A survay about Fractional Order Calculus.

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Here we will review derivations of fractional calculus.

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I. INTRODUCTION

II. FRACTIONAL ORDER INTEGRATION

A. Useful relations

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt \tag{1}$$

$$B(\alpha, \beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1} dt \tag{2}$$

$$B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \tag{3}$$

$$\alpha\Gamma(\alpha) = \Gamma(\alpha+1), \Gamma(0) = \infty, \Gamma(1) = 1 \tag{4}$$

$$(\alpha+n)\cdots\alpha\Gamma(\alpha) = \Gamma(\alpha+n+1) \tag{5}$$

B. Cachy formula

$${}_a\mathbf{I}_t^n[F] = \int_a^t d\tau_1 \int_a^{\tau_1} d\tau_2 \int_a^{\tau_2} d\tau_3 \cdots d\tau_{n-1} \int_a^{\tau_{n-1}} F(\tau_n) d\tau_n \tag{6}$$

$$\begin{aligned} {}_a\mathbf{I}_t^n[F] &= \int_a^t \underbrace{d\tau_1}_{d\nu} \underbrace{{}_a\mathbf{I}_{\tau_1}^{n-1}[F]}_{u} = \tau_1 {}_a I_{\tau_1}^{n-1}[F]|_a^t - \int_a^t \tau_1 d\tau_1 {}_a\mathbf{I}_{\tau_1}^{n-2}[F] \\ &= t \times {}_a\mathbf{I}_t^{n-1}[F] - \int_a^t \tau_1 d\tau_1 {}_a\mathbf{I}_{\tau_1}^{n-2}[F] = \int_a^t t d\tau_1 {}_a\mathbf{I}_{\tau_1}^{n-2}[F] - \int_a^t \tau_1 d\tau_1 {}_a\mathbf{I}_{\tau_1}^{n-2}[F] \\ &= \int_a^t \underbrace{d\tau_1(t-\tau_1)}_{d\nu} \underbrace{{}_a\mathbf{I}_{\tau_1}^{n-2}[F]}_{u} \end{aligned} \tag{7}$$

$$\begin{aligned} {}_a\mathbf{I}_t^n[F] &= \underbrace{-\frac{1}{2}(t-\tau_1)^2 {}_a I_{\tau_1}^{n-2}[F]|_a^t}_{=0} + \frac{1}{2}\int_a^t (t-\tau_1)^2 d\tau_1 {}_a\mathbf{I}_{\tau_1}^{n-3}[F] \\ &= \frac{1}{2}\int_a^t (t-\tau_1)^2 d\tau_1 {}_a\mathbf{I}_{\tau_1}^{n-3}[F] \\ &\vdots \\ &= \frac{1}{(n-1)!}\int_a^t (t-\tau_1)^{n-1} F(\tau_1) d\tau_1 \end{aligned} \tag{8}$$

$${}_a\mathbf{I}_t^n[F] = \frac{1}{\Gamma(n)}\int_a^t (t-\tau_1)^{n-1} F(\tau_1) d\tau_1 \tag{9}$$

C. Arbitrary order integration

Based on Cachy integration formula for repeated integrations one may extend the integer order integration to arbitrary order integration as follows,

$$ {}_a\mathbf{I}_t^{\alpha}[F] = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} F(\tau) d\tau \tag{10} $$

III. FRACTIONAL ORDER DIFFERENTATION

A. Reimann-Liouvile FOD

$$ {}_a^{\mathbf{RL}}\mathbf{D}_t^{\alpha}\mathbf{F}(t) = \frac{d^m}{dt^m}\, {}_a\mathbf{I}_t^{m-\alpha}[\mathbf{F}] = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_a^t (t-\tau)^{m-\alpha-1} \mathbf{F}(\tau) d\tau \tag{11} $$

where $m-1 \preceq \alpha \preceq m$.

B. Caputo FOD

$$ {}_a^{\mathbf{C}}\mathbf{D}_t^{\alpha}\mathbf{F}(t) = {}_a\mathbf{I}_t^{m-\alpha} \frac{d^m}{dt^m}[\mathbf{F}] = \frac{1}{\Gamma(m-\alpha)} \int_a^t (t-\tau)^{m-\alpha-1} \frac{d^m}{d\tau^m} \mathbf{F}(\tau) d\tau \tag{12} $$

where $m-1 \preceq \alpha \preceq m$.

C. Gronvald-Letnikov FOD and FOI

$$ {}_a^{\mathbf{GL}}\mathbf{D}_t^{\alpha}\mathbf{F}(t) = \lim_{h\to 0} \frac{(1-\hat{\mathbf{T}}_h)^{\alpha}}{h^{\alpha}} \mathbf{F}(t) \tag{13} $$

Using Tylor expansion of $(1-x)^{\alpha}$,

$$ \begin{aligned} (1-\hat{\mathbf{T}}_h)^{\alpha} &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \times \prod_{i=1}^{i=k} (\alpha - i + 1) \hat{\mathbf{T}}_h^n \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \underbrace{\Gamma(\alpha-n+1) \prod_{i=1}^{i=k} (\alpha-i+1)}_{\Gamma(\alpha+1)} \times \frac{1}{\Gamma(\alpha-n+1)} \hat{\mathbf{T}}_h^n \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-n+1)\Gamma(n+1)} \hat{\mathbf{T}}_h^n = \sum_{n=0}^{\infty} (-1)^n \binom{\alpha}{n} \hat{\mathbf{T}}_h^n \end{aligned} $$

Thus the GL FOD becomes,

$$ {}_a^{\mathbf{GL}}\mathbf{D}_t^{\alpha}\mathbf{F}(t) = \frac{1}{h^{\alpha}} \lim_{h\to 0} \sum_{n=0}^{N} (-1)^n \binom{\alpha}{n} \mathbf{F}(t-nh) \tag{14} $$

with $h = t/N$. One may define GL integration by just $\alpha \to -\alpha$,

$$
\begin{aligned}
(1-\hat{\mathbf{T}}_h)^{-\alpha} &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \times \prod_{i=1}^{i=k} (-\alpha - i + 1)\hat{\mathbf{T}}_h^n \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \underbrace{\Gamma(\alpha) \prod_{i=1}^{i=k} (\alpha + i - 1)}_{\Gamma(\alpha+n)} \times \frac{1}{\Gamma(\alpha)} \hat{\mathbf{T}}_h^n \\
&= \sum_{n=0}^{\infty} (-1)^n \times (-1)^n \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-n+1)\Gamma(n+1)} \hat{\mathbf{T}}_h^n \\
&= \sum_{n=0}^{\infty} (-1)^n \binom{\alpha+n-1}{n} \hat{\mathbf{T}}_h^n
\end{aligned}
$$

Thus GL FOI becomes,

$$
{}_a^{\mathbf{GL}}\mathbf{I}_t^{\alpha} = {}_a^{\mathbf{GL}}\mathbf{D}_t^{-\alpha}\mathbf{F}(t) = \lim_{h\to 0} h^{\alpha} \sum_{n=0}^{N} (-1)^n \binom{\alpha}{n} \mathbf{F}(t-nh) \tag{15}
$$

with $h = t/N$.

D. The relation between RL, Caputo and GL FOD

It is instructive to compare **Caputo** and **R**L, for $\mathbf{F}(t) = t^{\nu}$.
R*eimann-Liouvile*:

$$
\begin{aligned}
{}_a^{\mathbf{RL}}\mathbf{D}_t^{\alpha} t^{\nu} &= \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_a^t (t-\tau)^{m-\alpha-1} \tau^{\nu} d\tau \\
&= \frac{1}{\Gamma(m-\alpha)} \underbrace{\int_0^1 (1-x)^{m-\alpha-1} x^{\nu+1-1} dx}_{\mathbf{B}(m-\alpha,\nu+1)} \frac{d^m}{dt^m} t^{m+\nu-\alpha} \\
&= \frac{1}{\Gamma(m-\alpha)} \frac{\Gamma(m-\alpha)\Gamma(\nu+1)}{\Gamma(m-\alpha+v+1)} \frac{d^m}{dt^m} t^{m+\nu-\alpha} \\
&= \frac{\overbrace{\prod_{i=1}^{m} (i+\nu-\alpha)\Gamma(\nu-\alpha+1)}^{\Gamma(m-\alpha+v+1)} \Gamma(\nu+1)}{\Gamma(\nu-\alpha+1)\Gamma(m-\alpha+v+1)} t^{\nu-\alpha} \\
&= \frac{\Gamma(\nu+1)}{\Gamma(\nu-\alpha+1)} t^{\nu-\alpha}
\end{aligned} \tag{16}
$$

***C**aputo*:

$$
\begin{aligned}
{}_a^{\mathbf{C}}\mathbf{D}_t^{\alpha} t^{\nu} &= \frac{1}{\Gamma(m-\alpha)} \int_a^t (t-\tau)^{m-\alpha-1} \frac{d^m}{d\tau^m} \tau^{\nu} d\tau \;\; \Rightarrow \frac{\mathbf{d^m}}{\mathbf{d\tau^m}} \tau^{\nu} = \mathbf{0}, \;\textbf{if}\; \nu \in \mathbf{N}, \;\textbf{and}\; \nu < \mathbf{m} \\
&= \frac{1}{\Gamma(m-\alpha)} \int_a^t (t-\tau)^{m-\alpha-1} \tau^{\nu-m} d\tau \times \prod_{i=0}^{m-1} (\nu - i) \\
&= \frac{1}{\Gamma(m-\alpha)} \underbrace{\int_0^1 (1-x)^{m-\alpha-1} x^{\nu-m+1-1} dx}_{\mathbf{B}(m-\alpha,\nu-m+1)} \times \prod_{i=0}^{m-1} (\nu - i) \times t^{\nu-\alpha} \\
&= \frac{1}{\Gamma(m-\alpha)} \frac{\Gamma(m-\alpha)\overbrace{\Gamma(\nu-m+1)\prod_{i=0}^{m-1}(\nu-i)}^{\Gamma(\alpha+1)}}{\Gamma(\nu-\alpha+1)} \times t^{\nu-\alpha} \\
&= \frac{\Gamma(\nu+1)}{\Gamma(\nu-\alpha+1)} t^{\nu-\alpha} && (17) \\
& && (18)
\end{aligned}
$$

Therefore for **Caputo** FOD we have,

$$
{}_a^{\mathbf{C}}\mathbf{D}_t^{\alpha} t^{\nu} = \begin{cases} 0 & \text{if } \nu \in N \text{ and } \nu < m \\ \frac{\Gamma(\nu+1)}{\Gamma(\nu-\alpha+1)} t^{\nu-\alpha} & \text{otherwise} \end{cases} \tag{19}
$$

with $m - 1 < \nu < m$.

$$
\begin{aligned}
{}_a^{\mathbf{C}}\mathbf{D}_t^{\alpha} \mathbf{F}(t) &= \sum_{\nu=0}^{\infty} \frac{\mathbf{F}^{(\nu)}(0)}{\nu!} \times {}_a^{\mathbf{C}}\mathbf{D}_t^{\alpha} t^{\nu} \\
&= \sum_{\nu=m}^{\infty} \frac{\mathbf{F}^{(\nu)}(0)}{\nu!} \times \frac{\Gamma(\nu+1)}{\Gamma(\nu-\alpha+1)} t^{\nu-\alpha} \\
&= \sum_{\nu=m}^{\infty} \frac{\mathbf{F}^{(\nu)}(0)}{\Gamma(\nu-\alpha+1)} t^{\nu-\alpha}
\end{aligned} \tag{20}
$$

and for **RL** we have,

$$
{}_a^{\mathbf{RL}}\mathbf{D}_t^{\alpha} \mathbf{F}(t) = \sum_{\nu=0}^{\infty} \frac{\mathbf{F}^{(\nu)}(0)}{\Gamma(\nu-\alpha+1)} t^{\nu-\alpha} \tag{21}
$$

Finally we have,

$$
{}_a^{\mathbf{RL}}\mathbf{D}_t^{\alpha} \mathbf{F}(t) - {}_a^{\mathbf{C}}\mathbf{D}_t^{\alpha} \mathbf{F}(t) = \sum_{\nu=0}^{m-1} \frac{\mathbf{F}^{(\nu)}(0)}{\Gamma(\nu-\alpha+1)} t^{\nu-\alpha} \tag{22}
$$

Taking fractional integration on both sides of Eq. 22,

$$
\begin{aligned}
{}_a\mathbf{I}_t^{\alpha} \left[{}_a^{\mathbf{RL}}\mathbf{D}_t^{\alpha} \mathbf{F}(t) - {}_a^{\mathbf{C}}\mathbf{D}_t^{\alpha} \mathbf{F}(t) \right] &= \sum_{\nu=0}^{m-1} \frac{\mathbf{F}^{(\nu)}(0)}{\Gamma(\nu-\alpha+1)} {}_a\mathbf{I}_t{}^{\alpha} [t^{\nu-\alpha}] \\
&= \sum_{\nu=0}^{m-1} \frac{\mathbf{F}^{(\nu)}(0)}{\Gamma(\nu-\alpha+1)} \frac{\Gamma(\nu-\alpha+1)}{\Gamma(\nu+1)} t^{\nu} \\
&= \sum_{\nu=0}^{m-1} \frac{\mathbf{F}^{(\nu)}(0)}{\Gamma(\nu+1)} t^{\nu}
\end{aligned} \tag{23}
$$

IV. FOURIER AND LAPLACE TRANSFORMATIONS

$$
\begin{aligned}
\mathscr{L}\{\mathbf{F}\} &= \int_0^\infty {}_a^{\mathbf{RL}}\mathbf{D}_t^\alpha \mathbf{F}(t) e^{-st} dt \\
&= \frac{1}{\Gamma(m-\alpha)} \int_0^\infty e^{-st} dt \overbrace{\frac{d^m}{dt^m} \int_a^t (t-\tau)^{m-\alpha-1} \mathbf{F}(\tau) d\tau}^{\mathbf{U}^{(n)(t)}} \\
&= \frac{1}{\Gamma(m-\alpha)} \int_0^\infty \underbrace{e^{-st}}_{\nu} \underbrace{\mathbf{U}^{(n)}(t) dt}_{du} \\
&= \frac{1}{\Gamma(m-\alpha)} \int_0^\infty \underbrace{e^{-st}}_{\nu} \underbrace{\mathbf{U}^{(n)}(t) dt}_{du}
\end{aligned}
\tag{24}
$$

V. NUMERICAL METHODS OF SOLUTIONS OF ORDINARY DIFFERENTIAL EQUATIONS

A. Multi linear methods for the solutions of ODEs

B. Runge-Cutta

VI. NUMERICAL METHODS FOR SOLVING FRACTIONAL ORDER DIFFERENTIAL EQUATIONS(FODE)

A. Multi linear methods for the solutions of FODEs

B. Runge-Cutta

VII. HIGHER ORDER FINITE DIFFERENCE FOD

A. Higher order finite difference for integer order derivatives

B. Higher order finite difference for FOD

VIII. NUMERICAL IMPLEMENTATION

IX. TRANSIENT SIMULATION OF ELECTRICAL CIRCUITS(PASSIVE PARTS ONLY)

A. Simulation of parts without FO components

B. Simulation of parts with FO components

C. Implementation of transient algorithem

D. Inclusion of SPICE net lists

X. SOME APPLICATIONS OF FOC IN THE ELECTROCHEMISTRY

XI. POSSIBLE INTEGRATION INTO QUCS

ACKNOWLEDGMENTS

XII. APPENDIX