Homework Assignment 1

Numerical analysis for PDE's

Davoud Saffar Shamshirgar - Thomas Frachon

1.

2. a. ???

   b. We write the scheme under the semi-discrete form

$$u^{n+1}(x) = \frac{1}{2}\left[(u^n(x+\Delta x) + u^n(x-\Delta x)) + \lambda(u^n(x+\Delta x) + u^n(x-\Delta x))\right] \tag{4.1}$$

   with $\lambda = \frac{\Delta t}{\Delta x}$. Using the Fourier transform of 4.1 and we obtain

$$\begin{aligned}\widehat{u}^{n+1}(\xi) &= \frac{1}{2}\left[(\widehat{u}^n(\xi)e^{i\Delta x\xi} + \widehat{u}^n(\xi)e^{-i\Delta x\xi}) + \lambda(\widehat{u}^n(\xi)e^{i\Delta x\xi} - \widehat{u}^n(\xi)e^{-i\Delta x\xi}))\right] \\ &= A(\xi)\,\widehat{u}(\xi)\end{aligned}$$

   with $|A(\xi)|^2 = \cos(s)^2 + \lambda^2\sin(s)^2 = 1 - (1-\lambda^2)\sin(s)^2$ and $s = \xi\Delta x$.
   Therefore the scheme is L^2-stable if $|\lambda| \leq 1$. Finally, one notice that $|A(\pi)| = 1$ and hence (9) does not hold. Thus the Lax-Friedrichs scheme in not a dissipative scheme.

   c. Doing exactly the same analysis of the Lax-Wendroff scheme we get

$$|A(\xi)| = |1 + \lambda i\sin(s) - 2\lambda^2\sin(\frac{s}{2})^2|$$

   Then noticing that $0 \leq \sin(s) = s - \frac{s^3}{3} + O(s^5) \leq s$, we obtain

$$|A(\xi)| \leq 1 - \frac{\lambda^2}{4}s^4$$

   Finally we can conclude that the Lax-Wendroff scheme is dissipative of order 4.