

A survey about Fractional Order Calculus.

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Here we will review derivations of fractional calculus.

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I. INTRODUCTION

II. FRACTIONAL ORDER INTEGRATION

A. Useful relations

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt \quad (1)$$

$$B(\alpha, \beta) = \int_0^1 t^{\alpha-1} (1-t)^{\beta-1} dt \quad (2)$$

$$B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \quad (3)$$

$$\alpha\Gamma(\alpha) = \Gamma(\alpha+1), \Gamma(0) = \infty, \Gamma(1) = 1 \quad (4)$$

$$(\alpha+n)\cdots\alpha\Gamma(\alpha) = \Gamma(\alpha+n+1) \quad (5)$$

B. Cachy formula

$${}_a\mathbf{I}_t^n[F] = \int_a^t d\tau_1 \int_a^{\tau_1} d\tau_2 \int_a^{\tau_2} d\tau_3 \cdots d\tau_{n-1} \int_a^{\tau_{n-1}} F(\tau_n) d\tau_n \quad (6)$$

$$\begin{aligned} {}_a\mathbf{I}_t^n[F] &= \int_a^t \underbrace{d\tau_1}_{d\nu} \underbrace{{}_a\mathbf{I}_{\tau_1}^{n-1}[F]}_u = \tau_1 {}_a\mathbf{I}_{\tau_1}^{n-1}[F]|_a^t - \int_a^t \tau_1 d\tau_1 {}_a\mathbf{I}_{\tau_1}^{n-2}[F] \\ &= t \times {}_a\mathbf{I}_t^{n-1}[F] - \int_a^t \tau_1 d\tau_1 {}_a\mathbf{I}_{\tau_1}^{n-2}[F] = \int_a^t t d\tau_1 {}_a\mathbf{I}_{\tau_1}^{n-2}[F] - \int_a^t \tau_1 d\tau_1 {}_a\mathbf{I}_{\tau_1}^{n-2}[F] \\ &= \int_a^t \underbrace{d\tau_1(t-\tau_1)}_{d\nu} \underbrace{{}_a\mathbf{I}_{\tau_1}^{n-2}[F]}_u \end{aligned} \quad (7)$$

$$\begin{aligned} {}_a\mathbf{I}_t^n[F] &= \underbrace{-\frac{1}{2}(t-\tau_1)^2 {}_a\mathbf{I}_{\tau_1}^{n-2}[F]|_a^t}_{=0} + \frac{1}{2} \int_a^t (t-\tau_1)^2 d\tau_1 {}_a\mathbf{I}_{\tau_1}^{n-3}[F] \\ &= \frac{1}{2} \int_a^t (t-\tau_1)^2 d\tau_1 {}_a\mathbf{I}_{\tau_1}^{n-3}[F] \\ &\vdots \\ &= \frac{1}{(n-1)!} \int_a^t (t-\tau_1)^{n-1} F(\tau_1) d\tau_1 \end{aligned} \quad (8)$$

$${}_a\mathbf{I}_t^n[F] = \frac{1}{\Gamma(n)} \int_a^t (t-\tau_1)^{n-1} F(\tau_1) d\tau_1 \quad (9)$$

C. Arbitrary order integration

Based on Cuchy integration formula for repeated integrations one may extend the integer order integration to arbitrary order integration as follows,

$${}_a\mathbf{I}_t^\alpha[F] = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} F(\tau) d\tau \quad (10)$$

III. FRACTIONAL ORDER DIFFERENTIATION

A. Reimann-Liouville FOD

$${}^{\text{RL}}\mathbf{D}_t^\alpha \mathbf{F}(t) = \frac{d^m}{dt^m} {}_a\mathbf{I}_t^{m-\alpha}[\mathbf{F}] = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_a^t (t-\tau)^{m-\alpha-1} \mathbf{F}(\tau) d\tau \quad (11)$$

where $m-1 \preceq \alpha \preceq m$.

B. Caputo FOD

$${}_a^{\text{C}}\mathbf{D}_t^\alpha \mathbf{F}(t) = {}_a\mathbf{I}_t^{m-\alpha} \frac{d^m}{dt^m}[\mathbf{F}] = \frac{1}{\Gamma(m-\alpha)} \int_a^t (t-\tau)^{m-\alpha-1} \frac{d^m}{d\tau^m} \mathbf{F}(\tau) d\tau \quad (12)$$

where $m-1 \preceq \alpha \preceq m$.

C. Gronvald-Letnikov FOD and FOI

$${}_a^{\text{GL}}\mathbf{D}_t^\alpha \mathbf{F}(t) = \lim_{h \rightarrow 0} \frac{(1 - \hat{\mathbf{T}}_h)^\alpha}{h^\alpha} \mathbf{F}(t) \quad (13)$$

Using Tylor expansion of $(1-x)^\alpha$,

$$\begin{aligned} (1 - \hat{\mathbf{T}}_h)^\alpha &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \times \prod_{i=1}^{i=k} (\alpha - i + 1) \hat{\mathbf{T}}_h^n \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \underbrace{\Gamma(\alpha - n + 1) \prod_{i=1}^{i=k} (\alpha - i + 1)}_{\Gamma(\alpha+1)} \times \frac{1}{\Gamma(\alpha - n + 1)} \hat{\mathbf{T}}_h^n \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha - n + 1) \Gamma(n + 1)} \hat{\mathbf{T}}_h^n = \sum_{n=0}^{\infty} (-1)^n \binom{\alpha}{n} \hat{\mathbf{T}}_h^n \end{aligned}$$

Thus the GL FOD becomes,

$${}_a^{\text{GL}}\mathbf{D}_t^\alpha \mathbf{F}(t) = \frac{1}{h^\alpha} \lim_{h \rightarrow 0} \sum_{n=0}^N (-1)^n \binom{\alpha}{n} \mathbf{F}(t - nh) \quad (14)$$

with $h = t/N$. One may define GL integration by just $\alpha \rightarrow -\alpha$,

$$\begin{aligned}
(1 - \hat{\mathbf{T}}_h)^{-\alpha} &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \times \prod_{i=1}^{i=k} (-\alpha - i + 1) \hat{\mathbf{T}}_h^n \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \underbrace{\Gamma(\alpha) \prod_{i=1}^{i=k} (\alpha + i - 1)}_{\Gamma(\alpha+n)} \times \frac{1}{\Gamma(\alpha)} \hat{\mathbf{T}}_h^n \\
&= \sum_{n=0}^{\infty} (-1)^n \times (-1)^n \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha - n + 1) \Gamma(n + 1)} \hat{\mathbf{T}}_h^n \\
&= \sum_{n=0}^{\infty} (-1)^n \binom{\alpha + n - 1}{n} \hat{\mathbf{T}}_h^n
\end{aligned}$$

Thus GL FOI becomes,

$${}_a^{\text{GL}} \mathbf{I}_t^\alpha = {}_a^{\text{GL}} \mathbf{D}_t^{-\alpha} \mathbf{F}(t) = \lim_{h \rightarrow 0} h^\alpha \sum_{n=0}^N \binom{\alpha + n - 1}{n} \mathbf{F}(t - nh) \quad (15)$$

with $h = t/N$.

D. The relation between RL, Caputo and GL FOD

It is instructive to compare **Caputo** and **RL**, for $\mathbf{F}(t) = t^\nu$.
Reimann-Liouville:

$$\begin{aligned}
{}_a^{\text{RL}} \mathbf{D}_t^\alpha t^\nu &= \frac{1}{\Gamma(m - \alpha)} \frac{d^m}{dt^m} \int_a^t (t - \tau)^{m-\alpha-1} \tau^\nu d\tau \\
&= \frac{1}{\Gamma(m - \alpha)} \underbrace{\int_0^1 (1 - x)^{m-\alpha-1} x^{\nu+1-1} dx}_{\mathbf{B}(m-\alpha, \nu+1)} \frac{d^m}{dt^m} t^{m+\nu-\alpha} \\
&= \frac{1}{\Gamma(m - \alpha)} \frac{\Gamma(m - \alpha) \Gamma(\nu + 1)}{\Gamma(m - \alpha + \nu + 1)} \frac{d^m}{dt^m} t^{m+\nu-\alpha} \\
&= \frac{1}{\Gamma(m - \alpha + \nu + 1)} \underbrace{\prod_{i=1}^m (i + \nu - \alpha) \Gamma(\nu - \alpha + 1) \Gamma(\nu + 1)}_{\Gamma(m - \alpha + \nu + 1)} t^{\nu-\alpha} \\
&= \frac{\Gamma(\nu + 1)}{\Gamma(\nu - \alpha + 1)} t^{\nu-\alpha} \quad (16)
\end{aligned}$$

Caputo:

$$\begin{aligned}
{}_a^C D_t^\alpha t^\nu &= \frac{1}{\Gamma(m-\alpha)} \int_a^t (t-\tau)^{m-\alpha-1} \frac{d^m}{d\tau^m} \tau^\nu d\tau \Rightarrow \frac{d^m}{d\tau^m} \tau^\nu = 0, \text{ if } \nu \in \mathbb{N}, \text{ and } \nu < m \\
&= \frac{1}{\Gamma(m-\alpha)} \int_a^t (t-\tau)^{m-\alpha-1} \tau^{\nu-m} d\tau \times \prod_{i=0}^{m-1} (\nu-i) \\
&= \frac{1}{\Gamma(m-\alpha)} \underbrace{\int_0^1 (1-x)^{m-\alpha-1} x^{\nu-m+1-1} dx}_{\mathbf{B}(m-\alpha, \nu-m+1)} \times \prod_{i=0}^{m-1} (\nu-i) \times t^{\nu-\alpha} \\
&\quad \underbrace{\Gamma(\alpha+1)}_{\Gamma(m-\alpha) \Gamma(\nu-m+1) \prod_{i=0}^{m-1} (\nu-i)} \\
&= \frac{1}{\Gamma(m-\alpha)} \frac{\Gamma(m-\alpha) \Gamma(\nu-m+1) \prod_{i=0}^{m-1} (\nu-i)}{\Gamma(\nu-\alpha+1)} \times t^{\nu-\alpha} \\
&= \frac{\Gamma(\nu+1)}{\Gamma(\nu-\alpha+1)} t^{\nu-\alpha}
\end{aligned} \tag{17}$$

(18)

Therefore for **Caputo** FOD we have,

$${}_a^C D_t^\alpha t^\nu = \begin{cases} 0 & \text{if } \nu \in \mathbb{N} \text{ and } \nu < m \\ \frac{\Gamma(\nu+1)}{\Gamma(\nu-\alpha+1)} t^{\nu-\alpha} & \text{otherwise} \end{cases} \tag{19}$$

with $m-1 < \nu < m$.

$$\begin{aligned}
{}_a^C D_t^\alpha \mathbf{F}(t) &= \sum_{\nu=0}^{\infty} \frac{\mathbf{F}^{(\nu)}(0)}{\nu!} \times {}_a^C D_t^\alpha t^\nu \\
&= \sum_{\nu=m}^{\infty} \frac{\mathbf{F}^{(\nu)}(0)}{\nu!} \times \frac{\Gamma(\nu+1)}{\Gamma(\nu-\alpha+1)} t^{\nu-\alpha} \\
&= \sum_{\nu=m}^{\infty} \frac{\mathbf{F}^{(\nu)}(0)}{\Gamma(\nu-\alpha+1)} t^{\nu-\alpha}
\end{aligned} \tag{20}$$

and for **RL** we have,

$${}_a^{\mathbf{RL}} D_t^\alpha \mathbf{F}(t) = \sum_{\nu=0}^{\infty} \frac{\mathbf{F}^{(\nu)}(0)}{\Gamma(\nu-\alpha+1)} t^{\nu-\alpha} \tag{21}$$

Finally we have,

$${}_a^{\mathbf{RL}} D_t^\alpha \mathbf{F}(t) - {}_a^C D_t^\alpha \mathbf{F}(t) = \sum_{\nu=0}^{m-1} \frac{\mathbf{F}^{(\nu)}(0)}{\Gamma(\nu-\alpha+1)} t^{\nu-\alpha} \tag{22}$$

Taking fractional integration on both sides of Eq. 22,

$$\begin{aligned}
{}_a \mathbf{I}_t^\alpha [{}_a^{\mathbf{RL}} D_t^\alpha \mathbf{F}(t) - {}_a^C D_t^\alpha \mathbf{F}(t)] &= \sum_{\nu=0}^{m-1} \frac{\mathbf{F}^{(\nu)}(0)}{\Gamma(\nu-\alpha+1)} {}_a \mathbf{I}_t^\alpha [t^{\nu-\alpha}] \\
&= \sum_{\nu=0}^{m-1} \frac{\mathbf{F}^{(\nu)}(0)}{\Gamma(\nu-\alpha+1)} \frac{\Gamma(\nu-\alpha+1)}{\Gamma(\nu+1)} t^\nu \\
&= \sum_{\nu=0}^{m-1} \frac{\mathbf{F}^{(\nu)}(0)}{\Gamma(\nu+1)} t^\nu
\end{aligned} \tag{23}$$

IV. FOURIER AND LAPLACE TRANSFORMATIONS

$$\begin{aligned}
\mathcal{L}\{\mathbf{D}_t^\alpha \mathbf{F}\} &= \int_0^\infty \mathbf{D}_t^\alpha \mathbf{F}(t) e^{-st} dt \\
&= \frac{1}{\Gamma(m-\alpha)} \int_0^\infty e^{-st} dt \overbrace{\frac{d^m}{dt^m} \int_a^t (t-\tau)^{m-\alpha-1} \mathbf{F}(\tau) d\tau}^{\mathbf{U}^{(n)}(t)} \\
&= \frac{1}{\Gamma(m-\alpha)} \int_0^\infty \underbrace{e^{-st}}_\nu \underbrace{\mathbf{U}^{(n)}(t) dt}_{du} \\
&= \frac{1}{\Gamma(m-\alpha)} \left[\mathbf{U}^{(n-1)}(0) + s \int_0^\infty \underbrace{e^{-st}}_\nu \underbrace{\mathbf{U}^{(n-1)}(t) dt}_{du} \right] \\
&= \frac{1}{\Gamma(m-\alpha)} \left[\mathbf{U}^{(n-1)}(0) + s \mathbf{U}^{(n-2)}(0) + s^2 \int_0^\infty \underbrace{e^{-st}}_\nu \underbrace{\mathbf{U}^{(n-2)}(t) dt}_{du} \right] \\
&\vdots \\
&= \frac{1}{\Gamma(m-\alpha)} \left[\sum_{i=0}^{m-1} \mathbf{U}^{(i)}(0) s^{m-i-1} + s^n \sum_{i=0}^\infty \frac{\mathbf{F}^{(\nu)}(a)}{\nu!} \int_0^\infty e^{-st} \int_a^t (t-\tau)^{m-\alpha-1} \tau^\nu d\tau \right] \\
&= \frac{1}{\Gamma(m-\alpha)} \left[\sum_{i=0}^{n-1} \mathbf{U}^{(i)}(0) s^{m-i-1} + s^m \sum_{\nu=0}^\infty \frac{\mathbf{F}^{(\nu)}(a)}{\nu!} \int_0^\infty e^{-st} t^{m-\alpha+\nu} dt \underbrace{\int_0^1 (1-x)^{m-\alpha-1} x^{\nu+1-1} dx}_{\mathbf{B}(m-\alpha, \nu+1)} \right] \\
&= \frac{1}{\Gamma(m-\alpha)} \left[\sum_{\nu=0}^{m-1} \mathbf{U}^{(i)}(0) s^{m-i-1} + s^m \sum_{\nu=0}^\infty \frac{\mathbf{F}^{(\nu)}(a)}{\nu!} \mathbf{B}(m-\alpha, \nu+1) \underbrace{\int_0^\infty e^{-st} (st)^{m-\alpha+\nu} d(st) \times s^{\alpha-m-\nu-1}}_{\Gamma(m-\alpha+\nu+1)} \right] \\
&= \frac{1}{\Gamma(m-\alpha)} \left[\sum_{\nu=0}^{m-1} \mathbf{U}^{(i)}(0) s^{m-i-1} + s^m \sum_{\nu=0}^\infty \frac{\mathbf{F}^{(\nu)}(a)}{\nu!} \frac{\Gamma(\nu+1)\Gamma(m-\alpha)}{\Gamma(m-\alpha+\nu+1)} \Gamma(m-\alpha+\nu+1) s^{\alpha-m-\nu-1} \right] \\
&= \frac{1}{\Gamma(m-\alpha)} \sum_{\nu=0}^{m-1} \mathbf{U}^{(i)}(0) s^{m-i-1} + s^\alpha \underbrace{\sum_{\nu=0}^\infty \frac{\mathbf{F}^{(\nu)}(a)}{s^{\nu+1}}}_{\mathbf{F}(s)}
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}\{\mathbf{C}_a \mathbf{D}_t^\alpha \mathbf{F}\} &= \int_0^\infty \mathbf{C}_a \mathbf{D}_t^\alpha \mathbf{F}(t) e^{-st} dt \\
&= \sum_{\nu=0}^\infty \frac{\mathbf{F}^{(\nu)}(a)}{\nu!} \int_0^\infty \mathbf{C}_a \mathbf{D}_t^\alpha t^\nu e^{-st} dt \\
&= \sum_{\nu=m}^\infty \frac{\mathbf{F}^{(\nu)}(a)}{\nu!} \frac{\Gamma(\nu+1)}{\Gamma(\nu-\alpha+1)} \underbrace{\int_0^\infty (st)^{\nu-\alpha} e^{-st} d(st)}_{\Gamma(\nu-\alpha+1)} \times s^{\alpha-\nu-1} \\
&= s^\alpha \sum_{\nu=m}^\infty \frac{\mathbf{F}^{(\nu)}(a)}{s^{\nu+1}} \\
&= \underbrace{\sum_{\nu=0}^\infty \frac{\mathbf{F}^{(\nu)}(a)}{s^{\nu+1}}}_{\mathbf{F}(s)} - \sum_{\nu=0}^{m-1} \mathbf{F}^{(\nu)}(a) s^{\alpha-\nu-1}
\end{aligned}$$

V. NUMERICAL METHODS OF SOLUTIONS OF ORDINARY DIFFERENTIAL EQUATIONS

$$\dot{y}(t) = \mathbf{F}(y(t), t) \quad (24)$$

A. Multi-Linear methods for the solutions of ODEs

To solve Eq. 24, by assuming we have the values of $y(t)$ for $t < t_0$, one may integrate both sides of Eq. 24, over t from t_0 to $t_0 + h$. To perform, numerically, the integration $y(t)$ and $\mathbf{F}(t)$ could be approximated by a suitable interpolation function. Obviously, the larger the order of the interpolation will allow a larger choose of h . For the interpolation of the right hand side of Eq. 24, there is two option for the choose of the interpolation point: (1) $y(t_0 + h)$ being expluded from the interpolation reference points (predictor). (2) $y(t_0 + h)$ being included in the interpolation reference points (corrector). For the later, the solution of the resulting discrete equations must be proceed by iteration. The best starategy for solving Eq. 24 is to get an estimation of the $y(t_0 + h)$ through a predictor equation, then correcting the estimation through corrector. Thus a general form of the discritized equation proceeds as: The predictor equation,

$$\sum_{r=0}^N a_r y_{n+1-r} = \sum_{r=0}^N b_r \mathbf{F}_{n-r} \quad (25)$$

And for the corrector equation, we have:

$$\sum_{r=0}^N a_r y_{n+1-r} = \sum_{r=0}^N b_r \mathbf{F}_{n+1-r} \quad (26)$$

where $y_i = y(t_i)$ and $\mathbf{F}_i = \mathbf{F}(y(t_i), t_i)$.

$$\int_{t_n}^{t_{n+1}} y(\tau) \tau = \int_{t_n}^{t_{n+1}} \tilde{\mathbf{F}}_s(\tau) d\tau, \text{ with } s=\text{Pr or Corr} \quad (27)$$

where

$$\tilde{\mathbf{F}}_{Pr}(\tau) = \sum_{i=0}^N \mathbf{F}(t_{n-i}) L_{N-i}(\tau) \quad (28)$$

and

$$\tilde{\mathbf{F}}_{Corr}(\tau) = \sum_{i=0}^N \mathbf{F}(t_{n+1-i}) L_{N-i}(\tau) \quad (29)$$

$$L_i(t) = \frac{\prod_{k=0, k \neq i}^N (t - t_k)}{\prod_{k=0, k \neq i}^N (t_i - t_k)} \quad (30)$$

For the left hand side the simplest case is the Reiman integration, thus Predictor and Corrector equations reads,

Predictor:

$$y^{(P)}(t_{n+1}) = y(t_n) + \sum_{i=0}^N \mathbf{F}(t_{n-i}, y_{n-i}) \int_{t_n}^{t_{n+1}} L_{N-i}(\tau) d\tau \quad (31)$$

Predictor:

$$y(t_{n+1}) = y(t_n) + \sum_{i=0}^N \mathbf{F}(t_{n+1-i}, y_{n+1-i}^{(P)}) \int_{t_n}^{t_{n+1}} L_{N-i}(\tau) d\tau \quad (32)$$

For a four point approximation, without loss of generality, we may consider $t_{3-i} = -ih$. Thus we have,

$$\begin{aligned} L_0(t) &= \frac{(t+h)(t+2h)(t+3h)}{(0+h)(0+2h)(0+3h)} = \frac{1}{6h^3}(t^3 + 6ht^2 + 11h^2t + 6h^3) \\ L_1(t) &= \frac{(t-0)(t+2h)(t+3h)}{(-h+0)(-h+2h)(-h+3h)} = -\frac{1}{2h^3}(t^3 + 5ht^2 + 6h^2t) \\ L_2(t) &= \frac{(t-0)(t+h)(t+3h)}{(-2h+0)(-2h+h)(-2h+3h)} = \frac{1}{2h^3}(t^3 + 4ht^2 + 3h^2t) \\ L_3(t) &= \frac{(t-0)(t+h)(t+2h)}{(-3h+0)(-3h+h)(-3h+2h)} = -\frac{1}{6h^3}(t^3 + 3ht^2 + 2h^2t) \end{aligned}$$

For the corrector, the integration must be performed in the domain of $(-h, 0)$. Performing the integrations we have,

$$-h\mathbf{I}_0[L_0] = \frac{3h}{8}, -h\mathbf{I}_0[L_1] = \frac{19h}{24}, -h\mathbf{I}_0[L_2] = \frac{-5h}{24}, -h\mathbf{I}_0[L_3] = \frac{h}{24} \quad (33)$$

For the predictor the integration must be performed in the domain $(0, h)$,

$$_0\mathbf{I}_h[L_0] = \frac{3h}{8}, _0\mathbf{I}_h[L_1] = \frac{19h}{24}, _0\mathbf{I}_h[L_2] = \frac{-5h}{24}, _0\mathbf{I}_h[L_3] = \frac{h}{24} \quad (34)$$

B. Runge-Cutta

VI. NUMERICAL METHODS FOR SOLVING FRACTIONAL ORDER DIFFERENTIAL EQUATIONS(FODE)

A. Multi linear methods for the solutions of FODEs

B. Runge-Cutta

VII. HIGHER ORDER FINITE DIFFERENCE FOD

A. Higher order finite difference for integer order derivatives

B. Higher order finite difference for FOD

VIII. NUMERICAL IMPLEMENTATION

IX. TRANSIENT SIMULATION OF ELECTRICAL CIRCUITS(PASSIVE PARTS ONLY)

A. Simulation of parts without FO components

B. Simulation of parts with FO components

C. Implementation of transient algorithm

D. Inclusion of SPICE net lists

X. SOME APPLICATIONS OF FOC IN THE ELECTROCHEMISTRY

XI. POSSIBLE INTEGRATION INTO QUCS

ACKNOWLEDGMENTS

XII. APPENDIX
