

# A survey about Fractional Order Calculus.

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Here we will review derivations of fractional calculus.

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## I. INTRODUCTION

## II. FRACTIONAL ORDER INTEGRATION

### A. Cachy formula

$${}_a\mathbf{I}_t^n[F] = \int_a^t d\tau_1 \int_a^{\tau_1} d\tau_2 \int_a^{\tau_2} d\tau_3 \cdots d\tau_{n-1} \int_a^{\tau_{n-1}} F(\tau_n) d\tau_n \quad (1)$$

$$\begin{aligned} {}_a\mathbf{I}_t^n[F] &= \int_a^t \underbrace{d\tau_1}_{d\nu} \underbrace{{}_a\mathbf{I}_{\tau_1}^{n-1}[F]}_u = \tau_1 {}_aI_{\tau_1}^{n-1}[F]|_a^t - \int_a^t \tau_1 d\tau_1 {}_a\mathbf{I}_{\tau_1}^{n-2}[F] \\ &= t \times {}_a\mathbf{I}_t^{n-1}[F] - \int_a^t \tau_1 d\tau_1 {}_a\mathbf{I}_{\tau_1}^{n-2}[F] = \int_a^t t d\tau_1 {}_a\mathbf{I}_{\tau_1}^{n-2}[F] - \int_a^t \tau_1 d\tau_1 {}_a\mathbf{I}_{\tau_1}^{n-2}[F] \\ &= \int_a^t \underbrace{d\tau_1(t-\tau_1)}_{d\nu} \underbrace{{}_a\mathbf{I}_{\tau_1}^{n-2}[F]}_u \end{aligned} \quad (2)$$

$$\begin{aligned} {}_a\mathbf{I}_t^n[F] &= \underbrace{-\frac{1}{2}(t-\tau_1)^2 {}_aI_{\tau_1}^{n-2}[F]|_a^t + \frac{1}{2} \int_a^t (t-\tau_1)^2 d\tau_1 {}_a\mathbf{I}_{\tau_1}^{n-3}[F]}_{=0} \\ &= \frac{1}{2} \int_a^t (t-\tau_1)^2 d\tau_1 {}_a\mathbf{I}_{\tau_1}^{n-3}[F] \\ &\vdots \\ &= \frac{1}{(n-1)!} \int_a^t (t-\tau_1)^{n-1} F(\tau_1) d\tau_1 \end{aligned} \quad (3)$$

$${}_a\mathbf{I}_t^n[F] = \frac{1}{\Gamma(n)} \int_a^t (t-\tau_1)^{n-1} F(\tau_1) d\tau_1 \quad (4)$$

### B. Arbitrary order integration

Based on Cachy integration formula for repeated integrations one may extend the integer order integration to arbitrary order integration as follows,

$${}_a\mathbf{I}_t^\alpha[F] = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} F(\tau) d\tau \quad (5)$$

### C. General relations

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt \quad (6)$$

$$B(\alpha, \beta) = \int_0^1 t^{\alpha-1} (1-t)^{\beta-1} dt \quad (7)$$

$$B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)} \quad (8)$$

$$\alpha\Gamma(\alpha) = \Gamma(\alpha + 1), \Gamma(0) = \infty, \Gamma(1) = 1 \quad (9)$$

$$(\alpha + n) \cdots \alpha\Gamma(\alpha) = \Gamma(\alpha + n + 1) \quad (10)$$

### III. FRACTIONAL ORDER DIFFERENTIATION

#### A. Reimann-Liouville FOD

$$_a^{\text{RL}}\mathbf{D}_t^\alpha \mathbf{F}(t) = \frac{d^m}{dt^m} {}_a^{\text{I}}\mathbf{I}_t^{m-\alpha}[\mathbf{F}] = \frac{1}{\Gamma(m-\alpha)} \int_a^t (t-\tau)^{m-\alpha-1} \mathbf{F}(\tau) d\tau \quad (11)$$

where  $m - 1 \preceq \alpha \preceq m$ .

#### B. Caputo FOD

$$_a^{\text{C}}\mathbf{D}_t^\alpha \mathbf{F}(t) = {}_a^{\text{I}}\mathbf{I}_t^{m-\alpha}[\mathbf{F}] = \frac{1}{\Gamma(m-\alpha)} \int_a^t (t-\tau)^{m-\alpha-1} \frac{d^m}{d\tau^m} \mathbf{F}(\tau) d\tau \quad (12)$$

where  $m - 1 \preceq \alpha \preceq m$ .

#### C. Gronvald-Letnikov FOD and FOI

$$_a^{\text{GL}}\mathbf{D}_t^\alpha \mathbf{F}(t) = \lim_{h \rightarrow 0} \frac{(1 - \hat{\mathbf{T}}_h)^\alpha}{h^\alpha} \mathbf{F}(t) \quad (13)$$

Using Tylor expansion of  $(1 - x)^\alpha$ ,

$$\begin{aligned} (1 - \hat{\mathbf{T}}_h)^\alpha &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \times \prod_{i=1}^{i=k} (\alpha - i + 1) \hat{\mathbf{T}}_h^n \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \underbrace{\Gamma(\alpha - n + 1) \prod_{i=1}^{i=k} (\alpha - i + 1)}_{\Gamma(\alpha+1)} \times \frac{1}{\Gamma(\alpha - n + 1)} \hat{\mathbf{T}}_h^n \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha - n + 1) \Gamma(n + 1)} \hat{\mathbf{T}}_h^n = \sum_{n=0}^{\infty} (-1)^n \binom{\alpha}{n} \hat{\mathbf{T}}_h^n \end{aligned}$$

Thus the GL FOD becomes,

$$_a^{\text{GL}}\mathbf{D}_t^\alpha \mathbf{F}(t) = \frac{1}{h^\alpha} \lim_{h \rightarrow 0} \sum_{n=0}^N (-1)^n \binom{\alpha}{n} \mathbf{F}(t - nh) \quad (14)$$

with  $h = t/N$ . One may define GL integration by just  $\alpha \rightarrow -\alpha$ ,

$$\begin{aligned}
(1 - \hat{\mathbf{T}}_h)^{-\alpha} &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \times \prod_{i=1}^{i=k} (-\alpha - i + 1) \hat{\mathbf{T}}_h^n \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \underbrace{\Gamma(\alpha) \prod_{i=1}^{i=k} (\alpha + i - 1)}_{\Gamma(\alpha+n)} \times \frac{1}{\Gamma(\alpha)} \hat{\mathbf{T}}_h^n \\
&= \sum_{n=0}^{\infty} (-1)^n \times (-1)^n \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-n+1)\Gamma(n+1)} \hat{\mathbf{T}}_h^n \\
&= \sum_{n=0}^{\infty} (-1)^n \binom{\alpha+n-1}{n} \hat{\mathbf{T}}_h^n
\end{aligned}$$

Thus GL FOI becomes,

$${}_a^{\text{GL}} \mathbf{I}_t^\alpha = {}_a^{\text{GL}} \mathbf{D}_t^{-\alpha} \mathbf{F}(t) = \lim_{h \rightarrow 0} h^\alpha \sum_{n=0}^N (-1)^n \binom{\alpha}{n} \mathbf{F}(t - nh) \quad (15)$$

with  $h = t/N$ .

D. The relation between RL, GL, and Caputo FOD

#### IV. FOURIER AND LAPLACE TRANSFORMATIONS

### V. NUMERICAL METHODS OF SOLUTIONS OF ORDINARY DIFFERENTIAL EQUATIONS

A. Multi linear methods for the solutions of ODEs

B. Runge-Cutta

### VI. NUMERICAL METHODS FOR SOLVING FRACTIONAL ORDER DIFFERENTIAL EQUATIONS(FODE)

A. Multi linear methods for the solutions of FODEs

B. Runge-Cutta

### VII. HIGHER ORDER FINITE DIFFERENCE FOD

A. Higher order finite difference for integer order derivatives

B. Higher order finite difference for FOD

### VIII. NUMERICAL IMPLEMENTATION

### IX. TRANSIENT SIMULATION OF ELECTRICAL CIRCUITS(PASSIVE PARTS ONLY)

A. Simulation of parts without FO components

B. Simulation of parts with FO components

C. Implementation of transient algorithem

D. Inclusion of SPICE net lists

### X. SOME APPLICATIONS OF FOC IN THE ELECTROCHEMISTRY

### XI. POSSIBLE INTEGRATION INTO QUCS

### ACKNOWLEDGMENTS

### XII. APPENDIX

