mathbf{F}(t) = {}_a\mathbf{I}_t^{m-\alpha} \frac{d^m}{dt^m}[\mathbf{F}] = \frac{1}{\Gamma(m-\alpha)} \int_a^t (t-\tau)^{m-\alpha-1} \frac{d^m}{d\tau^m} \mathbf{F}(\tau) d\tau \tag{12}
$$

where $m - 1 \preceq \alpha \preceq m$.

### C. Gronvald-Letnikov FOD and FOI

$$
{}_a^{\mathbf{GL}}\mathbf{D}_t^\alpha \mathbf{F}(t) = \lim_{h\to 0} \frac{(1-\hat{\mathbf{T}}_h)^\alpha}{h^\alpha} \mathbf{F}(t) \tag{13}
$$

Using Tylor expansion of $(1-x)^\alpha$,

$$
\begin{aligned}
(1-\hat{\mathbf{T}}_h)^\alpha &= \sum_{n=0}^\infty \frac{(-1)^n}{n!} \times \prod_{i=1}^{i=k} (\alpha - i + 1) \hat{\mathbf{T}}_h^n \\
&= \sum_{n=0}^\infty \frac{(-1)^n}{n!} \underbrace{\Gamma(\alpha-n+1) \prod_{i=1}^{i=k} (\alpha-i+1)}_{\Gamma(\alpha+1)} \times \frac{1}{\Gamma(\alpha-n+1)} \hat{\mathbf{T}}_h^n \\
&= \sum_{n=0}^\infty (-1)^n \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-n+1)\Gamma(n+1)} \hat{\mathbf{T}}_h^n = \sum_{n=0}^\infty (-1)^n \binom{\alpha}{n} \hat{\mathbf{T}}_h^n
\end{aligned}
$$

Thus the GL FOD becomes,

$$
{}_a^{\mathbf{GL}}\mathbf{D}_t^\alpha \mathbf{F}(t) = \frac{1}{h^\alpha} \lim_{h\to 0} \sum_{n=0}^N (-1)^n \binom{\alpha}{n} \mathbf{F}(t-nh) \tag{14}
$$

with $h = t/N$. One may define GL integration by just $\alpha \to -\alpha$,

$$
\begin{aligned}
(1 - \hat{\mathbf{T}}_h)^{-\alpha} &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \times \prod_{i=1}^{i=k} (-\alpha - i + 1) \hat{\mathbf{T}}_h^n \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \underbrace{\Gamma(\alpha) \prod_{i=1}^{i=k} (\alpha + i - 1)}_{\Gamma(\alpha+n)} \times \frac{1}{\Gamma(\alpha)} \hat{\mathbf{T}}_h^n \\
&= \sum_{n=0}^{\infty} (-1)^n \times (-1)^n \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha - n + 1)\Gamma(n + 1)} \hat{\mathbf{T}}_h^n \\
&= \sum_{n=0}^{\infty} (-1)^n \binom{\alpha + n - 1}{n} \hat{\mathbf{T}}_h^n
\end{aligned}
$$

Thus GL FOI becomes,

$$
{}_a^{\mathbf{GL}}\mathbf{I}_t^\alpha = {}_a^{\mathbf{GL}}\mathbf{D}_t^{-\alpha}\mathbf{F}(t) = \lim_{h \to 0} h^\alpha \sum_{n=0}^{N} \binom{\alpha + n - 1}{n} \mathbf{F}(t - nh) \tag{15}
$$

with $h = t/N$.

### D. The relation between RL, Caputo and GL FOD

It is instructive to compare **Caputo** and **R**L, for $\mathbf{F}(t) = t^\nu$.
***R****eimann-Liouvile*:

$$
\begin{aligned}
{}_a^{\mathbf{RL}}\mathbf{D}_t^\alpha t^\nu &= \frac{1}{\Gamma(m - \alpha)} \frac{d^m}{dt^m} \int_a^t (t - \tau)^{m-\alpha-1} \tau^\nu d\tau \\
&= \frac{1}{\Gamma(m - \alpha)} \underbrace{\int_0^1 (1 - x)^{m-\alpha-1} x^{\nu+1-1} dx}_{\mathbf{B}(m-\alpha,\nu+1)} \frac{d^m}{dt^m} t^{m+\nu-\alpha} \\
&= \frac{1}{\Gamma(m - \alpha)} \frac{\Gamma(m - \alpha)\Gamma(\nu + 1)}{\overbrace{\Gamma(m - \alpha + v + 1)}^{\Gamma(m-\alpha+v+1)}} \frac{d^m}{dt^m} t^{m+\nu-\alpha} \\
&= \frac{\overbrace{\prod_{i=1}^{m} (i + \nu - \alpha)\Gamma(\nu - \alpha + 1)}^{} \Gamma(\nu + 1)}{\Gamma(\nu - \alpha + 1)\Gamma(m - \alpha + v + 1)} t^{\nu-\alpha} \\
&= \frac{\Gamma(\nu + 1)}{\Gamma(\nu - \alpha + 1)} t^{\nu-\alpha}
\end{aligned} \tag{16}
$$

*Caputo*:

$$\begin{aligned}
{}_a^\mathbf{C}\mathbf{D}_t^\alpha t^\nu &= \frac{1}{\Gamma(m-\alpha)}\int_a^t (t-\tau)^{m-\alpha-1}\frac{d^m}{d\tau^m}\tau^\nu d\tau \;\Rightarrow\; \frac{\mathbf{d^m}}{\mathbf{d\tau^m}}\tau^\nu = \mathbf{0},\; \text{if } \nu \in \mathbf{N},\; \text{and } \nu < \mathbf{m} \\
&= \frac{1}{\Gamma(m-\alpha)}\int_a^t (t-\tau)^{m-\alpha-1}\tau^{\nu-m}d\tau \times \prod_{i=0}^{m-1}(\nu-i) \\
&= \frac{1}{\Gamma(m-\alpha)}\underbrace{\int_0^1 (1-x)^{m-\alpha-1}x^{\nu-m+1-1}dx}_{\mathbf{B}(m-\alpha,\nu-m+1)} \times \prod_{i=0}^{m-1}(\nu-i) \times t^{\nu-\alpha} \\
&= \frac{1}{\Gamma(m-\alpha)}\frac{\Gamma(m-\alpha)\,\overbrace{\Gamma(\nu-m+1)\prod_{i=0}^{m-1}(\nu-i)}^{\Gamma(\alpha+1)}}{\Gamma(\nu-\alpha+1)} \times t^{\nu-\alpha} \\
&= \frac{\Gamma(\nu+1)}{\Gamma(\nu-\alpha+1)}t^{\nu-\alpha} \tag{17}
\end{aligned}$$

$$\tag{18}$$

Therefore for **Caputo** FOD we have,

$$
{}_a^\mathbf{C}\mathbf{D}_t^\alpha t^\nu = \begin{cases} 0 & \text{if } \nu \in N \text{ and } \nu < m \\ \frac{\Gamma(\nu+1)}{\Gamma(\nu-\alpha+1)}t^{\nu-\alpha} & \text{otherwise} \end{cases} \tag{19}
$$

with $m-1 < \nu < m$.

$$\begin{aligned}
{}_a^\mathbf{C}\mathbf{D}_t^\alpha\mathbf{F}(t) &= \sum_{\nu=0}^\infty \frac{\mathbf{F}^{(\nu)}(0)}{\nu!}\times_a^\mathbf{C}\mathbf{D}_t^\alpha t^\nu \\
&= \sum_{\nu=m}^\infty \frac{\mathbf{F}^{(\nu)}(0)}{\nu!}\times\frac{\Gamma(\nu+1)}{\Gamma(\nu-\alpha+1)}t^{\nu-\alpha} \\
&= \sum_{\nu=m}^\infty \frac{\mathbf{F}^{(\nu)}(0)}{\Gamma(\nu-\alpha+1)}t^{\nu-\alpha} \tag{20}
\end{aligned}$$

and for **RL** we have,

$$
{}_a^\mathbf{RL}\mathbf{D}_t^\alpha\mathbf{F}(t) = \sum_{\nu=0}^\infty \frac{\mathbf{F}^{(\nu)}(0)}{\Gamma(\nu-\alpha+1)}t^{\nu-\alpha} \tag{21}
$$

Finally we have,

$$
{}_a^\mathbf{RL}\mathbf{D}_t^\alpha\mathbf{F}(t) -{}_a^\mathbf{C}\mathbf{D}_t^\alpha\mathbf{F}(t) = \sum_{\nu=0}^{m-1}\frac{\mathbf{F}^{(\nu)}(0)}{\Gamma(\nu-\alpha+1)}t^{\nu-\alpha} \tag{22}
$$

Taking fractional integration on both sides of Eq. 22,

$$\begin{aligned}
{}_a\mathbf{I}_t^\alpha\left[{}_a^\mathbf{RL}\mathbf{D}_t^\alpha\mathbf{F}(t) -{}_a^\mathbf{C}\mathbf{D}_t^\alpha\mathbf{F}(t)\right] &= \sum_{\nu=0}^{m-1}\frac{\mathbf{F}^{(\nu)}(0)}{\Gamma(\nu-\alpha+1)}{}_a\mathbf{I}_t^\alpha[t^{\nu-\alpha}] \\
&= \sum_{\nu=0}^{m-1}\frac{\mathbf{F}^{(\nu)}(0)}{\Gamma(\nu-\alpha+1)}\frac{\Gamma(\nu-\alpha+1)}{\Gamma(\nu+1)}t^\nu \\
&= \sum_{\nu=0}^{m-1}\frac{\mathbf{F}^{(\nu)}(0)}{\Gamma(\nu+1)}t^\nu \tag{23}
\end{aligned}$$

## IV.  FOURIER AND LAPLACE TRANSFORMATIONS

$$\mathscr{L}\{{}^{\mathbf{RL}}_a\mathbf{D}^\alpha_t\mathbf{F}\} = \int_0^\infty {}^{\mathbf{RL}}_a\mathbf{D}^\alpha_t\mathbf{F}(t)e^{-st}dt$$

$$= \frac{1}{\Gamma(m-\alpha)}\int_0^\infty e^{-st}dt\,\overbrace{\frac{d^m}{dt^m}\int_a^t(t-\tau)^{m-\alpha-1}\mathbf{F}(\tau)d\tau}^{\mathbf{U}^{(m)}(t)}$$

$$= \frac{1}{\Gamma(m-\alpha)}\int_0^\infty \underbrace{e^{-st}}_{\nu}\underbrace{\mathbf{U}^{(m)}(t)dt}_{du}$$

$$= \frac{1}{\Gamma(m-\alpha)}\left[\mathbf{U}^{(m-1)}(0) + s\int_0^\infty \underbrace{e^{-st}}_{\nu}\underbrace{\mathbf{U}^{(m-1)}(t)dt}_{du}\right]$$

$$= \frac{1}{\Gamma(m-\alpha)}\left[\mathbf{U}^{(m-1)}(0) + s\mathbf{U}^{(m-2)}(0) + s^2\int_0^\infty \underbrace{e^{-st}}_{\nu}\underbrace{\mathbf{U}^{(m-2)}(t)dt}_{du}\right]$$

$$\vdots$$

$$= \frac{1}{\Gamma(m-\alpha)}\left[\sum_{\nu=0}^{m-1}\mathbf{U}^{(\nu)}(0)s^{m-\nu-1} + s^m\sum_{\nu=0}^{\infty}\frac{\mathbf{F}^{(\nu)}(a)}{\nu!}\int_0^\infty e^{-st}\int_a^t(t-\tau)^{m-\alpha-1}\tau^\nu d\tau\right]$$

$$= \frac{1}{\Gamma(m-\alpha)}\left[\sum_{\nu=0}^{m-1}\mathbf{U}^{(\nu)}(0)s^{m-\nu-1} + s^m\sum_{\nu=0}^{\infty}\frac{\mathbf{F}^{(\nu)}(a)}{\nu!}\int_0^\infty e^{-st}t^{m-\alpha+\nu}dt\underbrace{\int_0^1(1-x)^{m-\alpha-1}x^{\nu+1-1}dx}_{\mathbf{B}(m-\alpha,\nu+1)}\right]$$

$$= \frac{1}{\Gamma(m-\alpha)}\left[\sum_{\nu=0}^{m-1}\mathbf{U}^{(\nu)}(0)s^{m-\nu-1} + s^m\sum_{\nu=0}^{\infty}\frac{\mathbf{F}^{(\nu)}(a)}{\nu!}\mathbf{B}(m-\alpha,\nu+1)\underbrace{\int_0^\infty e^{-st}(st)^{m-\alpha+\nu}d(st)}_{\Gamma(m-\alpha+\nu+1)}\times s^{\alpha-m-\nu-1}\right]$$

$$= \frac{1}{\Gamma(m-\alpha)}\left[\sum_{\nu=0}^{m-1}\mathbf{U}^{(\nu)}(0)s^{m-\nu-1} + s^m\sum_{\nu=0}^{\infty}\frac{\mathbf{F}^{(\nu)}(a)}{\nu!}\frac{\Gamma(\nu+1)\Gamma(m-\alpha)}{\Gamma(m-\alpha+\nu+1)}\Gamma(m-\alpha+\nu+1)s^{\alpha-m-\nu-1}\right]$$

$$= \frac{1}{\Gamma(m-\alpha)}\sum_{\nu=0}^{m-1}\mathbf{U}^{(\nu)}(0)s^{m-\nu-1} + s^\alpha\underbrace{\sum_{\nu=0}^{\infty}\frac{\mathbf{F}^{(\nu)}(a)}{s^{\nu+1}}}_{\mathbf{F}(s)}$$

$$\mathscr{L}\{{}_a^{\mathbf{C}}\mathbf{D}_t^\alpha \mathbf{F}\} = \int_0^\infty {}_a^{\mathbf{C}}\mathbf{D}_t^\alpha \mathbf{F}(t)e^{-st}dt$$

$$= \sum_{\nu=0}^\infty \frac{\mathbf{F}^{(\nu)}(a)}{\nu!} \int_0^\infty {}_a^{\mathbf{C}}\mathbf{D}_t^\alpha t^\nu e^{-st}dt$$

$$= \sum_{\nu=m}^\infty \frac{\mathbf{F}^{(\nu)}(a)}{\nu!} \frac{\Gamma(\nu+1)}{\Gamma(\nu-\alpha+1)} \underbrace{\int_0^\infty (st)^{\nu-\alpha}e^{-st}d(st)}_{\Gamma(\nu-\alpha+1)} \times s^{\alpha-\nu-1}$$

$$= s^\alpha \sum_{\nu=m}^\infty \frac{\mathbf{F}^{(\nu)}(a)}{s^{\nu+1}}$$

$$= \underbrace{\sum_{\nu=0}^\infty \frac{\mathbf{F}^{(\nu)}(a)}{s^{\nu+1}}}_{\mathbf{F}(s)} - \sum_{\nu=0}^{m-1} \mathbf{F}^{(\nu)}(a)s^{\alpha-\nu-1}$$

## V.  SERIES EXPANSION SOLUTION OF FODES

The simplest example is

$$\tag{24} {}_a^{\mathbf{C}}\mathbf{D}_t^\alpha \mathbf{F}(t) + \lambda \mathbf{F}(t) = 0$$

Suppose, $\mathbf{F}(t) = \sum_{\nu=0}^\infty a_\nu t^{\nu\alpha}$.

$$\tag{25} \sum_{\nu=0}^\infty a_\nu {}_a^{\mathbf{C}}\mathbf{D}_t^\alpha t^{\nu\alpha} + \lambda \sum_{\nu=0}^\infty a_\nu t^{\nu t} = \sum_{\nu=0}^\infty \left[ a_{\nu+1}\frac{\Gamma((\nu+1)\alpha+1)}{\Gamma(\nu\alpha+1)} + \lambda a_\nu \right] t^{\nu\alpha} = 0$$

$$\tag{26} a_{\nu+1} = -\lambda \frac{\Gamma(\nu\alpha+1)}{\Gamma((\nu+1)\alpha+1)} a_\nu$$

$$a_{\nu+1} = -\lambda \frac{\Gamma(\nu\alpha+1)}{\Gamma((\nu+1)\alpha+1)} a_\nu$$

$$a_{\nu+1} = -\lambda \frac{\Gamma(\nu\alpha+1)}{\Gamma((\nu+1)\alpha+1)} \times -\lambda \frac{\Gamma(\nu\alpha+1)}{\Gamma((\nu+1)\alpha+1)} a_{\nu-1}$$

$$\tag{27} a_{\nu+1} = -\lambda \frac{\Gamma(\nu\alpha+1)}{\Gamma((\nu+1)\alpha+1)} \times -\lambda \frac{\Gamma(\nu\alpha+1)}{\Gamma((\nu+1)\alpha+1)} a_{\nu-1} \cdots \times -\lambda \frac{\Gamma(1)}{\Gamma(\alpha+1)} a_0$$

$$\tag{28} a_\nu = \frac{(-\lambda)^\nu}{\Gamma(\nu\alpha+1)} a_0$$

thus the solution of Eq. 24 becomes,

$$\tag{29} \mathbf{F}(t) = \mathbf{F}(0)\mathbf{E}_\alpha(-\lambda t^\alpha)$$

The $\mathbf{F}(0)$ fixes the initial condition for Eq. 24. the function,

$$\tag{30} \mathbf{E}_\alpha(x) = \sum_{\nu=0}^\infty \frac{x^\nu}{\Gamma(\nu\alpha+1)}$$

is dubed the one parameter Mittag-Leffer function. One may also employ the Laplace transform for solving Eq. 24,

$$\begin{aligned} \mathscr{L}\{{}_a^{\mathbf{C}}\mathbf{D}_t^\alpha \mathbf{F}(t) + \lambda \mathbf{F}(t)\} &= 0 \\ -s^{1-\alpha}\mathbf{F}(0) + s^\alpha \mathbf{F}(s) + \lambda \mathbf{F}(s) &= 0 \end{aligned} \tag{31}$$

$$\mathbf{F}(s) = \mathbf{F}(0)\frac{s^{1-\alpha}}{\lambda + s^\alpha} \tag{32}$$

Thus the solution of Eq. 24, becomes,

$$\mathbf{F}(t) = \mathscr{L}^{-1}\{\mathbf{F}(s)\} = \mathbf{F}(0)\int \frac{s^{1-\alpha}}{\lambda + s^\alpha}ds \tag{33}$$

therefore one may infer,

$$\int \frac{s^{1-\alpha}}{\lambda + s^\alpha}ds = \mathbf{E}_\alpha(-\lambda t^\alpha) \tag{34}$$

This means, as long as the $\Re[\lambda] < 0$, $\lim_{t\to+\infty}\mathbf{F}(t) \to 0$, otherwise if $\Re[\lambda] = 0$ it oscillates permanently and if $\Re[\lambda] > 0$ the solution diverges.

## VI.   NUMERICAL METHODS OF SOLUTIONS OF ORDINARY DIFFERENTIAL EQUATIONS

$$\dot{y}(t) = \mathbf{F}(y(t), t) \tag{35}$$

### A.   Multi-Linear methods for the solutions of ODEs

To solve Eq. 35, by assuming we have the values of $y(t)$ for $t < t_0$, one may integrate both sides of Eq. 35, over $t$ from $t_0$ to $t_0 + h$. To perform, numerically, the integration $y(t)$ and $\mathbf{F}(t)$ could be approximated by a suitable interpolation function. Obviously, the larger the order of the interpolation will allow a larger choose of $h$. For the interpolation of the right hand side of Eq. 35, there is two option for the choose of the interpolation point: (1) $y(t_0 + h)$ being expluded from the interpolation reference points (predictor). (2) $y(t_0+h)$ being included in the interpolation reference points (corrector). For the later, the solution of the resulting discrete equations must be proceed by iteration. The best strategy for solving Eq. 35 is to get an estimation of the $y(t_0 + h)$ through a predictor equation, then correcting the estimation through corrector. Thus a general form of the discritized equation proceeds as: The predictor equation,

$$\sum_{r=0}^{N} a_r y_{n+1-r} = \sum_{r=0}^{N} b_r \mathbf{F}_{n-r} \tag{36}$$

And for the corrector equation, we have:

$$\sum_{r=0}^{N} a_r y_{n+1-r} = \sum_{r=0}^{N} b_r \mathbf{F}_{n+1-r} \tag{37}$$

where $y_i = y(t_i)$ and $\mathbf{F}_i = \mathbf{F}(y(t_i), t_i)$.

$$\int_{t_n}^{t_{n+1}} y(\tau)\tau = \int_{t_n}^{t_{n+1}} \tilde{\mathbf{F}}_s(\tau)d\tau, \textbf{ with s=Pr or Corr} \tag{38}$$

where

$$\tilde{\mathbf{F}}_{Pr}(\tau) = \sum_{i=0}^{N} \mathbf{F}(t_{n-i})L_{N-i}(\tau) \tag{39}$$

and

$$\tilde{\mathbf{F}}_{Corr}(\tau) = \sum_{i=0}^{N} \mathbf{F}(t_{n+1-i})L_{N-i}(\tau) \tag{40}$$

$$L_i(t) = \frac{\prod_{k=0,k\neq i}^{N}(t - t_k)}{\prod_{k=0,k\neq i}^{N}(t_i - t_k)} \tag{41}$$

For the left hand side the simplest case is the Reiman integration, thus Predictor and Corrector equations reads,

Predictor:

$$y^{(P)}(t_{n+1}) = y(t_n) + \sum_{i=0}^{N} \mathbf{F}(t_{n-i}, y_{n-i}) \int_{t_n}^{t_{n+1}} L_{N-i}(\tau)d\tau \tag{42}$$

Predictor:

$$y(t_{n+1}) = y(t_n) + \sum_{i=0}^{N} \mathbf{F}(t_{n+1-i}, y_{n+1-i}^{(P)}) \int_{t_n}^{t_{n+1}} L_{N-i}(\tau)d\tau \tag{43}$$

For a four point approximation, without loss of generality, we may consider $t_{3-i} = -ih$. Thus we have,

$$
\begin{aligned}
L_0(t) &= \frac{(t+h)(t+2h)(t+3h)}{(0+h)(0+2h)(0+3h)} = \frac{1}{6h^3}(t^3 + 6ht^2 + 11h^2t + 6h^3) \\
L_1(t) &= \frac{(t-0)(t+2h)(t+3h)}{(-h+0)(-h+2h)(-h+3h)} = -\frac{1}{2h^3}(t^3 + 5ht^2 + 6h^2t) \\
L_2(t) &= \frac{(t-0)(t+h)(t+3h)}{(-2h+0)(-2h+h)(-2h+3h)} = \frac{1}{2h^3}(t^3 + 4ht^2 + 3h^2t) \\
L_3(t) &= \frac{(t-0)(t+h)(t+2h)}{(-3h+0)(-3h+h)(-3h+2h)} = -\frac{1}{6h^3}(t^3 + 3ht^2 + 2h^2t)
\end{aligned}
$$

For the corrector, the integration must be performed in the domain of $(-h, 0)$. Performing the integrations we have,

$$_{-h}\mathbf{I}_0[L_0] = \frac{3h}{8} \;,\; _{-h}\mathbf{I}_0[L_1] = \frac{19h}{24} \;,\; _{-h}\mathbf{I}_0[L_2] = \frac{-5h}{24} \;,\; _{-h}\mathbf{I}_0[L_3] = \frac{h}{24} \tag{44}$$

For the predictor the integration must performed in the domain $(0, h)$,

$$_0\mathbf{I}_h[L_0] = \frac{55h}{24} \;,\; _0\mathbf{I}_h[L_1] = \frac{-59h}{24} \;,\; _0\mathbf{I}_h[L_2] = \frac{37h}{24} \;,\; _0\mathbf{I}_h[L_3] = \frac{h}{24} \tag{45}$$

### B. Runge-Cutta

## VII. NUMERICAL METHODS FOR SOLVING FRACTIONAL ORDER DIFFERENTIAL EQUATIONS(FODE)

For this section we will exactly follow Deithelm approach. The suggested equation for dealing with practical applications is the following,

$$^{\mathbf{C}}_a\mathbf{D}^\alpha_t y(t) = \mathbf{F}(y, t) \tag{46}$$

by fraction integrating the both sides of Eq. 46, we have,

$$y(t) = \sum_{0}^{[\alpha]-1} \frac{t^\nu}{\nu!} y^{(\nu)}(0) + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{\mathbf{F}(y(\tau), \tau)}{(t-\tau)^{\alpha-1}} d\tau \tag{47}$$

The next step is to evaluate the integral in Eq. 47. Following Deithelm, one may slice the hole integration into slices which each slice could be evaluated by qudrature rule. In Deithelm they used the linear interpolation of integrand between two point in each slice.

$$\int_0^{t_{k+1}} (t-\tau)^{\alpha-1} \mathbf{F}(\tau)d\tau \simeq \int_0^{t_{k+1}} (t-\tau)^{\alpha-1} \tilde{\mathbf{F}}(\tau)d\tau = \sum_{j=0}^{k+1} a_{j,k+1} \mathbf{F}(t_j) \tag{48}$$

with

$$a_{j,k+1} = \frac{h^\alpha}{\alpha(\alpha+1)} \begin{cases} k^{\alpha+1} - (k-\alpha)(k+1)^\alpha & \text{if } j = 0 \\ (k-j+2)^{\alpha+1} + (k-j)^{\alpha+1} - 2(k-j+1)^{\alpha+1} & \text{if } 1 \preceq j \preceq k \\ 1 & \text{if } j = k+1 \end{cases} \tag{49}$$

The corrector for One step Adams-Moulton becomes,

$$y_{k+1} = \sum_0^{[\alpha]-1} \frac{t^\nu}{\nu!} y^{(\nu)}(0) + \frac{1}{\Gamma(\alpha)} \left[ a_{k+1,k+1}\mathbf{F}(t_{k+1}, y_{k+1}^P) + \sum_{j=0}^k a_{j,k+1}\mathbf{F}(t_j, y_j) \right] \tag{50}$$

replacing the intgral in Eq. 47 with rectangle rule will eliminat the share of the $y_{k+1}$, thus the equation will serve as a predictor estimation of the solution at $k+1$th point. In this case we have:

$$\int_0^{t_{k+1}} (t-\tau)^{\alpha-1}\mathbf{F}(\tau)d\tau \simeq \int_0^{t_{k+1}} (t-\tau)^{\alpha-1}\tilde{\mathbf{F}}(\tau)d\tau = \sum_{j=0}^{k+1} b_{j,k+1}\mathbf{F}(t_j) \tag{51}$$

with $b_{j,k+1} = \frac{h^\alpha}{\alpha}((k+j-1)^\alpha - (k-j)^\alpha)$.

$$y_{k+1}^P = \sum_0^{[\alpha]-1} \frac{t^\nu}{\nu!} y^{(\nu)}(0) + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^k b_{j,k+1}\mathbf{F}(t_j, y_j) \tag{52}$$

### A. Multi linear methods for the solutions of FODEs

### B. Runge-Cutta

## VIII. NUMERICAL IMPLEMENTATION

## IX. TRANSIENT SIMULATION OF ELECTRICAL CIRCUITS(PASSIVE PARTS ONLY)

### A. Simulation of parts without FO components

### B. Simulation of parts with FO components

### C. Implementation of transient algorithem

### D. Inclusion of SPICE net lists

## X. SOME APPLICATIONS OF FOC IN THE ELECTROCHEMISTRY

## XI. POSSIBLE INTEGRATION INTO QUCS

## XII. HIGHER ORDER FINITE DIFFERENCE FOR INTEGER ORDER DERIVATIVES

## XIII. FINITE DIFFERENCE FOD

## ACKNOWLEDGMENTS

## XIV. APPENDIX