Homework Assignment 2

Numerical analysis for PDE's

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1. In this task, we aim to derive boundary conditions such that the following IVP is well-posed.

$$u_t = -u_{xx} + u_{xxxx}, \quad x \in D = [0,1], \ t \geq 0. \tag{1.1}$$

Let us assume that the boundary condition has the form of $u(x,0) = f(x)$ where $x \in D$. In Fourier space, the differential equation in (1.1) can be written as

$$\frac{\partial \hat{u}(\omega,t)}{\partial t} = (\omega^2 + \omega^4)\hat{u}(\omega,t), \tag{1.2}$$

$$\hat{u}(\omega,0) = \hat{f}(\omega). \tag{1.3}$$

Hence,

$$\hat{u}(\omega,t) = e^{(\omega^2+\omega^4)t}\hat{f}(\omega), \tag{1.4}$$

and

$$|\hat{u}(\omega,t)| = |e^{(\omega^2+\omega^4)t}\hat{f}(\omega)|, \tag{1.5}$$

2.

3. a. Considering the the following second order PDE

$$\mathfrak{L} = -(u_{xx} + 2u_{xy} + u_{yy}) = 0, \tag{1.6}$$

and the following schemes

$$\mathfrak{L}_{\Delta_1} = \frac{1}{h^2}(2u_{i+1,j} + 2u_{i,j+1} + 2u_{i-1,j} + 2u_{i,j-1} - 6u_{i,j} - u_{i+1,j-1} - u_{i-1,j+1}), \tag{1.7}$$

$$\mathfrak{L}_{\Delta_2} = \frac{1}{h^2}(u_{i+1,j+1} - 2u_{i,j} - u_{i-1,j-1}), \tag{1.8}$$

we can show that the schemes above are consistent. Note that the schemes have the following stencils. To show that these schemes are both consistent, we start by doing a Taylor expansion.

$$u(x+ih, y+jh) = u(x,y) + h(iu_x + ju_y) + \frac{h^2}{2}(i^2u_{xx} + iju_{xy} + j^2u_{yy}) + \mathcal{O}(h^3). \tag{1.9}$$

Using equation (1.8), we can write the schemes above as

$$\mathfrak{L}_{\Delta_1} - \mathfrak{L} = C_1h^2 + \mathcal{O}(h^3), \tag{1.10}$$

$$\mathfrak{L}_{\Delta_2} - \mathfrak{L} = C_2h^2 + \mathcal{O}(h^3). \tag{1.11}$$

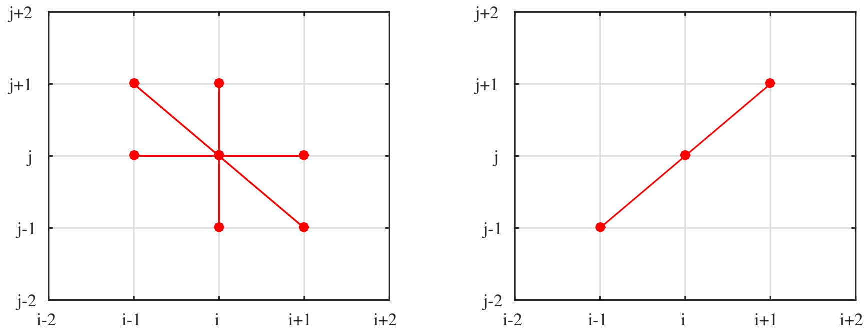

Figure 1.1 – Stencil for the scheme 1 (left) and scheme 2 (right).

From this, it is easy to verify that

$$\lim_{h\to 0} \mathfrak{L}_{\Delta_i} - \mathfrak{L} = 0, \quad i = 1,2.$$

This proves the consistency of both methods.

b. To find the exact solution of PDE $\mathfrak{L}$ in (1.5), we note that the PDE can be written as

$$\mathfrak{L} = f_x + f_y = 0, \quad f = u_x + u_y. \tag{1.12}$$

Now, let us consider two different set of boundary conditions;

i. $u(x, y) = C, \quad (x, y) \in \partial D.$
ii. $u(x, y) = \alpha \sin(6\pi(x - y)), \quad (x, y) \in \partial D, \quad \alpha \in \mathbb{R}.$

In both cases $f(x, y) = 0$ where $(x, y) \in \partial D$. The PDE in (1.11) can be considered as a steady state 2D Transport equation. This PDE has a general solution of the form

$$f(x, y) = g(x - y).$$

Considering the boundary conditions above, for each case, the solution of the PDE can be written as

i. $u(x, y) = C, \quad (x, y) \in D.$
ii. $u(x, y) = \alpha \sin(6\pi(x - y)), \quad (x, y) \in D, \quad \alpha \in \mathbb{R}.$

These solutions can be used as references to compute the error committed by each scheme.

c. The solutions of the PDE in (1.5) using the two schemes in part a, and with the sinusoidal boundary condition and $\alpha = 1$ are shown in figures 1.2 and 1.3. Different solutions are computed with three different steps sizes $h = 1/40, 1/80,$ and $1/120$. As figures state, the scheme 1 does not capture the exact solution at the center of the domain even with very small step sizes. In contrast, scheme 2 converges to the exact solution with almost any steps size.

For the second experiment, we choose

$$\alpha = \begin{cases} 1, & x \le y, \\ 0, & x > y. \end{cases} \tag{1.13}$$

The solutions of the PDE in (1.5) using the two schemes in part a, and with the sinusoidal boundary condition and α as in (1.12) are shown in figures 1.4 and 1.5. As the error figure in 1.4 shows, this scheme introduces a small dispersion in the solution for the right bottom triangular part of the domain. This is clearly not the case for the second scheme.

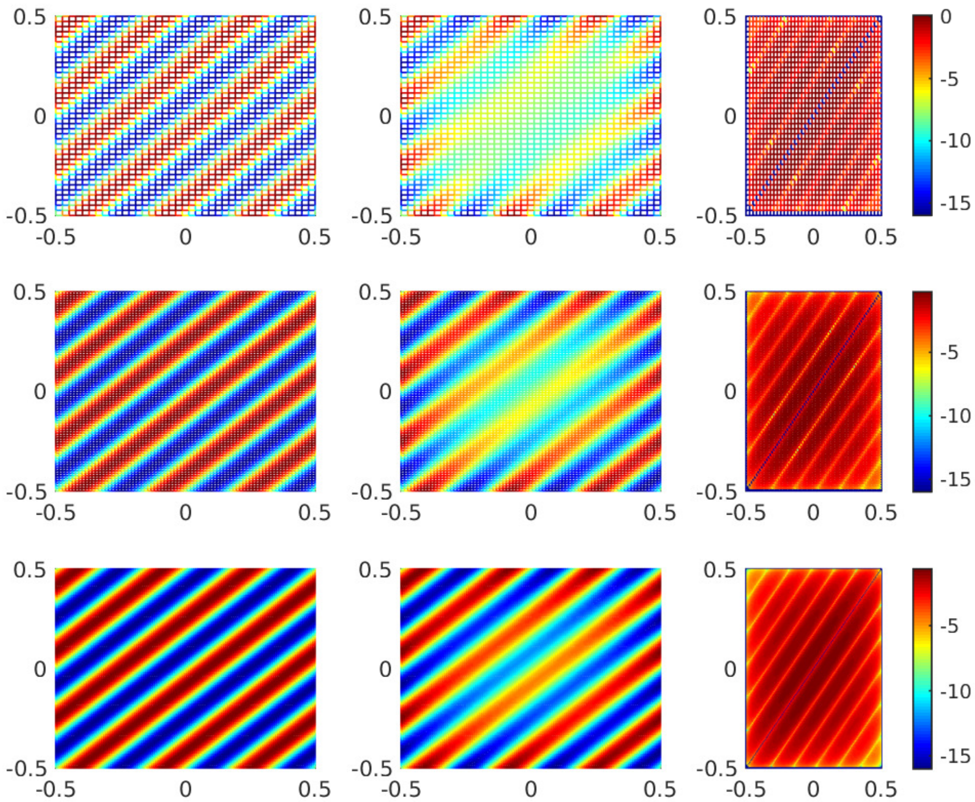

Figure 1.2 – (Top to Bottom) The exact solution (left), numerical solution (middle) and the committed error (right) of the PDE in (1.5) with the sinusoidal boundary condition and $\alpha = 1$ using the scheme 1 for $h = 1/40, 1/80, 1/120$.

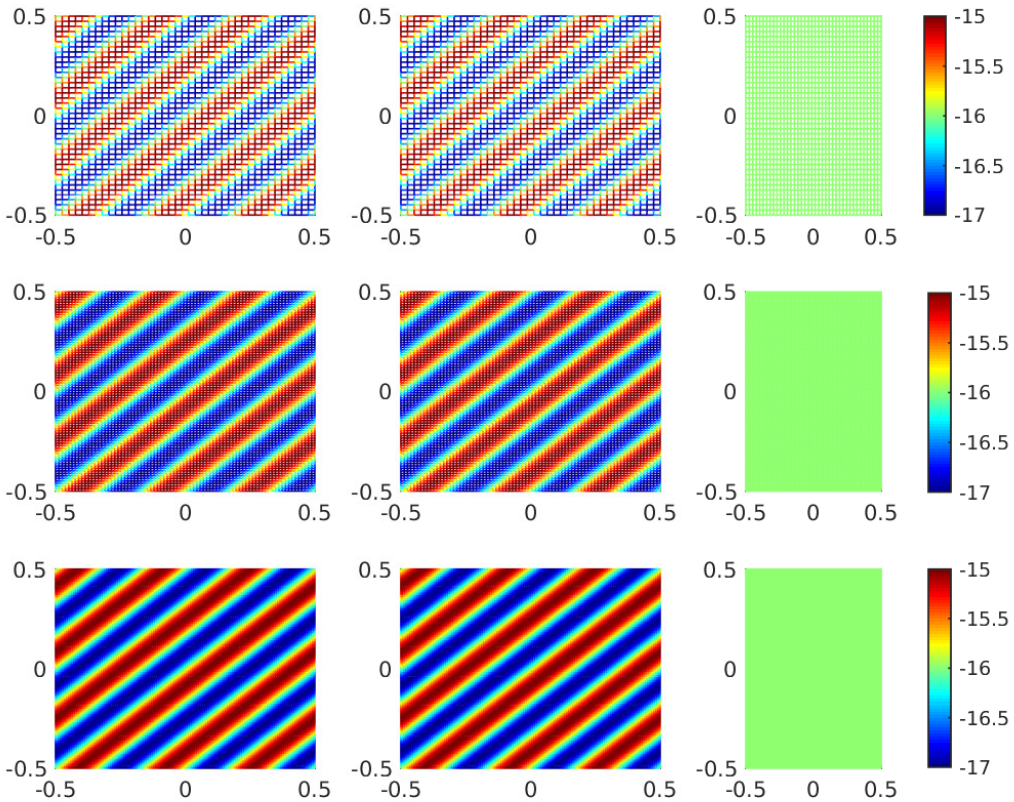

Figure 1.3 – (Top to Bottom) The exact solution (left), numerical solution (middle) and the committed error (right) of the PDE in (1.5) with the sinusoidal boundary condition and $\alpha = 1$ using the scheme 2 for $h = 1/40, 1/80, 1/120$.

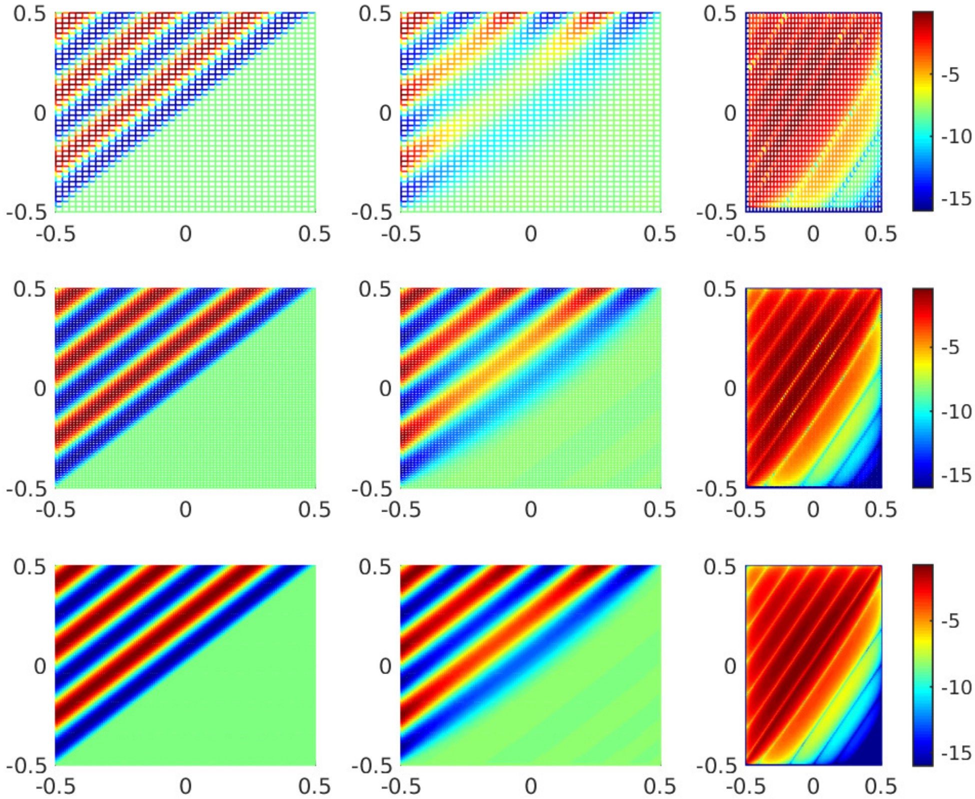

Figure 1.4 – (Top to Bottom) The exact solution (left), numerical solution (middle) and the committed error (right) of the PDE in (1.5) with the sinusoidal boundary condition and α as in (1.12) using the scheme 1 for $h = 1/40, 1/80, 1/120$.

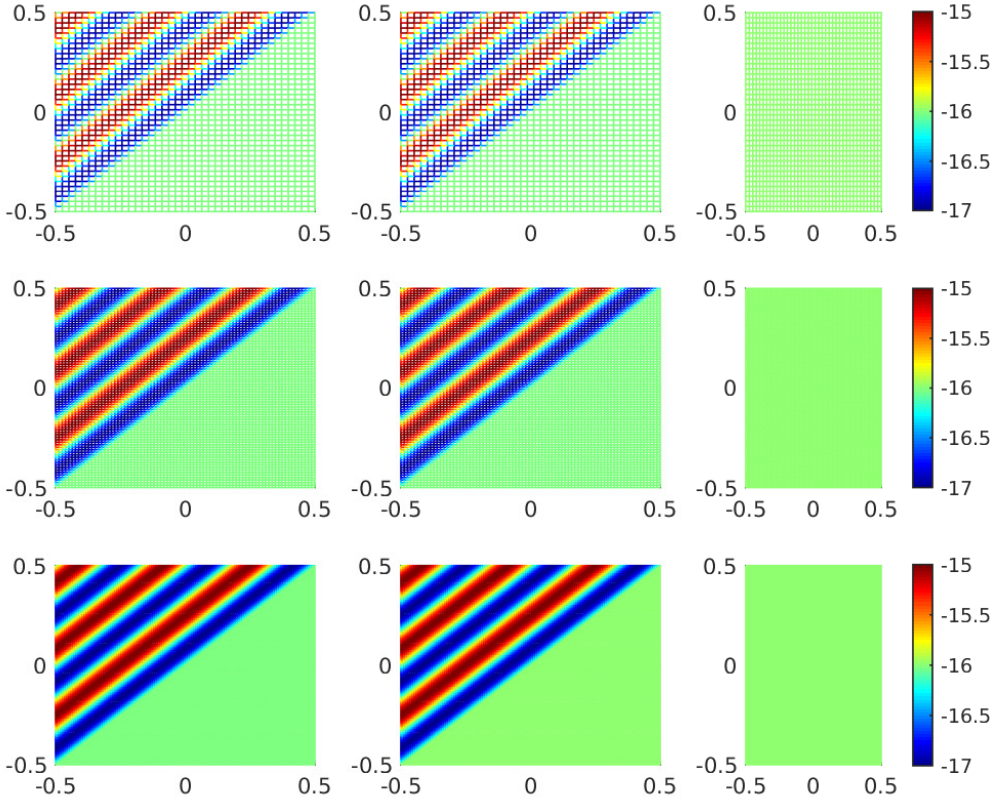

Figure 1.5 – (Top to Bottom) The exact solution (left), numerical solution (middle) and the committed error (right) of the PDE in (1.5) with the sinusoidal boundary condition and α as in (1.12) using the scheme 2 for $h = 1/40, 1/80, 1/120$.