Homework Assignment 2

Numerical analysis for PDE's

Thomas Frachon - Davoud Saffar Shamshirgar

1. In this task, we aim to derive boundary conditions such that the following IVP is well-posed.

$$u_t = -u_{xx} + u_{xxxx}, \quad x \in D = [0,1], \ t \geq 0. \tag{1.1}$$

Let us assume that the boundary condition has the form of $u(x,0) = f(x)$ where $x \in D$. In Fourier space, the differential equation in (1.1) can be written as

$$\frac{\partial \hat{u}(\omega,t)}{\partial t} = (\omega^2 + \omega^4)\hat{u}(\omega,t), \tag{1.2}$$

$$\hat{u}(\omega,0) = \hat{f}(\omega). \tag{1.3}$$

Hence,

$$\hat{u}(\omega,t) = e^{(\omega^2+\omega^4)t}\hat{f}(\omega), \tag{1.4}$$

and

$$\|\hat{u}(\omega,t)\| = \|e^{(\omega^2+\omega^4)t}\hat{f}(\omega)\|. \tag{1.5}$$

For the problem to be well-posed, we need that $\|\hat{u}(\omega,t)\| \leq \|\hat{f}\|$ for any ω. But this does not hold unless $\omega = 0$. This states that the problem is ill-posed for any given boundary condition.

But now, let us consider Neumann boundary conditions instead. For this, we start by multiplying the differential equation by u and integrating by part. We have,

$$\begin{aligned} \frac{1}{2}\frac{\partial}{\partial t}\|u\|_{L^2}^2 &= -\int_\Omega u.u_{xx}\,\mathrm{d}x + \int_\Omega u.u_{xxxx}\,\mathrm{d}x \\ &= \|u_x\|_{L^2}^2 + [u.u_x]_0^1 + \|u_{xx}\|_{L^2}^2 - [u_x.u_{xx}]_0^1 + [u.u_{xxx}]_0^1. \end{aligned} \tag{1.6}$$

Given the following Neumann boundary conditions

$$\begin{cases} u_x(0) = u_x(1) & = 0, \\ u_{xxx}(0) = u_{xxx}(1) & = 0, \end{cases} \tag{1.7}$$

we obtain,

$$\frac{1}{2}\frac{\partial}{\partial t}\|u\|_{L^2}^2 \leq \|u_x\|_{L^2}^2 + \|u_{xx}\|_{L^2}^2 \leq \|u\|_{H^2}^2. \tag{1.8}$$

Since the vector space is finite, all the norms are equivalent and therefore, it exists a constant C_1 such that

$$\|u\|_{H^2}^2 \le C_1 \|u\|_{L^2}^2.$$

This states that

$$\frac{\partial}{\partial t}\|u\|_{L^2}^2 \le C\|u\|_{L^2}^2. \tag{1.9}$$

Therefore, the operator $P(t,x,\partial/\partial x,\ldots,\partial^4/\partial x^4)$ is semi-bounded and the differential equation (1.1) is well-posed.

2. We consider the following stochastic problem

$$u_t = \max(2u_{xx}+u_{yy}, u_{xx}+2u_{yy}), \quad x,y\in[0,1], t\ge 0.$$

We can write,

$$u_t = u_{xx}+u_{yy}+\max(u_{xx},u_{yy}), \quad x,y\in[0,1], t\ge 0.$$

Discretizing using Forward Euler in time and central difference in space, we get

$$\begin{aligned} u_{i,j}^{n+1} &= (1-6\alpha)u_{i,j}^n + \alpha\left(\sum_{k=0,1} u_{i,j\pm e_k}^n\right) + \alpha\max(u_{i-1,j}^n + u_{i+1,j}^n, u_{i,j-1}^n + u_{i,j+1}^n) \\ &\le (1-6\alpha)u_{i,j}^n + \alpha\left(\sum_{k=0,1} u_{i,j\pm e_k}^n\right) + \alpha\max(u_{i-1,j}^n, u_{i,j-1}^n) + \alpha\max(u_{i+1,j}^n, u_{i,j+1}^n), \end{aligned} \tag{1.10}$$

where $\alpha = \frac{\Delta t}{\Delta h^2} \in [0,1]$, $\Delta x = \Delta y$. This is a convex combination of the involved terms. Therefore

$$|u_{i,j}^{n+1}| \le \max(|u_{i,j}^n|, |u_{i\pm1,j}^n|, |u_{i,j\pm1}^n|),$$

which gives

$$\begin{aligned} \max_{i,j}|u_{i,j}^{n+1}| &\le \max_{i,j}(|u_{i,j}^n|, |u_{i\pm1,j}^n|, |u_{i,j\pm1}^n|) \\ &\le \max_{i,j}(|u_{i,j}^n|). \end{aligned} \tag{1.11}$$

This assures the stability in the max norm.

Now we define $Z_{i,j}^n := U_{i,j}^n - u_{i,j}^n$ where $U_{i,j}^n = U(x_i, y_j)$. Inserting this into (1.10), we obtain

$$Z_{i,j}^{n+1} = (1-6\alpha)Z_{i,j}^n + \alpha\left(\sum_{k=0,1} Z_{i,j\pm e_k}^n\right) + \alpha\max(Z_{i-1,j}^n + Z_{i+1,j}^n, Z_{i,j-1}^n + Z_{i,j+1}^n) + \mathcal{O}(\Delta t\Delta x^2 + \Delta t^2),$$

and consequently using (1.11) we get

$$\begin{aligned} \|Z^n\|_\infty &\le \|Z^{n-1}\|_\infty + \mathcal{O}(\Delta t\Delta x^2 + \Delta t^2) \\ &\le \|Z^0\|_\infty + \mathcal{O}(n\Delta t\Delta x^2 + n\Delta t^2), \quad n\Delta t \le T. \end{aligned}$$

Hence

$$\|U(x_i, y_j, t^n) - u_{i,j}^n\| \le t\mathcal{O}(\Delta x^2 + \Delta t), \quad 0 \le \Delta t \le \frac{\Delta x^2}{2}.$$

3. a. Considering the the following second order PDE

$$\mathfrak{L} = -(u_{xx} + 2u_{xy} + u_{yy}) = 0, \tag{1.12}$$

and the following schemes

$$\mathfrak{L}_{\Delta_1} = \frac{1}{h^2}(2u_{i+1,j} + 2u_{i,j+1} + 2u_{i-1,j} + 2u_{i,j-1} - 6u_{i,j} - u_{i+1,j-1} - u_{i-1,j+1}), \tag{1.13}$$

$$\mathfrak{L}_{\Delta_2} = \frac{1}{h^2}(u_{i+1,j+1} - 2u_{i,j} - u_{i-1,j-1}), \tag{1.14}$$

we can show that the schemes above are consistent. Note that the schemes have the following stencils. To show that these schemes are both consistent, we start by doing a Taylor expansion.

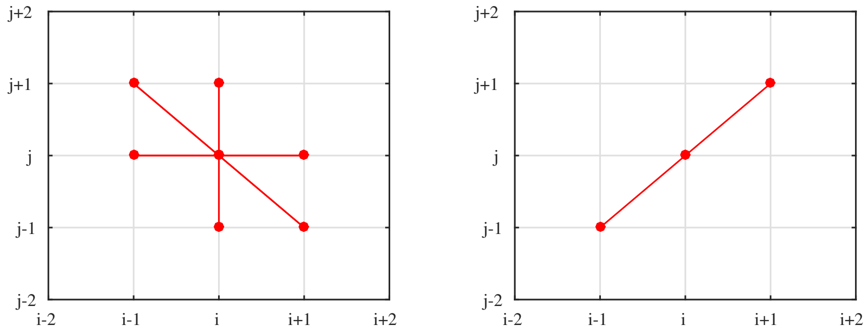

Figure 1.1 – Stencil for the scheme 1 (left) and scheme 2 (right).

$$u(x+ih, y+jh) = u(x,y) + h(iu_x + ju_y) + \frac{h^2}{2}(i^2u_{xx} + iju_{xy} + j^2u_{yy}) + \mathcal{O}(h^3). \tag{1.15}$$

Using equation (1.15), we can write the schemes above as

$$\mathfrak{L}_{\Delta_1} - \mathfrak{L} = C_1h^2 + \mathcal{O}(h^3), \tag{1.16}$$

$$\mathfrak{L}_{\Delta_2} - \mathfrak{L} = C_2h^2 + \mathcal{O}(h^3). \tag{1.17}$$

From this, it is easy to verify that

$$\lim_{h\to 0} \mathfrak{L}_{\Delta_i} - \mathfrak{L} = 0, \quad i = 1,2.$$

This proves the consistency of both methods.

b. To find the exact solution of PDE $\mathfrak{L}$ in (1.12), we note that the PDE can be written as

$$\mathfrak{L} = f_x + f_y = 0, \quad f = u_x + u_y. \tag{1.18}$$

Now, let us consider two different set of boundary conditions;

i. $u(x,y) = C, \quad (x,y) \in \partial D.$
ii. $u(x,y) = \alpha\sin(6\pi(x-y)), \quad (x,y) \in \partial D, \quad \alpha \in \mathbb{R}.$

In both cases $f(x,y) = 0$ where $(x,y) \in \partial D$. The PDE in (1.18) can be considered as a steady state 2D Transport equation. This PDE has a general solution of the form

$$f(x,y) = g(x-y).$$

Considering the boundary conditions above, for each case, the solution of the PDE can be written as

i. $u(x,y) = C, \quad (x,y) \in D.$

ii. $u(x, y) = \alpha \sin(6\pi(x - y)), \quad (x, y) \in D, \quad \alpha \in \mathbb{R}$.

These solutions can be used as references to compute the error committed by each scheme.

c. The solutions of the PDE in (1.12) using the two schemes in part a, and with the sinusoidal boundary condition and $\alpha = 1$ are shown in figures 1.2 and 1.3. Different solutions are computed with three different steps sizes $h = 1/40, 1/80$, and $1/120$. As figures state, the scheme 1 does not capture the exact solution at the center of the domain even with very small step sizes. In contrast, scheme 2 converges to the exact solution with almost any steps size.

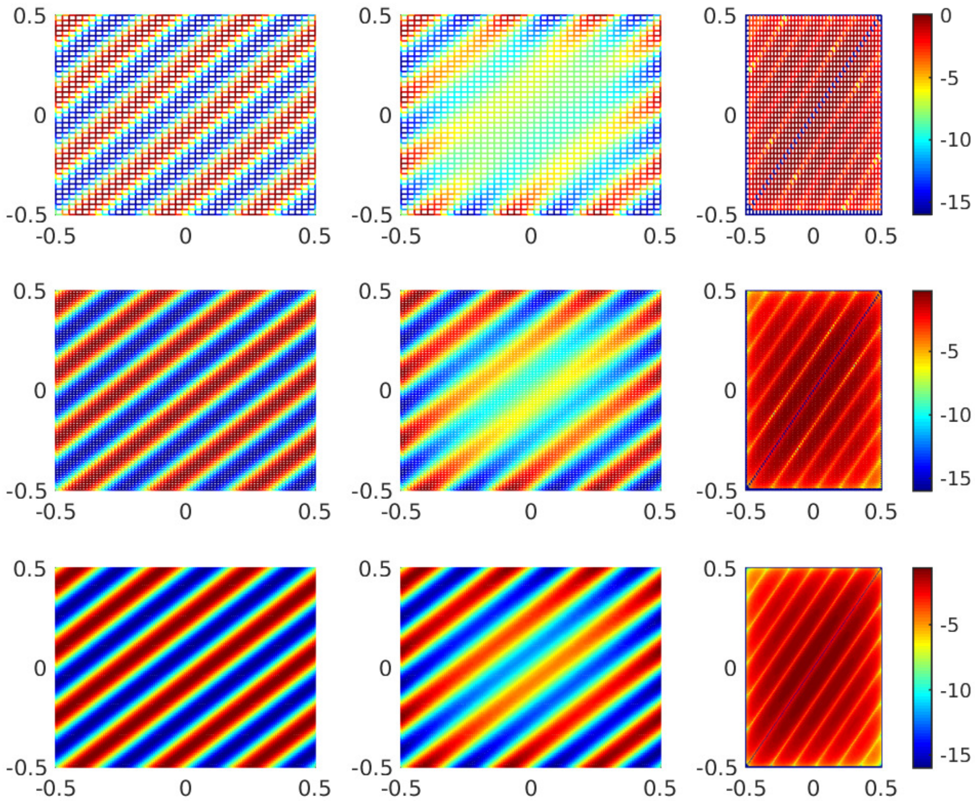

Figure 1.2 – (Top to Bottom) The exact solution (left), numerical solution (middle) and the committed error (right) of the PDE in (1.12) with the sinusoidal boundary condition and $\alpha = 1$ using the scheme 1 for $h = 1/40, 1/80, 1/120$.

For the second experiment, we choose

$$\alpha = \begin{cases} 1, & x \leq y, \\ 0, & x > y. \end{cases} \tag{1.19}$$

The solutions of the PDE in (1.12) using the two schemes in part a, and with the sinusoidal boundary condition and α as in (1.19) are shown in figures 1.4 and 1.5. As the error figure in 1.4 shows, this scheme introduces a small dispersion in the solution for the right bottom triangular part of the domain. This is clearly not the case for the second scheme.

Due to the pattern of the boundary condition, the second scheme works significantly better for both cases of α, since the stencil is aligned with the patterns. This can also be seen from the error plot in figure 1.5. On the other hand, the first scheme, averages solutions in an incorrect pattern and therefore we get a damping effect at the center of the domain. This can be also observed by computing the next term of the Taylor expansions in part a.

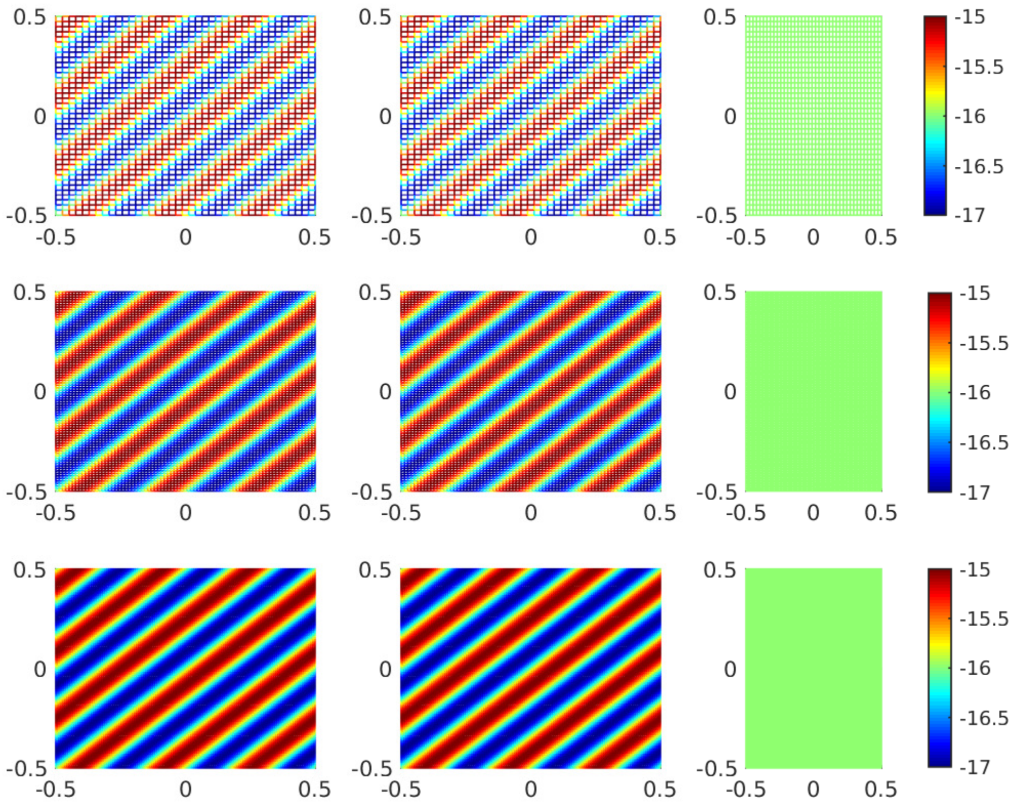

Figure 1.3 – (Top to Bottom) The exact solution (left), numerical solution (middle) and the committed error (right) of the PDE in (1.12) with the sinusoidal boundary condition and $\alpha = 1$ using the scheme 2 for $h = 1/40, 1/80, 1/120$.

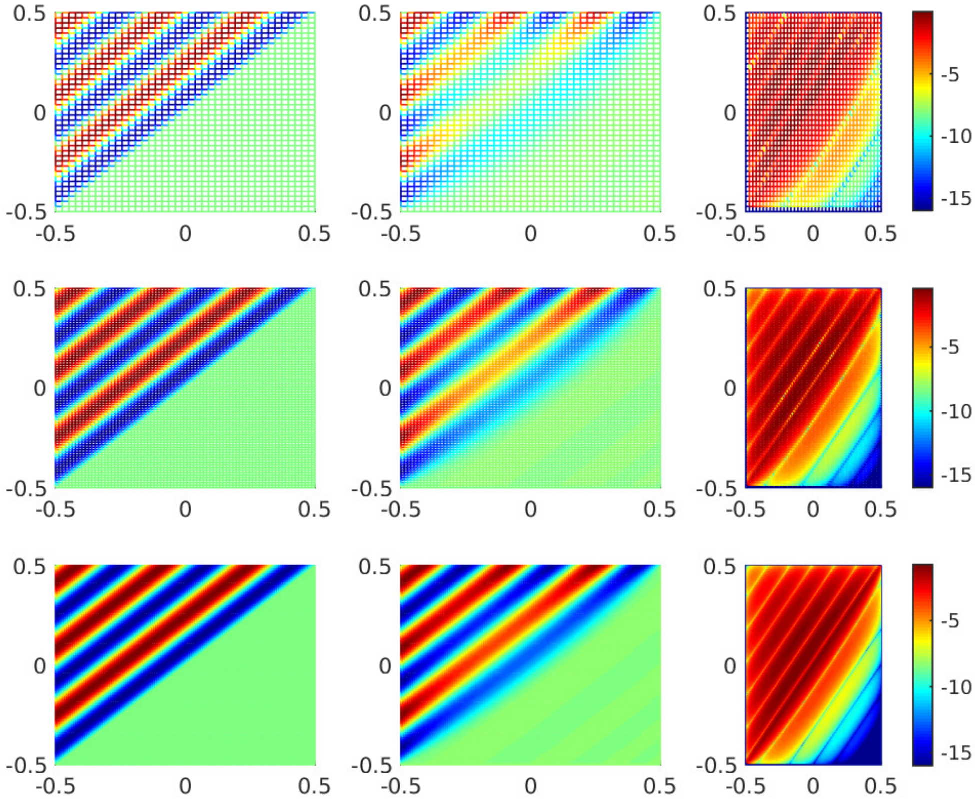

Figure 1.4 – (Top to Bottom) The exact solution (left), numerical solution (middle) and the committed error (right) of the PDE in (1.12) with the sinusoidal boundary condition and α as in (1.19) using the scheme 1 for $h = 1/40, 1/80, 1/120$.

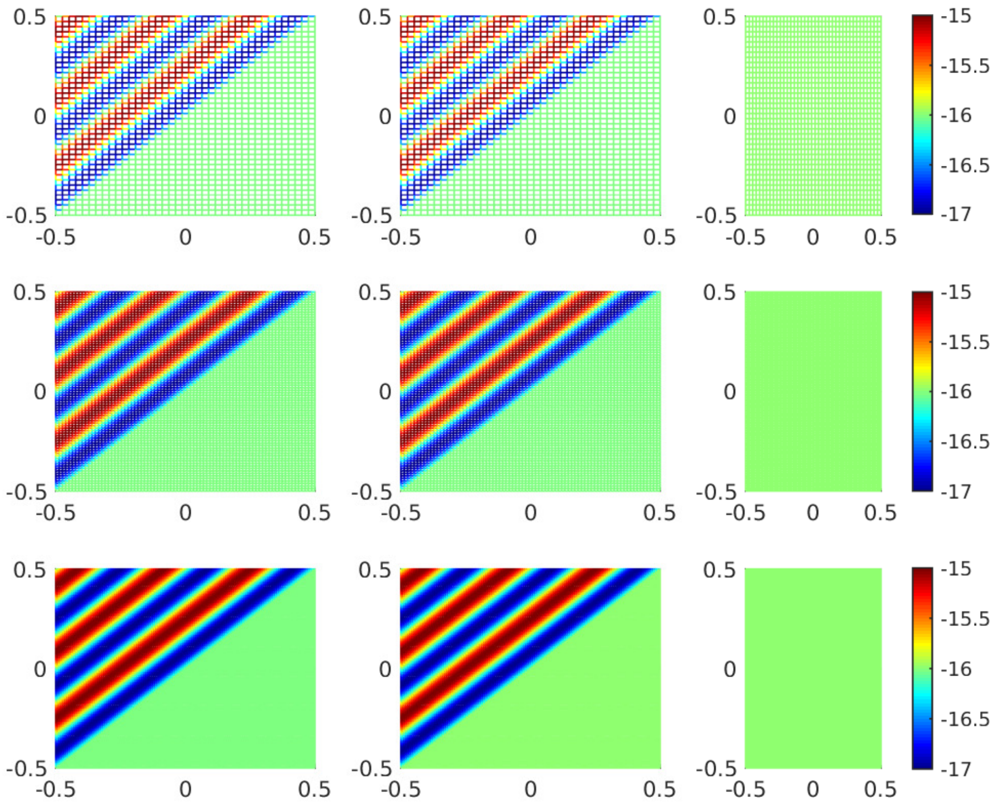

Figure 1.5 – (Top to Bottom) The exact solution (left), numerical solution (middle) and the committed error (right) of the PDE in (1.12) with the sinusoidal boundary condition and α as in (1.19) using the scheme 2 for $h = 1/40, 1/80, 1/120$.