Mechanics Research Communications 56 (2014) 31–36

Contents lists available at ScienceDirect

Mechanics Research Communications

journal homepage: www.elsevier.com/locate/mechrescom

Fractional viscoplasticity

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ARTICLE INFO

Article history:
Received 3 April 2013
Received in revised form
28 September 2013
Accepted 15 November 2013
Available online 23 November 2013

Keywords:
Viscoplasticity
Fractional calculus
Phenomenological models

ABSTRACT

In this paper we generalize the Perzyna's type viscoplasticity using fractional calculus. We call such model *fractional viscoplasticity*. The main objective of this research is to propose a new way of description of permanent deformation in a material body, especially under extreme dynamic conditions. In this approach the fractional calculus can be understood as a tool enabling the introduction of material heterogeneity/multi-scale effects to the constitutive model.

This newly developed phenomenological model is represented in the Euclidean space living more general setup for future work. The definition of the directions of a viscoplastic strains stated as a *fractional gradient* of plastic potential plays the fundamental role in the formulation. Moreover, the fractional gradient provides the non-associative plastic flow without necessity of additional potential assumption.

1. Introduction

Despite the fact that in most recent years micromechanical models describing material behavior are widely considered (cf. Stupkiewicz and Petryk, 2010; Coenen et al., 2012; Tasan et al., 2012) there is still need for new concepts in phenomenology particularly for models dedicated for extreme dynamic events (Sumelka and Łodygowski, 2011; Rusinek et al., 2007). In principal, micromechanics provides naturally deeper insight into the physical phenomena being considered but from the other point of view it appears that such formulations are still not suitable for extreme dynamic processes where wave effects play fundamental role and also current software/hardware capabilities are not good enough. On the other hand the most important drawback in phenomenology, in contrary to micromechanics, is that many material parameters need to be identified for practical applications when many phenomena are considered (e.g. thermo-mechanical coupling including anisotropic description of damage or phase transformations) (Eftis et al., 2003; Glema et al., 2009; Sumelka, 2009). Hence, the crucial tasks for research in the are of phenomenology is to simplify material functions considered or develop additional techniques such as soft computing reducing the number of considered material parameters (Sumelka and Łodygowski, 2013a). It is shown that to some extend fractional viscoplasticity can be viewed as a solution to described circumstances. In other words fractional calculus can be understood as a tool for introduction of material heterogeneity/multi-scale effects to the constitutive model (Sumelka, 2013a).

Independently of chosen technique describing experimentally observed body behavior the fundamental question arises: do we use correct mathematical tools for the description of the material body deformation? More precisely in aspect of the subject of this paper: are commonly used differential operators in the particular model correctly assumed to be of integer order or one should choose more general one, namely the differential operators of an arbitrary order? The answer to such question is not obvious. Considering many successful applications of fractional calculus in Fluid Flow, Rheology, Dynamical Processes in Self-Similar and Porous Structures, Diffusive Transport Akin to Diffusion, Electrical Networks, Probability and Statistics, Control Theory of Dynamical Systems, Viscoelasticity, Electrochemistry of Corrosion, Chemical Physics, Optics and others (Podlubny, 1999; Tarasov, 2008; Mainardi, 2010 and cited therein) one can be more than sure that in the theories describing permanent deformation of a body, such as viscoplasticity/plasticity, the use of fractional calculus should be appropriate.

In several papers (Sumelka, 2012a,b) the original idea of *fractional viscoplasticity* is introduced. Fractional viscoplasticity is generalization of classical Perzyna's type viscoplasticity (Perzyna, 1963) using fractional calculus. The fundamental role in the formulation plays the definition of the directions of a viscoplastic strains given as a *fractional gradient* of plastic potential. In this way one obtains a flexible tool that controls viscoplastic strain evolution (magnitude and directions) without necessity of adding explicitly new phenomena to the constitutive structure. Moreover, by introducing the new parameter to the model (order of derivative) we simultaneously obtain the non-associative plastic flow

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0093-6413/$ – see front matter © 2013 Elsevier Ltd. All rights reserved.
http://dx.doi.org/10.1016/j.mechrescom.2013.11.005

(in general) without necessity of additional potential assumption. This way we we decrease the number of material functions and simultaneous parameters. Classical Perzyna's solution is obtained as a special case - when order of fractional gradient is assumed to be equal to one.

As opposed to the previous works (Sumelka, 2012a,b) this paper provides complete description of this topic. Different fractional differential operator for fractional gradient of plastic potential is applied as well. Thus, fractional viscoplasticity can be redefined using different definitions of fractional derivative. In this sense there should exist optimal definition of derivative for a specific material, such as steel, rubber or concrete. As an example, the definition of classical Riesz–Feller fractional operator (Feller, 1952) (not discussed here) has an origin in processes with Lévy stable probability distribution. In this sense it should be possible to define fractional differential operator in such a way that it carries information about the distribution of grains sizes in a particular metal.

The paper is divided into three main parts.

In Section 2 fundamental concepts of fractional calculus are presented to justify assumptions imposed during the fractional viscoplasticity definition.

The fractional viscoplasticity in Euclidean space, leaving more general setup for future work, is defined in Section 3 along with the fractional viscoplatic strain gradient of Caputo's type. Because the fractional derivative is defined on interval (contrary to standard definition of derivative in a point) so called “short memory” principle (Podlubny, 1999) is utilized to make bounds of this interval with clear physical interpretation.

In Section 4 illustrative example showing the dependence of the direction of the viscoplastic flow plotted against the order of fractional gradient is discussed to prove that in general the non-associative plastic flow without necessity of additional potential assumption is obtained.

2. Fractional calculus – fundamental concepts

The theory of derivatives of non-integer order was initiated on 30th of September 1695 when Leibniz showed his concerns about the L'Hospital's derivative of order one and a half (Leibniz, 1962). The breakthrough sentence by Leibniz stated: “It will lead to a paradox from which one day useful consequences will be drawn”. Since that day fractional calculus became an individual branch of pure mathematics with many successful applications. It was discussed in many comprehensive encyclopedic-type monographs e.g. (Samko et al., 1993; Podlubny, 1999; Kilbas et al., 2006; Leszczyński, 2011).

Although there are numerous definitions for fractional differential operators they share the common attribute: they are defined on an interval in contrary to the integer order differential operators defined in a single point. The most commonly used are those defined by generalization of n-fold integration or n-fold derivative. To understand the idea let us consider the n-fold integration of a function f which is given by

$$f^{(-n)}(t) = \frac{1}{\Gamma(n)} \int_a^t (t-\tau)^{n-1} f(\tau) d\tau, \quad t > a, \quad n \in \mathbb{N}, \tag{1}$$

where Γ is the Euler gamma function defined as (a is arbitrary)

$$\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt. \tag{2}$$

Notice that if in Eq. (2) we apply $\alpha = n \in \mathbb{N}\backslash\{0\}$ we have $\Gamma(n) = (n-1)\,!$. Now, if we replace in Eq. (1) n with an arbitrary $\alpha > 0$ we obtain (left) fractional integral operator in Riemann–Liouville (RL) sense

$${}_aI_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau) d\tau, \quad t > a, \quad n \in \mathbb{R}^+. \tag{3}$$

Based on relation Eq. (3) one can define the following fractional derivatives (left sided)

$${}_a^{RL}D_t^\alpha f(t) = D^m({}_aI_t^{m-\alpha} f)(t), \tag{4}$$

$${}_a^{C}D_t^\alpha f(t) = {}_aI_t^{m-\alpha} (D^m f)(t), \tag{5}$$

where $m = [\alpha] + 1$, ${}_a^{RL}D_t^\alpha f(t)$ and ${}_a^{C}D_t^\alpha f(t)$ defines fractional derivatives in Riemann–Liouville (RL) and Caputo (C) sense, respectively.

As already mentioned the derivatives of an arbitrary order (even complex) are defined on an interval, thus one can define so called left and right sided derivatives. Considering Caputo (C) type derivative (the one used during fractional viscoplasticity definition) as an example, the explicit definitions are: left-sided Caputo's derivative for $t > a$ and $n = [\alpha] + 1$

$${}_a^{C}D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau; \tag{6}$$

and right-sided Caputo derivative for $t < b$ and $n = [\alpha] + 1$

$${}_t^{C}D_b^\alpha f(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_t^b \frac{f^{(n)}(\tau)}{(\tau-t)^{\alpha-n+1}} d\tau, \tag{7}$$

where $\alpha > 0$ denotes the real order of the derivative, D denotes ‘derivative’ and a, t, b are so called terminals. Notice that both definitions include integration over the interval (a, t) or (t, b), respectively. The terminals a and b can be chosen arbitrarily. Nevertheless, for fractional viscoplasticity definition we will use so called “short memory” principle (Podlubny, 1999) for terminal definition for clearer physical interpretation. It is clear that terminals must not be constant during the deformation – e.g. they can be a function of state variables.

As an illustrative example let us consider Caputo derivative of a power function $f(t) = (t - a)^\nu$, in this case we have

$${}_a^{C}D_t^\alpha (t-a)^\nu = \frac{\Gamma(\nu+1)}{\Gamma(-\alpha+\nu+1)} (t-a)^{\nu-\alpha}, \tag{8}$$

or optionally for $f(t) = (b - t)^\nu$ we have

$${}_t^{C}D_b^\alpha (b-t)^\nu = \frac{\Gamma(\nu+1)}{\Gamma(-\alpha+\nu+1)} (b-t)^{\nu-\alpha}, \tag{9}$$

and important result in case $f(t) = C = const$.

$${}_t^{C}D_b^\alpha C = {}_a^{C}D_t^\alpha C = 0. \tag{10}$$

In general any type of the fractional derivative of a constant function is **not** equal to zero (Caputo's derivative is an exception Eq. (10)). It is also fundamental that using Caputo's type derivative one needs standard (like in the classical differential equations) initial and/or boundary conditions, while for other types of fractional derivatives (e.g. RL) they are of a different type dependently of chosen definition.

Finally, let us define the Caputo's type derivative for interval $t \in (a, b)$. We call such derivative Riesz–Caputo (RC) derivative cf. (Frederico and Torres, 2010). This type of fractional derivative is crucial for further definition of directions of viscoplastic strains. Since any linear combination of derivatives Eqs. (6) and (7) defines new derivative (Samko et al., 1993), we put for $t \in (a, b) \subseteq \mathbb{R}$, $\alpha > 0$ and $n - 1 < \alpha < n$ (Agrawal, 2007) (when α is an integer, the usual definition of a derivative is used cf. (Agrawal, 2007; Frederico and Torres, 2010))

$${}_a^{RC}D_b^\alpha f(t) = \frac{1}{2}\left({}_a^{C}D_t^\alpha f(t) + (-1)^n\,{}_t^{C}D_b^\alpha f(t)\right). \tag{11}$$

In the remaining part of the paper the RC derivative is denoted as D^α with the possibility of writing variable under the D in case of partial differentiation of multivariate functions. For example $\underset{X_1}{D^\alpha} f$ means the partial fractional derivative of f with respect to the variable X_1 over the interval which should be explicitly defined before $X_1 \in (a, b)$. It is important that for $\alpha = 1$ we have

$$ {}_a^{RC}D_b^1 f(t) = \frac{\mathrm{d}}{\mathrm{d}t} f(t), \tag{12} $$

this property enables to recover classical Perzyna model from fractional viscoplasticity.

3. Fractional viscoplasticity model

It is important to emphasize that the results presented in this section represent the most simple formulation of Perzyna's type viscoplasticity to outline the ides of fractional viscoplasticity. It is clear, however, that further development of the formulation to finite deformations, thermal effects and anisotropic damage (cf. recent progress in Perzyna's type viscoplasticity by the author in Perzyna (2008),Sumelka (2009),Glema et al. (2009),Sumelka and Łodygowski (2011),Sumelka (2013c),Sumelka and Łodygowski (2013b) is straightforward but not discussed here.

Let us generalize the pioneer results given by Perzyna in Perzyna (1963). It should be emphasized that the presented formulation assumes classical kinematic description. However, it is possible to extend the overall concept by introducing the non-local fractional continua concept (Sumelka, 2013b). In this paper we omit such considerations for clarity.

Under small strain assumption we accept classical additive decomposition of total strain, namely (Lubliner, 1990)

$$ \boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^e + \boldsymbol{\varepsilon}^{vp}, \tag{13} $$

where $\boldsymbol{\varepsilon}$ stands for total strain and $\boldsymbol{\varepsilon}^e$, $\boldsymbol{\varepsilon}^{vp}$ denote elastic and viscoplastic parts, respectively. Utilizing Eq. (13) we have at once its rate form such as

$$ \dot{\boldsymbol{\varepsilon}} = \dot{\boldsymbol{\varepsilon}}^e + \dot{\boldsymbol{\varepsilon}}^{vp}. \tag{14} $$

As usual, the elastic strains are governed by Hooke's law

$$ \dot{\boldsymbol{\sigma}}^e = \mathcal{L}^e : \dot{\boldsymbol{\varepsilon}}^e, \tag{15} $$

where $\dot{\boldsymbol{\sigma}}^e$ denotes Cauchy's stresses and $\mathcal{L}^e$ denotes elastic constitutive tensor.

Definition of rate of viscoplastic strains needs detailed explanation. Thus, as in classical viscoplasticity we postulate that the rate of viscoplastic strains can be written as

$$ \dot{\boldsymbol{\varepsilon}}^{vp} = \Lambda \mathbf{p}, \tag{16} $$

where Λ is a scalar multiplier (the intensity of viscoplastic flow) and $\mathbf{p}$ represented as the second order tensor defines the direction of the flow (Euclidean norm of $\mathbf{p}$ is one). The explicit definition of Λ is assumed as Perzyna (1963)

$$ \Lambda = \gamma < \Phi(F) >, \tag{17} $$

where γ is the viscosity parameter, $<\cdot>$ denotes the Macaulay brackets, Φ is an overstress function and F is a statical yield function. We postulate at once that

$$ F = \frac{f}{\kappa} - 1, \tag{18} $$

where $f = \sqrt{J_2'}$, J_2' is a second invariant of stress deviator $\boldsymbol{\sigma}'$ and κ is the yield stress in simple shear. It is clear that when $f \le \kappa$ the rate of viscoplastic strain is zero and material yields for $f > \kappa$.

Let us consider the direction of yield defined by $\mathbf{p}$. It is common to assume in the classical viscoplasticity that the directions of the yield are treated as normal to yield surface (associated flow). Thus, F is a plastic potential, namely

$$ \mathbf{p} = \frac{\partial F}{\partial \boldsymbol{\sigma}} \left\| \frac{\partial F}{\partial \boldsymbol{\sigma}} \right\|^{-1} = \frac{\boldsymbol{\sigma}'}{\sqrt{J_2'}}. \tag{19} $$

Recall that Eq. (19) induces the viscoplastic strains being coaxial with stress deviator and of deviatoric type (does not cause volume change and are normal to yield surface). Hence, the volume change is controlled only by elastic deformation. In order to control volume change in viscoplastic range (in classical case) one needs to assume different potential in Eq. (19) (non-associated flow). Therefore, additional material function is needed what causes that the number of material parameters is greater.

The situation in which Eq. (19) does not hold for a particular real material is common. Materials such as the cast iron (Hjelm, 1994), granular/polymer, crushable foam (Gibson et al., 1982; Deshpande and Fleck, 2000), or geomaterials (Drucker and Prager, 1952) and many others provide the necessity of non-associate flow assumption. Hence, as previously mentioned, for classical models one needs to postulate additional potential (and simultaneously material parameters) to describe their behavior. As an alternative solution, we can apply fractional gradient of plastic potential. This way we obtain a flexible tool that controls viscoplastic strain evolution (their magnitude and directions) without necessity of adding explicitly new phenomena to the constitutive structure. Heterogeneity/multi-scale effects of a real material can be somehow introduced to the constitutive model by the order and type of fractional derivative applied.

Let us now reformulate the relation given by Eq. (19) using fractional calculus (Sumelka, 2012a). In such case we have

$$ \mathbf{p} = \underset{\boldsymbol{\sigma}}{D^\alpha} F \left\| \underset{\boldsymbol{\sigma}}{D^\alpha} F \right\|^{-1}. \tag{20} $$

Eq. (20) defines new class of viscoplasticity (plasticity) which we call *fractional viscoplasticity*, giving a classical formulation as a special case for $\alpha = 1$. As shown in the next section utilizing definition in Eq. (20) allows to obtain the non-associated flow (without necessity of additional potential assumption). It is important that D^α is defined on an interval, thus flow direction Eq. (20) keeps somehow more information about f than classical relation Eq. (19) which definition is given strictly in a point of interest. It can be concluded that using the fractional calculus one can introduce to a phenomenological model a complex information about the real material behavior not as in standard way using set of additional material functions but through the proper selection of the type of fractional derivative. Thus, simultaneously the number of material parameters can be reduced.

At the end of this section let us emphasize that using fractional differential operators allows us to obtain different physical units in contrary to standard one. As an example in Eq. (19) the unit of the first term is $[\partial F/\partial \boldsymbol{\sigma}] = [Pa]$ while in Eq. (20) we have $\left[\underset{\boldsymbol{\sigma}}{D^\alpha} F\right] = [Pa]^{2-\alpha}$. Nevertheless, in both cases, the proper normalization by norms $||\partial F/\partial \boldsymbol{\sigma}||^{-1}$ (cf. Eq. (19)) and $\left\|\underset{\boldsymbol{\sigma}}{D^\alpha} F\right\|^{-1}$, respectively causes that $[\mathbf{p}] = [-]$.

4. Example – fractional viscoplastic flow

Let us point out that the presented results are general and can be repeated for arbitrarily selected yield surface. It is clear that for complicated yield surfaces definition a numerical approach is necessary for fractional derivatives - cf. utilization of the approximations of Caputo's operator based on Grünwald-Letnikov one

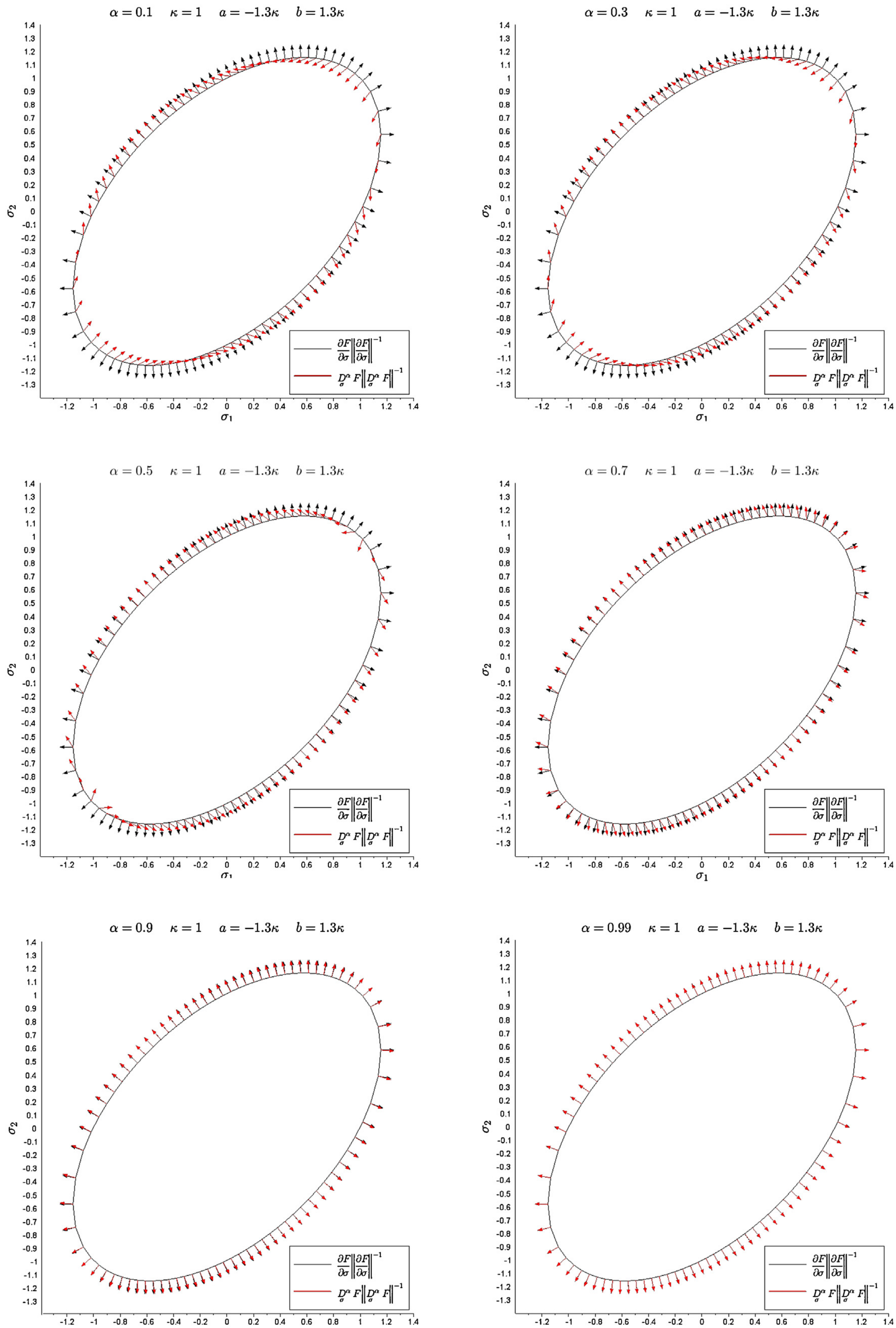

Fig. 1. The comparison of fractional vs. classical viscoplastic strains directions on HMH yield surface in $\sigma_1 - \sigma_2$ plane (for fractional one α changes while terminals a and b remain constant).

(Samko et al., 1993; Podlubny, 1999; Kilbas et al., 2006) discussed in Diethelm et al. (2005), Odibat (2006), Leszczyński (2011). Due of simplicity of the proceeded example the analytical solution is also obtained.

Let us consider a yield surface of the Huber–Mises–Hencky (HMH) (Huber, 2004) type written in a stress principal directions

$$F = 0 \rightarrow \sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \sigma_1\sigma_2 - \sigma_2\sigma_3 - \sigma_3\sigma_1 - 3\kappa^2 = 0, \tag{21}$$

where σ_i denotes the principal values of stress tensor. Classical directions of viscoplastic flow are

$$\mathbf{p} = \left\|\frac{\partial F}{\partial \boldsymbol{\sigma}}\right\|^{-1} \begin{bmatrix} 2\sigma_1 - (\sigma_2+\sigma_3) & 0 & 0 \\ 0 & 2\sigma_2 - (\sigma_1+\sigma_3) & 0 \\ 0 & 0 & 2\sigma_3 - (\sigma_1+\sigma_2) \end{bmatrix}, \tag{22}$$

The obtained viscoplastic strains do not cause volume change ($\varepsilon_{rr}^{vp} = 0$) due to coaxiality with stress deviator.

Due to simplicity of condition in Eq. (21) we are able to give analytical solution concerning fractional partial differentiation of F according to Eq. (20) (cf. Eq. (11)). In order to obtain analytical solution we utilize the relations in Eqs (8) and (9).

First, let us emphasize that yield condition Eq. (21) can be reformulated as:

- $(\sigma_i - a)^2 + [2a - (\sigma_j + \sigma_k)](\sigma_i - a) + \sigma_j^2 + \sigma_k^2 - \sigma_j\sigma_k - a\sigma_j - a\sigma_k + a^2 - 3\kappa^2 = 0$ for $i, j, k \in \{1, 2, 3\}$ but $i \neq j \neq k$ and $\sigma_i > a$, or
- $(b - \sigma_i)^2 + [(\sigma_j + \sigma_k) - 2b](b - \sigma_i) + \sigma_j^2 + \sigma_k^2 - \sigma_j\sigma_k - b\sigma_j - b\sigma_k + b^2 - 3\kappa^2 = 0$ for $i, j, k \in \{1, 2, 3\}$ but $i \neq j \neq k$ and $\sigma_i < b$.

Now, fractional gradient of F is represented as:

$$\mathbf{p} = \left\|\underset{\sigma}{D^\alpha} F\right\|^{-1} \begin{bmatrix} \underset{\sigma_1}{D^\alpha} F & 0 & 0 \\ 0 & \underset{\sigma_2}{D^\alpha} F & 0 \\ 0 & 0 & \underset{\sigma_3}{D^\alpha} F \end{bmatrix}, \tag{23}$$

where

$$\underset{\sigma_i}{D^\alpha} F = \frac{1}{2}\left({}_{a}^{C}D_{\sigma_i}^{\alpha} F + (-1)^n {}_{\sigma_i}^{C}D_{b}^{\alpha} F \right). \tag{24}$$

In Eq. (24) we have

$${}_{a}^{C}D_{\sigma_i}^{\alpha} F = \frac{\Gamma(3)}{\Gamma(3-\alpha)}(\sigma_i - a)^{2-\alpha} + [2a - (\sigma_j + \sigma_k)]\frac{\Gamma(2)}{\Gamma(2-\alpha)}(\sigma_i - a)^{1-\alpha}, \tag{25}$$

$${}_{\sigma_i}^{C}D_{b}^{\alpha} F = \frac{\Gamma(3)}{\Gamma(3-\alpha)}(b - \sigma_i)^{2-\alpha} + [(\sigma_j + \sigma_k) - 2b]\frac{\Gamma(2)}{\Gamma(2-\alpha)}(b - \sigma_i)^{1-\alpha}, \tag{26}$$

and $i, j, k \in \{1, 2, 3\}$ but $i \neq j \neq k$ and $a < \sigma_i < b$.

Notice that for $\alpha = 1$ (independently of chosen values of terminals a and b) from Eq. (24) classical solution is recovered. Due to class of function F we further restrict the order of fractional derivative in Eq. (24) to $\alpha \in (0, 1)$.

Let us now discuss the influence of α and terminals a and b on the directions of viscoplastic flow.

In Figs. 1 and 2 the comparison of classical and fractional flow directions is presented. In both figures the HMH condition (cf. Eq.(21)) on the $\sigma_1 - \sigma_2$ plane is presented.

In Fig. 1 we observe how the order of fractional gradient α influences the flow directions, simultaneously keeping terminals a and b constant (we have applied $a = -1.3\kappa$ and $b = 1.3\kappa$ but in real applications terminals can be a function of current overstress level).

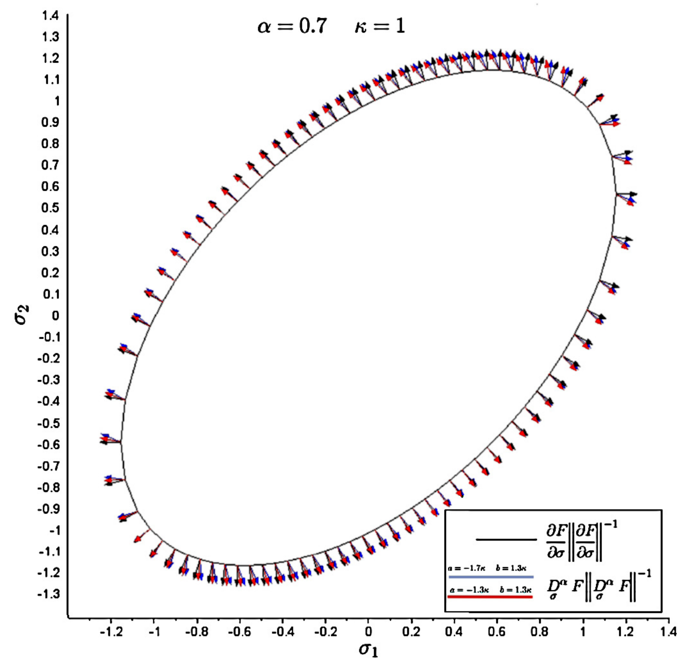

Fig. 2. The comparison of fractional vs. classical viscoplastic strains directions on HMH yield surface in $\sigma_1 - \sigma_2$ plane (for fractional one α and terminal b remain constant while terminal a changes).

Once more it should be emphasized that classical flow directions are defined in a point while fractional one takes into account the information from the internal (a, b). It is clearly seen in Fig. 1 that not all solutions are physical and when $\alpha \rightarrow 1$ classical Perzyna viscoplasticity model is recovered.

In Fig. 2 the evolution of fractional flow directions for constant order of fractional gradient α but for changing terminals a and b is presented. Thus, as an example, if one assumes that terminals depend somehow on the current level of viscoplastic deformation we see that fractional gradient gives flexible tool to fit to the experimental data.

Despite it was not considered here, let us point out that one can use different α and terminals a and b dependently on the direction being considered cf. Eqs (25) and (26). This particular case shows the possibility of adding anisotropy induced by microstructure and/or level of viscoplastic flow.

As a concluding point, let us shortly comment on selected consequences of application of this formulation to initial/boundary value problem. Using the concept of fractional viscoplasticity it is possible to control in a flexible way the level of volume change in the viscoplastic range of material behavior. It is clear that different kinematics will result different distributions of crucial quantities such as level of strains, stresses, temperatures etc. Finally, it is fundamental that overall we have only three additional material parameters comparing with classical Perzyna's viscoplasticity, namely order α and terminals a and b, which do not have to be constant and purely depend on specific material.

5. Conclusions

In this paper the generalization of the classical Perzyna's type viscoplasticity using fractional calculus is presented. Presented model is called *fractional viscoplasticity*. It is shown that non-associated flow is obtained using those new formulation without necessity of additional potential assumption, thus potentially we are able to decrease the number of material functions and simultaneous parameters for a particular material model. Hence the heterogeneity/multi-scale effects of a real material can be introduced into the constitutive model depending on the order and type of applied fractional derivative.

Formal considerations are illustrated in the example utilizing Huber–Mises–Hencky (HMH) yield condition. With the fundamental result that classical formulation is recovered as a special case of introduced generalization. It should be emphasized that the presented results are general and can be repeated for any known yield surface.

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