Modified nodal analysis: an essential addition to electrical circuit theory and analysis

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The known limitations of classical mesh and nodal methods of analysing linear electrical circuits are described before considering an established modification of the nodal approach. The method, known as 'modified nodal analysis', has none of the limitations of the basic nodal technique and is well suited both to symbolic and numeric analysis of complex circuits using modern matrix-based software. The simplicity of incorporating into the matrix equations all types of passive and active circuit elements is demonstrated and examples are used to illustrate further the efficacy of the method. It is emphasised that the absence of this circuit analysis technique from many academic engineering courses is totally at variance with its widespread application in modern circuit simulation packages.

Since the days of Maxwell the development of methods for the solution of electrical networks with large numbers of elements has proceeded systematically. Maxwell's cyclic current, so named in his honour, was fundamental to the method of mesh analysis, which was the earliest method of formulating equations for electrical circuits and is now more commonly identified with Kirchhoff's voltage law (KVL). Nodal analysis was subsequently introduced as the topological dual of mesh analysis and is identified with Kirchhoff's current law (KCL). The nodal method has become the mainstay of analysis of large electrical systems principally because it has two overriding advantages over mesh analysis. Firstly, problems of crossovers in nonplanar networks are eliminated thus avoiding the need for tree-graph theory to formulate the equations. Secondly, the number of equations is smaller with the nodal approach, reflecting the fact that the number of nodes in a network is generally less than the number of branches. Notwithstanding these advantages nodal analysis has certain limitations which complicate its application in some types of circuit.

It is the purpose of this paper to consider briefly the salient features of the more favoured nodal analysis, which has been in use for more than 100 years, and then to review a known method, derived from nodal analysis, which, combined with modern computing capabilities, is a much more powerful analytical tool. The method, known as 'modified nodal analysis[1–3], is applied quite widely in electrical circuit simulation packages. However, many undergraduate courses and texts do not acknowledge this key development so that engineers entering the profession have been denied the opportunity of studying it.

Classical nodal analysis

In classical nodal analysis currents are not explicitly calculated so the number of unknowns, namely the voltages and currents of branch elements, is halved. A further reduction is made by specifying nodal voltages with respect to a reference node so that voltages between nodes are expressed as the difference between the associated node voltages. Thus, there are as many unknowns as there are non-trivial nodes.

A consistent set of simultaneous equations is obtained by invoking Kirchhoff's current law for each node in turn. The currents are expressed in terms of all the branch currents leaving a node equated to the forcing currents entering the node. Voltage sources, transformers and dependent sources do not enter into nodal analysis in a natural way and have to be transformed using a variety of algorithms, such as Norton's theorem, to get the equations into suitable form (although it must be emphasised that ideal voltage sources and transformers cannot be transformed). Such transformations result invariably in loss of information because topologically connected nodes are reduced in number. Also, certain variables, such as source or transformer currents, have to be obtained by post-processing. It is these aspects of nodal analysis, requiring non-standard pre- and post-processing procedures to complete a solution, which have been a constant source of confusion and irritation and which are mastered only through long experience.

Before developing the theory of modified nodal analysis it is helpful first to review the properties of nodal analysis. Thus, in classical nodal analysis in which matrix methods are employed, each row in the matrix represents a constraint equation, namely the Kirchhoff current equation for that node. On the left-hand side of the equation, the unknown nodal voltages are multiplied by the admittance of the associated passive components that are connected to the node and on the right-hand side is the known forcing current injected into that node. Since there are as many nodes as unknown nodal voltages a consistent set of equations is obtained and a solution can be effected. If the numbering of the nodal voltages is consistent with the order of entry of the equations, the nodal admittance matrix is symmetrical. This means that the element at the intersection of row m and column n is the same as the element at the intersection of row n and column m. It is important to remember, especially when developing the modified nodal analysis algorithm, that the rows and columns have quite different meanings.

Modified nodal analysis (MNA) was formulated in the mid-1970s and developed subsequently for the analysis of analogue filters and the simulation of electronic circuits; it is used in one form or another in many modern simulation packages, such as SPICE. It is not well known in the analysis of power systems. However, it has many advantages over classical nodal analysis, principal amongst which are the following:

(*a*) All linear circuit elements, both passive and active, ideal and non-ideal, are accommodated by MNA with virtually no pre- and post-processing.
(*b*) The passive (admittance) element of the matrix is symmetrical, which greatly facilitates the checking of input and output data.
(*c*) Diakoptics, or partitioning of large systems, is very easy to define and apply.

Modified nodal analysis

In MNA, the first step is to remove completely all non-natural elements in order to write a conventional set of nodal equations. These elements are voltage sources, transformers and dependent sources (other than voltage-controlled current sources). The resulting admittance matrix differs from that obtained by using classical nodal analysis in which, for example, the nodes on either side of a voltage source become topologically identical and are short-circuited together.

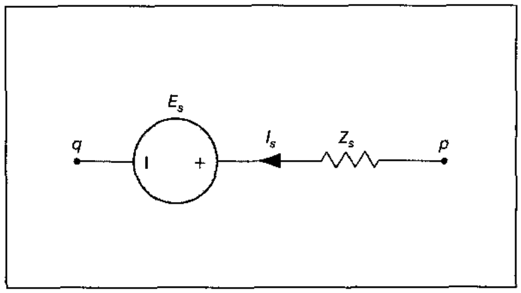

Fig. 1 Independent voltage source

In MNA such nodes retain their identity. So, having removed the non-natural elements, the resulting system is described by the basic equation:

$$\boldsymbol{YV} = \boldsymbol{I} \tag{1}$$

The process of reintroducing the non-natural elements is described in the following.

Voltage sources

Introducing an ideal voltage source, E_s, and series impedance, Z_s, between nodes p and q, as in Fig. 1, affects the original set of equations in two ways. Firstly, it introduces a new constraint equation defining the relationship between the two nodal voltages V_p and V_q. Secondly, it permits an unknown current I_s to flow. Thus the new constraint equation increases the number of rows in the matrix by one and the equations for nodes p and q are modified to take account of the additional currents in those two nodes. This maintains the consistency of the set of simultaneous equations. Assuming that the positive end of the voltage source E_s is connected to node p and that the unknown current I_s is directed toward the positive terminal of the source and hence flows away from node p and toward node q, then the nodal equations for p and q are modified and a new equation or row added, namely:

$$\left[\begin{array}{ccc|c} & & & 1 \\ & \boldsymbol{Y} & & \\ & & & -1 \\ \hline 1 & & -1 & -Z_s \end{array}\right]\begin{bmatrix} V_p \\ \\ V_q \\ I_s \end{bmatrix} = \begin{bmatrix} I_p \\ \\ I_q \\ E_s \end{bmatrix} \tag{2}$$

Note that for completeness the forcing currents I_p and I_q entering nodes p and q, respectively, have been included, although they are zero when voltage sources only are present in the circuit. Clearly, the consequence of introducing the voltage source is to augment the original matrix by one row and one column, the column being the transpose of the row, and the internal impedance, with negative sign, being at the intersection of the two. A short-circuit may be simulated by setting Z_s and E_s equal to zero and can be used as an ideal ammeter inserted between nodes p and q.

Using Gaussian elimination to remove the augmented row and column, by pivoting at $-Z_s$,

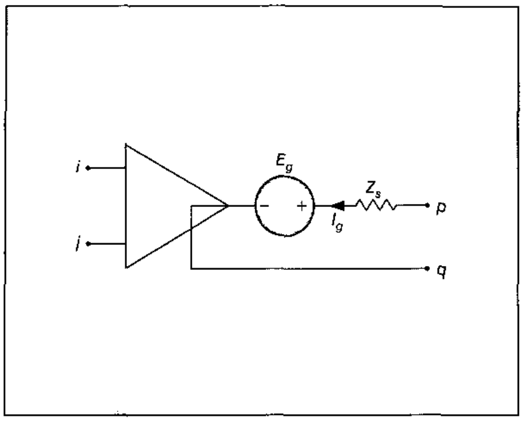

Fig. 2 Voltage-dependent voltage source

modifies eqn. 2 as follows:

$$\begin{bmatrix} Y_{11}+Y_s & Y_{12}-Y_s \\ Y_{21}-Y_s & Y_{22}+Y_s \end{bmatrix}\begin{bmatrix} V_p \\ V_q \end{bmatrix}=\begin{bmatrix} I_p+Y_sE_s \\ I_q-Y_sE_s \end{bmatrix} \tag{3}$$

where Y_{11}, Y_{12} etc. are the elements of $\mathbf{Y}$ in eqn. 2.

This is the classical nodal matrix that would have been obtained if the source voltage and source impedance had been subjected to a Norton transformation, which would not have been possible if the source impedance had been zero. In this case, the matrix could still have been reduced by first removing the last row by pivoting about the unit value in either of the columns and then removing the last column by pivoting about the unit value in the corresponding rows. In this case one of the nodes, p or q, would disappear, because they are topologically identical when joined by an ideal voltage source. This principle of equivalence is demonstrated further with a simple example in the section on applications of MNA.

The following additional features of the voltage source are worthy of note:

(*a*) If one of the nodes, p or q, is taken as a reference then the corresponding unit values in the matrix will be zero.

(*b*) Both ideal and non-ideal sources are accommodated by a single equation, i.e. Z_s can be zero or finite.

(*c*) It is possible to access the junction of the source impedance and source voltage by entering the impedance separately as an admittance in the nodal matrix and then entering the (ideal) voltage source.

Voltage-dependent voltage source

The most common example of a voltage-dependent voltage source is an operational amplifier. A schematic representation is shown in Fig. 2 for which the constraint equation is:

$$\begin{aligned} &V_p - V_q = G(V_i - V_j) + Z_sI_g \text{ or} \\ &-V_i + V_j + DV_p - DV_q - DZ_sI_g = 0 \\ &\text{where } D = 1/G \text{ and } E_g = G(V_i - V_j) \end{aligned} \tag{4}$$

The equations are arranged to use the inverse of the gain G in order to facilitate the analysis of ideal amplifiers in which the gain is infinite. It is assumed that the input impedance of the amplifier is infinite. If not, the impedance may be entered as a passive impedance externally. The action of introducing the amplifier results in an unknown current of magnitude I_g entering the amplifier. Thus the constraint equation augments the system matrix by one row and the current augments the matrix by one column, maintaining the consistency of the system equations. The structure of the augmented row and column is:

$$\left[\begin{array}{cccc:c} & & & & 0 \\ & \mathbf{Y} & & & 0 \\ & & & & 1 \\ & & & & -1 \\ \hdashline -1 & 1 & D & -D & -DZ_s \end{array}\right]\begin{bmatrix} V_i \\ V_j \\ V_p \\ V_q \\ I_g \end{bmatrix}=\begin{bmatrix} I_i \\ I_j \\ I_p \\ I_q \\ 0 \end{bmatrix} \tag{5}$$

Note that eqn. 4 is for a non-inverting amplifier if node p is chosen as the output node and an inverting amplifier if node q is chosen.

If now the additional row and column are removed by pivoting at $-DZ_s$, then the classical nodal matrix is recovered and takes the form of eqn. 6:

$$\begin{bmatrix} & & \\ & \mathbf{Y} & \\ & & \end{bmatrix}\begin{bmatrix} V_i \\ V_j \\ V_p \\ V_q \end{bmatrix}=\begin{bmatrix} I_i \\ I_j \\ I_p+E_gY_s \\ I_q-E_gY_s \end{bmatrix} \tag{6}$$

In the right-hand matrix, $E_gY_s = GY_s(V_i - V_j)$ and therefore the terms V_i and V_j can be incorporated into the left-hand matrix, resulting in an asymmetric modification to the original admittance matrix. Such modification can make it extremely difficult to set up the admittance matrix by inspection, even for relatively simple circuits, so that one of the very

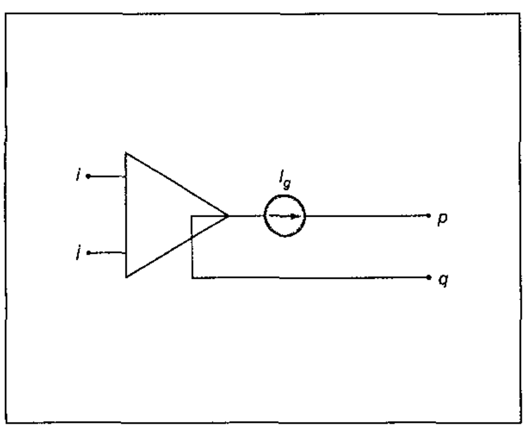

Fig. 3 Voltage-dependent current source

useful features of the nodal method is lost.

Voltage-dependent current source

A voltage-dependent current source, shown schematically in Fig. 3, may, in its simplest form, be a transistor or may be a more complex circuit arrangement. It should be noted however that this dependent source can be accommodated directly by classical nodal analysis.

The constraint equation is:

$$I_g = G_m(V_i - V_j) \text{ or}$$
$$V_i - V_j - R_m I_g = 0 \tag{7}$$
$$\text{where } R_m = 1/G_m$$

which is defined in terms of R_m rather than G_m in order to consider ideal sources in which the transconductance is infinite.

The structure of the matrix equation is:

$$\left[\begin{array}{cccc:c} & & & & 0 \\ & \mathbf{Y} & & & 0 \\ & & & & -1 \\ & & & & 1 \\ \hdashline 1 & -1 & 0 & 0 & -R_m \end{array}\right]\begin{bmatrix} V_i \\ V_j \\ V_p \\ V_q \\ I_g \end{bmatrix} = \begin{bmatrix} I_i \\ I_j \\ I_p \\ I_q \\ 0 \end{bmatrix} \tag{8}$$

It should be noted that if a non-inverting configuration is required, node p should be chosen as output node; node q should be chosen for an inverted output.

Current-dependent current source

The most common, and simple, realisation of such a source is a transistor operated as a current-dependent element, but it can of course be realised in many forms; Fig. 4 shows a general schematic representation.

An essential constraint is that the voltage between nodes i and j shall be zero. This allows an unknown current I_{in} to flow into node i and out of node j. The output current, being proportional to I_{in}, is known. However in order to permit an ideal amplifier with infinite current gain to be analysed, the output current I_o will be designated as unknown. Thus, the input current $I_{in} = DI_o$, where D is the reciprocal of the gain, and the structure of the augmented row and column becomes:

$$\left[\begin{array}{cccc:c} & & & & D \\ & \mathbf{Y} & & & -D \\ & & & & -1 \\ & & & & 1 \\ \hdashline 1 & -1 & 0 & 0 & 0 \end{array}\right]\begin{bmatrix} V_i \\ V_j \\ V_p \\ V_q \\ I_o \end{bmatrix} = \begin{bmatrix} I_i \\ I_j \\ I_p \\ I_q \\ 0 \end{bmatrix} \tag{9}$$

The device will be non-inverting if node p is chosen as output. For a grounded emitter transistor, which is inverting, the base would be node i, the emitter would be nodes j and p (the same node) and node q would be the collector (output).

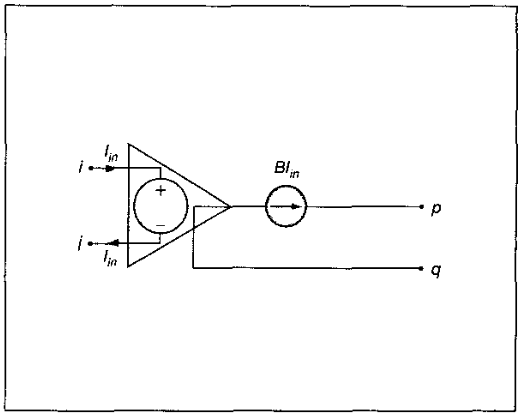

Fig. 4 Current-dependent current source

Current-dependent voltage source

This is the one case in which the element has to be defined in terms of two augmented rows and two new unknowns. The reason is that the input current has to be defined as an unknown by including a zero voltage source as was done for the current-dependent current source. However the output voltage source is also accompanied by an unknown current which is not related to the input current. Unlike the above examples of dependent sources the current-dependent voltage source is not realised in any natural physical device. However, if necessary, such an element can be defined by two augmenting rows and columns.

Two-winding transformers

The introduction of a transformer connected between four nodes, as in Fig. 5, creates one new constraint equation which is balanced by one additional unknown, namely the secondary current. The primary current will be determined by the ampere-turn balance equation. The constraint equation is:

$$N(V_i - V_j + NZ_tI_t) = V_p - V_q, \text{ or} \tag{10}$$
$$-NV_i + NV_j + V_p - V_q - N^2Z_tI_t = 0$$

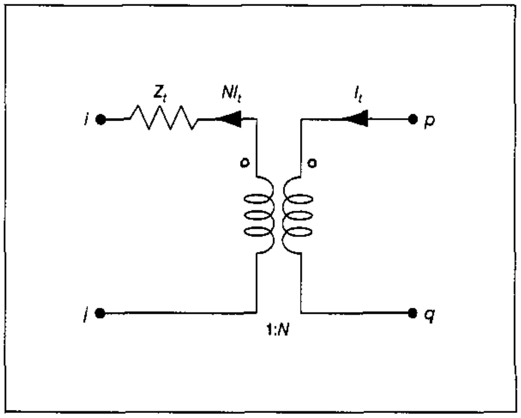

Fig. 5 Two-winding voltage transformer

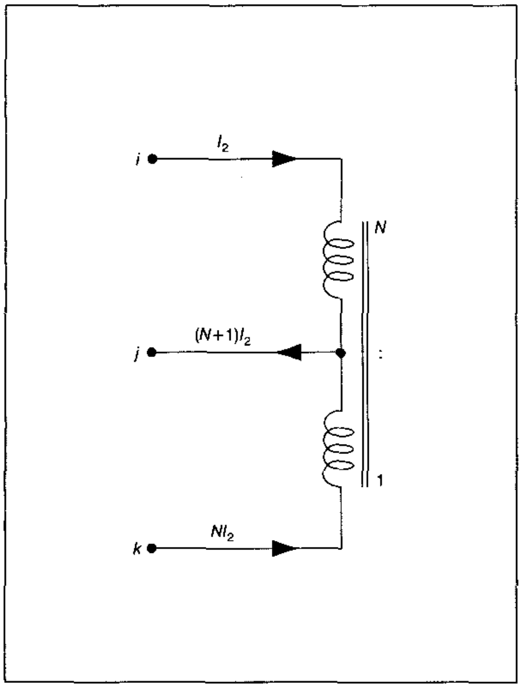

Fig. 6 Single (tapped) winding voltage transformer

In addition, currents $-NI_t$, NI_t, I_t and $-I_t$ leave nodes i, j, p and q, respectively. This means that the system matrix must be augmented by adding a row:

$$\left[\begin{array}{cccc:c} & & & & -N \\ & \mathbf{Y} & & & N \\ & & & & 1 \\ & & & & -1 \\ \hdashline -N & N & 1 & -1 & -N^2Z_t \end{array}\right]\begin{bmatrix}V_i\\V_j\\V_p\\V_q\\I_t\end{bmatrix}=\begin{bmatrix}I_i\\I_j\\I_p\\I_q\\0\end{bmatrix} \qquad (11)$$

It must also be augmented by a column to accommodate the variable 'I_t' with the column being the transpose of the row and with the parameter $-N^2Z_t$ being at the intersection of the row and column. With the normal consistent numbering between node voltages and node numbers, this will ensure that the currents will appear at the correct nodes with the correct polarity. An ideal transformer may be simulated by setting the impedance Z_t to zero.

If Z_t is non-zero, the augmented row and column may be removed using Gaussian elimination and choosing the parameter $-N^2Z_t$ as pivot. If this is done, and it is assumed for clarity that the elements of $\mathbf{Y}$ in eqn. 11 are zero, it will automatically generate the standard EMTP[4] nodal equation for a floating two-winding transformer, namely:

$$\begin{bmatrix} Y_t & -Y_t & -Y_t/N & Y_t/N \\ -Y_t & Y_t & Y_t/N & -Y_t/N \\ -Y_t/N & Y_t/N & Y_t/N^2 & -Y_t/N^2 \\ Y_t/N & -Y_t/N & -Y_t/N^2 & Y_t/N^2 \end{bmatrix}\begin{bmatrix}V_1\\V_2\\V_3\\V_4\end{bmatrix}=\begin{bmatrix}I_1\\I_2\\I_3\\I_4\end{bmatrix} \qquad (12)$$

Here the two-winding transformer of ratio N is connected between nodes 1, 2, 3 and 4 and $Y_t = 1/Z_t$. This conversion is only possible if Z_t is non-zero, precluding the use of the EMTP model when studying the behaviour of ideal transformers. No such restriction applies to eqn. 11, where $Z_t = 0$ does not create a problem.

Having established the modified matrix for a simple transformer it is then possible to introduce the concept of a phase-shifting transformer. Such a device is encountered in symmetrical component analysis whereby an ideal star-delta (or zig-zag) transformer has the effect of introducing a phase shift from one side to the other without any loss of power. The accompanying matrix is modified to the extent that the augmenting row in eqn. 11 becomes:

$$-N\angle\phi,\ N\angle\phi,\ 1, -1, -N^2Z_t \qquad (13)$$

and the column becomes the complex conjugate of the row:

$$-N\angle-\phi,\ N\angle-\phi,\ 1\ -1, -N^2Z_t \qquad (14)$$

Auto-transformers

An auto-transformer is the same as a two-winding transformer with one node 'shared' by primary and secondary. This is easily taken into account by summing the elements of the augmented row and column to the appropriate nodes. A simple arrangement is shown in Fig. 6 for which the constraint equation is:

$$\begin{aligned}&(V_i - V_j) = N(V_j - V_k), \text{ or}\\ &V_i - (N+1)V_j + NV_k = 0\end{aligned} \qquad (15)$$

Thus, the matrix is augmented as in eqn. 16, in which the additional column is, again, the transpose of the voltage constraint row, as may be seen from the magnitude and direction of current flow in Fig. 6:

$$\left[\begin{array}{ccc:c} & & & 1 \\ & \mathbf{Y} & & -(N+1) \\ & & & N \\ \hdashline 1 & -(N+1) & N & 0 \end{array}\right]\begin{bmatrix}V_i\\V_j\\V_k\\I_2\end{bmatrix}=\begin{bmatrix}I_i\\I_j\\I_k\\0\end{bmatrix} \qquad (16)$$

Entering data to the augmented rows and columns may be made identical to that for a two-winding transformer with discrete nodes. The augmented row and column are first set to zero. The entries are then summed into the appropriate locations. This eliminates the need for an auto-transformer to be defined as a special case.

Grounded tap-changing auto-transformer

In this case there are only two nodes for each transformer. The matrix equation is the following:

$$\begin{bmatrix} \mathbf{Y} & & \vdots & -(N+1) \\ & & \vdots & N \\ \text{----} & \text{----} & \text{----} & \text{----} \\ -(N+1) & N & \vdots & 0 \end{bmatrix} \begin{bmatrix} V_j \\ V_k \\ I_2 \end{bmatrix} = \begin{bmatrix} I_j \\ I_k \\ 0 \end{bmatrix} \qquad (17)$$

In the particular case that $N = 0$ the solution is trivial and correct, namely $V_k = V_j$.

Multiwinding ideal transformers

A three-winding transformer has two constraint equations, one defining the voltage transformation between primary and secondary and the other the primary to tertiary transformation. The two new unknowns are the currents in the secondary and tertiary windings. The voltage in the primary is derived from ampere-turn balance in all the windings, so that the matrix is augmented finally by two rows and two columns. These, in fact, will show the arrangement to be equivalent to two separate two-winding transformers. Such equivalence can be extended to a transformer with W windings, for which the matrix is extended by W–1 rows, to accommodate the voltage ratio constraints, and W–1 columns, to accommodate the W–1 unknown currents. As in the case of the auto-transformer this procedure eliminates the need to enter this arrangement as a special case.

Applications of modified nodal analysis

Example1: Relationship between MNA and classical nodal analysis

There is a fundamental relationship between MNA and nodal analysis which is best demonstrated by way of a simple example in which two separate three-node subsets of a system are connected together by a single voltage source between nodes 3 and 4. The MNA matrix equation is:

$$\begin{bmatrix} Y_{11} & -Y_{12} & -Y_{13} & 0 & 0 & 0 & 0 \\ -Y_{12} & Y_{22} & -Y_{23} & 0 & 0 & 0 & 0 \\ -Y_{13} & -Y_{23} & Y_{33} & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & Y_{44} & -Y_{45} & -Y_{46} & -1 \\ 0 & 0 & 0 & -Y_{45} & Y_{55} & -Y_{56} & 0 \\ 0 & 0 & 0 & -Y_{46} & -Y_{56} & Y_{66} & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \\ I \end{bmatrix} = \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \\ I_6 \\ E \end{bmatrix} \qquad (18)$$

Two successive reductions may be made. Firstly, I is eliminated using the fourth equation (pivoting about the term '–1' at the intersection of row 4 and column 7) and, secondly, V_4 is eliminated using the last equation (pivoting about the term '–1' at the intersection of row 7 and column 4). The result is:

$$\begin{bmatrix} Y_{11} & -Y_{12} & -Y_{13} & 0 & 0 \\ -Y_{12} & Y_{22} & -Y_{23} & 0 & 0 \\ -Y_{13} & -Y_{23} & Y_{33}+Y_{44} & -Y_{45} & -Y_{46} \\ 0 & 0 & -Y_{45} & Y_{55} & -Y_{56} \\ 0 & 0 & -Y_{46} & -Y_{56} & Y_{66} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_5 \\ V_6 \end{bmatrix} = \begin{bmatrix} I_1 \\ I_2 \\ I_3+I_4+Y_{44}E \\ I_5-Y_{45}E \\ I_6-Y_{46}E \end{bmatrix} \qquad (19)$$

This may be recognised as a classical nodal equation incorporating a voltage source. The source has been implicitly transformed by Norton's theorem and node 4 has disappeared because nodes 3 and 4 are topologically the same by virtue of the voltage source connecting them.

The following points may be noted:

(*a*) Pre-processing of both the system nodal matrix and the right-hand-side known vector are required before the matrix equation can be solved.

(*b*) The nodal matrix has to be modified by adding together rows 3 and 4 and then columns 3 and 4. The Norton transformation of the voltage source and the resultant additional driving currents in the right-hand-side vector presents problems. There are, confusingly, many different methods for explaining how this is to be done and it is quite common for polarity mistakes to be made.

(*c*) Both the voltage of node 4 and the current in the voltage source are suppressed, and have to be calculated by post-processing after the remaining nodal voltages have been determined.

(*d*) MNA is totally straightforward, the matrices are easy to compile and no knowledge of special theorems is required nor is any pre- or post-processing required.

There is another important property of MNA that has significant ramifications in the analysis of very large systems. Before augmenting eqn. 18 with the constraints for the voltage source, the pure nodal matrix could have been reduced to order 2 by eliminating rows and columns 1, 2, 5 and 6 by standard Gaussian elimination (pivoting on the diagonal elements 1, 2, 5 and 6). This is a very efficient process because the two halves of the system are decoupled. The reduced equation, after reintroducing the constraint equation for the voltage source, is:

$$\begin{bmatrix} Y'_{33} & 0 & 1 \\ 0 & Y'_{44} & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} V_3 \\ V_4 \\ I \end{bmatrix} = \begin{bmatrix} I'_1 \\ I'_2 \\ E \end{bmatrix} \qquad (20)$$

V_3, V_4 and I may now be found and the remaining voltages found by back substitution in the usual way. By postponing the introduction of the voltage source, 'fill-ins' have been substantially reduced. In large systems this dramatically improves the efficiency of the solution. It also directs the ordering of the elimination of nodes in a sparsity algorithm. Processing of the augmented rows and columns due to voltage sources, transformers and dependent sources should be postponed until the nodal component of the MNA matrix has been reduced by Gaussian elimination to those nodes associated with these elements. It should be noted that this procedure is an example of diakoptics[8] although there is no need to store the subcomponents of the nodal matrix separately unless storage is limited.

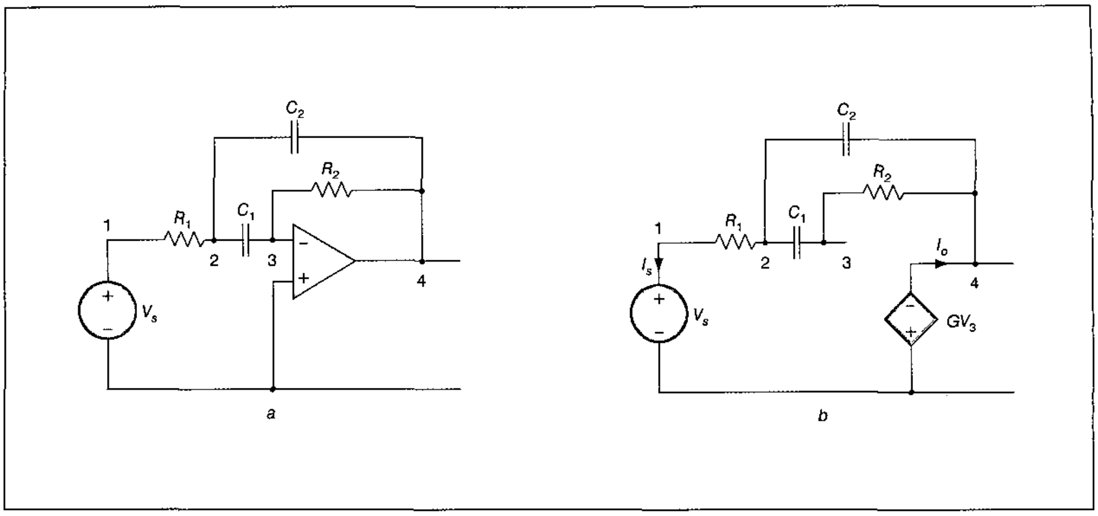

Fig. 7 (*a*) Active bandpass filter; (*b*) its equivalent circuit

Example 2: Dependent sources

A Sallen and Key inverting bandpass filter is shown in Fig. 7*a*. The equivalent circuit in Fig. 7*b* shows the operational amplifier as a voltage-dependent voltage source with an infinite input impedance, zero output impedance and voltage gain G. In fact the impedance parameters are easily accommodated but are omitted here for clarity. The resulting matrix equation follows from the two simple constraint equations which are:

$$\begin{aligned} &V_1 = V_s \\ &V_3 = DV_4 \\ &\text{where } D = 1/G \end{aligned} \tag{21}$$

(Note that if G were assumed infinite, then $D = 0$ and $V_3 = 0$.)

The matrix equation incorporates these constraints to become:

$$\begin{bmatrix} G_1 & -G_1 & 0 & 0 & 1 & 0 \\ -G_1 & (G_1+S_1+S_2) & -S_1 & -S_2 & 0 & 0 \\ 0 & -S_1 & (S_1+G_2) & -G_2 & 0 & 0 \\ 0 & -S_2 & -G_2 & (G_2+S_2) & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -D & 0 & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ I_s \\ I_o \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ V_s \\ 0 \end{bmatrix} \tag{22}$$

where

$$G_1 = 1/R_1, G_2 = 1/R_2, S_1 = j\omega C_1, S_2 = j\omega C_2 \tag{23}$$

As emphasised earlier, the admittance part of the matrix retains its symmetry about the diagonal and the augmented rows and columns are made possible by incorporating the currents in the two voltage sources as additional unknowns. This hybrid nature of the matrix equation makes it possible to define more easily, in this example, the effective input and output impedances. Classical nodal analysis could be applied only by invoking a Norton equivalent of the input voltage source and resistance R_1 in which the input current into node 2 would be V_s/R_1. Also, incorporating the dependent source (the amplifier) requires additional processing, which destroys the symmetry of the admittance part of the matrix. Of course, for hand analysis, there is obvious benefit in reducing the order of the matrix using whatever processing achieves that effect. However, one of the principle virtues of MNA lies in its simplicity and the ease with which numerical analysis can be undertaken using computer-based matrix methods. That is not to say that symbolic analysis based on MNA has any less value. Indeed, a number of modern symbolic circuit simulators use MNA in one form or another[5] but further consideration of such methods is beyond the scope of this paper.

Example 3: Star-delta transformer

Multiphase transformers present students with logistical problems in defining a consistent set of equations for solution. The star-delta configuration is particularly troublesome. To demonstrate the ease with which such a problem may be defined using MNA, consider the circuit arrangement of Fig. 8 in which the transformation ratio from primary to secondary is assumed to be 1:2. For the 9 designated nodes and the associated admittances there will be a 9 × 9 admittance matrix, which is then supplemented with 6 rows and associated columns for the voltage sources and the 3-phase transformer. In this instance the example can be simplified without loss of clarity by considering only one primary phase, say red, and the associated secondary (delta) components. Thus, the simple constraint equations are:

$$\begin{aligned} &V_1 = E_a \\ &V_3 - V_6 - 2V_2 = 0 \end{aligned} \tag{24}$$

and the associated matrix elements are:

$$\begin{bmatrix} Y_s & -Y_s & 0 & \cdots & 0 & \cdots & 1 & \cdots & 0 & \cdots & 0 \\ -Y_s & Y_s & 0 & \cdots & 0 & \cdots & 0 & \cdots & -2 & \cdots & 0 \\ 0 & 0 & Y_1 & \cdots & 0 & \cdots & 0 & \cdots & 1 & \cdots & -1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & -2 & 1 & \cdots & -1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ \vdots \\ V_6 \\ \vdots \\ I_a \\ \vdots \\ I_1 \\ \vdots \\ I_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ \vdots \\ E_a \\ \vdots \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad (25)$$

where, on the left-hand side of the equation, each column of the system matrix corresponds with each of the components identified in the unknown column matrix. By following the above procedure the remaining two phases can be entered into the appropriate locations of the matrix.

The ease of setting up the system matrix is apparent and, again, attention is drawn to the high degree of sparsity which, given a sparse matrix routine, minimises computation by bypassing trivial operations. A simple numerical example serves to illustrate the power of this approach. Thus, consider a balanced source in which the three phase voltages are each 100 V and the source resistors are each 1 Ω, and consider an unbalanced secondary in which the three resistors connecting the delta to ground are $1/Y_1 = 10$ Ω, $1/Y_2 = 20$ Ω and $1/Y_3 = 40$ Ω. The solution automatically delivers the primary and secondary currents as well as the transformer primary voltages from which the secondary voltages are then easily derived. For this particular case the various components are:

$V_2 = 91{\cdot}71\angle 0{\cdot}45$, $V_5 = 95{\cdot}49\angle{-119{\cdot}13}$,
$V_8 = 94{\cdot}25\angle 118{\cdot}68$
$V_3 = 69{\cdot}33\angle{-19{\cdot}56}$, $V_6 = 120{\cdot}64\angle{-168{\cdot}21}$,
$V_9 = 144{\cdot}36\angle 100{\cdot}02$

$I_1 = 4{\cdot}16\angle{-5}$, $I_2 = 2{\cdot}37\angle{-137{\cdot}78}$, $I_3 = 3{\cdot}088\angle 140{\cdot}63$
$I_a = 8{\cdot}32\angle{-5}$, $I_b = 4{\cdot}75\angle{-137{\cdot}78}$, $I_c = 6{\cdot}18\angle 140{\cdot}63$ (26)

The unbalance in the currents is due of course to the fact that the delta secondary is grounded through unequal resistors.

Implications for numerical analysis

It may appear from the foregoing that MNA requires more storage and computation than conventional nodal analysis. This is not so if a good sparsity routine[6–8] is used for loading the system data and solving the equations. This stems from the fact that, although there are more nodes, the MNA equations are much more sparse than their conventional nodal equivalents with the result that the number of numerical operations for the two methods is the same.

A particularly subtle benefit derives from the fact that MNA can result in a dramatic reduction in numerical processing if the equations are carefully ordered so that augmented rows and columns are dealt with last, as indicated in example 3. A further increase in efficiency may be achieved if voltage sources that change from one study to the next are ordered so that they are dealt with last. An example of this would be where a large complex system comprises passive and active elements but the configuration and number of such elements may vary due to switching operations in, for example, an electrical power system. The switches can be represented simply as zero voltage sources. If the ordering is such that switches are entered last, then the partitioned form of the system matrix takes the form:

$$\begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{D} \end{bmatrix} \quad (27)$$

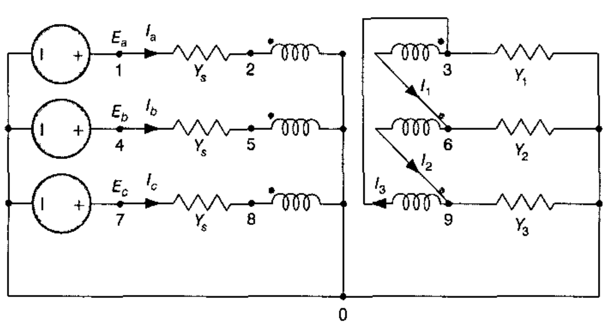

Fig. 8 Three-phase, star-delta, voltage transformer

Here A is that part of the system matrix that does not contain any switches, although it may contain, for example, voltage sources and transformers. B represents the elements acting on the unknown currents flowing in the switches, C represents the constraint equations for the switches and D contains the resistance values of the switches, which may or may not be zero. If triangularisation is applied from the top left, then the operations on A are independent of B, C and D and may be computed once and then stored. Furthermore, there are only two non-zero elements in each column of B and each row of C. By entering the nodes to which the switches are connected to be at the bottom of B, then the nodes in the upper half of B and the left half of C will be zero. Consequently, in the triangularisation process, no operations will be required on the elements in the upper half of B. This process will reduce fill-ins and will also achieve a major saving in computation and could well be considered one of the overriding advantages of MNA compared with other methods.

Conclusions

Modified nodal analysis removes all of the limitations of the classical nodal method and is most appropriate for the symbolic and numeric analysis of linear electrical circuits. The only overhead in adopting MNA is in the increased order of the matrix equation, but this is countered by the fact that such equations are generally highly sparse and amenable therefore to efficient solution using modern matrix-based computing packages. These make use of powerful sparsity routines whereby only nonzero elements are stored and trivial operations (such as multiplication by, and addition of, zero) are avoided. The fact that MNA is applied widely in commercial circuit simulators, and has been for some years, emphasises the importance of this powerful analytical tool and yet most engineering courses, and many associated academic texts, fail even to acknowledge the topic. The reason for this is difficult to understand given the inescapable advantages of the method, amongst which is the overriding benefit of specifying all circuit elements, without exception, by an irreducible number of parameters. For example, the necessary parameters for the ubiquitous voltage source are the positive and negative terminations (nodes) and the source magnitude and phase. Similarly, a two-winding transformer is defined simply by the start and finish nodes of the primary and secondary windings and the associated turns ratio.

From an educational viewpoint MNA enables students to focus on the fundamentals of circuit analysis rather than spend valuable time learning how to transform circuit elements and manipulating equations. But perhaps the most important benefit for the student is the fact that ideal circuits can be analysed that have no representation in classical nodal analysis. This is particularly important in understanding the behaviour of ideal transformers, which do not have a classical nodal equivalent circuit, it being necessary to include series impedance, so that the device then becomes non-ideal. Also, once the matrix equation has been established, the student has access to all the required unknowns without resorting to further processing.

Where large systems are being analysed, MNA indicates the most efficient ordering for the Gaussian triangularisation process, which conserves sparsity and minimises the number of arithmetic operations in obtaining the solution.

It is hoped that this short paper will help stimulate some development in the teaching of circuit analysis with the aim of filling the gap that has developed between some undergraduate and graduate courses and established engineering practice.

Finally, it is hoped also that more practising engineers who use traditional methods will consider MNA as an alternative technique.

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