Project

An embedded boundary method for the wave equation with discontinuous coefficients

Numerical analysis for PDE's

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1 introduction

The main goal of this manuscript is two present a second order accurate embedded boundary method for a two dimensional wave equation with discontinuous coefficients. The method is based on the Finite difference method with some special treatments to handle the discontinuity at the interface. The method can be used together with the Dirichlet and Neumann boundary conditions. The method discretizes the second order hyperbolic equation directly. Any second order hyperbolic equations can be reformulated to a system of first order PDEs. Different methods have been proposed to treat these systems such as the h-Box method by Berger et. al. [2] in the first order formulations. For linear wave propagations, an staggered grid is used, however these type of grids cannot be used simply for any complex boundaries that intersect the grid arbitrarily.

There exist other methods based on unstructured grids such as Finite volume or Discontinuous Galerkin method to solve the wave equation with jumps in the boundary. Other authors have considered immersed finite element method which is based on the unfitted mesh.

2 Outline of the algorithm

We consider a scalar second order wave equation in a two dimensional domain (see figure 2) with a piecewise constant coefficient,

$$\rho(\mathbf{x}) = \left\{ \begin{array}{ll} \rho_I, & \mathbf{x} \in \Omega_I, \\ \rho_{II}, & \mathbf{x} \in \Omega_{II}. \end{array} \right.$$

On each subdomain Ω_I and Ω_{II}, we consider the solutions $u(\mathbf{x}, t)$ and $w(\mathbf{x}, t)$ respectively satisfying the following equations

$$u_{tt} = \frac{1}{\rho_I} \Delta u + F(\mathbf{x}, t), \quad \mathbf{x} \in \Omega_I, \quad t \geq 0, \tag{2.1}$$

$$w_{tt} = \frac{1}{\rho_{II}} \Delta w + F(\mathbf{x}, t), \quad \mathbf{x} \in \Omega_I I, \quad t \geq 0. \tag{2.2}$$

and the initial conditions

$$u(\mathbf{x}, 0) = U_0(\mathbf{x}), \quad u_t(\mathbf{x}, 0) = U_1(\mathbf{x}), \mathbf{x} \in \Omega_I, \tag{2.3}$$

$$w(\mathbf{x}, 0) = W_0(\mathbf{x}), \quad w_t(\mathbf{x}, 0) = W_1(\mathbf{x}), \mathbf{x} \in \Omega_{II}, \tag{2.4}$$

where $F(\mathbf{x}, t)$ is a forcing term. Now assume a smooth interface Γ between the two subdomains such that the solutions u and w match by two jump conditions

$$u = w, \qquad \mathbf{x} \in \Gamma, \quad t \geq 0, \qquad (2.5)$$

$$\frac{1}{\rho_I} \frac{\partial u}{\partial n} = -\frac{1}{\rho_{II}} \frac{\partial w}{\partial n}, \qquad \mathbf{x} \in \Gamma, \quad t \geq 0. \qquad (2.6)$$
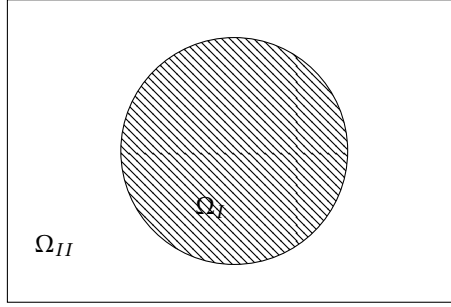

Figure 2.1 – The Domain is divided into two subdomains Ω_I and Ω_{II}.

The Laplacian is discretized using a second order accurate approximation given by

$$\Delta_h u_{i,j}^n := \frac{1}{h^2} \left(u_{i\pm1,j}^n + u_{i,j\pm1}^n - 4u_{i,j}^n \right), \quad \mathbf{x}_{i,j} \in \Omega_I. \qquad (2.7)$$

A similar formula can be written for the solution on subdomain Ω_{II}. Moreover we need to define a set of ghost points for the interior subdomain Ω_I, (see figure 2)

$$G_I = \{(i, j), \mathbf{x}_{i,j} \notin \Omega_I, \text{ but at least one of } \mathbf{x}_{i\pm1,j} \in \Omega_I \text{ or } \mathbf{x}_{i,j\pm1} \in \Omega_I\}. \qquad (2.8)$$

The corresponding expression can be used for the ghost points in Ω_{II}. Note that the solution at the ghost points are obtained using the jump conditions (2.5) and (2.6). We can derive a second order approximation for the value and the derivative at the interface using four Lagrange interpolations (see figure 1 in [1]).
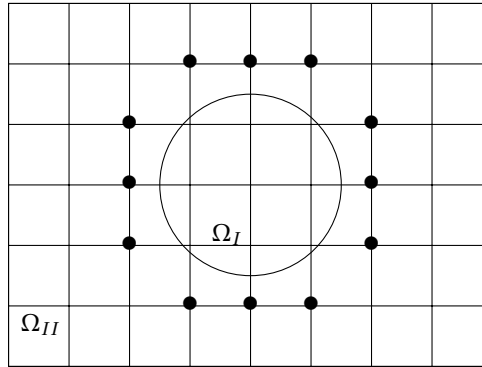

Figure 2.2 – The Domain is divided into two subdomains Ω_I and Ω_{II}.

Consider the following one dimensional wave equation with discontinuous wave propagation speed.

$$u_{tt} = u_{xx}, \qquad x_{\min} \leq x \leq 0, \qquad t \geq 0, \qquad (2.9)$$

$$w_{tt} = c^2 w_{xx}, \qquad 0 \leq x \leq x_{\max}, \qquad t \geq 0, \qquad (2.10)$$

$$u(x,0) = U_0(x), \quad u_t(x,0) = U_1(x), \qquad x_{\min} \leq x \leq 0, \qquad (2.11)$$

$$w(x,0) = W_0(x), \quad w_t(x,0) = W_1(x), \qquad 0 \leq x \leq x_{\max}. \qquad (2.12)$$

where $x_{\min} < 0$ and $x_{\max} > 0$, subject to the Dirichlet boundary conditions

$$u(x_{\min}, t) = \sin(\pi t), \quad w(x_{\max}, t) = 0, \quad t \geq 0. \tag{2.13}$$

We also implemented the method using a Dirichlet boundary condition at $x_{\min}$ and Homogeneous Neumann boundary condition at $x_{\max}$.

$$u(x_{\min}, t) = \sin(\pi t), \quad w_x(x_{\max}, t) = 0, \quad t \geq 0. \tag{2.14}$$

The jump conditions at the interface are

$$
\begin{aligned}
u(0, t) &= w(0, t), \\
u_x(0, t) &= c^2 w_x(0, t).
\end{aligned} \tag{2.15}
$$

Also we shift the grid to make sure that the discontinuity is not at one of the grid points. As in (**??**) we discretize the system of equations in (2.12).

References

[1] H-.O., Kreiss, and N.A. Petersson, *An embedded boundary method for the wave equation with discontinuous coefficients*, SIAM J. Sci. Comput., 28 (2006), No. 6, pp. 2054–2074.

[2] M.J., Berger, C. Helzel, R.J., LeVeque, *h-Box methods for the approximations of hyperbolic conservation laws on irregular grids,* SIAM J. Numer. Anal., 41 (2003), pp. 893–918.