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SAT® MATH

VOLUME I

Every SAT Math Topic, Patiently Explained
Includes Over 900 Practice Problems

J. Ernest Gotta, Daniel Kirchheimer, and
George Rimakis

1600.io SAT® Math

Volume I

A DIFFICULT SUM.

HARRY is pulling his hair, as if that would help him out of the difficulty. The figures won't come right, do what he will, and he has worked at it an hour. Many little boys would give it up; but Harry is persevering, and intends to go over the sum until he finds out his mistake.

My Pretty Present. (New York: Thomas Nelson & Sons, 1789)

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Joe, you made this possible, and your unwavering loyalty and commitment cannot be forgotten. Megan, the only reason I've done anything is for you. John and Paige, I couldn't wish for more fun and intelligent kids. Mom and Dad, you inspire me to be a good person, a good father, and a good teacher. Nettie, Billy, Steffie, Keith, Danny, Jared, Matt, Sean, and Brandon, you shaped the humor needed to survive writing a book this long. Mustafa, Radhika, Mahoney, Timo, Elijah, Carolina, Arun, Lenny, Corbett, Will, McKayla, Sarah, Annesha, Selena, Lana, and Tanay, I'm honored to have worked alongside you. Prof. Goloubeva and Mr. Speidel, you showed me what it was to be a math educator. My former students, you inspired many of the weirdos found in the problems.

J. Ernest

My contributions to this book are dedicated to my amazing parents, who celebrated learning every day and who made me believe that I could do anything; to my beautiful, patient, and indulgent wife Jill and my brilliant son Matthew, who both endured a ceaseless barrage of terrible dad jokes (but I repeat myself) while I worked on this text; and to my extraordinary daughter Danielle, the world's best 3rd-grade teacher, who knew long before I did what a joy it is to help educate young people.

Dan

This project is dedicated to all the students reading this book right now who dare to become smarter. You've made it this far: remember that the pursuit of knowledge, with its ups and downs, is always a worthy adventure. I would also like to dedicate this book to my parents, who have supported me in everything I have ever done; to my sister Katherine, who shared her love of math with me at an early age; and to all of the teachers and professors I have had the good fortune to learn from. Without their support and inspiration, I don't think that this book would have been possible.

George

We could use up two Eternities in learning all that is to be learned about our own world and the thousands of nations that have arisen and flourished and vanished from it. Mathematics alone would occupy me eight million years.

–Mark Twain

Table of Contents

Preface

Why did we decide to write an SAT math book? Amazon is groaning under the weight of such works. Does the world really need another? Our answer was an emphatic "yes." We felt that existing books either relied too much on tricks and tactics for trying to game the test, or the topics they taught weren't well-aligned with the current style of SAT, or the instruction was cursory, confusing, and scattershot, and these realizations crystallized our shared determination to wade into this crowded field and craft a contribution based on our collective wisdom.

Though the notion of creating a math book had been bandied about among us for some time, credit for initiating the project goes to Joe Zirkel, founder of Clarion Prep, who pushed the boat away from shore. Using our amazing math skills, we computed that it would take no more than six months to complete, and we only missed that estimate by a factor of four, which is close enough for jazz. Despite (or, perhaps, because of) the intensity and length of this process, the three authors forged an extraordinary working relationship that exceeded our expectations, and though the hours (and days, and months) have been long, the gratification we received along the way has been immense, and we could not be prouder of each other's efforts, contributions, willingness to accept constructive criticism, and good humor along the way. We set out to create the finest SAT math book possible, and we hope the reader finds that we have indeed succeeded.

Finally, a word about us. Collectively, we've helped thousands of students of all abilities, backgrounds, and educational levels prepare for the SAT. We operate under the conceit that we see things that others don't, and with the humility that drives us to believe there are always better ways to teach. This book, in its structure, tone, and approach, is a distillation of everything we've observed, learned, and implemented over the years with great success, and we think it's the best SAT math book ever written.

Acknowledgements

We must offer our profound thanks to the thousands of wonderful students who contributed to this book through their innumerable interactions with us as we helped them educate themselves. This encompasses the legions of people who have entrusted to us a part of their futures whether through our years of private tutoring, as members of 1600.io, as participants in our rich and diverse Discord server, via their posts and comments on Reddit, in the form of participation in our live classes, through their thousands of constructive comments on our explanatory videos, via countless DM conversations, and in their emails. This book is, in many ways, a distillation of everything we've learned from all those interactions, and our gratitude is deep and heartfelt. We couldn't have done it without you.

Introduction

This is not an SAT book.

That is, it's not a traditional "test prep" book that discourages students from actually becoming better-educated but instead touts a "secret method" or an array of "tricks" that are purported to help the student "game the test." We think that's insulting and fundamentally counter to the central reason students take the SAT, which is to further their education by gaining admission to the best possible college, and the reason students want to go to college is to become better-educated still. Education isn't merely an annoying chore; education is the point, the larger goal, and a good SAT score helps students along the path to a great education.

One underlying assumption that appears to be built into many books and test-prep programs, and that pervades the thoughts and discussions among students studying for the test, is that the SAT is like a complicated safe, and if you can just figure out the combination, you can unlock it and ace the test—that even though the SAT purports to assess students' skills, if you know the "secrets," you don't have to actually *have* those skills in order to score well.

This is complete nonsense. It's insulting and demeaning to students, and it serves to unjustifiably undermine their self-image with respect to their intelligence and potential. The reality is that the SAT is an exquisitely well-crafted test, and it does an extremely good job of assessing exactly what it tries to assess. And what it tries to assess is a thoughtfully chosen and appropriately broad set of basic academic skills. Therefore, the best way to have your score indicate that you have mastery of all those skills is *to actually have those skills*. And it sends a terrible message that without tricks and a gaming-the-test approach, you can't get a good score, because it tells you that you can't *really* learn; you have to resort to clever tactics in hopes you can dupe the test into indicating you're smart when you're really not.

We think better of you. We know for a certainty that given the right guidance, you can be a great reader, you can learn all the grammatical and writing structure principles necessary, and, most relevantly for this work, you can really learn the math—all of it. You're not dumb, you're not un-educable, and you're not so desperate that you'll grasp for any cultish trick to try to avoid actually learning what's being tested. We believe in you, and we don't want you to accept the message that all these side-door methods subtly send you—that you're just not smart enough to learn the material, so you have to find some way to fool the test (and colleges) about your abilities. Don't be lured into the "tricks" and "secrets" cult; think more of yourself. You're better than that.

Our philosophy is radically different from that of most test preparation resources. As alluded to earlier, we view the SAT not as a *goal*, but rather as a *tool* that provides colleges with an assessment of students' level of proficiency in three core academic areas. It follows that what students should be focused on isn't getting any particular score on the test, but rather on becoming as well-educated as possible in the areas the SAT assesses. The test will then do its job and reflect that level of achievement as a natural consequence. This, in our view, is by far the best way to approach the test: go right through it rather than attempting to sneak around it. The test is far too good; it can't be tricked into reporting that you're better at math, reading, or writing than you really are, and any plan to "beat the test" is doomed, as well as being a cop-out, a surrender, an acceptance of defeat. If, instead, you acquire the skills assessed by the test, you can free yourself of all the confusing static about strategies and tricks, and you can be confident that if you put in a smart, honest effort, the test will reward you with a fair result.

Lest the reader think that we've conjured up our philosophy merely to be contrarian, we'd like to make it clear that we, too, once believed that it was a winning mindset to view the test as a challenge that could be gamed through a set of strategies, but our long experience has proven that to be a fool's errand. We now know that we were not seeing the deeper truth about the test, and it is absolutely clear to us that the approach we are now preaching, as embodied in this work, is without question the better one.

The Essence of the SAT

Many students have a core misconception about the SAT, due no doubt to their scholastic experiences with tests. It is essential to understand that the SAT in general, and the math section in particular, is not a test of *facts*; it's a test of *processes and principles*. This distinguishes it from many of the exams students take in school: it doesn't demand *memorization of information*; it requires mental muscle memory gained through *performing processes*. It's about *doing* things, not *knowing* things. It's not a history test that demands the recitation of names and dates; it's not an English test that relies on remembering the hidden meanings in an assigned work of literature; it's not a vocabulary test that's susceptible to memorizing word lists; and it's not a math or science formula regurgitation exam. Many students don't realize these essential facts about the test, and their studying proceeds along all sorts of tangents and detours that don't make any sense. In short, you should not feel tormented by the fear of forgetting something when you open the test booklet; just focus on *doing*, and the rest will take care of itself.

The math section of the SAT does have a certain format, and the problems that appear tend to fit into certain general templates, so familiarity with the format and templates is valuable. This is not at all a trick or a secret; the College Board™ itself specifically encourages students to learn their way around the test by taking practice exams and by using Khan Academy for practice. But, importantly, there aren't any very difficult math problems on the SAT (the reading section is another story!). That's right: There are no problems that take ten minutes and dozens of steps to solve, that require some weird math skill you've never been taught (if you use this text, of course), or that demand some special "math smarts" insight that either you have or you don't. Indeed, the problems that seem to be the hardest often turn out to have very simple solutions; don't mistake a failure to recognize how to solve a particular problem for an assessment that the problem is therefore extremely difficult.

We're going to repeat this, because it's so important: *There are no very hard math problems on the SAT*. The difficulty is *intentionally* capped, and the complete set of skills needed to solve *any problem that has ever appeared on an administered SAT* is well-defined by the College Board itself in a public document. So, you might not know how to approach a specific problem; you may be completely baffled by it. You might be flummoxed by several problems, or by nearly *all* the problems, if you're just starting your studying and you've had terrible math teachers. But you must understand that this does not mean that the problems are inherently *difficult*; it just means you don't *yet* know how to solve them. That's all. And you *can* learn how to do them—all of them. We will teach you everything you need to know.

What's in This Text, and Why

The structure and content of this work is unlike that of any other math text or SAT prep book you've ever seen. On the one hand, it's not a collection of "tricks," intended to add a few points here and there while avoiding the matter of actually educating you. On the other, it's not just a general-purpose math textbook, though it covers every one of the topics that are at the core of the American high school mathematics curriculum.

Instead, this text teaches every math skill that has appeared on the SAT, and it is intimately linked to that test. We've been guided by the College Board's official specifications for the math section of the SAT and we've exhaustively analyzed every single released test problem—well over a thousand of them—to ensure that the skills needed to solve them are thoroughly explained herein, and we've crafted over 900 example and practice problems that exercise those skills. That means that students can have the confidence that they are learning *all the skills needed* to solve the problems they're likely to see on future tests.

The size of this work might be intimidating, and that's understandable. However, it is crucial for the reader to know that the reason this text is so hefty is because it is complete, thorough, and provides multi-level instruction throughout. It doesn't just cover some of the topics on the SAT; it covers every single one of them. It has hundreds of example problems so that students aren't just thrown into practice sets after an explanation; these examples are very carefully worked through with detailed comments along the way (and, often, with alternative solutions, too). There are a dozen tests' worth of practice problems, each carefully crafted to maximize their value. Every concept is explained in great detail, complete with illustrative diagrams, summary boxes to highlight essential principles, and a concise but complete topic summary that concludes each chapter. So, the size of this text is a *good* thing, because it means that no corners have been cut and no compromises have been made, but there are multiple levels of summarization available, too, so if you want the shorter version, or are looking for a quick refresher, we've got that covered. Once you dive in, we think you'll realize that the heft of this tome is a consequence of the depth and quality of the instruction we've provided.

As noted, this isn't an SAT strategy guide. More specifically, we don't engage in digressions about test-taking tactics or time management. This text is intended to teach you the skills you need to do the math, so we've focused our attention on clear, in-depth explanations and extensive, high-quality example and practice questions. However, we decided that while having a wealth of examples and practice problems is necessary, it's not enough. Most math problems on the SAT center around one topic—finding the solutions to a quadratic equation, for example—but they necessarily also require the use of several constituent skills. Furthermore, just as math expressions are like molecules assembled from atoms such as numbers and variables (a parallel we draw herein), many SAT math topics are also like molecules; they're composed of elemental skills (the "atoms" of the topic) that are combined to form the higher-level skill. This means that trying to solve complete SAT-style problems before mastering the underlying atomic skills can be frustrating and inefficient.

In response to this realization, we invented the SkillDrill™. SkillDrills are sets of micro-problems that focus on just one atomic skill, such as finding the two numbers that sum to the *b* coefficient and that multiply to the *c* coefficient in a quadratic expression, which is an essential atomic skill for the larger skill of factoring (which itself is a component of solving a quadratic). Another such SkillDrill just requires the student to decompose an improper fraction into a mixed number; this is never tested on its own, but it's an atomic skill essential for solving certain exponent problems. By working through these highly-specific drills, students engage in the essential repetitive practice needed to become comfortable with the "atomic toolkit" that serves as the foundation for proficiency with the larger topic—solving quadratics, manipulating exponential expressions, and so on. That this approach is highly effective is not mere speculation: we've tested it extensively, and the results confirm the value of this methodology. As with so much of education, a firm and complete foundation provides the solid support needed for building upon.

Every topic here not only has SAT-style example and practice problems and SkillDrills; we've also provided curated lists at the end of each topic section of *every SAT problem related to that topic that has ever appeared on a released test*, and we've marked the particularly challenging problems we've inducted into the 1600.io Hall of Fame with a special indicator. This provides the student with an additional resource—real test problems—for practicing the skills they just learned.

The linkage between the text and the real tests doesn't only go in one direction. To close the loop, we've provided an appendix that allows a student to look up any math problem from any released SAT and to find the topic or topics in this work that relate to that problem. This means that after a student takes a practice test, they can look up our detailed, patient, thorough, and empathetic guide for acquiring the mathematics techniques needed to solve the questions they struggled with without having to hunt through the text for relevant information.

Driving all of the ideas detailed above is a simple set of principles that we hold and that we kept foremost in our minds as we were crafting this text:

- We want you to understand concepts and techniques no one was ever able to explain to you before
- We want you to realize that math topics you thought were really hard are actually easy
- We want you to truly understand aspects of math that you had blindly accepted on faith

These principles have a common root: the goal of instilling in students the confidence that mastery of a subject produces, because mastery plus confidence equals 800.

A final word about the structure of the text: There is an expectation that students have a foundation in elementary algebra and that they are familiar with the various symbols that are used at the level of high school mathematics, and there are a few questions on each test that probe those elementary skills. In support of that, we start off the instruction with what we came to call "Chapter 0," now named Foundations, and it is a concise collection of the most basic math skills that are necessary for all the more-focused topics that follow. It is *essential* that students feel comfortable with these building-block skills before moving into the topic instruction, so give the Foundations chapter a read-through to be sure you're on solid footing before proceeding further.

How to Use This Text

There are many ways to leverage the instruction in this work. Here are a few of the primary scenarios we anticipate, though we expect that nearly all students will end up using a hybrid approach:

1. **Read and work through the text cover-to-cover:**

   If your math skills are particularly shaky or you never even learned the high school math topics assessed by the SAT, or you just want to be absolutely sure you've covered all the bases, you can simply go through the entire text, reading all the instructional material, doing all the SkillDrills, and solving all the practice problems. Practically speaking, most students will end up skipping or skimming portions of many topics when they realize they are already knowledgable in the subject being discussed, but there are explanations for every aspect of every topic in case you need help.

2. **Use the summaries as a refresher:**

   If you have a reasonably solid foundation but are rusty, you can focus on the condensed topic and subtopic explanations that are broken out in the boxed text and chapter recaps, referring to the full explanations as needed, and then you can work the practice problems to check your skills.

3. **Focus on your weaknesses:**

   Students can identify the major topics in which they are less proficient and then work through just the chapters that cover those areas.

4. **Use the text as an instructional reference:**

   Some students prefer to take practice tests or to use other resources for working problems; they can use the text as a reference tool when they discover a topic or a concept with which they are unfamiliar. This work makes that mode of use easy, because we've thoughtfully provided an appendix that maps every released tests' problems to the location in the text that explains the relevant topics, so when you're confused by a practice problem, you can get right to the explanation of the associated math principles.

5. **Work the problems for practice:**

   If you feel that your concept knowledge is basically sound, but you need more practice, you can use the hundreds of practice problems to solidify your skills. You can concentrate on the topics that need the most work, and any needed instructional support is available right where the practice problems are presented; additionally, individual video explanations are available at our website **1600.io** for every practice problem in this work.

6. **As a sleep aid:**

   See scenario 1.

Beyond the Book

As thorough as it is, this text does not stand alone; it's part of the 1600.io ecosystem of educational resources. We have extensive expertise in video-based standardized test instruction, and we put that experience to use by offering explanatory videos for *every practice problem and SkillDrill* for those who want to supplement their self-studying with a familiar video resource wherein George walks you through solving each problem. Other supporting resources will also be available at our website 1600.io, providing a rich set of tools that support students' efforts to master the math section of the SAT.

Wait, What's a "Wormhole?"

At the start of some of the example problem solutions, you'll see this curious symbol:

Sprinkled throughout the SAT, hiding in plain sight, are members of a class of math problems that have a non-obvious, accelerated solution that we've dubbed a **wormhole**. A real wormhole is a warp in the fabric of space that brings two distant locations close together; the wormhole allows travel from one such location to the other very quickly, as the normal distance between the points does not have to be traversed. So it is too with wormhole solutions: they let the student go from the question to the answer with just a few easy steps, rather than requiring them to travel through the usual lengthier solving procedures for that general type of problem.

Importantly, **nearly all wormhole problems can also be solved with more generalized methods**. What this means is that wormhole problems are really like several little tests, all packed into one. Though from a scoring standpoint, a right answer is a right answer, no matter how you get there, the solving procedure that takes advantage of a wormhole will proceed much more rapidly (and with a smaller probability of error due to the simpler procedure) than will the longer, more generalized procedure. That rewards the student with more than just the points for the answer; in effect, it grants the student a bit of bonus time that can be used on some other problems, or the time can be used for double-checking answers once the first pass through the section is done. Also, as alluded to earlier, the simpler procedure is less susceptible to error; the more steps there are in a process, the greater the likelihood an error will creep in. Wormhole problems are thus very powerful in sifting the test-takers by ability, because one problem can classify students into those who spot the wormhole, those who don't but who nonetheless solve the problem correctly with a generalized method, and those who fail to solve the problem.

Taking advantage of wormholes is not a test-taking trick. It doesn't attempt to evade the test's purpose in assessing the student's ability to efficiently solve math problems—quite the contrary, in fact: one of the test's many facets is that it allows students who have the aptitude and mental acuity to see things that others might not to demonstrate that ability and to have it pay off in their score.

Because this underappreciated aspect of the test is important, we've got a chapter dedicated to it, where example problems that are inspired by real SAT problems are presented and explained. Don't overlook it; a few precious seconds saved, or errors avoided, could be the difference between a good score and a great one. Also, it (rightly) makes you feel clever, and that builds confidence, which can fuel your efforts throughout the test.

A Few Words of Advice

While this text is not going to delve into test-taking tactics, we do have a few words of advice (and caution) that students should heed.

Calculators: Tools of the Devil

We strongly discourage the use of calculators while learning the math skills assessed by the SAT for anything other than time-consuming arithmetic. We're fond of declaring that calculators are tools of the devil because they produce a reliance on the device that causes math skills to atrophy (if they ever developed at all).

The skillful student will have no difficulty completing the math sections on the test without resorting to the use of a calculator for solving or graphing, and there is hardly a worse feeling than finding out during the test that your calculator won't turn on, and you're so reliant on it that you can't solve some problems without it. Don't be that person.

When studying, if you find yourself tempted to turn to your calculator to do something you don't know how to do—or don't *want* to do—just say no. You'll be far better off.

Plugging-in Is Surrender

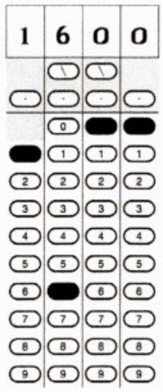
There is an alarmingly widespread belief that the best way to solve certain types of problems is to pick some value and substitute it for the variable in the provided expression or equation to see if a unique match with an answer choice is obtained. This is a terrible solving method, because it's nondeterministic (it's not guaranteed to produce the answer) and it's a cop-out—a way of avoiding actually learning and understanding the math needed to simply solve the problem directly. We strongly urge the reader not to resort to this technique. Note, importantly, that we are *not* referring to the practice of testing the provided answer choices to see which one(s) produce a valid equation; in some cases, that is a perfectly appropriate method for solving a multiple-choice math problem. We are instead directing our scorn at the very different procedure whereby the student cooks up some value—0, 1, 2, for example—and plugs it in. Please think better of yourself than to employ this method.

Writing Out Your Work Is Smart; Mental Math Is Dumb

One of the chief score killers in the math section of the SAT is sloppy arithmetic mistakes: dropping a sign, adding instead of subtracting, failure to distribute across all terms in an equation, and so on. One of the leading causes of these errors is students' unhealthy tendency to use mental math to compress solving steps rather than taking the effort to carefully (and legibly) write out their work as they go. Students perceive it as some sort of show of prowess to be able to do math in their heads rather than write it out, and, on the other side, they view it as some sort of failing to rely on written-out steps when doing a math problem that they feel they could probably handle without that process. This is dumb, and you don't want to do dumb things. **The best way to eliminate sloppy mistakes is to write out your work.**

A Note about Student-Produced Responses ("Grid-ins")

Many of the practice problems in this text and on the test require the student to write their answer rather than select it from a set of choices. Keep in mind the constraints on these grid-in answers; they must be enterable into an answer form like this:



Here are the restrictions that apply:

- You cannot enter a negative number (so if you get a negative answer, you made a mistake!)
- You cannot enter a zero in the first column (that is, as the first digit in a four-character answer); you are *never* required to enter a leading zero, so this should never cause any confusion
- You have a maximum of four character positions including a decimal point or a fraction slash
- Unless the problem specifies otherwise, you can either round or truncate decimal answers that don't fit into the grid, so $\frac{5}{3}$ could be entered as 5/3, 1.66, or 1.67, but you must use all four character positions for such values
- You cannot enter a mixed number such as $2\frac{1}{2}$; you must either enter the decimal equivalent (2.5) or an improper fraction (5/2)
- It doesn't matter in which position you start entering your answer as long as you enter the complete value (unless it's a value with more decimal digits than would fit into the four positions, in which case you must start with the first position)

Help Us Help You

We've made this work as good as we can make it, but all the same, we know it can be made better. If you have ideas, suggestions, criticisms, or if you spot errors, we very much would like to hear what you have to say. Email us at **MathBook@1600.io**, use the contact form at our website 1600.io, chat with us through our site, or DM us in our Discord server or through Reddit. This text could not have been written without your input, and that feedback is also essential to improving it. To ensure that all readers get the benefit of any corrections as soon as they are recognized, we will maintain an Errata page on our website 1600.io to log these changes.

Get Started!

You're holding this book in your hands. Take the next step forward: get a sharp pencil and some scrap paper, get focused, and turn the page. You're about to get smarter.

Foundations

0

This chapter covers the foundational arithmetic and algebraic knowledge that is required in order to get the most out of this book. The instruction in this text progresses in a methodical manner, working its way from writing simple expressions all the way through most of Algebra II and covering the necessary topics in statistics and geometry. However, you need to be proficient with skills such as manipulating fractions, simple factoring, and isolating variables in order to get started. Some skills in this chapter will be covered in more depth in later chapters when necessary, but others are going to be assumed at all points going forward. Even if you consider yourself reasonably well-grounded in basic techniques, look over this chapter to be sure you're well-equipped to move ahead.

0.1 Definitions and Fundamentals

First, we need to differentiate between **expressions** and **equations**. **Expressions**, strictly speaking, consist only of numerical and variable terms being added, subtracted, multiplied, and so on. For example, $40 + 2x$ is an expression. When we set this value equal to another expression, we can make an equation. **Equations** consist of two expressions that are set equal to each other using an equals sign (=). For example, $40 + 2x = 80 - 4x$ is an equation.

The expressions on both sides of the equals sign are exactly equivalent to each other (that's what equals means), even if they don't *look* exactly the same. Therefore, the expressions can switch sides of the equals sign; the equation $40 + 2x = 80 - 4x$ could also be written as $80 - 4x = 40 + 2x$.

One of the most essential foundational principles of mathematics is that **because the expressions on both sides of an equation are equivalent, we can perform the same operation on both of those expressions and the resulting equation will still be valid**. This can be analogized to a scale that has an assortment of weights on each side that produce a perfect balance; if we then add the same amount of weight to each side of the scale, the balance will be maintained. It is *absolutely crucial* that you understand and accept this principle so that you to have confidence in all that follows.

Obviously, you need to be familiar with how to add, subtract, multiply, and divide numbers (positive and negative) in order to solve any of the problems on this test.

A few basic arithmetic terms that will be used frequently herein:

Integers are all of the **natural numbers** (positive counting numbers), the negatives of these numbers, and zero, e.g. $\ldots -4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots$.

Non-negative numbers are all positive numbers (not just integers) and 0.

Non-positive numbers are all negative numbers (not just integers) and 0.

The result of an addition is called the **sum**. The result of a subtraction is called the **difference**. The result of a multiplication is called the **product**. The result of a division is called the **quotient**.

Important Symbols

The following table shows some of the symbols you will encounter on this test and in this text, some of which are discussed in greater detail in chapters in the book when needed. Consider this a quick primer on what to expect.

Name	Symbol	Usage
Equals	$=$	This symbol is used in equations to show that two values or expressions are equal. The equation $a = b$ is read as "a equals b" or "a is equal to b"
Not Equals	$\neq$	This symbol is used to show that two values or expressions are NOT equal. For example, $3 \neq 5$ because 3 is not equal to 5.
Approximately Equals	$\approx$	Used to show that a value is approximate. For example, $a \approx 3.14$ means that a is approximately—but not exactly—3.14.
Infinity	∞	An indefinitely large number.
Less Than	$<$	The value on the left (on the smaller side of the symbol) is less than the value on the right (on the larger side of the symbol). Mathematical statements involving less-than signs (and the next three symbols as well) are called **inequalities**. The inequality $x < 5$ means that x can be any value from $-\infty$ up to, but **not** including, 5.
Less Than Or Equal To	$\leq$	The value on the left is less than or equal to (no more than) the value on the right. For example, $x \leq 5$, means that x can be any value from $-\infty$ up to 5, including 5 itself.
Greater Than	$>$	The value on the left (on the larger side of the symbol) is greater than the value on the right (on the smaller side of the symbol). The inequality $x > 5$ means that x can be any value greater than 5 up to ∞.
Greater Than Or Equal To	$\geq$	The value on the left is greater than or equal to (no less than) the value on the right. For example, $x \geq 5$ means that x can be any value from 5 (including 5 itself) up to ∞.
Absolute Value	$\|a\|$	Produces a non-negative value indicating how far a number is from 0. ($\|-5\| = 5$)
Root	$\sqrt[n]{x}$	Indicates the n-th root of the operand x; when n is absent, its implied value is 2, so $\sqrt{x}$ indicates the second or square root of x. (See Chapter 6.)
Braces	$\{m, n, \ldots\}$	Braces indicate that elements m, n, etc. belong to a set.
Plus-Minus	$\pm$	Indicates that both the positive and negative values should be considered.
Ratio	$a : b$	Shows the relative value of a compared to b. (See Chapter 12.)
Percent	%	A relative measure expressed in hundredths of some reference value. (See Chapter 13.)
Line Segment	$\overline{AB}$	Used to refer to a line segment between points A and B.
Parallel	$\ell \parallel k$	Used to show that lines ℓ and k are parallel to each other (spaced a constant distance apart).
Perpendicular	$\ell \perp k$	Used to show that lines ℓ and k are perpendicular to each other (intersecting at a right angle).
Angle	$\angle A$ or $\angle ABC$	Used to refer to an angle in a figure, such as a triangle.
Pi	π	This Greek letter is used to represent the ratio of a circle's circumference to its diameter. $\pi \approx 3.14$ (See Chapter 20)
Theta	θ	This Greek letter is often used to represent the measurement of an angle.
Arc	$\overset{\frown}{AB}$ or $\overset{\frown}{ACB}$	Used to denote an arc on a circle.

What You Should Already Know About Fractions

The **numerator** is the expression on the top of a fraction. The **denominator** is the expression on the bottom of a fraction.

$$\frac{\text{numerator}}{\text{denominator}}$$

Any number can be represented as a fraction where the numerator is the number itself and the denominator is 1 because any number divided by 1 is equal to itself. For example, 2 can be written as $\frac{2}{1}$.

Fractions can be **reduced** if the numerator and denominator share a common divisor (factor) other than 1. For example, the fraction $\frac{4}{6}$ can be reduced to $\frac{2}{3}$ because the numerator, 4, and the denominator, 6, share a common factor of 2. Dividing both the numerator and denominator by 2 produces the reduced fraction. Other examples are $\frac{3}{9} = \frac{1}{3}$ and $\frac{7}{7} = \frac{1}{1} = 1$.

When a fraction cannot be reduced any further, it is said to be **in lowest terms**. Note that reducing is **not** the same as simplifying, which refers to making expressions simpler (a somewhat subjective concept); try to use the correct terms for these totally different procedures.

To multiply a fraction by a single value, **multiply just the numerator** of the fraction by the value. For example, $6\left(\frac{4}{5}\right) = \frac{6(4)}{5} = \frac{24}{5}$. Note that this is equivalent to multiplying $\frac{6}{1}$ and $\frac{4}{5}$ (see the next principle).

To multiply two fractions, multiply the numerators to form the numerator of the product and multiply the denominators to form the denominator of the product. For example, $\frac{1}{4}\left(\frac{3}{5}\right) = \frac{1(3)}{4(5)} = \frac{3}{20}$.

Dividing by a fraction is the same as multiplying by the **reciprocal** of the fraction (formed by "flipping" the fraction, swapping the positions of the numerator and denominator).

$$\frac{6}{\frac{1}{2}} = 6\left(\frac{2}{1}\right) = \frac{12}{1} = 12$$

When dividing one fraction by another, the same principle applies; multiply the fraction in the numerator by the reciprocal of the fraction in the denominator.

$$\frac{\frac{2}{3}}{\frac{1}{2}} = \frac{2}{3}\left(\frac{2}{1}\right) = \frac{4}{3}$$

You can **add or subtract fractions *only* when they have the same denominator**. When doing so, the addition or subtraction operation is applied to the numerators, while the denominator stays the same. For example, $\frac{1}{5} + \frac{2}{5} = \frac{1+2}{5} = \frac{3}{5}$.

When the denominators are *not* the same, you can make a **common denominator** (make the denominators match) in order to then be able to add or subtract the fractions. To create the common denominator, you have to multiply both the numerator and denominator of one or both fractions by values that will make the denominators of both fractions equal; that is, you will multiply one or both fractions by 1 expressed as a fraction (such as $\frac{2}{2}$) so the original fractions' *values* don't change, but the way they're *written* does. The simplest case arises when the denominator of one fraction is a multiple of the denominator of the other fraction as in the example below.

$$\frac{2}{3} + \frac{1}{6} = \frac{2}{3}\left(\frac{2}{2}\right) + \frac{1}{6} = \frac{4}{6} + \frac{1}{6} = \frac{5}{6}$$

If neither denominator is a multiple of the other, then you can try to make the common denominator of the fractions the **lowest (or least) common multiple** of the two denominators. Often the lowest common multiple is just the product of the two denominators. For example, in the example below, the denominators are 3 and 4, whose lowest common multiple is 12, which is equal to 3 times 4.

$$\frac{1}{3}-\frac{1}{4}=\frac{1}{3}\left(\frac{4}{4}\right)-\frac{1}{4}\left(\frac{3}{3}\right)=\frac{4}{12}-\frac{3}{12}=\frac{1}{12}$$

In the following example, the denominators, 8 and 12, share a lowest common multiple of 24 because both can go into 24 evenly (finding this value will result in smaller numbers than would using the product of 8 and 12, which is 96, as the common denominator).

$$\frac{3}{8}+\frac{5}{12}=\frac{3}{8}\left(\frac{3}{3}\right)+\frac{5}{12}\left(\frac{2}{2}\right)=\frac{9}{24}+\frac{10}{24}=\frac{19}{24}$$

When fractions appear in an equation rather than just in a lone expression, you can avoid making common denominators by eliminating fractions entirely (multiplying both sides of the equation by the denominators of any fractions so the denominators all get canceled).

Mixed numbers consist of a whole number part and a fraction part. For example, "two and a half" can be represented by the mixed number $2\frac{1}{2}$; however, mixed numbers are confusing when working in a mathematical context, and you can't fill them in as answers on the test anyway, so avoid them except in certain situations involving exponents as explained in Chapter 6.

If you are presented with a mixed number, you should immediately convert it into a fraction; because the absolute value of the fraction will be greater than or equal to 1 due to the presence of the whole number in the original mixed number, this represents a particular type of fraction called an **improper fraction** (a fraction in which the numerator has a larger absolute value than does the denominator). Do so by converting the whole number part into a fraction that shares a common denominator with the fraction part of the mixed number and then adding the two parts together.

$$3\frac{1}{4}=3+\frac{1}{4}=3\left(\frac{4}{4}\right)+\frac{1}{4}=\frac{12}{4}+\frac{1}{4}=\frac{13}{4}$$

0.2 Solving Equations/Isolating Variables

When we want to solve equations for a variable, symbolic constant, or expression, such as x, we need to isolate the term or expression whose value we want to find. To do so, we simply need to undo any mathematical operations that are being performed on the term we want to isolate. To undo an operation, we perform the inverse operation; addition and subtraction are inverse operations, and multiplication and division are inverse operations. Exponentiation and taking a root are also inverse operations, but there are some special considerations when performing those operations as explained in Chapter 9.

Let's start with the very basic equation $2x = 8$ and solve for x. The variable x is being multiplied by the **coefficient** 2, so we should perform the inverse operation to that multiplication and divide both sides of the equation by 2 to determine that $x = 4$.

Let's go one step further and solve the equation $-2x - 3 = 7$. We need to "peel away" the other parts of the expression on the left side of the equation to isolate x. The coefficient of -2 is most closely grouped with the x term, so we will come to that last. The outermost operation is the subtraction between the two terms, so first, we need to add 3 to both sides of the equation in order to cancel the -3. The equation becomes $-2x = 10$, and then we can divide both sides of the equation by -2 to get $x = -5$.

$$\begin{aligned} -2x - 3 &= 7 \\ -2x - 3 + \mathbf{3} &= 7 + \mathbf{3} \\ -2x \cancel{-3} \cancel{+3} &= 10 \\ -2x &= 10 \\ \frac{-2x}{\mathbf{-2}} &= \frac{10}{\mathbf{-2}} \\ \frac{\cancel{-2}x}{\cancel{-2}} &= \frac{10}{-2} \\ x &= -5 \end{aligned}$$

Example 0.2-1

1 0.2

What value of x satisfies the equation $4x + 4 = 24$?

A) 4

B) 5

C) 7

D) 24

Solution

1. Subtract 4 from both sides of the equation to cancel the +4, which is the outermost operation.

$$\begin{aligned} 4x + 4 &= 24 \\ 4x + 4 - \mathbf{4} &= 24 - \mathbf{4} \\ 4x &= 20 \end{aligned}$$

2. Divide both sides of the equation by 4 to cancel the multiplication of x by 4 in order to solve for x.

$$\begin{aligned} 4x &= 20 \\ \frac{4x}{\mathbf{4}} &= \frac{20}{\mathbf{4}} \\ x &= 5 \end{aligned}$$

3. The answer is B.

If the **coefficient** of the variable (the constant that is multiplying the variable) you need to isolate is in the form of a fraction, you can first multiply both sides of the equation by the coefficient's denominator and then divide both sides of the equation by the numerator. However, the best approach is to multiply both sides of the equation by the **reciprocal** of the fraction, which isolates the variable in one step, as it combines the multiplication and division operations. Note that this procedure is equivalent to dividing both sides of the equation by the fraction; division by a fraction can be implemented as multiplication by the reciprocal of the fraction.

Example 0.2-2

2 1600.io 0.2

$$\frac{3}{5}t = \frac{7}{3}$$

What value of t is the solution of the equation above?

Solution 1

1. To solve for t, we want to divide both sides of the equation by the coefficient of t, which is $\frac{3}{5}$. We can accomplish this by multiplying both sides by the reciprocal of $\frac{3}{5}$, which is $\frac{5}{3}$.

$$\frac{3}{5}t = \frac{7}{3}$$

$$\mathbf{\frac{5}{3}}\left(\frac{3}{5}\right)t = \mathbf{\frac{5}{3}}\left(\frac{7}{3}\right)$$

$$\frac{\cancel{15}}{\cancel{15}}t = \frac{35}{9}$$

$$t = \frac{35}{9}$$

2. The answer is $\frac{35}{9}$.

Solution 2

1. Multiply both sides of the equation by 5, the denominator of the fraction on the left side of the equation.

$$\frac{3}{5}t = \frac{7}{3}$$

$$\cancel{\mathbf{5}}\left(\frac{3}{\cancel{5}}\right)t = \mathbf{5}\left(\frac{7}{3}\right)$$

$$3t = \frac{35}{3}$$

2. Divide both sides of the equation by 3 (or multiply by $\frac{1}{3}$) to isolate and solve for t.

$$3t = \frac{35}{3}$$

$$\frac{\mathbf{1}}{\cancel{\mathbf{3}}}(\cancel{3}t) = \mathbf{\frac{1}{3}}\left(\frac{35}{3}\right)$$

$$t = \frac{35}{9}$$

3. The answer is $\frac{35}{9}$.

Section 0.2 Suggested Problems from Real Tests

- Test 3-C-7
- Test 6-NC-17
- Test 9-C-1
- Oct 2019-NC-1
- Mar 2020-NC-19
- Oct 2020-C-1

0.3 Substitution

If you are given the value of a constant or a particular value of a variable, you can replace that constant or variable with the given value. For example, if we are given the equation $3x + a = 13$ and told that $a = 1$, we can **substitute** 1 for a to make the equation $3x + 1 = 13$. From there, we can solve for x by subtracting 1 from both sides of the equation to get $3x = 12$ and then dividing both sides by 3 to arrive at $x = 4$.

Example 0.3-1

1 1600.io 0.3

If $\dfrac{x-2}{4} = k$ and $k = 3$, what is the value of x ?

A) 3

B) 7

C) 12

D) 14

Solution

1. Substitute 3 for k.

$$\frac{x-2}{4} = k$$
$$\frac{x-2}{4} = 3$$

2. Multiply both sides of the equation by 4.

$$\frac{x-2}{4} = 3$$
$$\cancel{4}\left(\frac{x-2}{\cancel{4}}\right) = \mathbf{4}(3)$$
$$x - 2 = 12$$

3. Add 2 to both sides of the equation to solve for x.

$$x - 2 = 12$$
$$x \cancel{-2} \cancel{+\mathbf{2}} = 12 + \mathbf{2}$$
$$x = 14$$

4. The answer is D.

Section 0.3 Suggested Problems from Real Tests

- Test 1-NC-1
- Test 4-C-5
- Test 8-C-32
- Apr 2018-NC-16

0.4 Solving for Expressions; Distributing and Factoring

Many questions on the test ask you to solve for an expression rather than for a lone variable, and usually this is a clue that you might be able to take a more direct route to the answer if you are thoughtful. However, in this section we are just interested in solving for the lone variables and substituting their values into the solve-for expression (we'll discuss the optimizations in later chapters). Most importantly, don't solve for a variable and assume that is the answer; **always double check what the problem is asking you for**.

Example 0.4-1

1 1600.io 0.4

If $3x + 5 = 7$, what is the value of $6x + 8$?

A) 2

B) 5

C) 12

D) 14

Solution

1. Subtract 5 from both sides of the equation.

$$3x + 5 = 7$$
$$3x\cancel{+5}\cancel{-\mathbf{5}} = 7 - \mathbf{5}$$
$$3x = 2$$

2. Divide both sides of the equation by 3 to solve for x, whose value needs to be substituted into the solve-for expression.

$$3x = 2$$
$$\frac{\cancel{3}x}{\cancel{\mathbf{3}}} = \frac{2}{\mathbf{3}}$$
$$x = \frac{2}{3}$$

3. Now that we have the value of x, we can substitute $\frac{2}{3}$ for x in the solve-for expression, which is $6x + 8$.

$$6x + 8 = 6\left(\frac{2}{3}\right) + 8$$
$$6x + 8 = \frac{12}{3} + 8$$
$$6x + 8 = 4 + 8$$
$$6x + 8 = 12$$

4. The answer is C.

In the following example, we must first translate a sentence into an equation and then solve for the desired expression. When translating a sentence, handle one phrase at a time. Place the equals sign immediately when you see a phrase like "is equal to" or "the result is," and anything that comes after that phrase should go on the other side of the equals sign.

Example 0.4-2

2 0.4

When 6 times the number x is added to 8, the result is 2.
What number results when 3 times x is added to 4 ?

A) -1

B) 1

C) 2

D) 7

Solution

1. The phrase "6 times the number x" translates to $6x$, and that is added to 8, so the first expression we should write is $8+6x$ (note that because you can perform addition in any order—it's **commutative**—this could also be written as $6x+8$, which you might find more natural).
2. The phrase "the result is 2" means that we should set the first expression we wrote equal to 2.

$$8 + 6x = 2$$

3. Subtract 8 from both sides of the equation.

$$\begin{aligned} 8 + 6x &= 2 \\ \cancel{8} + 6x \cancel{-\mathbf{8}} &= 2 - \mathbf{8} \\ 6x &= -6 \end{aligned}$$

4. Divide both sides of the equation by 6 to solve for x.

$$\begin{aligned} 6x &= -6 \\ \frac{\cancel{6}x}{\cancel{\mathbf{6}}} &= \frac{-6}{\mathbf{6}} \\ x &= -1 \end{aligned}$$

5. Don't forget that we need to construct the solve-for expression, which is specified in the second sentence of the question. Based on the phrase "3 times x is added to 4," the solve-for expression is $4 + 3x$ (you could also write this as $3x + 4$).
6. Substitute -1 for x in the solve-for expression.

$$\begin{aligned} 4 + 3x &= 4 + 3(-1) \\ 4 + 3x &= 4 - 3 \\ 4 + 3x &= 1 \end{aligned}$$

7. The answer is B.

In the following example, the equation contains two unknown quantities represented by letters, so we cannot solve for the value of either one by itself. Instead, we must take advantage of an existing grouping and isolate the grouped expression to solve.

Example 0.4-3

3 1600.io 0.4

If $5(a+b)=7$, what is the value of $a+b$?

A) $\frac{5}{7}$

B) $\frac{7}{5}$

C) 5

D) 7

Solution

1. The solve-for expression consists of two unknown terms added to each other. Because we have only one equation, but two unknown values, a and b, we cannot find the value of either one individually. However, because the solve-for expression $a+b$ appears in the original equation, we can isolate the solve-for expression by simply dividing both sides of the equation by 5.

$$5(a+b)=7$$

$$\frac{\cancel{5}(a+b)}{\cancel{5}}=\frac{7}{5}$$

$$a+b=\frac{7}{5}$$

2. The answer is B.

Distributing and Factoring Constants

When an expression involving addition or subtraction is multiplied by a constant (the expression is often enclosed in parentheses as in the previous example) or when an expression is divided by a constant (which is the same as multiplying by a fraction), to perform the multiplication or division the constant must be **distributed** to all the terms in the expression, meaning every term in the expression must be multiplied by the constant, and then all those products (results of multiplication) must be added together. For example, given the expression $3(x+2)$, we can distribute the 3 to both terms in the parenthetical expression (multiply both x and 2 by 3) to rewrite the expression as $3x+3(2)=3x+6$.

If you have to distribute a negative number, just **make sure the negative part is also distributed** (using extra parentheses around negative terms helps reinforce this). For example, $-2(x-4)=-2x-2(-4)=-2x+8$; notice how we placed -4 in parentheses to remind us to be careful about multiplying two negative numbers.

Example 0.4-4

4 0.4

Which of the following is equivalent to $4(x+3)-8$?

A) $4x-5$

B) $4x-4$

C) $4x+4$

D) $12x-8$

Solution

1. Distribute the 4 to both terms in the parentheses, then combine the numerical constants.

$$\begin{gathered} 4(x+3)-8 \\ 4x+4(3)-8 \\ 4x+12-8 \\ 4x+4 \end{gathered}$$

2. The answer is C.

Distribution also occurs if we divide an expression by a constant: all the terms in the expression must be divided by the constant. For example, if we want to divide the expression $8x+4$ by 4, we simply have to divide both of the terms in the expression by 4 (split the numerator into two fractions to visualize this).

$$\frac{8x+4}{4}=\frac{8x}{4}+\frac{4}{4}=2x+1$$

Alternatively, you can think about this as multiplying the expression in the numerator by $\frac{1}{4}$ (the reciprocal of 4) and then distributing the multiplication.

$$\frac{8x+4}{4}=\frac{1}{4}(8x+4)=\frac{1}{4}(8x)+\frac{1}{4}(4)=\frac{8x}{4}+\frac{4}{4}=2x+1$$

Another way to think about and handle the operation is to realize that since both terms in the expression $8x+4$ include multiples of 4, we can **factor** 4 out of both terms.

$$8x+4=4(2x)+4(1)=4(2x+1)$$

Note, importantly, that **factoring is the inverse operation of distributing**:

Factoring

$$8x+4=4(2x+1)$$

Distributing

Factoring constants out and then dividing by the factor to isolate the remaining expression is often used to produced wormhole-like shortcut solutions to problems like the following example (see the Wormholes chapter for a thorough definition of that term and an in-depth discussion of this principle). Realizing that we can factor as the first step leads to an optimization that helps us find the solve-for expression immediately. These processes are covered in later chapters, but the clue that factoring is the best approach is the need to make the coefficient of the variable term in the given equation match the coefficient of the variable term in the solve-for expression (this can be used as a mini optimization on some of the real-test problems that are suggested in the previous section but is more applicable to the ones suggested in this section). Note that the following problem has two unknown values, so we cannot solve for the individual values but must instead solve directly for an expression containing both of those unknowns.

Example 0.4-5

5 1600.io 0.4

$$6ax - 3 = 30$$

Based on the equation above, what is the value of $2ax - 1$?

A) 5

B) 6

C) 10

D) 15

Solution 1

1. We can factor 3 out of both terms on the left side of the equation, which will produce the solve-for expression as the other factor.

$$6ax - 3 = 30$$
$$3(2ax - 1) = 30$$

2. Divide both sides of the equation by 3 to isolate and solve for the solve-for expression.

$$3(2ax - 1) = 30$$
$$\frac{\cancel{3}(2ax - 1)}{\cancel{3}} = \frac{30}{\mathbf{3}}$$
$$2ax - 1 = 10$$

3. The answer is C.

Solution 2

1. Add 3 to both sides of the equation.

$$\begin{aligned} 6ax - 3 &= 30 \\ 6ax &= 33 \end{aligned}$$

2. Divide both sides of the equation by 6 to solve for ax (we cannot solve for either unknown value individually, but the term ax appears in the solve-for expression, so solving for ax is sufficient) and reduce the resulting fraction.

$$\begin{aligned} 6ax &= 33 \\ \frac{\cancel{6}ax}{\cancel{6}} &= \frac{33}{\mathbf{6}} \\ ax &= \frac{11}{2} \end{aligned}$$

3. Substitute $\frac{11}{2}$ for ax in the solve-for expression.

$$\begin{aligned} 2ax - 1 &= 2\left(\frac{11}{2}\right) - 1 \\ 2ax - 1 &= 11 - 1 \\ 2ax - 1 &= 10 \end{aligned}$$

4. The answer is C.

Section 0.4 Suggested Problems from Real Tests

- Test 1-C-4
- Test 2-NC-1
- Test 2-C-6
- Test 3-NC-2
- Test 3-NC-4
- Test 5-C-4
- Test 6-NC-6
- Test 7-NC-2
- Test 7-NC-16
- Test 9-NC-17
- Test 9-C-3
- Test 10-C-4
- May 2017-C-3
- Mar 2018-C-6
- Apr 2018-C-1
- May 2018-C-3
- May 2019 (US)-C-6
- Oct 2019-C-5
- Mar 2021-C-17

0.5 Combining Like Terms

We already know that numbers can be combined with addition or subtraction, but the same goes for variable terms as well as constants and expressions. For example, $x + x = 2x$ because all we've done is add two x terms together. Similarly, we can add $2x$ and $3x$ to make $5x$, and we can subtract $6x$ from $4x$: $4x - 6x = -2x$. Constants can be treated the same way, so $4a + 3a = 7a$, and like expressions can also be combined: $9(x + 4) - 7(x + 4) = 2(x + 4)$.

Note that **two different variables, constants, or expressions cannot be combined in any way like this**. For example, $2x + 5y \neq 7xy$ (or any other nonsense way people can think of to combine two incompatible things).

Example 0.5-1

1 1600.io 0.5

If $3x + 5 = 2x - 3$, what is the value of $x + 9$?

A) -8

B) -1

C) 1

D) 8

Solution

1. Subtract 5 from both sides of the equation in order to eliminate constants on the left side of the equation. Combine constants on the right side of the equation.

$$3x + 5 = 2x - 3$$
$$3x\cancel{+5}\cancel{\mathbf{-5}} = 2x - 3 \mathbf{- 5}$$
$$3x = 2x - 8$$

2. Subtract $2x$ from both sides of the equation in order to combine the x-terms on the left side of the equation.

$$3x = 2x - 8$$
$$3x - \mathbf{2x} = \cancel{2x} - 8\cancel{\mathbf{-2x}}$$
$$x = -8$$

3. Substitute -8 for x in the solve-for expression.

$$x + 9 = -8 + 9$$
$$x + 9 = 1$$

4. The answer is C.

In the following example, in order to combine the variable terms, we also have to either make use of common denominators or multiply both sides of the equation by a value that will eliminate the fractions entirely.

Example 0.5-2

2 1600.io 0.5

$$\frac{1}{2}x - 1 = 2 - x$$

What value of x satisfies the equation above?

A) $\frac{1}{2}$

B) 2

C) 3

D) $\frac{9}{2}$

Solution 1

1. Multiply both sides of the equation by 2 in order to eliminate any fractions, and then distribute.

$$\frac{1}{2}x - 1 = 2 - x$$

$$\mathbf{2}\left(\frac{1}{2}x - 1\right) = \mathbf{2}(2 - x)$$

$$\cancel{2}\left(\frac{1}{\cancel{2}}x\right) - 2(1) = 2(2) - 2(x)$$

$$x - 2 = 4 - 2x$$

2. Add 2 to both sides of the equation in order to eliminate constants on the left side of the equation. Combine constants on the right side of the equation.

$$x - 2 = 4 - 2x$$

$$x \cancel{- 2 + \mathbf{2}} = 4 - 2x + \mathbf{2}$$

$$x = 6 - 2x$$

3. Add $2x$ to both sides of the equation in order to combine x-terms on the left side of the equation.

$$x = 6 - 2x$$

$$x + \mathbf{2x} = 6 \cancel{- 2x + \mathbf{2x}}$$

$$3x = 6$$

4. Divide both sides of the equation by 3 in order to solve for x.

$$3x = 6$$

$$\frac{\cancel{3}x}{\cancel{\mathbf{3}}} = \frac{6}{\mathbf{3}}$$

$$x = 2$$

5. The answer is B.

Solution 2

1. Add 1 to both sides of the equation in order to eliminate constants on the left side of the equation. Combine constant terms on the right side of the equation.

$$\frac{1}{2}x - 1 = 2 - x$$

$$\frac{1}{2}x\cancel{-1}\cancel{+\mathbf{1}} = 2 - x + \mathbf{1}$$

$$\frac{1}{2}x = 3 - x$$

2. Add x to both sides of the equation in order to combine the x-terms on the left side of the equation. Make a common denominator (write the second x-term's implied coefficient of 1 as $\frac{2}{2}$) in order to allow the adding of the coefficients.

$$\frac{1}{2}x = 3 - x$$

$$\frac{1}{2}x + \boldsymbol{x} = 3\cancel{-x}\cancel{+\boldsymbol{x}}$$

$$\frac{1}{2}x + \frac{\mathbf{2}}{\mathbf{2}}x = 3$$

$$\frac{1+2}{2}x = 3$$

$$\frac{3}{2}x = 3$$

3. To solve for x, divide both sides of the equation by $\frac{3}{2}$; this is accomplished by multiplying by the reciprocal of $\frac{3}{2}$, which is $\frac{2}{3}$ (this method is much neater than writing out division by a fraction).

$$\frac{3}{2}x = 3$$

$$\frac{\cancel{\mathbf{2}}}{\cancel{\mathbf{3}}}\left(\frac{\cancel{3}}{\cancel{2}}\right)x = \frac{\mathbf{2}}{\cancel{\mathbf{3}}}(\cancel{3})$$

$$x = 2$$

4. The answer is B.

The next problem has us combining many like terms, which follows exactly the same principles as are used in the simpler examples above. For example, $x + 2x + 3x + 4x = 10x$ because the coefficients add to 10 $(1 + 2 + 3 + 4 = 10)$.

Example 0.5-3

3 1600.io 0.5

$$5(p + 1) + 9(p - 1) = 11p$$

What value of p is the solution of the equation above?

Solution

1. Distribute the constants 5 and 9 through their respective parenthetical expressions on the left side of the equation. Combine like terms on the left side of the equation.

$$5(p+1)+9(p-1)=11p$$
$$5p+5+9p-9=11p$$
$$14p-4=11p$$

2. Add 4 to both sides of the equation in order to eliminate constants on the left side of the equation (f you wanted to, you could instead subtract $14p$ from both sides of the equation in order to combine p-terms on the right side of the equation, but it is good practice to maintain a positive coefficient for the variable that you want to solve for, and following the steps shown below will allow us to do that).

$$14p-4=11p$$
$$14p\cancel{-4+\mathbf{4}}=11p+\mathbf{4}$$
$$14p=11p+4$$

3. Subtract $11p$ from both sides of the equation in order to combine p-terms on the left side of the equation.

$$14p=11p+4$$
$$14p-\mathbf{11p}=\cancel{11p}+4\cancel{-\mathbf{11p}}$$
$$3p=4$$

4. Divide both sides of the equation by 3 in order to solve for p.

$$3p=4$$
$$\frac{\cancel{3}p}{\cancel{\mathbf{3}}}=\frac{4}{\mathbf{3}}$$
$$p=\frac{4}{3}$$

5. The answer is $\frac{4}{3}$.

Section 0.5 Suggested Problems from Real Tests

- Test 3-NC-17
- Test 5-NC-17
- Test 5-C-33
- Test 6-C-32
- Test 8-NC-1
- Test 10-NC-1
- Test 10-C-27
- Apr 2017-C-1
- May 2017-NC-18
- May 2017-C-24
- Mar 2018-NC-6
- Mar 2018-NC-9
- Mar 2018-C-14
- Mar 2018-C-35
- Apr 2018-C-6
- May 2018-NC-14
- Mar 2019-NC-4
- Mar 2019-NC-16
- Mar 2019-C-3
- Apr 2019-NC-6
- Apr 2019-C-1
- Apr 2019-C-13
- May 2019 (US)-NC-2
- May 2019 (Int)-NC-5
- May 2019 (Int)-NC-18
- May 2019 (Int)-C-4
- Oct 2019-C-22
- Mar 2020-C-1
- Oct 2020-NC-17
- Mar 2021-NC-3

0.6 Unknown Values in Denominators

If an unknown value is in the denominator of a fraction, you should eliminate the fraction by multiplying both sides of the equation by the denominator (just as you would when numbers are in the denominator). For example, if you are given the equation $\frac{8}{x} = 160$ and are asked to solve for x, simply multiply both sides of the equation by x to get $8 = 160x$, and then divide both sides of the equation by 160 to solve for x: $x = \frac{8}{160} = \frac{1}{20}$.

Note that sometimes, having an unknown in the denominator can lead to extraneous solutions (which are covered in Chapter 9) because the denominator might end up being equal to 0, but those problems are easily spotted on the test, and none of those are included in the scope of this section. Therefore, in the cases covered here, simply multiplying by the denominator to eliminate the fractions works well.

When an expression appears in the denominator, we cannot simply split the fraction into parts as we could when the numerator is an expression and the denominator is just a number. For example, $\frac{2x+2}{2} = \frac{2x}{2} + \frac{2}{2} = x + 1$, but $\frac{2}{x+2} \neq \frac{2}{x} + \frac{2}{2}$, so be careful not to make that mistake moving forward.

In the following example, the unknown value appears in the denominator of two fractions (and the denominators aren't the same so we can't just combine the fractions). Instead of making a common denominator, we just need to multiply both sides of the equation by *both* of those denominators to eliminate both fractions.

Example 0.6-1

1 1600.io 0.6

$$\frac{1}{x} + \frac{2}{x-2} = 0$$

What value of x satisfies the equation above?

Solution

1. Our first goal is to eliminate fractions, so first multiply both sides of the equation by x, which is the denominator of the first fraction.

$$\frac{1}{x} + \frac{2}{x-2} = 0$$

$$\boldsymbol{x}\left(\frac{1}{x} + \frac{2}{x-2}\right) = \boldsymbol{x}(0)$$

$$\cancel{x}\left(\frac{1}{\cancel{x}}\right) + x\left(\frac{2}{x-2}\right) = 0$$

$$1 + \frac{2x}{x-2} = 0$$

2. Multiply both sides of the equation by $x - 2$, which is the denominator of the remaining fraction. Combine like terms.

$$1 + \frac{2x}{x-2} = 0$$

$$\mathbf{(x-2)}\left(1 + \frac{2x}{x-2}\right) = \mathbf{(x-2)}(0)$$

$$(x-2)(1) + \cancel{(x-2)}\left(\frac{2x}{\cancel{x-2}}\right) = 0$$

$$x - 2 + 2x = 0$$

$$3x - 2 = 0$$

3. Add 2 to both sides of the equation in order to isolate the x-term.

$$3x - 2 = 0$$

$$3x \cancel{-2} \cancel{+\mathbf{2}} = 0 + \mathbf{2}$$

$$3x = 2$$

4. Divide both sides of the equation by 3 to solve for x.

$$3x = 2$$

$$\frac{\cancel{3}x}{\cancel{\mathbf{3}}} = \frac{2}{\mathbf{3}}$$

$$x = \frac{2}{3}$$

5. The answer is $\frac{2}{3}$.

Occasionally, you will see a fraction in which both the numerator and denominator are unknown. If you know the value of a fraction $\frac{x}{y}$, then the value of the reciprocal, $\frac{y}{x}$, is equal to the reciprocal of the original value (this should come as no surprise). For example, if $\frac{x}{y} = \frac{2}{3}$, then $\frac{y}{x} = \frac{3}{2}$; we've just taken the reciprocal of both sides of an equation. Looked at another way (literally), if you stand on your head, the equation is still true.

In the following example, we cannot find the individual values of a and b, but we can solve for an expression that is useful to us by taking advantage of the reciprocal.

Example 0.6-2

2 1600.io 0.6

If $\frac{3a}{b} = \frac{1}{3}$, what is the value of $\frac{b}{a}$?

A) $\frac{1}{9}$

B) 1

C) 3

D) 9

Solution 1

1. Multiply both sides of the equation by $\frac{1}{3}$ (this is the same as dividing by 3) in order to isolate the expression $\frac{a}{b}$, which is the reciprocal of the solve-for expression.

$$\frac{3a}{b} = \frac{1}{3}$$

$$\frac{\mathbf{1}}{\cancel{\mathbf{3}}}\left(\frac{\cancel{3}a}{b}\right) = \frac{\mathbf{1}}{\mathbf{3}}\left(\frac{1}{3}\right)$$

$$\frac{a}{b} = \frac{1}{9}$$

2. Since $\frac{a}{b} = \frac{1}{9}$, then the reciprocals of both sides are also equal.

$$\frac{b}{a} = \frac{9}{1} = 9$$

3. The answer is D.

Solution 2

1. Multiply both sides of the equation by b to eliminate the fraction on the left side of the equation.

$$\frac{3a}{b} = \frac{1}{3}$$

$$\cancel{\boldsymbol{b}}\left(\frac{3a}{\cancel{b}}\right) = \boldsymbol{b}\left(\frac{1}{3}\right)$$

$$3a = \frac{b}{3}$$

2. Multiply both sides of the equation by 3 to eliminate the fraction on the right side of the equation.

$$3a = \frac{b}{3}$$

$$\mathbf{3}(3a) = \cancel{\mathbf{3}}\left(\frac{b}{\cancel{3}}\right)$$

$$9a = b$$

3. Divide both sides of the equation by a to create and solve for the solve-for expression.

$$9a = b$$

$$\frac{9\cancel{a}}{\cancel{\boldsymbol{a}}} = \frac{b}{\boldsymbol{a}}$$

$$9 = \frac{b}{a}$$

4. The answer is D.

Section 0.6 Suggested Problems from Real Tests

- Test 1-NC-8
- Test 3-NC-5
- Test 4-NC-10
- Test 8-NC-12
- Test 9-NC-7
- Mar 2018-NC-17
- May 2018-NC-18

Linear Relationships 1

1.1 Demystifying Linear Relationships

You might believe that you don't understand some of the math concepts on the SAT, but if I had a dollar for every student who wrongly thought they didn't understand linear relationships...

Let's say it's been a thousand students; then, you'd probably be able to figure out pretty quickly that I would have $1,000. If it had been twice as many students, I'd have twice as much money. If this makes sense, then you already get the essence of **linear relationships**. We often encounter real-life situations in which two quantities are related in such a way that a change in one of them results in a corresponding change in the other. If you get paid hourly, then working twice as many hours will double your pay. If your new school is only half as far away as your old one, it will take you half as long to get there if you travel at the same speed.

The phrase "linear relationships" doesn't necessarily make sense as a result of this knowledge because the phrase comes from the shapes of the graphs of these relationships, and we will get to those eventually. To get started though, we just need to be able to model these relationships by writing mathematical expressions that allow us to calculate values without simply listing a long series of numbers. You don't have to understand lines in an abstract way in order to express these relationships. Before we get into an algebraic understanding of lines, we can use common sense to work our way through some questions based on this type of relationship.

For example, Mindy lives within walking distance of three buffet restaurants that have different price structures.

The first buffet charges $20 to enter and dine, no matter how much food she eats. Whether Mindy eats 4 plates of food or 20 plates of food, she will pay a fixed amount of $20 to eat there.

$$\text{Total Cost} = \text{Entrance Charge}$$

The second buffet only charges by the plate (there is no entrance fee). Each plate of food Mindy eats costs $4. Here, Mindy will pay $4 times the number of plates of food she eats. If she eats one plate of food, her bill will be $4; if she eats 4 plates, her bill will be $\$4 \times 4 = \$16$. If she eats p plates of food, her bill will be $4p$ dollars.

$$\text{Total Cost} = \text{Price Per Plate} \times \text{Number of Plates}$$

The third buffet combines the approaches of both of the other buffets. This third buffet imposes a smaller entrance charge than does the first buffet: entrance to the third buffet only costs $5. The third buffet also charges less per plate than does the second buffet: each plate of food costs $2. Here, Mindy will pay $5 plus $2 per plate of food. In order to enter the restaurant, she pays $5; if she has a heart attack while waiting in line and gets 0 plates of food, the whole trip will still cost her only $5 (aside from medical bills). If she eats one plate of food, the trip will cost $2 for the one plate of food ($2 times 1 plate) plus the $5 entrance fee (a total of $7). If she eats two plates of food, the cost will be $4 for the two plates of food ($2 times 2 plates) plus the $5 entrance charge for a total of $9. If she eats p plates of food, her bill will be $2p$ ($2 times p plates) plus the $5 entrance charge for a total of $2p + 5$ dollars.

$$\text{Total Cost} = (\text{Price Per Plate} \times \text{Number of Plates}) + \text{Entrance Charge}$$

The different styles of buffet demonstrate different facets of linear relationships. In the simplest case, there can just be a constant amount that never changes (the entrance charge). More commonly, the value produced can be changed by some amount (here, the price per plate) every time a variable (the number of plates) is changed. And finally, in the most general situation, there can be a combination of both of these attributes.

1.2 Writing Linear Expressions

On the test, you will be responsible for writing mathematical expressions to represent linear relationships.

Example 1.2-1

1 1.2

One pound of peanuts costs \$3. At this price, how much will p pounds of peanuts cost?

A) $3p$

B) $3+p$

C) $\frac{3}{p}$

D) $\frac{p}{3}$

Solution

1. Since each pound costs \$3, multiply 3 by the number of pounds, p, to find the total cost, which is $3p$.
2. The answer is A.

Notes

If you need to verify that this works, just list out a few values to see that this is true.

Pounds of Peanuts	Cost	Pattern Representation
1	\$3	3(1)
2	\$6	3(2)
3	\$9	3(3)

Notice in each row that the value in the Cost column is three times the value in the Pounds of Peanuts column. If p is the number of pounds of peanuts, then three times p, or $3p$, is the cost of those p pounds of peanuts.

Now suppose that George starts a pen collection with 20 pens that were gifted to him by his grandfather. He plans on buying 3 new pens every month to add to his collection. How could we write an expression to show the total number of pens he would have after m months?

The number of new pens George would have after m months follows the same pattern we have already talked about. If he adds 3 pens a month for m months, he would have $3m$ *new* pens. All we have to do is add the original 20 pens to find out the total number of pens he has after m months: $3m + 20$.

Let's look at this situation as represented by a table to see how this expression makes sense as a way of representing the pattern of how his collection grows.

Number of Months	Total Number of Pens	Pattern Representation
0	20	$20 + 3(0)$
1	23	$20 + 3(1)$
2	26	$20 + 3(2)$
m	$20 + 3m$	$20 + 3(m)$

Example 1.2-2

2 1600.io 1.2

One of the requirements for becoming a journalist was the ability to write notes at a rate of 100 words per minute. Lewis was able to write only 40 words per minute when he started reporting, but through practice, he was able to increase his speed by 2 words per minute each week. Which of the following represents the number of words per minute that Lewis could write after w weeks of practice?

A) $2 + 40w$

B) $40 + 2w$

C) $100 + 2w$

D) $100 - 2w$

Solution

1. Since Lewis's speed increases by 2 words per minute every week, the amount his speed increases is equal to 2 times the number of weeks, w. His speed increases by $2w$ overall.
2. Since Lewis started at a speed of 40 words per minute, to find his overall speed, we need to add his speed increase to his original starting speed. Lewis's overall speed after w weeks of practice is $40 + 2w$.
3. The answer is B.

Notes

You may have noticed that in this example, the constant term (40) appears before the variable term ($2w$), but it's important to understand that the ordering of the terms does not matter: $40 + 2w$ is equivalent to $2w + 40$. Linear expressions on the test can appear in either arrangement. By convention, variable terms appear first when writing expressions, but the ordering has no mathematical consequences.

Note that in this problem, you're given some information that you do not need: the requirement that a journalist be able to write at 100 words per minute. This happens occasionally on word problems (though very rarely on strictly mathematical problems).

You should understand that just because you don't use a given quantity or other piece of information, that does not mean that you are not properly solving the problem.

SkillDrill 1.2-1

Directions: Based on the situation described, write a mathematical expression that can be used to answer the question.

1. Emily earns \$12 for every pizza she sells. How much money will she have if she sells p pizzas?

2. The Tooth Fairy gains one tooth for every house she visits. How many teeth will she have after visiting h houses?

3. Brandon started the day with \$50 in his wallet, and he will get another \$20 from each of his aunts. How much money will he have after he visits a of his aunts?

4. Charlie has 84 baseball cards and he buys 3 more every weekend. How many cards will he have after w weekends?

5. Priscilla has to eat 20 hamburgers to win a prize. It takes her 3 minutes to eat each hamburger. If she has already eaten h hamburgers, how much *longer* will it take her to eat the remaining hamburgers?

Section 1.2 Suggested Problems from Real Tests 🏆 Hall of Fame

- Test 4-C-2
- Test 8-C-1
- Apr 2017-C-21 🏆
- Mar 2018-NC-2 🏆
- Apr 2018-C-28
- May 2018-NC-1
- May 2019 (US)-C-12 🏆

Section 1.2 Practice Problems

1 1600.io 1.2

Saquon spent a total of \$450 on his hotel room during his vacation. Each day of his vacation, he bought a dinner at a restaurant for \$28. If Saquon bought d dinners, how much money, in dollars, did Saquon spend during his vacation on his hotel room and dinners?

A) $450 + 56d$

B) $450 + 28d$

C) $450 + 14d$

D) $28d$

2 1600.io 1.2

An artisan is crafting a custom chess set by carving each of the 32 pieces in the set by hand. It takes the artisan 40 minutes to carve each piece. If p of the pieces are already carved, which of the following represents the number of additional minutes needed to finish carving the set of chess pieces?

A) $32(40 - p)$

B) $32(p - 40)$

C) $40(32 - p)$

D) $40(p - 32)$

1.3 Writing and Solving Linear Equations

We have seen a few problems in which we wrote mathematical expressions that represent a certain type of simple pattern or relationship. **Expressions**, strictly speaking, consist only of numerical and variable terms being added, subtracted, multiplied, etc. For example, $40 + 2w$ is an expression. When we set this value equal to another expression, we can make an equation. **Equations** consist of two expressions that are set equal to each other using an equals sign (=). For example, if we wanted to make an equation to represent Lewis's writing speed, we could set s, the speed that Lewis can write words, equal to the expression $40 + 2w$, which tells us Lewis's writing speed after w weeks of practice. In equation form, the relationship is written $s = 40 + 2w$. Note that the linear expression could appear on either side of the equals sign; the equation $s = 40 + 2w$ could also be written as $40 + 2w = s$.

Choose Variable Letters that Make Sense

When writing your own equations, you should always choose letters that tell you what a variable stands for. For example, in the case of Lewis's writing speed, we used s to represent the *speed* that he writes (we were told to use w for the number of weeks).

Avoid always using x and y when writing your own equations for real world word problems, because—although x and y are usually the most familiar variable letters to most students—it is very easy to forget what each letter represents (unless you're working with xylophones and yodelers).

In the case of Lewis, if we write $y = 40 + 2x$ it is not apparent what x and y represent, but when we write $s = 40 + 2w$ it is much easier to see that s represents the speed and w represents the number of weeks he practiced.

To take the next step into writing linear *equations*, all we need to do is set the kinds of *expressions* we have been building equal to some other quantity or variable that represents the total.

Example 1.3-1

1 1.3

A drone, initially hovering 80 feet above the ground, begins to lose altitude at a rate of 3 feet per second. Which of the following equations represents the drone's altitude above the ground a, in feet, t seconds after the drone begins to lose altitude?

A) $a = 80 - 3$

B) $a = 80 + 3t$

C) $a = 80 - 3t$

D) $a = 80t - 3$

Solution

1. The altitude, or height, goes down by 3 feet every second (that is, the change in height is -3 feet every second), so to find the total change in the height, we have to multiply -3 by the number of seconds, t. The change in height after t seconds is $-3t$.

2. Since the drone's altitude started at 80 feet and *decreased* by $3t$ feet after t seconds, we start with 80 and *subtract* $3t$, then set it equal to a, the altitude of the drone after t seconds. The final equation used to find the height after t seconds is $a = 80 - 3t$.

3. The answer is C.

Notes

This is the first time we've seen a negative change in the value in a linear relationship. If you are unsure whether the change should be added or subtracted in this example, think about the way the height changes. A negative change in the height should be reflected in the equation by subtracting the change in height from the initial height.

Here, we have an ordinary linear relationship written as a constant first, then a variable term: $a = 80 - 3t$. You might already be familiar with Slope-Intercept Form for a linear equation, which, by convention, places the variable term first, followed by the constant (the "y-intercept", which we will explain thoroughly a bit later); here, that would be $a = -3t + 80$. Note, importantly, that this has *no effect whatsoever on the relationship*; it's just a matter of writing the terms in a different order which happens to make sense for the situation in the problem.

Having equations to represent this type of situation is useful for when we need to calculate values that we would not easily be able to count to. For example, it's easy to add the cost of a single item two or three times, but when we are buying hundreds or thousands of items, we can use the equations to easily find particular values in a way that involves no repetition.

Example 1.3-2

2 1600.io 1.3

A partially filled aquarium contains 20 gallons of water. A hose is turned on, and water flows into the tank at a rate of 4 gallons per minute. How many gallons of water will be in the aquarium after 30 minutes?

Solution

1. We have to start by writing the equation to represent the situation. Let w be the total amount of water in the aquarium after being filled for m minutes. The amount of water starts at 20 gallons and then increases by 4 gallons every minute, so the equation that represents the situation is $w = 4m + 20$ (again, note that we could have written this as $20 + 4m$, which you might feel matches the real-world situation more naturally; it makes no difference).

2. We want to know how much water there will be in the aquarium after 30 minutes, so we should replace m with 30 and find the value of w.

$$w = 4m + 20$$
$$w = 4(30) + 20$$
$$w = 120 + 20$$
$$w = 140$$

3. The answer is 140.

We now know how to find final values for situations where we know what to substitute into the equations: we can find the cost of a buffet when we know how many plates of food are eaten, and we can find the number of gallons of water in an aquarium after filling it for a certain amount of time. But what if we were told the total amount Mindy spent at a buffet, and we were then asked to figure out how many plates of food she ate? What if we were told how much water was in the aquarium after a certain amount of time, and we were then asked how many minutes the hose must have been filling the aquarium in order for there to be that much water? Essentially, we are being asked to substitute a different value for a different variable in an equation that models a linear relationship. The following example illustrates this concept.

Example 1.3-3

3 1600.io 1.3

The monthly membership fee for a food delivery service is \$22.50. The cost of dinners is included in the membership fee, but there is an additional fee of \$2.50 for each dessert. In April, Gus's membership and dessert fees were \$32.50. How many desserts did Gus have delivered that month?

A) 2

B) 3

C) 4

D) 5

Solution

1. We have to start by writing the equation that represents the total cost. Let's use t for the total amount of money Gus pays each month and d for the number of desserts. There is a constant charge of \$22.50, and the cost goes up by \$2.50 for every dessert Gus orders that month, so the equation is $t = 2.50d + 22.50$.

2. We know that his total charge in April was \$32.50, so we should replace t with 32.50.

$$t = 2.50d + 22.50$$
$$32.50 = 2.50d + 22.50$$

3. Now that we have substituted the total cost for April in the correct place, we can solve for d, the number of desserts Gus ordered that month.

$$32.50 = 2.50d + 22.50$$
$$10 = 2.50d$$
$$\frac{10}{2.50} = d$$
$$4 = d$$

4. The answer is C.

Notes

A problem like this could appear on either the Calculator or No Calculator section of the test. Don't panic when you see decimals and fractions. You know how many times \$2.50 goes into 10 dollars, just like you know how many times 25 cents goes into \$1.

When we're working with a linear equation of the form $y = mx + b$, we can be given any three of the four components (the value of x, the value of y, the coefficient m of x, and the value of the constant b) that comprise these equations and use them to determine the value of the remaining component. As long as you understand what type of information is represented by each component, you can easily work through problems that involve these types of relationships regardless of which values are given and which are unknown. If you write a nice equation, with well-chosen letters, it should be easy to substitute values for the information that you know in order to solve for the variable whose value you would like to find.

SkillDrill 1.3-1

Directions: Based on the situation described, write a mathematical equation that can be used to answer the question. If you have to choose your own variable, make sure to choose a letter that makes sense. Solve the equation to find the unknown value.

1. Emily earns \$12 for every pizza she sells. How much money, in dollars, will she have if she sells 5 pizzas?

2. The Tooth Fairy gains one tooth for every house she visits. How many teeth will she have after visiting 31 houses?

3. Brandon started the day with \$50 in his wallet, and he received another \$20 from each of his aunts during the day. If he has \$110 at the end of the day, how many aunts did he visit?

4. Charlie has 84 baseball cards and he buys 3 more every weekend. How many cards will he have after 2 weekends?

5. Priscilla has to eat 20 hamburgers to win a prize. It takes her 3 minutes to eat each hamburger. If she has been eating for 15 minutes, how many more hamburgers does she still have to eat?

Section 1.3 Suggested Problems from Real Tests

🏆 Hall of Fame

- Test 1-C-10
- Test 2-C-3
- Test 2-C-8
- Test 4-C-1
- Test 4-C-31
- Test 5-C-7
- Test 5-C-16
- Test 5-C-18 🏆
- Test 6-C-4
- Test 7-NC-6 🏆
- Test 8-NC-19
- Test 9-NC-4
- Test 10-NC-2
- Test 10-C-1
- Apr 2017-C-19
- May 2017-C-5
- May 2017-C-15
- May 2017-C-20 🏆
- Mar 2018-C-9
- May 2018-C-5
- May 2018-C-33
- Mar 2019-NC-2
- Apr 2019-NC-1
- Apr 2019-C-4
- Apr 2019-C-31 🏆
- May 2019 (US)-C-1
- May 2019 (Int)-C-2
- May 2019 (Int)-C-3
- May 2019 (Int)-C-36
- Oct 2019-C-32
- Mar 2020-NC-5
- Oct 2020-NC-1

Section 1.3 Practice Problems

1 1.3

The Finer Things Club (FTC) in a certain high school plans to increase its membership by a total of n members per year. There were m members in the FTC at the beginning of this year. Which function best models the total number of members, y, the FTC plans to have as members x years from now?

A) $y = nx + m$

B) $y = nx - m$

C) $y = mx + n$

D) $y = mx - n$

2 1.3

A home security company charges homeowners a onetime setup fee of \$220 plus d dollars for each month of home monitoring. If a homeowner paid \$640 for the first 12 months, including the setup fee, what is the value of d ?

A) 25

B) 35

C) 40

D) 50

3 1.3

An eagle takes off from a perch that is 10 feet above the ground. If the eagle rises at a constant rate of 23 feet per second, which of the following equations gives the height, h, in feet, of the eagle s seconds after it starts flying upwards?

A) $h = 10s + 23$

B) $h = 10s + \frac{23}{10}$

C) $h = 23s + 10$

D) $h = 23s + \frac{23}{10}$

4 1.3

Charles is planning to ride his bike 200 miles to prove something to himself. If he plans to ride 15 miles per hour, which of the following represents the remaining distance d, in miles, that Charles will have to ride his bike to achieve his goal after riding for n hours?

A) $d = 200 + 15n$

B) $d = 200n - 15$

C) $d = 15n - 200$

D) $d = 200 - 15n$

1.4 Function Notation/Inputs and Outputs

Let's look at an alternative way of writing equations that uses a different notation to represent the total amount. We call equations written in this new form **functions**. With functions, an input value is plugged into the function, some operations are performed on the input value, and then an output value is produced. We can represent the relationships we have talked about in this chapter using function notation, so let's write an equation as a function to see what functions look like.

For example, if 30 cubic feet of water is being pumped into a pool every minute, we could use v to represent the volume of water and t to represent the amount of time the pool is being filled. We could say that the volume of water in the pool is a **function** of time, t, because the volume of water is dependent only on the amount of time that has passed. Instead of just using v to represent the volume, we can use the notation $v(t)$ (pronounced "v of t") to show that v is a function of time. The equation representing this relationship could be written as $v(t) = 30t$ (the volume of water in the pool is 30 gallons per minute times the number of minutes the water flows into the pool).

The t in the parentheses is a placeholder for an input value just as variables are always placeholders. If we wanted to know how much water there would be after 3 minutes, we would use 3 for the value of the function's input variable t. Using function notation, we may be asked to find the value of $v(3)$ (pronounced "v of 3"), which means the value of v when t is 3. Whatever value replaces the variable in the parentheses gets substituted in for the variable everywhere it appears in the expression that defines the function, so in this case we replace each instance of t with 3.

$$v(t) = 30t$$
$$v(3) = 30(3)$$
$$v(3) = 90$$

Realize that **the function notation $v(3)$ does not means "v times 3."** Many students get this confused because this is the first time they have seen parentheses used for a purpose other than multiplication. The test will let you know that v is a function by using the word "function" in the question. Even if the word "function" is not explicitly stated, it is more likely that $v(3)$ is function notation because v times 3 (or 3 times v) would normally be written as $3v$.

If a function called $f(x)$ is defined by the equation $f(x) = 3x + 2$, we can find $f(4)$, the output value when the input value is 4, by substituting 4 for x regardless of whether the function represents some real-world situation or it's a purely abstract mathematical construct; we simply substitute the input value and evaluate the expression to get the output value.

SkillDrill 1.4-1

Directions: Given the three functions below, $f(x)$, $g(x)$, and $h(x)$, find the value you are asked to find.

$$f(x) = 2x + 1 \qquad g(x) = -3x + 4 \qquad h(x) = \frac{1}{2}x - 2$$

Note that answers here can be negative, which is not the case with the test's Student Produced Response ("grid-in") questions.

1. $f(1)$ **2.** $g(0)$ **3.** $h(4)$ **4.** $f(-2)$

5. $g(4)$ **6.** $h(5)$ **7.** $f(10)$ **8.** $g(-2)$

9. $h(2)$ **10.** $f(0)$ **11.** $g(-3)$ **12.** $h(12)$

Directions: Using the same three functions listed above, find the input value x that will result in the given output value of the function.

13. $f(x) = 5$ **14.** $g(x) = 10$ **15.** $h(x) = 4$ **16.** $f(x) = 14$

17. $g(x) = -20$ **18.** $h(x) = -31$ **19.** $f(x) = 0$ **20.** $g(x) = 4$

21. $h(x) = -4$ **22.** $f(x) = 11$ **23.** $g(x) = 1$ **24.** $h(x) = \frac{1}{2}$

Example 1.4-1

1 1600.io 1.4

$$f(x) = 4x + 1$$

The function f is defined above. What is the value of $f(-2)$?

A) -8

B) -7

C) 7

D) 9

Solution

1. Replace each x with -2.

$$\begin{aligned} f(x) &= 4x + 1 \\ f(-2) &= 4(-2) + 1 \\ f(-2) &= -8 + 1 \\ f(-2) &= -7 \end{aligned}$$

2. The answer is B.

If you are told the value of a function for a particular x-value, you should substitute the x-value, but also replace the $f(x)$ term with the value of the function.

Example 1.4-2

2 1600.io 1.4

$$f(x) = \frac{1}{2}x + b$$

In the function above, b is a constant. If $f(4) = 5$, what is the value of $f(-4)$?

A) -5

B) -2

C) 1

D) 3

Solution

1. When we are told that $f(4) = 5$, we know that the value of the function is 5 when x is 4. Replace each x with 4.

$$f(x) = \frac{1}{2}x + b$$

$$f(4) = \frac{1}{2}(4) + b$$

$$f(4) = 2 + b$$

2. Since we know that $f(4) = 5$, we can replace $f(4)$ with 5. Remember that $f(4) = 5$ does NOT mean "f times 4 equals 5," but instead tells us that the value of the function is 5 when x is 4. Making this replacement enables us to solve for b.

$$f(4) = 2 + b$$
$$5 = 2 + b$$
$$3 = b$$

3. Now that we have the value of b, we can replace b with 3 in the given function.

$$f(x) = \frac{1}{2}x + b$$

$$f(x) = \frac{1}{2}x + 3$$

4. Since we want to find $f(-4)$, which denotes the value of the function when x is -4, we should replace each instance of x with -4.

$$f(x) = \frac{1}{2}x + 3$$

$$f(-4) = \frac{1}{2}(-4) + 3$$

$$f(-4) = -2 + 3$$

$$f(-4) = 1$$

5. The answer is C.

Function notation is used in a variety of problem types on the test, so it's essential that you become familiar and comfortable with it. In general, when dealing with the graphs of functions in the xy-plane (which begins in the next chapter), the expression $f(x)$ refers to the y-value for a given x-value. For example, $f(4)$ is the y-value of the function when the x-value is 4 (this may be a hint that plugging in 4 for x is a useful step in that hypothetical problem).

Section 1.4 Suggested Problems from Real Tests

🏆 Hall of Fame

- Test 1-C-10
- Test 2-C-3
- Test 2-C-8
- Test 4-C-1
- Test 4-C-31
- Test 5-C-7
- Test 5-C-16
- Test 5-C-18 🏆
- Test 6-C-4
- Test 7-NC-6 🏆
- Test 8-NC-19
- Test 9-NC-4
- Test 10-NC-2
- Test 10-C-1
- Apr 2017-C-19
- May 2017-C-5
- May 2017-C-15
- May 2017-C-20 🏆
- Mar 2018-C-9
- May 2018-C-5
- May 2018-C-33
- Mar 2019-NC-2
- Apr 2019-NC-1
- Apr 2019-C-4
- Apr 2019-C-31 🏆
- May 2019 (US)-C-1
- May 2019 (Int)-C-2
- May 2019 (Int)-C-3
- May 2019 (Int)-C-36
- Oct 2019-C-32
- Mar 2020-NC-5
- Oct-2020-NC-5
- Oct 2020-C-11

Section 1.4 Practice Problems

1 1600.io 1.4

$$f(x) = 3(x - 4) + 1$$

For the function f defined above, what is the value of $f(4)$?

A) -2

B) 0

C) 1

D) 4

2 1600.io 1.4

A tank initially contains 4,678 cubic inches of orange juice. A pump begins emptying the orange juice at a constant rate of 63 cubic inches per minute. Which of the following functions best approximates the volume $v(t)$, in cubic inches, of orange juice in the tank t minutes after pumping begins, for $0 \leq t \leq 74$?

A) $v(t) = 4{,}678 - 63t$

B) $v(t) = 4{,}678 - 74t$

C) $v(t) = 4{,}678 + 63t$

D) $v(t) = 4{,}678 + 74t$

3 1600.io 1.4

$$b(t) = \frac{3(6t - 10k)}{2} - 120$$

The number of people who bike on a public trail can be modeled by the function b above, where k is a constant and t is the air temperature in degrees Fahrenheit (°F) for $70 < t < 100$. If 480 people are predicted to bike on the trail when the temperature is 80°F, what is the value of k ?

A) 8

B) 10

C) 28

D) 30

4 1600.io 1.4

$$f(x) = bx + 4$$

In the function above, b is a constant. If $f(1) = 3$, what is the value of $f(-3)$?

A) -3

B) -1

C) 1

D) 7

CHAPTER 1 RECAP

- Total Amount = (Amount per Thing × Number of Things) + Starting Amount
- **Expressions** consist only of numerical and variable terms being added, subtracted, multiplied, etc. For example, $40+2w$ is an expression. When we set this value equal to another expression, we can make an equation. **Equations** consist of two expressions that are set equal to each other using an equals sign (=).
- When writing your own expressions and equations, use letters that make sense for the variables so that you don't mistake what the letters stand for.
- The four components of a linear equation in the form $y = mx + b$ are the y-value (also called an output value when an equation is represented as a function), the x-value (the input value when working with function notation), the coefficient m of the x-term, and the constant b. We can solve for any one of these when we know the other three. Make sure you substitute given values for the the correct component.
- Functions are a type of equation for which an input value, often x, is plugged into an equation, producing an output value $f(x)$, which is read as "f of x."

  Note that $f(x)$ means the output value of a function when x is plugged into the function. It *does not* mean "f times x." For example, $f(3)$ means the value of the function when 3 is substituted for x.
- Whatever goes into the parentheses in a function is the input value. Replace all instances of the input variable with whatever is in the parentheses in order to find the function's value.

  If you are given the value of $f(x)$ and you're asked to find the corresponding value of x, you should replace the expression $f(x)$ with that value and then solve for the x-value that produced that output.

Additional Problems

1 1600.io 1.4

$$f(x) = \frac{(x+8)}{5}$$

For the function f defined above, what is the value of $f(7) - f(-3)$?

A) 3

B) $\frac{12}{5}$

C) 2

D) $\frac{4}{5}$

2 1600.io 1.3

A magazine debuted with 1,329 subscribers. The magazine's growth plan assumes that 270 new subscribers will be added each quarter (every 3 months) for the first 10 years. If an equation is written in the form $y = ax + b$ to represent the number of subscribers, y, subscribed to the magazine x quarters after the magazine debuted, what is the value of b?

3 1600.io 1.2

John earns \$14.70 per hour at his part-time job. When he works z hours, he earns $14.70z$ dollars. Which of the following expressions gives the amount, in dollars, John will earn if he works $5z$ hours?

A) $5 + 14.70z$

B) $14.70(z + 5)$

C) $5z + 14.70z$

D) $5(14.70z)$

4 1600.io 1.3

Cranjis rented a motorcycle for one day from a company that charges \$100 per day plus \$0.35 per mile driven. If she was charged a total of \$149 for the rental and mileage, for how many miles of driving was Cranjis charged? (Assume there is no tax.)

A) 35

B) 140

C) 285

D) 425

5 1600.io 1.3

The acceleration due to gravity, in feet per second per second (ft/s^2), on Jupiter is 7 ft/s^2 less than 3 times the acceleration due to gravity on Venus. If the acceleration due to gravity on Jupiter is 80 ft/s^2, what is the acceleration due to gravity, in ft/s^2, on Venus?

6 1600.io 1.3

The equation $y = 0.2x$ models the relationship between the number of different exercises a certain fitness model performs, y, during an x-minute workout session. How many exercises did the fitness model perform if the session lasted 40 minutes?

A) 2

B) 5

C) 8

D) 20

7 1600.io 1.3

$$p = 17f - (6f + t)$$

The profit, p, in dollars, from producing and selling f frisbees is given by the equation above, where t is a constant. If 300 frisbees are produced and sold for a profit of \$2,150, what is the value of t ?

8 1600.io 1.4

$$c(x) = mx + 360$$

A company's total cost $c(x)$, in euros (€), to produce x pantaloons is given by the function above, where m is a constant and $x > 0$. The total cost to produce 50 pantaloons is €660. What is the total cost, in euros, to produce 500 pantaloons? (Disregard the € sign when writing your answer.)

9 1600.io 1.4

$$f(x) = \frac{x + 2}{3}$$

For the function f defined above, what is the value of $f(-2)$?

A) $-\frac{2}{3}$

B) 0

C) $\frac{1}{3}$

D) 1

10 1600.io 1.3

$$T = 2{,}000 + 20a$$

In the equation above, T represents Penny's total points in a video game, where a represents the number of apples she has caught in the game and 2,000 represents a starting bonus for choosing to play on a harder difficulty. If Penny scored a total of 4,400 points, how many apples did she catch in the game?

A) 100

B) 120

C) 220

D) 440

11 1.2

The top floor of a trapezoidal building has 4 rooms. There are 45 floors in total. If each floor has 3 more rooms than the floor above it, which expression gives the total number of rooms on the bottom floor?

A) $4 + 3(45 - 1)$

B) $4 + 3(45)$

C) $45(4 + 3)$

D) $4 + 3^{45}$

12 1.3

Paige was 37 inches tall the day she turned 4 years old, and she was 41 inches tall the day she turned 5 years old. If Paige's height increases by the same amount each year between the ages of 4 and 9, how many inches tall will she be the day she turns 8 years old?

Questions 13 and 14 refer to the following information.

Species of tree	Growth factor
Hackberry	3.1
Cottonwood	1.6
Silver maple	2.0
Norway maple	3.0
Blue spruce	2.1
Ponderosa pine	2.3
Crabapple	2.6
Honeylocust	2.7

One method of calculating the approximate age, in years, of a tree of a particular species is to multiply the diameter of the tree, in inches, by a constant called the growth factor for that species. The table above gives the growth factors for eight species of trees.

13 1.2

According to the information in the table, what is the approximate age of a Norway maple tree with a diameter of 18 inches?

A) 36 years

B) 42 years

C) 54 years

D) 58 years

14 1.2

If a ponderosa pine tree and a hackberry tree each now have a diameter of 10 inches, which of the following will be closest to the difference, in inches, of their diameters 10 years from now?

A) 1.0

B) 1.1

C) 1.2

D) 1.3

15 1.3

A water company charges Franklin \$0.04 per gallon of water he uses in his house. If Franklin was charged \$40 by the water company, how many gallons of water did Franklin use?

A) 0.01

B) 100

C) 160

D) 1,000

16 1.4

The function g is defined as $g(x) = \frac{4x}{5} + 4$. What is the value of $g(-50)$?

A) -46

B) -44

C) -36

D) -6

17 1.3

On its opening day, a bicycle shop had an inventory of 42 bikes. During the first 3 months, 21 additional bikes were purchased by the shop each week, and the sales team sold an average of 14 bikes per week. During the first three months, which of the following equations best models the bike inventory, b, at the shop t weeks after opening?

A) $b = -7t + 42$

B) $b = \frac{2}{3}t + 42$

C) $b = 7t + 42$

D) $b = 14t + 42$

18 1.3

A company that makes plastic toys purchases a 3D printer for \$4,360. The printer depreciates in value at a constant rate for 8 years, after which it is considered to have no monetary value. How much is the printer worth 2 years after it is purchased?

A) \$1,090

B) \$2,180

C) \$3,270

D) \$3,815

Answer Key

SkillDrill 1.2-1

1. $12p$
2. h
3. $50 + 20a$
4. $84 + 3w$
5. $3(20 - h)$ or $60 - 3h$

Section 1.2 Practice Problems

1. B
2. C

SkillDrill 1.3-1

1. 60
2. 31
3. 3
4. 90
5. 15

Section 1.3 Practice Problems

1. A
2. B
3. C
4. D

SkillDrill 1.4-1

1. 3
2. 4
3. 0
4. -3
5. -8
6. $\frac{1}{2}$
7. 21
8. 10
9. -1
10. 1
11. 13
12. 4
13. 2
14. -2
15. 12
16. $\frac{13}{2}$ or 6.5
17. 8
18. -58
19. $\frac{-1}{2}$ or -0.5
20. 0
21. -4
22. 5
23. 1
24. 5

Section 1.4 Practice Problems

1. C
2. A
3. A
4. D

Additional Problems

1. C
2. 1329
3. D
4. B
5. 29
6. C
7. 1150
8. 3360
9. B
10. B
11. A
12. 53
13. C
14. B
15. D
16. C
17. C
18. C

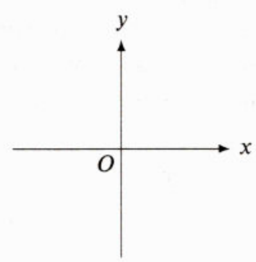
Slope-Intercept Form

2

2.1 Why They're Called Linear Relationships

Now it's time to see where linear relationships get their name. To start, you need to be familiar with the ***xy*-plane**.

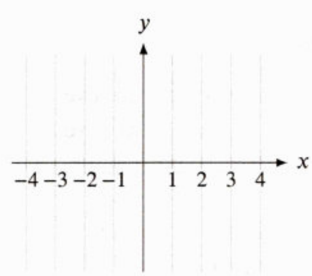
The xy-plane consists of two lines called **axes**. The ***x*-axis** goes from left to right (horizontal). The ***y*-axis** goes from bottom to top (vertical).



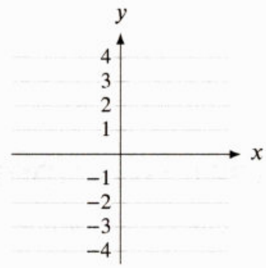
Typically, the point where the two axes cross is called the **origin** (usually marked with an O), and it is the point where both the x-value and y-value are equal to 0. When working with graphs of mathematical functions, the x- and y-axes will comply with this arrangement, but it's important to note that data sets represented as scatter plots, lines of best fit, and other statistical information are sometimes shown on graphs where the axes do **not** cross at their 0 values, and you must be alert for this situation, as it comes up often on the test.

There are often additional vertical and horizontal lines drawn on the plane that form a grid; these are evenly spaced lines that mark successive x- or y-values.

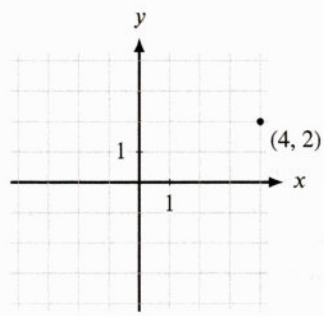
Because the y-axis goes straight up through the origin, which has an x-value of 0, all the points on the y-axis also have an x-value of 0. To the right of the y-axis are positive values of x, and to the left of the y-axis are negative values of x. The other vertical gridlines are usually (but not always!) used to denote the successive whole number values of x: 1, 2, 3, etc. To find the x-value of a point from a graph in that common arrangement, just count how many gridlines there are to the left or right of the y-axis.



The x-axis goes straight through the origin from left to right, and because the origin has a y-value of 0, all the points on the x-axis also have a y-value of 0. Above the x-axis are positive values of y, and below the x-axis are negative values of y. The other horizontal gridlines are usually (but not always!) used to denote the successive whole number values of y: 1, 2, 3, etc. To find the y-value of a point from a graph like that, count how many gridlines there are above or below the x-axis.

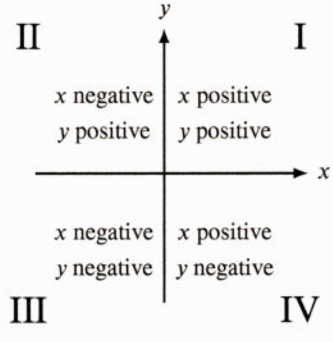


Points on this xy-plane will often be provided using the notation (x, y) where the first number tells us the x-value, and the second number tells us the corresponding y-value. For example, the point $(4, 2)$ is 4 lines to the right of the y-axis because the x-value is 4, and it's 2 lines above the x-axis because the y-value is 2. Because the *order* in which the two elements appear is important—the first value is the x-value, and the second, the y-value—this notation represents what is called an **ordered pair**. The ordered pair that represents the origin is $(0, 0)$ because both the x-and y-values are 0.



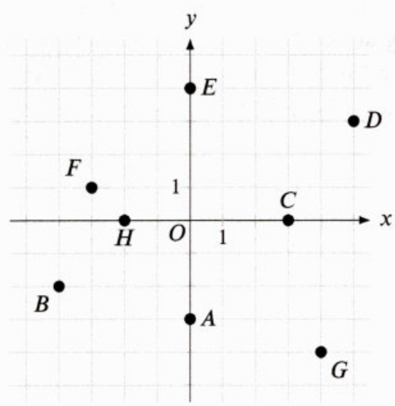
Another way you may be given the coordinates of points is based on function notation. For example, if you are told that $f(4) = 2$, then you know that $(4, 2)$ is a point on the graph of the function because the function's y-value is equal to 2 when the x-value is equal to 4. In other words, when plotting a function f on the xy-plane, $y = f(x)$.

The x- and y-axes divide the xy-plane into four **quadrants** that are numbered with Roman numerals starting in the upper right hand corner and going counter-clockwise.



SkillDrill 2.1-1

Directions: Write the ordered pair of x- and y-coordinates for each labeled point in the figure below.



1. Point A

2. Point B

3. Point C

4. Point D

5. Point E

6. Point F

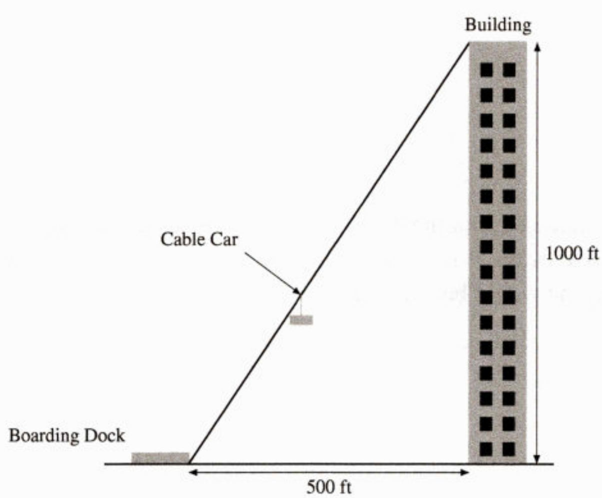
7. Point G

8. Point H

Slopes

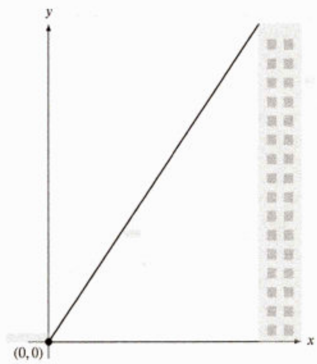
Now that we know about how points are represented on the xy-plane, let's relate this idea to a real world situation. We will plot some data points based on a scenario and see how they form a **line**, and linear relationships, when graphed in the xy-plane, produce **lines**—that's precisely why these relationships are "linear" (line-ar).

Molly is taking a ride on a cable car to the top of a tall building in a big city. The boarding platform is at ground level, and the building, which is located 500 feet away from the boarding dock, is 1,000 feet tall.

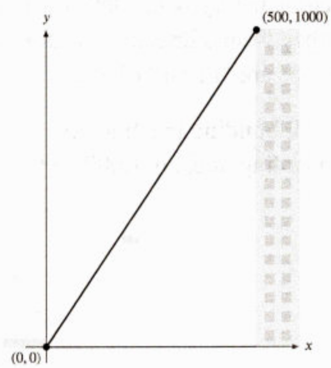


If Molly wanted to get to the top of the building without riding the cable car, she would have to walk horizontally 500 ft along the ground and then ride an elevator vertically 1,000 ft. The cable car, on the other hand, travels a straight line path from the dock to the top of the building that covers both the horizontal distance and vertical distance at the same time.

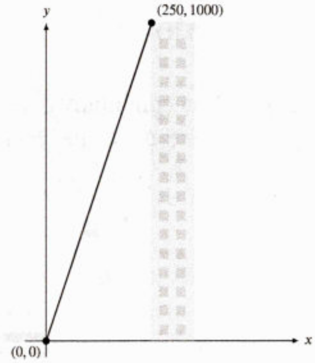
We can represent the cable on the xy-plane and use it to find the equation of the linear relationship between the height of the cable car and its horizontal distance from the boarding dock. If we use x to represent Molly's horizontal distance from the dock and y to represent her height above the ground, then when Molly is first starting her ride, we can say that 0 is an x-value and 0 is the corresponding y-value (she is at the boarding dock). Therefore, if we were to graph this point, we would graph the point $(0, 0)$, which is the origin.



In order to define a specific line on a graph, we need at least two points so we can draw a line through both of them. When Molly is at the top of the building, she is 500 feet away horizontally (the x-value is 500) from the boarding dock, and she is 1,000 ft above the boarding dock (the y-value is 1000), so another point on the graph is $(500, 1000)$. When we graph this point and draw a line connecting the two points we have labelled, we can see a representation of the vertical distance from the boarding dock as a function of the horizontal distance from the boarding dock.



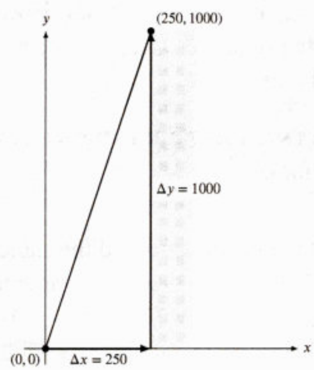
If the boarding dock were built closer to the base of the building, this would affect how slanted the line graph would be. The cable car would have to cover the same vertical distance in a smaller horizontal space. For example, if the boarding dock were only 250 ft away from the building, the tow line would be twice as steep.



We can write an equation to represent Molly's height as a function of the distance from the boarding dock based on the new, steeper configuration. Her starting height is 0, but we also need to find the amount that her height changes as she moves away from the boarding dock. When we look at any two points on the line, we can find how much Molly moves upward for every foot she moves forward. Let's use the points $(0,0)$ and $(250,1000)$.

We can draw two arrows that show how far away the end point at the top of the building lies from the starting point horizontally (in the x-direction) and vertically (in the y-direction). How far across we go represents the *change* in x-value; we can use the Greek letter delta (Δ) to mean "change in," so we write the change in x-value as Δx, pronounced "delta x." Similarly, how far up or down we go (that is, the movement parallel to the y-axis) represents the *change* in y-value; we write that change in y-value as Δy, pronounced "delta y."

Between the two points in our picture, $(0,0)$ and $(250,1000)$, we can draw in two arrows (the over-and-up arrows) showing the change in the x- and y-values.



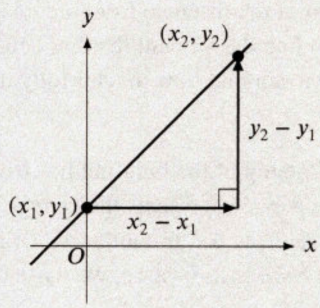
The y-value goes up by 1,000, so $\Delta y = 1000$. The x-value increases by 250, so $\Delta x = 250$. The word that we use to describe the steepness of such a graphed line is **slope**, and we define slope as the amount y changes relative to a change in x, so slope is calculated simply by dividing Δy (the change in y) by Δx (the change in x). For this line, the slope is therefore $\frac{1000}{250}$ or 4.

Slope

Slope is a quantity that shows the "steepness" of a line. It is commonly denoted by the variable m, and it is calculated using the following formula, where Δy is the change in y-values and Δx is the change in x-values.

$$m = \frac{\Delta y}{\Delta x}$$

Alternatively, the change in y-values can be more formally written as $y_2 - y_1$, where y_1 is the y-value of the first point and y_2 is the y-value of the second point. The change in x-values can be written as $x_2 - x_1$, where x_1 is the x-value of the first point and x_2 is the x-value of the second point. This equation is referred to as the **Slope Formula**.



$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

While this second form of the equation is not incorrect and is sometimes useful, in general, a quicker and less error-prone method of finding the slope is by traveling from one point to the other by drawing the over-and-up arrows (as in the example above) and determining their lengths to find Δy and Δx based on the graph alone. It is very easy to mess up signs and values when substituting the coordinates of two points into the Slope Formula. However, if the points are far apart, it could be very time-consuming to do all the incremental counting required using the traveling method, and errors can creep in, so in these circumstances, the Slope Formula is the better choice.

When given points on a line, you should always start by finding the slope whenever possible. It is very rare that the slope is not important to solving a problem involving lines.

For completeness, we will demonstrate using the Slope Formula to find the same information, though we recommend drawing the over-and-up arrows on the graph to find the slope directly. The first point we will choose is $(0, 0)$, which means that $x_1 = 0$ and $y_1 = 0$. The second point is $(250, 1000)$, which means that $x_2 = 250$ and $y_2 = 1000$. Plug these values into the slope formula and be careful that you are consistent about which point is (x_1, y_1) and which point is (x_2, y_2).

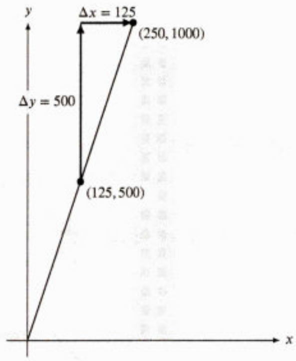
$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m = \frac{1000 - 0}{250 - 0}$$

$$m = \frac{1000}{250}$$

$$m = 4$$

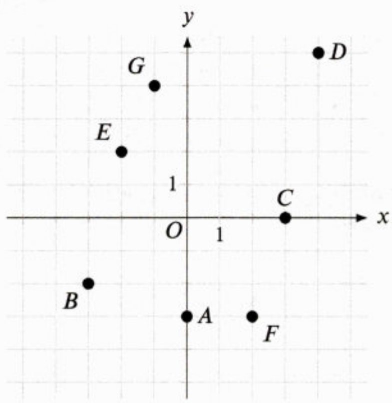
The slope of a particular line will be the same no matter which two points are chosen or which point is (x_1, y_1) and which point is (x_2, y_2). What if, instead of (0, 0) and (250, 1000), we chose the points (250, 1000) and (125, 500) (this is the midpoint of the tow line because it is half of the total x distance and half of the total y-distance between the boarding dock and the top of the building)? Note also that it doesn't matter whether we draw the over-and-up arrows above or below the line in question, as long as we make sure to go from left to right (it is not wrong to go from right to left, but you have to be careful to take negative signs into account because Δx would be negative if we go from right to left).



The y-value goes up by 500, so Δy is 500. The x-value goes up by 125, so Δx is 125. The slope is $\frac{500}{125}$, which still simplifies to 4.

SkillDrill 2.1-2

Directions: Find the slope of the line containing the two points found in the figure below.



1. Points A and C

2. Points A and D

3. Points C and D

4. Points B and E

5. Points C and F

6. Points D and F

7. Points B and D

8. Points B and G

Slope-Intercept Form

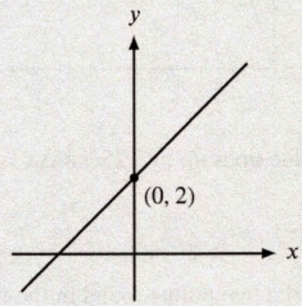
The next thing to determine when you are trying to find the equation of a line based on a graph is the points where the line crosses the axes. These crossing points are called **intercepts**.

Specifically, we are interested in the point where the line crosses the y-axis. For any linear graph, the point where the line crosses the y-axis is called the ***y*-intercept**.

y-intercept

The *y*-intercept is the point where a line crosses the *y*-axis. Another way of thinking about the y-intercept is that it's **the value of *y* when the *x*-value is 0** (the y-intercept is the value of $f(0)$). This quantity is often represented by the variable b.

In the graph below, the y-intercept is 2 because the line passes through the point $(0, 2)$, which is on the y-axis.



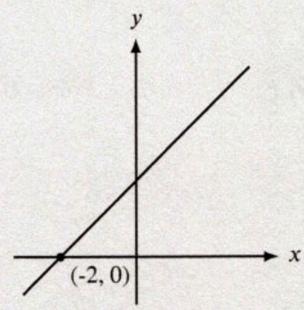
In real world problems, the y-intercept represents a starting or base amount when the x-value is 0. For example, if your cell phone plan charges you a fixed fee for the line each month plus an additional amount depending on how much data you use, the y-intercept of the function representing this linear relationship will be that base fee for the line; that's what you'd pay even if you use 0 gigabytes of data in a given month. As another example, in the last chapter, we talked about George's numerous pens, and the number of pens he started his collection with is 20, so 20 is the y-intercept.

When you need to solve for the *y*-intercept of the graph of an equation or function, substitute 0 for *x* and solve for *y*.

x-intercept

The *x*-intercept is the point where a line crosses the *x*-axis, and it can also be thought of as the value of *x* when the *y*-value is 0.

In the graph below, the x-intercept is -2 because the line passes through the point $(-2, 0)$, which is on the x-axis.



When you need to solve for the *x*-intercept of the graph of an equation or function, substitute 0 for *y* and solve for *x*.

All we have done is given new names to concepts we were already familiar with. The starting amount is the y-intercept, and the amount the values change is the slope. We can use this information to define a general linear equation that can be used even when we don't have a real world context to work from. It's called the **Slope-Intercept Form** of a line, and it's one of the most useful representations of lines on this test.

Slope-Intercept Form

For a line with slope m and y-intercept b, the Slope-Intercept Form of linear equations is the following:

$$y = mx + b$$

Note that any of the letters in the Slope-Intercept Form can change; the letters used above are the ones most commonly used by convention, but sometimes different letters that are more meaningful in the context of a specific problem are used instead for clarity. The arrangement and meaning of the terms are what really matter.

In the case of Molly on the cable car, we can find the equation that represents the linear relationship between her height above the ground and her distance (horizontally) from the boarding dock based on just two points on the line. We used the points $(0, 0)$ and $(250, 1000)$ to find that the slope of the line is 4. Also, we know the y-intercept of the line because the cable passes through the point $(0, 0)$; since the x-value of the point is 0, this point is the y-intercept of the line. Since we know the slope is 4 and the y-intercept is 0, the equation that represents the linear relationship in this example is $y = 4x + 0$, or just $y = 4x$.

SkillDrill 2.1-3

Directions: Given a linear equation in Slope-Intercept Form, identify the value of the slope, m, and the y-intercept, b.

1. $y = 2x + 1$

2. $y = 3x + 6$

3. $y = x - 7$

4. $y = 5x$

Directions: Write the equation of the line in Slope-Intercept Form given the value of the slope, m, and the y-intercept, b.

5. $m = 5, b = -4$

6. $m = 8, b = 1$

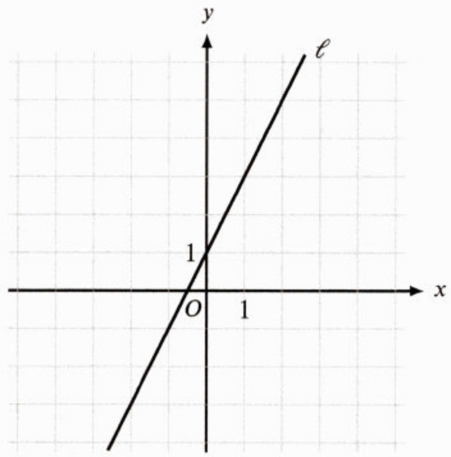
7. $m = 1, b = 0$

8. $m = 2, b = -2$

Let's walk through an example problem that requires us to write the equation of the line without any real world context. To do so, we will need to pull points from the graph in order to find the equation of the line. Sometimes points on a line are marked by black dots that are labelled with the x- and y-value of the point. Often, however, you will have to determine points on a line by using the gridlines to read the x- and y-values of the point. **Make sure to use points where the line crosses through the intersection of two gridlines whenever possible so that you know the exact coordinates of the point on the line**.

Example 2.1-1

1 2.1



Which of the following is an equation of line ℓ in the xy-plane above?

A) $x = 2$

B) $y = 1$

C) $y = x + 2$

D) $y = 2x + 1$

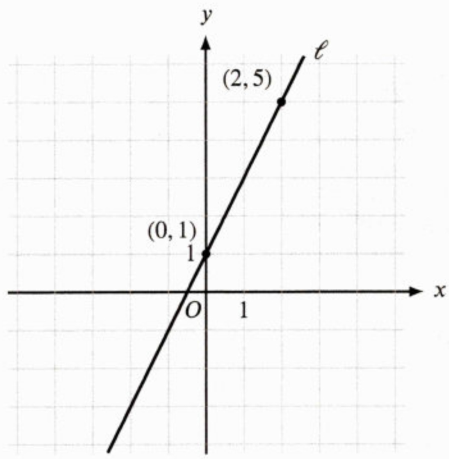
Solution

1. We know the equation of a line is of the form $y = mx + b$, and we need to find m, which represents the slope, and b, which represents the y-intercept (you can find them in any order depending on which is easier).

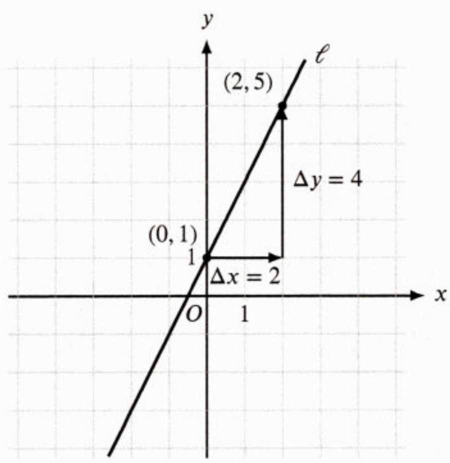
2. It is clear that the line passes through the point $(0, 1)$, which tells us that the y-intercept of the line is 1. We can replace b in the general Slope-Intercept Form with 1.

$$y = mx + b$$
$$y = mx + 1$$

3. We can find the slope by picking any two points on the line and finding the changes in x- and y-values between those points. Make sure to pick points that are located at the crossing of two gridlines. The best one to start with is the point $(0, 1)$ because this is the y-intercept and this tells us that the y-intercept is 1. We can pick any other point on the line that is also where two grid lines meet. Let's use $(2, 5)$ (the point $(1, 3)$ is also a good choice).



4. Draw the over-and-up arrows to find Δy and Δx.



5. The y-value increases by 4, so $\Delta y = 4$, and the x-value increases by 2, so $\Delta x = 2$. Therefore, the slope, which is equal to $\frac{\Delta y}{\Delta x}$, is $\frac{4}{2}$ or 2.
6. Replace m with 2 in the general Slope-Intercept Form to find the equation of the line.

$$y = 2x + 1$$

7. The answer is D.

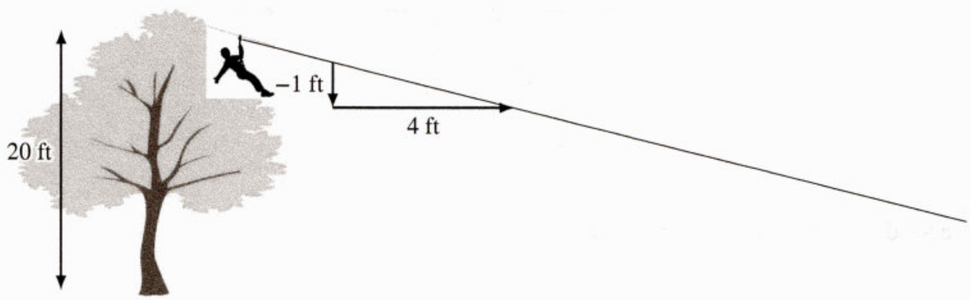
Notes

You can also use the Slope Formula to find the slope of the line, but usually drawing the over-and-up arrows is the preferred method, especially when the picture is already halfway drawn for you.

Slopes Aren't Always Positive Integers

The slopes of lines are not always positive; they can be positive, negative, or zero. Also, a slope does not need to be an integer.

David owns an adventure course with a zipline on which the rider starts 20 ft in the air, and the zipline is sloped such that the height drops by 1 ft for every 4 ft that a rider travels horizontally.

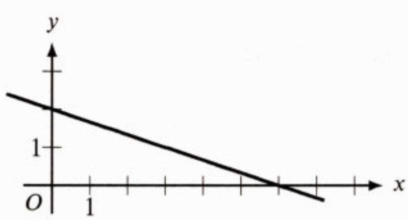


In this example, we can say that the y-intercept is 20 because the zipliner starts at a height of 20 feet off the ground when he has moved 0 ft away from the tree. The slope is $\frac{-1}{4}$ because the rider's height goes down by 1 ft (that is, it changes by -1 ft) for every 4 ft he moves away from the tree. The equation of the line is $y = \frac{-1}{4}x + 20$.

Note that a negative fractional slope might be written in the form $\frac{-1}{4}$ or $-\frac{1}{4}$, but the value is the same; in the example below, the second form is used, so don't let that confuse you. Also note that when we draw the over-and-down arrows, we are still working from left to right so that Δx (the "over") will always be positive and Δy can be positive or negative depending on how the y-values change between the two points.

Example 2.1-2

2 2.1



Which of the following could be an equation for the graph shown in the xy-plane above?

A) $y = -\frac{1}{3}x + 6$

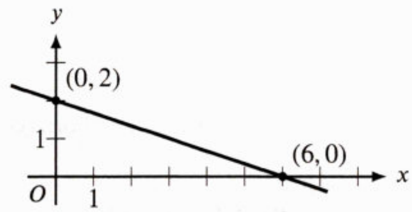
B) $y = -3x + 2$

C) $y = -\frac{1}{3}x + 2$

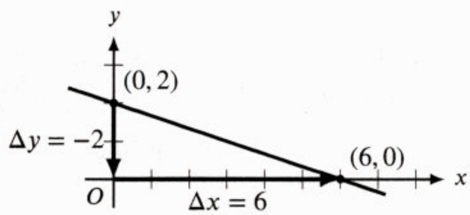
D) $y = -3x + 6$

Solution

1. We need to pick points to find the equation of the line. In this case, since there are no gridlines, the only points we can use are the y-intercept, which is the point $(0, 2)$, and the x-intercept, which is the point $(6, 0)$. Since we know from the graph that the y-intercept is 2, we can eliminate choices A and D, which show 6 as the y-intercept.



2. Draw the over-and-down arrows to show the change in the y-values and the change in the x-values.



The y-value decreases by 2, so $\Delta y = -2$. The x-value increases by 6, so $\Delta x = 6$. Therefore, the slope is $\frac{-2}{6}$, or $\frac{-1}{3}$ when we reduce the fraction.

3. Since the slope is $\frac{-1}{3}$ and the y-intercept is 2, the equation of the line is $y = \frac{-1}{3}x + 2$.

4. The answer is C.

Notes

You could have used the Slope Formula to find the slope; we'll use $(0, 2)$ as point 1 and $(6, 0)$ as point 2, but you could have chosen the reverse assignment, as it has no effect on the result.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m = \frac{0 - 2}{6 - 0}$$

$$m = \frac{-2}{6}$$

$$m = \frac{-1}{3}$$

Once again, this method is not incorrect, but it is easy to mess up. If you are going to choose to use the Slope Formula, make sure your work is neatly written out, with all steps shown.

As you can see, it's possible to encounter linear relationships that do not have positive integer slopes, but finding the slope of these lines is just as easy when you are given a graph on which to find the slope. If you are given a real world situation in which the y-variable decreases as the x-variable increases, you do not necessarily need to graph the points to find the slope or equation of the line. For example, if the average temperature in a certain country drops by 1°C every 10 days during the winter, then in a linear equation representing the average temperature in this country during winter, we can say that the slope must be $\frac{-1}{10}$ because the temperature decreases by 1 when the number of days increases by 10.

Vertical and Horizontal Lines

One last topic that is related to slopes of lines is the slopes of vertical and horizontal lines.

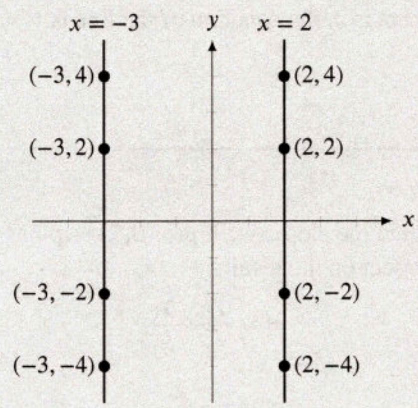
Vertical Lines

Vertical lines are lines where the x-value never changes. They have equations like $x = 5$, which tells us that the x-value is always 5 and the y-value does not depend on the x-value at all.

The slope of these lines is undefined, and looking at the Slope Formula will tell us why. Since the x-value does not change, $\Delta x = 0$. When we substitute 0 for Δx in the Slope Formula, we have a problem because it is impossible to divide by 0.

$$m = \frac{\Delta y}{0} = \text{undefined}$$

From this, it should be clear that vertical lines have equations written in the form $x = c$, where c is a constant value. Note, importantly, that this equation is not in Slope-Intercept form, because that form cannot represent a vertical line. For any point on vertical lines, the x-value is fixed, but each point will have a different y-value.



We can use the Slope Formula to verify that the slope between any two of these points is undefined. From the line $x = 2$, let's use $(2, 4)$ as (x_1, y_1) and $(2, 2)$ as (x_2, y_2).

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2 - 4}{2 - 2} = \frac{-2}{0} = \text{undefined}$$

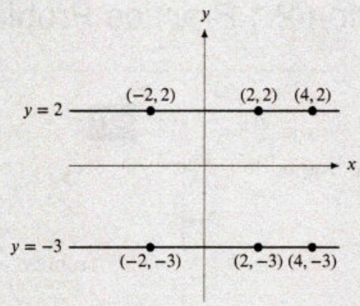
Horizontal Lines

Horizontal lines are lines where the y-value never changes. They have equations like $y = -2$, which tells us that the y-value is always -2 no matter what the x-value is.

The slope of these lines is 0. Since the y-value does not change, $\Delta y = 0$. When we substitute 0 for Δy in the Slope Formula, the numerator is 0, and 0 divided by a number is still 0.

$$m = \frac{0}{\Delta x}$$

From this, it should be clear that horizontal lines have equations written in the form $y = c$, where c is a constant value. Note that, unlike the equation for a vertical line, the horizontal line equation is in Slope-Intercept form, as it is equivalent to $y = 0x + c$.

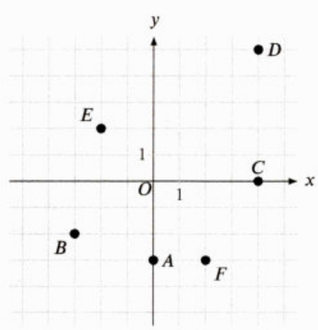


We can use the Slope Formula to verify that the slope between any two of these points is equal to 0. From the line $y = 2$, let's use $(4, 2)$ as (x_1, y_1) and $(-2, 2)$ as (x_2, y_2).

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2-2}{4-(-2)} = \frac{0}{6} = 0$$

SkillDrill 2.1-4

Directions: Based on the points in the figure below, find the equation of the line in Slope-Intercept Form (or $x = c$ form for vertical lines) that contains the two listed points, and find the coordinates of the x-intercept (if there is an x-intercept).



1. Points A and B

2. Points B and D

3. Points E and D

4. Points E and F

5. Points A and F

6. Points C and D

7. Points B and C

8. Points E and C

Section 2.1 Suggested Problems from Real Tests

🏆 Hall of Fame

- Test 1-NC-12
- Test 1-C-16
- Test 2-C-25 🏆
- Test 3-NC-8
- Test 3-C-17
- Test 3-C-26
- Test 4-NC-7
- Test 4-C-8
- Test 5-NC-1
- Test 6-C-14
- Test 7-C-28
- Test 7-C-32
- Test 8-NC-2
- Test 8-C-7
- Test 9-C-8
- Test 9-C-15
- Test 10-NC-4
- Test 10-C-8
- May 2017-NC-1
- Apr 2018-NC-5
- May 2018-NC-3
- May 2018-C-13
- Mar 2019-NC-12
- Mar 2019-C-8
- Apr 2019-NC-4
- Apr 2019-C-2 🏆
- Apr 2019-C-22 🏆
- May 2019 (Int)-NC-8
- May 2019 (Int)-NC-10
- Oct 2019-NC-6
- Oct 2019-C-7
- Oct 2019-C-33
- Mar 2020-NC-16
- Oct 2020-C-33
- Mar 2021-NC-4
- Mar 2021-C-16
- Mar 2021-C-35

Section 2.1 Practice Problems

1 1600.io 2.1

In the xy-plane, what is the y-intercept of the line with the equation $y = 3x - 2$?

A) 3

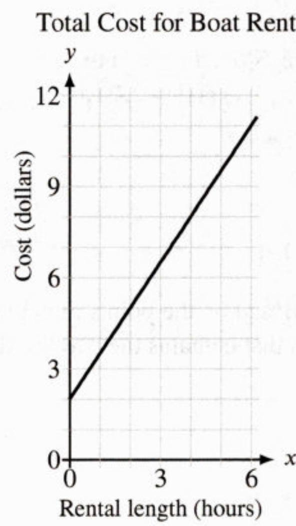
B) $\frac{1}{3}$

C) $-\frac{1}{3}$

D) -2

2 1600.io 2.1

The line graphed in the xy-plane below models the total cost, in dollars, for a paddleboat rental, y, in a certain city park based on the number of hours the paddleboat is rented, x.



According to the graph, what is the cost for each additional hour, in dollars, of the rental?

A) \$0.50

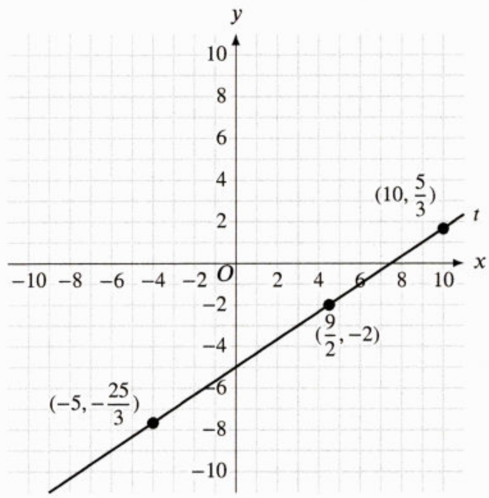
B) \$1.00

C) \$1.50

D) \$2.00

3 2.1

Line t is shown in the xy-plane below.



What is the slope of line t ?

4 2.1

The graph of $y = f(x)$ is a line in the xy-plane that passes through the point $(0, 7)$ and has a slope of 3. Which of the following equations could define the function f ?

A) $f(x) = -\frac{1}{7}x + 3$

B) $f(x) = -\frac{1}{3}x + 7$

C) $f(x) = 3x + 7$

D) $f(x) = 7x + 3$

2.2 Analyzing Graphs of Linear Equations

In a way, we have already walked through the process of making a graphical representation of a linear relationship, but we used real world examples like hills and ziplines, which both have natural slopes and automatically conjure up a mental image of a sloped line. However, we need to be able to identify the graphs of linear equations based on the equations themselves, without the aid of any real world context.

Luckily, the Slope-Intercept Form of lines provides us with enough useful information to identify the graph of a linear equation based on the slope and y-intercept, which are the two key visual features of line graphs.

Graphing Slope and y-intercept

When you have to identify the graph that corresponds to a linear equation, the key features to look out for are the slope and y-intercept.

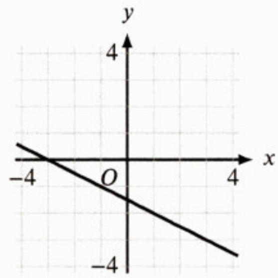
Start by eliminating any choices that do not pass through the correct y-intercept value. Then check that the line slopes in the correct direction (positive or negative slope). Finally, if there are still multiple remaining choices, make sure that the slope is the correct value (i.e. if the slope is supposed to be -2, make sure that the graph goes down by 2 units for every 1 unit it goes to the right).

Example 2.2-1

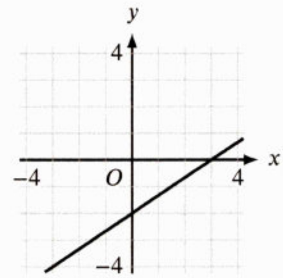
1 1600.io 2.2

Which of the following is the graph of the equation $y = -2x + 3$ in the xy-plane?

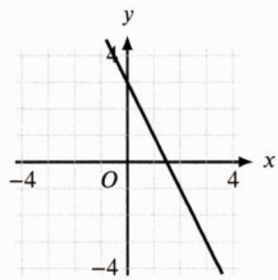
A)



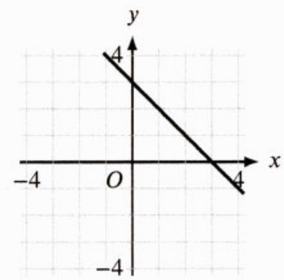
B)



C)



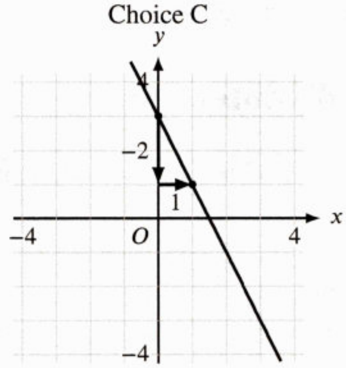
D)



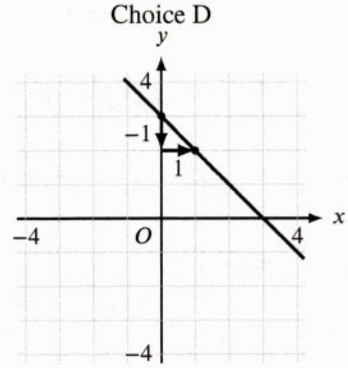
Solution

1. The y-intercept of the given equation is 3, so eliminate choices A and B, which do not pass through the y-axis at $y = 3$ (that is, at the point $(0, 3)$).
2. The slope of the line is negative (it's -2), but the remaining choices (C and D) have negative slopes, so we cannot eliminate either of them immediately. We need to choose the graph that has a slope of -2.
   Looking at the two remaining choices, we can start from the y-intercept and see how much the graphs go down and over to see which has a slope of -2. The correct answer will be the graph that goes down by 2 units for every 1 unit it goes over.

Choice C



Choice D



3. The graph of choice C is the one that has a slope of -2 because the graph goes down by 2 units for every 1 unit it goes over; the graph of choice D goes down by 1 unit for every 1 unit it goes over, so its slope is -1, and that does not match the slope we're looking for.
4. The answer is C.

Section 2.2 Suggested Problems from Real Tests

🏆 Hall of Fame

- Test 6-NC-5
- Apr 2017-NC-8
- Test 9-C-20
- Mar 2018-C-8
- Apr 2018-NC-2
- Mar 2020-C-27 🏆
- Oct 2020-C-12
- Mar 2021-NC-1

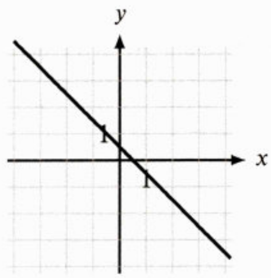
Section 2.2 Practice Problems

1 2.2

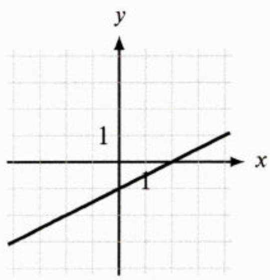
Which of the following is the graph of the equation

$y = \frac{1}{2}x - 1$ in the xy-plane?

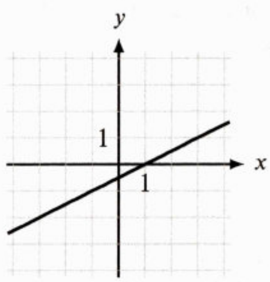
A)



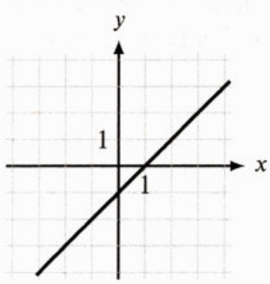
B)



C)



D)



2 2.2

The function f is defined for all real numbers, and the graph of $y = f(x)$ in the xy-plane is a line with a positive slope. Which of the following must be true?

I. If $a < b$, then $f(a) > f(b)$

II. If $a < 0$, then $f(a) < 0$

III. If $a > 0$, then $f(a) > 0$

A) None

B) I only

C) II and III only

D) I, II, and III

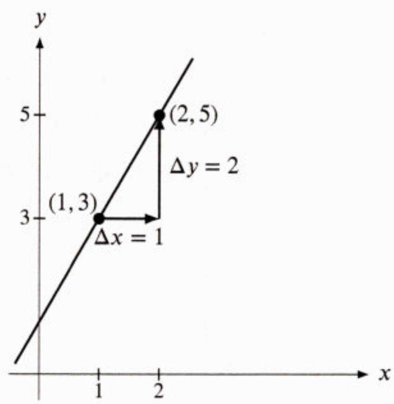
2.3 Using Points and Tables

You don't need an actual graph to find the equation of a line. Sometimes you are just given two or more points and are told to find the equation of the line that goes through those points. **Any two points are sufficient to find the equation of a line.**

For example, let's find the slope of the line that goes through $(1, 3)$ and $(2, 5)$ using the Slope Formula by replacing y_2 with 5 and x_2 with 2 (from the point $(2, 5)$) and replacing y_1 with 3 and x_1 with 1 (from the point $(1, 3)$).

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$m = \frac{5 - 3}{2 - 1}$$
$$m = \frac{2}{1}$$
$$m = 2$$

Alternatively, we can draw a quick picture, plotting the points $(1, 3)$ and $(2, 5)$, and then using the over-and-up arrows to find the slope from Δy and Δx.



The x-values of the points increase by 1 (from 1 to 2) so $\Delta x = 1$. The y-values of the points increase by 2 (from 3 to 5) so $\Delta y = 2$. The slope of the line is $\frac{2}{1} = 2$. If we want to find the equation of this line in Slope-Intercept Form, we can replace m with 2 because the slope is 2.

$$y = mx + b$$
$$y = 2x + b$$

We can plug in the x and y-values of either given point, and once we have those three values (m, x, and y) substituted into the equation, we can solve for b, the y-intercept of the line. Let's use the point $(1, 3)$. **Note that any point on the line can be used**, but you can make smarter, easier choices if you choose points that have 0 or 1 as a value of the x- or y-coordinates because those values result in the simplest arithmetic.

$$y = 2x + b$$
$$3 = 2(1) + b$$
$$3 = 2 + b$$
$$1 = b$$

Since the slope is 2 and the y-intercept is 1, the equation of the line is $y = 2x + 1$.

Plugging In Points to Find Unknown Values

Once you have the slope (or y-intercept) of a line and you know a point on the line, the next logical step is to substitute the coordinates of the point into the Slope-Intercept Form. Replace the x variable with the x-value of the point you know, and replace the y variable with the y-value of the point you know.

You will have three of the four unknown values filled in and you can then solve for the remaining unknown value. In the previous example, we replaced m with the slope that we calculated, leaving us with the equation $y = 2x + b$.

Next comes the step where many students freeze up because they do not realize that they should substitute the coordinates of one of the points they know in order to figure out the y-intercept and complete the equation.

This tip does not just apply to lines but for all equations that will come up on the test: **if you are trying to fill in unknown values in an equation, you should substitute the coordinates of any points that are given to you by the text, graph, or table** (tables of points are discussed below).

SkillDrill 2.3-1

Directions: Based on the points given, find the equation of the line in Slope-Intercept Form.

1. $(-2, 3)$ and $(4, 1)$

2. $(0, -5)$ and $(2, 3)$

3. $(5, 14)$ and $(1, 11)$

4. $(-8, 0)$ and $(2, 0)$

5. $(-3, 4)$ and $(3, -4)$

6. $(5, -3)$ and $(3, 1)$

7. $(10, 2)$ and $(20, 3)$

8. $(8, 4)$ and $(-1, 5)$

Example 2.3-1

1 2.3

The graph of a line in the xy-plane passes through the point $(5, 2)$ and crosses the x-axis at the point $(7, 0)$. The line crosses the y-axis at the point $(0, b)$. What is the value of b?

Solution 1

1. Recognize that this problem asks us to find the y-value of the point $(0, b)$, which is when the x-value is 0. **This is just another way of asking for the y-intercept of the line**. This means that the symbol b in the problem represents the y-value of the y-intercept, just as it does in the template $y = mx + b$ in the Slope-Intercept Form of a linear equation.

2. First we will find the slope of the line using the Slope Formula. The point (x_1, y_1) is $(5, 2)$, and the point (x_2, y_2) is $(7, 0)$.

$$m = \frac{0-2}{7-5}$$

$$m = \frac{-2}{2}$$

$$m = -1$$

3. Since we know the slope and a point on the line, we can substitute these values into the general Slope-Intercept Formula to find the y-intercept of the line. Replace m with -1, then, based on the point $(7, 0)$, substitute 7 for x and 0 for y.

$$\begin{aligned} y &= mx + b \\ y &= (-1)x + b \\ y &= -x + b \\ 0 &= -(7) + b \\ 0 &= -7 + b \\ 7 &= b \end{aligned}$$

4. Since the slope is -1 and the y-intercept is 7, the equation of the line is $y = -x + 7$. Since the y-intercept is equal to 7, you can stop here and finalize 7 as your answer.

5. If you did not notice at the beginning of the problem that you were being asked for the y-intercept, which is the y-value of the point where the x-value is 0, you can find the y-value of the point $(0, b)$ by substituting 0 for x and b for y in the equation of this line (which is $y = -x + 7$).

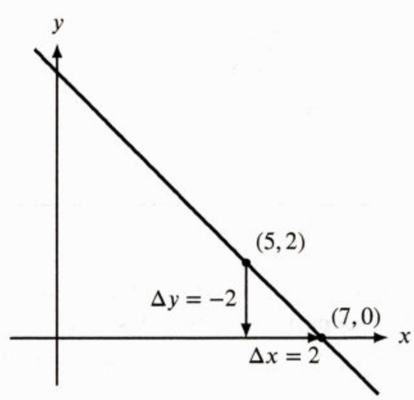
$$\begin{aligned} y &= -x + 7 \\ (b) &= -(0) + 7 \\ b &= 7 \end{aligned}$$

6. The answer is 7.

Solution 2

1. Recognize that this problem asks us to find the y-value of the point $(0, b)$, which is when the x-value is 0. This is just another way of asking for the y-intercept of the line.

2. First we will graph the points $(5, 2)$ and $(7, 0)$, and then use the picture to find the slope by drawing in the over-and-down arrows showing the change in y and the change in x.



3. The y-value decreases by 2, so $\Delta y = -2$, and the x-value increases by 2, so $\Delta x = 2$. Therefore, the slope, which is Δy divided by Δx, is $\frac{-2}{2}$ or -1.

4. Since we know the slope and a point on the line, we can substitute these values into the general Slope-Intercept Formula to find the y-intercept of the line. Replace m with -1, then, based on the point $(7, 0)$, substitute 0 for y and 7 for x.

$$y = mx + b$$
$$y = (-1)x + b$$
$$y = -x + b$$
$$0 = -(7) + b$$
$$0 = -7 + b$$
$$7 = b$$

5. Since the slope is -1 and the y-intercept is 7, the equation of the line is $y = -x + 7$. Since the y-intercept is equal to 7, you can stop here and finalize 7 as your answer.

6. If you did not notice at the beginning of the problem that you were being asked for the y-intercept, you can find the y-value of the point $(0, b)$ by substituting 0 for x and b for y in the equation of this line (which is $y = -x + 7$).

$$y = -x + 7$$
$$(b) = -(0) + 7$$
$$b = 7$$

7. The answer is 7.

Since we know how to find the equation of a line when we are given two points, we can also find the equation of a line when we are given a table of values that represent the x- and y- coordinates of a set of points on a line. The points will be represented as a column (or row) of x-values and a column (or row) of y-values that correspond to those x-values. Let's look at the following table of values.

x	y
-1	-2
0	2
1	6
2	10

Each row is a point, with the x-value in the left column and the y-value in the right column. For example, the first row shows us that $(-1, -2)$ is one point on the line. Similarly, the second row shows us that $(0, 2)$ is another point on the line. Also notice that $(0, 2)$ is the y-intercept of the line because the x-value of the point is 0. To find the equation of the line, we would start by using any two points (rows) from the table to find the slope of the line.

Remember that we found the slope from a graph by drawing a right triangle and finding Δy and Δx directly from the picture. We can similarly avoid formally plugging points into the Slope Formula on these problems. First, choose any two points from the table. By subtraction or counting, see how much the x- and y-values increase or decrease between those two points (finding the value of Δx and Δy, respectively). You can easily notate this by marking the change in x and y-values next to the table with brackets of some kind. Divide Δy by Δx to find the slope.

Let's use the first two points (rows). The y-value increases by 4 (from -2 to 2), so $\Delta y = 4$. The x-value increases by 1, so $\Delta x = 1$.

$\Delta x = 1$

x	y
-1	-2
0	2
1	6
2	10

$\Delta y = 4$

Since Δy is 4 and Δx is 1, the slope of this line is $\frac{4}{1}$ or 4. As it happens, we also know that the y-intercept is 2 because $(0, 2)$ is a point given to us in the table, so the equation of the line is $y = 4x + 2$.

Note that **you can always find the equation of a line when you are given any two points on the line (or the slope and any single point)**, so we could have used any two of the points in the table above to find the same equation of the line. If the y-intercept is not shown in the table, then you can substitute the slope (for m) and the x- and y coordinates of any point on the line (for x, and y, respectively) into the Slope-Intercept Form, $y = mx + b$, in order to solve for the y-intercept (b).

Using Tables of Points on Lines

Each row (or column) in a table is a point on a line. **Many tables show you the *y*-intercept of the line. Look out for the point where the *x*-value is 0.**

There is no need to use the Slope Formula when you are given a table of points on a line. **Subtract or count to see how much the *x*- and *y*-values increase or decrease between any two points to find Δx and Δy, repectively.**

Example 2.3-2

2   2.3

Shipping Charges

Book Weight (pounds)	Shipping Charge ($)
2	2
4	3
8	5
16	9

The table above shows the shipping charges for an online retailer that sells books. There is a linear relationship between the shipping charge and the weight of the books. Which equation can be used to determine the total shipping charge y, in dollars, for an order with a book weight of x pounds?

A) $y = \frac{1}{2}x$

B) $y = \frac{1}{2}x + 1$

C) $y = x$

D) $y = x + 2$

Solution

1. The left column contains x-values and the right column contains y-values. Each row in the table is a point on the line. The first row tells us that $(2, 2)$ is a point, and the second row tells us that $(4, 3)$ is a point. We are told that the relationship between the cost and weight is linear, so we can use these points to find the slope of the line that describes the relationship.

2. Between the first and second row, the x-value increases by 2 (from 2 to 4), so $\Delta x = 2$, and the y-value increases by 1 (from 2 to 3), so $\Delta y = 1$.

$\Delta x = 2$ $\Delta y = 1$

Book Weight (pounds)	Shipping Charge ($)
2	2
4	3
8	5
16	9

Since $\Delta y = 1$ and $\Delta x = 2$, the slope, m, which is defined as $\frac{\Delta y}{\Delta x}$, is $\frac{1}{2}$.
Alternatively, substitute the coordinates of the two points into Slope Formula to find the slope of the line.

$$m = \frac{y_2 - y_1}{y_2 - y_1}$$

$$m = \frac{3-2}{4-2}$$

$$m = \frac{1}{2}$$

3. Since we know that the slope is $\frac{1}{2}$, we can replace m with $\frac{1}{2}$ in the general Slope-Intercept equation.

$$y = \frac{1}{2}x + b$$

4. Choose any point from the table to substitute into the equation so we can solve for b. Let's use $(2, 2)$. Replace the x-value with 2, and replace the y-value with 2.

$$y = \frac{1}{2}x + b$$

$$2 = \frac{1}{\cancel{2}}(\cancel{2}) + b$$

$$2 = 1 + b$$

5. Subtract 1 from both sides of the equation to solve for b, the y-intercept.

$$1 = b$$

6. Replace b with 1 in the general Slope-Intercept equation to find the equation of the line.

$$y = \frac{1}{2}x + 1$$

7. The answer is B.

Notes

If you are really comfortable working directly on the table, you can add an extra row to the table to work out what the y-value would be when the x-value is 0 in order to find the y-intercept more quickly.

	Book Weight (pounds)	Shipping Charge ($)	
	0	b	
$\Delta x = 2$	2	2	$\Delta y = 1$
$\Delta x = 2$	4	3	$\Delta y = 1$
$\Delta x = 4$	8	5	$\Delta y = 2$
$\Delta x = 8$	16	9	$\Delta y = 4$

The value of b has to be 1 because between the first row and the second row, the y-value has to increase by 1 because the x-value increases by 2 (the slope still has to be $\frac{1}{2}$).

Note also that the slope can be calculated using any two rows in the table. Even when Δx is different, the corresponding Δy will also change in such a way that the ratio of Δy to Δx (the slope) will always be constant because the slope of a line is the same everywhere by definition.

SkillDrill 2.3-2

Directions: Based on the points given in the tables below, find the equation of each line in Slope-Intercept Form and find the value of k either by counting the change in values between rows or by substituting into your equation the value of the known coordinate for the row (point) in which k appears.

1.

x	y
0	−5
1	−3
2	−1
3	k

2.

x	y
−2	10
−1	6
0	k
1	−2

3.

x	y
2	3
4	6
k	15
14	21

4.

x	y
−5	2
−1	1
3	0
k	−3

Section 2.3 Suggested Problems from Real Tests 🏆 Hall of Fame

- Test 1-C-22
- Test 3-C-4
- Test 3-C-10
- Test 3-C-11
- Test 4-NC-20 🏆
- Test 5-C-2
- Test 5-C-23
- Test 5-C-28
- Test 6-C-15
- Test 6-C-25
- Test 7-NC-19
- Test 8-C-25
- Test 9-NC-5
- Test 9-C-14
- Test 10-NC-3
- Mar 2018-NC-18
- Apr 2018-NC-19
- Apr 2018-C-18
- Apr 2018-C-27
- Mar 2019-C-12
- Mar 2019-C-36
- May 2019 (US)-NC-3
- May 2019 (US)-NC-10
- May 2019 (US)-C-19 🏆
- May 2019 (Int)-NC-6
- May 2019 (Int)-C-24
- May 2019 (Int)-C-35 🏆
- Oct 2019-NC-9
- Oct 2019-C-26
- Mar 2020-NC-1
- Mar 2020-NC-9
- Mar 2020-C-8
- Oct 2020-NC-16
- Mar 2021-C-13
- Mar 2021-C-29 🏆
- Mar 2021-C-38

Section 2.3 Practice Problems

1 2.3

For a function f, $f(3) = 18$ and $f(5) = 26$. If the graph of $y = f(x)$ is a line in the xy-plane, what is the slope of the line?

2 2.3

x	$h(x)$
1	1
3	5
5	9
7	13

For the linear function h, the table above shows several values of x and their corresponding values of $h(x)$. Which of the following defines h?

A) $h(x) = 2x - 3$

B) $h(x) = 2x - 1$

C) $h(x) = 4x - 3$

D) $h(x) = 4x - 7$

3 2.3

In the xy-plane, line k intersects the y-axis at the point $(0, -4)$ and passes through the point $(3, 2)$. If the point $(15, w)$ lies on line k, what is the value of w?

4 2.3

x	$f(x)$
1	3
3	17
4	24

Some values of the linear function f are shown in the table above. Which of the following defines f?

A) $f(x) = x + 3$

B) $f(x) = 3x + 7$

C) $f(x) = 7x + 3$

D) $f(x) = 7x - 4$

2.4 Interpreting Linear Equations

Now that we know about slope and y-intercept and we can read and create graphs of linear equations, we can come back to real world problems to round out the topic of lines in Slope-Intercept Form. Some of the questions people struggle with the most are based on interpreting the *meaning* of the slope and y-intercept of linear equations that represent real world situations.

Remember the four parts of the equation that we talked about in the last chapter, and relate each part to a real world situation. The y-intercept (the value of b in the Slope-Intercept Form template) is the starting amount. The slope (m in the template) is the rate of change (how the y-value changes when the x-value changes) and is usually indicate by words like "each" or "per." Recall that slopes can be positive (to show an increase in a real world quantity) or negative (to show a decrease).

Example 2.4-1

1 1600.io 2.4

$$1500 - 32t = 220$$

In 1954, the population of a frog species was 1500. The population t years after 1954 was 220, and t satisfies the equation above. Which of the following is the best interpretation of the number 32 in this context?

A) The population t years after 1954

B) The value of t when the population was 220

C) The difference between the population in 1954 and the population t years after 1954

D) The average decrease in the population per year from 1954 to t years after 1954

Solution 1

1. The equation given to us is a linear equation. The terms are on the opposite side of the equals sign from what we are used to, but the pattern still fits: instead of being in the form $y = mx + b$, this line is in the form $b + mx = y$. The letter t is being used to represent the time, in years, instead of the generic x, in order to add meaning to the variable.
2. The coefficient of the x-, or in this case t-, variable is the slope, so, for this problem, we know that the slope is -32.
3. The slope is the change in y divided by the change in x. Since the y-value represents the population and the x-value (or t-value, in this case) is the number of years, **the slope of this line is the change in the population each year**. We choose to represent -32 as $\frac{-32}{1}$ in order to show the slope as a fraction that represents the change in y divided by the change in x.

   $$\frac{\Delta y}{\Delta t} = \frac{\text{change in population}}{\text{change in years}} = \frac{-32}{1}$$

   As you can see the population decreases by 32 every time the year increases by 1.
4. The answer is D.

Notes

For this example, let's identify each of the four parts of the equation and what those mean in the real world context.

The value 1500 is the y-intercept, or the starting population when the variable t is 0, which corresponds to the year 1954. The population in 1954 was 1500 and has been decreasing every year since then, until it finally reached a value of 220 after t years. The value 220 is like a particular y-value: it is the population that still exists after t years, so there exists a point $(t, 220)$, though note that we are never asked to find the value of t in this problem (spoiler alert: it's 40, which you should be able to figure out by solving the original equation for t). The variable t is acting as the x-variable: t represents the number of years since 1954, and there is a change in the population every year. If we want to know the remaining population after t years, we would have to subtract the population loss from the starting amount. The total loss would be found by multiplying the yearly decrease in population by the number of years since 1954. Since t is the number of years since 1954, then 32 must be the amount that the population decreases each year. Also recall that based on the general Slope-Intercept Form, the slope of the line is -32, which represents how much the population changes each year.

Example 2.4-2

2 2.4

Corinne is a computer technician. Each week, she receives a batch of computers that need repairs. The number of computers that she has left to fix at the end of each day can be estimated with the equation $C = 30 - 6d$, where C is the number of computers left and d is the number of days she has worked that week. What is the meaning of the value 30 in this equation?

A) Corinne will complete the repairs within 30 days.

B) Corinne starts each week with 30 computers to fix.

C) Corinne repairs computers at a rate of 30 per hour.

D) Corinne repairs computers at a rate of 30 per day.

Solution

1. We can see that this is a linear equation. We know that the y-variable is C, the number of computers left to fix after d days of work. We know that the x-variable is d, the number of days Corinne has worked.

2. The number of computers she has left to fix at the beginning of the week would be how many computers she has left to fix at the start of the week, or when $d = 0$. One way to contextualize the number 30 is to substitute 0 for d and see that when she has not worked any days yet, she still has 30 computers left to fix.

$$C = 30 - 6d$$
$$C = 30 - 6(0)$$
$$C = 30$$

3. Alternatively, you can just realize that the y-intercept (which is 30 based on the given linear equation) can be interpreted as the starting amount and immediately realize that she starts the week with 30 computers to fix.

4. The answer is B.

Some of the most difficult real-world linear equation interpretation questions involve complicated-looking fractional or decimal slopes. Remember that the slope is the change in y-values (Δy) divided by the change in x-values (Δx), so the change in y-values is measured in the same units as the y-value and the change in the x-values is measured in the same units as the x-value. Therefore, the slope will tell you how many units the y-value increases (or decreases) as the x-value increases by a certain number of units.

For example, if the weight, w, in pounds (or lbs), of a certain cactus that is currently in a pot of soil weighing 15 lbs increases by 13 lbs as the height, h, in inches, of the cactus increases by 2 in, then the linear equation representing the weight of the cactus and pot of soil (acting as the y-value) as a function of the height of the cactus (acting as the x-value) is $w = \frac{13}{2}h + 15$, where the slope is measured in units of lbs/in.

Because we know that the slope must be constant, we can use a simple proportion to see what change in the x-values (the change in height in this case) would produce a certain change in the y-values (the weight in this case). See the chapter on ratios and proportions if you're fuzzy on this type of setup.

$$\frac{\text{Change in Weight}}{\text{Change in Height}} = \frac{\Delta y}{\Delta x}$$

$$\frac{13}{2} = \frac{\Delta y}{\Delta x}$$

So, for example, if we wanted to know how much taller the cactus would have to grow in order to increase the weight by 1 lb, we would just substitute 1 for the change in weight (Δy), and solve for the corresponding change in height (Δx) by cross multiplying and then solving for Δx.

$$\frac{13}{2} = \frac{\Delta y}{\Delta x}$$

$$\frac{13}{2} = \frac{1}{\Delta x}$$

$$13(\Delta x) = 2(1)$$

$$13(\Delta x) = 2$$

$$\Delta x = \frac{2}{13}$$

In this example, if the height of the cactus increases by $\frac{2}{13}$ in, the weight of the cactus will increase by 1 lb.

We can verify this if we choose to input any two weights that are 1 lb different from each other into the equation $w = \frac{13}{2}h + 15$, and see that the heights corresponding to those weights are $\frac{2}{13}$ inches different from each other. Let's use the weights 15 lbs (the y-intercept, which makes our job easier because it corresponds to a cactus height of 0 in) and 16 lbs (which is 1 lb heavier than the other weight).

For the first plant and pot weighing 15 lbs, substitute 15 for w and solve for h (which will end up being 0).

$$w = \frac{13}{2}h + 15$$

$$15 = \frac{13}{2}h + 15$$

$$0 = \frac{13}{2}h$$

It should be obvious what h will be, but for completeness, let's multiply both sides of the equation by $\frac{2}{13}$ in order to solve for h.

$$\frac{2}{13}(0) = \frac{\cancel{2}}{\cancel{13}}\left(\frac{\cancel{13}}{\cancel{2}}h\right)$$
$$0 = h$$

The first plant and pot weighs 15 lbs when the height of the cactus is 0 in. For the second plant and pot, which weighs 16 lbs (1 lb more than the other plant and pot), substitute 16 for w and solve for h (which we predict should be $\frac{2}{13}$ in, which is $\frac{2}{13}$ in more than 0 in, the height of the other potted cactus).

$$w = \frac{13}{2}h + 15$$
$$16 = \frac{13}{2}h + 15$$
$$1 = \frac{13}{2}h$$

Multiply both sides of the equation by $\frac{2}{13}$ in order to solve for h.

$$\frac{2}{13}(1) = \frac{\cancel{2}}{\cancel{13}}\left(\frac{\cancel{13}}{\cancel{2}}h\right)$$
$$\frac{2}{13} = h$$

As we can see, in comparison to substituting x- and y-values into the linear equation, using the proportion based on the slope (which is constant for lines) was a much quicker and easier way to find corresponding changes in the x- and y-values. Problems that ask you to find how much one quantity changes in response to a specific change in the other are routine on the SAT, so it's important to become comfortable with using the proportion technique to rapidly solve such problems.

You don't always have to use the proportion in interpretation questions, but **you should be aware of the units of measurement for the x-and y-values so that you can easily interpret what the slope means in a real-world linear equation problem**. If we were just told that the relationship between the weight, w, of a potted cactus (measured in lbs) and the height, h, of the potted cactus (measured in in) is represented by the equation $w = \frac{13}{2}h + 15$, then we know that the meaning of the fraction $\frac{13}{2}$ is that the weight increases by 13 lbs for every 2 in increase in height.

Be aware that this slope could also be interpreted as a $\frac{13}{2}$-pound increase in weight for every 1-in increase in height, a 130-pound increase in weight for every 20-in increase in height, or even a 1-lb increase in weight for every $\frac{2}{13}$-in increase in height because all of the changes would still produce the same slope.

$$\text{Slope} = \frac{\Delta y}{\Delta x} = \frac{\text{Change in Weight}}{\text{Change in Height}} = \frac{13}{2} = \frac{\frac{13}{2}}{1} = \frac{130}{20} = \frac{1}{\frac{2}{13}}$$

Example 2.4-3

3 1600.io 2.4

$$d(t) = \frac{1}{248}t + 1{,}857$$

A geologist estimates that, as a result of erosion, the depth of the Grand Canyon has been increasing at a constant rate since it was first formed. The function above is used by the geologist to model the depth $d(t)$, in meters, of the canyon t years after the year 2000. According the the function, which of the following statements is true?

A) Every 248 years the depth of the canyon increases by 0.1 meters.

B) Every 2,480 years the depth of the canyon increases by 10 meters.

C) Every 100 years the depth of the canyon increases by 2.48 meters.

D) Every year the depth of the canyon increases by 248 meters.

Solution

1. In this equation, the d-value (acting as the y-value) is the depth of the canyon, measured in meters, and the t-value (acting as the x-value) is the number of years since the year 2000. Therefore, the slope, which is $\frac{\Delta d}{\Delta t}$, is measured in units of m/yr. As such, the fraction $\frac{1}{248}$ tells us that the depth of the canyon increases by 1 m every 248 yr.
2. Choice A is clearly wrong because we know that the depth increase by 1 full meter and not by 0.1 m every 248 yr. Eliminate choice A.
3. Choice D is frankly outlandish if you think about it, but more directly to the point, it inverts the units of the numerator and denominator of the slope, so eliminate choice D.
4. Choice C is not nearly as freakish as choice D; however, if we know that it takes 248 years for the depth to increase by 1 m, then the depth could not increase by more than 1 m in a smaller amount of time (only 100 years versus 248 years). Eliminate choice C.
5. The only remaining choice is B, which isn't immediately recognizable with our correct interpretation of the slope; however, if we set up the fraction representing the slope, $\frac{\Delta d}{\Delta t}$, we will see that a change in depth of 10 m (Δd) for a change in years of 2,480 yr still reduces to the correct slope.

$$\frac{\Delta d}{\Delta t} = \frac{10}{2{,}480}$$

$$\frac{\Delta d}{\Delta t} = \frac{1}{248}$$

6. The answer is B.

Notes

You could do this problem slowly by substituting values into the equation to check each answer choice. For example, to check choice A, you could input two values of t that are 248 years different from one another and see if the corresponding depths are 0.1 m different from one another (they will not be because this answer is not correct). To check choice B, you could input two values of t that are 2,480 yr different from one another and see if the corresponding depths are 10 m different from one another (they will be because this answer is correct). And so on...

Section 2.4 Suggested Problems from Real Tests

🏆 Hall of Fame

- Test 1-NC-4
- Test 1-NC-6
- Test 1-C-15
- Test 2-C-35
- Test 3-NC-15 🏆
- Test 3-C-8
- Test 4-C-17 🏆
- Test 4-C-32
- Test 5-NC-8
- Test 5-NC-13 🏆
- Test 6-NC-1
- Test 6-C-2
- Test 6-C-13
- Test 7-C-17
- Test 8-NC-13
- Apr 2017-NC-14
- May 2017-NC-11
- May 2017-C-19
- May 2017-C-29
- Mar 2018-C-24
- Apr 2018-NC-11
- Apr 2018-C-23 🏆
- May 2018-NC-13
- May 2018-C-15
- Test 10-C-29
- Apr 2019-NC-13
- Apr 2019-C-7
- May 2019 (US)-C-10
- Oct 2019-NC-10
- Oct 2019-C-9
- Oct 2020-C-20
- Mar 2021-NC-9
- Mar 2021-C-21

Section 2.4 Practice Problems

1 1600.io 2.4

$$a = 20t + 130$$

Gregory made an initial deposit into his vacation fund jar. Each week thereafter he deposited a fixed amount to the jar. The equation above models the amount a, in dollars, that Gregory has deposited after t weekly deposits. According to the model, how many dollars was Gregory's initial deposit? (Disregard the \$ sign when writing your answer.)

2 1600.io 2.4

$$4{,}200 - 35k = 175$$

In 1806, the population of a fish species was 4,200. The population k years after 1806 was 175, and k satisfies the equation above. Which of the following is the best interpretation of the number 35 in this context?

A) The difference between the population in 1806 and the population k years after 1806

B) The average decrease in the population per year from 1806 to k years after 1806

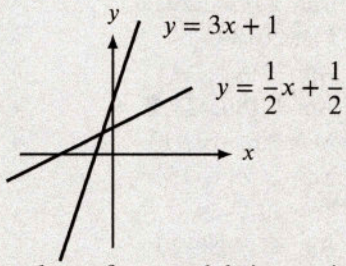
C) The population k years after 1806

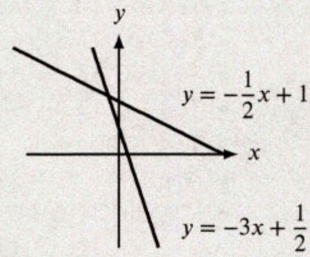
D) The value of k when the population was 175

CHAPTER 2 RECAP

- Slope-Intercept Form: $y = mx + b$
- Slope Formula: $m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{\Delta y}{\Delta x}$
- A positive slope means that the y-value increases as x-value increases. Steeper lines have higher slopes because the y-value goes up more as the x-value increases. A line with a slope of 3 is steeper than a line with a slope of $\frac{1}{2}$.

  A negative slope means that the y-value decreases as the x-value increases. Steeper lines have lower (more negative) slopes because the y-value decreases more as the x-value increases. A line with a slope of -3 is steeper than a line with a slope of $-\frac{1}{2}$.





- Horizontal lines have a slope of zero and their equations are of the form $y = c$, where c is a constant (the y-value is the same for all points on the line).

  Vertical lines have an undefined slope and their equations are of the form $x = c$, where c is constant (the x-value is the same for all points on the line). Note that this is **not** a Slope-Intercept equation, because a vertical line cannot be represented in that form because the slope is undefined.
- The y-intercept is where a line crosses the y-axis. One useful way of thinking about this (especially for real world problems) is that the y-intercept is the starting amount or the value of y when the x-value is 0. To find the y-intercept when it is not clearly marked on a graph, substitute the slope and x- and y-coordinates of a point into the Slope-Intercept Form and solve for b.

  The x-intercept is where a line crosses the x-axis. Another useful way of thinking about this (especially for real world problems) is that this is the value of x when the y-value is equal to 0. Unlike the y-intercept, this value is not visible in any common representation of linear equations, but to find it when it is not clearly marked on a graph, substitute the slope and y-intercept into the Slope-Intercept Form, then set the y-value equal to 0 and solve for x.
- You can find the equation of any line as long as you know any two points on the line or know the slope and any one point. If you are given the slope and y-intercept, then you already have everything you need to make the equation.
- Using using up-and-over arrows with graphical representations to find Δy and Δx, or counting differences between coordinate values between rows of tables, is often less error prone than substituting the coordinates of points into the long from of the Slope Formula. There is no mathematical difference between methods, but graphical methods are harder to mess up.
- When asked to interpret the meaning of a coefficient or constant in a Slope-Intercept Form linear equation, first determine which one of the four components of a linear equation it is (the y-value, the slope, the x-value, or the y-intercept).

  The y-intercept is the "starting amount" and is measured in the same units as the y-value.

  The slope is the rate of change of the y-value as the x-value changes and is measured in the units of the y-value divided by the units of the x-value (look for keywords like "per" which indicate division).

Additional Problems

1 1600.io 2.1

A line in the xy-plane has a slope of 0. Which of the following could be the equation of the line?

A) $x = 0$

B) $y = x$

C) $y = 0$

D) $y = -x$

2 1600.io 2.1

The line $y = kx + 5$, where k is a constant, is graphed in the xy-plane. If the line contains the point (c, d), where $c \neq 0$ and $d \neq 0$, what is the slope of the line in terms of c and d?

A) $\dfrac{d-5}{c}$

B) $\dfrac{c-5}{d}$

C) $\dfrac{4-d}{c}$

D) $\dfrac{4-c}{d}$

3 1600.io 2.1

Which of the following is an equation of the line in the xy-plane that has slope -5 and passes through the point $(0, 2)$?

A) $y = -5x - 2$

B) $y = -5x + 2$

C) $y = -5(x + 2)$

D) $y = -5(x - 3)$

4 1600.io 2.3

$$C = 85h + 100$$

The equation above gives the amount C, in dollars, an HVAC technician charges for a job that takes h hours. Ms. Porter and Mr. Green each hired this technician. The technician worked 3 hours longer on Ms. Porter's job than on Mr. Green's job. How much more did the technician charge Ms. Porter than Mr. Green?

A) \$85

B) \$100

C) \$170

D) \$255

5 2.3

The graph in the xy-plane of the linear function f contains the point $(1, 6)$. For every increase of 3 units in x, $f(x)$ increases by 4 units. Which of the following equations defines the function?

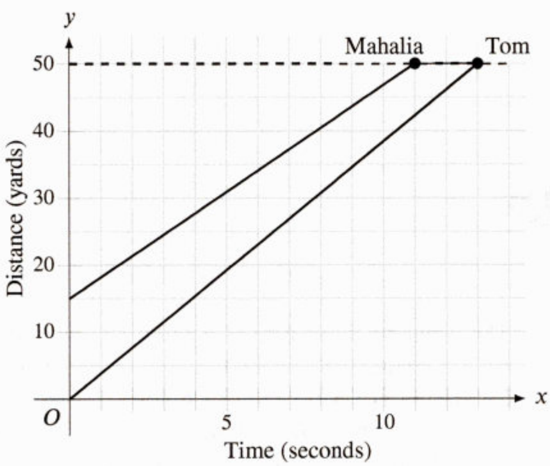
A) $f(x) = -\frac{4}{3}x + 10$

B) $f(x) = -\frac{3}{4}x + \frac{21}{4}$

C) $f(x) = \frac{3}{4}x + 2$

D) $f(x) = \frac{4}{3}x + \frac{14}{3}$

6 2.4



The graph above shows the positions of Tom and Mahalia during a race. Tom and Mahalia each rollerbladed at a constant rate, and Mahalia was given a head start to shorten the distance she needed to skate. Tom finished the race in 13 seconds, and Mahalia finished the race in 11 seconds. According to the graph, Mahalia was given a head start of how many yards?

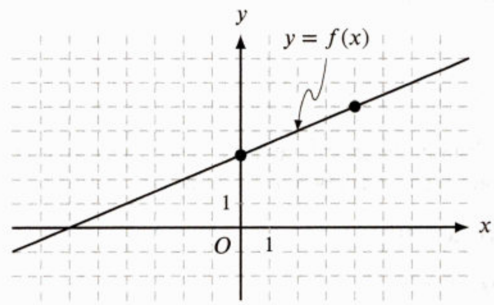
A) 3

B) 10

C) 15

D) 20

7 2.1



The graph of the function f is shown in the xy-plane above. The function f is defined by the equation $f(x) = \frac{a}{b}x + c$ for positive constants a, b, and c, where $\frac{a}{b}$ is a fraction in lowest terms. Which of the following orders a, b, and c from least to greatest?

A) $a < b < c$

B) $a < c < b$

C) $b < c < a$

D) $c < a < b$

8 2.3

In the xy-plane, the points $(-1, 4)$ and $(2, -1)$ lie on the graph of which of the following linear functions?

A) $f(x) = -\frac{5}{3}x + \frac{7}{3}$

B) $f(x) = -\frac{3}{5}x + 1$

C) $f(x) = \frac{1}{2}x - 2$

D) $f(x) = 5x + 9$

9 1600.io 2.4

The equation $p = 14.7 + 15.4d$ approximates the pressure p, in pounds per square inch, exerted on a peanut at a depth of d inches (in) below the surface of a chocolate syrup vat. What is the increase in depth that is necessary to increase the pressure by 1 pound per square inch?

A) $\frac{1}{14.7}$ in

B) $\frac{1}{15.4}$ in

C) 14.7 in

D) 15.4 in

10 1600.io 2.3

x	$f(x)$
4	15
8	7

The table above shows two pairs of values for the linear function f. The function can be written in the form $f(x) = ax + b$, where a and b are constants. What is the value of $a + b$?

11 1600.io 2.4

$$A = 205 + 24.14m$$

The equation above can be used to estimate the body surface area A, in square inches, of a child with mass m, in pounds mass, where $7 \leq m \leq 67$. Which of the following statements is consistent with the equation?

A) For each increase of 205 pounds in mass, A increases by approximately 1 square inch.

B) For each increase of 1 pound in mass, A increases by approximately 205 square inches.

C) For each increase of 1 pound in mass, A increases by approximately 24.14 square inches.

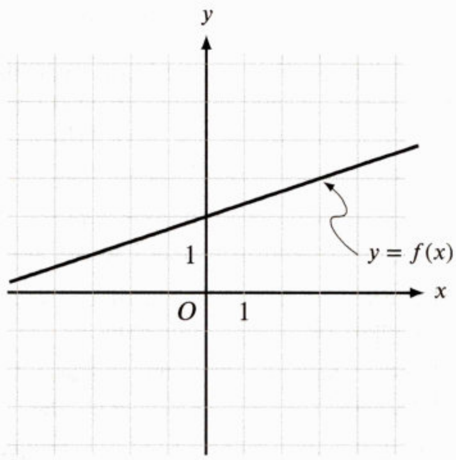
D) For each increase of 24.14 pounds in mass, A increases by approximately 1 square inch.

12 1600.io 2.3

x	y
1	-2
k	10
5	n

The table above shows the coordinates of three points on a line in the xy-plane, where k and n are constants. If the slope of the line is 6, what is the value of $n - k$?

13 2.3



The graph of the linear function f is shown in the xy-plane above. The slope of the graph of the linear function g is 3 times the slope of the graph of f. If the graph of g passes through the point $(0, -8)$, what is the value of $g(6)$?

A) -8

B) -6

C) -2

D) 10

14 2.1

A line in the xy-plane passes through the origin and has a slope $\frac{1}{10}$. Which of the following points lies on the line?

A) $(20, 2)$

B) $(10, 10)$

C) $(1, 10)$

D) $(0, 10)$

15 2.3

The function f is linear, $f(1) = 20$, and $f(4) = 11$. If $f(x) = mx + b$, where m and b are constants, what is the value of b ?

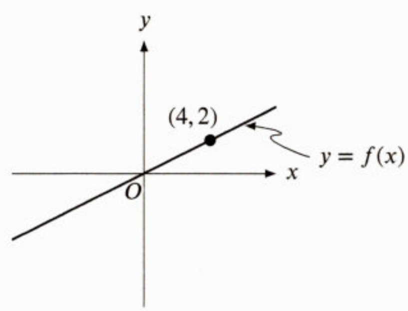
A) $\frac{61}{3}$

B) 23

C) 17

D) $\frac{34}{3}$

16 2.1



In the xy-plane above, a point (not shown) with coordinates (s, t) lies on the graph of the linear function f. If s and t are positive integers, what is the ratio of t to s ?

A) 1 to 2

B) 2 to 3

C) 2 to 1

D) 3 to 2

17 1600.io 2.4

$$s = 32.2t$$

The equation above can be used to approximate the speed s, in feet per second (ft/s), of an object t seconds after being dropped into a free fall. Which of the following is the best interpretation of the number 32.2 in this context?

A) The speed, in ft/s, of the object when it hits the ground

B) The initial speed, in ft/s, of the object when it is dropped

C) The speed, in ft/s, of the object t seconds after it is dropped

D) The increase in speed, in ft/s, of the object for each second after it is dropped

18 1600.io 2.3

x	$f(x)$
0	−2
3	7
7	19

Some values of the linear function f are shown in the table above. What is the value of $f(5)$?

A) 10

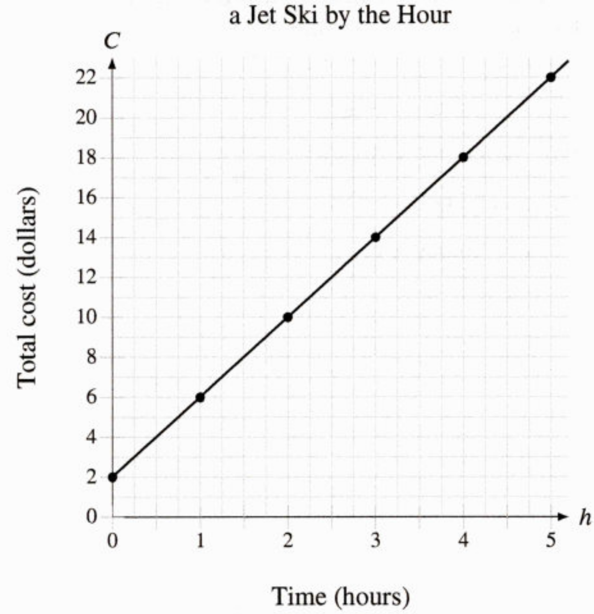
B) 13

C) 14

D) 15

Questions 19 and 20 refer to the following information.

Total Cost of Renting a Jet Ski by the Hour



The graph above displays the total cost C, in dollars, of renting a jet ski for h hours.

19 1600.io 2.4

What does the C-intercept represent in the graph?

A) The total number of jet skis rented

B) The total number of hours the jet ski is rented

C) The increase in cost to rent the jet ski for each additional hour

D) The initial cost of renting the jet ski

20 1600.io 2.1

Which of the following represents the relationship between h and C ?

A) $h = 4C$

B) $C = h + 2$

C) $C = 2h$

D) $C = 4h + 2$

21 2.3

A business owner purchased a vehicle valued at \$90,000. The values of the vehicle depreciates by the same amount each year so that after 5 years the value will be \$50,000. Which of the following equations gives the value, v, of the machine, in dollars, t years after it was purchased, for $0 \leq t \leq 5$?

A) $v = 50{,}000 - 8{,}000t$

B) $v = 90{,}000 - 50{,}000t$

C) $v = 90{,}000 - 8{,}000t$

D) $v = 90{,}000 + 8{,}000t$

22 2.4

$$3n + 10 = 46$$

A snake had a length of 10 inches when it was hatched. The equation above can be used to find how many months n it took the snake to reach a length of 46 inches. Which of the following is the best interpretation of the number 3 in this context?

A) The number of months it took the snake to triple its length

B) The length, in inches, of the snake when the snake was 1 month old

C) The average number of months it takes similar snakes to grow 46 inches

D) The average number of inches that the snake grew per month

23 2.3

x	$h(x)$
0	1
1	3
3	7

The table above shows selected values for the function h. In the xy-plane, the graph of $y = h(x)$ is a line. What is the value of $h(6)$?

A) 9

B) 11

C) 13

D) 14

24 2.3

x	$f(x)$
0	c
1	$-2c$
2	$-5c$

For the linear function f, the table above gives some values of x and their corresponding values $f(x)$, where c is a constant. Which of the following equations defines f ?

A) $f(x) = -5cx - 5c$

B) $f(x) = -3cx + c$

C) $f(x) = -cx + c$

D) $f(x) = cx + c$

25 2.3

Which of the following is an equation of the line in the xy-plane that contains the points $(1, 4)$ and $(4, 16)$?

A) $y = 4x$

B) $y = x + 3$

C) $y = 3x + 4$

D) $y = \frac{1}{4}x$

26 2.3

Population of Hardlocke, Texas

Year	Population
1900	711
1910	687

The table above shows the population of Hardlocke, Texas, for the years 1900 and 1910. If the relationship between population and year is linear, which of the following functions P models the population of Hardlocke t years after 1900?

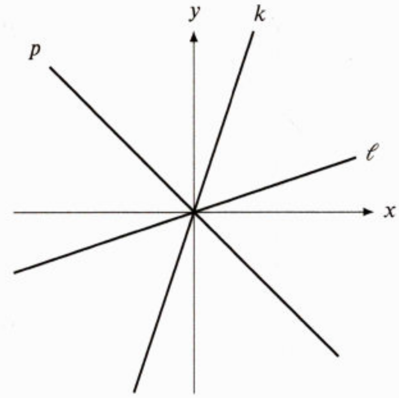
A) $P(t) = 711 - 2.4t$

B) $P(t) = 711 - 24t$

C) $P(t) = 711 - 2.4(t - 1900)$

D) $P(t) = 711 - 24(t - 1900)$

27 2.1



In the xy-plane, lines k, ℓ, and p are shown. Which of the following lists the slopes from least to greatest?

A) The slope of p, the slope of ℓ, the slope of k

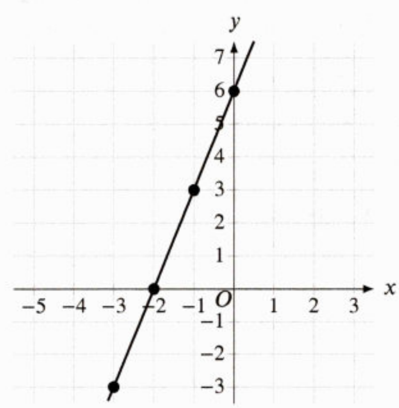
B) The slope of ℓ, the slope of p, the slope of k

C) The slope of k, the slope of ℓ, the slope of p

D) The slope of p, the slope of k, the slope of ℓ

28 2.1

The graph of $y = mx + b$, where m and b are constants, is shown in the xy-plane.

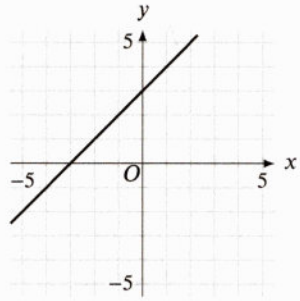


What is the value of m ?

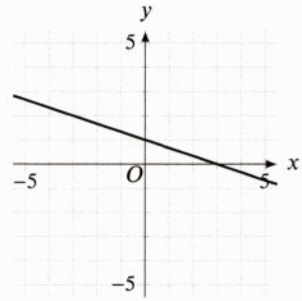
29 2.2

Which of the following is the graph of the equation $y = 3x + 1$ in the xy-plane?

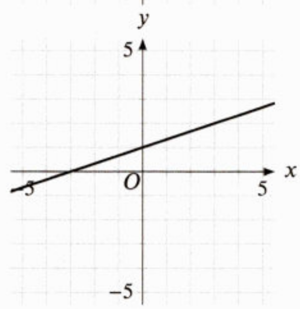
A)



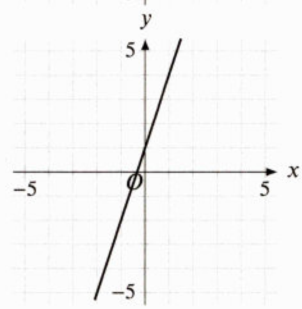
B)



C)



D)



30 2.3

Speed (mph)	Pulse (bpm)
4	65
8	77
12	89

The table lists selected values of Gina's rollerblading speed, in miles per hour (mph), and her corresponding pulse, in beats per minute (bpm). There is a linear relationship between Gina's speed, x, and her pulse, $f(x)$. Which of the following equations describes $f(x)$?

A) $f(x) = 12x + 65$

B) $f(x) = 12x + 53$

C) $f(x) = 3x + 65$

D) $f(x) = 3x + 53$

31 2.4

Milk production in a certain area dropped from 10 million cartons in 2000 to 2.7 millions cartons in 2018. Assuming that the milk production decreased at a constant rate, which of the following linear functions f best models the production, in millions of cartons, t years after the year 2000?

A) $f(t) = \frac{73}{180}t + 10$

B) $f(t) = \frac{27}{180}t + 10$

C) $f(t) = -\frac{73}{180}t + 10$

D) $f(t) = -\frac{27}{180}t + 10$

Answer Key

SkillDrill 2.1-1

1. $(0,-3)$
2. $(-4,-2)$
3. $(3,0)$
4. $(5,3)$
5. $(0,4)$
6. $(-3,1)$
7. $(4,-4)$
8. $(-2,0)$

SkillDrill 2.1-2

1. 1
2. 2
3. 5
4. 4
5. 3
6. 4
7. 1
8. 3

SkillDrill 2.1-3

1. $m=2, b=1$
2. $m=3, b=6$
3. $m=1, b=-7$
4. $m=5, b=0$
5. $y=5x-4$
6. $y=8x+1$
7. $y=x$
8. $y=2x-2$

SkillDrill 2.1-4

1. $y=\frac{-1}{3}x-3, x_{int}=(-9,0)$
2. $y=x+1, x_{int}=(-1,0)$
3. $y=\frac{1}{2}x+3, x_{int}=(-6,0)$
4. $y=\frac{-5}{4}x-\frac{1}{2}, x_{int}=\left(\frac{-2}{5},0\right)$
5. $y=-3$, x_{int} does not exist
6. $x=4, x_{int}=(4,0)$
7. $y=\frac{2}{7}x-\frac{8}{7}, x_{int}=(4,0)$
8. $y=\frac{-1}{3}x+\frac{4}{3}, x_{int}=(4,0)$

Section 2.1 Practice Problems

1. D
2. C
3. $\frac{2}{3}$, .666, or .667
4. C

Section 2.2 Practice Problems

1. B
2. A

SkillDrill 2.3-1

1. $y=\frac{-1}{3}x+\frac{7}{3}$
2. $y=4x-5$
3. $y=\frac{3}{4}x+\frac{41}{4}$
4. $y=0$
5. $y=\frac{-4}{3}x$
6. $y=-2x+7$
7. $y=\frac{1}{10}x+1$
8. $y=\frac{-1}{9}x+\frac{44}{9}$

SkillDrill 2.3-2

1. $y=2x-5, k=1$
2. $y=-4x+2, k=2$
3. $y=\frac{3}{2}x, k=10$
4. $y=\frac{-1}{4}x+\frac{3}{4}, k=15$

Section 2.3 Practice Problems

1. 4
2. B
3. 26
4. D

Section 2.4 Practice Problems

1. 130
2. B

Additional Problems

1. C
2. A
3. B
4. D
5. D
6. C
7. A
8. A
9. B
10. 21
11. C
12. 19
13. C
14. A
15. B
16. A
17. D
18. B
19. D
20. D
21. C
22. D
23. C
24. B
25. A
26. A
27. A
28. 3
29. D
30. D
31. C

Standard Form/Parallel and Perpendicular Lines

3

3.1 Standard Form

Jimmy wants to buy notebooks that cost \$4 each. We've learned already how we could calculate the total amount that Jimmy would spend on notebooks by multiplying the cost of each notebook by the number of notebooks he buys. If Jimmy buys n notebooks, then he will spend $4n$ dollars.

Suppose he also needs to buy pens that cost \$2 each. The amount Jimmy would spend on p pens would be $2p$ dollars. Even though he is buying two different items, we can make one equation that represents the total amount he spends on both items by adding the total cost of the notebooks and pens together. If t is the total cost of the supplies, then the equation representing Jimmy's total cost is $2p + 4n = t$.

Let's say that Jimmy can only spend a total of \$40 on supplies. In the equation, the value of t would be 40:

$$2p + 4n = 40$$

Jimmy wants to use this equation to figure out how many pens and notebooks he can buy.

The first thing to note is that the more he buys of one thing, the fewer he could buy of the other. If he buys 0 notebooks, he can buy 20 pens because $2(\mathbf{20}) + 4(\mathbf{0}) = 40$. If he buys 1 notebook, he still has enough money for 18 pens ($2(\mathbf{18}) + 4(\mathbf{1}) = 40$), but if he buys 2 notebooks, he can only buy 16 pens ($2(\mathbf{16}) + 4(\mathbf{2}) = 40$).

As soon as we limit the total amount of money he is going to spend, the number of pens and notebooks has to strike a balance: the two values are related, much in the same way x and y-values are related to each other for the points on a line. That is because the equation we just wrote is a linear equation, but it's not in Slope-Intercept Form; it's in what we call **Standard Form**.

Standard Form Linear Equation

The following is the equation of a line in Standard Form, where A, B, and C are constants:

$$Ax + By = C$$

In a lot of word problems, it is easier to write the equation in Standard Form than in Slope-Intercept Form due to how the information is given to you. Once you have the equation, you can use it to solve for any values that are unknown.

SkillDrill 3.1-1

Directions: Write a Standard Form linear equation to represent the given situation. In order to make sure that you use variable letters that make sense (this is an important habit), we will provide them for you.

1. Bella gets paid \$2 for every backflip she does and \$3 for every front flip she does. In one day, she makes \$56 doing backflips and front flips. (Use b for the number of backflips and f for the number of front flips.)
2. In an eating contest, Ferdinand takes 1 minute to eat each hot dog and 3 minutes to eat each cake. It took Ferdinand 60 minutes to eat h hot dogs and c cakes.
3. It takes Paula 8 minutes to drink a small water bottle and 20 minutes to drink a big water bottle. In one week, it took her a total of 112 minutes to drink s small water bottles and b big water bottles.
4. A company loses \$100 for every inappropriate comment they make on social media and loses \$2.50 every time one of their employees exposes the hazardous working conditions in the company's overseas sweatshops. In one year, the company loses \$197.50 from c inappropriate comments and h reports of hazardous working conditions.
5. Jeremetrius gains 5 subscribers for every reaction video he posts and loses 10 subscribers for every prank video he posts. In one week, Jeremetrius loses a total of 42 subscribers. (Use r for the number of reaction videos and p for the number of prank videos.)

Example 3.1-1

1 3.1

Nick spends \$57 per month on chips. A 10-bag pack of chips costs \$8.50, and a single bag of chips costs \$1.20. If p represents the number of 10-bag packs Nick buys in a month, and s represents the number of single bags Nick buys in a month, which of the following equations best represents the relationship between p and s ?

A) $p + s = 8.50 + 1.20$

B) $p + s = 57$

C) $1.20p + 8.50s = 57$

D) $8.50p + 1.20s = 57$

Solution

1. We need to write an equation that represents the total amount of money Nick spends on chips in a month. To do this, we should start by finding the amount he spends on p 10-bag packs and s single bags of chips.
2. The cost of p 10-bag packs is equal to the cost per 10-bag pack (\$8.50) times the number of 10-bag packs (p), which is $8.50p$.
3. The cost of s single bags is equal to the cost per single bag (\$1.20) times the number of single bags (s), which is $1.20s$.
4. We know the total cost was \$57, so we can add the costs of buying p 10-bag packs ($8.50p$) and s single bags ($1.20s$) and set the sum equal to 57.

$$8.50p + 1.20s = 57$$

5. The answer is D.

For many word problems, it is easier to write linear equations in Standard Form; however, when lines are graphed, it is usually easier to write the equations first in Slope-Intercept Form and then convert the equations into Standard Form if necessary to match the format of the answer choices.

Converting from Slope-Intercept Form to Standard Form

We can convert from Slope-Intercept Form to Standard Form through simple algebraic steps by moving the x-variable to the same side of the equation as the y-variable.

We want to convert from a line in Slope-Intercept Form ($y = mx + b$) to a line in Standard Form ($Ax + By = C$), so we should subtract mx from both sides of the equation.

$$y = mx + b$$
$$-mx + y = b$$

The equation above is now in Standard Form ($Ax + By = C$) where $A = -m$, $B = 1$, and $C = b$.

For example, the line $y = -2x + 3$ is in Slope-Intercept Form, but by adding $2x$ to both sides of the equation (subtracting $-2x$ from both sides of the equation) we arrive at the same linear equation in Standard Form: $2x + y = 3$.

When the slope of a line in Slope-Intercept Form is a fraction, you may also wish to (or have to in order to match answer choices) **multiply both sides of the equation by the denominator of the x-coefficient to eliminate the fraction**.

For example, we can rewrite the linear equation $y = \frac{2}{3}x - 4$ in Standard Form in multiple equivalent ways. Start by subtracting $\frac{2}{3}x$ from both sides of the equation: $-\frac{2}{3}x + y = -4$. This is already in Standard Form, but it is not necessarily the stopping point (the answer choices for many questions asking you for the Standard Form equation of a line based on a graph will generally not have any fractional coefficients). We can multiply both sides of the equation by 3 (the denominator of the x-coefficient) in order to eliminate the fractional coefficient.

$$-\frac{2}{3}x + y = -4$$
$$3\left(-\frac{2}{3}x + y\right) = 3(-4)$$
$$\cancel{3}\left(-\frac{2}{\cancel{3}}x\right) + 3y = -12$$
$$-2x + 3y = -12$$

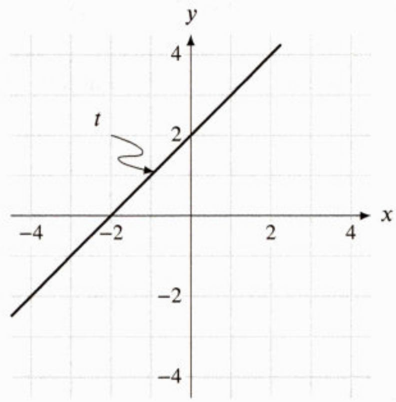
Similarly, we can (if necessary to match answer choices) multiply both sides of the equation by -1 in order to rewrite the equation into yet another equivalent representation.

$$-2x + 3y = -12$$
$$-1(-2x + 3y) = -1(-12)$$
$$-1(-2x) - 1(3y) = 12$$
$$2x - 3y = 12$$

This highlights the fact that while there is only **one** Slope-Intercept Form of a linear equation because one side of the equation must be just y, there are **infinite** equivalent Standard Form versions of the same equation because there are no constraints on the form or value of the variables' coefficients.

Example 3.1-2

2 3.1



Which of the following is an equation of line t in the xy-plane above?

A) $x + y = 2$

B) $x + y = -2$

C) $x - y = 2$

D) $x - y = -2$

Solution 1

1. All of the answer choices are in Standard Form, $Ax + By = C$, where the value of A is 1 (because the coefficient of x is 1 in every choice). Therefore, we can simplify the equation of this line to $x + By = C$.

2. We can plug in points on the line to find the value of B and C. If we start with the point $(-2, 0)$, the y-term will drop out of the equation, helping us find the value of C easily.

$$\begin{aligned} x + By &= C \\ -2 + B(0) &= C \\ -2 &= C \end{aligned}$$

3. Since the value of C is -2, the equation of this line is $x + By = -2$ (limiting us to choices B and D), and we can solve for B by plugging in the point $(0, 2)$.

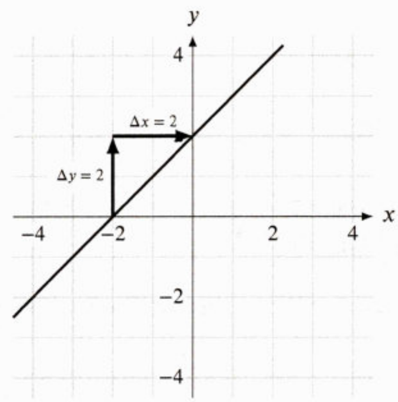
$$\begin{aligned} x + By &= -2 \\ 0 + B(2) &= -2 \\ 2B &= -2 \\ B &= -1 \end{aligned}$$

4. Since the value of B is -1, the equation of this line is $x - y = -2$.

5. The answer is D.

Solution 2

1. Start by finding the equation of the line in Slope-Intercept Form, which can then be converted into Standard Form in order to match the answer choices.
2. Since the graph goes through the point $(0, 2)$, we know that the y-intercept of the line is 2.
3. We can draw in over-and-up arrows that show the changes in the y- and x-values so that we can find the slope of the line. Let's use the points $(0, 2)$ and $(-2, 0)$.



   Since the slope is equal to $\dfrac{\Delta y}{\Delta x}$, the slope of this line is $\dfrac{2}{2}$ or 1.

4. We can write the equation of the line in Slope-Intercept Form ($y = mx + b$). We know that the slope is 1 and the y-intercept is 2, so we can plug in 1 for m (remember when the coefficient of a variable is 1, you do not have to write the coefficient) and 2 for b.

$$y = mx + b$$
$$y = (1)x + (2)$$
$$y = x + 2$$

5. In order to match the form of the answer choices, subtract x from both sides of the equation in order to rewrite the equation in Standard Form.

$$y = x + 2$$
$$-x + y = 2$$

6. None of the answer choices match this form. Since all of the answer choices start with a positive x instead of $-x$, multiply both sides of the equation by -1.

$$-x + y = 2$$
$$-1(-x + y) = -1(2)$$
$$-1(-x) - 1(y) = -2$$
$$x - y = -2$$

7. The answer is D.

Just as lines written in Slope-Intercept Form can be rewritten in Standard Form, lines written in Standard Form can be converted into Slope-Intercept Form.

Converting from Standard Form to Slope-Intercept Form

Sometimes we are given lines in Standard Form, but it is more useful for the problem to have them in Slope-Intercept Form. We can convert from Standard Form to Slope-Intercept Form through simple algebraic steps by isolating the y-variable.

We want to convert from a line in Standard Form ($Ax + By = C$) to a line in Slope-Intercept Form ($y = mx + b$), so we will complete the two following steps:

1. Subtract Ax from both sides of the equation: $By = -Ax + C$
2. Divide both sides of the equation by B: $y = \frac{-A}{B}x + \frac{C}{B}$

Remember, the slope of a line expressed in Slope-Intercept Form is the coefficient of x, so looking at the equation above, you can see that the **slope of a line in Standard Form is $\frac{-A}{B}$**.

Similarly, the y-intercept of a line expressed in Slope-Intercept Form is the constant term, so the **y-intercept of a line in Standard Form is $\frac{C}{B}$**.

You do not need to memorize this conversion formula; what's essential is that you understand the *procedure* involved, which is simply isolating the y term with no coefficient on one side of the equation. As long as you know that principle (and your algebra is solid), you can perform this conversion.

SkillDrill 3.1-2

Directions: Convert each linear equation from Standard Form to Slope-Intercept Form.

1. $4x - 2y = 8$ **2.** $9x - 3y = -21$ **3.** $-3x - 2y = 7$ **4.** $-5x - 6y = -18$

Directions: Convert each linear equation from Slope-Intercept Form to Standard Form, eliminating fractional coefficients if any. There are multiple correct answers; the answer key gives simple versions, with lowest non-fractional coefficients and non-negative x-coefficients.

5. $y = -5x - 10$ **6.** $y = 2x + 4$ **7.** $y = \frac{-1}{3}x - \frac{2}{3}$ **8.** $y = \frac{5}{2}x + 7$

Let's look back at the example of Jimmy buying pens and notebooks. We said if he buys p pens for \$2 each and n notebooks for \$4 each and spends a total of t dollars, then the equation that represents the relationship is $2p + 4n = t$. While this is a linear equation in Standard Form, it doesn't represent a specific line because t, the total amount that can be spent, is a variable, not a constant. As soon as we have a specific total amount that Jimmy can spend, the equation will represent a specific line that reflects the relationship.

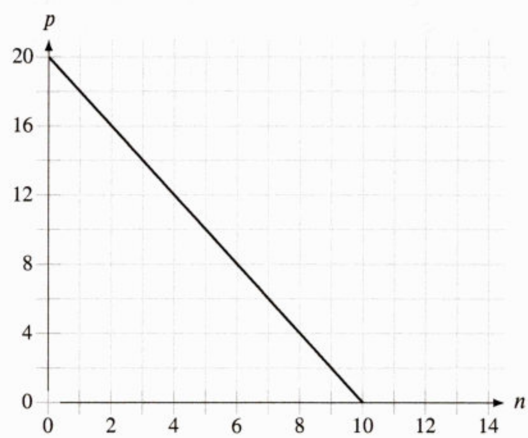
For example, when we said that Jimmy has to spend exactly \$40, we have the Standard Form linear equation $2p + 4n = 40$, and now we have a linear equation for a specific line (which can be graphed) for which the number of pens and notebooks must be in balance so that regardless of how many of each are bought, the total cost will come to \$40.

It might be easier to visualize this if we write the equation in Slope-Intercept Form, which makes it easy to graph the line so we can select some points and plug them into the Standard Form equation; converting the equation to Slope-Intercept Form lets us find the value of p (which will be on the vertical axis) for any value of n (the horizontal axis), which lets us plot points and draw the line through them. Let's find the number of pens he buys as a function of the number of notebooks he buys by solving for p in terms of n, converting the equation into Slope-Intercept Form (where p is like the y-value and n is like the x-value). That is, we'd like an equation into which we can input the number of notebooks, and the equation will then produce the number of pens; to accomplish that, we will isolate the number of pens p on one side so when we evaluate the other side, we'll know the number of pens.

$$\begin{aligned} 2p + 4n &= 40 \\ 2p &= -4n + 40 \\ \frac{1}{\cancel{2}}(\cancel{2}p) &= \frac{1}{2}(-4n + 40) \\ p &= \frac{-4}{2}n + \frac{40}{2} \\ p &= -2n + 20 \end{aligned}$$

Now that the line is in Slope-Intercept Form, we can verify what we said before about how the number of pens is affected by the number of notebooks. The y-intercept is 20, because if Michael bought 0 notebooks, he could buy 20 pens (that is, n would equal zero, so the equation would be $p = 20$). The slope, given as the coefficient of the variable n, is -2, which means that the number of pens he can buy decreases by 2 every time he buys another notebook. This makes perfect sense, because a notebook costs \$2 and a pen costs \$1.

Let's now graph the line and pick a couple of points, so we can show how all of the points on the line also satisfy the Standard Form equation and provide ordered pairs of the number of pens and notebooks Jimmy can buy that will result in a total cost of \$40.



Remember that the cost of the supplies can be represented with the Standard Form equation $2p + 4n = t$. If we plug in any points from the graph into the Standard Form equation, we should see that the cost will add up to \$40. The two obvious points to look at are the p- and n-intercepts, $(0, 20)$ and $(10, 0)$, respectively. If we plug in the point $(0, 20)$, where $n = 0$ and $p = 20$, we will see that the cost of the supplies, t, is \$40.

$$\begin{aligned} 2p + 4n &= t \\ 2(20) + 4(0) &= t \\ 40 &= t \end{aligned}$$

If we plug in the point $(10, 0)$, where $n = 10$ and $p = 0$, we will see agin that the cost of the supplies, t, is \$40.

$$\begin{aligned} 2p + 4n &= t \\ 2(0) + 4(10) &= t \\ 40 &= t \end{aligned}$$

Any point that we pick on the line will show that the total cost of supplies is \$40. Let's try $(5, 10)$, where $n = 5$ and $p = 10$.

$$\begin{aligned} 2p + 4n &= t \\ 2(10) + 4(5) &= t \\ 20 + 20 &= t \\ 40 &= t \end{aligned}$$

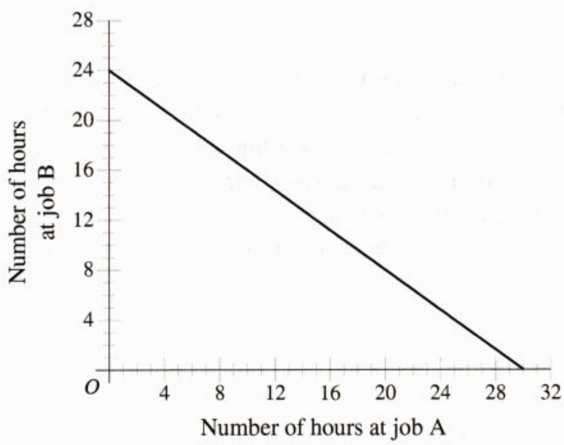
Now let's check the point $(8, 4)$, where $n = 8$ and $p = 4$.

$$\begin{aligned} 2p + 4n &= t \\ 2(4) + 4(8) &= t \\ 8 + 32 &= t \\ 40 &= t \end{aligned}$$

As soon as the total cost of the supplies was set at \$40, a single line was established, which could be graphed with the aid of a conversion to Slope-Intercept Form. Every point on the line could be plugged in to the Standard Form equation to show that any combination of notebooks and pens that appears as a point on the line would result in a total cost of \$40 (note that a point with a non-counting number coordinate value such as $(1.5, 17)$ is on the line and satisfies the equation because $2(17) + 4(1.5) = 34 + 6 = 40$, but Jimmy probably can't buy one and half notebooks).

Example 3.1-3

3 3.1



To earn money for piano lessons, Carlita works two part-time jobs: A and B. She earns \$8 an hour working at job A and \$10 an hour working at job B. In one week, Carlita earned a total of s dollars for working at the two part-time jobs. The graph above represents all possible combinations of numbers of hours Carlita could have worked at the two jobs to earn s dollars. What is the value of s ?

A) 192

B) 240

C) 300

D) 720

Solution

1. Normally, when there is a linear graph presented above a question, it is useful to find the equation of the line in Slope-Intercept Form. However, for this problem, that step would probably not be very useful to understanding the problem (it is not necessarily a waste of time or useless, and it should still generally be considered a good first step for many line problems).
   We can see that the horizontal axis (usually the x-axis) tells us the number of hours that Carlita worked at job A, which we will denote as a, and the vertical axis (usually the y-axis) tells us the number of hours that Carlita worked at job B, which we will denote as b. We are told that the line in the graph represents all possible combinations of hours that Carlita could have worked in order to earn a specific (but unknown) amount of money: s dollars.
   If we want to figure out the value of s, we need only pick any point (a, b) on this line and figure out how much money Carlita would make for working a hours at job A and b hours at job B (any and every point on the line tells us a combination of hours she could have worked to make s dollars). First, we need to write the Standard Form equation representing the line, which is easier to do based on the question's text than based on the Slope-Intercept Form of the line.

2. If Carlita works a hours at job A for \$8 an hour, then the total money she makes from job A is $8a$ dollars. If she works b hours at job B for \$10 an hour, then the total money she makes from job B is $10b$ dollars. In total, the amount of money she makes from working a hours at job A and b hours at job B, which is said to be s dollars, is equal to $8a + 10b$. Therefore, we can write the Standard Form linear equation $8a + 10b = s$ to represent the total amount of money she makes in a week.

3. Now that we have the Standard Form equation of the line (where s is still unknown), we can find the value of s by plugging in the a- and b-values of any point shown on the line. The easiest points to use would be either the a-intercept $(30, 0)$ (where the line hits the horizontal axis), or the b-intercept $(0, 24)$ (where the line hits the vertical axis). Note that any points on the line would work, but the lack of gridlines in the graph makes it harder to accurately choose any other points, and by choosing a point where one of the coordinate values is 0, we simplify the arithmetic we're about to perform. Let's check both intercepts just to verify that the total will be the same. First, let's use the point $(30, 0)$ by plugging in 30 for a and 0 for b.

$$\begin{aligned} 8a + 10b &= s \\ 8(30) + 10(0) &= s \\ 240 &= s \end{aligned}$$

We can comfortably choose choice B as the correct answer at this point, but for completeness, let's also check that the point $(0, 24)$ also produces the same value of s by plugging in 0 for a and 24 for b.

$$\begin{aligned} 8a + 10b &= s \\ 8(0) + 10(24) &= s \\ 240 &= s \end{aligned}$$

4. The answer is B.

Section 3.1 Suggested Problems from Real Tests 🏆 Hall of Fame

- Test 1-NC-3
- Test 2-NC-16 🏆
- Test 2-C-1
- Test 6-C-35
- Test 7-NC-1
- Test 7-C-25
- Test 7-C-26 🏆
- Test 9-NC-3
- Test 10-NC-9
- Test 10-NC-16
- Test 10-C-23
- Mar 2018-NC-5
- Apr 2018-C-34
- May 2018-NC-6
- Mar 2019-NC-1
- Mar 2019-NC-15 🏆
- Apr 2019-NC-19
- Apr 2019-C-27
- May 2019 (US)-C-21
- May 2019 (US)-C-23
- May 2019 (Int)-NC-14 🏆
- May 2019 (Int)-C-12
- Oct 2019-NC-8
- Mar 2020-C-13 🏆
- Mar 2020-C-21
- Oct 2020-NC-7
- Mar 2021-NC-5

Section 3.1 Practice Problems

1 3.1

Kenzie burns 3 calories per minute lifting weights and 15 calories per minute swimming. Which of the following equations represents the total number of calories, T, Kenzie has burned after lifting weights for 30 minutes and swimming for m minutes?

A) $T = 45m + 30$

B) $T = 15m + 90$

C) $T = 3m + 450$

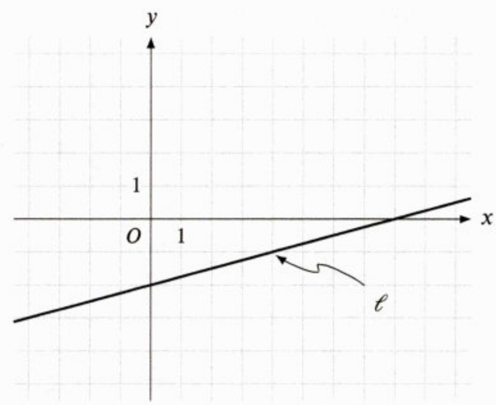
D) $T = 18m + 30$

2 3.1

$$T = 4b + 10s$$

A manufacturer produces units of two different products. The equation above shows the total manufacturing cost T, in dollars, for manufacturing b units of the big product and manufacturing s units of the small product. If the total manufacturing cost was $5,670 and 800 units of the big product were manufactured, how many units of the small product were manufactured?

3 3.1



Line ℓ is shown in the xy-plane above. Which of the following is an equation of line ℓ?

A) $8x + 2y = 0$

B) $8x - 2y = 0$

C) $4x + y = 8$

D) $x - 4y = 8$

4 3.1

What are the slope and the y-intercept of the graph in the xy-plane of the equation $2x - 5y - 7 = 0$?

A) The slope is $-\frac{2}{5}$, and the y-intercept is $(0, -\frac{7}{5})$.

B) The slope is $-\frac{2}{5}$, and the y-intercept is $(0, \frac{7}{5})$.

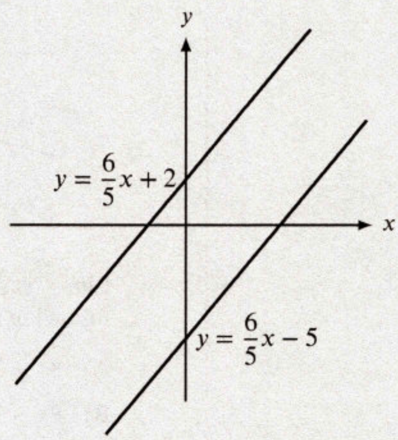
C) The slope is $\frac{2}{5}$, and the y-intercept is $(0, -\frac{7}{5})$.

D) The slope is $\frac{2}{5}$, and the y-intercept is $(0, \frac{7}{5})$.

3.2 Parallel and Perpendicular Lines

One of the main reasons we may need to convert from Standard Form to Slope-Intercept Form is to solve problems that ask us about parallel or perpendicular lines. These problems usually need you to make use of the lines' slopes, which are most easily found when the lines are written in Slope-Intercept Form.

Parallel Lines



Parallel lines are the same distance apart everywhere, and thus they never intersect. Parallel Lines have the same slope. The line $y = \mathbf{\frac{6}{5}}x + 2$ is parallel to the line $y = \mathbf{\frac{6}{5}}x - 5$ because they have the same slope.

So we can see what parallel lines look like in Standard Form, convert both lines into Standard Form.

$$y = \frac{6}{5}x + 2 \qquad\qquad y = \frac{6}{5}x - 5$$

$$\frac{-6}{5}x + y = 2 \qquad\qquad \frac{-6}{5}x + y = -5$$

$$5\left(\frac{-6}{5}x + y\right) = 5(2) \qquad\qquad 5\left(\frac{-6}{5}x + y\right) = 5(-5)$$

$$\cancel{5}\left(\frac{-6}{\cancel{5}}x\right) + 5y = 10 \qquad\qquad \cancel{5}\left(\frac{-6}{\cancel{5}}x\right) + 5y = -25$$

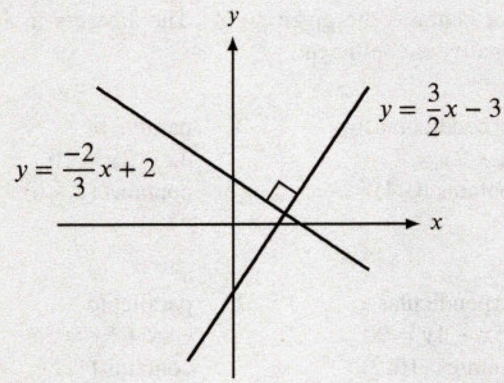
$$-6x + 5y = 10 \qquad\qquad -6x + 5y = -25$$

We now have both equations in Standard Form, and we should notice that the x- and y-coefficients match in both equations.

$$\mathbf{-6}x + \mathbf{5}y = 10$$
$$\mathbf{-6}x + \mathbf{5}y = -25$$

If we were given these two lines in Standard Form, we would know that the lines are parallel because they x- and y-coefficients match. You can also verify this by remembering that the slope of a line in Standard Form, $Ax + By = C$, is equal to $\frac{-A}{B}$, so the slopes will match as long as the x- and y-coefficients are the same (or are in the same ratio to each other, which is covered more in the next chapter).

Perpendicular Lines



Perpendicular lines cross at a right angle. The slope of each line in a pair of perpendicular lines is the negative reciprocal of the slope of the other line. The **negative reciprocal** is found by swapping the values in the numerator and denominator of a slope and multiplying by -1. For example, a line with a slope of $\frac{-2}{3}$ is perpendicular to a line with a slope of $\frac{3}{2}$, which was found by flipping the fraction and multiplying by -1. Similarly, a line with a slope of 4 (or $\frac{4}{1}$) is perpendicular to a line with a slope of $\frac{-1}{4}$.

To consider what perpendicular lines look like when written in Standard Form, let's convert the perpendicular lines $y = 4x+2$ and $y = \frac{-1}{4}x - 3$ into Standard Form.

$$y = 4x + 2$$
$$-4x + y = 2$$

$$y = \frac{-1}{4}x - 3$$
$$\frac{1}{4}x + y = -3$$
$$4\left(\frac{1}{4}x + y\right) = 4(-3)$$
$$\cancel{4}\left(\frac{1}{\cancel{4}}x\right) + 4y = -12$$
$$x + 4y = -12$$

These two lines in Standard Form are the following.

$$-4x + y = 2$$
$$x + 4y = -12$$

If we look at the two perpendicular lines in Standard Form, it looks like the x- and y-coefficients have swapped positions, and one of them has been multiplied by -1. In a way, you can think of this as a "horizontal negative reciprocal"—just as the negative reciprocal is found by swapping the numerator and denominator and multiplying by -1, the "horizontal negative reciprocal" here is found by swapping the x- and y-coefficients and multiplying one of them by -1. Remember again that the slope of a line in Standard Form, $Ax + By = C$, is equal to $\frac{-A}{B}$, so swapping the coefficients and multiplying one of them by -1 (to get the line $Bx - Ay = C$) effectively makes the slope equal to $\frac{B}{A}$, which is the negative reciprocal of $\frac{-A}{B}$.

SkillDrill 3.2-1

Directions: Given a linear equation in Standard Form, write the equation in both Standard Form and Slope-Intercept Form of the parallel or perpendicular line that contains the given point. The answers in Standard Form are written with lowest non-fractional coefficients and a non-negative x-coefficient.

1. parallel to $3x - 2y = 7$ contains $(0, 4)$

2. perpendicular to $3x - 2y = 7$ contains $(0, 4)$

3. parallel to $4x + 2y = 10$ contains $(2, -6)$

4. perpendicular to $4x + 2y = 10$ contains $(-1, 10)$

5. parallel to $-5x - 4y = 90$ contains $(5, 4)$

6. perpendicular to $-5x - 4y = 90$ contains $(10, 2)$

7. parallel to $-8x + 5y = 35$ contains $(1, 1)$

8. perpendicular to $-8x + 5y = 35$ contains $(-4, 0)$

Example 3.2-1

1 1600.io 3.2

Lines t and w are parallel in the xy-plane. The equation of line t is $4x + 7y = 14$, and line w passes through $(7, 6)$. What is the value of the y-intercept of line w?

Solution 1

1. Remember that parallel lines in Slope-Intercept Form should have the same x- and y-coefficients (or be in the same ratio), so we know that for line w in Standard Form, $Ax + By = C$, the value of A should be 4 and the value of B should be 7 (we do not know C, but that is part of what is needed to solve the problem).

$$\begin{aligned} Ax + By &= C \\ 4x + 7y &= C \end{aligned}$$

2. In order to solve for C, we should plug in the only point on line w that we know (which we are given), $(7, 6)$. Plug in 7 for x, and plug in 6 for y.

$$\begin{aligned} 4x + 7y &= C \\ 4(7) + 7(6) &= C \\ 28 + 42 &= C \\ 70 &= C \end{aligned}$$

3. Since $C = 70$, we can substitute 70 in for C to find the Standard Form equation of line w.

$$\begin{aligned} 4x + 7y &= C \\ 4x + 7y &= 70 \end{aligned}$$

4. The y-intercept is the y-value of the point on the line where $x = 0$, so plug in 0 for x in order to find the y-intercept of line w.

$$\begin{aligned} 4x + 7y &= 70 \\ 4(0) + 7y &= 70 \\ 7y &= 70 \\ y &= 10 \end{aligned}$$

5. The answer is 10.

Solution 2

1. First rewrite the equation of line t in Slope-Intercept Form, so we can find its slope (note that we don't actually need the entire equation including the constant that represents the y-intercept, because all we need to find is the slope, but we'll perform the conversion for completeness).

$$\begin{aligned} 4x + 7y &= 14 \\ 7y &= -4x + 14 \\ y &= \frac{-4}{7}x + 2 \end{aligned}$$

2. The slope of line t is $\frac{-4}{7}$, and since line w is parallel to line t, the slope of line w is also $\frac{-4}{7}$. Replace m with $\frac{-4}{7}$ in the general Slope-Intercept equation to begin writing the equation of line w.

$$y = \frac{-4}{7}x + b$$

3. Since we know that $(7, 6)$ is a point on line w, we can plug in 7 for x and 6 for y in order to solve for b, the y-intercept of line w.

$$\begin{aligned} y &= \frac{-4}{7}x + 2 \\ 6 &= \frac{-4}{7}(7) + b \\ 6 &= -4 + b \\ 10 &= b \end{aligned}$$

4. The answer is 10.

Notes

In Step 4 of Solution 1, instead of plugging in 0 for x to find the y-intercept, you could remember that the y-intercept of a line in Standard Form $Ax + By = C$ is equal to $\frac{C}{B}$ (but this doesn't save much time, is easy to screw up, and doesn't reinforce the fact the y-intercept of an equation is the y-value of the equation when the x-value is 0).

Example 3.2-2

2 3.2

$$-4x + 3y = 8$$

In the xy-plane, the graph of which of the following equations is perpendicular to the graph of the equation above?

A) $3x + 4y = 8$

B) $3x - 4y = 8$

C) $4x + 3y = 8$

D) $4x - 3y = 8$

Solution 1

1. First we need to write the original equation in Slope-Intercept Form to find its slope.

$$-4x + 3y = 8$$
$$3y = 4x + 8$$
$$y = \frac{4}{3}x + \frac{8}{3}$$

2. The slope of the given line is $\frac{4}{3}$, so the slope of a perpendicular line is the negative reciprocal of $\frac{4}{3}$. Flip the fraction and multiply by -1 to determine that the slope of a perpendicular line is $\frac{-3}{4}$.

3. Start converting each answer choice from Standard Form to Slope-Intercept Form until you find one with a slope of $\frac{-3}{4}$. Let's begin with choice A: $3x + 4y = 8$.

$$4y = -3x + 8$$
$$y = \frac{-3}{4}x + 2$$

4. Luckily, the first choice was the correct choice, but if you want to double check, you could put all of the other lines in Slope-Intercept Form. None of the other lines will have a slope of $\frac{-3}{4}$.

5. The answer is A.

Solution 2

1. Instead of converting to Slope-Intercept Form, use the fact that the slope of a line in Standard Form, $Ax + By = C$, is equal to $\frac{-A}{B}$. In the given equation, $-4x + 3y = 8$, we know that $A = -4$ and $B = 3$, so the slope is equal to $\frac{-(-4)}{3}$ or $\frac{4}{3}$.
2. The slope of the given line is $\frac{4}{3}$, so the slope of a perpendicular line is the negative reciprocal of $\frac{4}{3}$. Flip the fraction and multiply by -1 to determine that the slope of a perpendicular line is $\frac{-3}{4}$.
3. Find the slope of the lines in the answer choices using the same method in order to see which choice has a slope of $\frac{-3}{4}$. Let's begin with choice A: $3x + 4y = 8$.
4. For this equation, $A = 3$ and $B = 4$, so the slope of the line is equal to $\frac{-3}{4}$, which is the slope of a line perpendicular to the line $-4x + 3y = 8$, which has a slope of $\frac{4}{3}$.
5. The answer is A.

Notes

Once you become familiar with finding the equation of a line perpendicular to a given line with both lines in Standard Form, you'll start to think about looking for (or creating) an equation where the coefficients of x and y are swapped and one of them is negated; inspect the coefficients in the given line equation and in answer choice A to see this relationship. This procedure streamlines the solving process, because you never actually need to determine the slopes explicitly.

Section 3.2 Suggested Problems from Real Tests

🏆 Hall of Fame

- Test 2-NC-6
- Test 2-C-28 🏆
- Test 4-NC-8
- Test 5-C-11
- Test 9-NC-20
- Test 9-C-32
- May 2017-C-36
- Mar 2018-NC-4
- Mar 2018-C-30
- Apr 2018-C-19
- May 2018-NC-16
- May 2018-C-35
- Mar 2019-C-24
- Oct 2020-C-28

Section 3.2 Practice Problems

1 3.2

Which of the following statements is true about the graph of the equation $3y + 5x = -12$ in the xy-plane?

A) It has a negative slope and a positive y-intercept.

B) It has a negative slope and a negative y-intercept.

C) It has a positive slope and a positive y-intercept.

D) It has a positive slope and a negative y-intercept.

2 3.2

In the xy-plane, line ℓ has a slope of -3. If line k is perpedicular to line ℓ, which of the following could be an equation of line k ?

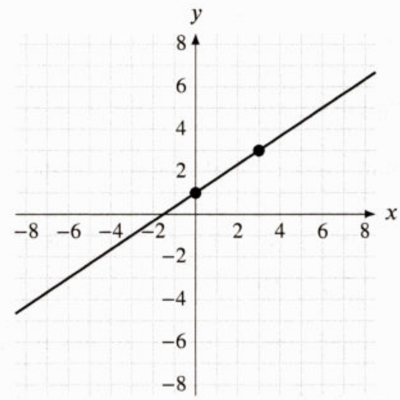
A) $-9x + 3y = 10$

B) $-2x + 6y = 12$

C) $-x - 3y = 4$

D) $12x + 4y = 7$

3 3.2



A line is shown in the xy-plane above. A second line (not shown) is parallel to the line shown and passes through the points $(0, 3)$ and $(4, c)$, where c is a constant. What is the value of c ?

CHAPTER 3 RECAP

- The Standard Form of Linear Equations is $Ax + By = C$.
- Standard Form can be converted into Slope-Intercept Form algebraically or by using the formula template $y = \frac{-A}{B}x + \frac{C}{B}$.
- The slope of a line in Standard Form is $\frac{-A}{B}$.
- The y-intercept of a line in Standard Form is $\frac{C}{B}$.
- Parallel lines are the same distance apart everywhere and thus they never intersect. Parallel Lines have the same slope.
- Perpendicular lines cross at a right angle. The slope of each line in a pair of perpendicular lines is the **negative reciprocal** of the slope of the other line. The negative reciprocal is found by swapping the values in the numerator and denominator of a slope and multiplying by -1.
- Parallel lines in Standard Form will have the same x- and y-coefficients (or the ratio of x- and y-coefficients will be the same).
- Perpendicular lines in Standard Form will have the x- and y-coefficients swapped, and one of the coefficients will be multiplied by -1 (or the ratio of the x- and y-coefficients will be flipped and multiplied by -1).

Additional Problems

1 1600.io 3.1

The line with the equation $\frac{3}{4}x + \frac{1}{5}y = 1$ is graphed in the xy-plane. What is the x-coordinate of the x-intercept of the line?

2 1600.io 3.1

$$x + y = 20$$

The equation above relates the number of minutes, x, Kevin spends showering each day and the number of minutes, y, he spends eating breakfast each day. In the equation, what does the number 20 represent?

A) The number of minutes spent showering each day

B) The number of minutes spent eating breakfast each day

C) The number of minutes spent eating breakfast for each minute spent showering

D) The total number of minutes spent showering and eating breakfast each day

3 1600.io 3.1

Gaston bought two types of candies: red candies that cost \$0.60 each and green candies that each cost z times as much as a red candy. If the cost of 3 red candies and 1 green candy was \$3, what is the value of z ?

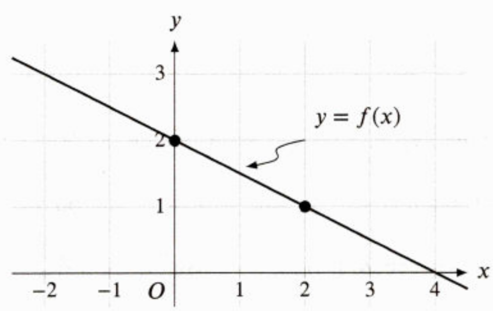
4 1600.io 3.2

In the xy-plane, line k passes through the point $(5, 7)$ and is parallel to the line with the equation $y = \frac{7}{5}x - \frac{11}{5}$. What is the slope of line k ?

5 3.1

The headmaster of a wizarding school awarded a total of 1200 bonus points to the most upstanding students. The points were awarded in amounts of 50 points or 200 points. If at least one 50-point bonus and at least one 200-point bonus were awarded, what is one possible number of 50-point bonuses awarded?

6 3.2



The graph of the linear function f is shown in the xy-plane above. The graph of the linear function g (not shown) is perpendicular to the graph of f and passes through the point $(2, 6)$. What is the value of $g(0)$?

7 3.1

The graph of the equation $ax + ky = -4$ is a line in the xy-plane, where a and k are constants. If the line contains the points $(2, 0)$ and $(0, -2)$, what is the value of k ?

A) -4

B) -2

C) 2

D) 4

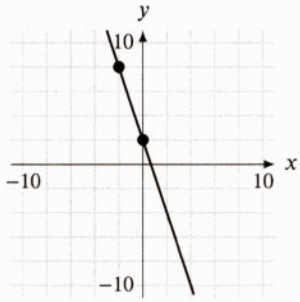
8 3.2

Lines z and w are parallel in the xy-plane. The equation of line z is $2x + 5y = 10$, and line w passes through $(-4, 6)$. What is the value of the y-intercept of line w ?

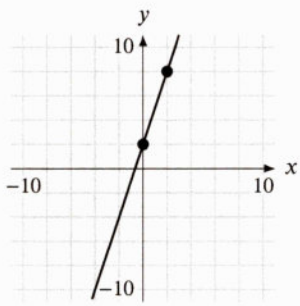
9 1600.io 3.1

Which of the following is the graph of $y + 3x = 2$ in the xy-plane?

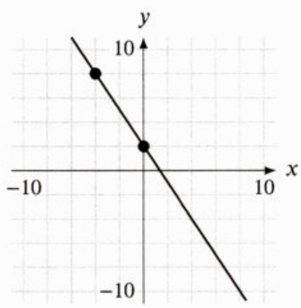
A)



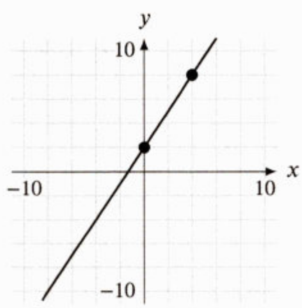
B)



C)



D)



10 1600.io 3.1

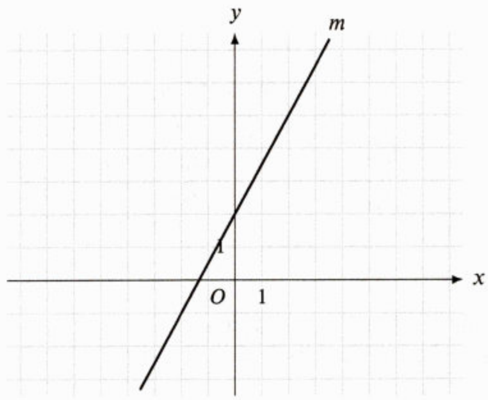
$$-3x + 5y = -2$$

The graph of the equation above in the xy-plane is a line. What is the x-coordinate of the x-intercept of the line?

11 1600.io 3.2

In the xy-plane, line ℓ has a y-intercept of -2 and is perpendicular to the line with the equation $y = -\frac{3}{5}x$. If the point $(6, b)$ is on line ℓ, what is the value of b ?

12 3.2



In the xy-plane above, line m is perpendicular to line ℓ (not shown). Which of the following could be an equation of line ℓ ?

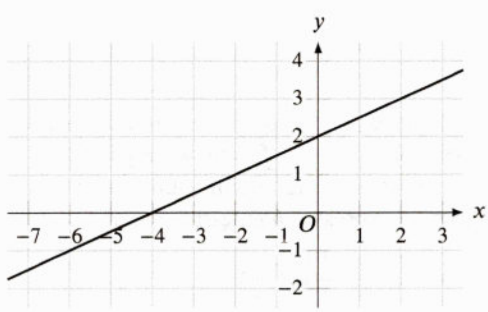
A) $2x + 3y + 6 = 0$

B) $2x - 3y + 6 = 0$

C) $3x - 2y + 2 = 0$

D) $3x + 2y + 2 = 0$

13 3.1



An equation of the graph shown is $ax + by = -8$, where a and b are constants. What is the value of b ?

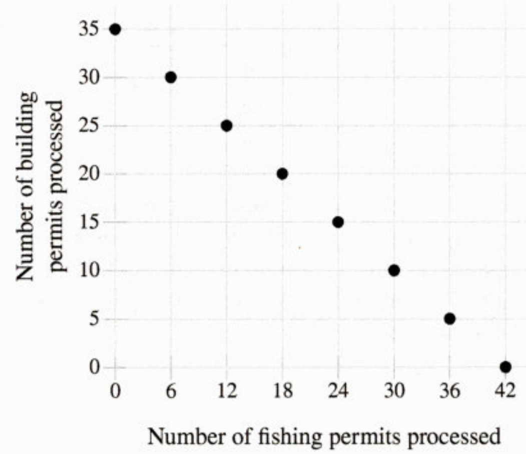
A) -4

B) -2

C) 2

D) 4

14 3.1



For his job, Kelvin spent a total of n minutes processing building permits and fishing permits. It takes Kelvin 30 minutes to process a building permit and 25 minutes to process a fishing permit. The graph above represents all possible combinations for the number of building permits and the number of fishing permits that Kelvin could have processed in the n minutes. What is the value of n ?

A) 1,470

B) 1,260

C) 1,050

D) 875

15 3.1

The graph of a line in the xy-plane has a negative slope and intersects the y-axis at a point that has a positive y-coordinate. Which of the following could be an equation of the line?

A) $-7x + 4y = -10$

B) $-7x + 4y = 10$

C) $7x + 4y = -10$

D) $7x + 4y = 10$

16 3.1

x	y
$2b$	0
$4b$	$-b$
$6b$	$-2b$

Some values of x and their corresponding values of y are shown in the table above, where b is a constant and $b \neq 0$. If there is a linear relationship between x and y, which of the following equations represents the relationship?

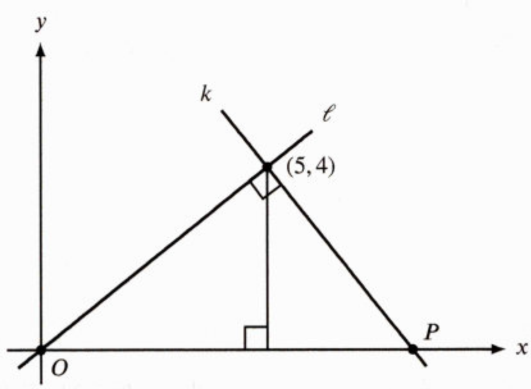
A) $x + 2y = b$

B) $x + 2y = 2b$

C) $2x - y = -6b$

D) $2x - y = -b$

17 3.2



In the xy-plane above, lines k and ℓ are perpendicular. What is the x-coordinate of point P ?

A) 8.2

B) 8.25

C) 8.5

D) 8.8

Questions 18 and 19 refer to the following information.

Cost per Pound of Nuts

Nut	Dollars	Euros
Cashew	4.00	4.40
Brazil	3.00	3.30
Macadamia	4.00	4.40

The table above gives the typical costs per pound of nut, expressed in both dollars and euros, of the three nuts that a vendor sells.

18 3.1

If x dollars is equivalent to k euros, of the following, which best represents the relationship between x and k ?

A) $k = 0.91x$

B) $xk = 1.1$

C) $k = 1.1x$

D) $x = 1.1k$

19 3.1

If a client pays \$300 for a mixture of nuts consisting entirely of c pounds of cashews, b pounds of brazil nuts, and m pounds of macadamia nuts, which of the following expresses b in terms of c and m ?

A) $b = 100 - \frac{4}{3}(c + m)$

B) $b = 100 - \frac{4}{3}(c - m)$

C) $b = 100 + \frac{4}{3}(c + m)$

D) $b = 100 + \frac{3}{4}(c + m)$

Answer Key

SkillDrill 3.1-1

1. $2b + 3f = 56$

2. $h + 3c = 60$

3. $8s + 20b = 112$

4. $-100c - 2.50h = -197.50$

5. $5r - 10p = -42$

SkillDrill 3.1-2

1. $y = 2x - 4$

2. $y = 3x + 7$

3. $y = \frac{-3}{2}x - \frac{7}{2}$

4. $y = \frac{-5}{6}x + 3$

5. $5x + y = -10$

6. $2x - y = -4$

7. $x + 3y = -2$

8. $5x - 2y = -14$

Section 3.1 Practice Problems

1. B

2. 247

3. D

4. C

SkillDrill 3.2-1

1. $y = \frac{3}{2}x + 4$, $3x - 2y = -8$

2. $y = \frac{-2}{3}x + 4$, $2x + 3y = 12$

3. $y = -2x - 2$, $2x + y = -2$

4. $y = \frac{1}{2}x + \frac{21}{2}$, $x - 2y = -21$

5. $y = \frac{-5}{4}x + \frac{41}{4}$, $5x + 4y = 41$

6. $y = \frac{4}{5}x - 6$, $4x - 5y = 30$

7. $y = \frac{8}{5}x - \frac{3}{5}$, $8x - 5y = 3$

8. $y = \frac{-5}{8}x - \frac{5}{2}$, $5x + 8y = -20$

Section 3.2 Practice Problems

1. B

2. B

3. $\frac{17}{3}$, 5.66, or 5.67

Additional Problems

1. $\frac{4}{3}$ or 1.33

2. D

3. 2

4. $\frac{7}{5}$ or 1.4

5. 4, 8, 12, 16, or 20

6. 2

7. C

8. $\frac{22}{5}$ or 4.4

9. A

10. $\frac{2}{3}$, .666, or .667

11. 8

12. A

13. A

14. C

15. D

16. B

17. A

18. C

19. A

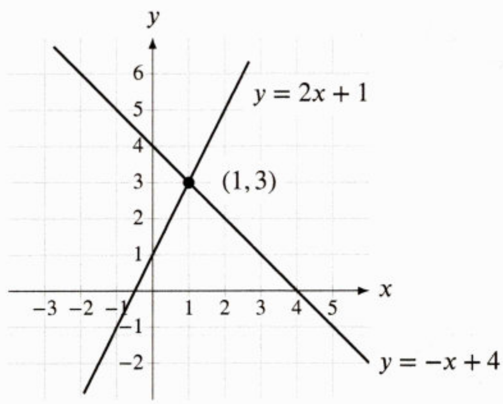
Systems of Linear Equations

4

4.1 Introduction to Systems of Linear Equations

A **system of linear equations** is a collection of two linear equations with the same set of variables. In previous chapters, we talked about the information that can be gleaned from a single linear equation and its graph, but what happens when we have two lines in the same system?

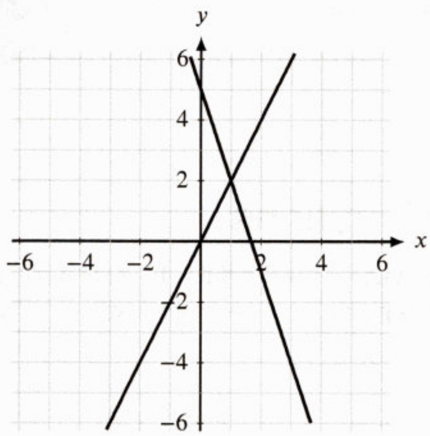
Let's look at a graph of two distinct lines: $y = 2x + 1$ and $y = -x + 4$.



Notice that the two lines intersect at the point $(1, 3)$. This intersection point exists as a point on the graphs of both functions. Remember that the graph of a linear equation represents all of the points that are solutions to that linear equation. When two intersecting lines are graphed together to represent a system of linear equations, their intersection point gives one pair of x- and y-values that satisfies both equations. The intersection point is called a **solution to the system of equations** because it is a solution to both of the equations in the system.

Example 4.1-1

1 1600.io 4.1



The lines in the xy-plane above are the graphs of two linear equations. What is the solution (x, y) to the system formed by the equations?

A) $(0, 5)$

B) $(0, 0)$

C) $(1, 2)$

D) $(2, 1)$

Solution

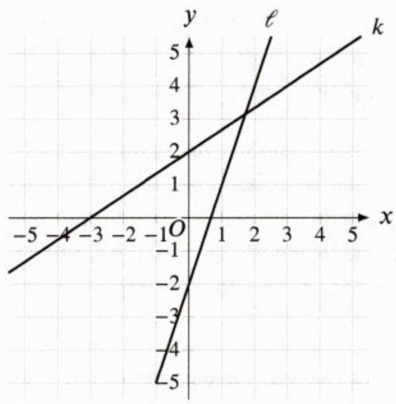
1. Since the solution point for a system of two linear equations is the point where the two lines intersect, we should check the gridlines to see if the intersection point is easily found.
2. Because the lines intersect at a meeting of gridlines, we can clearly see that the lines intersect at the point $(1, 2)$, so the x- and y-value pair that constitutes the solution to the system of linear equations is represented by the point $(1, 2)$.
3. The answer is C.

Section 4.1 Suggested Problems from Real Tests

- Mar 2018-NC-1
- Mar 2019-C-5
- Apr 2019-C-15
- May 2019 (US)-C-14

Section 4.1 Practice Problems

1 4.1



Lines ℓ and k in the xy-plane above are the graphs of the equations in a system. How many solutions does the system of equations have?

A) None

B) One

C) Two

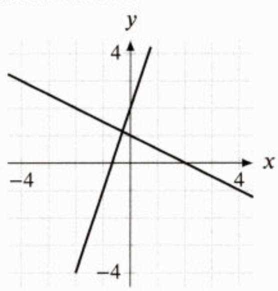
D) More than two

2 4.1

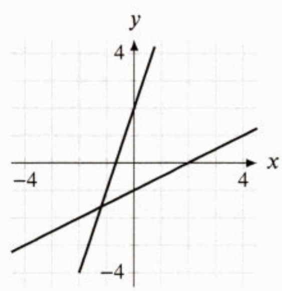
$$-x + 2y = -2$$
$$3x - y = -2$$

Which of the following graphs in the xy-plane could be used to solve the system of equations above?

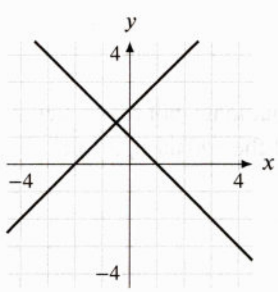
A)



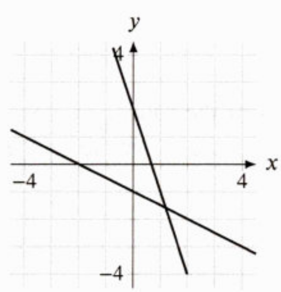
B)



C)



D)



4.2 Using Substitution to Solve Systems of Linear Equations

Let's come back to our opening example with the lines $y = 2x + 1$ and $y = -x + 4$. We determined from the graph that the point $(1, 3)$ existed on both lines. But suppose the lines were not graphed on a carefully laid out grid, and we were asked to find the solution to the system if we only had the equations themselves. Perhaps there is a more mathematical approach to solving this system than sloppily drawing the lines ourselves and hoping we could draw a perfectly proportioned grid that would enable us to correctly identify the solution to the system (**NEVER** attempt to solve system of equations problem this way).

Instead of simply considering the solution to a system of linear equations as an intersection point on a graph, we can also consider that (as mentioned earlier) a solution to the system is also a pair of x- and y-values that satisfies both equations. Algebraically, we can solve the system using substitution. Since we are looking for an x- and y-coordinate pair that is a solution to both lines, the same x- and y-values that satisfy one of the equations also satisfy the other equation. When we are looking for this solution, we know that the y-value is the same in both equations (and so are the x-values), so we can plug one in for the other, which is very easy when both equations are written in Slope-Intercept Form.

When both lines are in Slope-Intercept Form, as in our current example, they are both solved for y in terms of x. For example, we know from one of our equations that y must be equal to $2x + 1$, and we know from the other equation that y must also be equal to $-x + 4$.

Here's what we're given:

$$y = 2x + 1$$
$$y = -x + 4$$

Let's just rewrite the first equation by switching the sides of the terms:

$$2x + 1 = y$$

Arranging our two equations:

$$\begin{aligned} 2x + 1 = {} & y \\ & y = -x + 4 \end{aligned}$$

Now, it should be apparent that

$$2x + 1 = -x + 4$$

because equality is transitive: generally, if we're given

$$y = u$$
$$y = v$$

we know that

$$u = v$$

Sometimes, this procedure is casually referred to as "setting the equations equal to each other," but that's not really right: we're not setting the two *equations* equal; we're setting the two *expressions* that are on the opposite side of the equals sign from y equal to each other, so be sure you understand that.

This procedure can equivalently be thought of as **substituting** an expression in terms of x for y in an equation, which has the effect of eliminating y from the equation, leaving x as the only variable. So, we have

$$y = 2x + 1$$

We also have

$$y = -x + 4$$

so we can substitute $-x + 4$ for y in the first equation:

$$\begin{aligned} y &= 2x + 1 \\ -x + 4 &= 2x + 1 \end{aligned}$$

Of course, we could do the same thing but switching the equations, so we could substitute $2x + 1$ for y in the second equation:

$$2x + 1 = -x + 4$$

The result is the same, except the sides of the expressions are swapped, which has no mathematical importance. Let's proceed with the solving using the first unified equation we made.

Now that the two expressions are set equal to each other, we can collect and combine all the x-terms on one side of the equation and all the constant terms on the other side of the equation in order to solve for x.

$$\begin{aligned} -x + 4 &= 2x + 1 \\ 4 &= 3x + 1 \\ 3 &= 3x \\ 1 &= x \end{aligned}$$

By setting the two expressions that represent y equal to each other or, equivalently, by substituting the expression that gives the value of y from one equation in for y in the other equation, we have determined the x-value of the solution point. Now that we know that $x = 1$ is *part* of the solution, we can **substitute this value for x in the equation for either line** (remember that the solution to the system is the coordinate pair that exists on **both** lines in the system) in order to find the y-value of the solution.

$$\begin{aligned} y &= 2x + 1 \\ y &= 2(1) + 1 \\ y &= 2 + 1 \\ y &= 3 \end{aligned}$$

By using substitution (instead of locating the intersection point on a graph), we have found the pair of x- and y-variables that form the solution to the system of equations.

SkillDrill 4.2-1

Directions: Use substitution to find the intersection point that is the solution to the system of linear equations.

1. $y = 8x + 2$
$y = -2x + 12$

2. $y = -3x$
$y = 2x + 25$

3. $y = x - 7$
$y = 3x - 14$

4. $y = 9x + 6$
$y = -3x + 5$

In the following example, we will solve for the solution to a system of linear equations by setting the expressions representing y equal to each other (we will have to write the second equation based on the question prompt before we can solve).

Example 4.2-1

1 1600.io 4.2

$$y = x + 4$$

An equation of line ℓ in the xy-plane is shown above. Another line, k, has a slope equal to triple the slope of ℓ and a y-intercept equal to triple the y-intercept of ℓ. At which point (x, y) do lines ℓ and k intersect?

A) $(-4, 0)$

B) $\left(-\frac{1}{4}, 0\right)$

C) $\left(0, \frac{1}{4}\right)$

D) $(0, 4)$

Solution

1. Remember, the point of intersection of two lines represents the solution to the system the lines represent. Write the equation of the second line in the system based on the information given in the question. The slope of line k is triple (three times) the slope of line ℓ, and as the slope of line ℓ is 1, the slope of line k is 3. The y-intercept of line k is triple the y-intercept of line ℓ, and as the y-intercept of line ℓ is 4, the y-intercept of line k is 12. Since its slope is 3 and its y-intercept is 12, the equation of line k is $y = 3x + 12$.
2. Both equations are solved for y in terms of x. Since the y-values have to be equal at the intersection point, we can use substitution and set the right sides of both equations equal to each other.
$$x + 4 = 3x + 12$$
3. Combine the x-terms on one side of the equation and the constant terms on the other side of the equation, then solve for x.
$$\begin{aligned} x + 4 &= 3x + 12 \\ 4 &= 2x + 12 \\ -8 &= 2x \\ -4 &= x \end{aligned}$$
4. There is only one choice in which the x-value of the point is -4, so there is no need to solve for the y-value of the intersection point (by plugging the x-value into either of the original equations in the system), but you can if you like to play it safe.
$$\begin{aligned} y &= x + 4 \\ y &= 4 + 4 \\ y &= 0 \end{aligned}$$
5. The answer is A.

It is not necessary for both equations to be in Slope-Intercept Form (or, more generally, to be solved in terms of the same variable, whether it's x or y) in order to use substitution. **Whenever at least one of the equations in a system is solved for one variable in terms of another (one variable is isolated on one side of the equation), substitution will probably be the best method of solution** (we will discuss other cases later in the chapter). In the next example, neither linear equation is written in Slope-Intercept Form. DON'T PANIC; both of the equations are still linear, and we can still use substitution even when the linear equations are not written in either of the traditional forms of linear equations that we have discussed.

Example 4.2-2

2 4.2

$$\frac{1}{4}(x+3y)=\frac{9}{4}$$
$$x=3y$$

The system of equations above has solution (x, y). What is the value of y?

Solution

1. Start by getting rid of the fractions in the first equation (a smart first step in most situations). Multiply both sides of the first equation by 4.

$$\cancel{4}\left(\frac{1}{\cancel{4}}\right)(x+3y)=\cancel{4}\left(\frac{9}{\cancel{4}}\right)$$
$$x+3y=9$$

2. Since the second equation tells us that x is equal to $3y$, substitute $3y$ in for x in the simplified version of the first equation ($x+3y=9$), then solve for y.

$$x+3y=9$$
$$(3y)+3y=9$$
$$6y=9$$
$$y=\frac{9}{6}$$
$$y=\frac{3}{2}$$

3. The answer is $\frac{3}{2}$.

Notes

It was not essential that the fractions be eliminated as the first step; you could have started by substituting $3y$ for x in the first equation, which would result in $\frac{1}{4}(6y)=\frac{9}{4}$. At that point, you could multiply through by 4 to get $6y=9$ and then proceed with solving for y from there.

Section 4.2 Suggested Problems from Real Tests

🏆 Hall of Fame

- Test 1-NC-11
- Test 2-NC-9
- Test 2-C-34
- Test 3-C-18
- Test 3-C-24 🏆
- Test 4-NC-3
- Test 4-C-6
- Test 5-NC-18
- Test 6-C-10
- Test 7-C-16
- Apr 2017-C-8 🏆
- May 2017-NC-3
- May 2017-NC-4
- May 2017-C-34
- Mar 2018-C-34 🏆
- May 2018-NC-17
- May 2018-C-8
- Mar 2019-C-37 🏆
- May 2019 (US)-NC-16
- Oct 2019-NC-19
- Oct 2019-C-11
- Mar 2020-NC-17
- Oct 2020-NC-11
- Oct 2020-C-30

Section 4.2 Practice Problems

1 1600.io 4.2

$$7x + y = 53$$
$$x = 6$$

If (x, y) is the solution to the given system of equations, what is the value of y ?

2 1600.io 4.4

$$x - 4y = 16$$
$$4y = 12$$

If (x, y) is the solution to the system of equations above, what is the value of x ?

A) -4

B) 4

C) 28

D) 64

3 1600.io 4.2

$$x + 5y = 16$$
$$y = 3x$$

If the ordered pair (x, y) satisfies the system of equations above, what is the value of x ?

4 1600.io 4.2

In the xy-plane, the graph of $y = 3x - 2$ intersects the graph of $y = 2x + 4$ at the point (a, b). What is the value of a ?

A) -2

B) 2

C) 6

D) 16

5 1600.io 4.2

The graph of a line in the xy-plane has slope 4 and contains the point $(1, 10)$. The graph of a second line passes through the points $(1, 4)$ and $(2, 5)$. If the two lines intersect at the point (a, b), what is the value of $b - a$?

A) -3

B) -1

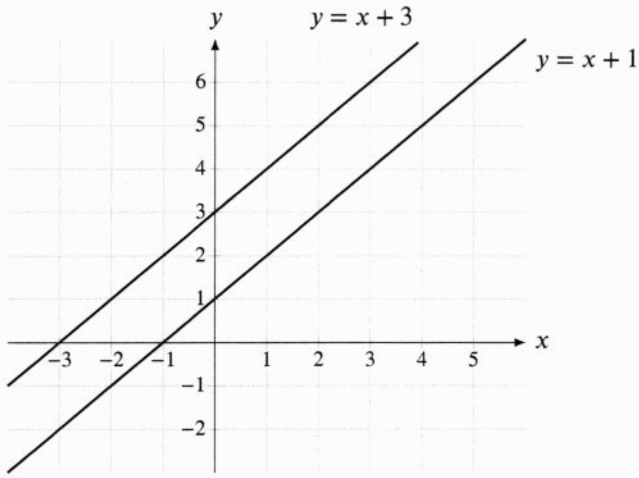
C) 2

D) 3

4.3 Number of Solutions to Systems of Linear Equations Part I

We know that a solution to a system of two linear equations is an x- and y-value pair that satisfies both linear equations, and the graphical representation of that solution is a point with that x- and y-value that lies on both lines. **When the lines intersect at a point, a single solution to the system exists**. But what happens if the lines in a system never intersect? There can be no solution to the system because there is no common point that exists on both lines.

The only way for two lines to never intersect (that is, for the system of equations to have no solution) is for them to be parallel: the two lines must have **the same slope and different y-intercepts**. For example, let's consider the lines $y = x + 1$ and $y = x + 3$.



There are no x- and y-coordinate pairs that exist on both lines (there is no intersection point). Therefore, there is no solution to the system of equations.

If we tried to solve this system using substitution to create the equation $x + 1 = x + 3$, we would arrive at an impossible situation (which indicates that there is no solution) when we attempt to solve for x. When we subtract x from both sides of the equation (in an attempt to group the x-terms on one side of the equation), we end up with no x terms and have the irreconcilable, untrue equation $1 = 3$. Since 1 is obviously never equal to 3, there is no value of x for which $x + 1$ equals $x + 3$; there is no solution to the system of equations (no matter what value of x we plug into the equations).

Most of the time, problems about systems of linear equations with no solutions are disguised in the form of a single equation with only one variable. For example, if we are given the equation

$$ax + 5 = 2x + 3$$

and told that there are no solutions, and we are asked to find the value of a, then we need to understand that **this is actually a system of equations problem, but the first step of substitution is already done for us** (the expressions representing y for two separate linear equations have already been set equal to each other).

Essentially, it is as if we have been given a system of linear equations in which one equation is $y = ax + 5$ (based on the left side of the equation) and the other equation is $y = 2x + 3$ (based on the right side of the equation). If these two lines have no solution, then they must be parallel. **In order for the two lines to be parallel, their slopes must be the same, and their y-intercepts must be different.**

Since the slope of the first line is a, and the slope of the second line is 2, it must be true that $a = 2$ in order for the lines to have the same slope.

In essence, you do not have to actually split the equation into two separate equations; you simply need to **match the x-coefficients**

(the slopes) on both sides of the equation (and verify that the constants, or y-intercepts, are not the same).

Indeed, if $a = 2$, we can see that plugging 2 in for a in the original single equation given to us, $ax + 5 = 2x + 3$, will once again produce an impossible situation.

$$ax + 5 = 2x + 3$$
$$(2)x + 5 = 2x + 3$$
$$5 = 3????$$

SkillDrill 4.3-1

Directions: All of the following equations have no solution. Find the value of a.
Challenge questions are boxed.

1. $ax + 2 = 4x - 7$

2. $x + 2 = 2a(x - 7)$

3. $a(3x + 2) + 2x = 7x - 7$

4. $\boxed{ax + 2 = -7}$

Example 4.3-1

1 4.3

$$a(-4x + 2) + 3x = 4x - 1$$

The equation above has no solutions, and a is a constant.
What is the value of a ?

A) -1

B) $-\frac{1}{2}$

C) $-\frac{1}{4}$

D) 1

Solution

1. Our goal is to get both sides of the equation into the form $mx + b$ (as if the y-expressions for two lines have already been set equal to each other) so we can simply match the x-coefficients (and verify that the constants are not the same).

2. Distribute a on the left side of the equation.

$$a(-4x + 2) + 3x = 4x - 1$$
$$-4ax + 2a + 3x = 4x - 1$$

3. Subtract $3x$ from both sides of the equation to group the x-terms with numerical coefficients on the right side of the equation.

$$-4ax + 2a + 3x = 4x - 1$$
$$-4ax + 2a = x - 1$$

4. In order for there to be no solutions, the x-coefficients (which represent the slopes) must be the same (and the constants must not be equal). Set the x-coefficients ($-4a$ and 1) equal to each other to solve for a.

$$-4a = 1$$
$$a = -\frac{1}{4}$$

5. The answer is C.

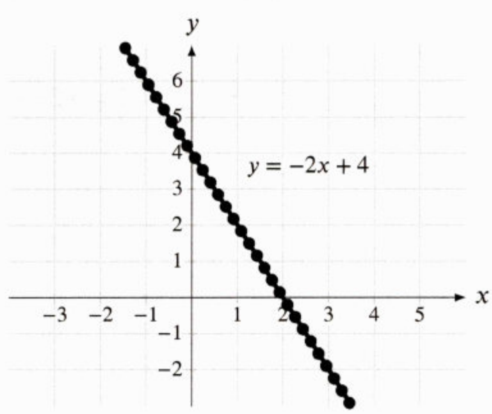
Notes

You can double check the value of a by making sure that the constant terms (the y-intercepts) DO NOT match (if the slopes were the same and the y-intercepts were the same, the lines would be the same line instead of parallel lines). If $a = \frac{-1}{4}$, then the constant on the left side of the equation is $2\left(\frac{-1}{4}\right)$ or $\frac{-1}{2}$, which is not equal to the constant term on the other side of the equation (-1). The value we found for a passes this test, so we know that the two lines are parallel and are not the same line.

In Step 3, we subtracted $3x$ from both sides in order to avoid combining the x-coefficients ($-4a$ and 3) and factoring out x. This would not be wrong but is probably confusing for some students. In general, you want to try to keep unknown constants such as a, b, or c on one side of the equation (separate from numerical constants) when you have to match coefficients.

We've seen that a system of equations can have one solution where there's a single pair of values (x, y) that satisfies both equations, and we showed that the graphical representation of this is the point of intersection of the two lines that represent the equations. If the lines do not cross anywhere—they are parallel—there's no point that lies on both lines, and thus there is no solution.

There is only one other relationship between two linear equations: they're actually the same, and thus they produce the same line when graphed; such lines are called coincident lines. In that case, every solution to one of the equations is also a solution to the other equation; graphically, this is evident because the two lines are in fact the same, so every point lies on both, and is thus a solution to both. Because there are an infinite number of points on a line, this means that **there are infinitely many solutions to the system of equations in which the lines are exactly the same**. In the graph below, the line $y = -2x + 4$ is extra thick to show that the same line has been graphed twice (one on top of the other) and several of the infinite solution points are shown along the line.



If we were given a system of linear equations in which both equations were the same, say $y = -2x + 4$ and $y = -2x + 4$ (it should be obvious the lines are the same if they are given in the same form, but let's continue anyway), we could try to solve the system with substitution.

$$-2x + 4 = -2x + 4$$

When we try to group the x-terms, they will once again cancel each other (as with a system of parallel lines), but this time we will be left with the undeniably true equation $4 = 4$. Since 4 is obviously always equal to 4 no matter what the value of x is, there are infinitely many solutions to the system of equations—no matter what value of x or y we plug into the equations, the two lines will always be equal because they are the same line.

Most of the time, problems about systems of linear equations with infinitely many solutions are disguised in the form of a single equation with only one variable. For example, if we are given the equation $ax + b = -3x + 5$ and told that there are infinitely many solutions, and we are asked to find the value of a, we need to understand that this is actually a systems of equations problem, but the first step of substitution is already done for us (the expressions that equal y for two separate linear equations have already been set equal to each other).

Essentially, it is as if we have been given a system of linear equations in which one equation is $y = ax + b$ (based on the left side of the equation) and the other equation is $y = -3x + 5$ (based on the right side of the equation). If these two lines have infinitely many solutions, then they must be exactly the same line. **In order for the two lines to be the same, their slopes *and* their y-intercepts must be the same.**

Since the slope of the first line is a, and the slope of the second line is -3, it must be true that $a = -3$ in order for the lines to have the same slope. Since the y-intercept of the first line is b, and the y-intercept of the second line is 5, it must be true that $b = 5$ for the lines to have the same y-intercept.

In essence, you do not have to actually split the equation into two separate equations; you simply need to **match the x-coefficients (the slopes) and the constant terms (the y-intercepts) on both sides of the equation when dealing with a system that has infinitely many solutions**.

Indeed, if $a = -3$ and $b = 5$, we can see that plugging -3 in for a and 5 in for b in the original single equation given to us, $ax + b = -3x + 5$, will once again produce a statement that is always true for any value of x.

$$\begin{gathered} ax + b = -3x + 5 \\ -3x + 5 = -3x + 5 \\ 5 = 5 \end{gathered}$$

SkillDrill 4.3-2

Directions: Each of the following equations has infinite solutions. Find the values of a and b, reducing any fractions to lowest terms.

1. $ax - b = 4x - 7$ **2.** $a(x + 2) = 6x + b$ **3.** $ax + 2 = b(3x - 1)$ **4.** $\boxed{abx + b = 3(5x + 3)}$

Example 4.3-2

2 1600.io 4.3

$$ax - 5(4 - 3x) = -20$$

In the equation above, a is a constant. For what value of a does the equation have infinitely many solutions?

A) -15

B) -3

C) 3

D) 5

Solution

1. Our goal is to get both sides of the equation into the form $mx + b$ (as if the expressions representing y for two lines have already been set equal to each other) so we can simply match the x-coefficients and constant terms on both sides of the equation (satisfying the conditions for infinitely many solutions).
2. Distribute -5 on the left side of the equation.
$$ax - 5(4 - 3x) = -20$$
$$ax - 20 + 15x = -20$$
3. Subtract $15x$ from both sides of the equation in order to have an x-term on both sides of the equation.
$$ax - 20 + 15x = -20$$
$$ax - 20 = -15x - 20$$
4. In order for there to be infinitely many solutions, the x-coefficients (a and -15), which represent the slopes, must be the same (so too must the constants, which, as expected, already match). Therefore $a = -15$.
5. The answer is A.

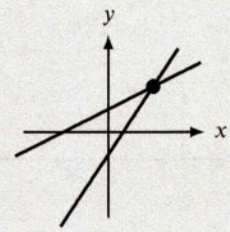
Notes

In Step 3, we subtracted $15x$ from both sides in order to avoid combining the x-coefficients (a and 15) and factoring out x. This would not be wrong but is probably confusing for some students. In general, you want to try to keep unknown constants such as a, b, or c on one side of the equation (separate from numerical constants) when you have to match coefficients.

Systems of Linear Equations with One Solution

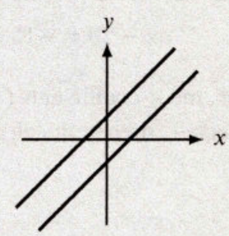
In a system of equations with one solution, the lines intersect at one point, with the coordinates of the intersection point representing the x- and y-values of the solution. Remember, a solution represents a pair of x- and y-values that satisfies both equations, which, in graphical terms, means these values define a single point that lies on both lines; a point that's on both lines is a point of intersection.

***Any* two lines that have different slopes will intersect once.** Note that the lines do not have to be perpendicular.



Systems of Linear Equations with No Solution

In a system of equations with no solutions, the lines are parallel, so they do not intersect. Because a solution is an x-value and a y-value that satisfies both linear equations, and the graphical representation of that solution is a point with that x- and y-value that lies on both lines, for a single solution to exist, the lines must intersect, with the point of intersection representing that solution. The only way two lines can never intersect is for the two lines to be parallel; **to be parallel, two lines must have the *same* slope and they must have *different* y-intercepts** (otherwise they'd be the same lines).

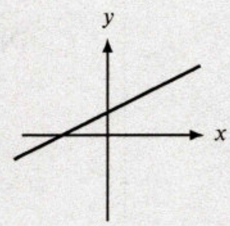


When the lines representing two linear equations are parallel, you will see when inspecting the equations in Slope-Intercept Form that the lines will have the same slope, but different y-intercepts. For example, the lines $y = x + 1$ and $y = x - 1$ are parallel and thus will never intersect because they have the same slope but different y-intercepts.

Systems of Linear Equations with Infinite Solutions

In a system of equations with infinitely many solutions, the lines must be exactly the same, and thus they produce the same line when graphed. Every solution to one of the equations is also a solution to the other equation; graphically, this is evident because the two lines are in fact the same (they're coincident), so every point on one line is common to the other line and is thus a solution to both. Because there are an infinite number of points on a line, this means that there are infinitely many solutions to the system of equations in this situation.

When two linear equations in Slope-Intercept Form are the same, you will see that **the lines will have the *same* slope and *same* y-intercept**. In the following picture, it may look like only one line is graphed, but in reality, the same line is graphed twice, and the two lines are actually drawn on top of each other.



Section 4.3 Suggested Problems from Real Tests 🏆 Hall of Fame

- Test 7-C-9
- Test 9-NC-8
- Test 10-C-35 🏆
- Apr 2018-NC-10
- May 2018-NC-4
- May 2018-C-28
- Mar 2019-C-27
- May 2019 (US)-NC-1
- May 2019 (Int)-NC-17
- May 2019 (Int)-C-29
- Oct 2019-NC-20
- Oct 2019-C-30
- Mar 2020-NC-10
- Mar 2020-C-29
- Oct 2020-NC-13
- Mar 2021-NC-20

Section 4.3 Practice Problems

1 1600.io 4.3

$$y = -\frac{1}{3}x + 1$$
$$y = ax - 4$$

In the system of equations above, a is a constant. If the system of equations has no solution, what is the value of a ?

A) -3

B) $-\frac{1}{3}$

C) 0

D) 3

2 1600.io 4.3

$$4k(x - 6) = x - 6$$

In the equation above, k is a constant. If the equation has infinitely many solutions, what is the value of k ?

3 1600.io 4.3

$$3ax - 12 = 2(x + 4) + 4(x - 3)$$

In the equation above, a is a constant. If no value of x satisfies the equation, what is the value of a ?

A) 0

B) 1

C) 2

D) 3

4 1600.io 4.3

$$5(x + b) = ax + 2c$$

In the equation above, a, b, and c are constants. If the equation has infinitely many solutions, which of the following must be equal to c ?

A) $5a$

B) $5b$

C) $\frac{5a}{2}$

D) $\frac{5b}{2}$

4.4 Solving Systems of Linear Equations by Elimination/Combination

The intersection of two graphs is one way to think about the solution to a system of linear equations. However, it is probably more useful to think about the solution to a system of equations as an ordered pair (x, y) of values that satisfies both equations because most systems of linear equations on the test are based on the Standard Form of lines and do not lend themselves to graphing and finding intersections (unlike lines in Slope-Intercept Form).

We will discuss the methods of solving systems of linear equations shortly, but for now let's briefly focus on writing our own system of linear equations in Standard Form (rather than in Slope-Intercept Form as in the previous example).

For many of the problems on the test, you need to write your own system of linear equations in Standard Form. Most of these are found in the form of word problems about how much money is made from selling a certain amount of two different items. The following example reflects something you are sure to see on every test. You will have to write one equation that represents the total number of items sold and one equation that represents the amount of money made from selling those things. It is easiest to write these equations in Standard Form based on how they are presented (not to mention that the answer choices will almost always be in Standard Form).

Example 4.4-1

1 4.4

A butcher sold 89 pounds of meat that consisted of b pounds of beef and p pounds of pork. The butcher sold the beef for \$2.39 per pound and the pork for \$1.39 per pound and collected a total of \$172.71. Which of the following systems of equations can be used to find the number of pounds of beef that were sold?

A) $$\begin{aligned} b + p &= 172.71 \\ 2.39b + 1.39p &= 89 \end{aligned}$$

B) $$\begin{aligned} b + p &= 89 \\ 2.39b + 1.39p &= 172.71 \end{aligned}$$

C) $$\begin{aligned} b + p &= 89 \\ 1.39b + 2.39p &= 172.71 \end{aligned}$$

D) $$\begin{aligned} b + p &= 172.71 \\ 1.39b + 2.39p &= 89 \end{aligned}$$

Solution

1. We need to write one equation that describes the total number of pounds of meat sold. We know that b is the number of pounds of beef, p is the number of pounds of pork, and 89 is the total number of pounds of meat sold. So one of the equations is the following:

$$b + p = 89$$

2. We also need to write an equation that describes the amount of money the butcher made. Since beef is sold for \$2.39 per pound and b pounds of beef are sold, the total amount of money made from beef is $2.39b$. Since pork is sold for \$1.39 per pound and p pounds of pork are sold, the total amount of money made from pork is $1.39p$. Add these two amounts and set them equal to 172.71, which is the total amount of money the butcher made from selling this meat.

$$2.39b + 1.39p = 172.71$$

3. The two equations that describe the number of pounds of meat sold and the amount of money made from selling the meat are the following:

$$\begin{aligned} b + p &= 89 \\ 2.39b + 1.39p &= 172.71 \end{aligned}$$

4. The answer is B.

In the previous example, we only had to write the system of linear equations, but notice that the question indicated that we were writing these equations in order to hypothetically solve for the number of pounds of beef that were sold.

How would we actually solve one of these systems of linear equations (find the values that satisfy both of the equations in the system)? In general, the process of solving systems of equations having two variables is performed by reducing the system of equations to a single equation with only one variable (and temporarily eliminating the other variable). Once we know the value of one of the variables, we can substitute the value back into either of the original equations (remember that the solution values will satisfy both of the original equations) and solve for the value of the remaining variable.

There are several ways to isolate the single variable and reduce the system to a single equation (namely, Elimination and Substitution), but the majority of the system of linear equations questions on the test (especially those in which the equations are written in Standard Form) can and should be solved using the **Elimination** method, which is based on the procedure of adding and subtracting entire linear equations; this technique can be leveraged to eliminate one of the variables. Before we outline the method, let's first understand why we can combine linear equations with addition or subtraction.

Combining Equations

Equations can be added together or subtracted from each other because both sides of an equation are equal. If we add the left sides of two equations and we add the right sides of two equations, the resulting equation will still be valid.

$$\begin{array}{rr} & a+b = 4 \\ + & c+d = 3 \\ \hline & (a+b)+(c+d) = 7 \end{array}$$

Since $a + b$ is 4 and $c + d$ is 3, then we can substitute 4 for $a + b$ and 3 for $c + d$ and verify that the result is 7.

$$\begin{aligned} (a+b)+(c+d) &= 7 \\ 4 \quad + \quad 3 \quad &= 7 \\ 7 \quad &= 7 \end{aligned}$$

Note also that we can multiply both sides of any equation by a constant and the resulting equation is still the same equation as before, just with scaled coefficients and constant. Multiplying equations by constants is sometimes a necessary step in solving by elimination.

In order to solve linear equations by Elimination, we need to build off of the notion that we can combine linear equations by addition and subtraction. **Our goal is to add the equations in such a way that one of the variables will be eliminated.**

Solving Systems of Linear Equations by Elimination

Consider the following system of linear equations.

$$\begin{aligned} x + 2y &= 5 \\ x - 2y &= 3 \end{aligned}$$

Notice that the coefficients of y in the two equations are negatives of each other (the coefficient is 2 in the top equation and -2 in the bottom equation). If we add the two equations, then the y-terms will cancel each other out (their coefficients will sum to 0), eliminating the y-variables and leaving us with a single equation with only one variable remaining (the x-variable). We can use this new equation to solve for x.

$$\begin{array}{rr} & x + 2y = 5 \\ + & x - 2y = 3 \\ \hline & 2x + 0y = 8 \end{array}$$

$$\Downarrow$$

$$\begin{aligned} 2x &= 8 \\ x &= 4 \end{aligned}$$

Now that we have the value of x, we can plug it into either of the original two equations and solve for y. Let's use the first equation, $x + 2y = 5$.

$$\begin{aligned} x + 2y &= 5 \\ 4 + 2y &= 5 \\ 2y &= 1 \\ y &= \frac{1}{2} \end{aligned}$$

We sometimes need to multiply one or both equations in a system by a constant so that one of the variables will be canceled out when we add the the two equations to each other.

Consider the following system of linear equations.

$$\begin{aligned} 2x + 3y &= 6 \\ x - y &= 3 \end{aligned}$$

Let's solve for x by eliminating the y-variables. In order to eliminate the y-variable and solve for x, we need the y-coefficients to cancel each other. The easiest way to do that in this case is to multiply the bottom equation by 3 so that the top equation has the term $3y$ and the bottom equation has the term $-3y$. When we add the equations together, the y-terms will cancel.

$$\begin{aligned} 2x + 3y &= 6 \\ 3(x - y) &= 3(3) \\ &\Downarrow \\ 2x + 3y &= 6 \\ 3x - 3y &= 9 \end{aligned}$$

Now, we can add the two equations, and the y-variables will cancel, allowing us to solve for x.

$$\begin{array}{rr} & 2x + 3y = 6 \\ + & 3x - 3y = 9 \\ \hline & 5x + 0y = 15 \\ & \Downarrow \\ & 5x = 15 \\ & x = 3 \end{array}$$

We could have also eliminated the y terms by multiplying the top equation by $\frac{1}{3}$, but we chose the easier route of working with whole numbers instead of fractions. When it comes to systems of equations, working with whole-number coefficients is almost always the better option.

We could also have solved directly for y by multiplying both sides of the second equation by -2 in order for the x-coefficients to cancel when the equations are added, thus using the elimination strategy in order to solve for each variable separately instead of solving for one using elimination and then using substitution to find the other as above.

In the following example, we will find the solution of a system using Elimination, but we will have to multiply *both* of the equations by constants in order to eliminate a variable. As a generalized procedure that will allow you to eliminate a variable, multiply the first equation by the coefficient of that variable in the second equation, and multiply the second equation by the coefficient of that variable in the first equation; this will result in that variable having the same coefficient in both equations, allowing that variable to be eliminated (if one coefficient is a and the other is b, you'll end up with coefficients ab and ba, which have the same value because multiplication is commutative). If both of the coefficients have the same sign, multiply one of the equations by -1 as well before adding the two equations so the variable terms will cancel each other when summed.

Example 4.4-2

2 1600.io 4.4

$$\begin{aligned} -5x + 2y &= 10 \\ 4x + 5y &= 25 \end{aligned}$$

If (x, y) is the solution to the system of equations above, what is the value of x ?

Solution

1. Since we want to solve for x, we need to eliminate the y-terms. To do that while avoiding fractions as much as possible, we need to multiply both equations by constants that will cause the y-coefficients to be opposites that will cancel when we add the two equations. Multiply the **top** equation by 5 (which is the y-coefficient in the **bottom** equation) and multiply the **bottom** equation by 2 (which is the y-coefficient in the **top** equation) so that the y-coefficients will both be 10.

$$\begin{aligned} 5(-5x + 2y) &= 5(10) \\ 2(4x + 5y) &= 2(25) \end{aligned}$$

$$\Downarrow$$

$$\begin{aligned} -25x + 10y &= 50 \\ 8x + 10y &= 50 \end{aligned}$$

2. We need to eliminate the y-variables, so multiply the top equation by -1 so the coefficients of the y-terms will cancel when they are added (we could have multiplied the bottom equation by -1, but we notice that if we use the top equation, the sign of the resulting x term will be positive, which is always preferred).

$$\begin{aligned} -1(-25x + 10y) &= -1(50) \\ 8x + 10y &= 50 \end{aligned}$$

$$\Downarrow$$

$$\begin{aligned} 25x - 10y &= -50 \\ 8x + 10y &= 50 \end{aligned}$$

3. Combine the equations and solve for x.

$$\begin{array}{r} 25x - 10y = -50 \\ + \quad 8x + 10y = 50 \\ \hline 33x + 0y = 0 \end{array}$$

$$\Downarrow$$

$$\begin{aligned} 33x &= 0 \\ x &= 0 \end{aligned}$$

4. The answer is 0.

Notes

Steps 1 and 2 could have been combined. If you notice that both of the coefficients of the variable you want to eliminate have the same sign (both positive or both negative), include a factor of -1 when multiplying one of the equations by the coefficient of the variable in the other equation. For example, in this problem, we could have multiplied the top equation by -5 instead of 5 if we kept in mind that we want the y-coefficients to be negatives of each other.

Alternatively, you could forego the multiplication by -1 and simply subtract the equations rather than adding them; just take care that you keep all the signs straight during the operation.

SkillDrill 4.4-1

Directions: Use elimination to find the intersection point that is the solution to the system of equations.

1. $2x + y = 6$
$x + y = 5$

2. $x - 3y = 4$
$x + 6y = 13$

3. $-2x - 3y = 13$
$4x + 9y = -35$

4. $4x + 5y = 2$
$2x - 3y = 1$

Example 4.4-3

3 4.4

A group of 140 people are going to a wedding, where there are 30 tables. Some of the tables seat 4 people each, and the rest seat 6 people each. Assuming all the seats are filled and every person gets to sit in a seat, exactly how many of the tables were 4-person tables?

A) 10

B) 15

C) 20

D) 24

Solution

1. We need to write an equation that represents the total number of tables. Use the letter f to represent the number of 4-person tables and s to represent the number of 6-person tables. Since $f + s$ is the total number of tables, and there are 30 total tables, our first equation is $f + s = 30$.
2. We need to write an equation that represents the total number of seats. Since there are f tables that seat 4 people, the number of seats from those f tables is $4f$. Since there are s tables that seat 6 people, the number of seats from those s tables is $6s$. The total number of seats is $4f + 6s$, and there are 140 seats, so our second equation is $4f + 6s = 140$.
3. The system of equations is the following:

$$\begin{aligned} f + s &= 30 \\ 4f + 6s &= 140 \end{aligned}$$

4. Since we want to solve for f, we should multiply both sides of the top equation by -6 so the y-coefficients will cancel when we combine the equations.

$$\begin{aligned} -6(f+s) &= -6(30) \\ 4f+6s &= 140 \\ &\Downarrow \\ -6f-6s &= -180 \\ 4f+6s &= 140 \end{aligned}$$

5. Combine the equations and solve for f.

$$\begin{array}{r} -\,6f - 6s = -180 \\ +\quad 4f + 6s = 140 \\ \hline -2f + 0s = -40 \\ \Downarrow \\ -2f = -40 \\ f = 20 \end{array}$$

6. The answer is C.

Notes

Take note of the fact that we did not have to find the value of s at any point in order to find the value of f. The true advantage of elimination over other methods of solving systems of equations is that there is no confusion about which variable you are actually solving for first. The answer is in the name "elimination": you eliminate the variable you don't want to solve for, leaving only the variable whose value you're seeking.

Occasionally, there is a test problem that asks you for the value of an expression rather than for the value of one variable. Most of the time, the expression has terms with both x- and y-variables. In almost all of these cases, it is much simpler to find the value of that expression directly rather than to first find the value of one variable, substitute, find the value of the other variable, then construct the desired expression and evaluate it. An example will illustrate this clearly. Usually, simply adding or subtracting the equations from each other will make the coefficients in the combined equation match up perfectly with the expression that you are asked to find.

For a deeper dive into the subject of problems for which there is a rapid, direct route to a solution that bypasses the more general approach, see the "Wormholes" chapter.

Example 4.4-4

4 1600.io 4.4

$$\begin{aligned} 2x+3y &= 100 \\ 4x+3y &= 360 \end{aligned}$$

In the system of equations above, what is the value of $6x+6y$?

Solution 1

1. Notice that you are asked to solve for $6x + 6y$. You could solve this by finding the values of x and y independently and then plugging those values into the expression $6x + 6y$ and evaluating it, but notice that if we just combine the equations by adding them, the two x-terms will add up to $6x$, and the two y-terms will add up to $6y$.

$$\begin{array}{rr} & 2x + 3y = 100 \\ + & 4x + 3y = 360 \\ \hline & 6x + 6y = 460 \end{array}$$

2. The answer is 460.

Solution 2

1. We can choose to solve for either variable first if we decide to solve for them individually. Based on the starting conditions of the problem, it makes sense to eliminate y first because its coefficients already match. Multiply the top equation by -1, and then add the two equations in order to eliminate y and solve for x.

$$-1(2x + 3y) = -1(100)$$
$$4x + 3y = 360$$
$$\Downarrow$$
$$\begin{array}{rr} & -2x - 3y = -100 \\ + & 4x + 3y = 360 \\ \hline & 2x + 0y = 260 \end{array}$$
$$\Downarrow$$
$$2x = 260$$
$$x = 130$$

2. Now that we know the value of x, we can plug the value into either equation in order to solve for y. Let's use the top equation (because the coefficients are smaller).

$$2x + 3y = 100$$
$$2(130) + 3y = 100$$
$$260 + 3y = 100$$
$$3y = -160$$
$$y = \frac{-160}{3}$$

3. Now that we have the values of both x and y, we can substitute the values into the expression $6x + 6y$ in order to find its value. Plug in 130 for x and $\frac{-160}{3}$ for y.

$$6x + 6y = 6(130) + 6\left(\frac{-160}{3}\right)$$
$$6x + 6y = 780 - 320$$
$$6x + 6y = 460$$

4. The answer is 460.

Notes

Solution 2 is not horribly long, but it involves messy fractions and large numbers that can be completely avoided if you know the test well and look for the shortcut provided by Solution 1.

Section 4.4 Suggested Problems from Real Tests

🏆 Hall of Fame

- Test 1-NC-9
- Test 1-NC-18
- Test 1-C-19
- Test 2-NC-2
- Test 3-NC-6 🏆
- Test 3-NC-19
- Test 3-C-30 🏆
- Test 4-NC-19
- Test 5-C-6
- Test 5-C-12
- Test 7-NC-3
- Test 7-C-11
- Test 7-C-33 🏆
- Test 8-NC-10
- Test 8-NC-18
- Test 8-C-6
- Test 8-C-27 🏆
- Test 9-NC-1
- Test 9-C-17
- Test 10-NC-19
- Test 10-C-25
- Apr 2017-NC-1
- Apr 2017-NC-7
- Apr 2017-C-36
- May 2017-NC-17
- May 2017-C-9
- Mar 2018-C-26
- Apr 2018-NC-1
- Apr 2018-C-14
- May 2018-C-11
- Mar 2019-NC-18 🏆
- Apr 2019-NC-18
- Apr 2019-C-33
- May 2019 (US)-NC-17
- May 2019 (Int)-NC-2
- May 2019 (Int)-NC-19
- May 2019 (Int)-C-11
- Oct 2019-NC-4
- Oct 2019-C-24
- Mar 2020-C-18
- Mar 2020-C-33
- Mar 2020-C-36 🏆
- Oct 2020-C-38 🏆
- Mar 2021-NC-19
- Mar 2021-C-18

Section 4.4 Practice Problems

1 1600.io 4.4

The Hartford ferry charges $9 for an adult and $1 for a child to ride one way. During a certain 8-hour shift, a ticket booth attendant collects $1,051 from 155 passengers. Which of the following systems of equations could be used to determine the number of adult riders, A, and the number of child riders, C, during this 8-hour shift?

A) $9A + C = 8(1{,}051)$
$A + C = 8(155)$

B) $8(9A) + 8(C) = 1{,}051$
$8(A + C) = 155$

C) $9A + C = 155$
$A + C = 1{,}051$

D) $9A + C = 1{,}051$
$A + C = 155$

2 1600.io 4.4

$$5m + 3p = 24$$
$$m + p = 8$$

If (m_1, p_1) is the solution to the system of equations above, what is the value of p_1 ?

3 1600.io 4.4

The sum of two different numbers x and y is 80, and the difference when the smaller number is subtracted from the larger number is 40. What is the value of xy ?

A) 120

B) 320

C) 1,200

D) 3,200

4 1600.io 4.4

$$5x + 6y = 6$$
$$3x + 8y = 9$$

If (x, y) is the solution to the system of equations above, what is the value of $2x - 2y$?

A) -15

B) -3

C) 3

D) 15

5 1600.io 4.4

$$x + y = 13$$
$$x - 3y = -3$$

According to the system of equations above, what is the value of x ?

A) 2

B) 4

C) 9

D) 11

4.5 Number of Solutions to a System of Linear Equations Part II

We already know that the number of solutions to a system of linear equations can be thought of in terms of the slopes and y-intercepts of the lines in the system and the number of intersections of those lines: nonparallel lines will intersect once (one solution), parallel lines (lines with the same slope but different y-intercepts) will never intersect (0 solutions), and coincident lines (two equal lines with the same slope and y-intercept) share every point in common (infinitely many solutions). However, it is not so readily apparent (based on slope and y-intercept) how many solutions there will be to a system of linear equations in Standard Form because the slope and y-intercept do not exist as individual coefficients or constants in the equations.

Remember that lines in Standard Form can be easily rewritten in Slope-Intercept Form (though this is not a necessary step in the problems we are going to discuss), and that the slope and y-intercept are directly related to the constants and coefficients of the line in Standard Form.

For example, let's look at the symbolic Standard Form line $A_1x + B_1y = C_1$ and convert it into Slope-Intercept Form.

$$A_1x + B_1y = C_1$$
$$B_1y = -A_1x + C_1$$
$$y = \frac{-A_1}{B_1}x + \frac{C_1}{B_1}$$

The slope of a Standard Form line $A_1x + B_1y = C_1$ is equal to $\frac{-A_1}{B_1}$, and the y-intercept is equal to $\frac{C_1}{B_1}$.

Let's form a system of linear equations by also considering another line in Standard Form, $A_2x + B_2y = C_2$, whose slope is equal to $\frac{-A_2}{B_2}$ and whose y-intercept is equal to $\frac{C_2}{B_2}$ (based on the same conversion that was done for the line $A_1x + B_1y = C_1$).

We can use what we know about the number of solutions based on slope and y-intercept to draw some conclusions about how we can handle questions about the number of solutions to a system of linear equations in Standard Form.

We know that any two lines with **different slopes** (lines that are not parallel) will have **one solution**. In order for the system of equations consisting of the two lines mentioned above ($A_1x + B_1y = C_1$ and $A_2x + B_2y = C_2$) to have one solution, the ratios $\frac{-A_1}{B_1}$ and $\frac{-A_2}{B_2}$, which represent the slopes of their respective lines, must NOT be equal (we can actually omit the negative signs because they will cancel out anyway when the two ratios are set equal).

$$\frac{-A_1}{B_1} \neq \frac{-A_2}{B_2}$$
$$\frac{A_1}{B_1} \neq \frac{A_2}{B_2}$$

As long as the ratio of the x-coefficient to the y-coefficient is different for both equations, the system will have one solution (remember that if these ratios are not the same, it indicates that the slopes are not the same, despite the missing negative signs). For example, given the lines $4x + 2y = 5$ and $-3x + y = -5$, we can tell that the system will have one solution because the ratio of x- to y-coefficients is not the same for both equations: $\frac{4}{2} \neq \frac{-3}{1}$.

In order for the system to have **no solutions**, the lines would have to have the **same slope and different y-intercepts**, so we would know that **the ratio of x- to y-coefficients would have to be the same for both equations** (note again that the negative signs can be omitted): $\frac{A_1}{B_1} = \frac{A_2}{B_2}$ We also know that the y-intercepts of the lines must not be equal, and because the y-intercepts are equal to the ratio of the constant terms to the y-coefficients, these ratios must be different for both equations $\left(\frac{C_1}{B_1} \neq \frac{C_2}{B_2}\right)$.

For example, given the lines $4x + 2y = 5$ and $10x + 5y = 3$, we can tell that the system will have no solutions because not only is the ratio of the x- to y-coefficients the same for both equations $\left(\frac{4}{2} = \frac{10}{5} \text{ which reduces to } 2 = 2\right)$, which means that the slopes of these two line are the same, but the ratio of constant terms to y-coefficients is also not the same $\left(\frac{5}{2} \neq \frac{3}{5}\right)$, which means that the y-intercepts of these two lines are different. This means the lines are parallel and therefore have no intersection point, and thus there is no solution to the system.

Finally, in order for the system to have **infinitely many solutions**, the lines have to be **exactly the same, with the same slope and y-intercept**. Even if the coefficients in the equations are different, the coefficients and constant in one equation may have the same *ratio* among them as do the coefficients and constant in the other equation.

If the ratios of the x- and y-coefficients are equal $\left(\frac{A_1}{B_1} = \frac{A_2}{B_2}\right)$, which means that the slopes of these two lines are the same for both equations, and the ratios of the constant terms and y-coefficients are equal $\left(\frac{C_1}{B_1} = \frac{C_2}{B_2}\right)$, which means that the y-intercepts of these two lines are equal for both equations, then the lines are the same, and the system will have infinitely many solutions.

For example, given the lines $4x + 2y = 5$ and $10x + 5y = \frac{25}{2}$, we can see that the system will have infinitely many solutions because not only is the ratio of the x- to y-coefficients is the same for both equations $\left(\frac{4}{2} = \frac{10}{5}\right)$, but the ratio of the constant terms to the y-coefficients is also the same for both equations.

$$\frac{5}{2} = \frac{\frac{25}{2}}{5}$$

$$\frac{5}{2} = \frac{25}{10}$$

$$\frac{5}{2} = \frac{5}{2}$$

Systems of Linear Equations with One Solution

There will be **one solution** to a system of equations in Standard Form when the **ratio of x- to y-coefficients is different**.

For example, the lines $2x + 3y = 4$ and $-4x + 6y = 4$ will have one solution because the ratio of x- to y-coefficients for the first equation is $\frac{2}{3}$, but the ratio of x- to y-coefficients for the second equation is $\frac{-4}{6}$, which reduces to $\frac{-2}{3}$, which is not equal to $\frac{2}{3}$.

The y-intercept doesn't matter in this situation.

Systems of Linear Equations with No Solution

There will be **no solution** to a system of linear equations in Standard Form when the **ratio of x- to y-coefficients is the same for both equations, but the ratio of constant terms to y-coefficients is different for both equations**.

For example, the lines $2x + 3y = 4$ and $2x + 3y = 6$ will be parallel because the left sides of the equations are the same (so the ratio of the x- to y-coefficients is the same) but the right sides are different. If you want, you can convert the lines into Slope-Intercept Form to verify that they are parallel (they will have the same slope and different y-intercepts).

$$2x + 3y = 4 \quad \Rightarrow \quad y = \frac{-2}{3}x + 2$$

$$2x + 3y = 6 \quad \Rightarrow \quad y = \frac{-2}{3}x + 3$$

Similarly, the lines $2x+3y = 4$ and $4x+6y = 10$ are also parallel lines (with the same slope), but how is this possible? Though the coefficients might be different in the two equations, the *ratio* of the coefficients is the same.

For the lines $2x + 3y = 4$ and $4x + 6y = 10$, the ratio of x-coefficient to y-coefficient for the first equation is 2 to 3 or $\frac{2}{3}$. The ratio of the x-coefficient to y-coefficient for the second equation is 4 to 6 or $\frac{4}{6}$, which reduces to $\frac{2}{3}$. Since these ratios match, these lines have the same slope.

Notice also that the ratio of constant to y-coefficient is 4 to 3 for the first equation but 6 to 2, which reduces to 3 to 1, for the second equation. These ratios aren't equal, so the y-intercepts are different.

Systems of Linear Equations with Infinite Solutions

There will be **infinite solutions** to a system of linear equations in Standard Form when **the ratio among the two variables' coefficients and the constant are the same in both equations**. In the simplest case, these values will simply be equal in both equations, but things will rarely be that simple, and you will usually have to test the ratios among the corresponding components of the equations to see if they are all identical.

For example, the lines $2x + 3y = 4$ and $2x + 3y = 4$ are very obviously the same line. However, the lines $2x + 3y = 4$ and $4x + 6y = 8$ are also the same line. In this example, it's easy to realize this; all the terms in the second equation are just double those in the first, so you can multiply both sides of the first equation by 2 to get the line $4x + 6y = 8$, which is exactly the same as the second equation.

In the general case, when you can't tell whether one equation is a scaled version of the other because the numbers are larger and/or are in non-integral ratios, you can scale one or both equations so as to match one of the terms, and then you can see if the other two terms also end up matching, which would indicate that the equations have the same solution set and thus that there are infinite solutions to the system.

You *could* choose any pair of corresponding coefficients, or the constants, and form a fraction that represents the scaling factor of one of the equations relative to the other, and then multiply the other two values by this scaling factor to see if the results match the corresponding elements in the other equation, which would indicate that the equations are the same. However, in some situations you will have to deal with fractions if you use this method, and fractions can make things messy. Alternatively, you can use one of the techniques used when performing elimination: you can pick any pair of corresponding coefficients, or the constants, and perform the complementary multiplication operations so the selected terms end up with exactly the same coefficients (be sure to preserve the signs of the coefficients when performing this procedure). Then, you can inspect the other terms to see if they, too, are now identical in both equations as a result of this equalizing procedure; if they are, the equations are the same and there are infinite solutions.

We need to determine if the ratio among these three components is the same in both equations; in the first equation in the example above, it's $2:3:4$, and in the second, it's $4:6:8$. Though the answer might be obvious here due to the simple numbers, let's work through it more rigorously to show the procedure so you're ready if you're faced with a less-pleasant situation.

Let's select the x-coefficients as our reference. We're going to perform what we call complementary intermultiplication, which is an appropriate term because we said so. We'll multiply the first equation by the x-coefficient of the second equation (which is 4), and we'll multiply the second equation by the x-coefficient of the first equation (which is 2). This will, of course, result in matching coefficients for the x terms in both equations (because $ab = ba$), but we need to determine if the *other* two terms end up matching as well.

$$\begin{gathered}4(2x + 3y = 4)\\2(4x + 6y = 8)\\\Downarrow\\8x + 12y = 16\\8x + 12y = 16\end{gathered}$$

All the terms match, which means the equations represent the exact same relationship (the lines represented by the equations are coincident), and therefore there are infinite solutions to the system.

You can also realize that **the ratios of the x- to y-coefficients and the ratios of the constant terms to the y-coefficients is the same for the two equations**.

The ratio of the x- to y-coefficients for the first equation is $\frac{2}{3}$. The ratio of the x- to y-coefficients for the second equation is $\frac{4}{6}$, which reduces to $\frac{2}{3}$, so this ratio matches in the two equations, indicating that the slopes are the same. Similarly, the ratio of the constant terms to y-coefficient is $\frac{4}{2}$, which reduces to 2, and in the second equation, this ratio is equal to $\frac{8}{4}$, which also reduces to 2; this means the y-intercepts are the same as well. Since both of these ratios are the same for both equations, the lines are the same (they have the same slope and y-intercept) and will therefore have infinitely many solutions.

Finally, yet another option to check for infinitely many solutions is to convert both equations to Slope-Intercept Form and see if the slope and y-intercept are the same.

If you are told that a system has no solutions, but you need to determine the value of one of the coefficients, you can use the fact that the ratio of x- to y-coefficients must be equal to set up a simple proportion that will allow you to solve for the unknown quantity. For example, if you are given the following system with no solutions,

$$\begin{aligned}12x - 4y &= 2\\ax + 2y &= 7\end{aligned}$$

you can set up the following proportion which leverages the known ratio between the y-coefficients (we like to place the unknown in the numerator of the fraction on the left to facilitate solving):

$$\frac{a}{12} = \frac{2}{-4}$$

Solving for a,

$$a = 12\left(\frac{2}{-4}\right)$$

$$a = 2(-3)$$

$$a = -6$$

The use of a proportion keeps things organized, and more generally, writing down your work instead of relying on mental math is the better habit to maintain.

If you are told that a system has infinite solutions, not only will the x- and y-coefficients be in the same ratio, but the constants will be, too, so you can use the same procedure described above to determine the value of an unknown constant. If we have a system with infinite solutions such as this,

$$\begin{aligned} 4x + 6y &= 12 \\ ax - 3y &= b \end{aligned}$$

you can set up a proportion to find either a or b by making use of the known ratio of the y-coefficients. We showed the procedure for finding a, the x-coefficient, above; here's what the proportion involving b would look like:

$$\frac{b}{12} = \frac{-3}{6}$$

Solving for b,

$$b = 12\left(\frac{-3}{6}\right)$$

$$b = -3(2)$$

$$b = -6$$

Generally, you will use a known ratio (whether of a pair of coefficients or of the constants) to set up a proportion to solve for an unknown value (coefficient or constant).

SkillDrill 4.5-1

Directions: All of the following systems of linear equations have no solution. Find the value of a.

1. $8x - 2y = 1$
$ax + y = 3$

2. $5x - 15y = 45$
$ax - 3y = 8$

3. $2x + 3y = 6$
$ax - y = 1$

4. $5x + 2y = 10$
$x + ay = 4$

Directions: All of the following systems of linear equations have infinite solutions. Find the values of a and b.

5. $3x - 9y = 12$
$ax + by = 4$

6. $2x + 3y = 6$
$ax - 3y = b$

7. $x + 5y = 8$
$ax - by = 1$

8. $-6x + 5y = 6$
$4ax + 2y = b$

Example 4.5-1

1 1600.io 4.5

$$-5x + y = 12$$
$$ax + 3y = 32$$

In the system of equations above, a is a constant. For which of the following values of a does the system have no solution?

A) -15

B) -10

C) -5

D) 0

Solution 1

1. In order for a system of linear equations to have no solution, the two lines must be parallel but have different y-intercepts. When the lines are in Standard Form, we can tell that they are parallel when the ratio of x- to y-coefficients is the same for both equations.

2. The ratio of x- to y-coefficients for the first equation is -5 to 1, or $\frac{-5}{1}$. The ratio of x- to y-coefficients for the second equation is a to 3, or $\frac{a}{3}$. Create a proportion by setting the two fractions equal to each other and solve for a. Remember to cross-multiply.

$$\frac{-5}{1} = \frac{a}{3}$$
$$-5(3) = a$$
$$-15 = a$$

3. The answer is A.

Solution 2

1. In order for a system of linear equations to have no solution, the two lines must be parallel but have different y-intercepts. When the lines are in Standard Form, we can take the approach of making x- and y-coefficients match in both equations. We just have to verify the constants on the right side of the equations are different from each other (technically, we know this will be true because the problem is set up so that there will be no solutions, and thus the constant terms will definitely not match).

2. We do not know how to make the x-coefficients match, since we do not know the value of a, but we can make the y-coefficients match by multiplying both sides of the first equation by 3. This is the first step in trying to get the left sides of the equations to look identical.

$$3(-5x + y) = 3(12)$$
$$ax + 3y = 32$$
$$\Downarrow$$
$$-15x + 3y = 36$$
$$ax + 3y = 32$$

3. The y-coefficients match, and the constants on the right side of the equation are different (again, this should be true given that the problem is set up to ensure there will be no solutions to the system of equations for a certain value of a), so there will be no solution when the x-coefficients are the same (making the lines parallel), which means that the value of a is -15.

4. The answer is A.

Solution 3

1. In order for a system of linear equations to have no solution, the two lines must be parallel but have different y-intercepts. We can convert both lines into Slope-Intercept Form and then check what value of of a would make the slopes equal.

2. Convert the first equation into Slope-Intercept Form.

$$-5x + y = 12$$
$$y = 5x + 12$$

3. Convert the second equation into Slope-Intercept Form.

$$ax + 3y = 32$$
$$3y = -ax + 32$$
$$y = \frac{-a}{3}x + \frac{32}{2}$$

4. The y-intercepts do not match, which is expected since the problem is about two lines that never intersect. In order for the lines to be parallel, the slopes must be the same. Set the two slopes equal to each other to find what value of a would make the slopes equal. The slope of the first line is 5. The slope of the second line is $\frac{-a}{3}$.

$$5 = \frac{-a}{3}$$
$$15 = -a$$
$$-15 = a$$

5. The answer is A.

Example 4.5-2

2 4.5

$$ax + by = 4$$
$$3x + 6y = 60$$

In the system of equations above, a and b are constants.

If the system has infinitely many solutions, what is the value of $\frac{a}{b}$?

Solution 1

1. In order for a system of linear equations to have infinitely many solutions, the two lines must be exactly the same. The shortest way to handle this one is to remember that if the lines are in Standard Form, then the ratio of x and y-coefficients will be the same in both equations, so $\frac{a}{b} = \frac{3}{6}$, or (reducing the fraction) $\frac{a}{b} = \frac{1}{2}$.

2. The answer is $\frac{1}{2}$.

Solution 2

1. Since the problem states that there are infinitely many solutions, the two equations must be equal. If you are confused by the fact that the constants 4 and 60 don't match, then you can multiply both sides of the first equation by 15 (because $\frac{60}{4} = 15$) so that the 4 will match the 60 on the right side of the equation.

$$\begin{aligned} 15(ax + by) &= 15(4) \\ 3x + 6y &= 60 \end{aligned}$$

$$\Downarrow$$

$$\begin{aligned} 15ax + 15by &= 60 \\ 3x + 6y &= 60 \end{aligned}$$

2. Now that the constants match, the next step is setting the ratios of x and y-coefficients for both equations equal to each other because the equations must be the same. Set $\frac{15a}{15b}$ equal to $\frac{3}{6}$, and reduce the fractions to solve for $\frac{a}{b}$.

$$\frac{\cancel{15}a}{\cancel{15}b} = \frac{3}{6}$$

$$\frac{a}{b} = \frac{1}{2}$$

3. The answer is $\frac{1}{2}$.

Notes

In Solution 1, we ignored the fact that the constant terms didn't match in the original forms of the equation. If the problem states that there are infinitely many solutions, you don't have to worry about making the constant terms match first before you start comparing the ratios of the coefficients.

The problem can also be done by converting both equations into Slope-Intercept Form and setting the two slopes equal to each other, then solving for $\frac{a}{b}$.

Section 4.5 Suggested Problems from Real Tests

- Test 2-NC-20
- Test 3-NC-9
- Test 6-C-11
- Test 8-C-33
- Test 9-C-36
- Test 10-NC-15
- Apr 2017-C-23
- May 2019 (US)-C-29
- Oct 2020-NC-19
- Oct 2020-C-21
- Mar 2021-C-28

Section 4.5 Practice Problems

1 4.5

In the system of equations below, a and c are constants.

$$\frac{1}{3}x + \frac{1}{5}y = \frac{1}{15}$$
$$ax + y = c$$

If the system of equations has an infinite number of solutions (x, y), what is the value of a?

A) $-\frac{5}{3}$

B) $-\frac{1}{3}$

C) $\frac{1}{3}$

D) $\frac{5}{3}$

2 4.5

In the xy-plane, the equations $x + 5y = 8$ and $4x + 20y = c$ represent the same line for some constant c. What is the value of c?

3 4.5

$$kx - 2y = 3$$
$$7x + 4y = 4$$

In the system of equations above, k is a constant and x and y are variables. For what value of k will the system of equations have no solution?

A) $-\frac{21}{4}$

B) $-\frac{7}{2}$

C) $\frac{7}{2}$

D) $\frac{21}{4}$

4 4.5

$$3x + 2y = 10$$
$$6x + cy = 21$$

In the system of equations above, c is a constant. For what value of c will there be no solution (x, y) to the system of equations?

A) 2

B) 3

C) 4

D) 6

CHAPTER 4 RECAP

- Solutions to systems of linear equations can be thought of as points where the graphs of the lines intersect. At these intersection points, both lines have the same x- and y- values.
- A system of linear equations has **one solution when the two lines are *not* parallel and cross at one point**.
- A system of linear equations has **no solutions when the lines are parallel**, because the lines do not intersect. If the equations of the lines are in Slope-Intercept Form, the equations of parallel lines will display the ***same* slope and *different* y-intercepts**.
- A system of linear equations has **infinite solutions when the two lines are exactly the same**. The **slopes and y-intercepts must be the same**.
- Set expressions for y equal to each other (substitution) to find the solution point when both equations are already in Slope-Intercept Form.
- Use substitution when the equations are in different forms, particularly when one variable is already solved for in terms of the other. Substitution is RARELY preferable to elimination for problems on the test.
- Use elimination to solve most systems of linear equations, particularly when both equations are in Standard Form. Multiply one or both equations by numbers that will cause the coefficients of the variable you want to eliminate to cancel out when the equations are combined through addition or subtraction.
- Use combination without eliminating either variable if possible when the problem asks for an expression involving terms with both variables.
- A system of linear equations in Standard Form has **one solution when the ratio of x- to y-coefficients is NOT the same for both equations** (this correlates to the fact that the slopes are not the same).
- A system of linear equations in Standard Form has **no solutions when the ratio of x- to y-coefficients is the same for both equations** (which correlates to the slopes being the same) **but the ratio of constant terms to y-coefficients is NOT the same for both equations** (which correlates to the y-intercepts being different).
- A system of linear equations in Standard Form has **infinitely many solutions when the ratio of x- to y-coefficients is the same for both equations** (which correlates to the slopes being the same) **and when the ratio of constant terms to y-coefficients is the same for both equations** (which correlates to the y-intercepts being the same).
- When asked to find unknown coefficient values in systems of linear equations in Standard Form that have no solutions or infinitely many solutions, use proportions of the ratios of the coefficients and constants to solve for unknown values.

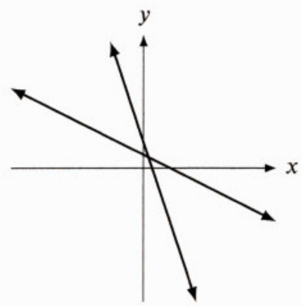
  Alternatively, multiply one or both equations by scale factors so that the any known coefficients or constants in the same position can be matched between the two equations.
- When you are asked about the number of solutions to a system of linear equations in Standard Form, you can always convert Standard Form linear equations into Slope-Intercept Form to make it easier to match the slope and y-intercept.

Additional Problems

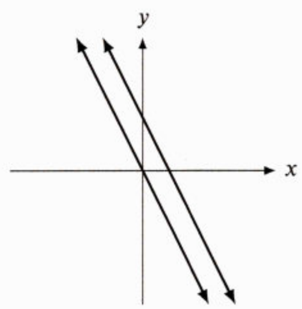
1 4.1

Which of the following is a graph of a system of equations with no solution?

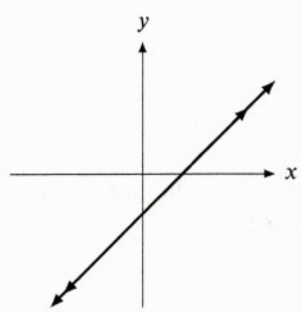
A)



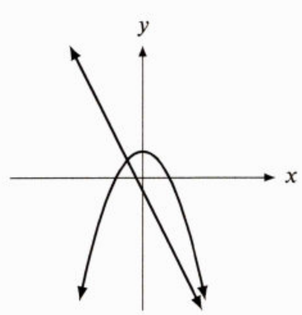
B)



C)



D)



2 4.4

An aquarium sells two types of tickets. The standard ticket, for admission only, costs \$20. The premium ticket, which includes admission and a swim with angelfish, costs \$40. One Tuesday, the aquarium sold a total of 400 tickets and collected a total of \$9,000 from ticket sales. Which of the following systems of equations can be used to find the number of standard tickets, s, and premium tickets, p, sold on that Tuesday?

A) $$\begin{aligned} s + p &= 400 \\ 20s + 40p &= 9{,}000 \end{aligned}$$

B) $$\begin{aligned} s + p &= 400 \\ 40s + 20p &= 9{,}000 \end{aligned}$$

C) $$\begin{aligned} 20s + 40p &= 400 \\ s + p &= 9{,}000 \end{aligned}$$

D) $$\begin{aligned} 40s + 20p &= 400 \\ s + p &= 9{,}000 \end{aligned}$$

3 4.3

$$ax = 3$$

In the equation above, a is a constant. For which of the following values of a will the equation have no solution?

A) 0

B) 1

C) 3

D) 6

4 1600.io 4.4

$$\frac{1}{3}y = 6$$
$$x - \frac{1}{3}y = 2$$

The system of equations above has solution (x, y). What is the value of x ?

A) 4

B) $\frac{17}{3}$

C) 8

D) 20

5 1600.io 4.4

Marcus sells only hats and scarves on his website. Hats sell for \$15 each, and scarves sell for \$10 each. If Marcus sold 30 pieces of clothing and his sales totaled \$400, how many scarves did Marcus sell?

6 1600.io 4.3

$$12x - 4x(c - 2) = 3x$$

In the equation above, c is a constant. If the equation has infinitely many solutions, what is the value of c ?

A) $\frac{11}{4}$

B) $\frac{13}{4}$

C) $\frac{15}{4}$

D) $\frac{17}{4}$

7 1600.io 4.4

$$x = y - 5$$
$$\frac{x}{2} + 4y = 2$$

Which ordered pair (x, y) satisfies the system of equations shown above?

A) $(-5, 0)$

B) $(-4, 1)$

C) $(4, 0)$

D) $(5, 10)$

8 4.2

$$\frac{x}{y} = 4$$
$$3(y+2) = x$$

If (x, y) is the solution of the system of equations above, what is the value of y?

A) 2

B) 6

C) 12

D) 24

9 4.4

$$2x + 5y = 800$$
$$5x + 2y = 1500$$

Based on the system of equations above, what is the value of $7x + 7y$?

10 4.2

$$4(x+y) = 56$$
$$\frac{x}{2} = 5$$

If (x, y) is a solution to the system of equations above, what is the value of y?

A) 16

B) 4

C) −4

D) −16

11 4.4

If $3w + 5t = 21$ and $5w - t = 7$, what is the value of $4w + 2t$?

A) 14

B) 18

C) 24

D) 28

12 1600.io 4.4

At a food truck, each empanada has 40 more calories than each taco. If 1 empanada and 8 tacos have a total of 1,570 calories, how many calories does an empanada have?

13 1600.io 4.4

$$x - 4y = 24$$
$$0.25x + y = 4$$

The solution to the system of equations above is (x, y). What is the value of x ?

A) -4

B) 4

C) 20

D) 40

14 1600.io 4.4

$$-x + y = -4.75$$
$$x + 5y = 11.75$$

If (x, y) satisfies the system of equations above, what is the value of y ?

15 1600.io 4.4

In the xy-plane, the point (p, r) lies on the line with equation $y = 2x + b$, where b is a constant. The point with coordinates $(4p, 3r)$ lies on the line with the equation $y = 3x + b$. If $p \neq 0$, what is the value of $\frac{r}{p}$?

A) 5

B) 2

C) $\frac{1}{2}$

D) $\frac{1}{5}$

16 4.3

The equation $7x - 4 = a(x - b)$, where a and b are constants, has no solutions. Which of the following must be true?

I. $a = 7$

II. $b = 4$

III. $b \neq \frac{4}{7}$

A) None

B) I only

C) I and II only

D) I and III only

17 4.2

In a wedding hall, during a reception, guests are seated at tables. If 6 guests are assigned to each table, 3 additional tables will be needed to seat all of the guests. If 12 guests are assigned to each table, 3 tables will not be used. How many guests will be attending the wedding reception?

18 4.5

$$\frac{2}{3}x - \frac{4}{5}y = 10$$
$$ax - by = 15$$

The system of equations above has no solutions. If a and b are constants, what is the value of $\frac{a}{b}$?

19 4.2

$$\frac{1}{4}y = \frac{21}{20} - \frac{1}{5}x$$
$$8y = 9x$$

In the xy-plane, the lines that correspond to the system of equations above intersect at the point (a, b). What is the value of $\frac{a}{b}$?

20 1600.io 4.4

The score on a history test is obtained by subtracting the number of incorrect answers from 4 times the number of correct answers. If a student answered all 25 questions and obtained a score of 65, how many questions did the student answer correctly?

21 1600.io 4.2

In one month, Dan and John spent a combined 500 hours writing a math book. If Dan spent 50 more hours writing the book than John did, how many hours did John spend writing the book?

22 1600.io 4.4

$$2x + y = a$$
$$-x + 2y = 4$$

In the system of equations above, a is a constant. What is the y-value of the solution to the system in terms of a?

A) $\dfrac{a+8}{5}$

B) $\dfrac{2a-4}{5}$

C) $\dfrac{4a+4}{5}$

D) $\dfrac{-3a-1}{5}$

23 1600.io 4.2

Ms. Morrissey has a bag containing n pieces of chocolate to distribute to the students in her second grade class. If she gives each student 2 pieces of chocolate, she will have 6 pieces of chocolate left over. In order to give each student 3 pieces of chocolate, she will need an additional 16 pieces of chocolate. How many students are in the class?

A) 10

B) 16

C) 19

D) 22

Answer Key

Section 4.1 Practice Problems

1. B
2. B

SkillDrill 4.2-1

1. $(1, 10)$
2. $(-5, 15)$
3. $\left(\frac{7}{2}, \frac{-7}{2}\right)$ or $(3.5, -3.5)$
4. $\left(\frac{-1}{12}, \frac{21}{4}\right)$ or $(-0.08\overline{33}, 5.25)$

Section 4.2 Practice Problems

1. 11
2. C
3. 1
4. C
5. D

SkillDrill 4.3-1

1. 4
2. $\frac{1}{2}$ or 0.5
3. $\frac{5}{3}$ or $1.\overline{66}$
4. 0

SkillDrill 4.3-2

1. $a = 4, b = 7$
2. $a = 6, b = 12$
3. $a = -6, b = -2$
4. $a = \frac{5}{3}, b = 9$

Section 4.3 Practice Problems

1. B
2. $\frac{1}{4}$ or .25
3. C
4. D

SkillDrill 4.4-1

1. $(1, 4)$
2. $(7, 1)$
3. $(-2, -3)$
4. $\left(\frac{1}{2}, 0\right)$ or $(0.5, 0)$

Section 4.4 Practice Problems

1. D
2. 8
3. C
4. B
5. C

SkillDrill 4.5-1

1. -4
2. 1
3. $\frac{-2}{3}$ or $-0.\overline{66}$
4. $\frac{5}{2}$ or 2.5
5. $a = 1, b = -3$
6. $a = -2, b = -6$
7. $a = \frac{1}{8}, b = \frac{-5}{8}$
   or
   $a = 0.125, b = -0.625$
8. $a = \frac{-3}{5}, b = \frac{12}{5}$
   or
   $a = -0.6, b = 2.4$

Section 4.5 Practice Problems

1. D
2. 32
3. B
4. C

Additional Problems

1. B
2. A
3. A
4. C
5. 10
6. D
7. B
8. B
9. 2300
10. B
11. A
12. 210
13. C
14. $\frac{7}{6}$, 1.16, or 1.17
15. A
16. D
17. 72
18. $\frac{5}{6}$ or .833
19. $\frac{8}{9}$, .888, or .889
20. 18
21. 225
22. A
23. D

Linear Inequalities and Absolute Value

5

5.1 Writing and Solving Linear Inequalities

Developing children are known to test their boundaries as a way of learning their limits. How many cookies can you sneak out of the kitchen before your stomach aches? How loud can you be before your dad loses his mind? Though the breaking point is important, the goal of this testing is to figure out what the safe zone is—as long as you eat fewer cookies than your limit, your mission can be considered a success.

When we work with linear expressions, sometimes we need to figure out boundaries, maximums, and minimums instead of just finding one particular value: we are interested in **inequalities** rather than equations. Writing and solving linear inequalities is extremely similar to writing and solving linear equations (in either Slope-Intercept Form or Standard Form). There are just a few key differences, the first being that we use inequality signs rather than equals signs.

Inequality Signs

Linear inequalities are written using one of four signs.

- **Less Than**
  The less-than sign is the following:
  $$<$$
  It signifies that the left side of the inequality is less than the right side. For example, $x < 8$ means that x is less than 8 (note that x cannot be equal to 8).

- **Less Than or Equal To**
  The less-than-or-equal-to sign is the following:
  $$\leq$$
  It signifies that the left side of the inequality is less than or equal to the right side. Another way the relationship might be phrased is that the value of the left side must be "at most" a certain number (or "no greater than" a certain number). The left side can match the right side, but cannot exceed its value. For example, $x \leq 8$ means that x is less than or equal to 8 (the maximum value of x is 8, but x can be any number that is no more than 8, including 8 itself).

- **Greater Than**
  The greater-than sign is the following:
  $$>$$
  It signifies that the left side of the inequality is greater than the right side. For example, $x > 8$ means that x is greater than 8 (note that x cannot be equal to 8).

- **Greater Than or Equal To**
  The greater-than-or-equal-to sign is the following:
  $$\geq$$
  It signifies that the left side of the inequality is greater than or equal to the right side. Another way the relationship might be phrased is that the value of the left side must be "at least" a certain number (or "no less than" a certain number). The left side can match the right side, but cannot be less than the right side. For example, $x \geq 8$ means that x is greater than or equal to 8 (the minimum value of x is 8, but x can be any number that is no less than 8, including 8 itself).

People often get confused which sign to use, but you only need to know two things. First, the small side of the inequality sign (the pointy end) is pointing to the smaller value side of the inequality, and the big side of the inequality (the open end) is on the same side as the greater value side of the inequality. Second, if the sign has a line underneath it, it means the two sides can also be equal.

SkillDrill 5.1-1

Directions: Write an inequality to represent the information given.

1. Bella gets paid \$2 for every backflip she does and \$3 for every front flip she does. In one day, she makes less than \$56 doing backflips and front flips. (Use b for the number of backflips and f for the number of front flips.)

2. In an eating contest, Ferdinand takes 1 minute to eat each hot dog and 3 minutes to eat each cake. It took Ferdinand at least 60 minutes to eat h hot dogs and c cakes.

3. A number x is at least three more than twice a number y.

4. Five less than three times a number n is no more than 35.

5. It takes Paula 8 minutes to drink a small water bottle and 20 minutes to drink a big water bottle. This week, she wants to spend fewer than 112 minutes drinking s small water bottles and b big water bottles.

Example 5.1-1

1 1600.io 5.1

A cookware store is having a sale on knives and pans. During the sale, the cost of each knife is \$20, and the cost of each pan is \$30. Gordon can spend at most \$240 at the store. If Gordon buys k knives and p pans, which of the following must be true?

A) $20k + 30p \leq 240$

B) $20k + 30p \geq 240$

C) $30k + 20p \leq 240$

D) $30k + 20p \geq 240$

Solution

1. Start by writing an expression for the total cost of Gordon's items, but instead of writing an equation, we will form an inequality.
2. Since he is buying k knives for \$20 each, the total cost of k knives is $20k$. Since he is buying p pans for \$30 each, the cost of p pans is $30p$. Therefore, Gordon's total cost for knives and pans is $20k + 30p$.
3. The total amount he spends has an upper limit; he can spend at most \$240, so the total cost of buying k knives and p pans must be less than or equal to \$240. Set the expression $20k + 30p$ less than or equal to 240 in the inequality.

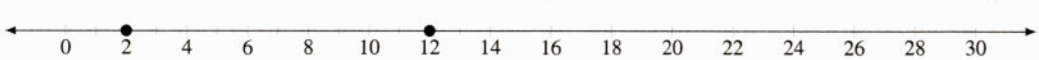
$$20k + 30p \leq 240$$

4. The answer is A.

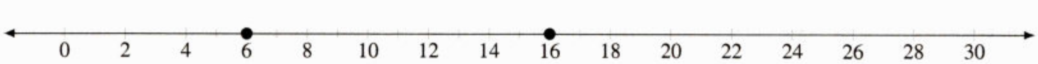
Solving Linear Inequalities

When you have to solve inequalities, most operations can be applied just as you would apply them when solving equations, though there are a few notable exceptions.

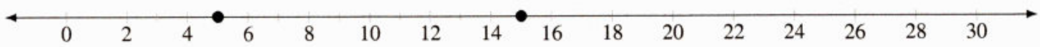
Let's consider the true inequality $2 < 12$ and look at how performing operations on this inequality can result in other true inequalities. We will also graph the points 2 and 12 on a number line and verify that the inequality makes sense.



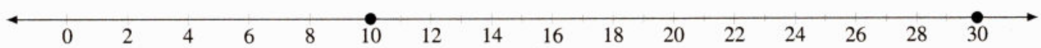
It is clear on the number line that 2 is further to the left than 12 (2 is less than 12). If we add 4 to both sides of the inequality, we have the inequality $6 < 16$, which is still true (the two original points are simply shifted, but maintain their positions relative to each other).



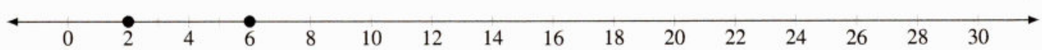
If we subtract 1 from both sides of this inequality, we have the inequality $5 < 15$, which is still true (again, the points have merely shifted).



If we multiply both sides by 2, we have the inequality $10 < 30$, which is still true. Unlike when we added and subtracted, the points have not only shifted, but the distance between the points has grown due to the scale factor of 2 (they are now twice as far apart).

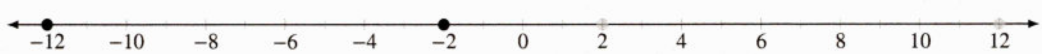


If we divide both sides of this inequality by 5, we have the inequality $2 < 6$, which is still true. Once again, the points have not only shifted, but the distance between them has shrunk (by a scale factor of 5).

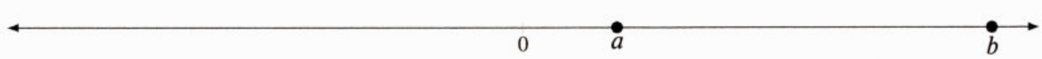


All of the operations we applied so far have produced valid inequalities. But what if we had multiplied both sides of the inequality by 0? The inequality breaks down because $0 \nless 0$ (0 is not less than 0).

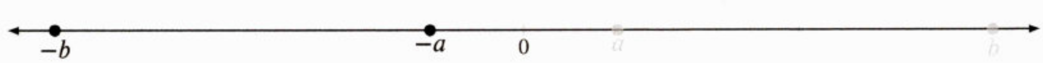
Perhaps even more interesting is what happens when we multiply (or divide) an inequality by a negative number. Let's start back at the inequality $2 < 12$. If we multiply (or divide) both sides of the inequality by -1 (essentially mirroring both points, 2 and 12, about 0), we will see once again that the resulting inequality would not be true: $-2 \nless -12$ (notice that -2 is further to the right than -12).



When multiplying (or dividing) an inequality by a negative number, you need to reverse the inequality sign in order to produce the correct inequality: $-2 > -12$. For a moment, let's consider what happens when we deal with some unknown constants, a and b, which for now, we will assume are both positive (though this is not important). If we know that $a < b$, then we can draw the following number line graph of the two values a and b.



If we multiply both sides of the inequality by -1, we mirror both points, and we can see that $-a > -b$.



We can also prove this algebraically using subtraction instead of multiplication. Starting with the inequality $a < b$, let's subtract a from both sides.

$$a < b$$
$$0 < b - a$$

Now, subtract b from both sides of the inequality.

$$0 < b - a$$
$$-b < -a$$

Once again, we see that $-b$ is less than $-a$. Essentially, we have not "flipped the inequality sign," but reversed the sides of the terms by normal algebraic manipulation. Regardless, it is sufficient to say that **when we multiply (or divide) an inequality by a negative number, we need to reverse the inequality sign**.

Solving inequalities that contain variables is just like solving equations, but **if one of your algebraic steps involves multiplying or dividing both sides of the inequality by a negative number, you must remember to reverse the inequality sign**.

For example, if we are given the inequality $-8x + 3 \leq 59$ and are asked to solve for x, we can start by subtracting 3 from both sides of the equation.

$$-8x + 3 \leq 59$$
$$-8x \leq 56$$

Divide both sides of the inequality by -8. When we do that, we have to reverse the less-than-or-equal-to sign into a greater-than-or-equal-to sign to get the correct inequality: $x \geq -7$.

Any value of x that is greater than or equal to -7 is a solution to this inequality.

SkillDrill 5.1-2

Directions: Solve the given inequality for x (with x on the left side of the inequality) and find the maximum or minimum *integer* value of x that satisfies the inequality.

1. $3x + 2 \leq 8$

2. $5x - 3 \geq 4x + 5$

3. $10(x - 2) < 3x$

4. $3x - 5 < 4x - 7$

5. $3x + 2 \leq 8x$

6. $2x + 10 \geq 4x + 30$

7. $2x - 4 < 3x - 4$

8. $6x + 3 < 4x + 15$

Example 5.1-2

2 1600.io 5.1

Which of the following numbers is NOT a solution of the inequality $6x - 9 \leq 7x - 6$?

A) -1

B) -2

C) -3

D) -4

Solution

1. Group the x-terms on the right side of the inequality by subtracting $6x$ from both sides.

$$\begin{aligned} 6x - 9 &\leq 7x - 6 \\ -9 &\leq x - 6 \end{aligned}$$

2. Group the constant terms on the left side of the inequality by adding 6 to both sides in order to isolate and solve for x.

$$\begin{aligned} -9 &\leq x - 6 \\ -3 &\leq x \end{aligned}$$

3. Any value of x that *IS* greater than or equal to -3 is a solution of the inequality. Since the question asks us to pick the choice that is *NOT* a solution, we need to pick the only choice that is less than -3 (that is, not greater than or equal to -3), which is -4.

4. The answer is D.

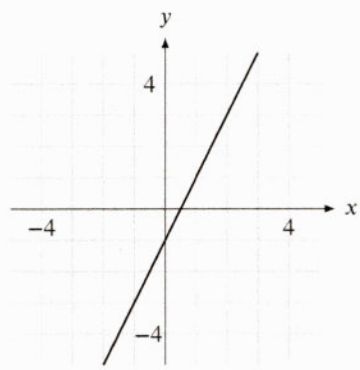
Notes

You could do this problem by plugging in each choice and looking for the one that does not produce a valid inequality, but this method wastes time and is potentially confusing, especially since this inequality is very easy to solve.

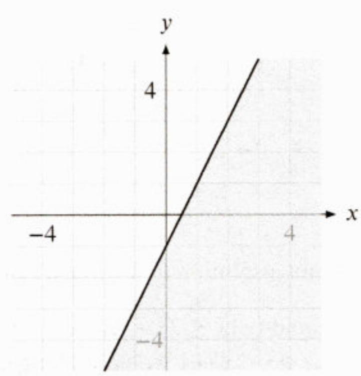
Graphing Linear Inequalities

Let's extend the concept of inequalities into two dimensions (with two variables, x and y) by considering what it means to have a linear inequality, and what it means to be a solution to the linear inequality.

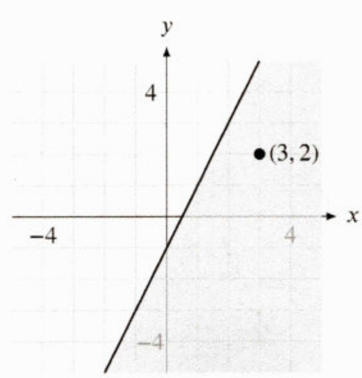
Let's consider the linear inequality (in Slope-Intercept Form) $y \leq 2x - 1$. We already know that if this were the linear equation $y = 2x - 1$, then the solution set could be represented with a line on the xy-plane that shows all possible pairs of x- and y-values that satisfy the equation.



In the inequality, all of these points where the y-value is equal to $2x - 1$ are still solutions to the inequality because it includes the values of y that are **equal** to $2x - 1$, but they are not the only solutions to the inequality. Any point that is **below** a point on this line is also a solution to the system because the y-values at these points are **less than** the value of $2x - 1$ for any given x-value, so the solution region is represented by the entire shaded region under the line including the line itself.



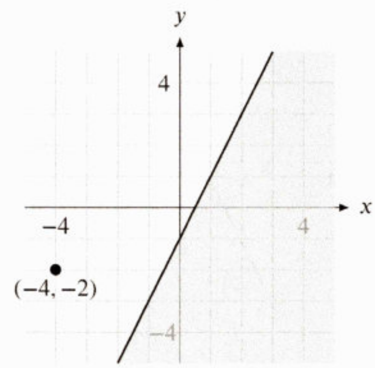
If we pick any point on the line or in the shaded region, we can plug the x- and y-values into the inequality to verify that it is still true. Let's use the point $(3, 2)$, which is located in the shaded region.



Let's plug in 3 for x and 2 for y in the linear inequality.

$$
\begin{aligned}
&y \le 2x - 1\\
&(2) \le 2(3) - 1\\
&2 \le 6 - 1\\
\checkmark\quad &2 \le 5
\end{aligned}
$$

It is true that $2 \le 5$, so the point $(3, 2)$ is a solution to the inequality. But what if we choose a point outside of the shaded region (and not on the line)? Let's use the point $(-4, -2)$.

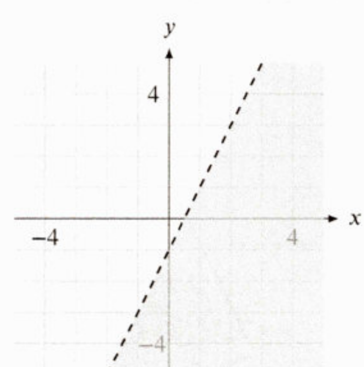


Let's plug in -4 for x and -2 for y in the linear inequality (note that we are simply plugging in negative values and not multiplying both sides by a negative, so we don't have to touch the inequality sign).

$$
\begin{aligned}
&y \le 2x - 1\\
&(-2) \le 2(-4) - 1\\
&-2 \le -8 - 1\\
\times\quad &-2 \le -9\\
&-2 \nleq -9
\end{aligned}
$$

Since -2 is not less than or equal to -9, this point is not a solution to the system of inequalities.

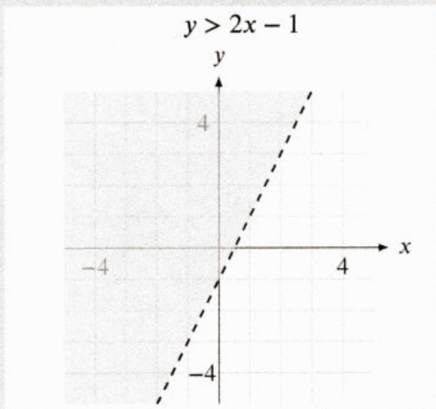
If the inequality were not a "less than or equal to" inequality ($y \le 2x - 1$), but instead were just a "less than" inequality ($y < 2x - 1$), the only difference would be that the line $y = 2x - 1$ would no longer be included in the solution set (because the y-values on the line are not strictly less than $2x - 1$ for any given value of x), so a dashed line instead of a solid line would be used to indicate the border of the region.

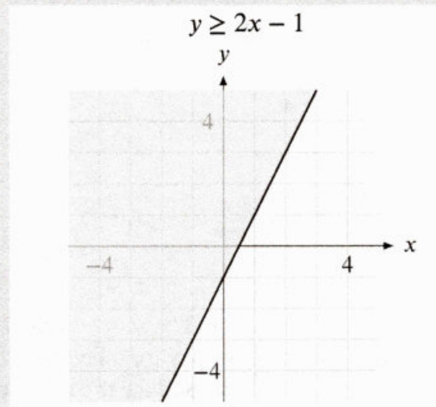


Graphing Linear Inequalities

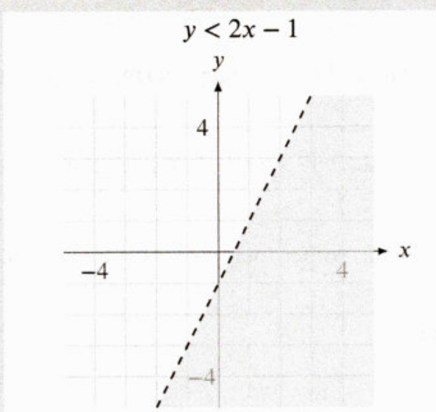
When the sign in the inequality is $\leq$ or $\geq$, you will graph the line as a solid line because the points on the line are included in the solutions to the inequality, but if the symbol is $<$ or $>$, you will use a dashed line to graph the line because the points on the line will not satisfy the inequality and therefore are excluded from the solution set.

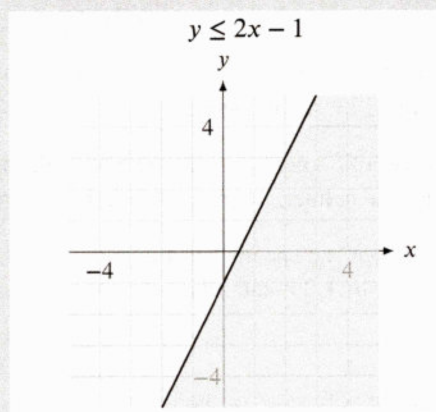
When we graph any linear inequality in Slope-Intercept Form with a greater-than or greater-than-or-equal-to sign, we shade above the line because any value of y that is above the y-values on the line is greater than the value of the y-coordinate of the point on the line below it, and thus will satisfy the inequality.





When we graph any linear inequality in Slope-Intercept Form with a less-than or less-than-or-equal-to sign, we shade below the line because any value of y that is below the y-values on the line will satisfy the inequality.





Most of the time, graphing is reserved for Systems of Linear Inequalities problems, which will be discussed later in the chapter. For now, it is sufficient that we know what it means for a point to be a solution to a linear inequality, and for many of the inequality problems, all we have to do is plug in values to check if the x- and y-values satisfy the inequality.

SkillDrill 5.1-3

Directions: For each inequality, see if each of the three following points is a solution to the inequality. List the Roman numerals of the points that are solutions to each inequality.

I. $(0, -5)$

II. $(1, 1)$

III. $(2, 4)$

1. $y < x$ **2.** $y \leq 2x$ **3.** $y < 2x - 5$ **4.** $y \geq 3x - 2$

Example 5.1-3

3 1600.io 5.1

Gretel rented an electric scooter. The scooter rental cost \$4 per hour, and she had to also pay for a helmet that costs \$8. Gretel spent more than \$42 for the rental and the helmet. If the scooter was available for only a whole number of hours, what was the minimum number of hours that Gretel could have rented the scooter?

Solution

1. Start by writing a linear expression for the total cost of Gretel's rental in Slope-Intercept Form, but instead of writing an equation, we will form an inequality.
2. There is a cost of \$4 per hour, so if she rents the scooter for h hours, she will spend $4h$ dollars plus the initial one-time fee of \$8 for the helmet. Therefore, the total cost of her rental was $4h + 8$.
3. The total amount she spends has a boundary; she spent more than \$42, so her total cost must be greater than \$42. Set the expression $4h + 8$ greater than 42 in the inequality.

$$4h + 8 > 42$$

4. Subtract 8 from both sides of the inequality.

$$4h > 34$$

5. Divide both sides of the inequality by 4 in order to solve for h.

$$4h > 34$$
$$h > \frac{34}{4}$$
$$h > \frac{17}{2}$$

6. Since $\frac{17}{2}$ is 8.5 and Gretel had to have rented the scooter for a whole number of hours, the least number of hours she could have rented the scooter that is still greater than 8.5 is 9.

7. The answer is 9.

Linear Inequalities can be written in Standard Form as well, and the math works the same (though graphing would require converting the inequality into Slope-Intercept Form, but graphing is rarely necessary). Simply plug in any given values and check if the points satisfy the inequality.

Example 5.1-4

4 1600.io 5.1

Which of the following ordered pairs (x, y) satisfies the inequality $6x - 2y < 8$?

I. $(0, -4)$

II. $(1, 1)$

III. $(2, 2)$

A) I only

B) II only

C) I and II only

D) II and III only

Solution

1. Divide both sides of the inequality by 2 (because 2 is the greatest common factor of all of the coefficients) in order to simplify the inequality and make it easier to plug in points and evaluate the results (this step is not essential, but it simplifies the arithmetic to come, and why shouldn't you make life a little easier for yourself?).

$$6x - 2y < 8$$
$$\frac{1}{2}(6x - 2y) < \frac{1}{2}(8)$$
$$3x - y < 4$$

2. For each of the given points, plug in the x-coordinate for x and the y-coordinate for y in the simplified form of the inequality in order to check if the resulting inequality is true (which means the point is in the solution set).

3. Start with point I, $(0, -4)$: plug in 0 for x and -4 for y.

$$\begin{aligned} 3x - y &< 4 \\ 3(0) - (-4) &< 4 \\ 0 + 4 &< 4 \\ \text{✗} \quad 4 &< 4 \\ 4 &\not< 4 \end{aligned}$$

Since 4 is not less than 4, this point does not satisfy the inequality. Eliminate choices A and C, which include point I.

4. By process of elimination, point II, $(1, 1)$ will satisfy the inequality because it is included in both of the remaining answers, but if you want to be thorough, plug in 1 for x and 1 for y.

$$\begin{aligned} 3x - y &< 4 \\ 3(1) - (1) &< 4 \\ 3 - 1 &< 4 \\ \checkmark \quad 2 &< 4 \end{aligned}$$

Point II satisfies the inequality because 2 is less than 4.

5. Check if point III, $(2, 2)$ satisfies the inequality by plugging in 2 for x and 2 for y.

$$\begin{aligned} 3x - y &< 4 \\ 3(2) - (2) &< 8 \\ 6 - 2 &< 4 \\ \text{✗} \quad 4 &< 4 \\ 4 &\not< 4 \end{aligned}$$

Since 4 is not less than 4, this point does not satisfy the inequality, so eliminate choice D, which includes point III.

6. The answer is B.

Notes

The points can simply be plugged into the inequality as provided (skipping Step 1), but here, as is sometimes the case with substitute-and-test problems, the inequality (or an equality, if that's what's being evaluated) can be simplified. While this adds a step up front (increasing the time needed and adding an opportunity for an error to be made), this investment is repaid with interest as it simplifies the substitution/evaluation procedure due to smaller (or no) coefficients and constants, making the method shown in the solution above more rapid and less error-prone.

When dealing with linear inequalities in Standard Form, sometimes you have to handle maximization or minimization problems. Remember a few chapters ago when we said that Standard Form linear equations can be though of as striking a balance between two variable quantities? The same holds true here: when two variable terms add up to a constant value, if one of those terms increases in value, the other term must decrease to compensate so the sum remains the same. In the following example, this means that in order to *maximize* one variable, we must *minimize* the other variable.

Example 5.1-5

5 1600.io 5.1

$$C = 20x + 6y$$

The formula above gives the monthly cost C, in dollars, of operating a printer when the technician works a total of x hours and when y reams of paper are used. If, in a particular month, it cost no more than \$3,800 to operate the printer and at least 100 reams of paper were used, what is the maximum number of hours the technician could have worked?

A) 160

B) 300

C) 1,800

D) 3,200

Solution

1. Start by rewriting the given equation as an inequality. The total cost is no more than (less than or equal to) \$3,800, so set the expression $20x + 6y$ less than or equal to 3,800.

$$20x + 6y \leq 3{,}800$$

2. Since we want to maximize the number of hours the technician worked, we need to minimize the number of reams of paper used (so that the cost of paper will be as low as possible, meaning that the maximum portion of the budget will go towards the technician's pay, which indicates that the technician worked as many hours as possible). Since they used at least 100 reams, then 100 is the lowest possible number of reams of paper in order to maximize the number of hours. Substitute 100 for y.

$$\begin{aligned} 20x + 6y &\leq 3{,}800 \\ 20x + 6(100) &\leq 3{,}800 \\ 20x + 600 &\leq 3{,}800 \\ 20x &\leq 3{,}200 \\ x &\leq 160 \end{aligned}$$

3. If $x \leq 160$, the maximum value x can have is 160, so the answer is A.

Be aware that in real world problems, the limit of an inequality is not necessarily the answer due to additional constraints, such as a requirement that the result must be a counting number (a non-negative integer). For example, if someone is buying b books that cost \$20 each, and they can spend at most \$75, then the inequality that represents the relationship is $20b \leq 75$. Solving for b, we see that $b \leq 3.75$, so as long as this person buys less than 3.75 books they will come in under budget. However, this person cannot buy three quarters of a book, so this person is limited to only buying 3 books. Be on the alert for problems that have an additional (sometimes implied but not directly stated) constraint that informs us that the answer must be an integer.

Section 5.1 Suggested Problems from Real Tests 🏆 Hall of Fame

- Test 1-C-11
- Test 1-C-28
- Test 1-C-32 🏆
- Test 2-NC-8 🏆
- Test 2-C-12 🏆
- Test 4-C-16
- Test 4-C-19
- Test 5-NC-16
- Test 5-C-21 🏆
- Test 6-C-5
- Test 7-C-5
- Test 8-C-20
- Test 10-C-31
- Apr 2017-NC-2
- Apr 2017-C-32
- May 2017-NC-8
- May 2017-C-1
- May 2017-C-13
- May 2018-C-17
- Mar 2019-NC-3
- Mar 2019-C-35
- Apr 2019-C-26
- Mar 2020-NC-4
- Mar 2020-C-15
- Oct 2020-C-13

Section 5.1 Practice Problems

1 1600.io 5.1

Fred harvests corn at a constant rate of 1.2 acres per day. He harvested 3 acres so far this month and plans to spend d days harvesting for the rest of the month. If Fred wants to harvest at least 10 acres of corn this month, which of the following inequalities best represents this situation?

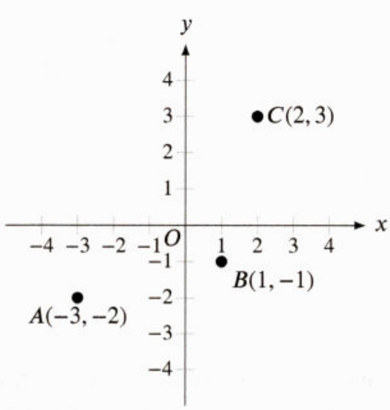
A) $3d + 1.2 \geq 10$

B) $1.2d + 3 \geq 10$

C) $1.2d - 3 \geq 10$

D) $1.2d \geq 10$

2 1600.io 5.1



The coordinates of points A, B, and C are shown in the xy-plane above. For which of the following inequalities will each of the points A, B, and C be contained in the solution region?

A) $x > -3$

B) $y > x - 2$

C) $y < x + 2$

D) $y < x + 1$

3 1600.io 5.1

$$C = 5x + y$$

The formula above gives the weekly cost C, in dollars, of operating a lemon press when the lemonade maker works a total of x hours and when y lemons are used. If, in a particular week, it cost at least \$400 to operate the lemon press and no more than 100 lemons were used, what is the least number of hours the lemonade maker could have worked?

A) 40

B) 60

C) 100

D) 400

4 1600.io 5.1

If $5p - 7 \geq 4$, what is the least possible value of $5p + 1$?

A) $\frac{11}{5}$

B) 5

C) 11

D) 12

5.2 Compound Linear Inequalities

A **Compound Linear Inequality** is when a linear expression is bounded on both sides by both an upper and lower limit. The value of the expression must be between two values. For example, the compound inequality $3 \leq x < 7$ means that x is greater than or equal to 3, but x is also less than 7. By convention, the lower bound is written on the left, and the upper bound is written on the right, so only the less-than and less-than-or-equal-to signs are used.

When working with compound inequalities, it can be tempting to split the inequality up into two separate inequalities, but most of the time, it is easier to just work with what you are given. Doing algebraic work on compound inequalities is simple: just apply the same operations to all three parts of the inequality. Extending the principle that applies to simpler linear inequalities that we discussed earlier, if you multiply or divide by a negative number, you have to remember to flip **both** inequality signs; when you do this, re-write the inequality in the conventional orientation, with the lower bound on the left, thus using only less-than and less-than-or-equal-to signs.

For example, let's solve for x in the compound inequality $2 < -3x + 2 < 8$.

$2 < -3x + 2 < 8$	
$0 < -3x < 6$	Subtract 2 from all three parts of the compound inequality.
$\frac{0}{-3} > x > \frac{6}{-3}$	Divide all three parts by -3 and flip the inequality signs.
$0 > x > -2$	
$-2 < x < 0$	Rewrite the compound inequality in traditional orientation with only less-than signs.

SkillDrill 5.2-1

Directions: Solve each compound inequality for x.

1. $2 < 5x + 2 < 10$ **2.** $-6 < 4x - 3 < 9$ **3.** $0 < -7x - 1 < 6$ **4.** $-4 < 3 - 2x < 7$

Example 5.2-1

I 5.2

Tickets for a school's winter concert cost $2.50 for students and $5 for adults. If Harold spends at least $15 but no more than $24 on x students tickets and 2 adult tickets, what is one possible value of x ?

Solution

1. Start by writing an expression to represent the total cost of tickets in Standard Form. Let's use s for the number of student tickets (even though they tell us to use x later), and let's use a for the number of adult tickets. If student tickets cost \$2.50 each, and Harold buys s student tickets, then the cost of those tickets is $2.5s$. Since adult tickets are \$5 each and Harold buys a of them, the price of those tickets is $5a$. Therefore, the total amount that Harold spends on tickets is $2.5s + 5a$.

2. The amount Harold spends must be between \$15 and \$24, inclusive, so we should set the expression $2.5s + 5a$ greater than or equal to 15, but also less than or equal to 24.

$$15 \le 2.5s + 5a \le 24$$

3. Since the problem tells us that Harold buys x student tickets, replace s with x. Since we know that Harold buys 2 adult tickets, replace a with 2.

$$\begin{aligned} 15 &\le 2.5s + 5a &&\le 24 \\ 15 &\le 2.5x + 5(2) &&\le 24 \\ 15 &\le 2.5x + 10 &&\le 24 \end{aligned}$$

4. Subtract 10 from all three parts of the inequality to isolate the x term.

$$\begin{aligned} 15 &\le 2.5x + 10 &&\le 24 \\ 5 &\le 2.5x &&\le 14 \end{aligned}$$

5. Divide all three parts of the inequality by 2.5 (if you like working without a calculator, you can convert 2.5 into the fraction $\frac{5}{2}$, then multiply all three parts by $\frac{2}{5}$ in order to isolate and solve for x).

$$\begin{aligned} 5 &\le 2.5x \le 14 \\ 2 &\le x \quad\ \le 5.6 \end{aligned}$$

or

$$\begin{aligned} 5 &\le 2.5x &&\le 14 \\ 5 &\le \frac{5}{2}x &&\le 14 \\ \frac{2}{\cancel{5}}(\cancel{5}) &\le \frac{\cancel{2}}{\cancel{5}}\left(\frac{\cancel{5}}{\cancel{2}}x\right) &&\le \frac{2}{5}(14) \\ 2 &\le x &&\le \frac{28}{5} \end{aligned}$$

6. For your answer, you can fill in any whole number (integer) value between 2 and 5.6, inclusive (the valid answers must be whole numbers because x represents a number of tickets, and you can't buy part of a ticket). A number like 3 is a safe choice if you are unsure whether the boundaries should be included or not, though 2 is still a valid answer because the inequality indicates that x can be greater than *or equal to* 2.

Section 5.2 Suggested Problems from Real Tests

🏆 Hall of Fame

- Test 1-C-31
- Test 3-C-31
- Test 6-C-18 🏆
- Test 6-C-19 🏆
- Test 7-NC-14 🏆
- May 2019 (US)-C-34 🏆
- Oct 2019-C-21
- Oct 2020-C-4

Section 5.2 Practice Problems

1 5.2

Jenny can shuck at least 20 dozen oysters per hour and at most 24 dozen oysters per hour. Based on this information, what is a possible amount of time, in hours, that it could take Jenny to shuck 120 dozen oysters?

2 5.2

A certain rhinoceros weighs 100 pounds at birth and gains more than 3 but less than 5 pounds per day during its first year. Which of the following inequalities represents all possible weights w, in pounds, for the rhinoceros 180 days after birth?

A) $640 < w < 1{,}000$

B) $540 < w < 900$

C) $300 < w < 500$

D) $280 < w < 900$

5.3 Systems of Linear Inequalities

We can write systems of linear inequalities just like we write systems of linear equations. The only difference is we also have to make sure that we use the correct inequality signs in all of the inequalities that we write.

Example 5.3-1

1 5.3

A maid service is buying window cleaner and floor polish from its supplier. The supplier will deliver no more than 450 pounds in a shipment. Each container of window cleaner weighs 5.65 pounds, and each container of floor polish weighs 8.4 pounds. The service wants to buy at least three times as many containers of floor polish as containers of window cleaner. Let w represent the number of containers of window cleaner, and let f represent the number of containers of floor polish, where w and f are non-negative integers. Which of the following systems of inequalities best represents this situation?

A) $8.4w + 5.65f \leq 450$
$f \geq 3w$

B) $8.4w + 5.65f \leq 450$
$3f \geq w$

C) $5.65w + 8.4f \leq 450$
$f \geq 3w$

D) $5.65w + 8.4f \leq 450$
$3f \geq w$

Solution

1. We need to write one inequality that describes the total number of pounds of product bought. We know that w is the number of containers of window cleaner, and each one weighs 5.65 pounds, so the total weight of window cleaner is $5.65w$. We know that f is the number of containers of floor polish, and each one weighs 8.4 pounds, so the total weight of floor polish is $8.4f$. Therefore, the total weight of the supplies is $5.65w + 8.4f$. The total weight must be no more than (less than or equal to) 450 pounds, so this expression should be set less than or equal to 450 to form the first linear inequality in the system.

   $$5.65w + 8.4f \leq 450$$

   Eliminate choices A and B, leaving only choices C and D.

2. This step is the crucial step in this problem, and it requires careful translation. We are told that the service wants to buy at least three times as many containers of floor polish as containers of window cleaner. The most common mistake made on questions like this is mistranslating this part into the wrong inequality. **Don't just try to write this relationship based on the order that the information appears in the sentence**. Because the number 3 is presented close to the words "floor polish," you might think that f should be multiplied by 3. However, this is incorrect.

   Visualize the situation before you write the inequality. For example, if you had one container of window cleaner, you would need at least 3 containers of floor polish. The number of floor polishes is larger, and it is three times the number of window cleaners, so therefore we would have to multiply the number of window cleaners by 3 in order to match the number of floor polishes. We have now rephrased the statement in a way that reflects its true meaning and helps us write the inequality: *the number of floor polish containers* ***is at least*** *(greater than or equal to) 3 times the number of window cleaner containers.*

   $$f \geq 3w$$

3. The set of inequalities that describe the given circumstances is the following:

   $$5.65w + 8.4f \leq 450$$
   $$f \geq 3w$$

4. The answer is C.

Notes

One straightforward way to approach the construction of inequalities (or equations) in scenarios like in Step 2 of the solution above is to start by setting up a proportion (they are covered in a later chapter); using proportions is really easy and you don't have to rely on intuition or plugging in actual values to figure out the solution. The given constraint that there must be at least three times as many containers of floor polish as containers of window cleaner simply means the ratio of the number of floor polish containers to the number of window cleaner containers must be at least 3 to 1. The phrase "at least" tells us to use a greater-than-or-equal-to sign.

$$\frac{\text{floor polish}}{\text{window cleaner}} \geq \frac{3}{1}$$

$$\frac{f}{w} \geq 3$$

Now, we just need to do some simple algebra to get the equation into the form of the answer choices. Multiply both sides of the equation by w.

$$f \geq 3w$$

Solving systems of linear inequalities can be confusing. Luckily, some problems gives you points as choices, and a lot of times, the easiest way to handle those problems is by plugging the answer choices into the two inequalities and seeing which points satisfy both inequalities. If a point only makes one of the inequalities true, then it is NOT a solution to the system.

Example 5.3-2

2 1600.io 5.3

$$y \geq -x + 3$$
$$y > 2x - 3$$

In the xy-plane, point A is contained in the graph of the solution set of the system of inequalities above. Which of the following could be the coordinates of point A ?

A) $(2, 1)$

B) $(3, 0)$

C) $(3, 3)$

D) $(3, 5)$

Solution

1. Try plugging the x-coordinate and y-coordinate of each answer choice into both inequalities to find the point that satisfies both inequalities.

2. Start with choice A, (2, 1); plug in 2 for x and 1 for y.

$$y \geq -x + 3$$
$$y > 2x - 3$$
$$\Downarrow$$
$$1 \geq -(2) + 3$$
$$1 > 2(2) - 3$$
$$\Downarrow$$
$$1 \geq -2 + 3$$
$$1 > 4 - 3$$
$$\Downarrow$$
$$1 \geq 1$$
$$1 > 1$$
$$\Downarrow$$
$$\checkmark \quad \mathbf{1 \geq 1}$$
$$\text{✗} \quad \mathbf{1 \not> 1}$$

Even though the first inequality is true, the second inequality is not because 1 is not greater than 1. Eliminate choice A.

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Building-block sub-skills are thoroughly exercised through our innovative **SkillDrill™ mini-problem sets** that build and reinforce crucial elemental math skills, empowering students with the tools they need to confidently approach any test problem

Multiple solution methods are exhaustively discussed, thereby providing students with the confidence that they can choose from among several approaches when solving

An extensive appendix **cross-references every problem on every released SAT** with the corresponding topic section in the book, allowing students to rapidly identify and learn the skills needed to solve any test problem they may encounter

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