Results in Physics 46 (2023) 106307

Contents lists available at ScienceDirect

Results in Physics

journal homepage: www.elsevier.com/locate/rinp

A Lie group treatment on a generalized evolution Fisher type equation with variable coefficients

Shao-Wen Yao [a], Mir Sajjad Hashemi [b], Mustafa Inc [c,d,*]

[a] School of Mathematics and Information Science, Henan Polytechnic University, Jiaozuo, Henan 454000, China
[b] Department of Computer Engineering, Biruni University, 34010 Istanbul, Turkey
[c] Firat University, Science Faculty, Department of Mathematics, 23119 Elazig, Turkey
[d] Department of Medical Research, China Medical University, 40402 Taichung, Taiwan

ARTICLE INFO

Keywords:
Heir-equation method
Non-classical symmetry
Fisher-type equation

ABSTRACT

In this study, a general second-order evolution equation of the Fisher-type, with time-dependent variable coefficients, is considered. This equation contains many well-known equations, and obtained results may be applicable in investigating other evolution equations. Lie symmetries and corresponding invariant solutions of the considered problem are studied by a non-traditional Lie symmetry method (LSM). Prolongation of the model is an essential part of the Lie symmetry method, which in the current work, we analyze by the heir-equations. Finally, different types of solutions depending on the variable coefficients, such as the exponential solutions, trigonometric solutions, Bessel solutions, and Mathieu solutions, are extracted.

Introduction

Non-linear partial differential equations (NLPDEs) play important role for the learning of nonlinear substantial models as well as used to describe complex phenomenon such as mechanical systems [1,2], electronics [3,4], optics [5,6], etc. This issue motivates many researchers to investigate and analyze the solutions of NLPDEs from numerical and analytical points of view. For the approximate techniques of NLPDEs, we can mention to the wavelet method [7,8], meshfree methods [9–11], and so on. Besides, there are some approaches that obtain exact solutions to differential equations. We know that exact solutions are more desirable than approximate ones. Recently, different analytical techniques are devised to consider the exact solutions of differential equations such as the LSM [12–14], the exponential rational function method [15,16], the Kudryashov method [17,18], invariant subspace method [19–22], reduction method [23], and so on.

The Fisher partial differential equation, also known as the Fisher–Kolmogorov equation, is a reaction–diffusion equation that describes the spatiotemporal dynamics of a population undergoing logistic growth with diffusion. The equation takes the form of a nonlinear parabolic equation and is used to model a wide range of phenomena in biology, ecology, and physics. The equation was first introduced by geneticist and statistician Ronald Fisher in 1937 as a model for the spread of advantageous genes in a population [24]. Since then, it has become a fundamental tool for understanding the dynamics of spatially extended populations and has been extensively studied in the field of mathematical biology.

In this work, we consider a second order evolution equation of Fisher-type equation (FTE) with variable coefficients

$$\varpi_t + \delta_1(t)\varpi\varpi_x + \delta_2(t)\varpi_{xx} + \delta_3(t)\varpi + \delta_4(t)\varpi^2 + \delta_5(t) = 0, \tag{1}$$

where $\delta_i(t),\ i = 1, \dots, 5$, are arbitrary time variable functions. The origin of this general equation rises from reaction diffusion equations with the following form [25]:

$$\varpi_t = D\varpi_{xx} + K(\varpi), \tag{2}$$

where ϖ is chemical concentration, D is diffusion coefficient, and $K(\varpi)$ represents the kinetics. Eq. (2) is called the Fisher equation, if $K(\varpi) = \varpi(1-\varpi)$. Moreover, Eq. (2), with $K(\varpi) = \kappa\varpi(1-\frac{\varpi}{\lambda})$, which denotes nonlinear growth rate, is famous in population dynamics, where $D > 0$, is a diffusion constant, $\kappa > 0$, is the linear growth rate, and $\lambda > 0$, is the carrying capacity of the environment [26]. A special case of Eq. (1), with $\delta_1(t) = 0$, $\delta_2(t) = 1$, $\delta_3(t) = b(t)$, $\delta_4(t) = -a(t)$, and $\delta_5(t) = 0$ is considered by the auto-Bäcklund transformation in [27].

The document is laid out as follows: In Section "Heir-equations and non-classical symmetries", we give a brief discussion about the heir-equation method and corresponding non-classical Lie symmetries. In Section "Heir-equations and exact solutions of Eq. (1)", we use the heir-equation method to find non-classical symmetries of Eq. (1). Results and discussions about the obtained solutions are investigated in Section "Results and discussion". Finally, the document's conclusion is shown in Section "Conclusion".

* Corresponding author at: Fırat University, Science Faculty, Department of Mathematics, 23119 Elazig, Turkey.
E-mail address: minc@firat.edu.tr (M. Inc).

https://doi.org/10.1016/j.rinp.2023.106307
Received 4 February 2023; Received in revised form 17 February 2023; Accepted 21 February 2023
Available online 25 February 2023

Heir-equations and non-classical symmetries

One of the most well-known methods with geometric inheritance for obtaining exact and group-invariant solutions to differential equations is the LSM. The literature suggests a number of modifications and developments to the traditional LSM, including the approximation symmetries, λ-symmetries, and others. However, Bluman and Cole's [28] non-classical symmetry approach is one of the most well-known. The determining equations in the traditional LSM are linear, but those in the non-traditional method are nonlinear. This problem results in some restrictions when using this method with differential equations. To get around this problem, Nucci [29] presented an iterative method that solves the equation's non-classical and classical symmetries. This technique is well-known as the heir-equation method (HEM).

Now, we give a brief introduction to the HEM. Let us consider:

$$\varpi_t = \mathcal{F}(X, \varpi_x, \varpi_{xx}), \tag{3}$$

where $X = [t, x, \varpi]$. If

$$\Gamma = \Theta_1(X)\frac{\partial}{\partial t} + \Theta_2(X)\frac{\partial}{\partial x} - Y(X)\frac{\partial}{\partial \varpi}, \tag{4}$$

is an infinitesimal generator correspond to Eq. (3), then

$$\Theta_1(X)\frac{\partial \varpi}{\partial t} + \Theta_2(X)\frac{\partial \varpi}{\partial x} = Y(X), \tag{5}$$

constitutes the invariant surface condition. To derive the first heir-equation, let us assume $\Theta_1 = 0$ and $\Theta_2 = 1$. Then (5) becomes[1]:

$$\varpi_x = \mathcal{G}^{[1]}(X) \tag{6}$$

Then, the well-known $\mathcal{G}^{[1]}$-equation is easily acquired [12]. Corresponding invariant surface condition is given by:

$$\xi_1(X, \mathcal{G}^{[1]})\mathcal{G}^{[1]}_t + \xi_2(X, \mathcal{G}^{[1]})\mathcal{G}^{[1]}_x + \xi_3(X, \mathcal{G}^{[1]})\mathcal{G}^{[1]}_\varpi = \mathcal{G}^{[2]}(X, \mathcal{G}^{[1]}). \tag{7}$$

If we assume $\xi_1 = 0$, $\xi_2 = 1$, and $\xi_3 = \mathcal{G}^{[1]}$, then Eq. (7) reduces into

$$\mathcal{G}^{[1]}_x + \mathcal{G}^{[1]}\mathcal{G}^{[1]}_\varpi = \mathcal{G}^{[2]}(X, \mathcal{G}^{[1]}). \tag{8}$$

We nominate this second heir-equation as the $\mathcal{G}^{[2]}$-equation. Obviously

$$\mathcal{G}^{[1]}_x + \mathcal{G}^{[1]}\mathcal{G}^{[1]}_\varpi \equiv \varpi_{xx} \equiv \mathcal{G}^{[2]}. \tag{9}$$

By keeping on the same procedure, we get the third heir-equation, called the $\mathcal{G}^{[3]}$-equation, which corresponds to:

$$\mathcal{G}^{[2]}_x + \mathcal{G}^{[1]}\mathcal{G}^{[2]}_\varpi + \mathcal{G}^{[2]}\mathcal{G}^{[2]}_{\mathcal{G}^{[1]}} \equiv \varpi_{xxx} \equiv \mathcal{G}^{[3]}(X, \mathcal{G}^{[1]}, \mathcal{G}^{[2]}). \tag{10}$$

Indeed, the proposed heir-equations inherit the symmetry algebra of the original equation. That is, the first prolongation of the considered original equation corresponds to the $\mathcal{G}^{[1]}$-equation, the second prolongation corresponds to the $\mathcal{G}^{[2]}$-equation, and the third prolongation corresponds to the $\mathcal{G}^{[3]}$-equation. The equivalence of the HEM with the generalized conditional symmetries method is investigated by Goard in [30].

In order to use the HEM, we assume $\Theta_1 = 1$, and get ϖ_t from (3) and replace it into (5), i.e.:

$$\mathcal{F}(X, \varpi_x, \varpi_{xx}) + \Theta_2(X)\varpi_x = Y(X). \tag{11}$$

Then, we generate the second prolongation for the $\mathcal{G}^{[2]}$-equation with $\mathcal{G}^{[2]} = \mathcal{G}^{[2]}(X, \mathcal{G}^{[1]})$, and replace $\varpi_x = \mathcal{G}^{[1]}$, $\varpi_{xx} = \mathcal{G}^{[2]}$ into (11), i.e.:

$$\mathcal{F}(X, \mathcal{G}^{[1]}, \mathcal{G}^{[2]}) = Y(X) - \Theta_2(X)\mathcal{G}^{[1]} \tag{12}$$

For Dini's theorem, we derive $\mathcal{G}^{[2]}$ in (12), e.g.:

$$\mathcal{G}^{[2]} = \left[\mathcal{S}_1(X, \mathcal{G}^{[1]}) + Y(X) - \Theta_2(X)\mathcal{G}^{[1]}\right]\mathcal{S}_2(X, \mathcal{G}^{[1]}) \tag{13}$$

where $\mathcal{S}_i(X, \mathcal{G}^{[1]})(i = 1, 2)$ are well-known. Hence, we acquired a particular solution of $\mathcal{G}^{[2]}$ which have to conclude an identity if replaced into the $\mathcal{G}^{[2]}$-equation. $\Theta_2 = \Theta_2(X)$ and $Y = Y(X)$ are the only unknowns.

Heir-equations and exact solutions of Eq. (1)

We employ the Maple package to acquire the heir-equations. The first heir-equation of Eq. (1), that is $\mathcal{G}^{[1]}$−equation, can be written as

$$\begin{aligned}&\delta_2(t)(\mathcal{G}^{[1]})^2\mathcal{G}^{[1]}_{\varpi\varpi} + 2\delta_2(t)\mathcal{G}^{[1]}\mathcal{G}^{[1]}_{x\varpi} + \delta_3(t)\mathcal{G}^{[1]}\\&+\delta_2(t)\mathcal{G}^{[1]}_{xx} - \mathcal{G}^{[1]}_\varpi\left(\delta_4(t)\varpi^2 + \delta_3(t)\varpi + \delta_5(t)\right)\\&+\delta_1(t)\varpi\mathcal{G}^{[1]}_x + \mathcal{G}^{[1]}_t + \delta_1(t)(\mathcal{G}^{[1]})^2 + 2\delta_4(t)\varpi\mathcal{G}^{[1]} = 0.\end{aligned}$$

and the $\mathcal{G}^{[2]}$−equation is given by

$$\begin{aligned}&\delta_2(t)(\mathcal{G}^{[2]})^2\mathcal{G}^{[2]}_{\mathcal{G}^{[1]}\mathcal{G}^{[1]}} + 2\delta_2(t)\mathcal{G}^{[1]}\mathcal{G}^{[2]}\mathcal{G}^{[2]}_{\varpi\mathcal{G}^{[1]}} + 2\delta_2(t)\mathcal{G}^{[2]}\mathcal{G}^{[2]}_{x\mathcal{G}^{[1]}}\\&+\delta_2(t)(\mathcal{G}^{[1]})^2\mathcal{G}^{[2]}_{\varpi\varpi} + 2\delta_2(t)\mathcal{G}^{[1]}\mathcal{G}^{[2]}_{x\varpi}\\&+\delta_2(t)\mathcal{G}^{[2]}_{xx} - \mathcal{G}^{[2]}_{\mathcal{G}^{[1]}}\left(2\delta_4(t)\varpi\mathcal{G}^{[1]} + \delta_1(t)(\mathcal{G}^{[1]})^2 + \delta_3(t)\mathcal{G}^{[1]}\right)\\&-\mathcal{G}^{[2]}_\varpi\left(\delta_5(t) + \delta_3(t)\varpi + \delta_4(t)\varpi^2\right)\\&+\mathcal{G}^{[2]}_t + \delta_1(t)\varpi\mathcal{G}^{[2]}_x + \mathcal{G}^{[2]}\left(2\delta_4(t)\varpi + 3\delta_1(t)\mathcal{G}^{[1]} + \delta_3(t)\right) + 2\delta_4(t)(\mathcal{G}^{[1]})^2 = 0.\end{aligned}$$

According to Eq. (13), the particular solution of the $\mathcal{G}^{[2]}$-equation is

$$\mathcal{G}^{[2]}(X, \mathcal{G}^{[1]}) = -\frac{\delta_1(t)\varpi\mathcal{G}^{[1]} + \delta_4(t)\varpi^2 - \Theta_2(X)\mathcal{G}^{[1]} + \delta_3(t)\varpi + \delta_5(t) + Y(X)}{\delta_2(t)}, \tag{14}$$

that yields an over-determined system in the unknowns Y, Θ_2.

In relation to $\mathcal{G}^{[1]}$, we acquire a third-degree polynomial. The three coefficients we refer to as Δ_i, $i = 0, \dots, 3$ are calculated. Now, we try to vanish these coefficients. From

$$\Delta_3 = \delta_2(t)\frac{\partial^2\Theta_2(X)}{\partial\varpi^2} = 0,$$

we have

$$\Theta_2(X) = \theta_1(t, x)\varpi + \theta_2(t, x), \tag{15}$$

where $\theta_1(t, x)$ and $\theta_2(t, x)$ are arbitrary functions. Now, substituting the (15) into $\Delta_2 = 0$, we get the following linear PDE with respect to the dependent variable ϖ:

$$\begin{aligned}&\delta_2^2(t)\frac{\partial^2}{\partial\varpi^2}Y(X) - 2\delta_2^2(t)\frac{\partial}{\partial x}\theta_1(t, x) - 2\left(\theta_1(t, x)\varpi + \theta_2(t, x)\right)\delta_2(t)\theta_1(t, x)\\&+2\delta_1(t)\delta_2(t)\theta_1(t, x)\varpi = 0.\end{aligned} \tag{16}$$

Solving the Eq. (16), which can be considered as a linear ODE with respect to ϖ, concludes

$$\begin{aligned}Y(X) =& \frac{\theta_1^2(t, x)\varpi^3}{3\delta_2(t)} - \frac{\theta_1(t, x)3\delta_1(t)\varpi^3}{\delta_2(t)} + \frac{\partial}{\partial x}\theta_1(t, x)\varpi^2 + \frac{\theta_1(t, x)\theta_2(t, x)\varpi^2}{\delta_2(t)}\\&+\theta_3(t, x)\varpi + \theta_4(t, x),\end{aligned} \tag{17}$$

Now, substituting (15) and (17) into the $\Delta_1 = 0$, gives a third order polynomial with respect to ϖ. We nominate the corresponding coefficients as $\Delta\Delta_i$, $i = 0, \dots, 3$. Vanishing the coefficient of ϖ^3, that is

$$\Delta\Delta_3 = -\delta_2(t)\left(2\theta_1^3(t, x) - \theta_1(t, x)\delta_1^2(t) - \theta_1^2(t, x)\delta_1(t)\right) = 0,$$

concludes

$$\theta_1(t, x) \in \left\{0, -\frac{\delta_1(t)}{2}, \delta_1(t)\right\}.$$

We proceed our computations with $\theta_1(t, x) = \delta_1(t)$. Then

$$\Delta\Delta_2 = -9\delta_2(t)\delta_1(t)\left(\theta_2(t, x)\delta_1(t) + \delta_2(t)\delta_4(t)\right) = 0,$$

yields

$$\theta_2(t, x) = -\frac{\delta_2(t)\delta_4(t)}{\delta_1(t)}. \tag{18}$$

[1] We have replaced $Y(X)$ with $\mathcal{G}^{[1]}(X)$ to avoid any ambiguity in the forthcoming clarification.

Substituting the acquired values into the $\Delta\Delta_1 = 0$, gives

$$-\delta_1(t)\delta_2^2(t)\left(\theta_3(t,x) + \delta_3(t)\right) = 0,$$

or, equivalently

$$\theta_3(t,x) = -\delta_3(t). \tag{19}$$

Finally,

$$-3\left(\frac{\delta_2(t)}{\delta_1(t)}\right)^2\left(3\delta_1^3(t)\theta_4(t,x) + 3\delta_1^3(t)\delta_5(t) - \delta_2(t)\delta_4(t)\delta_1'(t) + \delta_1(t)\delta_2(t)\delta_4'(t)\right) = 0,$$

gives

$$\theta_4(t,x) = \frac{-3\delta_1^3(t)\delta_5(t) + \delta_2(t)\delta_4(t)\delta_1'(t) - \delta_1(t)\delta_2(t)\delta_4'(t)}{3\delta_1^3(t)}. \tag{20}$$

Now, let us return to $\Delta_1 = 0$. This equation has

$$\left(2\delta_4^2(t)\delta_1(t)\delta_1'(t) - 2\delta_1^2(t)\delta_4(t)\delta_4'(t)\right)\varpi + \delta_1(t)\delta_1'(t)\delta_3(t)\delta_4(t) - \delta_4'(t)\delta_1^2(t)\delta_3(t)$$
$$-3\left(\delta_1'(t)\right)^2\delta_4(t) + \delta_1''(t)\delta_1(t)\delta_4(t) + 3\delta_1(t)\delta_1'(t)\delta_4'(t) - \delta_4''(t)\delta_1^2(t) = 0. \tag{21}$$

The right-hand side of Eq. (21), is a monomial. Vanishing the coefficient of ϖ, yields

$$\delta_1(t) = \lambda\delta_4(t), \tag{22}$$

where λ is an arbitrary constant. This solves the constant coefficient of Eq. (21), too.

After substituting the obtained results, we can write the following final desired values

$$\Theta_2(X) = \frac{\lambda^2\delta_4(t)\varpi - \delta_2(t)}{\lambda},$$
$$\Upsilon(X) = -\delta_4(t)\varpi^2 - \delta_3(t)\varpi - \delta_5(t). \tag{23}$$

Therefore, $\mathcal{G}^{[2]}$-equation (14), can be written as

$$\mathcal{G}^{[2]}(X,\mathcal{G}^{[1]}) = -\frac{\mathcal{G}^{[1]}}{\lambda}, \tag{24}$$

namely,

$$\varpi_{xx} = -\frac{\varpi_x}{\lambda}.$$

This exact solution is given by

$$\varpi(t,x) = \mathcal{F}_1(t) + \mathcal{F}_2(t)\exp\left(-\frac{x}{\lambda}\right), \tag{25}$$

where $\mathcal{F}_1(t)$ and $\mathcal{F}_2(t)$, are functions to be determined. In order to

To determine the $\mathcal{F}_1(t)$ and $\mathcal{F}_2(t)$, we substitute (25) into the original Eq. (1). Hence, we acquire

$$\left(\mathcal{F}_1(t)\mathcal{F}_2(t)\delta_4(t) + \mathcal{F}_2(t)\delta_3(t) + \mathcal{F}_2'(t) + \frac{\mathcal{F}_2(t)\delta_2(t)}{\lambda^2}\right)\exp\left(-\frac{x}{\lambda}\right)$$
$$+\mathcal{F}_1^2(t)\delta_4(t) + \mathcal{F}_1(t)\delta_3(t) + \mathcal{F}_1'(t) + \delta_5(t) = 0. \tag{26}$$

Indeed, Eq. (26), constitutes the following two ODEs with respect to the independent variable t:

$$\mathcal{F}_1(t)\mathcal{F}_2(t)\delta_4(t) + \mathcal{F}_2(t)\delta_3(t) + \mathcal{F}_2'(t) + \frac{\mathcal{F}_2(t)\delta_2(t)}{\lambda^2} = 0,$$
$$\mathcal{F}_1^2(t)\delta_4(t) + \mathcal{F}_1(t)\delta_3(t) + \mathcal{F}_1'(t) + \delta_5(t) = 0. \tag{27}$$

Case 1. If we assume, $\delta_5(t) = 0$, then the second equation of (27), concludes

$$\mathcal{F}_1(t) = \frac{\exp\left(-\int\delta_3(t)dt\right)}{\int\delta_4(t)\exp\left(-\int\delta_3(t)dt\right)dt + \mu},$$

where μ is an arbitrary constant. Substituting the acquired value of $\mathcal{F}_1(t)$ into the first equation of (27), yields

$$\mathcal{F}_2(t) = \rho\exp\left(-\int\left[\frac{\Psi(\lambda^2\delta_3(t) + \delta_2(t)) + \lambda^2\delta_4(t)\exp(-\int\delta_3(t)dt) + \lambda^2\mu\delta_3(t) + \mu\delta_2(t)}{\lambda^2(\mu+\Psi)}\right]dt\right),$$

where ρ is an arbitrary constant and $\Psi = \int\delta_4(t)\exp\left(-\int\delta_3(t)dt\right)dt$.

Therefore, in this case, the final solution can be written as is given in Box I. where $\Psi = \int\delta_4(t)\exp\left(-\int\delta_3(t)dt\right)dt$.

Case 2. In order to get additional exact solutions, we turn to (27), and assume $\delta_3(t) = \delta_5(t) = \delta_4(t)$. Then, second equation of (27), has

$$\mathcal{F}_1(t) = -\frac{1}{2}\left[1 + \sqrt{3}\tan\left(\frac{\sqrt{3}}{2}\left(\mu + \int\delta_4(t)dt\right)\right)\right],$$

where μ is an arbitrary constant. Similar to the previous case, if we substitute the acquired $\mathcal{F}_1(t)$, into the first equation of (27), we obtain

$$\mathcal{F}_2(t) = \rho\exp\left(-\frac{1}{2\lambda^2}\int\left[2\delta_2(t) + \lambda^2\delta_4(t)\left[1 - \sqrt{3}\tan\left(\frac{\sqrt{3}}{2}\left(\mu + \int\delta_4(t)dt\right)\right)\right]\right]dt\right),$$

where ρ is an arbitrary constant. Therefore, another exact solution has

$$\varpi(t,x) = -\frac{1}{2}\left[1 + \sqrt{3}\tan\left(\frac{\sqrt{3}}{2}\left(\mu + \int\delta_4(t)dt\right)\right)\right]$$
$$+\rho\exp\left(-\frac{1}{2\lambda^2}\int\left[2\delta_2(t) + \lambda^2\delta_4(t)\left[1 - \sqrt{3}\tan\left(\frac{\sqrt{3}}{2}\right.\right.\right.\right.$$
$$\left.\left.\left.\left.\times\left(\mu + \int\delta_4(t)dt\right)\right)\right]\right]dt\right)\exp\left(-\frac{x}{\lambda}\right). \tag{29}$$

Case 3. In this case, we assume $\delta_3(t) = \delta_3,\ \delta_4(t) = \delta_4,\ \delta_5(t) = \delta_5$. Then, second equation of (27), has

$$\mathcal{F}_1(t) = -\frac{1}{2\delta_4}\left[\delta_3 + \sqrt{4\delta_4\delta_5 - \delta_3^2}\tan\left(\frac{\sqrt{4\delta_4\delta_5 - \delta_3^2}}{2}(\mu + t)\right)\right],$$

where μ is an arbitrary constant. Similar to the previous case, if we substitute the acquired $\mathcal{F}_1(t)$, into the first equation of (27), we acquire

$$\mathcal{F}_2(t) = \rho\exp\left(-\frac{1}{2\lambda^2}\int\left[2\delta_2(t) + \lambda^2\delta_3\right.\right.$$
$$\left.\left.-\lambda^2\sqrt{4\delta_4\delta_5 - \delta_3^2}\tan\left(\frac{\sqrt{4\delta_4\delta_5 - \delta_3^2}}{2}(\mu + t)\right)\right]dt\right),$$

where ρ is an arbitrary constant. Therefore, another exact solution has

$$\varpi(t,x) = -\frac{1}{2\delta_4}\left[\delta_3 + \sqrt{4\delta_4\delta_5 - \delta_3^2}\tan\left(\frac{\sqrt{4\delta_4\delta_5 - \delta_3^2}}{2}(\mu + t)\right)\right]$$
$$+\rho\exp\left(-\frac{x}{\lambda} - \frac{1}{2\lambda^2}\int\left[2\delta_2(t) + \lambda^2\delta_3\right.\right.$$
$$\left.\left.-\lambda^2\sqrt{4\delta_4\delta_5 - \delta_3^2}\tan\left(\frac{\sqrt{4\delta_4\delta_5 - \delta_3^2}}{2}(\mu + t)\right)\right]dt\right). \tag{30}$$

Case 4. In this case, we assume $\delta_3(t) = \delta_3,\ \delta_4(t) = \delta_4,\ \delta_5(t) = \exp(t)$. Then, second equation of (27), has the following solution in terms of the Bessel functions[2]

$$\mathcal{F}_1(t) = \left(-\frac{\mu Y_{\delta_3+1}\left(2\sqrt{\delta_4}\exp(\frac{t}{2})\right)}{\sqrt{\delta_4}\left(Y_{\delta_3}\left(2\sqrt{\delta_4}\exp(\frac{t}{2})\right)\mu + J_{\delta_3}\left(2\sqrt{\delta_4}\exp(\frac{t}{2})\right)\right)}\right.$$

[2] $J_\nu(x)$, and $Y_\nu(x)$, are the Bessel functions of the first and second kinds, respectively. They satisfy Bessel's equation:

$$xy'' + xy' + (x^2 - \nu^2)y = 0.$$

$$\varpi(t,x)=\frac{\exp\left(-\int\delta_3(t)dt\right)}{\Psi+\mu}+\rho\exp\left(-\frac{x}{\lambda}-\int\left[\frac{\Psi(\lambda^2\delta_3(t)+\delta_2(t))+\lambda^2\delta_4(t)\exp(-\int\delta_3(t)dt)+\lambda^2\mu\delta_3(t)+\mu\delta_2(t)}{\lambda^2(\mu+\Psi)}\right]dt\right), \tag{28}$$

Box I.

$$-\frac{J_{\delta_3+1}\left(2\sqrt{\delta_4}\exp(\frac{t}{2})\right)}{\sqrt{\delta_4}\left(Y_{\delta_3}\left(2\sqrt{\delta_4}\exp(\frac{t}{2})\right)\mu+J_{\delta_3}\left(2\sqrt{\delta_4}\exp(\frac{t}{2})\right)\right)}\Bigg)\exp(\frac{t}{2}),$$

where μ is an arbitrary constant. Similar to the previous case, if we substitute the obtained $\mathcal{F}_1(t)$, into the first equation of (27), we obtain

$$\mathcal{F}_2(t)=\rho\exp\left(\frac{1}{\lambda^2}\int\left[\frac{\lambda^2\sqrt{\delta_4}(\mu Y_{\delta_3+1}(\vartheta)+J_{\delta_3+1}(\vartheta))}{\mu Y_{\delta_3}(\vartheta)+J_{\delta_3}(\vartheta)}-(\lambda^2\delta_3+\delta_2(t))\right]dt\right)$$

where ρ is an arbitrary constant, and $\vartheta=2\sqrt{\delta_4}\exp\left(\frac{t}{2}\right)$. Therefore, another exact solution has

$$\varpi(t,x)=\left(-\frac{\mu Y_{\delta_3+1}(\vartheta)}{\sqrt{\delta_4}\left(Y_{\delta_3}(\vartheta)\mu+J_{\delta_3}(\vartheta)\right)}-\frac{J_{\delta_3+1}(\vartheta)}{\sqrt{\delta_4}\left(Y_{\delta_3}(\vartheta)\mu+J_{\delta_3}(\vartheta)\right)}\right)\exp\left(\frac{t}{2}\right)+\rho\exp\left(-\frac{x}{\lambda}+\frac{1}{\lambda^2}\int\left[\frac{\lambda^2\sqrt{\delta_4}(\mu Y_{\delta_3+1}(\vartheta)+J_{\delta_3+1}(\vartheta))}{\mu Y_{\delta_3}(\vartheta)+J_{\delta_3}(\vartheta)}-(\lambda^2\delta_3+\delta_2(t))\right]dt\right), \tag{31}$$

where $\vartheta=2\sqrt{\delta_4}\exp\left(\frac{t}{2}\right)$.

Case 5. In this case, we assume $\delta_3(t)=\delta_3,\ \delta_4(t)=\delta_4,\ \delta_5(t)=\sin(t)$. Then, second equation of (27), has the following solution in terms of the Bessel functions[3]

$$\mathcal{F}_1(t)=-\frac{\delta_3}{2\delta_4}-\frac{1}{2\delta_4}\times\frac{MathieuSPrime(-\delta_3^2,-2\delta_4,-\frac{\pi}{4}+\frac{t}{2})\mu+MathieuCPrime(-\delta_3^2,-2\delta_4,-\frac{\pi}{4}+\frac{t}{2})}{MathieuS(-\delta_3^2,-2\delta_4,-\frac{\pi}{4}+\frac{t}{2})\mu+MathieuC(-\delta_3^2,-2\delta_4,-\frac{\pi}{4}+\frac{t}{2})},$$

where μ is an arbitrary constant. Similar to the previous case, if we substitute the obtained $\mathcal{F}_1(t)$, into the first equation of (27), we obtain

$$\mathcal{F}_2(t)=\rho\exp\Bigg(-\frac{1}{\lambda^2}\int\Bigg[\lambda^2\delta_3+2\delta_2(t)+\lambda^2\frac{MathieuSPrime(-\delta_3^2,-2\delta_4,-\frac{\pi}{4}+\frac{t}{2})\mu+MathieuCPrime(-\delta_3^2,-2\delta_4,-\frac{\pi}{4}+\frac{t}{2})}{MathieuS(-\delta_3^2,-2\delta_4,-\frac{\pi}{4}+\frac{t}{2})\mu+MathieuC(-\delta_3^2,-2\delta_4,-\frac{\pi}{4}+\frac{t}{2})}\Bigg]dt\Bigg),$$

where ρ is an arbitrary constant, and $\vartheta=2\sqrt{\delta_4}\exp\left(\frac{t}{2}\right)$. Therefore, another exact solution has the following form

$$\varpi(t,x)=-\frac{\delta_3}{2\delta_4}-\frac{1}{2\delta_4}\times\frac{MathieuSPrime(-\delta_3^2,-2\delta_4,-\frac{\pi}{4}+\frac{t}{2})\mu+MathieuCPrime(-\delta_3^2,-2\delta_4,-\frac{\pi}{4}+\frac{t}{2})}{MathieuS(-\delta_3^2,-2\delta_4,-\frac{\pi}{4}+\frac{t}{2})\mu+MathieuC(-\delta_3^2,-2\delta_4,-\frac{\pi}{4}+\frac{t}{2})}+\rho\exp\Bigg(-\frac{x}{\lambda}-\frac{1}{\lambda^2}\int\Bigg[\lambda^2\delta_3+2\delta_2(t)+\lambda^2\frac{MathieuSPrime(-\delta_3^2,-2\delta_4,-\frac{\pi}{4}+\frac{t}{2})\mu+MathieuCPrime(-\delta_3^2,-2\delta_4,-\frac{\pi}{4}+\frac{t}{2})}{MathieuS(-\delta_3^2,-2\delta_4,-\frac{\pi}{4}+\frac{t}{2})\mu+MathieuC(-\delta_3^2,-2\delta_4,-\frac{\pi}{4}+\frac{t}{2})}\Bigg]dt\Bigg). \tag{32}$$

[3] The Mathieu functions $MathieuC(a,q,x)$ and $MathieuS(a,q,x)$ are solutions of the Mathieu differential equation:

$$y''+(a-2q\cos(2x))y=0,$$

MathieuC and MathieuS are even and odd functions of x, respectively. Moreover, MathieuCPrime, and MathieuSPrime are the first derivatives with respect to x of the corresponding Mathieu functions.

Case 6. In this case, we assume $\mathcal{F}_1(t)=\mu$, and $\delta_5(t)=-\mu^2\delta_4(t)-\mu\delta_3(t)$. Then, from (27), we obtain

$$\mathcal{F}_2(t)=\rho\exp\left(-\frac{1}{\lambda^2}\int\left[\mu\lambda^2\delta_4(t)+\lambda^2\delta_3(t)+\delta_2(t)\right]dt\right),$$

where ρ is an arbitrary constant. Therefore, another exact solution has the following form

$$\varpi(t,x)=\mu+\rho\exp\left(-\frac{x}{\lambda}-\frac{1}{\lambda^2}\int\left[\mu\lambda^2\delta_4(t)+\lambda^2\delta_3(t)+\delta_2(t)\right]dt\right). \tag{33}$$

Results and discussion

Differential equations with constant coefficients are equations where the coefficients of the highest order derivative to its lower order derivatives are constants. These equations have well-defined methods for finding their solutions, and their behavior is predictable. In contrast, differential equations with variable coefficients are equations where the coefficients of the derivatives can vary as a function of the independent variable(more specially in temporal sense). These equations often require more sophisticated methods to find their solutions, and their behavior can be more complex and difficult to predict. Differential equations with variable coefficients arise in many physical and engineering problems, where the coefficients may represent physical properties that vary with position or time.

The solutions of the Fisher partial differential equation have various physical interpretations depending on the specific problem being modeled. In general, the solutions describe the spatiotemporal distribution of a population undergoing logistic growth with diffusion. The solution can represent the spread of an advantageous gene, the distribution of a species in a heterogeneous environment, or the dynamics of a disease outbreak in a population, among other applications. The equation predicts the formation of traveling wave solutions, which correspond to the propagation of a population front through space. These wave solutions have been observed in various natural systems, such as the spread of fire, insect outbreaks, and the invasion of species into new habitats. The solutions of the Fisher equation have important implications for understanding the spread of populations and the development of spatial patterns in biology and ecology.

Corresponding diffusion constant in (1), is $\delta_2(t)$. In Fig. 1, we compare the obtained solution (28), in the constant diffusion $\delta_2(t)=1$ and variable diffusion term $\delta_2(t)=\sinh(t)$. The main difference between these profiles is the "hump" and this is a direct result of the variable diffusion term produced by $\delta_2(t)$. The evolution of this hump occur near the initial data and it holds biological significance.

In Fig. 2, we compare the obtained solution (29), in the constant diffusion $\delta_2(t)=1$ and variable diffusion term $\delta_2(t)=t^2$.

In the Case 3, Eq. (1) is considered with the constant coefficients and variable diffusion coefficient $\delta_2(t)$. From the real-world application point of view, the nonlinear term $\delta_4(t)\varpi^2$, plays a central role in the bacterial evolution. Therefore, we try to compare the reported solutions in (30), with different values of bacterial evolution term coefficient δ_4. In Fig. 3, corresponding profiles and density plots of (30) with $\delta_4=20$, and $\delta_4=4$, are considered.

Fig. 4, which is corresponding to the Case 4, demonstrates two dimensional and density plots of (31) with the non-homogeneous term $\delta_5(t)=\exp(t)$, and evolution coefficient $\delta_4=2$.

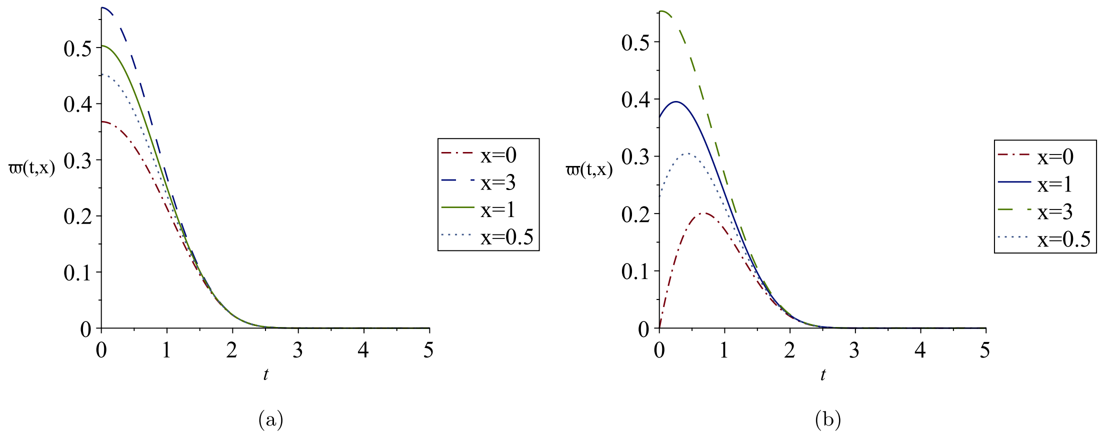


Fig. 1. Two dimensional profile of $\varpi(t,x)$, in (28) with respect to $\delta_3(t) = \delta_4(t) = \sinh(t)$, $\lambda = \rho = \mu = 1$, and (a) $\delta_2(t) = \sinh(t)$, (b) $\delta_2(t) = 1$.

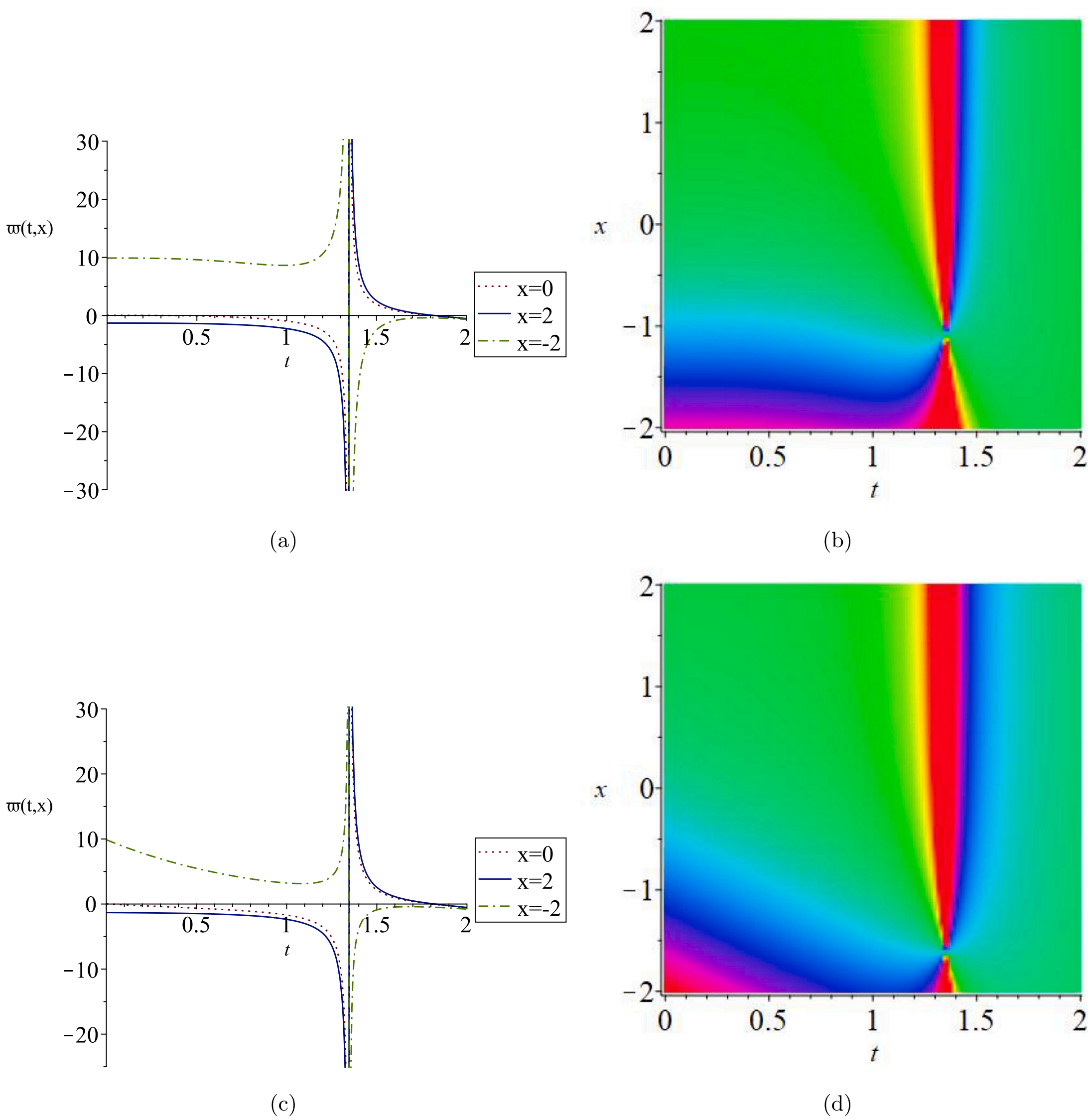


Fig. 2. Two dimensional profile and density plots of $\varpi(t,x)$, in (29) with respect to $\delta_4(t) = t^2$, $\lambda = \rho = \mu = 1$, and (a)–(b) $\delta_2(t) = t^2$, (c)–(d) $\delta_2(t) = 1$.

Fig. 5, which is corresponding to the Case 5, demonstrates two dimensional and density plots of (32) with $\delta_2(t) = \sinh(t)$, and $\delta_3 = 0$.

Fig. 6, which is corresponding to the Case 5, demonstrates two dimensional and density plots of (33) with $\delta_2(t) = \sinh(t)$, and $\delta_3 = 0$.

Conclusion

In this study, we investigated the non-classical Lie symmetries of the FTE by iterative technique, namely the heir-equation method. Instead

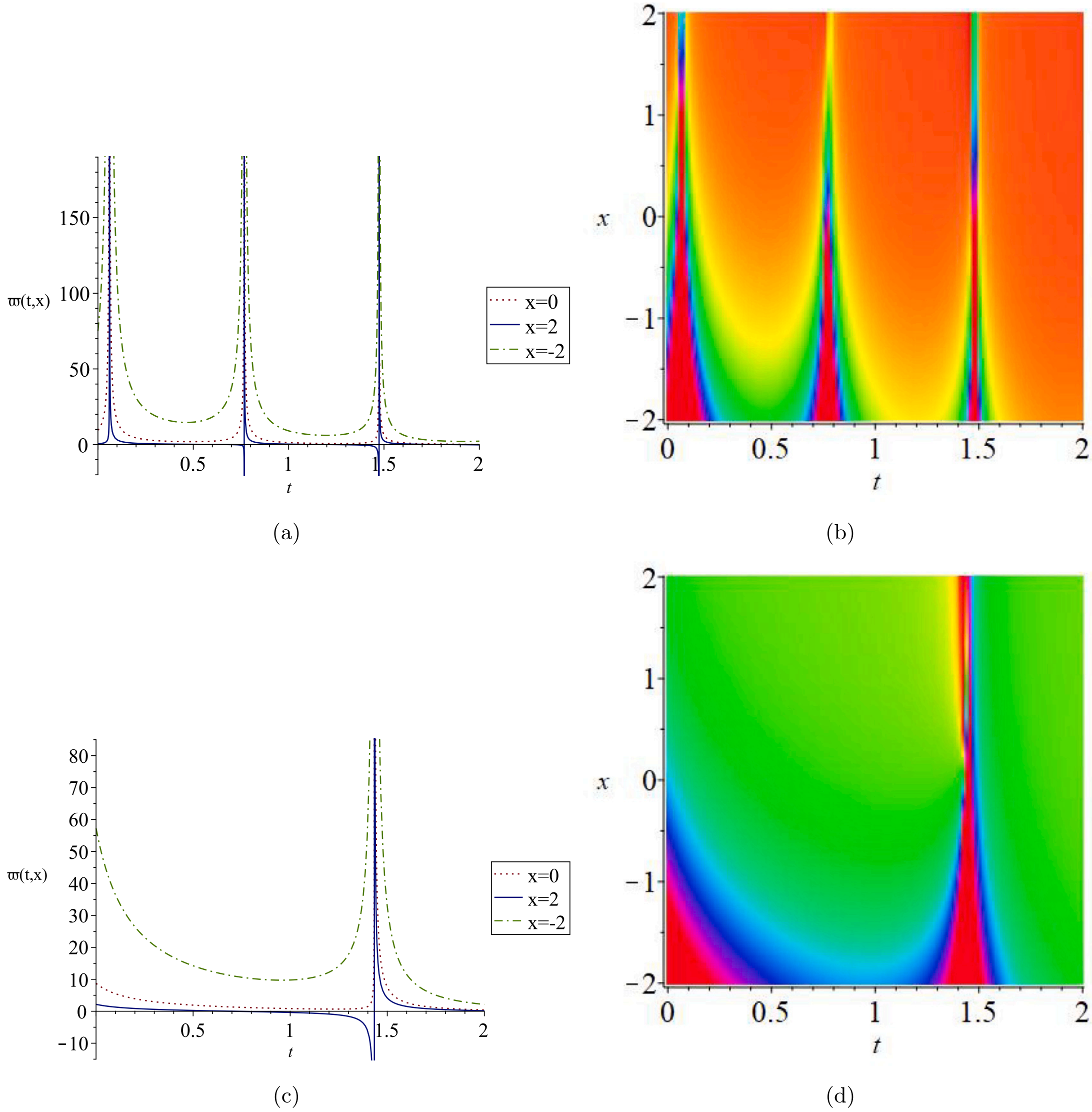


Fig. 3. Two dimensional profile and density plots of $\varpi(t,x)$, in (30) with respect to $\delta_3 = \delta_5 = \lambda = \rho = \mu = 1$, $\delta_2(t) = \sin(t)$, and (a)–(b) $\delta_4 = 20$, (c)–(d) $\delta_4 = 4$.

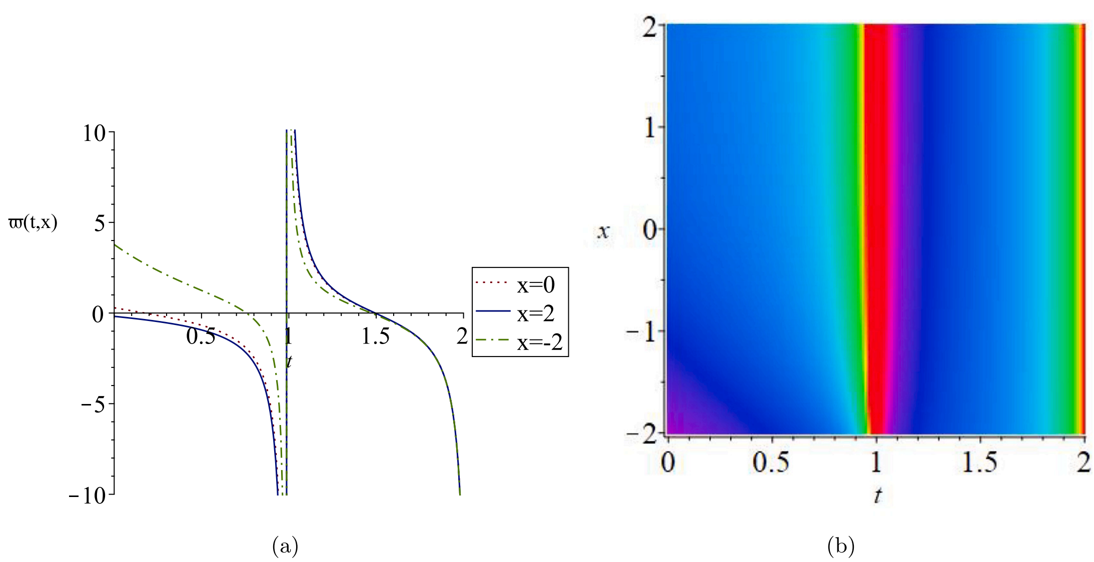
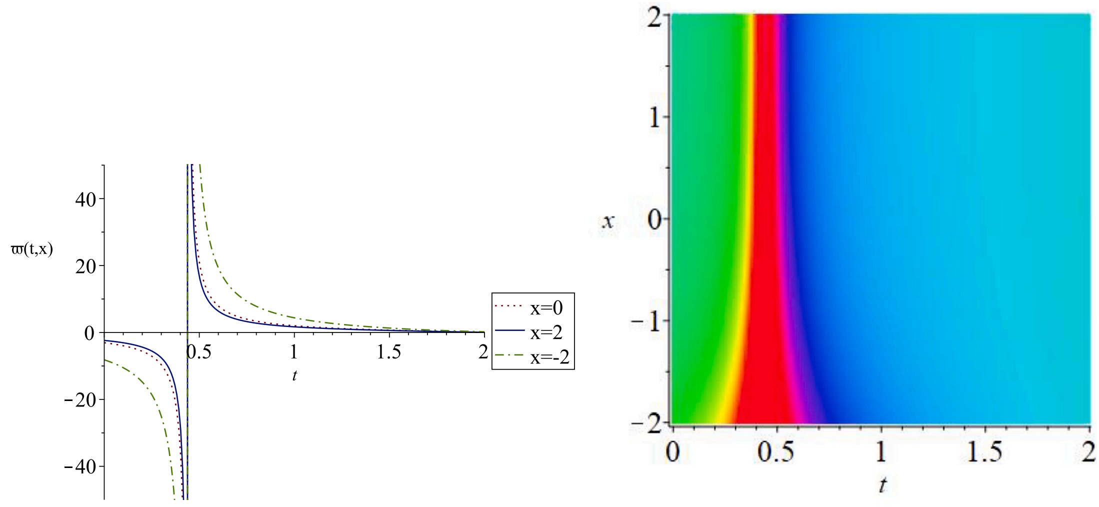
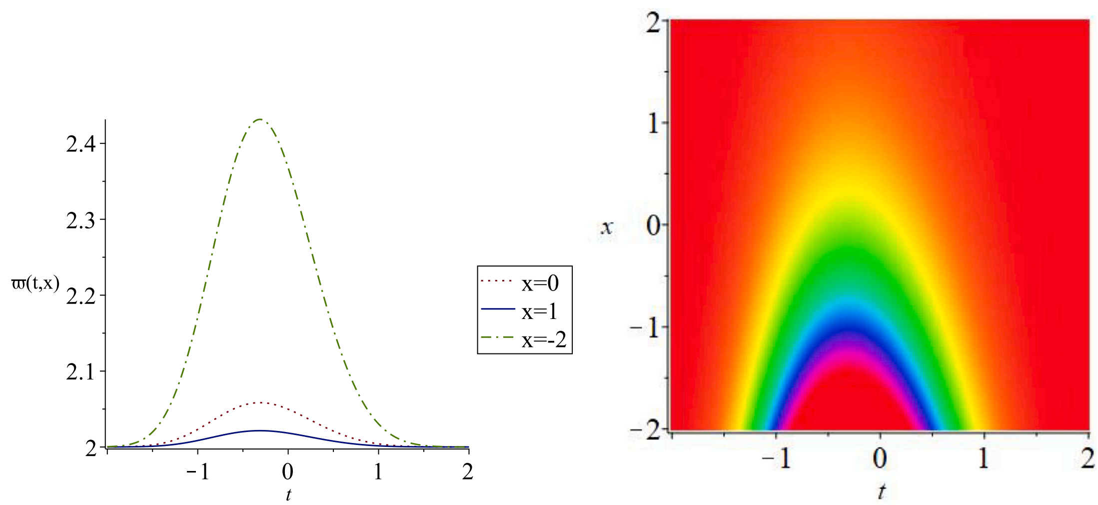


Fig. 4. Two dimensional profile and density plots of $\varpi(t,x)$, in (31) with respect to $\delta_3 = \lambda = \rho = \mu = 1$, $\delta_2(t) = \exp(t)$, and $\delta_4 = 2$.



(a) (b)

Fig. 5. Two dimensional profile and density plots of $\varpi(t,x)$, in (32) with respect to $\delta_3(t)=\delta_4(t)=\sinh(t)$, $\lambda=\rho=1$, $\mu=2$, and $\delta_2(t)=\mathrm{sech}(t)$.



(a) (b)

Fig. 6. Two dimensional profile and density plots of $\varpi(t,x)$, in (33) with respect to $\delta_3(t)=\delta_4(t)=\sinh(t)$, $\lambda=\rho=1$, $\mu=2$, and $\delta_2(t)=\mathrm{sech}(t)$.

of solving the nonlinear determining equations in the non-classical LSM, we obtained the heir-equations correspond to the appropriate prolongations by the aid of the Maple package. Then by solving a simple second order $\mathcal{G}^{[2]}$-equation, we obtained a series of the exact solutions. Some special exact solutions are reported in 6 cases with different types of solutions with respect to the time dependent coefficients of the original equation. Exact solutions are reported by this technique in terms of the exponential solutions, trigonometric solutions, Bessel solutions, and Mathieu solutions.

Funding

National Natural Science Foundation of China (No. 71601072), Key Scientific Research Project of Higher Education Institutions in Henan Province of China (No. 20B110006), and the Fundamental Research Funds for the Universities of Henan Province, China (No. NSFRF210314).

CRediT authorship contribution statement

Shao-Wen Yao: Funding acquisition, Researcher. **Mir Sajjad Hashemi:** Writing – original draft, Investigation. **Mustafa Inc:** Writing – review & editing, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

Acknowledgments

We are grateful to the editor and reviewers for their valuable comments which have significantly contributed to the improvement of the quality of our paper.

References

[1] Jajarmi A, Baleanu D, Vahid KZ, Pirouz HM, Asad J. A new and general fractional Lagrangian approach: A capacitor microphone case study. Results Phys 2021;31:104950.
[2] Li X, Lei M, Mu Q, Ren K. Thermo-mechanical modeling of thermal stress during multi-cycle intense pulsed light sintering of thick conductive wires on 3D printed dark substrate. Results Phys 2023;44:106192.
[3] Akgül A, Inc M, Hashemi MS. Group preserving scheme and reproducing kernel method for the Poisson–Boltzmann equation for semiconductor devices. Nonlinear Dynam 2017;88(4):2817–29.
[4] Souleymanou A, Ali KK, Rezazadeh H, Eslami M, Mirzazadeh M, Korkmaz A. The propagation of waves in thin-film ferroelectric materials. Pramana 2019;93(2):1–6.
[5] Rezazadeh H, Mirzazadeh M, Mirhosseini-Alizamini SM, Neirameh A, Eslami M, Zhou Q. Optical solitons of Lakshmanan–Porsezian–Daniel model with a couple of nonlinearities. Optik 2018;164:414–23.
[6] Rezazadeh H, Mirhosseini-Alizamini SM, Eslami M, Rezazadeh M, Mirzazadeh M, Abbagari S. New optical solitons of nonlinear conformable fractional Schrödinger-Hirota equation. Optik 2018;172:545–53.
[7] Cinar M, Secer A, Bayram M. Solving nonlinear fractional PDEs using novel wavelet collocation method. New Trends Math Sci 2022;10(1):71–6.
[8] Ozdemir N, Secer A, Bayram M. The Gegenbauer wavelets-based computational methods for the coupled system of Burgers' equations with time-fractional derivative. Mathematics 2019;7(6):486.
[9] Hashemi MS, Inc M, Seyfi N, Bayram M. Two reliable methods for solving the forced convection in a porous-saturated duct. Eur Phys J Plus 2020;135(1):1–18.
[10] Abbasbandy S, Roohani Ghehsareh H, Hashim I, Alsaedi A. A comparison study of meshfree techniques for solving the two-dimensional linear hyperbolic telegraph equation. Eng Anal Bound Elem 2014;47:10–20.
[11] Aslefallah M, Abbasbandy S, Shivanian E. Fractional cable problem in the frame of meshless singular boundary method. Eng Anal Bound Elem 2019;108:124–32.
[12] Hashemi MS, Baleanu D. Lie symmetry analysis of fractional differential equations. CRC Press; 2020.
[13] Hashemi MS, Inc M, Bayram M. Symmetry properties and exact solutions of the time fractional Kolmogorov-Petrovskii-Piskunov equation. Rev Mex de Física 2019;65(5):529–35.
[14] Aliyu AI, Inc M, Yusuf A, Bayram M, Baleanu D. Symmetry reductions, explicit solutions, convergence analysis and conservation laws via multipliers approach to the Chen–Lee–Liu model in nonlinear optics. Modern Phys Lett B 2019;33(04):1950035.
[15] Aksoy E, Kaplan M, Bekir A. Exponential rational function method for space–time fractional differential equations. Waves Random Complex Media 2016;26(2):142–51.
[16] Khodadad F, Mirhosseini-Alizamini SM, Günay B, Akinyemi L, Rezazadeh H, Inc M. Abundant optical solitons to the Sasa-Satsuma higher-order nonlinear Schrödinger equation. Opt Quantum Electron 2021;53(12):1–17.
[17] Raheel M, Inc M, Tala-Tebue E, Bayram M. Optical solitons of the Kudryashov equation via an analytical technique. Opt Quantum Electron 2022;54(6):1–12.
[18] Ozisik M, Cinar M, Secer A, Bayram M. Optical solitons with Kudryashov's sextic power-law nonlinearity. Optik 2022;261:169202.
[19] Hashemi MS. Invariant subspaces admitted by fractional differential equations with conformable derivatives. Chaos Solitons Fractals 2018;107:161–9.
[20] Sahadevan R, Bakkyaraj T. Invariant subspace method and exact solutions of certain nonlinear time fractional partial differential equations. Fract Calc Appl Anal 2015;18(1):146–62.
[21] Gazizov RK, Kasatkin AA. Construction of exact solutions for fractional order differential equations by the invariant subspace method. Comput Math Appl 2013;66(5):576–84.
[22] Chu Y-M, Inc M, Hashemi MS, Eshaghi S. Analytical treatment of regularized Prabhakar fractional differential equations by invariant subspaces. Comput Appl Math 2022;41(6):271.
[23] Hashemi MS. A novel approach to find exact solutions of fractional evolution equations with non-singular kernel derivative. Chaos Solitons Fractals 2021;152:111367.
[24] Fisher RA. The wave of advance of advantageous genes. Ann Eugen 1937;7(4):355–69.
[25] de Pablo A, Sánchez A. Travelling wave behaviour for a porous-Fisher equation. European J Appl Math 1998;9(3):285–304.
[26] Debnath L, Debnath L. Nonlinear partial differential equations for scientists and engineers. Springer; 2005.
[27] Gao Y-T, Tian B. Symbolic computation for the Fisher-type equation with variable coefficients. Internat J Modern Phys C 2001;12(08):1251–9.
[28] Bluman GW, Cole JD. The general similarity solution of the heat equation. J Math Mech 1969;18(11):1025–42.
[29] Nucci MC. Iterating the nonclassical symmeteries method. Physica D 1994;78(1–2):124–34.
[30] Goard J. Generalised conditional symmetries and Nucci's method of iterating the nonclassical symmetries method. Appl Math Lett 2003;16(4):481–6.