HW 1.3

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1 Calculation of Eigenmodes

$$M\ddot{x} = -Kx \tag{1}$$

for mass m_1:

$$m_1\ddot{x_1} = -kx_1 + k(x_2 - x_1) \tag{2}$$

$$m_1\ddot{x_1} = -2kx_1 + kx_2 + (0)x_3 \tag{3}$$

for mass m_2:

$$m_2\ddot{x_2} = -k(x_2 - x_1) + k(x_3 - x_1) \tag{4}$$

$$m_2\ddot{x_2} = kx_1 - 2kx_2 + kx_3 \tag{5}$$

for mass m_3:

$$m_3\ddot{x_3} = -k(x_3 - x_2) - kx_3 \tag{6}$$

$$m_3\ddot{x_3} = (0)x_1 - 2kx_2 + kx_3 \tag{7}$$

$$det(K - \omega^2 M) = 0 \tag{8}$$

$$K = \begin{bmatrix} 2k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & 2k \end{bmatrix} \tag{9}$$

$$M = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix} \tag{10}$$

$$det(K - \omega^2 M) = 0 = \begin{bmatrix} 2k - m\omega^2 & -k & 0 \\ -k & 2k - m\omega^2 & -k \\ 0 & -k & 2k - m\omega^2 \end{bmatrix} \tag{11}$$

$$= (2k - m\omega^2)[(2k - m\omega^2)^2 - (-k)(-k)] - (-k)[(-k)(2k - m\omega^2) - 0] = 0 \tag{12}$$

$$= (2k - m\omega^2)(m^2\omega^4 - 4km\omega^2 + 2k^2) = 0 \tag{13}$$

2 solve for frequencies to find eigenvectors

$$(2k - m\omega^2)(m^2\omega^4 - 4km\omega^2 + 2k^2) = 0 \tag{14}$$

ω_1:

$$\omega_1{}^2 = \frac{2k}{m} \tag{15}$$

$$K = \begin{bmatrix} 2k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & 2k \end{bmatrix} \tag{16}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{2k}{m} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix} \tag{17}$$

$$2x - y = 2x => y = 0 \tag{18}$$

$$-x - z = 0 => x = -z \tag{19}$$

$$a \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} ==> a^2 \begin{bmatrix} 1 & 0 & -1] \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = 1 + 1 = 2a^2 = 1 ==> a = \frac{1}{\sqrt{2}} \tag{20}$$

eigenvector:

$$e_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \tag{21}$$

this first eigenvector indicates how the eigenmodes can be excited. m_1 and m_3 are oscillating with the same amplitude perfectly out of phase while m_2 remains stationary

ω_2 :

$$\omega_2{}^2 = (2 + \sqrt{2})\frac{k}{m} a \begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix} \tag{22}$$

$$a^2 \begin{bmatrix} 1 & -\sqrt{2} & 1] \end{bmatrix} \begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix} \tag{23}$$

$$4a^2 = 1 ==> a = \frac{1}{2} \tag{24}$$

eigenvector:

$$e_2 = \frac{1}{2} \begin{bmatrix} -1 \\ \sqrt{2} \\ -1 \end{bmatrix} \tag{25}$$

For this eigenvector m_1 and m_3 are oscillating with the same amplitude and phase while m_2 oscillates out of phase by a factor of $\sqrt{2}$