Applied Mathematical Modelling 37 (2013) 5927–5937

Contents lists available at SciVerse ScienceDirect

Applied Mathematical Modelling

journal homepage: www.elsevier.com/locate/apm

Numerical approach for solving fractional relaxation–oscillation equation

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ARTICLE INFO

Article history:
Received 7 June 2012
Received in revised form 30 October 2012
Accepted 18 December 2012
Available online 28 December 2012

Keywords:
Fractional relaxation-oscillation equation
Fractional differential equation
Generalized Taylor series
Generalized Taylor matrix method
Numerical approximation

ABSTRACT

In this study, we will obtain the approximate solutions of relaxation–oscillation equation by developing the Taylor matrix method. A relaxation oscillator is a kind of oscillator based on a behavior of physical system's return to equilibrium after being disturbed. The relaxation–oscillation equation is the primary equation of relaxation and oscillation processes. The relaxation–oscillation equation is a fractional differential equation with initial conditions. For this propose, generalized Taylor matrix method is introduced. This method is based on first taking the truncated fractional Taylor expansions of the functions in the relaxation–oscillation equation and then substituting their matrix forms into the equation. Hence, the result matrix equation can be solved and the unknown fractional Taylor coefficients can be found approximately. The reliability and efficiency of the proposed approach are demonstrated in the numerical examples with aid of symbolic algebra program, Maple.

1. Introduction

The concept of fractional or non-integer order derivation and integration can be traced back to the genesis of integer order calculus itself [1]. Almost most of the mathematical theory applicable to the study of non-integer order calculus was developed through the end of 19th century. However it is in the past hundred years that the most intriguing leaps in engineering and scientific application have been found. The fractional differential equations (FDEs) have received considerable interest in recent years. FDEs have shown to be adequate models for various physical phenomena in areas like damping laws, diffusion processes, etc. For example, the nonlinear oscillation of earthquake can be modeled with fractional derivatives [2], the fluid-dynamic traffic model with fractional derivatives [3], psychology [4], modeling of viscoelastic dampers [5–7], self-similar protein dynamics [8], bioengineering [9], viscoelastically damped structures[10] and others [1,11–16]. Most FDEs do not have analytic solutions, so we need approximate approach. Solution techniques for FDEs have been studied extensively by many researchers such as collocation method [17–19], Adomian decomposition method [20,21], operational matrix method [22–24], variational iteration method [25], tau method [26]. A relaxation oscillator is a kind of oscillator based on a behavior of physical system's return to equilibrium after being disturbed [27,28]. There are many relaxation–oscillation models such as positive fractional derivative, fractal derivative, and fractional derivative [27–31]. The relaxation–oscillation equation is the primary equation of relaxation and oscillation processes.

The standard relaxation equation is

$$\frac{dy}{dx} + By = f(x), \tag{1}$$

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0307-904X/$ - see front matter
http://dx.doi.org/10.1016/j.apm.2012.12.015

where B denotes Elc, E is the elastic modulus, $f(x)$ denotes E multiplying the strain rate. When $f(x) = 0$, we have the analytic solution

$$y(x) = Ce^{-Bx}, \tag{2}$$

where C is a constant determined by the initial condition.

The standard oscillation equation

$$\frac{dy^2}{dx^2} + By = f(x), \tag{3}$$

where β equals $k/m = w^2$, k is the stiffness coefficient, m the mass, ω the angular frequency. When $f(x) = 0$, we have the analytic solution

$$y(x) = C\cos\sqrt{B}x + D\sin\sqrt{B}x, \tag{4}$$

where C and D are constants determined by the initial conditions.

The fractional derivatives are employed in the relaxation and oscillation models to represent slow relaxation and damped oscillation [28,29].

Fractional relaxation–oscillation model can be depicted as

$$D_x^{\beta}y(x) + Ay(x) = f(x), \quad x > 0, \tag{5}$$

$$y(0) = a \quad \text{if } 0 < \beta \leqslant 1, \tag{6}$$

or

$$y(0) = \lambda \text{ and } y'(0) = \mu \quad \text{if } (1 < \beta \leqslant 2, \tag{7}$$

where A is a positive constant. For $0 < \beta \leqslant 2$ this equation is called the fractional relaxation–oscillation equation. When $0 < \beta < 1$, the model describes the relaxation with the power law attenuation. When $1 < \beta < 2$, the model depicts the damped oscillation with viscoelastic intrinsic damping of oscillator [32,33].

This model has been applied in electrical model of the heart, signal processing, modeling cardiac pacemakers, predator–prey system, spruce–budworm interactions etc. [32–37].

We use the generalized Taylor matrix method (power of fractional number) instead of the standard Taylor matrix method (power of positive integer). Because, if exact solution of Eq. (5) can be written as a fractional Taylor series, then we don't obtain the fractional terms by approximate the standard Taylor matrix method. This method transform each part of equation into matrix form then, we get the linear algebraic equation. Solving this equation, we obtained the coefficients of the generalized Taylor series then so, we obtain the approximate solutions for various N. In this article, we seek the approximate solution of Eq. (5) with the fractional Taylor series [38] as $D_x^{k\alpha}y(x) \in C(a,\ b]$,

$$y_N(x) = \sum_{i=0}^{N} \frac{(x-a)^{i\alpha}}{\Gamma(i\alpha+1)}(D_x^{i\alpha}y(x))(a). \tag{8}$$

2. Basic definitions

In this section, we introduce the Caputo fractional differential operator and the generalized Taylor formula base on it.

Theorem 2.1. *The fractional derivative of f(x) in Caputo sense is defined as*

$${}_a{}^C D_x^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-\tau)^{n-\alpha-1} f^{(n)}(\tau)d\tau$$

for $n-1 < \alpha \leqslant n$, $n \in \mathrm{N}$, $x > 0$, $f \in C_{-1}^n$.Theorem 2.1 *The fractional derivative* ${}_aD_t^{\alpha}f(t)$ *of the power function* $f(t) = (t-a)^v$, *where v is a real number is*

$${}_aD_x^{\alpha}(x-a)^{v} = \frac{\Gamma(v+1)}{\Gamma(v-\alpha+1)}(x-a)^{v-\alpha}.$$

Theorem 2.2. *(Generalized Taylor Formula) Suppose that* $D_a^{k\alpha}f(x) \in C(a,b]$ *for k = 0, 1, ..., n + 1 where* $0 < \alpha \leqslant 1$, *then we have* [38].

$$f(x) = \sum_{i=0}^{n} \frac{(x-a)^{i\alpha}}{\Gamma(i\alpha+1)}(D_a^{i\alpha} f)(a) + \frac{(D_a^{(n+1)}f)(\xi)}{\Gamma((n+1)\alpha+1)}(x-a)^{(n+1)\alpha},$$

with $a \leqslant \xi \leqslant x$, $\forall x \in (a,\ b]$, and

$$D_a^{n\alpha} = D_a^{\alpha} \cdot D_a^{\alpha} \cdot D_a^{\alpha} \cdot \ldots D_a^{\alpha} \quad (n \text{ times}).$$

3. Fundamental relations

In this section, we consider the fractional relaxation–oscillation equation Eq. (5). We use the thought of Taylor matrix method [39–43] and combine the generalized Taylor formula (see Theorem 2.2) to establish the truncated Taylor expansions of each term in FDEs. The matrix representations of expansions are substituted into FDEs to form a solution algebraic system. Firstly we can write

$$D_a^{\beta} y(x) = D_a^{k\alpha} y(x), \tag{9}$$

where $k \in Z^{+}$, $\alpha \in (0, 1]$.

Consider the approximate solution $y_N(x)$ of Eq. (5) defined by a truncated Taylor series (8). We have the matrix form of the solution

$$[y_N(x)] = \mathbf{T}(x)\mathbf{Y} = \tilde{\mathbf{X}}\mathbf{M}_0\mathbf{Y}, \tag{10}$$

where

$$\tilde{\mathbf{X}}(t) = [\,1 \quad (x-a)^{\alpha} \quad (x-a)^{2\alpha} \quad \cdots \quad (x-a)^{N\alpha}\,],$$

$$\mathbf{M}_0 = \begin{bmatrix} \frac{1}{\Gamma(1)} & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{\Gamma(\alpha+1)} & 0 & \cdots & 0 \\ 0 & 0 & \frac{1}{\Gamma(2\alpha+1)} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{\Gamma(N\alpha+1)} \end{bmatrix} \quad \mathbf{Y} = \begin{bmatrix} D_*^{0\alpha} y(a) \\ D_*^{1\alpha} y(a) \\ D_*^{2\alpha} y(a) \\ \vdots \\ D_*^{N\alpha} y(a) \end{bmatrix},$$

where $D^0 y_N(x) = y_N(x)$.

Now, we consider the matrix form of $D_a^{k\alpha} y(x)$ in Eq. (9) where $\alpha \in [0,1]$ and $k = n, n-1, \ldots, 0$.

For $i = 1$, we obtained the matrix representation of the function $D_a^{1\alpha} y_N(x)$

$$D_a^{1\alpha} y_N(x) = D_a^{1\alpha} \tilde{\tilde{\mathbf{X}}}\mathbf{M}_0\mathbf{Y}, \tag{11}$$

and we compute the $D_a^{1\alpha}\mathbf{X}$, then

$$\begin{aligned} D_a^{1\alpha}\tilde{\tilde{\mathbf{X}}} &= [\,D_a^{\alpha} 1 \quad D_a^{\alpha}(x-c)^{\alpha} \quad D_a^{\alpha}(x-c)^{2\alpha} \quad \cdots \quad D_a^{\alpha}(x-c)^{N\alpha}\,] \\ &= [\,0 \quad \tfrac{\Gamma(\alpha+1)}{\Gamma(1)} \quad \tfrac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)}(x-c)^{\alpha} \quad \cdots \quad \tfrac{\Gamma(N\alpha+1)}{\Gamma((N-1)\alpha+1)}(x-c)^{(N-1)\alpha}\,] \quad = \tilde{\tilde{\mathbf{X}}}\mathbf{M}_1, \end{aligned}$$

where

$$\mathbf{M}_1 = \begin{bmatrix} 0 & \frac{\Gamma(\alpha+1)}{\Gamma(1)} & 0 & \cdots & 0 \\ 0 & 0 & \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{\Gamma(N\alpha+1)}{\Gamma((N-1)\alpha+1)} \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Thus, the matrix representation of $D_a^{1\alpha} y_N(x)$ can be written as

$$D_a^{1\alpha} y_N(x) = \tilde{\tilde{\mathbf{X}}}\mathbf{M}_1\mathbf{M}_0\mathbf{Y}. \tag{12}$$

Similiarly, for integer k, we obtain

$$D_a^{k\alpha} y_N(x) = \tilde{\tilde{\mathbf{X}}}\mathbf{M}_k\mathbf{M}_0\mathbf{Y} \quad 0 \leqslant k \leqslant n,$$

where

$$\mathbf{M}_k = \begin{bmatrix} 0 & \cdots & \frac{\Gamma(k\alpha+1)}{\Gamma((k-1)\alpha+1)} & \cdots & 0 \\ 0 & \cdots & 0 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \frac{\Gamma(N\alpha+1)}{\Gamma((N-k)\alpha+1)} \\ 0 & \cdots & 0 & \cdots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{bmatrix}.$$

The matrix representation of $D_a^{k\alpha}y(x)$ become

$$D_a^{k\alpha}y_N(x) = \tilde{\tilde{\mathbf{X}}}\,\mathbf{M}_k\mathbf{M}_0\mathbf{Y}. \tag{13}$$

For the inhomogeneous term, let assume that the $f(x)$ can be written as a truncated Taylor series

$$f(x) = \sum_{n=0}^{N} \frac{1}{\Gamma(n\alpha+1)} (D_a^{n\alpha}f(x))_{t=c}(x-a)^{n\alpha}. \tag{14}$$

The matrix form of Eq. (14)

$$[f(x)] = \tilde{\tilde{\mathbf{X}}}(x)\mathbf{M}_0\mathbf{F}, \tag{15}$$

where

$$F = [\,f(c) \quad (D_a^{\alpha}f(x))_{x=a} \quad (D_a^{2\alpha}f(x))_{x=a} \quad \cdots \quad (D_a^{N\alpha}f(x))_{x=a}\,]^T.$$

Thus, we obtain the fundemendal matrix form of Eq. (5)

$$(\mathbf{M}_k\mathbf{M}_0 + A\mathbf{M}_0)\mathbf{Y} = \mathbf{M}_0\mathbf{F}. \tag{16}$$

On the other hand, the matrix representation of the initial conditions Eqs. (6) and (7), we have

$$y(0) = D_a^{0\alpha}y_N(x) = \tilde{\tilde{\mathbf{X}}}(0)\mathbf{M}_0\mathbf{Y} = \lambda \textbf{ and } y'(0) = D_a^{i\alpha}y_N(x) = \tilde{\tilde{\mathbf{X}}}(0)\mathbf{M}_i\mathbf{M}_0\mathbf{Y} = \mu, \tag{17}$$

where $i\alpha = 1$.

Define $\mathbf{U}_i$, $i = 0, 1$ as:

$$\mathbf{U}_0 = \mathbf{X}(0)\mathbf{M}_0 = [\,u_{00} \quad u_{01} \quad u_{02} \quad \cdots \quad u_{0N}\,] = [\lambda], \tag{18}$$

$$\mathbf{U}_1 = \mathbf{X}(0)\mathbf{M}_i\mathbf{M}_0 = [\,u_{10} \quad u_{11} \quad u_{12} \quad \cdots \quad u_{1N}\,] = [\mu]. \tag{19}$$

4. Method of solution

Write Eq. (16) in the form

$$\mathbf{WY} = \mathbf{G}, \tag{20}$$

where

$$\mathbf{W} = [w_{ij}] = \mathbf{M}_k\mathbf{M}_0 + A\mathbf{M}_0, \quad i,j = 0,1\ldots,N \text{ and } \mathbf{G} = \mathbf{M}_0\mathbf{F}.$$

Consequently, to find the unknown Taylor coefficients $D_a^{k\alpha}y(a)$, $k = 0, 1, \ldots, N$, related to the approximate solution of the problem consisting of Eq. (5) and conditions (7), replacing the last two rows of the matrix (20) by the matrices (18) and (19), we have augmented matrix

$$[\mathbf{W}^*;\mathbf{G}^*] = \begin{bmatrix} w_{00} & w_{01} & \cdots & w_{0N} & ; & (D_*^{0\alpha}f(c))/\Gamma(0\alpha+1) \\ w_{10} & w_{11} & \cdots & w_{1N} & ; & (D_*^{1\alpha}f(c))/\Gamma(1\alpha+1) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ w_{N-2,0} & w_{N-2,1} & \cdots & w_{N-2,N} & \vdots & (D_*^{(N-2)\alpha}f(c))/\Gamma((N-2)\alpha+1) \\ u_0 & u_{01} & \cdots & u_{0N} & ; & \lambda \\ u_{10} & u_{11} & \cdots & u_{1N} & ; & \mu \end{bmatrix},$$

the corresponding matrix equation

$$\mathbf{W}^*\mathbf{A} = \mathbf{G}^*. \tag{21}$$

If det $W^* \neq 0$, we can write Eq. (21) as

$$\mathbf{Y} = (\mathbf{W}^*)^{-1}\mathbf{G}^*, \tag{22}$$

and the matrtix $\mathbf{Y}$ is uniquely determined. By Eqs. (8) and (10), the approximate solution is given by the truncated Taylor series

$$f(x) = \sum_{i=0}^{n} \frac{(x-a)^{i\alpha}}{\Gamma(i\alpha+1)} (D_a^{i\alpha} f(x))(a).$$

We can easily check the accuracy of the method. Since the truncated fractional Taylor series (8) is an approximate solution of Eq. (5), when the solution $y_N(x)$ and its fractional derivatives are substituted in Eq. (5), the resulting equation must be satisfied approximately [39–43]; that is, for $x = x_q \in [a, b], \quad q = 0, 1, 2, \ldots$

$$E_N(x_q) = |D_a^\beta y_N(x_q) + Ay_N(x_q) - f(x_q)| \cong 0.$$

5. Examples

In order to illustrate the effectiveness of the method proposed, three numerical examples are given in this section. Absolute errors between approximate solution y_N and the corresponding exact solutions y i.e. $N_e = |y_N - y|$ are considered.

Example 1. Let us consider the relaxation–oscillation equation [1,16,19]

$$D_a^{3/2} y(x) = -y(x),$$

with initial conditions

$$y(0) = 1 \quad , y'(0) = 0.$$

The exact solution is $E_\beta(-x^\beta)$. $E_\beta(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\beta k+1)}$ is called the Mittag Lefler Function of order β. Here $\frac{3}{2} = 3\alpha, \ 0 \leqslant x \leqslant 1$ and we seek the approximate solutions by fractional Taylor series, for $a = 0$,

$$y_6(x) = \sum_{k=0}^{6} \frac{x^{k\alpha}}{\Gamma(k\alpha+1)} \left(D_a^{k\alpha} y(x)\right)_{x=0}.$$

Fundamental matrix relation of this is

$$(\mathbf{M}_3\mathbf{M}_0 + \mathbf{M}_0)\mathbf{Y} = 0,$$

where

$$\mathbf{M}_3\mathbf{M}_0 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2/\sqrt{\pi} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4/3\sqrt{\pi} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \mathbf{M}_0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2/\sqrt{\pi} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4/3\sqrt{\pi} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 8/15\sqrt{\pi} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1/6 \end{bmatrix},$$

then, we obtained

$$\mathbf{W} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 2/\sqrt{\pi} & 0 & 0 & 2/\sqrt{\pi} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 4/3\sqrt{\pi} & 0 & 0 & 4/3\sqrt{\pi} \\ 0 & 0 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 8/15\sqrt{\pi} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1/6 \end{bmatrix}.$$

Also, we have the matrix representation of conditions,

$$y(0) = [1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]\mathbf{Y} = [1],$$

$$y'(0) = [0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0]\mathbf{Y} = [0],$$

then, augmented matrix becomes

$$[\mathbf{W}^*;\mathbf{F}^*]=\begin{bmatrix}1&0&0&1&0&0&0&;0\\0&2/\sqrt{\pi}&0&0&2/\sqrt{\pi}&0&0&;0\\0&0&1&0&0&1&0&;0\\0&0&0&4/3\sqrt{\pi}&0&0&4/3\sqrt{\pi}&;0\\0&0&0&0&1/2&0&0&;0\\1&0&0&0&0&0&0&;1\\0&0&1&0&0&0&0&;0\end{bmatrix},$$

and so we solve this equation, we obtained the coefficients of the Taylor series

$$\mathbf{Y}=[\,1\quad 0\quad 0\quad -1\quad 0\quad 0\quad 1\,].$$

Hence, for $N = 6$, the approximate solution of example 1 is given

$$y_6(x)=1-\frac{4x^{\frac{3}{2}}}{3\sqrt{\pi}}+\frac{x^3}{6}.$$

Comparison of numerical and exact solutions are shown in Table 1. Figs. 1 and 2 show the approximant and error comparisions, respectively for various N. Additionally, we tabulated the comparison results with Taylor collocation method [19] in Table 2.

Example 2. Consider the following problem

$$D_a^{1/2}y(x)=-y(x),$$

with initial condition

$$y(0)=1.$$

We assume that $\alpha=\frac{1}{2},\ 0\leqslant x\leqslant 1$ and we seek the approximate solutions by Taylor series, for $a = 0$,

$$y_6(x)=\sum_{k=0}^{6}\frac{x^{k\alpha}}{\Gamma(k\alpha+1)}(D_a^{k\alpha}y(x))_{x=0},$$

Fundamental matrix relation of this is

$$(\mathbf{M}_1+\mathbf{M}_0)\mathbf{Y}=0,$$

where

$$\mathbf{M}_1=\begin{bmatrix}0&1&0&0&0&0&0\\0&0&2/\sqrt{\pi}&0&0&0&0\\0&0&0&1&0&0&0\\0&0&0&0&4/3\sqrt{\pi}&0&0\\0&0&0&0&0&1/2&0\\0&0&0&0&0&0&8/15\sqrt{\pi}\\0&0&0&0&0&0&0\end{bmatrix}\quad \mathbf{M}_0=\begin{bmatrix}1&0&0&0&0&0&0\\0&2/\sqrt{\pi}&0&0&0&0&0\\0&0&1&0&0&0&0\\0&0&0&4/3\sqrt{\pi}&0&0&0\\0&0&0&0&1/2&0&0\\0&0&0&0&0&8/15\sqrt{\pi}&0\\0&0&0&0&0&0&1/6\end{bmatrix},$$

then, we obtained

Table 1
Numerical result for Example 1.

x	Exact solution	Present method					
		$N = 6$	$N_e = 6$	$N = 12$	$Ne = 12$	$N = 18$	$N_e = 18$
0.0	1.0000000	1.0000000	0.0000E-0	1.0000000	0.0000E-0	1.0000000	0.0000E-0
0.1	0.9763777	0.9763783	0.6031E-6	0.9763777	0.1000E-10	0.9763777	0.1000E-10
0.2	0.9340362	0.9340497	0.1358E-4	0.9340362	0.3700E-9	0.9340362	0.2000E-10
0.3	0.8808084	0.8808922	0.8375E-4	0.8808085	0.8500E-8	0.8808084	0.0000E-0
0.4	0.8200563	0.8203600	0.3037E-3	0.8200564	0.7390E-7	0.8200563	0.1100E-10
0.5	0.7540488	0.7548718	0.8230E-3	0.7540449	0.3876E-6	0.7540488	0.1100E-8
0.6	0.6845298	0.6863845	0.1854E-2	0.6845314	0.1517E-5	0.6845298	0.0500E-8
0.7	0.6129215	0.6166007	0.3679E-2	0.6129266	0.4800E-5	0.6129215	0.1900E-8
0.8	0.5404169	0.5470650	0.6648E-2	0.5404299	0.1300E-4	0.5404169	0.8100E-8
0.9	0.4680306	0.4792153	0.1118E-1	0.4680619	0.3129E-4	0.4680307	0.2780E-7
1.0	0.3966293	0.4144138	0.1778E-1	0.3966979	0.6858E-4	0.39662944	0.8300E-7

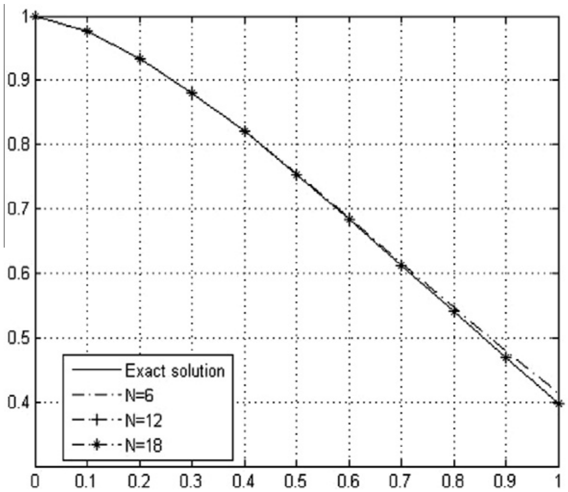

Fig. 1. Comparison of approximate solutions and exact solution.

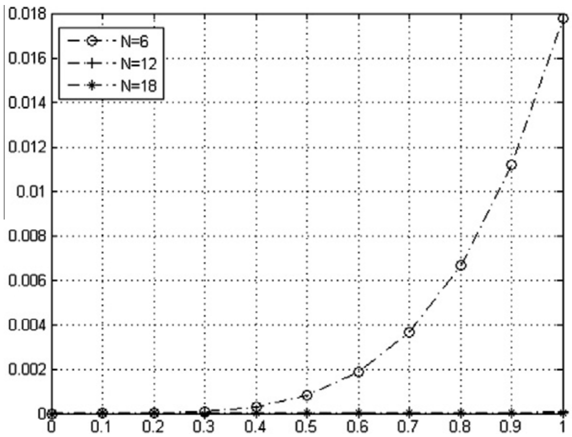

Fig. 2. Comparison of errors function.

Table 2
Comparison of numerical results with Present method.

x	Exact solution	Taylor col. met. ($N = 12$)	Present met. ($N = 12$)
0.0	1.0000000	0.00000E-0	0.000000E-0
0.2	0.93403621	0.44805E-7	0.370000E-14
0.4	0.82232699	0.60316E-7	0.739000E-11
0.6	0.68452989	0.67627E-7	0.151700E-9
0.8	0.54041695	0.69751E-7	0.130000E-8
1.0	0.39662936	0.66570E-7	0.685800E-7

$$\mathbf{W} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2/\sqrt{\pi} & 2/\sqrt{\pi} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4/3\sqrt{\pi} & 4/3\sqrt{\pi} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 8/15\sqrt{\pi} & 8/15\sqrt{\pi} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1/6 \end{bmatrix}.$$

Also, we have the matrix representation of conditions,

Table 3
Numerical result for Example 2.

x	Exact solution	Present method					
		$N = 6$	$N_e = 6$	$N = 12$	$N_e = 12$	$N = 18$	$N_e = 18$
0.0	1.0000000	1.0000000	0.0000E-0	1.0000000	0.0000E-0	1.0000000	0.0000E-0
0.1	0.7235784	0.7236019	0.2355E-4	0.7235784	0.2000E-9	0.7235784	0.1000E-9
0.2	0.6437882	0.6440406	0.2523E-3	0.6437882	0.1370E-7	0.6437882	0.9000E-9
0.3	0.5920184	0.5930206	0.1002E-2	0.5920185	0.1769E-6	0.5920184	0.4000E-9
0.4	0.5536062	0.5562613	0.2655E-2	0.5536073	0.1120E-5	0.5536062	0.2000E-9
0.5	0.5231565	0.5287949	0.5638E-2	0.5231612	0.4670E-5	0.5231565	0.1000E-8
0.6	0.4980245	0.5084380	0.1041E-1	0.4980395	0.1497E-5	0.4980245	0.6000E-8
0.7	0.4767027	0.4941725	0.1746E-1	0.4767427	0.4005E-5	0.4767027	0.2400E-7
0.8	0.4582460	0.4855662	0.2732E-1	0.4583398	0.9386E-5	0.4582461	0.8400E-7
0.9	0.4420214	0.4825183	0.4049E-1	0.4422202	0.1987E-4	0.4420216	0.2480E-6
1.0	0.4275835	0.4851336	0.5755E-1	0.4279722	0.3887E-4	0.4275842	0.6740E-6

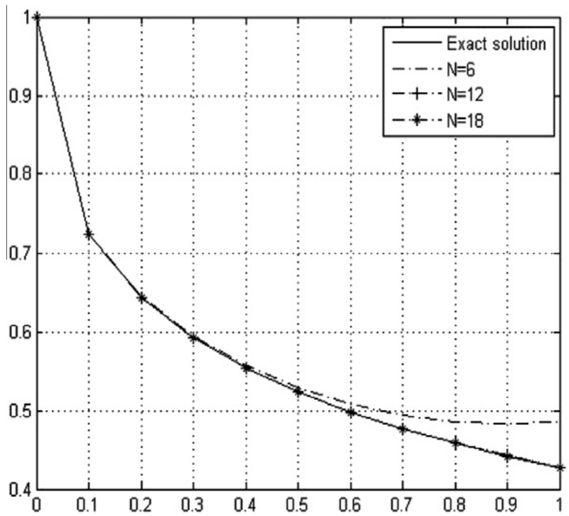

Fig. 3. Comparison of approximate solutions and exact solution.

$$y(0) = [1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]\mathbf{Y} = [1],$$

$$y'(0) = [0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0]\mathbf{Y} = [0],$$

then, augmented matrix becomes

$$[\mathbf{W}^*;\mathbf{F}^*] = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & ;0 \\ 0 & 2/\sqrt{\pi} & 2/\sqrt{\pi} & 0 & 0 & 0 & 0 & ;0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & ;0 \\ 0 & 0 & 0 & 4/3\sqrt{\pi} & 4/3\sqrt{\pi} & 0 & 0 & ;0 \\ 0 & 0 & 0 & 0 & 1/2 & 1/2 & 0 & ;0 \\ 1 & 0 & 0 & 0 & 0 & 8/15\sqrt{\pi} & 8/15\sqrt{\pi} & ;0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & ;1 \end{bmatrix},$$

and so we solve the this equation, we obtained the coefficients of the Taylor series

$$\mathbf{Y} = [1 \quad -1 \quad 1 \quad -1 \quad 1 \quad -1 \quad 1].$$

Hence, for $N = 6$, the approximate solution of example 1 is given

$$y_6(x) = 1 - \frac{2\sqrt{x}}{\sqrt{\pi}} + x - \frac{4x^{\frac{3}{2}}}{3\sqrt{\pi}} + \frac{x^2}{2} - \frac{8x^{\frac{5}{2}}}{15\sqrt{\pi}} + \frac{x^3}{6}.$$

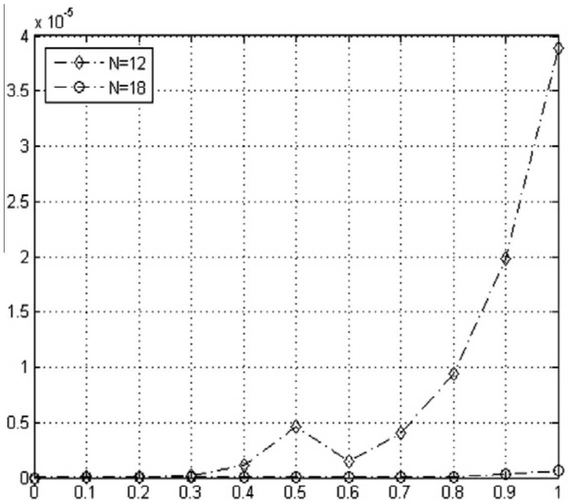

Fig. 4. Comparison of errors function.

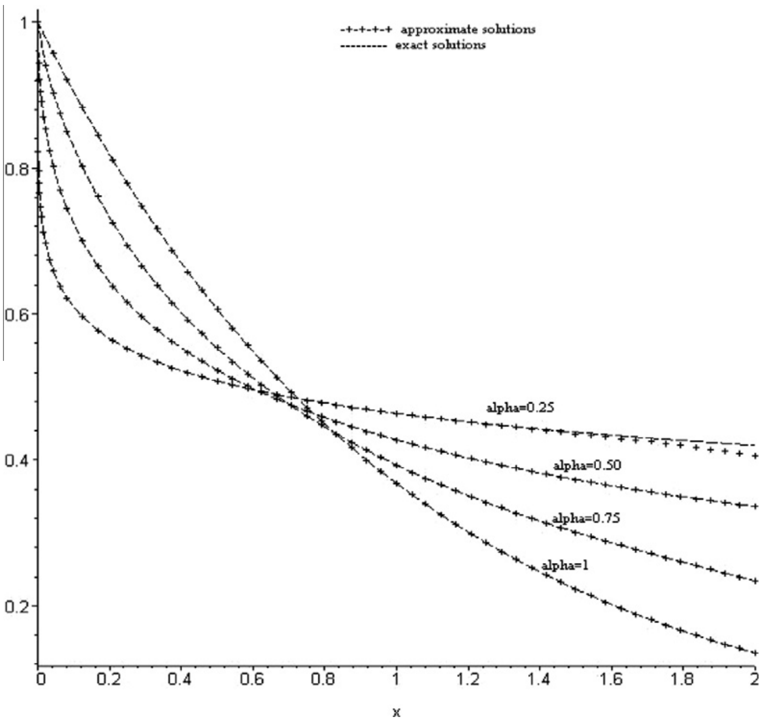

Fig. 5. Comparison of $y_N(x)$ for $N = 27$ and $\alpha = 0.25, 0.50, 0.75, 1$ with exact solution in Example 3.

Comparison of numerical results with the exact solution is shown in Table 3 and plotted the numerical results in Figs. 3 and 4 for various N.

Example 3. First consider the Eq. (5) for $\beta = 0.25, 0.5, 0.75$ and 1 with initial condition Eq. (6). Fig. 5 shows that the numerical results are consistent with the exact ones and as β approaches 1 the corresponding solutions of Eq. (5) approach that of integer-order differential equation. Fig. 6 illustrate the numerical solutions by present method and exact solution for $\beta = 1.25,\ 1.5$ and 2. Obviously the numerical results well with the exact ones. For $\beta = 2$, the Eq. (5) is the oscillation equation and the exact solution is $y(x) = \cos(x)$.

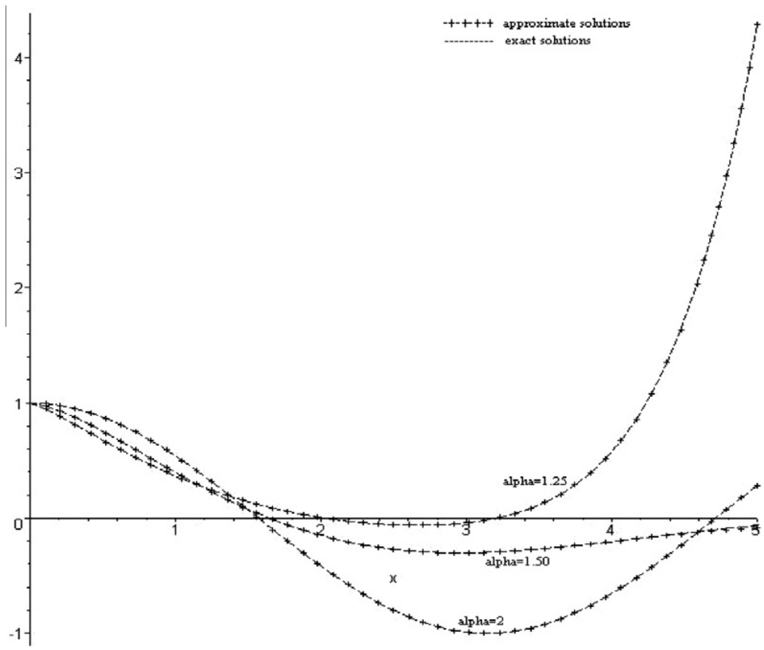

Fig. 6. Comparison of $y_N(x)$ for $N = 27$ and $\alpha = 1.25, 1.50, 2$ with exact solution in Example 3.

6. Conclusion

In this study, the Taylor matrix method has been generalized to obtain approximate solutions of fractional relaxation–oscillation equation. We have demonstrated the accuracy and efficiency of the present technique. The convergence of our method can be seen from Figs. 3 and 4. The better approximants may be obtained by increasing *value N* when the interval gets longer. This can be implied from Figs. 2 and 4. Table 2 shows that the present method is more accurate than collocation method [19]. The main advantage of method is that the approximate solutions can be calculated easily with the mathematics programs such as Matlab, Maple and Mathematica. The computation work here is finished by a computer code written in Maple 13. The present method is expected to be further employed to solve other similar problems.

Acknowledgments

We are very much indebted to both unknown reviewers for comments and remarks which led to improved presentation and quality of this paper.

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