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The use of He's variational iteration method for solving the telegraph and fractional telegraph equations

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SUMMARY

In this paper the variational iteration method is used to compute the solution for the linear, variable coefficient, fractional derivative and multi space telegraph equations. The method constructs a convergent sequence of functions, to approximate the exact solution with a few number of iterations without discretization. Numerical results and comparison with exact solutions are given for some examples in order to show its applicability and efficiency. Copyright © 2009 John Wiley & Sons, Ltd.

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KEY WORDS: telegraph equation; variational iteration method; semi-analytical solution

1. INTRODUCTION

Telegraph equations are hyperbolic partial differential equations that are applicable in several fields such as wave propagation [1], signal analysis [2], random walk theory [3], etc. In this paper we consider four different types of linear telegraph equations.

The first one is the classical linear telegraph equation:

$$\begin{gathered} u_{tt}+\alpha u_t+\beta u = u_{xx}+f(x,t) \\ u(x,0)=\phi, \quad u_t(x,0)=\psi \end{gathered} \tag{1}$$

where u can be considered as a function depending on distance (x) and time (t), α and β are constants depending on given problem and f, ϕ, ψ are known continuous functions. This type of telegraph equation applies to high-frequency transmission lines such as telegraph wires and radio frequency conductors. It is also applicable for designing high voltage transmission lines. Several numerical methods exist for solving this equation. Authors of [4] used an analytical approach to solve it.

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The second equation is the following linear telegraph equation with variable coefficients [5]:

$$
\begin{aligned}
r(x)u_{tt}(x,t) &= [p(x)u_x(x,t)]_x - q(x)u(x,t) + F(x,t), \quad 0\leqslant x\leqslant 1\\
au_x(0,t)+bu(0,t) &= 0, \quad t>0\\
cu_x(1,t)+du(1,t) &= 0, \quad t>0\\
u(x,0) &= f(x), \quad 0\leqslant x\leqslant 1\\
u_t(x,0) &= g(x), \quad 0\leqslant x\leqslant 1
\end{aligned}
\tag{2}
$$

where $|a|+|b|>0$, $|c|+|d|>0$, $r(x)>0$, $p(x)>0$ and p, r, q, f, g, F are continuous functions. This equation is solved in [5] by means of discrete eigenfunction method.

The third equation that we are going to solve numerically is the telegraph equation with fractional derivatives:

$$
\begin{aligned}
D_x^{\alpha}u(x,t) &= u_{tt}(x,t)+u_t(x,t)+u(x,t)+f(x,t), \quad t\geqslant 0, \quad 1<\alpha\leqslant 2\\
u(0,t) &= l_1(t), \quad t\geqslant 0\\
u_x(0,t) &= l_2(t), \quad t\geqslant 0\\
u(x,0) &= L(x), \quad 0<x<1
\end{aligned}
\tag{3}
$$

where α is the parameter of order of fractional derivative and D_x^{α} is the Caputo fractional differential operator defined as

$$
D_x^{\alpha}u(x,t) = \begin{cases} \dfrac{1}{\Gamma(m-\alpha)}\displaystyle\int_0^x (x-\theta)^{m-\alpha-1}\frac{\partial^m}{\partial\theta^m}u(\theta,t)\,\mathrm{d}\theta, & m-1<\alpha\leqslant m\\ \dfrac{\partial^m}{\partial x^m}u(x,t), & \alpha=m\in N \end{cases}
$$

This fractional type of telegraph equation is solved in [6, 7] by the Adomian decomposition method. Applications of this equation are pointed out in [6] and the references therein.

Finally, we discuss on the multi-dimensional telegraph equations such as the three-space-dimensional equation with variable coefficients [8, 9]. This equation can be shown by

$$
\begin{aligned}
&u_{tt}(x,y,z,t)+p(x,y,z,t)u_t+q(x,y,z,t)u(x,y,z,t)\\
&\quad = A(x,y,z,t)u_{xx}(x,y,z,t)+B(x,y,z,t)u_{yy}(x,y,z,t)\\
&\qquad +C(x,y,z,t)u_{zz}(x,y,z,t)+f(x,y,z,t), \quad 0<x,y,z<1, \quad t>0
\end{aligned}
\tag{4}
$$

with initial conditions,

$$
\begin{aligned}
u(x,y,z,0) &= \phi(x,y,z), \quad 0\leqslant x,y,z\leqslant 1\\
u_t(x,y,z,0) &= \psi(x,y,z), \quad 0\leqslant x,y,z\leqslant 1
\end{aligned}
$$

and boundary conditions,

$$
\begin{aligned}
u(x,y,z,t) &= g(x,y,z,t), \quad (x,y,z,t)\in\partial\Omega\times[0,T]\\
\Omega &= \{(x,y,z)|0\leqslant x,y,z\leqslant 1\}
\end{aligned}
$$

where A, B, C, p and q are in $\Omega\times[0,T]$ and ϕ, ψ and their partial derivatives up to order two are continuous and T is a positive integer. The two-dimensional version of this equation is solved numerically in [9]. It is worth pointing out that authors of [10] used the Adomian decomposition method [11] to obtain solutions of the fourth-order fractional diffusion-wave equation that is defined in a bounded space domain. The one-dimensional telegraph equation is investigated in [12].

A numerical technique is presented [12] for the solution of this equation that uses the Chebyshev cardinal functions. The method consists of expanding the required approximate solution as the elements of the Chebyshev cardinal functions. Using the operational matrix of derivative, the problem is reduced to a set of algebraic equations [12]. Authors of [13] developed a numerical scheme to solve this hyperbolic telegraph equation using collocation points and approximating directly the solution using the thin plate splines radial basis function. The scheme works in a similar fashion as finite difference methods. Authors of [14] combined a high-order compact finite difference scheme to approximate the spatial derivative and collocation technique for the time component to numerically solve the one-space-dimensional linear hyperbolic telegraph equation. A numerical technique is presented in [15] to solve this equation. This method is based on the shifted Chebyshev tau method. This method consists of expanding the required approximate solution as the elements of shifted Chebyshev polynomials. Using the operational matrices of integral and derivative, the problem can be reduced [15] to a set of linear algebraic equations. The two-dimensional version of the telegraph equation is investigated in [9] and a high-order accurate method is proposed for solving it. The proposed technique is based on a compact finite difference approximation of fourth order for discretizing spatial derivatives and collocation method for the time component. The resulted method is unconditionally stable and solves the two-dimensional linear telegraph equation with high accuracy. In Dehghan and Mohebbi's approach [9], the solution is approximated by a polynomial at each grid point whose coefficients are determined by solving a linear system of equations.

In the current investigation, the approach is different as we use the variational iteration method to solve four kinds of telegraph equations. In this paper we are going to find the solution of the telegraph equations by using this method. With demonstrating examples we will see that by few iterations we get approximate solutions that are very accurate in comparison with exact or other solutions, which are achieved by other methods, and it shows the efficiency of this method.

The article is organized as follows: In Section 2 we describe the variational iteration method. In Section 3 we use the method for Problems (1)–(4) in order to construct an iteration formula. In Section 4 we demonstrate the accuracy of method by considering numerical examples. Finally Section 5 concludes this article with a brief summary.

2. VARIATIONAL ITERATION METHOD

The variational iteration method is a method for solving linear and nonlinear problems. This method is introduced by the Chinese researcher He [16] by modifying the general Lagrange multiplier method [17]. The method constructs an iterative sequence of functions converging to exact solution. In the case of linear problems by determining exact Lagrange multiplier, approximate solution turns into exact solution and is available by only one iteration. The method works without discretization of problem and round off error does not affect it. The method is used in several well-known problems successfully such as delay differential equations [18], differential equations with fractional derivatives [19–21], autonomous differential equations [22], Helmholtz equations [23], Burgers and coupled Burgers equations [24], non-linear wave and diffusion equations [25], parabolic integro-differential equations arising in heat conduction in materials with memory [26] and some other problems such as [16, 21, 27, 28]. In [29] a convergence criterion based on Banach fixed point is given for this method. The variational iteration method is used in [30] for solving the Lane–Emden equation, in [31] to solve several problems in calculus of variations, in [32] to find the solution of an inverse problem in partial differential equations, in [33] to solve a system of nonlinear integro-differential equations arising in biology, in [34] to solve the Kline–Gordon equation, in [35] to find solution of the Fokker–Planck equation, in [36] to solve the Cauchy-reaction–diffusion partial differential equation, in [37] to solve the biological population equation. In addition the main advantage of the new approach in comparison with the mesh point techniques [38] is highlighted in these works. Wazwaz [39] employed this method to solve the nonlinear Goursat problem. In [40] the variational iteration method is compared with

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the Adomian decomposition method for homogeneous and nonhomogeneous advection problems. Author of [41] used this scheme to determine rational solutions for the KdV, $k(2,2)$, Burgers and cubic Boussinesq equations. Abbasbandy and Shirzadi [42] employed the variational iteration method to solve some eighth-order boundary-value problems. Some special cases of the mentioned equations that govern scientific and engineering experimentations are tested. In addition the initial approximation was selected as a polynomial with unknown constants, which was determined by considering the boundary conditions. Nonlinear Volterra's integro-differential equations are solved in [43] using the variational iteration technique. The variational iteration method is applied to find the solution of the wave equation subject to an integral conservation condition [44]. Authors of [45] modified the variational iteration method to solve a system of differential equations. The variation iteration technique is used in [46] to solve the generalized pantograph equation [47]. This scheme is employed in [48] to solve several variational problems.

Consider the following general nonlinear problem:

$$L(u(t))+N(u(t))=g(t) \tag{5}$$

where L is a linear operator, N is a nonlinear operator and $g(t)$ is a known analytical function. Variational iteration method constructs an iterative sequence called correction functional as

$$u_{n+1}(t)=u_n(t)+\int_0^t \lambda(L(u_n(s))+N(\widetilde{u}_n(s))-g(s))\,\mathrm{d}s \tag{6}$$

where λ is general Lagrange multiplier, which can be identified optimally via the variational theory [16, 24], $\widetilde{u}_n(s)$ is considered as restricted variation [16, 49], i.e. $\delta\widetilde{u}_n=0$, and the index n denotes the nth iteration.

3. ANALYSIS OF TELEGRAPH EQUATIONS

Considering Equation (1) with respect to Equation (6), we construct an iteration formula in t-direction, which is as follows:

$$u_{n+1}(x,t)=u_n(x,t)+\int_0^t \lambda(s)((u_n(x,s))_{ss}+\alpha(u_n(x,s))_s+\beta\widetilde{u_n(x,s)}-(\widetilde{u_n(x,s)})_{xx}-f(x,s))\,\mathrm{d}s$$

We now determine the Lagrange multiplier.

$$\begin{aligned}\delta u_{n+1}(x,t) &= \delta u_n(x,t)+\delta\int_0^t \lambda(s)((u_n(x,s))_{ss}+\alpha(u_n(x,s))_s+\beta\widetilde{u_n(x,s)}\\ &\quad -((\widetilde{u_n(x,s)})_{xx})-f(x,s))\,\mathrm{d}s=0\\ \delta u_{n+1} &= \delta u_n+\alpha\lambda(s)\delta u_n|_{s=t}+\lambda(s)\delta((u_n)_s)|_{s=t}-\lambda'(s)\delta(u_n)|_{s=t}\\ &\quad+\int_0^t(\lambda''(s)-\alpha\lambda'(s)+\beta\lambda(s))\delta u_n\,\mathrm{d}s=0\end{aligned}$$

$$\begin{aligned}\delta u_n &: (1-\lambda'(s)+\alpha\lambda(s))|_{s=t}=0\\ \delta((u_n)_s) &: \lambda(s)|_{s=t}=0\\ \delta u_n &: (\lambda''(s)-\alpha\lambda'(s))=0\end{aligned}$$

So we have

$$\lambda(s)=\frac{1}{\alpha}(\mathrm{e}^{\alpha(s-t)}-1)$$

and we obtain the following iteration formula:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \frac{1}{\alpha}(\mathrm{e}^{\alpha(s-t)}-1)((u_n(x,s))_{ss} + \alpha(u_n(x,s))_s + \beta u_n(x,s) - ((u_n(x,s))_{xx} - f(x,s))\,\mathrm{d}s \tag{7}$$

with initial approximation $u_0(x,t)$ such that $u_0(x,0)=\phi$ and $u_{0t}(x,0)=\psi$, which guarantees satisfaction of boundary conditions in each iteration. Considering Equation (2) we construct an iteration sequence the same as the above:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(s)(r(x)u_{ss}(x,s) - ([p(x)\widetilde{u_x(x,s)}]_x) + q(x)\widetilde{u(x,s)} - F(x,s))\,\mathrm{d}s$$

By taking variation with respect to u_n, we determine the Lagrange multiplier.

$$\delta u_{n+1}(x,t) = \delta u_n(x,t) + \delta\int_0^t \lambda(s)(r(x)u_{ss}(x,s) - ([p(x)\widetilde{u_x(x,s)}]_x) + q(x)\widetilde{u(x,s)} - F(x,s))\,\mathrm{d}s = 0$$

$$\delta u_n : 1 - r(x)\lambda'(s)|_{s=t} = 0$$

$$\delta((u_n)_s) : r(x)\lambda(s)|_{s=t} = 0$$

$$\delta u_n : r(x)\lambda''(s) = 0$$

So we have

$$\lambda(s) = \frac{1}{r(x)}(s-t)$$

and we get the iteration formula:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \frac{1}{r(x)}(s-t)(r(x)u_{ss}(x,s) - ([p(x)\widetilde{u_x(x,s)}]_x) + q(x)u(x,s) - F(x,s))\,\mathrm{d}s \tag{8}$$

We start the iteration formula with initial approximation

$$u_0(x,t) = f(x) + tg(x)$$

For Equation (3) we construct the following iteration formula:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^x \lambda(s)(D_s^\alpha u(s,t) - \widetilde{u_{tt}(s,t)} - \widetilde{u_t(s,t)} - \widetilde{u(s,t)} - f(s,t))\,\mathrm{d}s$$

By taking variation with respect to u_n when $\alpha=2$, we determine the Lagrange multiplier approximately.

$$\delta u_{n+1}(x,t) = \delta u_n(x,t) + \delta\int_0^x \lambda(s)(D_s^\alpha u(s,t) - \widetilde{u_{tt}(s,t)} - \widetilde{u_t(s,t)} - \widetilde{u(s,t)} - f(s,t))\,\mathrm{d}s = 0$$

Then we have

$$\lambda(s) = (s-t)u_{n+1}(x,t) = u_n(x,t) + \int_0^x (s-t)(D_s^\alpha u(s,t) - u_{tt}(s,t) - u_t(s,t) - u(s,t) - f(s,t))\,\mathrm{d}s \tag{9}$$

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with initial approximation $u_0(x,t)=l_1(t)+xl_2(t)$. It is obvious that by setting $\alpha=2$ the fractional telegraph equation reduces to the classical form equation (1), which is discussed above. Considering Equation (4) we construct the iteration formula for it. Thus we get

$$\begin{aligned} u_{n+1}(x,y,z,t) = u_n(x,y,z,t) + \int_0^t & \lambda(s)(u_{ss}(x,y,z,s)+p(x,y,z,s)u_s(x,y,z,s) \\ & + q(x,y,z,s)u(x,y,z,s) - A(x,y,z,s)u_{xx}(x,y,z,s) - B(x,y,z,s)u_{yy}(x,y,z,s) \\ & - C(x,y,z,s)u_{zz}(x,y,z,s) - f(x,y,z,s))\,\mathrm{d}s \end{aligned}$$

By taking variation with respect to u_n we determine the Lagrange multiplier.

$$\begin{aligned} \delta u_{n+1}(x,y,z,t) = \delta u_n(x,y,z,t) + \delta \int_0^t & \lambda(s)(u_{ss}(x,y,z,s)+p(x,y,z,s)u_s(x,y,z,s) \\ & + q(x,y,z,s)u\widetilde{(x,y,z},s) - A(x,y,z,s)(\widetilde{u_{xx}})(x,y,z,s) \\ & - B(x,y,z,s)(\widetilde{u_{yy}})(x,y,z,s) \\ & - C(x,y,z,s)(\widetilde{u_{zz}})(x,y,z,s) - f(x,y,z,s))\,\mathrm{d}s = 0 \end{aligned} \tag{10}$$

$$\begin{aligned} \delta u_n(s) : 1 + p\lambda(s) - \lambda'(s)|_{s=t} &= 0 \\ \delta((u_n)_s) : \lambda(s)|_{s=t} &= 0 \\ \delta u_n(s) : [\lambda''(s) - p(x,y,z,s)\lambda'(s) &= 0 \end{aligned} \tag{11}$$

By solving the above equations in each case we determine the Lagrange multiplier.

4. ILLUSTRATIVE EXAMPLES

We applied the method presented in this paper and solved five test problems to show the efficiency of the new approach.

4.1. Example 1

Let

$$\alpha=\beta=1, \quad f(x,t)=0, \quad \phi=\mathrm{e}^x, \quad \psi=-\mathrm{e}^x, \quad 0\leqslant x\leqslant 4$$

In Equation (1) then we have

$$\begin{aligned} u_{tt}+u_t+u &= u_{xx} \\ u(x,0)=\mathrm{e}^x, &\quad u_t(x,0)=-\mathrm{e}^x \end{aligned}$$

Now we have the iteration formula from (7) as

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t (\mathrm{e}^{(s-t)}-1)((u_n(x,s))_{ss} + (u_n(x,s))_s + u_n(x,s) - ((u_n(x,s))_{xx}))\,\mathrm{d}s$$

$$u_0(x,t) = (1-t)\mathrm{e}^x$$

$$u_1(x,t) = (1-t)\mathrm{e}^x + \int_0^t (\mathrm{e}^{(s-t)}-1)(0-\mathrm{e}^x+(1-s)\mathrm{e}^x-(1-s)\mathrm{e}^x)\,\mathrm{d}s = \mathrm{e}^{(x-t)}$$

which is the exact solution of the given equation.

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4.2. *Example 2*

Let

$$\alpha = 6, \quad \beta = 2, \quad f(x,t) = (2-\alpha+\beta)\,\mathrm{e}^{-t}\sin(x)$$

$$\phi = \sin(x), \quad \psi = -\sin(x), \quad 0 \leqslant x \leqslant \pi$$

In Equation (1) then we have

$$u_{tt} + 6u_t + 2u = u_{xx} - 2\mathrm{e}^{-t}\sin(x)$$

$$u(x,0) = \sin(x), \quad u_t(x,0) = -\sin(x)$$

In addition we have the iteration formula from (6) that can be written as

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \frac{1}{6}(\mathrm{e}^{6(s-t)} - 1)((u_n(x,s))_{ss} + 6(u_n(x,s))_s + 2u_n(x,s)$$
$$-((u_n(x,s))_{xx}) + 2\mathrm{e}^{-s}\sin(x))\,\mathrm{d}s$$

$$u_0(x,t) = (1-t)\sin(x)$$

$$u_1(x,t) = (1-t)\sin(x) + \frac{1}{360}(\mathrm{e}^{-6t} + 144\mathrm{e}^{-t} + 5(-29 + 6t(5+3t)))\sin(x)$$

and so on. The exact solution $u(x,t) = \mathrm{e}^{-t}\sin(x)$ and also $u_1(x,t)$ are plotted in Figure 1.

4.3. *Example 3*

Considering Equation (2) with

$$a = 1, \; b = -1, \; c = 1, \; d = -1, \quad f(x) = g(x) = \mathrm{e}^x, \quad r(x) = 1$$

$$p(x) = x^2, \quad q(x) = x, \quad F(x,t) = (1 - x - x^2)\mathrm{e}^{x+t}$$

Then we have

$$u_{tt} - x^2 u_{xx} - 2x u_x + x u = (1 - x - x^2)\mathrm{e}^{t+x}$$

and using (8) we get the following iteration formula:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t (s-t)((u_n(x,s))_{ss} - x^2(u_n(x,s))_{xx}$$
$$-2x(u_n(x,s))_x + x u_n(x,s) - (1 - x - x^2)\mathrm{e}^{t+x})\,\mathrm{d}s$$

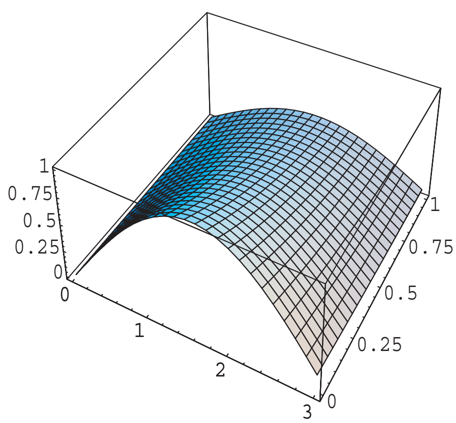

Figure 1. Exact and approximate solutions.

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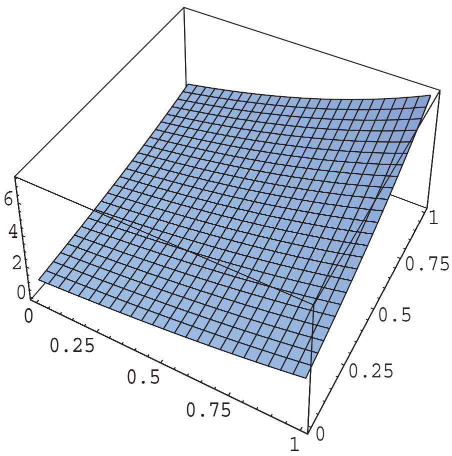

Figure 2. Exact and approximate solutions.

We start with

$$u_0(x,t)=e^x(1+t)$$

and by the above iteration formula we will have

$$\begin{aligned}
u_1(x,t) &= e^x(1+t)+(\tfrac{1}{6})e^x(t^2x(1+x)-6e^t(-1+x+x^20)+6(t+1)(-1+x+x^2))\\
u_2(x,t) &= u_1(x,t)+1/120e^x x(-120e^t(1+x(2+x)(4+x))+120(1+t)(1+x(2+x)(4+x))\\
&\quad +t^2(5t^2(2+x)(1+x(4+x))+t^3(2+x)(1+x(4+x))\\
&\quad +60(1+x(2+x)(4+x))+20t(1+x(2+x)(4+x))))
\end{aligned}$$

and so on. In Figure 2, the exact solution $u(x,t)=e^{(x+t)}$ and the approximate solution $u_2(x,t)$ are plotted.

4.4. Example 4

Consider Equation (3) with

$$\alpha=3/2,\quad l_1(t)=l_2(t)=e^{-t},\ L(x)=e^x,\ f(x,t)=0$$

Then we have

$$\begin{aligned}
D_x^\alpha u(x,t) &= u_{tt}(x,t)+u_t(x,t)+u(x,t)\\
u(0,t) &= e^{-t},\quad t\geqslant 0\\
u_x(0,t) &= e^{-t},\quad t\geqslant 0\\
u(x,0) &= e^x,\quad 0<x<1
\end{aligned}$$

With this assumption and using (9) the following iteration formula will be obtained:

$$\begin{aligned}
u_{n+1}(x,t) &= u_n(x,t)+\int_0^x (s-x)(D_s^\alpha u(s,t)-u_{tt}(x,t)-u_t(x,t)-u(x,t))\,ds\\
u_0(x,t) &= e^{-t}(1+x)
\end{aligned}$$

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$$u_1(x,t)=\mathrm{e}^{-t}(1+x)+1/6\mathrm{e}^{-t}x^2(3+x)$$

$$u_2(x,t)=\mathrm{e}^{-t}(1+x)+1/6\mathrm{e}^{-t}x^2(3+x)+(\mathrm{e}^{-t}x^2(-64\sqrt{x}(7+2x)$$
$$+7\sqrt{\pi}(60+20x+5x^2+x^3))/(840\sqrt{\pi})$$

and so on. The approximate solution $u_3(x,t)$ and the solution obtained by the Adomian decomposition method proposed in [6]

$$u(x,t)=\mathrm{e}^{-t}(1+x)+\mathrm{e}^{-t}((4x^{3/2})/(3\sqrt{\pi})+(8x^{5/2})/(15\sqrt{\pi})+x^3/6+x^4/24)$$

are plotted in Figure 3.

4.5. Example 5

Consider Equation (4) with

$$p,q=1,\ A=x,\ B=y,\ C=z,\quad f(x,y,z,t)=\mathrm{e}^{(x+y+z+t)}(3-x-y-z)$$
$$\phi(x,y,z)=\mathrm{e}^{(x+y+z)},\quad \psi(x,y,z)=\mathrm{e}^{(x+y+z)},\quad T=1$$
$$u_{tt}+u_t+u=xu_{xx}+yu_{yy}+zu_{zz}+\mathrm{e}^{(x+y+z+t)}(3-x-y-z)$$
$$u(x,y,z,0)=\mathrm{e}^{(x+y+z)},\quad u_t(x,y,z,0)=\mathrm{e}^{(x+y+z)}$$
$$u(0,y,z,t)=\mathrm{e}^{(y+z)},\quad u(x,0,z,t)=\mathrm{e}^{(x+z)},\quad u(x,y,0,t)=\mathrm{e}^{(x+y)}$$

By using Equations (10)–(11), we get $\lambda(s)=\mathrm{e}^{(s-t)}-1$ and we have the iteration formula:

$$u_{n+1}(x,y,z,t)=u_n(x,y,z,t)+\int_0^t(\mathrm{e}^{(s-t)}-1)(u_{ss}(x,y,z,s)$$
$$+u_s(x,y,z,s)+u(x,y,z,s)-xu_{xx}(x,y,z,s)$$
$$-yu_{yy}(x,y,z,s)-zu_{zz}(x,y,z,s)-\mathrm{e}^{(x+y+z+s)}(3-x-y-z))\,\mathrm{d}s$$

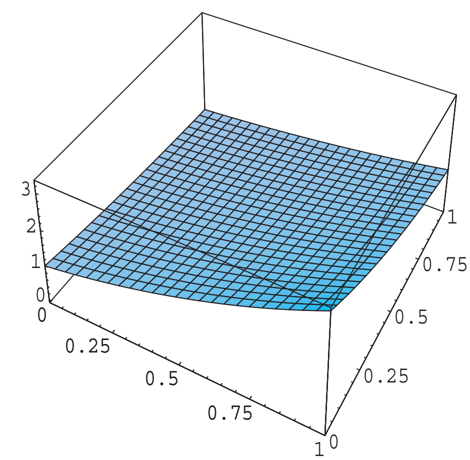

Figure 3. Exact and approximate solutions.

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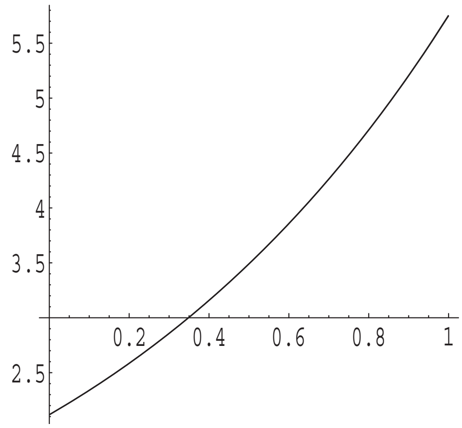

Figure 4. Exact and approximate solutions for $x=y=z=0.25$ and $0\leqslant t\leqslant 1$.

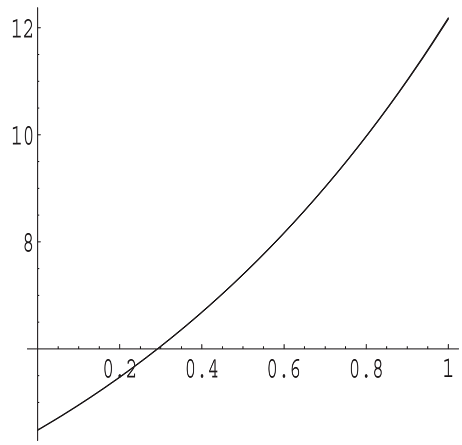

Figure 5. Exact and approximate solutions for $x=y=z=0.5$ and $0\leqslant t\leqslant 1$.

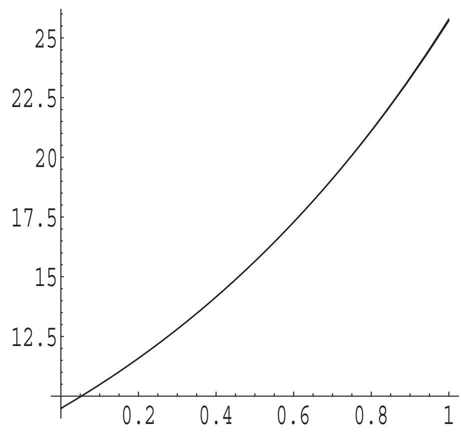

Figure 6. Exact and approximate solutions for $x=y=z=0.75$ and $0\leqslant t\leqslant 1$.

We start with initial approximation

$$u_0(x,y,z,t)=(1+t)\mathrm{e}^{(x+y+z)}$$

The approximate solution $u_2(x,y,z,t)$ and the exact solution $u(x,y,z,t)=\mathrm{e}^{(x+y+z+t)}$ for various values of x, y and z are plotted in Figures 4–7.

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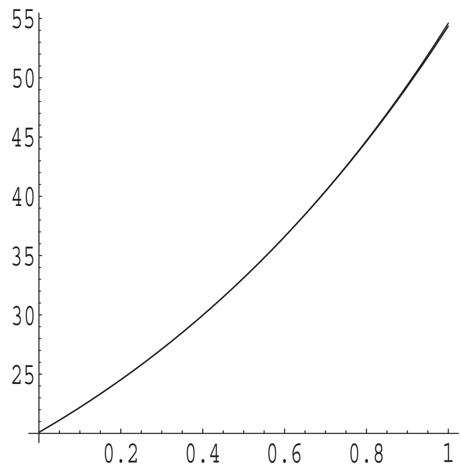

Figure 7. Exact and approximate solutions for $x=y=z=1$ and $0 \leqslant t \leqslant 1$.

5. CONCLUSION

In this paper the variational iteration method was used for solving the linear, variable coefficient, fractional derivative telegraph equations and also for the three-space-dimensional telegraph equation. We described the method and used it in some test examples in order to show its applicability and validity in comparison with exact and other numerical solutions. We achieved accurate approximations by using only a few number of iterations, which reveals efficiency of the new method.

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